IV. – STATICS AND DYNAMICS OF DEFORMABLE MEDIA.

48. Deformable medium. Natural state and deformed state. – The theories of the deformable line and the deformable surface that we discussed lead, in a very natural manner, to envisioning a more general deformable medium than the one that is habitually considered in the theory of elasticity, and seems, to us, to achieve the goal that was pursued by LORD KELVIN and HELMHOLTZ in the theories of light and magnetism.

Consider a space (M_0) that is described by a point M_0, whose coordinates x_0, y_0, z_0 with respect to three fixed rectangular axes Ox, Oy, Oz. We may regard these coordinates as functions of the three parameters ρ_1, ρ_2, ρ_3, which are chosen in an arbitrary manner; however, to simplify, we suppose that these coordinates are taken to be independent variables. Affix a tri-rectangular triad to each point M_0 of the space (M_0), whose axes $M_0x_0', M_0y_0', M_0z_0'$ have direction cosines $\alpha_0, \alpha_0', \alpha_0''; \beta_0, \beta_0', \beta_0''; \gamma_0, \gamma_0', \gamma_0''$ with respect to the axes Ox, Oy, Oz, and which are functions of the independent variables x_0, y_0, z_0.

The continuous three-dimensional set of all such triads $M_0x_0'y_0'z_0'$ will be what we call a *deformable medium.*

Give a displacement M_0M to a point M_0; let x, y, z be the coordinates of the point M with respect to the fixed triad $Oxyz$. In addition, endow the triad $M_0x_0'y_0'z_0'$ with a rotation that will ultimately bring its axes into agreement with those of a triad $Mx'y'z'$ that we affix to the point M. We define that rotation by giving the direction cosines $\alpha, \alpha', \alpha''; \beta, \beta', \beta''; \gamma, \gamma', \gamma''$ of the axes Mx', My', Mz' with respect to the fixed axes.

The continuous three-dimensional set of all such triads $Mx'y'z'$ will be what we call the *deformed state* of the deformable medium under consideration, which will be called the *natural state* in its original state.

49. Kinematical elements that relate to the states of the deformable medium. – For ease of notation, we sometimes introduce the letters ρ_1, ρ_2, ρ_3, instead of x_0, y_0, z_0 in the sequel, as expressed by the formulas:

$$x_0 = \rho_1, \qquad y_0 = \rho_2, \qquad z_0 = \rho_2,$$

so it will suffice to keep them in mind.

Denote the components of the velocity of the origin M_0 of the axes $M_0x_0', M_0y_0', M_0z_0'$ with respect to these axes by $\xi_i^{(0)}, \eta_i^{(0)}, \varsigma_i^{(0)}$ when ρ_i alone varies and plays the role of time. Likewise, let $p_i^{(0)}, q_i^{(0)}, r_i^{(0)}$ be the projections on these axes of the instantaneous rotation of the triad $M_0x_0'y_0'z_0'$ relative to the parameter ρ_i. We denote the analogous quantities for the triad $Mx'y'z'$ by ξ_i, η_i, ζ_i, and p_i, q_i, r_i when they, like the triad $M_0x_0'y_0'z_0'$, are referred to the fixed triad $Oxyz$.

The elements that we introduced before are calculated in the usual fashion; in particular, one has:

$$(43)\quad \begin{cases} \xi_i = \alpha \dfrac{\partial x}{\partial \rho_i} + \alpha' \dfrac{\partial y}{\partial \rho_i} + \alpha'' \dfrac{\partial z}{\partial \rho_i}, \\ \eta_i = \beta \dfrac{\partial x}{\partial \rho_i} + \beta' \dfrac{\partial y}{\partial \rho_i} + \beta'' \dfrac{\partial z}{\partial \rho_i}, \\ \varsigma_i = \gamma \dfrac{\partial x}{\partial \rho_i} + \gamma' \dfrac{\partial y}{\partial \rho_i} + \gamma'' \dfrac{\partial z}{\partial \rho_i}, \end{cases} \qquad (44)\quad \begin{cases} p_i = \sum \gamma \dfrac{\partial \beta}{\partial \rho_i} = -\sum \beta \dfrac{\partial \gamma}{\partial \rho_i}, \\ q_i = \sum \alpha \dfrac{\partial \gamma}{\partial \rho_i} = -\sum \gamma \dfrac{\partial \alpha}{\partial \rho_i}, \\ r_i = \sum \beta \dfrac{\partial \alpha}{\partial \rho_i} = -\sum \alpha \dfrac{\partial \beta}{\partial \rho_i}. \end{cases}$$

The linear element of the deformed medium (*M*), when referred to the independent variables x_0, y_0, z_0, is defined by the formula:

$$\begin{aligned} ds^2 &= (1+2\varepsilon_1)dx_0^2 + (1+2\varepsilon_2)dy_0^2 + (1+2\varepsilon_3)dz_0^2 \\ &+ 2\gamma_1 dy_0 dz_0 + +2\gamma_2 dz_0 dx_0 + +2\gamma_3 dx_0 dy_0, \end{aligned}$$

in which ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 are calculated by the following double formulas:

$$(45)\quad \begin{cases} \varepsilon_1 = \frac{1}{2}\left[\left(\dfrac{\partial x}{\partial x_0}\right)^2 + \left(\dfrac{\partial y}{\partial x_0}\right)^2 + \left(\dfrac{\partial z}{\partial x_0}\right)^2 - 1\right] = \frac{1}{2}(\xi_1^2 + \eta_1^2 + \varsigma_1^2 - 1), \\ \varepsilon_2 = \frac{1}{2}\left[\left(\dfrac{\partial x}{\partial y_0}\right)^2 + \left(\dfrac{\partial y}{\partial y_0}\right)^2 + \left(\dfrac{\partial z}{\partial y_0}\right)^2 - 1\right] = \frac{1}{2}(\xi_2^2 + \eta_2^2 + \varsigma_2^2 - 1), \\ \varepsilon_3 = \frac{1}{2}\left[\left(\dfrac{\partial x}{\partial z_0}\right)^2 + \left(\dfrac{\partial y}{\partial z_0}\right)^2 + \left(\dfrac{\partial z}{\partial z_0}\right)^2 - 1\right] = \frac{1}{2}(\xi_3^2 + \eta_3^2 + \varsigma_3^2 - 1), \\ \gamma_1 = \dfrac{\partial x}{\partial y_0}\dfrac{\partial x}{\partial z_0} + \dfrac{\partial y}{\partial y_0}\dfrac{\partial y}{\partial z_0} + \dfrac{\partial z}{\partial y_0}\dfrac{\partial z}{\partial z_0} = \xi_2\xi_3 + \eta_2\eta_3 + \varsigma_2\varsigma_3, \\ \gamma_2 = \dfrac{\partial x}{\partial z_0}\dfrac{\partial x}{\partial x_0} + \dfrac{\partial y}{\partial z_0}\dfrac{\partial y}{\partial x_0} + \dfrac{\partial z}{\partial z_0}\dfrac{\partial z}{\partial x_0} = \xi_3\xi_1 + \eta_3\eta_1 + \varsigma_3\varsigma_1, \\ \gamma_3 = \dfrac{\partial x}{\partial x_0}\dfrac{\partial x}{\partial y_0} + \dfrac{\partial y}{\partial x_0}\dfrac{\partial y}{\partial y_0} + \dfrac{\partial z}{\partial x_0}\dfrac{\partial z}{\partial y_0} = \xi_1\xi_2 + \eta_1\eta_2 + \varsigma_1\varsigma_2. \end{cases}$$

Denote the projections of the segment *OM* onto the axes Mx', My', Mz' by x', y', z', in such a way that the coordinates of the *fixed point O* with respect to these axes become $-x', -y', -z'$. We have the following well-known formulas:

$$(46)\quad \xi_i - \frac{\partial x'}{\partial \rho_i} - qr' + ry' = 0, \quad \eta_i - \frac{\partial y'}{\partial \rho_i} - rx' + pz' = 0, \quad \varsigma_i - \frac{\partial z'}{\partial \rho_i} - py' + qx' = 0,$$

which gives new expressions for ξ_i, η_i, ζ_i .

50. Expressions for the variations of the velocities of translation and rotation of the triad relative to the deformed state. – Suppose that one endows each of the triads of the deformed state with an infinitely small displacement that may vary in a continuous fashion with these triads. Denote the variations of $x, y, z;$ $x', y', z';$ $\alpha, \alpha', \cdots, \gamma''$ by δx, δy, $\delta z;$ $\delta x', \delta y', \delta z';$ $\delta\alpha, \delta\alpha', \cdots, \delta\gamma''$, respectively. The variations $\delta\alpha, \delta\alpha', \cdots, \delta\gamma''$ are expressed by formulas such as the following:

$$\delta\alpha = \beta\delta K' - \gamma\delta J', \tag{47}$$

by means of the three auxiliary functions $\delta I', \delta J', \delta K'$, which are the components of well-known instantaneous rotation that is attached to the infinitely small displacement in question with respect to Mx', My', Mz'. The variations δx, δy, δz are the projections of the infinitely small displacement felt by the point M onto Ox, Oy, Oz. The projections $\delta' x, \delta' y, \delta' z$ of this displacement onto Mx', My', Mz' are deduced immediately and have the values:

$$\delta' x = \delta x' + z'\delta J' - y'\delta K', \quad \delta' y = \delta y' + x'\delta K' - z'\delta I', \quad \delta' z = \delta z' + y'\delta I' - x'\delta J'. \tag{48}$$

We propose to determine the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$, δp_i, δq_i, δr_i felt by ξ_i, η_i, ζ_i, p_i, q_i, r_i, respectively. From the formulas (44), we have:

$$\delta p_i = \sum\left(\frac{\partial\beta}{\partial\rho_i}\delta\gamma + \gamma\frac{\partial\delta\beta}{\partial\rho_i}\right),$$
$$\delta q_i = \sum\left(\frac{\partial\gamma}{\partial\rho_i}\delta\alpha + \alpha\frac{\partial\delta\gamma}{\partial\rho_i}\right),$$
$$\delta r_i = \sum\left(\frac{\partial\alpha}{\partial\rho_i}\delta\beta + \beta\frac{\partial\delta\alpha}{\partial\rho_i}\right).$$

Replace $\delta\alpha$ by its value $\beta\delta K' - \gamma\delta J'$, and $\delta\alpha', \cdots, \delta\gamma''$ with their analogous values; we obtain:

$$\delta p_i = \frac{\partial\delta I'}{\partial\rho_i} + q_i\delta K' - r_i\delta J', \quad \delta q_i = \frac{\partial\delta J'}{\partial\rho_i} + r_i\delta I' - p_i\delta K', \quad \delta r_i = \frac{\partial\delta K'}{\partial\rho_i} + p_i\delta J' - q_i\delta I'. \tag{49}$$

Similarly, formulas (46) give us three formulas, the first of which is:

$$\delta\xi_i = \frac{\partial\delta\delta'}{\partial\rho_i} + q_i\delta z' - r_i\delta y' + z'\delta q_i - y'\delta r_i.$$

Replace δp_i, δq_i, δr_i with their values as given by formulas (49); we obtain:

(50)
$$\begin{cases} \delta\xi_i = \eta_i\delta K' - \varsigma_i\delta J' + \dfrac{\partial\delta' x}{\partial\rho_i} + q_i\delta' x - r_i\delta' y, \\ \delta\eta_i = \varsigma_i\delta I' - \xi_i\delta K' + \dfrac{\partial\delta' y}{\partial\rho_i} + r_i\delta' y - p_i\delta' z, \\ \delta\varsigma_i = \xi_i\delta J' - \eta_i\delta I' + \dfrac{\partial\delta' z}{\partial\rho_i} + p_i\delta' z - q_i\delta' x, \end{cases}$$

in which we have introduced the three symbols $\delta' x, \delta' y, \delta' z$ defined by formulas (48).

51. Euclidian action of deformation on a deformable medium. – We preserve the notations of sec. **49** and introduce the known quantity, Δ, which is defined by the formula:

$$\Delta = \frac{D(x,y,z)}{D(x_0,y_0,z_0)} = \begin{vmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0} \\ \dfrac{\partial y}{\partial x_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial y}{\partial z_0} \\ \dfrac{\partial z}{\partial x_0} & \dfrac{\partial z}{\partial y_0} & \dfrac{\partial z}{\partial z_0} \end{vmatrix},$$

and whose square, which is formed by the rule for multiplication of determinants, is expressed as a function of ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 by the formula:

$$\Delta^2 = \begin{vmatrix} 1+2\varepsilon_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & 1+2\varepsilon_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & 1+2\varepsilon_3 \end{vmatrix}.$$

Consider a function W of *two infinitely close positions* of the triad $Mx'y'z'$, i.e., a function from x_0, y_0, z_0 to x, y, z, $\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha'', \beta'', \gamma''$, and their first derivatives with respect to x_0, y_0, z_0. We propose to determine the form that W must take in order for the integral:

$$\iiint W dx_0 dy_0 dz_0,$$

when taken over an arbitrary portion of the space (M_0) to have null variation when one subjects the set of all triads of the deformable medium, taken in its deformed state, *to the same arbitrary infinitesimal transformation of the group of Euclidian displacements.*

By definition, this amounts to determining W in such a way that one has:

$$\delta W = 0,$$

when, on the one hand, the origin M of the triad $Mx'y'z'$ is subjected to an infinitely small displacement whose projections δx, δy, δz on the axes Ox, Oy, Oz are:

$$\text{(51)} \qquad \begin{cases} \delta x = (a_1 + \omega_2 z - \omega_3 y)\delta t, \\ \delta y = (a_2 + \omega_3 x - \omega_1 z)\delta t, \\ \delta z = (a_3 + \omega_1 y - \omega_2 x)\delta t, \end{cases}$$

where a_1, a_2, a_3, ω_1, ω_2, ω_3 are six arbitrary constants and δt is an infinitely small quantity that is independent of x_0, y_0, z_0, and when, on the other hand, the triad $Mx'y'z'$ is subjected to an infinitely small rotation whose components along the axes Ox, Oy, Oz are:

$$\omega_1 \delta t, \quad \omega_2 \delta t, \quad \omega_3 \delta t.$$

Observe that in the present case the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$; δp_i, δq_i, δr_i of the eighteen expressions ξ_i, η_i, ζ_i; p_i, q_i, r_i are null, since this results from the well-known theory of moving frames, and as we may, moreover, verify immediately by means of formulas (49) and (50) by replacing $\delta' x, \delta' y, \delta' z; \delta I', \delta J', \delta K'$ by their actual values. It results from this that we obtain a solution to the question by taking W to be an arbitrary function of x_0, y_0, z_0, and the eighteen expressions ξ_i, η_i, ζ_i; p_i, q_i, r_i. We shall now show that we thus obtain the general solution ([1]) of a problem that we now pose.

To that effect, we remark that the relations (44) permit us to express the first derivatives of the nine cosines $\alpha, \alpha', \cdots, \gamma''$ with respect to x_0, y_0, z_0 by means of these cosines and p_i, q_i, r_i using well-known formulas. On the other hand, formulas (43) permit us to think of expressing the nine cosines $\alpha, \alpha', \cdots, \gamma''$ by means of ξ_1, η_1, ζ_1, and the first derivatives of x, y, z with respect to x_0, or by means of ξ_2, η_2, ζ_2, and the first derivatives of x, y, z with respect to y_0, or, finally, by means of ξ_3, η_3, ζ_3, and the first derivatives of x, y, z with respect to z_0. Furthermore, it is useless in this case for us to make any hypothesis on the mode of solution because it is clear that we will not obtain a more general form than the one that we started with by supposing that the function W that we seek is an arbitrary function of x_0, y_0, z_0 and x, y, z, and their first derivatives with respect to x_0, y_0, z_0, and of ξ_i, η_i, ζ_i; p_i, q_i, r_i, which we indicate by using the notations $\rho_1 = x_0$, $\rho_2 = y_0$, $\rho_3 = z_0$, by writing:

$$W = W\left(\rho_i, x, y, z, \frac{\partial x}{\partial \rho_i}, \frac{\partial y}{\partial \rho_i}, \frac{\partial z}{\partial \rho_i}, \xi_i, \eta_i, \varsigma_i, p_i, q_i, r_i\right).$$

Since the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$; δp_i, δq_i, δr_i are non-null in the actual case one remarks that there is an instant, which we shall ultimately describe, for which we have, by virtue of formulas (51), the new form for W for any a_1, a_2, a_3, ω_1, ω_2, ω_3 :

[1] In all of what follows we suppose that *the medium is susceptible to all possible deformations*, so that, as a result *the deformed state may be taken absolutely arbitrarily.*

$$\frac{\partial W}{\partial x}\delta x+\frac{\partial W}{\partial y}\delta y+\frac{\partial W}{\partial z}\delta z+\sum\left(\frac{\partial W}{\partial \frac{\partial x}{\partial \rho_i}}\delta\frac{\partial x}{\partial \rho_i}+\frac{\partial W}{\partial \frac{\partial y}{\partial \rho_i}}\delta\frac{\partial y}{\partial \rho_i}+\frac{\partial W}{\partial \frac{\partial z}{\partial \rho_i}}\delta\frac{\partial z}{\partial \rho_i}\right)=0.$$

We replace δx, δy, δz with their values (51) and $\delta\frac{\partial x}{\partial \rho_i},\delta\frac{\partial y}{\partial \rho_i},\delta\frac{\partial z}{\partial \rho_i}$ with the values that one deduces by differentiation. We set the coefficients of a_1, a_2, a_3, ω_1, ω_2, ω_3; we obtain the following six conditions:

$$\frac{\partial W}{\partial x}=0,\qquad \frac{\partial W}{\partial y}=0,\qquad \frac{\partial W}{\partial z}=0,$$

$$\sum\left(\frac{\partial W}{\partial \frac{\partial y}{\partial \rho_i}}\frac{\partial z}{\partial \rho_i}-\frac{\partial W}{\partial \frac{\partial z}{\partial \rho_i}}\frac{\partial y}{\partial \rho_i}\right)=0,\qquad \sum\left(\frac{\partial W}{\partial \frac{\partial z}{\partial \rho_i}}\frac{\partial x}{\partial \rho_i}-\frac{\partial W}{\partial \frac{\partial x}{\partial \rho_i}}\frac{\partial z}{\partial \rho_i}\right)=0,$$

$$\sum\left(\frac{\partial W}{\partial \frac{\partial x}{\partial \rho_i}}\frac{\partial y}{\partial \rho_i}-\frac{\partial W}{\partial \frac{\partial y}{\partial \rho_i}}\frac{\partial x}{\partial \rho_i}\right)=0,$$

which are identities, if we assume that the expressions that figure in W have been reduced to the smallest number.

The first three show us, as one may easily foresee, that W is independent of x, y, z. The last three express that W depends on the first derivatives of x, y, z with respect to x_0, y_0, z_0 only by the intermediary of the quantities ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 that were defined by the formulas (45). Finally, we see that *the desired function W has the remarkable form:*

$$W(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i; p_i, q_i, r_i),$$

which is analogous to the one that we encountered before for the deformable line and the deformable surface.

If we multiply W by the volume element $dx_0dy_0dz_0$ of the space (M_0) then the product $Wdx_0dy_0dz_0$ so obtained is an invariant in the group of Euclidian displacements that is analogous to the volume element of the medium (M).

Just as the common value of the integrals:

$$\iiint_{S_0}|\Delta|\,dx_0dy_0dz_0,\qquad \iiint_S dxdydz,$$

taken over the interior of a surface S_0 of the medium (M_0) and the interior of the corresponding surface S of the medium (M), respectively, determines the *volume* of the domain bounded by the surface S. Likewise, if we associate, in the same spirit, the notion of the action for the passage from the natural state (M_0) to the deformed state (M) then we add the function W to the elements in the definition of a deformable medium, and we say that the integral:

$$\iiint_{S_0} W dx_0 dy_0 dz_0 ,$$

is the *action of deformation* for the interior of the surface S in the deformed medium.

On the other hand, we say that W is the *density* of the action of deformation *at a point* of the deformed medium when referred to the unit of volume of the undeformed medium, and that $\frac{W}{|\Delta|}$ is the density of that action at a point when referred to the unit of volume of the deformed medium.

52. The external force and moment. The external moment and effort. The effort and moment of deformation at a point of the deformed medium. – Consider an arbitrary variation of the action of deformation of the interior of a surface S in the medium (M), namely:

$$\delta \iiint_{S_0} W dx_0 dy_0 dz_0$$
$$= \iiint_{S_0} \sum \left(\frac{\partial W}{\partial \xi_i} \delta\xi_i + \frac{\partial W}{\partial \eta_i} \delta\eta_i + \frac{\partial W}{\partial \varsigma_i} \delta\varsigma_i + \frac{\partial W}{\partial p_i} \delta p_i + \frac{\partial W}{\partial q_i} \delta q_i + \frac{\partial W}{\partial r_i} \delta r_i \right) dx_0 dy_0 dz_0 .$$

By virtue of formulas (49) and (50) of sec. 50, we may write:

$$\begin{aligned}
\delta \iiint_{S_0} W dx_0 dy_0 dz_0 = \iiint_{S_0} \sum \Bigg\{ & \frac{\partial W}{\partial \xi_i} \left(\eta_i \delta K' - \varsigma_i \delta J' + \frac{\partial \delta' x}{\partial \rho_i} + q_i \delta' z - r_i \delta' y\right) \\
& + \frac{\partial W}{\partial \eta_i} \left(\varsigma_i \delta I' - \xi_i \delta K' + \frac{\partial \delta' y}{\partial \rho_i} + r_i \delta' x - p_i \delta' z\right) \\
& + \frac{\partial W}{\partial \varsigma_i} \left(\xi_i \delta J' - \eta_i \delta I' + \frac{\partial \delta' z}{\partial \rho_i} + p_i \delta' y - q_i \delta' x\right) \\
& + \frac{\partial W}{\partial p_i} \left(\frac{\partial \delta I'}{\partial \rho_i} + q_i \delta K' - r_i \delta J' \right) + \frac{\partial W}{\partial q_i} \left(\frac{\partial \delta J'}{\partial \rho_i} + r_i \delta I' - p_i \delta K' \right) \\
& + \frac{\partial W}{\partial r_i} \left(\frac{\partial \delta K'}{\partial \rho_i} + p_i \delta J' - q_i \delta I' \right) \Bigg\} dx_0 dy_0 dz_0 .
\end{aligned}$$

We apply the GREEN formula to the terms that explicitly refer to the derivative with respect to one of the variables ρ_1, ρ_2, ρ_3. If we let l_0, m_0, n_0 denote the direction cosines with respect to Ox, Oy, Oz of the exterior normal to the surface S_0 that bounds the medium before deformation and the area element of that surface by $d\sigma_0$ then this gives:

$$\delta\iiint_{S_0} W dx_0 dy_0 dz_0 = \iint_{S_0}\left\{\left(l_0\frac{\partial W}{\partial\xi_1}+m_0\frac{\partial W}{\partial\xi_2}+n_0\frac{\partial W}{\partial\xi_3}\right)\delta' x\right.$$
$$+\left(l_0\frac{\partial W}{\partial\eta_1}+m_0\frac{\partial W}{\partial\eta_2}+n_0\frac{\partial W}{\partial\eta_3}\right)\delta' y+\left(l_0\frac{\partial W}{\partial\varsigma_1}+m_0\frac{\partial W}{\partial\varsigma_2}+n_0\frac{\partial W}{\partial\varsigma_3}\right)\delta' z$$
$$+\left(l_0\frac{\partial W}{\partial p_1}+m_0\frac{\partial W}{\partial p_2}+n_0\frac{\partial W}{\partial p_3}\right)\delta I'+\left(l_0\frac{\partial W}{\partial q_1}+m_0\frac{\partial W}{\partial q_2}+n_0\frac{\partial W}{\partial q_3}\right)\delta J'$$
$$\left.+\left(l_0\frac{\partial W}{\partial r_1}+m_0\frac{\partial W}{\partial r_2}+n_0\frac{\partial W}{\partial r_3}\right)\delta K'\right\}d\sigma_0$$
$$-\iiint_{S_0}\left\{\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\xi_i}+q_i\frac{\partial W}{\partial\varsigma_i}-r_i\frac{\partial W}{\partial\eta_i}\right)\right]\delta' x\right.$$
$$+\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\eta_i}+r_i\frac{\partial W}{\partial\xi_i}-p_i\frac{\partial W}{\partial\varsigma_i}\right)\right]\delta' y$$
$$+\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\varsigma_i}+p_i\frac{\partial W}{\partial\eta_i}-q_i\frac{\partial W}{\partial\xi_i}\right)\right]\delta' z$$
$$+\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial p_i}+q_i\frac{\partial W}{\partial r_i}-r_i\frac{\partial W}{\partial q_i}+\eta_i\frac{\partial W}{\partial\varsigma_i}-\varsigma_i\frac{\partial W}{\partial\eta_i}\right)\right]\delta I'$$
$$+\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial q_i}+r_i\frac{\partial W}{\partial p_i}-p_i\frac{\partial W}{\partial r_i}+\varsigma_i\frac{\partial W}{\partial\xi_i}-\xi_i\frac{\partial W}{\partial\varsigma_i}\right)\right]\delta J'$$
$$\left.+\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial r_i}+p_i\frac{\partial W}{\partial q_i}-q_i\frac{\partial W}{\partial p_i}+\xi_i\frac{\partial W}{\partial\eta_i}-\eta_i\frac{\partial W}{\partial\xi_i}\right)\right]\delta K'\right\}dx_0dy_0dz_0.$$

Set:

$$F_0' = l_0\frac{\partial W}{\partial\xi_1}+m_0\frac{\partial W}{\partial\xi_2}+n_0\frac{\partial W}{\partial\xi_3},\qquad I_0' = l_0\frac{\partial W}{\partial p_1}+m_0\frac{\partial W}{\partial p_2}+n_0\frac{\partial W}{\partial p_3},$$
$$G_0' = l_0\frac{\partial W}{\partial\eta_1}+m_0\frac{\partial W}{\partial\eta_2}+n_0\frac{\partial W}{\partial\eta_3},\qquad J_0' = l_0\frac{\partial W}{\partial q_1}+m_0\frac{\partial W}{\partial q_2}+n_0\frac{\partial W}{\partial q_3},$$
$$H_0' = l_0\frac{\partial W}{\partial\varsigma_1}+m_0\frac{\partial W}{\partial\varsigma_2}+n_0\frac{\partial W}{\partial\varsigma_3},\qquad K_0' = l_0\frac{\partial W}{\partial r_1}+m_0\frac{\partial W}{\partial r_2}+n_0\frac{\partial W}{\partial r_3},$$
$$X_0' = \sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\xi_i}+q_i\frac{\partial W}{\partial\varsigma_i}-r_i\frac{\partial W}{\partial\xi_i}\right],$$
$$Y_0' = \sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\eta_i}+r_i\frac{\partial W}{\partial\xi_i}-p_i\frac{\partial W}{\partial\varsigma_i}\right],$$

$$Z'_0 = \sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\varsigma_i} + p_i\frac{\partial W}{\partial\eta_i} - q_i\frac{\partial W}{\partial\xi_i}\right],$$
$$L'_0 = \sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial p_i} + q_i\frac{\partial W}{\partial r_i} - r_i\frac{\partial W}{\partial q_i} + \eta_i\frac{\partial W}{\partial\varsigma_i} - \varsigma_i\frac{\partial W}{\partial\eta_i}\right],$$
$$M'_0 = \sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial q_i} + r_i\frac{\partial W}{\partial p_i} - p_i\frac{\partial W}{\partial r_i} + \varsigma_i\frac{\partial W}{\partial\xi_i} - \xi_i\frac{\partial W}{\partial\varsigma_i}\right],$$
$$N'_0 = \sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial r_i} + p_i\frac{\partial W}{\partial q_i} - q_i\frac{\partial W}{\partial p_i} + \xi_i\frac{\partial W}{\partial\eta_i} - \eta_i\frac{\partial W}{\partial\xi_i}\right],$$

we have:

$$\delta\iiint_{S_0} W dx_0 dy_0 dz_0 = \iint_{S_0}(F'_0\delta' x + G'_0\delta' y + H'_0\delta' z + I'_0\delta I' + J'_0\delta J' + K'_0\delta K')d\sigma_0$$
$$- \iiint_{S_0}(X'_0\delta' x + Y'_0\delta' y + Z'_0\delta' z + L'_0\delta I' + M'_0\delta J' + N'_0\delta K')dx_0 dy_0 dz_0 .$$

If we first direct our attention to the triple integral that figures in the expression for $\delta\iiint_{S_0} W dx_0 dy_0 dz_0$ then we call the segments that have their origin at M and whose projections onto the axes Mx', My', Mz' are X'_0, Y'_0, Z'_0 and L'_0, M'_0, N'_0, respectively, the *external force and external moment at the point M referred to the unit of volume of the undeformed medium.*

Next, directing our attention to the surface integral that figures in:

$$\delta\iiint_{S_0} W dx_0 dy_0 dz_0 ,$$

we call the segments that issue from the point M and have projections $-F'_0, -G'_0, -H'_0$ and $-I'_0, -J'_0, -K'_0$ on the axes Mx', My', Mz', respectively, the *external effort and external moment of deformation at the point M of the surface* S_0 that bounds the medium *referred to the unit of area of the surface* S_0. At a definite point M of (S) these last six quantities depend only on the direction of the exterior normal to the surface (S). They remain invariant if the region in question is varied and the direction of the exterior normal does not change, but they change sign if this direction is replaced by the opposite direction.

Suppose that one traces a surface (Σ) in the interior of the deformed medium that is bounded by the surface (S) in such a way that (Σ), together with a portion of surface (S), uniquely circumscribes a subset (A) of the medium, and let (B) denote the rest of the medium outside of the subset (A). Let (Σ_0) be the surface of (M_0) that corresponds to the surface (S) of (M), and let (A_0) and (B_0) be the regions of (M_0) that correspond to the regions (A) and (B) of (M). Mentally separate the two subsets (A) and (B). One may regard the two segments $(-F'_0, -G'_0, -H'_0)$ and $(-I'_0, -J'_0, -K'_0)$ that are determined by the point M and the direction of the normal to (Σ_0) that points towards the exterior of (A_0) as the external effort and moment of deformation at the point M of the frontier (Σ) of the region (A). Similarly, one may regard the two segments (F_0', G_0', H_0') and (I_0', J_0', K_0') as the external effort and moment of deformation at the point M of the frontier (Σ) of the region (B). By reason of that remark, we say that $-F_0', -G_0', -H_0'$ and $-I_0', -J_0', -K_0'$ are the components with respect to the axes Mx', My', Mz' of the *effort and moment of deformation that are exerted at M on the portion* (A) *of the medium* (M), and that F_0', G_0', H_0' and I_0', J_0', K_0' are the components with respect to the axes Mx', My', Mz' of the *effort and moment of deformation that are exerted at M on the portion* (B) *of the medium* (M).

The observation made at the end of secs. **9** and **34** on the subject of replacing the triad $Mx'y'z'$ by a triad that is invariantly related to it may be repeated here without modification.

53. Various ways of specifying the effort and moment of deformation. – Set:

$$A_i' = \frac{\partial W}{\partial \xi_i}, \qquad B_i' = \frac{\partial W}{\partial \eta_i}, \qquad C_i' = \frac{\partial W}{\partial \varsigma_i},$$
$$P_i' = \frac{\partial W}{\partial p_i}, \qquad Q_i' = \frac{\partial W}{\partial q_i}, \qquad R_i' = \frac{\partial W}{\partial r_i}.$$

A_i', B_i', C_i' and P_i', Q_i', R_i' represent the projections onto Mx', My', Mz' of the effort and moment of deformation, respectively, that are exerted at the point M on a surface that has an interior normal at the point M_0 that is parallel to the coordinate axis Ox, Oy, Oz that corresponds to the index i before deformation. Indeed, it suffices to recall that one has already agreed to replace the letters x_0, y_0, z_0, which correspond, by this notation, to the indices 1, 2, 3, respectively, with ρ_1, ρ_2, ρ_3. If you recall, that effort and moment of deformation are referred to the unit of area of the undeformed surface.

The new efforts and moments of deformation that we define are related to the elements introduced in the preceding section by the following relations:

$$F_0' = l_0 A_1' + m_0 A_2' + n_0 A_3', \qquad I_0' = l_0 P_1' + m_0 P_2' + n_0 P_3',$$
$$G_0' = l_0 B_1' + m_0 B_2' + n_0 B_3', \qquad J_0' = l_0 Q_1' + m_0 Q_2' + n_0 Q_3',$$
$$H_0' = l_0 C_1' + m_0 C_2' + n_0 C_3', \qquad K_0' = l_0 R_1' + m_0 R_2' + n_0 R_3',$$
$$\sum \left(\frac{\partial A_i'}{\partial \rho_i} + q_i C_i' - r_i B_i' \right) - X_0' = 0,$$
$$\sum \left(\frac{\partial B_i'}{\partial \rho_i} + r_i A_i' - p_i C_i' \right) - Y_0' = 0,$$
$$\sum \left(\frac{\partial C_i'}{\partial \rho_i} + p_i B_i' - q_i A_i' \right) - Z_0' = 0,$$

$$\sum\left(\frac{\partial P_i'}{\partial \rho_i} + q_i R_i' - r_i Q_i' + \eta_i C_i' - \xi_i B_i'\right) - L_0' = 0,$$
$$\sum\left(\frac{\partial Q_i'}{\partial \rho_i} + r_i P_i' - p_i R_i' + \varsigma_i A_i' - \xi_i C_i'\right) - M_0' = 0,$$
$$\sum\left(\frac{\partial R_i'}{\partial \rho_i} + p_i Q_i' - q_i P_i' + \xi_i B_i' - \eta_i A_i'\right) - N_0' = 0.$$

We propose to transform these relations into ones that are independent of the values of the quantities that we calculated by means of W that figure in them. Indeed, these relations pertain to the segments that are attached to the point M to which we gave the names. Instead of defining these segments by their projections on Mx', My', Mz', we may define them by their projections on the other axes; the latter projections will be coupled by relations that are transforms of the preceding ones.

Moreover, the transformed relations are obtained immediately if one remarks that the original formulas have simple and immediate interpretations ([1]) by the adjunction to these moving axes of axes that are parallel to them at the point O.

1. We confine ourselves to the consideration of fixed axes Ox, Oy, Oz. Denote the projections of the external force and external moment at an arbitrary point M of the deformed medium onto these axes by X_0, Y_0, Z_0, and L_0, M_0, N_0, respectively, and the projections of effort and moment of deformation on a surface whose interior normal has the direction cosines l_0, m_0, n_0 before deformation by F_0, G_0, H_0 and I_0, J_0, K_0, respectively. The projections of the effort (A_i', B_i', C_i') and the moment of deformation (P_i', Q_i', R_i') are denoted by A_i, B_i, C_i and P_i, Q_i, R_i, respectively. The transforms of the preceding relations are obviously:

$$F_0 = l_0 A_1 + m_0 A_2 + n_0 A_3, \qquad I_0 = l_0 P_1 + m_0 P_2 + n_0 P_3,$$
$$G_0 = l_0 B_1 + m_0 B_2 + n_0 B_3, \qquad J_0 = l_0 Q_1 + m_0 Q_2 + n_0 Q_3,$$
$$H_0 = l_0 C_1 + m_0 C_2 + n_0 C_3, \qquad K_0 = l_0 R_1 + m_0 R_2 + n_0 R_3,$$
$$\frac{\partial A_1}{\partial x_0} + \frac{\partial A_2}{\partial y_0} + \frac{\partial A_3}{\partial z_0} - X_0 = 0,$$
$$\frac{\partial B_1}{\partial x_0} + \frac{\partial B_2}{\partial y_0} + \frac{\partial B_3}{\partial z_0} - Y_0 = 0,$$
$$\frac{\partial C_1}{\partial x_0} + \frac{\partial C_2}{\partial y_0} + \frac{\partial C_3}{\partial z_0} - Z_0 = 0,$$
$$\frac{\partial P_1}{\partial x_0} + \frac{\partial P_2}{\partial y_0} + \frac{\partial P_3}{\partial z_0} + C_1\frac{\partial y}{\partial x_0} + C_2\frac{\partial y}{\partial y_0} + C_3\frac{\partial y}{\partial z_0} - B_1\frac{\partial z}{\partial x_0} - B_2\frac{\partial z}{\partial x_0} - B_3\frac{\partial z}{\partial x_0} - L_0 = 0,$$

[1] An interesting interpretation to note is the analogy with the one given by P. SAINT-GUILHEM in the context of the dynamics of triads.

$$\frac{\partial Q_1}{\partial x_0}+\frac{\partial Q_2}{\partial y_0}+\frac{\partial Q_3}{\partial z_0}+A_1\frac{\partial z}{\partial x_0}+A_2\frac{\partial z}{\partial y_0}+A_3\frac{\partial z}{\partial z_0}-C_1\frac{\partial x}{\partial x_0}-C_2\frac{\partial x}{\partial x_0}-C_3\frac{\partial x}{\partial x_0}-M_0=0,$$
$$\frac{\partial R_1}{\partial x_0}+\frac{\partial R_2}{\partial y_0}+\frac{\partial R_3}{\partial z_0}+B_1\frac{\partial x}{\partial x_0}+B_2\frac{\partial x}{\partial y_0}+B_3\frac{\partial x}{\partial z_0}-A_1\frac{\partial y}{\partial x_0}-A_2\frac{\partial y}{\partial x_0}-A_3\frac{\partial y}{\partial x_0}-N_0=0,$$

relations that are the three-dimensional generalizations of the two-dimensional equations of LORD KELVIN and TAIT.

2. Now observe that we may express the nine cosines $\alpha,\alpha',\cdots,\gamma''$ by means of three auxiliary functions; let λ_1, λ_2, λ_3 be three such auxiliary functions. Set:

$$\begin{aligned}\sum\gamma d\beta=-\sum\beta d\gamma&=\varpi_1'd\lambda_1+\varpi_2'd\lambda_2+\varpi_3'd\lambda_3,\\ \sum\alpha d\gamma=-\sum\gamma d\alpha&=\chi_1'd\lambda_1+\chi_2'd\lambda_2+\chi_3'd\lambda_3,\\ \sum\beta d\alpha=-\sum\alpha d\beta&=\sigma_1'd\lambda_1+\sigma_2'd\lambda_2+\sigma_3'd\lambda_3.\end{aligned}$$

The functions $\varpi_i',\chi_i',\sigma_i'$ of λ_1, λ_2, λ_3 so defined satisfy the relations:

$$\begin{aligned}&\frac{\partial\varpi_j'}{\partial\lambda_i}-\frac{\partial\varpi_i'}{\partial\lambda_j}+\chi_i'\sigma_j'-\chi_j'\sigma_i'=0,\\ &\frac{\partial\chi_j'}{\partial\lambda_i}-\frac{\partial\chi_i'}{\partial\lambda_j}+\sigma_i'\varpi_j'-\sigma_j'\varpi_i'=0, \qquad (i,j)=1,2,3.\\ &\frac{\partial\sigma_j'}{\partial\lambda_i}-\frac{\partial\sigma_i'}{\partial\lambda_j}+\varpi_i'\chi_i'-\varpi_i'\chi_i'=0,\end{aligned}$$

and one has:

$$\begin{aligned}p_i&=\varpi_1'\frac{\partial\lambda_1}{\partial\rho_i}+\varpi_2'\frac{\partial\lambda_2}{\partial\rho_i}+\varpi_3'\frac{\partial\lambda_3}{\partial\rho_i},\\ q_i&=\chi_1'\frac{\partial\lambda_1}{\partial\rho_i}+\chi_2'\frac{\partial\lambda_2}{\partial\rho_i}+\chi_3'\frac{\partial\lambda_3}{\partial\rho_i}, \qquad (\text{or } x_0=\rho_1,\ y_0=\rho_2,\ z_0=\rho_3)\\ r_i&=\sigma_1'\frac{\partial\lambda_1}{\partial\rho_i}+\sigma_2'\frac{\partial\lambda_2}{\partial\rho_i}+\sigma_3'\frac{\partial\lambda_3}{\partial\rho_i}.\end{aligned}$$

Let ϖ_i, χ_i, σ_i denote the projections onto the fixed axes Ox, Oy, Oz of the segment whose projections onto the axes Mx', My', Mz' are $\varpi_i',\chi_i',\sigma_i'$; we have:

$$\begin{aligned}\sum\alpha'd\alpha''=-\sum\alpha''d\alpha'&=\varpi_1d\lambda_1+\varpi_2d\lambda_2+\varpi_3d\lambda_3,\\ \sum\alpha''d\alpha=-\sum\alpha d\alpha''&=\chi_1d\lambda_1+\chi_2d\lambda_2+\chi_3d\lambda_3,\\ \sum\alpha d\alpha'=-\sum\alpha'd\alpha&=\sigma_1d\lambda_1+\sigma_2d\lambda_2+\sigma_3d\lambda_3,\end{aligned}$$

by virtue of which [1], the new functions ϖ_i, χ_i, σ_i of λ_1, λ_2, λ_3 satisfy the relations:

$$\frac{\partial\varpi_j}{\partial\lambda_i}-\frac{\partial\varpi_i}{\partial\lambda_j}=\chi_i\sigma_j-\chi_j\sigma_i,$$
$$\frac{\partial\chi_j}{\partial\lambda_i}-\frac{\partial\chi_i}{\partial\lambda_j}=\sigma_i\varpi_j-\sigma_j\varpi_i,\quad (i,j)=1,2,3.$$
$$\frac{\partial\sigma_j}{\partial\lambda_i}-\frac{\partial\sigma_i}{\partial\lambda_j}=\varpi_i\chi_i-\varpi_i\chi_i.$$

Again, we make the remark, which will be of use later on, that if one lets $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$ denote the variations of $\lambda_1, \lambda_2, \lambda_3$ that correspond to the variations $\delta\alpha,\delta\alpha',\cdots,\delta\gamma''$ of $\alpha,\alpha',\cdots,\gamma''$ then one will have:

$$\begin{aligned}
\delta I' &= \varpi_1'\delta\lambda_1+\varpi_2'\delta\lambda_2+\varpi_3'\delta\lambda_3,\\
\delta J' &= \chi_1'\delta\lambda_1+\chi_2'\delta\lambda_2+\chi_3'\delta\lambda_3,\\
\delta K' &= \sigma_1'\delta\lambda_1+\sigma_2'\delta\lambda_2+\sigma_3'\delta\lambda_3,\\
\delta I &= \alpha\delta I'+\beta\delta J'+\gamma\delta K' = \varpi_1\delta\lambda_1+\varpi_2\delta\lambda_2+\varpi_3\delta\lambda_3,\\
\delta J &= \alpha'\delta I'+\beta'\delta J'+\gamma'\delta K' = \chi_1\delta\lambda_1+\chi_2\delta\lambda_2+\chi_3\delta\lambda_3,\\
\delta K &= \alpha''\delta I'+\beta''\delta J'+\gamma''\delta K' = \sigma_1\delta\lambda_1+\sigma_2\delta\lambda_2+\sigma_3\delta\lambda_3,
\end{aligned}$$

in which δI, δJ, δK are the projections onto the fixed axes of the segment whose projections onto Mx', My', Mz' are $\delta I', \delta J', \delta K'$.

Now set:

$$\begin{aligned}
\mathcal{I}_0 &= \varpi_1'I_0'+\chi_1'J_0'+\sigma_1'K_0' = \varpi_1 I_0+\chi_1 J_0+\sigma_1 K_0,\\
\mathcal{J}_0 &= \varpi_2'I_0'+\chi_2'J_0'+\sigma_2'K_0' = \varpi_2 I_0+\chi_2 J_0+\sigma_2 K_0,\\
\mathcal{K}_0 &= \varpi_3'I_0'+\chi_3'J_0'+\sigma_3'K_0' = \varpi_3 I_0+\chi_3 J_0+\sigma_3 K_0,\\
\mathcal{L}_0 &= \varpi_1'L_0'+\chi_1'M_0'+\sigma_1'N_0' = \varpi_1 L_0+\chi_1 M_0+\sigma_1 N_0,\\
\mathcal{M}_0 &= \varpi_1'L_0'+\chi_1'M_0'+\sigma_1'N_0' = \varpi_1 L_0+\chi_1 M_0+\sigma_1 N_0,\\
\mathcal{N}_0 &= \varpi_1'L_0'+\chi_1'M_0'+\sigma_1'N_0' = \varpi_1 L_0+\chi_1 M_0+\sigma_1 N_0.
\end{aligned}$$

In addition, we introduce the following notations:

$$\Pi_i = \varpi_1'P_i'+\chi_1'Q_i'+\sigma_1'R_i' = \varpi_1 P_i+\chi_1 Q_i+\sigma_1 R_i,$$

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[1] These formulas may serve to define the functions ϖ_i, χ_i, σ_i, directly, and the substitution is defined by:

$$\begin{aligned}
\varpi_i &= \alpha\varpi_i'+\beta\chi_i'+\gamma\sigma_i',\\
\chi_i &= \alpha'\varpi_i'+\beta'\chi_i'+\gamma'\sigma_i', \qquad (i=1,2,3)\\
\sigma_i &= \alpha''\varpi_i'+\beta''\chi_i'+\gamma''\sigma_i'.
\end{aligned}$$

$$\mathrm{X}_i = \varpi'_2 P'_i + \chi'_2 Q'_i + \sigma'_2 R'_i = \varpi_2 P_i + \chi_2 Q_i + \sigma_2 R_i,$$
$$\Sigma_i = \varpi'_3 P'_i + \chi'_3 Q'_i + \sigma'_3 R'_i = \varpi_3 P_i + \chi_3 Q_i + \sigma_3 R_i,$$

then, instead of the latter system in which either P'_i, Q'_i, R'_i or P_i, Q_i, R_i figure, we have the following:

$$\mathcal{L}_0 = \sum_i \left[\frac{\partial \Pi_i}{\partial \rho_i} - P'_i\left(\frac{\partial \varpi'_i}{\partial \rho_i} + q_i\sigma'_1 - r_i\chi'_1\right) - Q'_i\left(\frac{\partial \chi'_i}{\partial \rho_i} + r_i\varpi'_1 - p_i\sigma'_1\right) - R'_i\left(\frac{\partial \sigma'_i}{\partial \rho_i} + p_i\chi'_1 - q_i\varpi'_1\right)\right.$$
$$\left. + A'_i(\chi'_1\varsigma_i - \sigma'_1\eta_i) + B'_i(\sigma'_1\xi_i - \varpi'_1\varsigma_i) + C'_i(\varpi'_1\eta_i - \chi'_1\xi_i)\right],$$

with two analogous equations. If one remarks that the functions ξ_i, η_i, ζ_i, p_i, q_i, r_i of λ_1, λ_2, λ_3, $\dfrac{\partial \lambda_1}{\partial \rho_i}, \dfrac{\partial \lambda_2}{\partial \rho_i}, \dfrac{\partial \lambda_3}{\partial \rho_i}$ give rise to the formulas:

$$\frac{\partial \xi_i}{\partial \lambda_j} + \chi'_j\varsigma_i - \sigma'_j\eta_i = 0, \qquad \frac{\partial p_i}{\partial \lambda_j} = \frac{\partial \varpi'_i}{\partial \rho_j} + q_j\sigma'_i - r_j\chi'_i,$$
$$\frac{\partial \eta_i}{\partial \lambda_j} + \sigma'_j\xi_i - \varpi'_j\varsigma_i = 0, \qquad \frac{\partial q_i}{\partial \lambda_j} = \frac{\partial \chi'_i}{\partial \rho_j} + r_j\varpi'_i - p_j\sigma'_i,$$
$$\frac{\partial \varsigma_i}{\partial \lambda_j} + \varpi'_j\eta_i - \chi'_j\xi_i = 0, \qquad \frac{\partial r_i}{\partial \lambda_j} = \frac{\partial \sigma'_i}{\partial \rho_j} + p_j\chi'_i - q_j\varpi'_i,$$

that result from the defining relations of the functions $\varpi'_i, \chi'_i, \sigma'_i$, and the nine identities that they verify, then one may give the preceding system the new form:

$$\mathcal{L}_0 = \sum_i \left[\frac{\partial \Pi_\blacklozenge}{\partial \rho_\blacklozenge} - A'_i\frac{\partial \xi_i}{\partial \lambda_1} - B'_i\frac{\partial \eta_i}{\partial \lambda_1} - C'_i\frac{\partial \varsigma_i}{\partial \lambda_1} - P'_i\frac{\partial p_i}{\partial \lambda_1} - Q'_i\frac{\partial q_i}{\partial \lambda_1} - R'_i\frac{\partial r_i}{\partial \lambda_1}\right],$$

with two analogous equations.

3. The preceding equations that we introduced also constitute the generalization of the ones we developed in an earlier work ([1]). We may transform them in such a way as to obtain the generalization of the well-known equations of the theory of elasticity that relate to effort. To that effect, it will suffice to reproduce the method we already employed in the work that we mentioned.

To abbreviate the writing, let $\mathcal{X}'_0, \mathcal{Y}'_0, \mathcal{Z}'_0$ and $\mathcal{L}'_0, \mathcal{M}'_0, \mathcal{N}'_0$ denote – for the moment – the left-hand sides of the transformation relations, which refer to X_0, Y_0, Z_0, L_0, M_0, N_0, respectively, and observe that one may summarize the twelve relations that we established by the following:

[1] E. and F. COSSERAT. – *Premier mémoire sur la théorie de l'élasticité; Annales de la Faculté des sciences de Toulouse* (1), **10**, pp. $\mathrm{I}_1 - \mathrm{I}_{116}$, 1896.

$$\iiint (\mathcal{X}_0'\lambda_1 + \mathcal{Y}_0'\lambda_2 + \mathcal{Z}_0'\lambda_3 + \mathcal{L}_0'\mu_1 + \mathcal{M}_0'\mu_2 + \mathcal{N}_0'\mu_3)\, dx_0 dy_0 dz_0$$
$$- \iint \{(F_0 - l_0 A_1 - m_0 A_2 - n_0 A_3)\lambda_1 + (G_0 - l_0 B_1 - m_0 B_2 - n_0 B_3)\lambda_2$$
$$+ (H_0 - l_0 C_1 - m_0 C_2 - n_0 C_3)\lambda_3 + (I_0 - l_0 P_1 - m_0 P_2 - n_0 P_3)\mu_1$$
$$+ (J_0 - l_0 Q_1 - m_0 Q_2 - n_0 Q_3)\mu_2 + (K_0 - l_0 R_1 - m_0 R_2 - n_0 R_3)\mu_3\} d\sigma_0 = 0,$$

in which $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3$ are arbitrary functions and the integrals are taken over the surface S_0 of the medium (M_0) and the domain bounded by it. If we apply GREEN'S formula then the relation that we wrote becomes the following one:

$$\iiint (X_0\lambda_1 + Y_0\lambda_2 + Z_0\lambda_3 + L_0\mu_1 + M_0\mu_2 + N_0\mu_3) dx_0 dy_0 dz_0$$
$$- \iint (F_0\lambda_1 + G_0\lambda_2 + H_0\lambda_3 + I_0\mu_1 + J_0\mu_2 + K_0\mu_3) d\sigma_0$$
$$+ \iiint \left(A_1 \frac{\partial\lambda_1}{\partial x_0} + A_2 \frac{\partial\lambda_1}{\partial y_0} + A_3 \frac{\partial\lambda_1}{\partial z_0} + B_1 \frac{\partial\lambda_2}{\partial x_0} + B_2 \frac{\partial\lambda_2}{\partial y_0} + B_3 \frac{\partial\lambda_2}{\partial z_0} \right.$$
$$\left. C_1 \frac{\partial\lambda_3}{\partial x_0} + C_2 \frac{\partial\lambda_3}{\partial y_0} + C_3 \frac{\partial\lambda_3}{\partial z_0} \right) dx_0 dy_0 dz_0$$
$$+ \iiint \left(P_1 \frac{\partial\mu_1}{\partial x_0} + P_2 \frac{\partial\mu_1}{\partial y_0} + P_3 \frac{\partial\mu_1}{\partial z_0} + Q_1 \frac{\partial\mu_1}{\partial x_0} + Q_2 \frac{\partial\mu_2}{\partial y_0} + Q_3 \frac{\partial\mu_2}{\partial z_0} \right.$$
$$\left. R_1 \frac{\partial\mu_3}{\partial x_0} + R_2 \frac{\partial\mu_3}{\partial y_0} + R_3 \frac{\partial\mu_3}{\partial z_0} \right) dx_0 dy_0 dz_0$$
$$- \iiint \left(C_1 \frac{\partial y}{\partial x_0} + C_2 \frac{\partial y}{\partial y_0} + C_3 \frac{\partial y}{\partial z_0} + B_1 \frac{\partial z}{\partial x_0} + B_2 \frac{\partial z}{\partial y_0} + B_3 \frac{\partial z}{\partial z_0} \right) \mu_1 dx_0 dy_0 dz_0$$
$$- \iiint \left(A_1 \frac{\partial z}{\partial x_0} + A_2 \frac{\partial z}{\partial y_0} + A_3 \frac{\partial z}{\partial z_0} + C_1 \frac{\partial x}{\partial x_0} + C_2 \frac{\partial x}{\partial y_0} + C_3 \frac{\partial x}{\partial z_0} \right) \mu_2 dx_0 dy_0 dz_0$$
$$- \iiint \left(B_1 \frac{\partial x}{\partial x_0} + B_2 \frac{\partial x}{\partial y_0} + B_3 \frac{\partial x}{\partial z_0} + A_1 \frac{\partial y}{\partial x_0} + A_2 \frac{\partial y}{\partial y_0} + A_3 \frac{\partial y}{\partial z_0} \right) \mu_3 dx_0 dy_0 dz_0 = 0.$$

We seek the transform of this latter relation when one takes the functions x, y, z of x_0, y_0, z_0 for the new variables. If one lets φ denote an arbitrary function of x_0, y_0, z_0 that becomes a function of x, y, z then the elementary formulas for the change of variables are:

$$\frac{\partial\varphi}{\partial x_0} = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial x_0} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial x_0} + \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial x_0},$$
$$\frac{\partial\varphi}{\partial y_0} = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial y_0} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial y_0} + \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial y_0},$$
$$\frac{\partial\varphi.}{\partial z_0} = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial z_0} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial z_0} + \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial z_0}.$$

Apply these formulas to the functions λ_1, λ_2, λ_3, μ_1, μ_2, μ_3. With S always denoting the surface of the medium (M) that corresponds to the surface S_0 of (M_0), we further denote the projections onto Ox, Oy, Oz of the external force and external moment applied to the point M by X, Y, Z, L, M, N, which are referred to the unit of volume of the deformed medium (M), and the projection onto Ox, Oy, Oz of the effort and the moment of deformation that are exerted at the point M of S by F, G, H, I, J, K referred to the unit of area on S. Finally, introduce the eighteen new auxiliary functions p_{xx}, p_{yx}, p_{zx}, p_{xy}, p_{yy}, p_{zy}, p_{xz}, p_{yz}, p_{zz}, q_{xx}, q_{yx}, q_{zx}, q_{xy}, q_{yy}, q_{zy}, q_{xz}, q_{yz}, q_{zz} by the formulas:

$$\begin{aligned}
\Delta p_{xx} &= A_1 \frac{\partial x}{\partial x_0} + A_2 \frac{\partial x}{\partial y_0} + A_3 \frac{\partial x}{\partial z_0}, &\quad \Delta q_{xx} &= P_1 \frac{\partial x}{\partial x_0} + P_2 \frac{\partial x}{\partial y_0} + P_3 \frac{\partial x}{\partial z_0},\\
\Delta p_{yx} &= A_1 \frac{\partial y}{\partial x_0} + A_2 \frac{\partial y}{\partial y_0} + A_3 \frac{\partial y}{\partial z_0}, &\quad \Delta q_{yx} &= P_1 \frac{\partial y}{\partial x_0} + P_2 \frac{\partial y}{\partial y_0} + P_3 \frac{\partial y}{\partial z_0},\\
\Delta p_{zx} &= A_1 \frac{\partial z}{\partial x_0} + A_2 \frac{\partial z}{\partial y_0} + A_3 \frac{\partial z}{\partial z_0}, &\quad \Delta q_{zx} &= P_1 \frac{\partial z}{\partial x_0} + P_2 \frac{\partial z}{\partial y_0} + P_3 \frac{\partial z}{\partial z_0},
\end{aligned}$$

and the analogous ones that are obtained by replacing:

$$A_1, A_2, A_3, p_{xx}, p_{yx}, p_{zx}, P_1, P_2, P_3, q_{xx}, q_{yx}, q_{zx}$$

with:

$$B_1, B_2, B_3, p_{xy}, p_{yy}, p_{zz}, Q_1, Q_2, Q_3, q_{xy}, q_{yy}, q_{zy},$$

and then by:

$$C_1, C_2, C_3, p_{xz}, p_{yz}, p_{zz}, R_1, R_2, R_3, q_{xz}, q_{yz}, q_{zz},$$

respectively.

We obtain the transformed relation:

$$\begin{aligned}
&\iiint (X\lambda_1 + Y\lambda_2 + Z\lambda_3 + L\mu_1 + M\mu_2 + N\mu_3)\,dxdydz\\
&-\iint (F\lambda_1 + G\lambda_2 + H\lambda_3 + I\mu_1 + J\mu_2 + K\mu_3)\,d\sigma\\
&+\iiint\left(p_{xx}\frac{\partial \lambda_1}{\partial x} + p_{yx}\frac{\partial \lambda_1}{\partial y} + p_{zx}\frac{\partial \lambda_1}{\partial z} + p_{xy}\frac{\partial \lambda_2}{\partial x} + p_{yy}\frac{\partial \lambda_2}{\partial y} + p_{zy}\frac{\partial \lambda_2}{\partial z}\right.\\
&\qquad\qquad\left. + p_{xz}\frac{\partial \lambda_3}{\partial x} + p_{yz}\frac{\partial \lambda_3}{\partial x} + p_{zz}\frac{\partial \lambda_3}{\partial x}\right) dxdydz\\
&+\iiint\left(q_{xx}\frac{\partial \mu_1}{\partial x} + q_{yx}\frac{\partial \mu_1}{\partial y} + q_{zx}\frac{\partial \mu_1}{\partial z} + q_{xy}\frac{\partial \mu_2}{\partial x} + q_{yy}\frac{\partial \mu_2}{\partial y} + q_{zy}\frac{\partial \mu_2}{\partial z}\right.\\
&\qquad\qquad\left. + q_{xz}\frac{\partial \mu_3}{\partial x} + q_{yz}\frac{\partial \mu_3}{\partial x} + q_{zz}\frac{\partial \mu_3}{\partial x}\right) dxdydz\\
&-\iiint \{(p_{yz} - p_{zy})\mu_1 + (p_{zx} - p_{xz})\mu_2 + (p_{xy} - p_{yx})\mu_3\}\,dxdydz = 0,
\end{aligned}$$

in which the integrals are taken over the surface S of the medium (M), and the domain bounded by it, with $d\sigma$ designating the area element of S.

Once more, apply GREEN'S formula to the terms that refer to the derivatives of λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 with respect to *x, y, z,* and let *l, m, n* denote the direction cosines of the exterior normal to the surface *S* with respect to the fixed axes. Since λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary, they become:

$$
\begin{aligned}
&F = lp_{xx} + mp_{yx} + np_{zx}, && I = lq_{xx} + mq_{yx} + nq_{zx}, \\
&G = lp_{xy} + mp_{yy} + np_{zy}, && J = lq_{xy} + mq_{yy} + nq_{zy}, \\
&H = lp_{xz} + mp_{yz} + np_{zz}, && K = lq_{xz} + mq_{yz} + nq_{zz}, \\
&\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} - X = 0, \\
&\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} - Y = 0, \\
&\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} - Z = 0, \\
&\frac{\partial q_{xx}}{\partial x} + \frac{\partial q_{yx}}{\partial y} + \frac{\partial q_{zx}}{\partial z} + p_{yz} - p_{zy} - L = 0, \\
&\frac{\partial q_{xy}}{\partial x} + \frac{\partial q_{yy}}{\partial y} + \frac{\partial q_{zy}}{\partial z} + p_{zx} - p_{xz} - M = 0, \\
&\frac{\partial q_{xz}}{\partial x} + \frac{\partial q_{yz}}{\partial y} + \frac{\partial q_{zz}}{\partial z} + p_{xy} - p_{yx} - N = 0.
\end{aligned}
$$

The significance of the eighteen new auxiliary functions p_{xx}, …, q_{xx}, … results immediately from the relations that we just found. Indeed, it is clear that the coefficients p_{xx}, p_{xy}, p_{xz} of *l* in the expressions for *F*, *G*, *H* represent the projections onto *Ox*, *Oy*, *Oz* of the effort that is exerted at the point *M* on the surface whose exterior normal is parallel to *Ox*, and that the coefficients q_{xx}, q_{xy}, q_{xz} of *l* in the expressions for *I*, *J*, *K* are the projections onto *Ox*, *Oy*, *Oz* of the moment of deformation at *M* relative to the same surface. The coefficients of *m* and of *n* give rise to an analogous interpretation in regard to surfaces whose interior normals are parallel to *Oy* and *Oz*.

The auxiliary functions that we just introduced and the equations that relate them do not appear to have been envisioned in a form that was that general up till now; to our knowledge, they have been considered only in the particular case in which the nine quantities q_{xx}, …, q_{zz} are null, and the first work to treat that question seems to be that of VOIGT ([1]).

[1] WALDEMAR VOIGT. – *Theoretische Studien über die Elasticitätsverhältnisse der Krystalle,* I, II, *Abhandlungen der königlichen Gesellschaft der Wissenschaften zu Göttingen,* Bd. 34, 1887. The first section, entitled: *Ableitung der Grundgleichungen aus der Annahme mit Polarität begabter Moleküle,* has 49 pages (3-52), the second one, entitled: *Untersuchung des elastische Verhaltens eines Cylinders aus krystallinscher Substanz, auf dessen Mantelfläche keine Kräfte wirken, wenn in seinem Innern wirkenden Spannungen längs der Cylinderaxe constant sind,* is 48 pages (53-100). One may likewise consult the work of VOIGT: *L'État actuel de nos connaissances sur l'élasticité des cristaux* (Report presented at the International Congress of Physics convened in Paris in 1900, T. I, pp. 277-347), in which he alludes to

In conclusion, we observe that if one performs a change of variables in the six equations that involve X, Y, Z, F, G, H in such a fashion as to introduce the original variables x_0, y_0, z_0 then one immediately finds equations whose first three constitute the generalization of the equations that were established by BOUSSINESQ.

54. External virtual work. Theorem analogous to those of Varignon and Saint-Guilhem. Remarks on the auxiliary functions that were introduced in the preceding section. –We give the name of *external virtual work* on the deformed medium (M) for an arbitrary virtual deformation, to the expression:

$$\delta\mathcal{T}_e = -\iint_{S_0}(F_0'\delta'x + G_0'\delta'y + H_0'\delta'z + I_0'\delta I' + J_0'\delta J' + K_0'\delta K')\,d\sigma_0$$
$$+\iiint_{S_0}(X_0'\delta'x + Y_0'\delta'y + Z_0'\delta'z + L_0'\delta I' + M_0'\delta J' + N_0'\delta K')\,dx_0\,dy_0\,dz_0.$$

We refer to the notations of sec. **50**, and let δI, δJ, δK denote the projections onto the fixed axes of the segment whose projections onto Mx', My', Mz' are $\delta I', \delta J', \delta K'$, in such a way that one has, for example:

$$-\delta I = \alpha''\delta\alpha' + \beta''\delta\beta' + \gamma''\delta\gamma' = -(\alpha'\delta\alpha'' + \beta'\delta\beta'' + \gamma'\delta\gamma''),$$

upon always supposing that the axes in question have the same orientation.

This being the case, suppose as in sec. **53** that one gives the arbitrary functions λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 the significance defined from the formulas:

$$\lambda_1 = \delta x,\ \ \lambda_2 = \delta y,\ \ \lambda_3 = \delta z, \qquad \mu_1 = \delta I,\ \ \mu_2 = \delta J,\ \ \mu_3 = \delta K.$$

We then see that the previously-obtained relations between the auxiliary functions that we introduced serves only to express the following condition:

When any of the virtual displacements in sec. **50** are given to the deformed medium the external virtual work $\delta\mathcal{T}_e$ is given, either by the relation:

$$\delta\mathcal{T}_e = -\iiint\left(p_{xx}\frac{\partial\delta x}{\partial x} + p_{yx}\frac{\partial\delta x}{\partial y} + p_{zx}\frac{\partial\delta x}{\partial z} + p_{xy}\frac{\partial\delta y}{\partial x} + p_{yy}\frac{\partial\delta y}{\partial y} + p_{zy}\frac{\partial\delta y}{\partial z}\right.$$
$$\left. + p_{xz}\frac{\partial\delta z}{\partial x} + p_{yz}\frac{\partial\delta z}{\partial y} + p_{zz}\frac{\partial\delta z}{\partial z}\right)dxdydz$$
$$-\iiint\left(q_{xx}\frac{\partial\delta I}{\partial x} + q_{yx}\frac{\partial\delta I}{\partial y} + q_{zx}\frac{\partial\delta I}{\partial z} + q_{xy}\frac{\partial\delta J}{\partial x} + q_{yy}\frac{\partial\delta J}{\partial y} + q_{zy}\frac{\partial\delta J}{\partial z}\right.$$

POISSON, *Mém. de l'Acad.,* T. XVIII, pp. 3, 1842 (see pp. 289). Also consult LARMOR, *On the propagation of a disturbance in a gyrostatically loaded medium (Proc. Lond. Math. Soc.,* Nov., 1891); LOVE, *Treatise on the Mathematical Theory of Elasticity (Camb. University Press,* 1st ed., 1892, 2nd ed., 1906); COMBEBIAC, *Sur les équations générales de l'élasticité, Bull. De la Soc. Math. De France,* T. XXX, pp. 108-110, and pp. 242-247, 1902.

$$+ q_{xz}\frac{\partial\delta K}{\partial x} + q_{yz}\frac{\partial\delta K}{\partial y} + q_{zz}\frac{\partial\delta K}{\partial z}\Bigg)dxdydz$$

$$+\iiint\{(p_{yz}-p_{zy})\delta I + (p_{zx}-p_{xy})\delta J + (p_{xy}-p_{yx})\delta K\}dxdydz,$$

where the integrals are taken over the deformed medium, or by the relation:

$$\begin{aligned}
\delta\mathcal{T}_e = -\iiint\Bigg(& A_1\frac{\partial\delta x}{\partial x_0} + A_2\frac{\partial\delta x}{\partial y_0} + A_3\frac{\partial\delta x}{\partial z_0} + B_1\frac{\partial\delta y}{\partial x_0} + B_2\frac{\partial\delta y}{\partial y_0} + B_3\frac{\partial\delta y}{\partial z_0} \\
& + C_1\frac{\partial\delta z}{\partial x_0} + C_2\frac{\partial\delta z}{\partial y_0} + C_3\frac{\partial\delta z}{\partial z_0}\Bigg)dx_0dy_0dz_0 \\
-\iiint\Bigg(& P_1\frac{\partial\delta I}{\partial x_0} + P_2\frac{\partial\delta I}{\partial y_0} + P_3\frac{\partial\delta I}{\partial z_0} + Q_1\frac{\partial\delta J}{\partial x_0} + Q_2\frac{\partial\delta J}{\partial y_0} + Q_3\frac{\partial\delta J}{\partial z_0} \\
& + R_1\frac{\partial\delta K}{\partial x_0} + R_2\frac{\partial\delta K}{\partial y_0} + R_3\frac{\partial\delta K}{\partial z_0}\Bigg)dx_0dy_0dz_0 \\
+\iiint\Bigg(& C_1\frac{\partial y}{\partial x_0} + C_2\frac{\partial y}{\partial y_0} + C_3\frac{\partial y}{\partial z_0} - B_1\frac{\partial z}{\partial x_0} - B_2\frac{\partial z}{\partial y_0} - B_3\frac{\partial z}{\partial z_0}\Bigg)\delta I dx_0dy_0dz_0 \\
+\iiint\Bigg(& A_1\frac{\partial z}{\partial x_0} + A_2\frac{\partial z}{\partial y_0} + A_3\frac{\partial z}{\partial z_0} - C_1\frac{\partial x}{\partial x_0} - C_2\frac{\partial x}{\partial y_0} - C_3\frac{\partial x}{\partial z_0}\Bigg)\delta J dx_0dy_0dz_0 \\
+\iiint\Bigg(& B_1\frac{\partial x}{\partial x_0} + B_2\frac{\partial x}{\partial y_0} + B_3\frac{\partial x}{\partial z_0} - A_1\frac{\partial y}{\partial x_0} - A_2\frac{\partial y}{\partial y_0} - A_3\frac{\partial y}{\partial z_0}\Bigg)\delta K dx_0dy_0dz_0,
\end{aligned}$$

in which the integrals are taken over the undeformed medium, because the formula we gave above:

$$\begin{aligned}
\delta\mathcal{T}_e = & -\iint_{S_0}(F_0'\delta' x + G_0'\delta' y + H_0'\delta' z + I_0'\delta I' + J_0'\delta J' + K_0'\delta K')d\sigma_0 \\
& + \iiint_{S_0}(X_0'\delta' x + Y_0'\delta' y + Z_0'\delta' z + L_0'\delta I' + M_0'\delta J' + N_0'\delta K')dx_0dy_0dz_0.
\end{aligned}$$

to serve as the definition of external virtual work may also be written:

$$\begin{aligned}
\delta\mathcal{T}_e = & -\iint_{S_0}(F_0\delta x + G_0\delta y + H_0\delta z + I_0\delta I + J_0\delta J + K_0\delta K)d\sigma_0 \\
& + \iiint_{S_0}(X_0\delta x + Y_0\delta y + Z_0\delta' + L_0\delta I + M_0\delta J + N_0\delta K)dx_0dy_0dz_0,
\end{aligned}$$

by virtue of the significance of X_0, Y_0, …, N_0, F_0, G_0, …, K_0, and likewise:

$$\delta\mathcal{T}_e = -\iint_S(F\delta x + G\delta y + H\delta z + I\delta I + J\delta J + K\delta K)d\sigma_0$$

$$+\iiint_S (X\delta x + Y\delta y + Z\delta' + L\delta I + M\delta J + N\delta K)dx_0 dy_0 dz_0,$$

by virtue of the significance of *X*, *Y*, …, *N*, *F*, *G*, …, *K*.

Start with the formula:

$$\iiint_{S_0} \delta W dx_0 dy_0 dz_0 + \delta \mathcal{T}_e = 0,$$

which is applied to an arbitrary portion of a medium that is bounded by a surface S_0.

Since δW must be identically null, by virtue of the invariance of *W* under the group of Euclidean displacements with the variations given by formulas (51), namely:

$$\delta x = (a_1 + \omega_2 z - \omega_3 y)dt,$$
$$\delta y = (a_2 + \omega_3 z - \omega_1 y)dt,$$
$$\delta z = (a_3 + \omega_1 z - \omega_2 y)dt,$$

and δI, δJ, δK by:

$$\delta I = \omega_1 \delta t, \qquad \delta J = \omega_2 \delta t, \qquad \delta K = \omega_3 \delta t,$$

and from this, and the expressions for $\delta \mathcal{T}_e$ on which we must insist [1], we conclude that one has:

$$\iint_{S_0} F_0 d\sigma_0 - \iiint_{S_0} X_0 dx_0 dy_0 dz_0 = 0,$$

$$\iint_{S_0} (I_0 + H_0 y - G_0 z)d\sigma_0 - \iiint_{S_0} (L_0 + Z_0 y - Y_0 z)dx_0 dy_0 dz_0 = 0,$$

and four analogous equations. These six formulas are easily deduced from the ones that one ordinarily writes by means of the principle of solidification.

One may imagine that the frontier S is variable in these formulas.

The auxiliary functions that were introduced in the preceding paragraphs are not the only ones that may be envisioned; if we confine ourselves to their consideration then we simply add a few obvious remarks.

By definition, we have introduced two systems of efforts and moments of deformation relative to a point *M* of the deformed medium. The first are the ones that are exerted on surfaces that have their normal parallel to one of the fixed axes *Ox*, *Oy*, *Oz* before deformation. The second are the ones that are exerted on surfaces that have their normal parallel to one of the same fixed axes *Ox*, *Oy*, *Oz*.

The formulas that we have indicated give the latter elements by means of the former; however, by an immediate solution, which we shall not stop to perform, one obtains, conversely, the former elements in terms of the latter.

Now suppose that we have introduced the function *W*. The former efforts and moments of deformation have the expressions we already gave, and one immediately deduces their expressions in terms of the latter from this. Nevertheless, in these calculations one may specify the functions that one must introduce according to the

[1] The passage from elements referred to the unit of volume of the undeformed medium and area of the frontier S_0 to the elements referred to unit of volume for the deformed medium and the area of the frontier *S* sufficiently immediate that it suffices to confine ourselves to the former as we have done, for example.

nature of the problem, and which will be, for example, x, y, z or x', y', z', and three parameters ([1]) $\lambda_1, \lambda_2, \lambda_3$ by means of which one expresses $\alpha, \alpha', \cdots, \gamma''$.

If one introduces $x, y, z, \lambda_1, \lambda_2, \lambda_3$, and if one continues to let W denote the function that depends on x_0, y_0, z_0, the first derivatives of x, y, z with respect to x_0, y_0, z_0 on $\lambda_1, \lambda_2, \lambda_3$, and their first derivatives with respect to x_0, y_0, z_0, and is obtained by replacing the different quantities $\xi_i, \eta_i, \zeta_i, p_i, q_i, r_i$ in the function $W(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i,)$ with their values as given by formulas (43) and (44), then one will have:

$$
\begin{aligned}
A_1 &= \frac{\partial W}{\partial \dfrac{\partial x}{\partial x_0}}, & A_2 &= \frac{\partial W}{\partial \dfrac{\partial x}{\partial y_0}}, & A_3 &= \frac{\partial W}{\partial \dfrac{\partial x}{\partial z_0}}, \\
B_1 &= \frac{\partial W}{\partial \dfrac{\partial y}{\partial x_0}}, & B_2 &= \frac{\partial W}{\partial \dfrac{\partial y}{\partial y_0}}, & B_3 &= \frac{\partial W}{\partial \dfrac{\partial y}{\partial z_0}}, \\
C_1 &= \frac{\partial W}{\partial \dfrac{\partial z}{\partial x_0}}, & C_2 &= \frac{\partial W}{\partial \dfrac{\partial z}{\partial y_0}}, & C_3 &= \frac{\partial W}{\partial \dfrac{\partial z}{\partial z_0}}, \\
\Pi_i &= \frac{\partial W}{\partial \dfrac{\partial \lambda_1}{\partial \rho_i}}, & \mathrm{X}_i &= \frac{\partial W}{\partial \dfrac{\partial \lambda_2}{\partial \rho_i}}, & \Sigma_i &= \frac{\partial W}{\partial \dfrac{\partial \lambda_3}{\partial \rho_i}}.
\end{aligned}
$$

55. Notion of energy of deformation. Theorem that leads to that of Clapeyron as a particular case. – Envision the two states, (M_0) and (M) of the deformable medium bounded by the surfaces (S_0) and (S), and consider an arbitrary sequence of states that start with (M_0) and end with (M). To that end, it suffices to consider functions $x, y, z, \alpha, \alpha', \cdots, \gamma''$ of x_0, y_0, z_0, and one variable h that reduce to $x_0, y_0, z_0, \alpha_0, \alpha'_0, \cdots, \gamma''_0$, respectively, when h is zero, and reduce to the values $x, y, z, \alpha, \alpha', \cdots, \gamma''$, respectively, for non-zero h relative to (M).

If we make the parameter h vary in a continuous fashion from 0 to h then we obtain a continuous deformation that permits us to pass from the state (M_0) to the state (M). For this continuous deformation, consider the *total work* performed by the forces and external moments that are applied to the different volume elements of the medium and by the efforts and moments of deformation that are applied to the surface elements of the frontier. To obtain this total work, it suffices to integrate the differential so obtained from 0 to h, starting with one of the expressions for δT_e in the preceding section and substituting the partial differentials that correspond to the increase dh in h for the variations of $x, y, z, \alpha, \alpha', \cdots, \gamma''$; the formula:

[1] For such auxiliary functions $\lambda_1, \lambda_2, \lambda_3$, one may take, for example, the components of the rotation that makes the axes Ox, Oy, Oz parallel to Mx', My', Mz', respectively.

$$\delta\mathcal{T}_e = -\iiint_{S_0} \delta W dx_0 dy_0 dz_0$$

gives the expression $-\iiint_{S_0} \frac{\partial W}{\partial h} dx_0 dy_0 dz_0$ for the value of $\delta\mathcal{T}_e$, and we obtain:

$$-\int_0^h \left(\iiint_{S_0} \frac{\partial W}{\partial h} dx_0 dy_0 dz_0 \right) dh = -\iiint_{S_0} (W_h - W_0) dx_0 dy_0 dz_0$$

for the total work. The work in question is independent of the intermediary states and depends only on the extreme states (M_0) and (M).

This leads us to introduce the notion of *energy of deformation,* which must be distinguished from that of the action of deformation that we previously envisioned. We say that $-W$ is the density of the *energy of deformation,* referred to the unit of volume of the undeformed medium.

The proposition that we must encounter, which determines the *total work* that is performed by the external forces and moments, as well as the efforts and moments of deformation that are applied to the frontier, gives CLAPEYRON'S *theorem* ([1]) when we consider an infinitely small deformation and specify the medium. Indeed, first introduce simply the hypothesis – and we refer to sec. **58** for the more general form – that W is a simple function of ε_1, ε_2, ε_3, λ_1, λ_2, λ_3. We may then envision the formulas:

$$\Omega_1 = \frac{\partial W}{\partial \varepsilon_1}, \qquad \Omega_2 = \frac{\partial W}{\partial \varepsilon_2}, \qquad \Omega_3 = \frac{\partial W}{\partial \varepsilon_3}, \qquad \Xi_1 = \frac{\partial W}{\partial \lambda_1}, \qquad \Xi_2 = \frac{\partial W}{\partial \lambda_2}, \qquad \Xi_3 = \frac{\partial W}{\partial \lambda_3},$$

as defining a change of variables that replaces the letters ε_1, ε_2, ε_3, λ_1, λ_2, λ_3 with the letters Ω_1, Ω_2, Ω_3, Ξ_1, Ξ_2, Ξ_3. By virtue of this change of variables, W becomes a function W' of Ω_1, Ω_2, Ω_3, Ξ_1, Ξ_2, Ξ_3.

Having said this, we pass to infinitely small deformations and put ourselves into the situation envisioned in sec. **31**, pp. 74-76, of our *Premier mémoire sur la théorie de l'élasticité;* W and W' become quadratic forms W_2 of e_1, e_2, e_3, g_1, g_2, g_3, and W_2', of $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$; the latter is, up to a factor of ¼, what one calls the *adjoint form* to W_2. When this is of issue, and in the case of infinitely small deformations, one obtains the following expression for the total work:

$$\iiint W_2 dx_0 dy_0 dz_0 .$$

---

[1] LAMÉ seems to have been credited with making CLAPEYRON'S theorem known in his Note to the *Comptes Rendus,* T. XXXV, pp. 459-464, 1852, then in his *Leçons sur la théorie mathématique de l'élasticité des corps solides,* (1st ed., 1852, 2nd ed., 1866); indeed, it was only in the 1st of February, 1858, that the following note appeared: CLAPEYRON, *Mémoire sur le travail des forces élastiques, dans un corps solide déformé par l'action de forces exterieures, Comtes rendus,* T. XLVI, pp. 208, 1858. Also consult TODHUNTER and PEARSON, *A History of the Theory of Elasticity,* etc., secs., **1041** and **1067-1070**.

To be more specific, if we suppose that we have ([1]):

$$W_2(e_i, g_i) = -\left(\frac{\lambda}{2} + \mu\right)(e_1 + e_2 + e_3)^2 - \frac{\mu}{2}(g_1^2 + g_2^2 + g_3^2 - 4e_2e_3 - 4e_3e_1 - 4e_1e_2),$$

then we have:

$$W_2'(\mathcal{N}_i, \mathcal{T}_i) = -\frac{1}{2}\left\{\frac{\mathcal{N}_1^2 + \mathcal{N}_2^2 + \mathcal{N}_3^2}{2\mu} - \frac{\lambda}{2\mu}\frac{(\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3)^2}{3\lambda + 2\mu} + \frac{\mathcal{T}_1^2 + \mathcal{T}_2^2 + \mathcal{T}_3^2}{\mu}\right\},$$

or:

$$W_2'(\mathcal{N}_i, \mathcal{T}_i) = -\frac{1}{2}\left\{\frac{1 + \dfrac{\lambda}{\mu}}{3\lambda + 2\mu}(\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3)^2 - \frac{1}{\mu}(\mathcal{N}_2\mathcal{N}_3 + \mathcal{N}_3\mathcal{N}_1 + \mathcal{N}_1\mathcal{N}_2 - \mathcal{T}_1^2 - \mathcal{T}_2^2 - \mathcal{T}_3^2)\right\}$$

One sees that one has recovered the result of LAMÉ precisely, if one remarks that the total work of the external forces and efforts on the frontier obviously reduces to the indicated expression in the case of infinitely small deformations.

56. Natural state of the deformable medium. – In the preceding we started with a natural state of a deformable medium and then we were given a state we called "deformed." We indicated the formulas that permit us to calculate external force and the analogous elements that are adjoined to the function *W* for the deformable medium and represent the action of deformation at a point.

As before, let us stop for a moment on this notion of *natural state.*

Up till now, the latter is a state that has not been subjected to any deformation. Imagine that the functions $x, y, z, \alpha, \alpha', \cdots, \gamma''$ that define the deformed state depend on one parameter, and that one recovers the natural state for a particular value of this parameter. The latter then seems to us to be a special case of a deformed state, and we are led to attempt to apply the notions relating to the latter to it.

Without changing the values of the elements that are defined by the formulas of sec. **52**, one may replace the function *W* with this function augmented by an arbitrary *definite* function of x_0, y_0, z_0, and, if one is inspired by the idea of *action* that we associate to the passage from the natural state (M_0) to the deformed state (M) then one may, if one prefers, suppose that *the function of* x_0, y_0, z_0 that is defined by the expression:

$$W(x_0, y_0, z_0, \xi_i^{(0)}, \eta_i^{(0)}, \varsigma_i^{(0)}, p_i^{(0)}, q_i^{(0)}, r_i^{(0)})$$

is identically null; however, the values obtained for the external force and the analogous elements with regard to the natural state will not necessarily be null. We say that they define the external force and the analogous elements relative to the natural state ([1]).

[1] E. and F. COSSERAT. – *Premier mémoire sur la théorie de l'élasticité,* pp. 77.

In our way of speaking, the natural state presents itself as the initial state of a sequence of deformed states, a state that we start with in order to study the deformation. As a result, one is led to demand that it is not possible to make one of the deformed states play the role that we have the natural state play, and that this must be true in such a way that the elements that we defined in sec. **52** (external force and moment, external effort and moment of deformation), which were calculated for the other deformed states, have the same values if one refers the first of these elements to the unit of volume of the deformed medium and the second of these to the unit of area of the deformed surface. This question may receive a response only if one introduces and specifies the notion of the action that corresponds to the passage from one deformed state to another state.

The simplest hypothesis consists of assuming that this latter action is obtained by subtracting the action that corresponds to the passage from the natural state (M_0) to the first deformed state (M') from the action that corresponds to the passage from the natural state to the second deformed state (M). With regard to (M'), if we denote the quantities that are analogous ([2]) to ξ_i, η_i, ζ_i, p_i, q_i, r_i *relative to* (M) by $\xi'_i, \eta'_i, \varsigma'_i,\ \ p'_i, q'_i, r'_i,$ then we are led to adopt the following expression for the action of the deformation relating to the passage from the state (M') to the state (M):

$$(52)\qquad \iiint_{S_0} \{W(x_0, y_0, z_0, \xi_i, \eta_i, \varsigma_i, p_i, q_i, r_i) - W(x_0, y_0, z_0, \xi'_i, \eta'_i, \varsigma'_i, p'_i, q'_i, r'_i)\} dx_0 dy_0 dz_0,$$

which one may write, if Δ' is the value of Δ for (M'):

$$(53)\qquad \iiint_{S_0} W'_0(x_0, y_0, z_0, \xi_i, \eta_i, \varsigma_i, p_i, q_i, r_i) \,|\, \Delta' \,|\, dx_0 dy_0 dz_0,$$

in which we have let S' denote the surface of (M') that corresponds to S_0 for (M_0), and $W'_0(x_0, y_0, z_0, \xi_i, \eta_i, \varsigma_i, p_i, q_i, r_i)$ denotes the expression:

$$\{W(x_0, y_0, z_0, \xi_i, \eta_i, \varsigma_i, p_i, q_i, r_i) - W(x_0, y_0, z_0, \xi'_i, \eta'_i, \varsigma'_i, p'_i, q'_i, r'_i)\} \frac{1}{|\Delta'|}.$$

Furthermore, from the remark made at the beginning of this paragraph, one may, if one prefers, substitute the following expressions for (33):

$$(53')\qquad \iiint_{S_0} W'(x_0, y_0, z_0, \xi_i, \eta_i, \varsigma_i, p_i, q_i, r_i) \,|\, \Delta' \,|\, dx_0 dy_0 dz_0,$$

---

[1] We may then speak of the force, effort, etc., since we regard the natural state as the limit of a sequence of states for which we know the force, effort, etc. Up till now, the force, effort, etc. were defined for us only when there was a deformation capable of manifesting and measuring them.

[2] One must remark that $\xi'_i, \eta'_i, \varsigma'_i, p'_i, q'_i, r'_i$ are not analogous to $\xi_i^{(0)}, \eta_i^{(0)}, \varsigma_i^{(0)}, p_i^{(0)}, q_i^{(0)}, r_i^{(0)}$, because they are not formed by means of the coordinates x', y', z' of (M') in the same way that $\xi_i^{(0)}, \eta_i^{(0)}, \varsigma_i^{(0)}$, $p_i^{(0)}, q_i^{(0)}, r_i^{(0)}$ are formed by means of x_0, y_0, z_0.

in which $W'(x_0, y_0, z_0, \xi_i, \eta_i, \varsigma_i, p_i, q_i, r_i)$ denotes the expression:

$$W(x_0, y_0, z_0, \xi_i, \eta_i, \varsigma_i, p_i, q_i, r_i)\frac{1}{|\Delta'|}.$$

If one remarks that one has, for example:

$$|\Delta'|\frac{\partial W'(x_0, y_0, z_0, \xi_i, \cdots, r_i)}{\partial \xi_i} = \frac{\partial W(x_0, y_0, z_0, \xi_i, \cdots, r_i)}{\partial \xi_i},$$

then it is clear that applying formulas that are analogous to those of sec. **52** to expressions (53) or (53′) and starting with (M') as the natural state, *but while supposing that* (M') *is referred to the system of coordinates* x_0, y_0, z_0, *and assuming that the formulas of sec.* **52** *are modified as a consequence*, will give the same values for the exterior force and moment relative to the state (M) referred to the unit of volume of (M), as well as the same values for the effort and the moment of deformation referred to the unit of area for (S).

Therefore we may consider (M) to be a deformed state for which (M') is a natural state, provided that the function W associated with the state (M) is actually ([1]) W_0' or W'.

Conforming to these indications, suppose, to fix ideas, that the external force and moment are given by means of simple functions of x_0, y_0, z_0 and elements that fix the position of the triad $Mx'y'z'$. Suppose, moreover, that the natural state is given. We may consider the equations of sec. **52** relating to the external force and moment to be partial differential equations in the unknowns x, y, z and the three parameters λ_1, λ_2, λ_3 by means of which one may express $\alpha, \alpha', \cdots, \gamma''$. The expressions ξ_i, η_i, ζ_i, p_i, q_i, r_i are then functions of $\frac{\partial x}{\partial \rho_i}, \frac{\partial y}{\partial \rho_i}, \frac{\partial z}{\partial \rho_i}, \lambda_1, \lambda_2, \lambda_3, \frac{\partial \lambda_1}{\partial \rho_i}, \frac{\partial \lambda_2}{\partial \rho_i}, \frac{\partial \lambda_3}{\partial \rho_i}$ (always setting $\rho_1 = x_0$, $\rho_2 = y_0$, $\rho_3 = z_0$) that one calculates by means of formulas (43) and (44).

Suppose that $X_0', Y_0', Z_0', L_0', M_0', N_0'$, or, what amounts to the same thing, X_0, Y_0, Z_0, L_0, M_0, N_0 are given functions of x_0, y_0, z_0, x, y, z, λ_1, λ_2, λ_3. The expression W is, after substituting for the values of ξ_i, η_i, ζ_i, p_i, q_i, r_i by means of formulas (43) and (44), a definite function of x_0, y_0, z_0, $\frac{\partial x}{\partial x_0}, \cdots, \frac{\partial z}{\partial z_0}, \lambda_1, \lambda_2, \lambda_3, \frac{\partial \lambda_1}{\partial x_0}, \cdots, \frac{\partial \lambda_3}{\partial z_0}$, which we continue to denote by W, and the equations of the problem may be written:

[1] As we said at the beginning of this section, this permits us to generalize the notion of natural state that we first introduced. Instead of making this word correspond to the idea of a particular state, we may, in a more general fashion, make it correspond to the idea of an arbitrary state, starting from which we may study the deformation. The fact that we introduced x_0, y_0, z_0 at the beginning of the theory seems to make (M_0) play a particular role; however, one must not consider x_0, y_0, z_0 as anything but the coordinates that serve to define the *different media*, and not only (M_0). One has chosen these coordinates in a particular fashion, and in relation to a particular medium, in order that one must, as a result, pay attention to (M_0) in the context of infinitely small deformations.

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial x}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial x}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial x}{\partial z_0}}=X_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial y}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial y}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial y}{\partial z_0}}=Y_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial z}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial z}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial z}{\partial z_0}}=Z_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial z_0}}-\frac{\partial W}{\partial \lambda_1}=\mathcal{L}_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial z_0}}-\frac{\partial W}{\partial \lambda_2}=\mathcal{M}_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial z_0}}-\frac{\partial W}{\partial \lambda_3}=\mathcal{N}_0,$$

in which $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are functions of x_0, y_0, z_0, x, y, z, λ_1, λ_2, λ_3 that result from the definitions of sec. **53**.

It results directly from the formulas of the preceding paragraphs that a more immediate way of defining X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ may be summarized in the relation:

$$\delta\iiint W dx_0 dy_0 dz_0 + \delta\mathcal{T}_e = 0,$$

i.e., in:

$$\delta\iiint W dx_0 dy_0 dz_0 = \iint (F_0\delta x + G_0\delta y + H_0\delta z + \mathcal{I}_0\delta\lambda_1 + \mathcal{J}_0\delta\lambda_2 + \mathcal{K}_0\delta\lambda_3)\,d\sigma$$
$$-\iiint (X_0\delta x + Y_0\delta y + Z_0\delta z + \mathcal{L}_0\delta\lambda_1 + \mathcal{M}_0\delta\lambda_2 + \mathcal{N}_0\delta\lambda_3)\,dx_0 dy_0 dz_0$$

57. Notions of hidden triad and hidden *W*. – In the study of deformable media, as in the study of deformable lines and surfaces, it is natural to pay particular attention to the *pointlike media* that are described by the deformable media. This amounts to envisioning *x, y, z* separately and considering $\alpha, \alpha', \cdots, \gamma''$ as simply auxiliary functions. This is what we likewise express by imagining that one ignores the existence of the triads that determine the deformable medium, and that one knows only the vertices of those triads. If we adopt that viewpoint in order to envision the partial differential equations that one is led to in this case then we may introduce the notion of *hidden triad,* and we are led to a resulting classification of the diverse circumstances that may be produced by the elimination the $\alpha, \alpha', \cdots, \gamma''$.

Therefore, a primary study that presents itself is that of the reductions that relate to the elimination of the $\alpha, \alpha', \cdots, \gamma''$. Likewise, in the corresponding particular cases in which the attention is directed almost exclusively to the pointlike media that are described by the deformed medium (M) one may sometimes abstract from (M_0), and, as a result, from the deformation that permits us to pass from (M_0) to (M).

As we already said for the deformable line and surface, the triad may be employed in another fashion. We may make particular hypotheses on it and the medium (M); all of this amounts to envisioning particular deformations of the free deformable line. If the relations that we impose are simple relations between $\xi_i, \eta_i, \zeta_i, p_i, q_i, r_i$, as will be the case in the applications that we shall study, we may account for these relations in the calculation of W and deduce more particular functions from W. The interesting question that this poses is that of introducing these particular forms simply, and to consider the general W that serves as the point of departure as being hidden, in some sense. We thus have a *theory that will be specific to the particular deformations brought to light by the given relations between* $\xi_i, \eta_i, \zeta_i, p_i, q_i, r_i$.

We confirm that by means of the theory of free deformable media one may therefore combine the particular cases and provide a common origin to the equations that are the result of special theories that one encounters in physics ([1]).

In the particular cases, one sometimes finds oneself in the proper circumstances to avoid the consideration of these deformations; in reality, they must sometimes be completed. This is what one may do in practical applications when one envisions infinitely small deformations.

Take the case in which the external force and moment refer only to the first derivatives of the unknowns x, y, z and $\lambda_1, \lambda_2, \lambda_3$; the second derivatives of these unknowns will be introduced into these partial differential equations only for W; however, the derivatives of x, y, z figure only in ξ_i, η_i, ζ_i, and those of $\lambda_1, \lambda_2, \lambda_3$ show up only in p_i, q_i, r_i. One therefore sees that if W depends only on ξ_i, η_i, ζ_i, or only on p_i, q_i, r_i, then there will be a reduction in the order of the derivatives that enter into the partial differential equations. Here, we examine the first of these two cases, which corresponds to the ordinary theory of elasticity for material media and to the theory of the various ethereal media that are envisioned in the doctrine of luminous waves.

58. Case in which W depends only on $x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i$, and is independent of p_i, q_i, r_i. How one recovers the equations that relate to the deformable body of the classical theory and to the media of hydrostatics. – Suppose that W depends only on the quantities $x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i$, and not on p_i, q_i, r_i. The equations of sec. **56**, which reduce to the following:

[1] All of our considerations heretofore may be applied just the same to material media as to various ethereal media. We have declared the word *matter* to be invalid, and what we expose is, as we said to begin with, a *theory of action for extension and movement.* To have a more complete idea of the notion of matter, we shall explain later on how one must approach the latter from the concept of *entropy* according to the profound viewpoint that LIPPMANN introduced into electricity.

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial x}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial x}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial x}{\partial z_0}}=X_0, \qquad \frac{\partial W}{\partial \lambda_1}+\mathcal{L}_0=0,$$
$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial y}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial y}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial y}{\partial z_0}}=Y_0, \qquad \frac{\partial W}{\partial \lambda_2}+\mathcal{M}_0=0,$$
$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial z}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial z}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial z}{\partial z_0}}=Z_0, \qquad \frac{\partial W}{\partial \lambda_3}+\mathcal{N}_0=0,$$

in which W depends only on x_0, y_0, z_0, $\frac{\partial x}{\partial x_0},\cdots,\frac{\partial z}{\partial z_0}$, λ_1, λ_2, λ_3, we show that if one takes the simple case in which X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are given functions ([1]) of x_0, y_0, z_0, x, y, z, $\frac{\partial x}{\partial x_0},\cdots,\frac{\partial z}{\partial z_0}$, λ_1, λ_2, λ_3 then the three equations may be solved for λ_1, λ_2, λ_3, and one finally obtains three partial differential equations that, from our hypotheses, refer to only the x_0, y_0, z_0, and to x, y, z, and their first and second derivatives.

First, envision the particular case in which the given functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are null; the same will be true for the corresponding values of the functions of one of the systems (L_0', M_0', N_0'), (L_0, M_0, N_0),(L, M, N). It results from this that the equations:

$$\frac{\partial W}{\partial \lambda_1}=0, \qquad \frac{\partial W}{\partial \lambda_2}=0, \qquad \frac{\partial W}{\partial \lambda_3}=0,$$

amount to:

$$C_1\frac{\partial y}{\partial x_0}+C_2\frac{\partial y}{\partial y_0}+C_3\frac{\partial y}{\partial z_0}-B_1\frac{\partial z}{\partial x_0}-B_2\frac{\partial z}{\partial y_0}-B_3\frac{\partial z}{\partial z_0}=0,$$
$$A_1\frac{\partial z}{\partial x_0}+A_2\frac{\partial z}{\partial y_0}+A_3\frac{\partial z}{\partial z_0}-C_1\frac{\partial x}{\partial x_0}-C_2\frac{\partial x}{\partial y_0}-C_3\frac{\partial x}{\partial z_0}=0,$$
$$B_1\frac{\partial x}{\partial x_0}+B_2\frac{\partial x}{\partial y_0}+B_3\frac{\partial x}{\partial z_0}-A_1\frac{\partial y}{\partial x_0}-A_2\frac{\partial y}{\partial y_0}-A_3\frac{\partial y}{\partial z_0}=0,$$

i.e.,

$$p_{yz}=p_{zy}, \qquad p_{zx}=p_{xz}, \qquad p_{xy}=p_{yx},$$

whose interpretation is immediate.

Haing said this, observe that if one of the two positions (M_0) and (M) is assumed to be *given*, and that if one deduces the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ from this, as in sec. **53**, then in the case in which these three functions are null one may arrive at this result accidentally,

[1] In order to simplify the exposition, and to indicate more easily what we are alluding to, we suppose that X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ do not refer to the derivatives of λ_1, λ_2, λ_3.

i.e., for a certain set of particular deformations; however, one may arrive at this result for any deformation (*M*) since it is a consequence of the nature of the medium (*M*), i.e., of the form of *W*.

Consider this latter case, which is particularly interesting; *W* is then a simple function ([1]) of ρ_1, ρ_2, ρ_3, and the six expressions ε_1, ε_2, ε_3, λ_1, λ_2, λ_3, which are defined by the formulas (45).

The equations deduced from sec. **52** and **53** reduce to either:

$$\sum_i \left(\frac{\partial A'_i}{\partial \rho_i} + q_i C'_i - r_i B'_i \right) = X'_0, \qquad F'_0 = l_0 A'_1 + m_0 A'_2 + n_0 A'_3,$$

$$\sum_i \left(\frac{\partial B'_i}{\partial \rho_i} + r_i A'_i - p_i C'_i \right) = Y'_0, \qquad G'_0 = l_0 B'_1 + m_0 B'_2 + n_0 B'_3,$$

$$\sum_i \left(\frac{\partial C'_i}{\partial \rho_i} + p_i B'_i - q_i A'_i \right) = Z'_0, \qquad H'_0 = l_0 C'_1 + m_0 C'_2 + n_0 C'_3,$$

in which one has:

$$\left.\begin{aligned} A'_i &= \xi_i \frac{\partial W}{\partial \varepsilon_i} + \xi_k \frac{\partial W}{\partial \gamma_j} + \xi_j \frac{\partial W}{\partial \gamma_k} \\ B'_i &= \eta_i \frac{\partial W}{\partial \varepsilon_i} + \eta_k \frac{\partial W}{\partial \gamma_j} + \eta_j \frac{\partial W}{\partial \gamma_k} \\ C'_i &= \varsigma_i \frac{\partial W}{\partial \varepsilon_i} + \varsigma_k \frac{\partial W}{\partial \gamma_j} + \varsigma_j \frac{\partial W}{\partial \gamma_k} \end{aligned}\right\} \qquad (i, j, k = 1, 2, 3).$$

or to ([2]):

$$\frac{\partial A_1}{\partial x_0} + \frac{\partial A_2}{\partial y_0} + \frac{\partial A_3}{\partial z_0} = X_0, \qquad F_0 = l_0 A_1 + m_0 A_2 + n_0 A_3,$$

$$\frac{\partial B_1}{\partial x_0} + \frac{\partial B_2}{\partial y_0} + \frac{\partial B_3}{\partial z_0} = Y_0, \qquad G_0 = l_0 B_1 + m_0 B_2 + n_0 B_3,$$

$$\frac{\partial C_1}{\partial x_0} + \frac{\partial C_2}{\partial y_0} + \frac{\partial C_3}{\partial z_0} = Z_0, \qquad H_0 = l_0 C_1 + m_0 C_2 + n_0 C_3,$$

in which one has:

$$A_1 = \Omega_1 \frac{\partial x}{\partial x_0} + \Xi_1 \frac{\partial x}{\partial y_0} + \Xi_2 \frac{\partial x}{\partial z_0},$$

$$A_2 = \Xi_3 \frac{\partial x}{\partial x_0} + \Omega_2 \frac{\partial x}{\partial y_0} + \Xi_1 \frac{\partial x}{\partial z_0},$$

$$A_3 = \Xi_2 \frac{\partial x}{\partial x_0} + \Xi_1 \frac{\partial x}{\partial y_0} + \Omega_3 \frac{\partial x}{\partial z_0},$$

[1] The triad is completely hidden; we may also conceive that we have a simple pointlike medium.

[2] Compare E. and F. COSSERAT. – *Premier Mémoire sur la théorie de l'élasticité,* pp. 45, 46, 65.

$$B_1 = \Omega_1 \frac{\partial y}{\partial x_0} + \Xi_1 \frac{\partial y}{\partial y_0} + \Xi_2 \frac{\partial y}{\partial z_0},$$
$$B_2 = \Xi_3 \frac{\partial y}{\partial x_0} + \Omega_2 \frac{\partial y}{\partial y_0} + \Xi_1 \frac{\partial y}{\partial z_0},$$
$$B_3 = \Xi_2 \frac{\partial y}{\partial x_0} + \Xi_1 \frac{\partial y}{\partial y_0} + \Omega_3 \frac{\partial y}{\partial z_0},$$
$$C_1 = \Omega_1 \frac{\partial z}{\partial x_0} + \Xi_1 \frac{\partial z}{\partial y_0} + \Xi_2 \frac{\partial z}{\partial z_0},$$
$$C_2 = \Xi_3 \frac{\partial z}{\partial x_0} + \Omega_2 \frac{\partial z}{\partial y_0} + \Xi_1 \frac{\partial z}{\partial z_0},$$
$$C_3 = \Xi_2 \frac{\partial z}{\partial x_0} + \Xi_1 \frac{\partial z}{\partial y_0} + \Omega_3 \frac{\partial z}{\partial z_0},$$

in which we set $\Omega_i = \frac{\partial W}{\partial \varepsilon_i}, \Xi_i = \frac{\partial W}{\partial \gamma_i}$, to abbreviate notation, or we get ([1]):

$$\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} = X, \qquad F = lp_{xx} + mp_{yx} + np_{zx},$$
$$\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} = Y, \qquad G = lp_{xy} + mp_{yy} + np_{zy},$$
$$\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} = Z, \qquad H = lp_{xz} + mp_{yz} + np_{zz},$$

in which one has:

$$p_{xx} = \frac{1}{\Delta}\left[\Omega_1\left(\frac{\partial x}{\partial x_0}\right)^2 + \Omega_2\left(\frac{\partial x}{\partial y_0}\right)^2 + \Omega_3\left(\frac{\partial x}{\partial z_0}\right)^2 + 2\Xi_1 \frac{\partial x}{\partial y_0}\frac{\partial x}{\partial z_0} + 2\Xi_2 \frac{\partial x}{\partial z_0}\frac{\partial x}{\partial x_0} + 2\Xi_3 \frac{\partial x}{\partial x_0}\frac{\partial x}{\partial y_0}\right],$$

and analogous formulas for p_{yz}, … Δ has the significance that we gave it in sec. **51**, which we shall recall in a moment.

As one sees, we recover the continuous deformable medium as it is treated in the ordinary theory of elasticity.

A particularly interesting case is obtained by looking for a form for *W* that gives the identities:

$$p_{yz} = 0, \qquad p_{yx} = 0, \qquad p_{xy} = 0,$$

for any $\frac{\partial x}{\partial x_0}, \cdots$ One finds that *W* must be a simple function of x_0, y_0, z_0, and the expression Δ, which is defined by the formulas ([1]):

[1] Compare E. and F. COSSERAT. – *Premier Mémoire sur la théorie de l'élasticité*, pp. 40, 44, 65.

$$\Delta = \frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)}, \qquad \Delta^2 = \begin{vmatrix} 1+2\varepsilon_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & 1+2\varepsilon_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & 1+2\varepsilon_3 \end{vmatrix},$$

from which one may see, upon remarking that if one refers to the previous formulas ([2]) that gave us p_{yz}, p_{yx}, p_{zx},… as a function of A_1, … then one has:

$$\frac{\dfrac{\partial W}{\partial \dfrac{\partial x}{\partial x_0}}}{\dfrac{\partial \Delta}{\partial \dfrac{\partial x}{\partial x_0}}} = \frac{\dfrac{\partial W}{\partial \dfrac{\partial x}{\partial y_0}}}{\dfrac{\partial \Delta}{\partial \dfrac{\partial x}{\partial y_0}}} = \frac{\dfrac{\partial W}{\partial \dfrac{\partial x}{\partial z_0}}}{\dfrac{\partial \Delta}{\partial \dfrac{\partial x}{\partial z_0}}},$$

and two analogous systems; since W is assumed to be a simple function of x_0, y_0, z_0, and Δ, one has, as a result:

$$p_{xx} = p_{yy} = p_{zz} = \frac{\partial W}{\partial \Delta}.$$

If we consider the particular case in which W depends only on Δ, and if we assume that we are given X, Y, Z expressed as functions of x, y, z then the equations in question, which are:

$$\frac{\partial p}{\partial x} = X, \qquad \frac{\partial p}{\partial y} = Y, \qquad \frac{\partial p}{\partial z} = Z, \qquad F = lp, \qquad G = mp, \qquad H = np,$$

upon setting $p = \dfrac{\partial W}{\partial \Delta}$, become those which serve as the basis for hydrostatics ([3]). The initial medium (M_0) appears only by way of Δ, and one may replace the unknown Δ with the unknown p that is related to it by the relation $p = \dfrac{\partial W}{\partial \Delta}$. If the function W, which is not given, is *hidden* then one has the preceding equations, in which p is an auxiliary function whose significance is well known.

It will suffice for us to indicate that the case in which the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are non-null comprises the theory of all the ethereal media that have been considered for the study of luminous waves from MACCULLAGH to LORD KELVIN, but here the theory of these media is completely mechanical. We likewise mention that the most general

[1] Compare E. and F. COSSERAT. – *Premier Mémoire sur la théorie de l'élasticité*, pp. 23, 24.

[2] These formulas are actually the ones on page 47 of our *Premier Mémoire sur la théorie de l'élasticité.*

[3] Compare DUHEM. – *Hydrodynamique, Elasticité, Acoustique.*

case, in which the trace of the derivatives of the action W with respect to the rotations p_i, q_i, r_i remains in the expression for the external moment leads in the most natural manner to the notion of *magnetic induction* that was introduced by MAXWELL.

59. The rigid body. – We have considered the particular case in which W does not depend on p_i, q_i, r_i, and different special cases of this case. One may arrive at the other media that were considered, at least in part, by the authors, either by the study of particular deformations, or by the study of new media that are defined by a theory of constraints that profits from the results that we already acquired.

For example, start with the simple case, in which the triad is *hidden,* i.e., by definition, it is a *pointlike* medium in which W is a function of x_0, y_0, z_0, ε_1, ε_2, ε_3, γ_1, γ_2, γ_3.

1. We may imagine that one pays attention only to the deformations of the medium for which one has:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0.$$

In the definitions of forces, etc., it suffices to introduce these hypotheses, and, if the forces are given, to introduce these six conditions. In the latter case, the *habitual* problems, which correspond to the given of the function W, and to the general case in which the ε_i, γ_i are non-null, may be posed only for particular givens.

If we suppose *only* that the function W_0 that is obtained by taking $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0$ in $W(\rho_1, \rho_2, \varepsilon_1, \ldots)$ is given, that one does not know the values of the derivatives of W with respect to $\varepsilon_1, \varepsilon_2, \ldots, \gamma_3$ for $\varepsilon_1 = \varepsilon_2 = \ldots = \gamma_3 = 0$, so that W is *hidden,* then we see that $p_{xx}, \ldots, p_{zz}$, for example, become six auxiliary functions that one must adjoin to x, y, z, in such a way that, for the case in which the forces that act on the volume elements are given, we have nine partial differential equations in nine unknowns in the case, to which one must adjoin accessory conditions.

Now we remark that one knows how to integrate the system:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0.$$

Since the deformation is supposed continuous, the integral corresponds to a displacement of the set of the medium; it thus remains for us to determine the six constants of integration and the auxiliary functions $p_{xx}, \ldots$

If the forces and efforts that act on the medium are given, and we suppose that $X, \ldots$ are known as functions of x, y, z then the six equations of sec. **54**, with the simplifications implied for the form of W, when applied to the entire body, determine the six integration constants. To complete the process, what remains is for us to *ultimately* determine $p_{xx}, \ldots$

If we leave aside the problem of this ultimate determination, then one sees that we recover the habitual problems of the mechanics of rigid bodies, in which one might ordinarily suppose that the hidden function W depends only on Δ.

2. We may imagine that we seek to define a medium whose definition already takes the conditions:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0$$

into account, *sui generis*.

In order to define the new medium, while thinking along the same lines as before, we further define $F_0', \cdots, N_0'$ by the identity:

$$\iiint_{S_0} \delta W dx_0 dy_0 dz_0 = \iint_{S_0} (F_0'\delta' x + \cdots + K_0'\delta K')d\sigma_0$$
$$- \iiint_{S_0} (X_0'\delta' x + \cdots + N_0'\delta K')dx_0 dy_0 dz_0.$$

However, this identity must no longer hold, by virtue of the fact that $\varepsilon_1 = \ldots = \gamma_3 = 0$. In other words, we envision a medium in which the theory must result from the *a posteriori* addition of the conditions $\varepsilon_1 = \ldots = \gamma_3 = 0$ to the knowledge of a function $W(x_0, y_0, z_0, \varepsilon_1, \varepsilon_2, \ldots, \gamma_3)$ and six auxiliary functions $\mu_1, \ldots, \mu_6$ of x_0, y_0, z_0, by means of the identity:

$$\iiint_{S_0} (\delta W + \mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \cdots + \mu_6\gamma_3)dx_0 dy_0 dz_0 = \iint_{S_0} (F_0'\delta' x + \cdots)d\sigma_0$$
$$- \iiint_{S_0} (X_0'\delta' x + \cdots)dx_0 dy_0 dz_0,$$

which amounts to setting $\varepsilon_1 = \ldots = \gamma_3 = 0$ in the general theory that preceded, in which one has replaced W with $W_1 = W + \mu_1\varepsilon_1 + \ldots + \mu_6\varepsilon_3$.

As one sees, we come down to the *theory of elastic media that correspond to the function W of* $x_0, y_0, z_0, \varepsilon_1, \varepsilon_2, \ldots, \gamma_3$ *when one restricts oneself to the study of deformations that correspond to* $\varepsilon_1 = \ldots = \gamma_3 = 0$. Therefore, if we consider the case of a *hidden W* then if we suppose that we known simply the value $W(x_0, y_0, z_0)$ that W and W_1 take simultaneously when $\varepsilon_1 = \ldots = \gamma_3 = 0$ then we recover the habitual theory of the rigid body.

Observe that if we account for the conditions $\varepsilon_1 = \ldots = \gamma_3 = 0$ in *W a priori* by a change of auxiliary functions then we are led to replace W with $\mu_1\varepsilon_1 + \ldots + \mu_6\varepsilon_3$ in the calculations that relate to the general medium, and we likewise find formulas that come down to the study of an elastic medium in which we are confined to studying deformations that correspond to $\varepsilon_1 = \ldots = \gamma_3 = 0$. Upon supposing that $\mu_1, \ldots, \mu_6$ are *unknown*, we once more come down to theory that comprises the habitual theory of the rigid body. From this latter viewpoint, we return to the exposition that one may make about the ideas of LAGRANGE. In particular, we may observe that in the case in which X_0, Y_0, Z_0 are given as the partial derivatives with respect to x, y, z of a function φ of x_0, y_0, z_0, x, y, z the equations in which X_0, Y_0, Z_0 figure are none other than the equations that one is led to when one seeks to determine the extremum of the integral:

$$\iiint \varphi dx_0 dy_0 dz_0,$$

given the conditions:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0.$$

3. We discuss a third procedure ([1]) for constituting a medium for which the theory always leads to the same equations, and which will be a limiting case of the original theory. This procedure agrees with the first one, and it may also be applied to the cases of the deformable line and surface.

Imagine that the W that serves to define the original medium is variable, and, to fix ideas, suppose that the values of $\varepsilon_1, \ldots, \gamma_3$ are developable in a MACLAURIN series in a neighborhood of zero by the formula:

$$W = W_1 + W_2 + \ldots + W_i + \ldots,$$

in which W_i represents the set of terms of the i^{th} degree. Assume that the coefficients of W_2 (which may depend on x_0, y_0, z_0) increase indefinitely in their variation. *If we want W to conserve a finite value* then we must suppose that $\varepsilon_1, \ldots, \gamma_3$ tend towards zero. In other words, we may then consider only deformations that satisfy $\varepsilon_1 = \ldots = \gamma_3 = 0$. In other words, the body that we approach in the limit may take only displacements of the set. We may suppose that one makes the derivatives $\frac{\partial W}{\partial \varepsilon_1}, \cdots$, which approach limits when W varies in a manner we shall describe, likewise vary as a consequence of a studied deformation for this medium.

To explain this in a more precise fashion, imagine that the coefficients of $W_1, W_2, \ldots$ depend on one parameter h, in such a way that when h tends towards zero the coefficients of W_2 increase indefinitely. To fix ideas, suppose that the latter coefficients are linear with respect to $\frac{1}{h}$. Likewise, imagine that x, y, z, which define the deformation in question, vary with h in such a way that $\varepsilon_1, \ldots$ tend to zero. In addition, we suppose that $\varepsilon_1, \ldots$ are infinitely small of first order with respect to h; for example, $\varepsilon_1, \ldots$might be developed in powers of h, and the first terms of that development are the ones in h. With these conditions, W tends to zero, and $\frac{\partial W}{\partial \varepsilon_1}, \cdots, \frac{\partial W}{\partial \gamma_3}$ tend to certain limits (which may be functions of x_0, y_0, z_0). Therefore if we consider the equations of sec. **52** that serve to define external force and moment then we are finally led to formulas that permit us to define them, and which are none other than equations of our point of departure, *in which the notion of the function W has disappeared,* and in which six auxiliary functions $F_0', G_0', H_0', I_0', J_0', K_0'$ figure.

60. Deformable media in motion. – The theory of motion for the deformable line and that of the motion of the deformable surface present themselves very naturally as special cases of the theory of the deformable surface and that of the deformable medium. To see this, it suffices to give one of the parameters ρ_i of the surface or medium the significance of time. As we will not envision the statics of media of dimension greater than three here, we must expose the theory of motion of a deformable medium directly in

[1] Compare THOMSON and TAIT. – *Treatise,* vol. I., Part. I, pp. 271, starting with the 11th line down.

what follows; however, we nevertheless give it a form that is entirely analogous to the one that we indicated for the dynamics of deformable line and the deformable surface.

Consider a space (M_0) that is described by a point M_0 whose coordinates are x_0, y_0, z_0 with respect to the three fixed rectangular axes *Ox, Oy, Oz,* and adjoin a trirectangular triad to each point M_0 of the space (M_0) whose axes $M_0x_0', M_0y_0', M_0z_0'$ have the direction cosines $\alpha_0, \alpha_0', \alpha_0''; \beta_0, \beta_0', \beta_0''; \gamma_0, \gamma_0', \gamma_0''$ with respect to the axes *Ox, Oy, Oz,* respectively, and which are functions of the independent variables x_0, y_0, z_0.

The continuous three-dimensional set of such triads $M_0x_0'y_0'z_0'$ may be considered as the position at a definite instant t of a deformable medium that is defined in the following fashion:

Give the point M_0 a displacement M_0M, which is a function of time t and the position of the point M_0, and is null for $t = t_0$. Let x, y, z be the coordinates of the point M, which we consider to be functions of x_0, y_0, z_0, t. In addition, endow the triad $M_0x_0'y_0'z_0'$ with a rotation that makes its axes finally agree with those of a triad $Mx'y'z'$ that we adjoin to the point M. We define that rotation by giving the direction cosines $\alpha, \alpha', \alpha''$; β, β', β''; $\gamma, \gamma', \gamma''$ of the axes Mx', My', Mz' with respect to the fixed axes *Ox, Oy, Oz.* Like x, y, z, these cosines will be functions of x_0, y_0, z_0, t.

The continuous three-dimensional set of triads $Mx'y'z'$, for a given value of time t, will be what we call the *deformed state* of the deformable medium considered at the instant t. The continuous four-dimensional set of triads $Mx'y'z'$ that is obtained by making t vary will be the *trajectory of the deformed state* of the deformable medium.

For ease of writing and notation in the sequel, we sometimes introduce, as we already did, the letters ρ_1, ρ_2, ρ_3, instead of x_0, y_0, z_0. We continue to denote the components of the velocity of the origin M_0 of the axes $M_0x_0', M_0y_0', M_0z_0'$ along these axes by $\xi_i^{(0)}, \eta_i^{(0)}, \varsigma_i^{(0)}$, *when* ρ_i *alone varies,* and the projections of the instantaneous rotation, relative to the parameter ρ_i, of the triad $M_0x_0'y_0'z_0'$ on these same axes by $p_i^{(0)}, q_i^{(0)}, r_i^{(0)}$. We denote the analogous expressions for the triad $Mx'y'z'$ by ξ_i, η_i, ζ_i, and p_i, q_i, r_i, when one refers them, like the triad $M_0x_0'y_0'z_0'$, to the fixed axes *Oxyz.*

When time t varies, and the motion of the triad $Mx'y'z'$ is referred to the fixed triad *Oxyz* then the origin M has a velocity whose components along the axes Mx', My', Mz' will be designated by ξ, η, ζ, and the instantaneous rotation of the triad $Mx'y'z'$ will be defined by the components p, q, r.

The elements that must introduce are calculated as in sec. **49**; first, one has the formulas:

$$(54)\qquad \begin{cases} \xi_i = \alpha\dfrac{\partial x}{\partial \rho_i} + \alpha'\dfrac{\partial y}{\partial \rho_i} + \alpha''\dfrac{\partial z}{\partial \rho_i}, \\ \eta_i = \beta\dfrac{\partial x}{\partial \rho_i} + \beta'\dfrac{\partial y}{\partial \rho_i} + \beta''\dfrac{\partial z}{\partial \rho_i}, \\ \varsigma_i = \gamma\dfrac{\partial x}{\partial \rho_i} + \gamma'\dfrac{\partial y}{\partial \rho_i} + \gamma''\dfrac{\partial z}{\partial \rho_i}, \end{cases} \qquad (55)\qquad \begin{cases} p_i = \sum \gamma\dfrac{\partial \beta}{\partial \rho_i} = -\sum \beta\dfrac{\partial \gamma}{\partial \rho_i}, \\ q_i = \sum \alpha\dfrac{\partial \gamma}{\partial \rho_i} = -\sum \gamma\dfrac{\partial \alpha}{\partial \rho_i}, \\ r_i = \sum \beta\dfrac{\partial \alpha}{\partial \rho_i} = -\sum \alpha\dfrac{\partial \beta}{\partial \rho_i}, \end{cases}$$

to which we adjoin the following:

$$(54')\quad \begin{cases} \xi = \alpha\dfrac{\partial x}{\partial t} + \alpha'\dfrac{\partial y}{\partial t} + \alpha''\dfrac{\partial z}{\partial t}, \\ \eta = \beta\dfrac{\partial x}{\partial t} + \beta'\dfrac{\partial y}{\partial t} + \beta''\dfrac{\partial z}{\partial t}, \\ \varsigma = \gamma\dfrac{\partial x}{\partial t} + \gamma'\dfrac{\partial y}{\partial t} + \gamma''\dfrac{\partial z}{\partial t}, \end{cases} \qquad (55')\quad \begin{cases} p = \sum \gamma\dfrac{\partial \beta}{\partial t} = -\sum \beta\dfrac{\partial \gamma}{\partial t}, \\ q = \sum \alpha\dfrac{\partial \gamma}{\partial t} = -\sum \gamma\dfrac{\partial \alpha}{\partial t}, \\ r = \sum \beta\dfrac{\partial \alpha}{\partial t} = -\sum \alpha\dfrac{\partial \beta}{\partial t}, \end{cases}$$

if one now introduces the distinction between the notations for the derivatives with respect to time depending on whether one takes x_0, y_0, z_0, t or x, y, z, t for the independent variables.

Suppose that one endows each of the triads of the trajectory of the deformed state with an infinitely small displacement that varies in a continuous fashion with these triads. With the same notations as in sec. **50**, we have:

$$(56)\qquad \delta\alpha = \beta\delta K' - \gamma\delta J',$$

$$(57)\qquad \delta' x = \delta x' + z'\delta J' - y'\delta K', \quad \delta' y = \delta y' + x'\delta K' - z'\delta I', \quad \delta' z = \delta z' + y'\delta I' - x'\delta J',$$

$$(58)\quad \begin{cases} \delta\xi_i = \eta_i\delta K' - \varsigma_i\delta J' + \dfrac{\partial \delta' x}{\partial \rho_i} + q_i\delta' z - r_i\delta' y, \\ \eta_i = \varsigma_i\delta I' - \xi_i\delta K' + \dfrac{\partial \delta' y}{\partial \rho_i} + r_i\delta' x - p_i\delta' z, \\ \varsigma_i = \xi_i\delta J' - \eta_i\delta I' + \dfrac{\partial \delta' z}{\partial \rho_i} + p_i\delta' y - q_i\delta' x, \end{cases} \quad (59)\quad \begin{cases} \delta p_i = \dfrac{\partial \delta I'}{\partial \rho_i} + q_i\delta K' - r_i\delta J', \\ \delta q_i = \dfrac{\partial \delta J'}{\partial \rho_i} + r_i\delta I' - p_i\delta K', \\ \delta r_i = \dfrac{\partial \delta K'}{\partial \rho_i} + p_i\delta J' - q_i\delta I', \end{cases}$$

$$(58')\quad \begin{cases} \delta\xi_i = \eta_i\delta K' - \varsigma_i\delta J' + \dfrac{\partial \delta' x}{\partial t} + q_i\delta' z - r_i\delta' y, \\ \eta_i = \varsigma_i\delta I' - \xi_i\delta K' + \dfrac{\partial \delta' y}{\partial t} + r_i\delta' x - p_i\delta' z, \\ \varsigma_i = \xi_i\delta J' - \eta_i\delta I' + \dfrac{\partial \delta' z}{\partial t} + p_i\delta' y - q_i\delta' x, \end{cases} \quad (59')\quad \begin{cases} \delta p_i = \dfrac{\partial \delta I'}{\partial t} + q_i\delta K' - r_i\delta J', \\ \delta q_i = \dfrac{\partial \delta J'}{\partial t} + r_i\delta I' - p_i\delta K', \\ \delta r_i = \dfrac{\partial \delta K'}{\partial t} + p_i\delta J' - q_i\delta I'. \end{cases}$$

61. Euclidean action of deformation and motion for a deformable medium in motion. – Consider a function W of *two infinitely close positions* of the triad $Mx'y'z'$, i.e., a function of x_0, y_0, z_0, t, and of $x, y, z, \alpha, \alpha', \cdots, \gamma''$, and their first derivatives with respect to x_0, y_0, z_0, t. We propose to determine the form that W must take in order for the quadruple integral:

$$\iiiint W dx_0 dy_0 dz_0 dt,$$

when taken over an arbitrary portion of space (M_0), and the time interval between two instants t_1 and t_2 to have a null variation when one subjects the set of all triads along what we are calling the trajectory of the deformable medium – taken its deformed state – *to the same arbitrary infinitesimal transformation of the group of euclidean displacements.*

By definition, this amounts to determining W in such a fashion that one has:

$$\delta W = 0$$

when, on the one hand, the origin M of the triad $Mx'y'z'$ is subjected to an infinitely small displacement whose projections δx, δy, δz on the axes Ox, Oy, Oz are:

$$\text{(60)} \qquad \begin{cases} \delta x = (a_1 + \omega_2 z - \omega_3 y)\delta t, \\ \delta y = (a_2 + \omega_3 x - \omega_1 z)\delta t, \\ \delta z = (a_3 + \omega_1 y - \omega_2 x)\delta t, \end{cases}$$

in which a_1, a_2, a_3, ω_1, ω_2, ω_3 are six arbitrary constants, and δt is an infinitely small quantity that is independent of x_0, y_0, z_0, t, and when, on the other hand, this triad $Mx'y'z'$ is subjected to an infinitely small rotation whose components along the Ox, Oy, Oz axes are:

$$\omega_1\,\delta t, \qquad \omega_2\,\delta t, \qquad \omega_3\,\delta t.$$

It suffices for us to repeat the reasoning that we made before, with several reprises, in order to see that *the desired function W has the remarkable form:*

$$W(x_0, y_0, z_0, t, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i, \xi, \eta, \zeta, p, q, r),$$

which is analogous to the one we encountered for the deformable line, surface, and medium at rest.

We say that the integral:

$$\int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt,$$

is the *action of deformation and motion* in the interior of the surface S of the deformed medium in motion and in the interval of time between the instants t_1 and t_2. On the other hand, we say that W is the *density* of the action of deformation and motion *at a point* of the deformed medium when taken *at a given instant*, and referred to the unit of volume of the undeformed medium and the unit of time. If we give Δ the same significance as in sec. **51** then $\frac{W}{|\Delta|}$ is the density of that action at a point and a given instant, when referred to the unit of volume of the deformed medium and the unit of time.

62. The external force and moments; the external effort and moment of deformation; the effort, moment of deformation, quantity of motion, and the moment of the quantity of motion of a deformable medium in motion at a given point and instant. – Consider an *arbitrary* variation of the action of deformation and movement in the interior of a surface (S) of the medium (M), and the time interval between the instants t_1 and t_2, namely:

$$\delta\int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt = \int_{t_1}^{t_2}\iiint_{S_0}\left\{\sum\left(\frac{\partial W}{\partial \xi_i}\delta\xi_i + \frac{\partial W}{\partial \eta_i}\delta\eta_i + \frac{\partial W}{\partial \varsigma_i}\delta\varsigma_i \right.\right.$$
$$\left. + \frac{\partial W}{\partial p_i}\delta p_i + \frac{\partial W}{\partial q_i}\delta q_i + \frac{\partial W}{\partial r_i}\delta r_i\right) + \frac{\partial W}{\partial \xi}\delta\xi + \frac{\partial W}{\partial \eta}\delta\eta + \frac{\partial W}{\partial \varsigma}\delta\varsigma$$
$$\left. + \frac{\partial W}{\partial p}\delta p + \frac{\partial W}{\partial q}\delta q + \frac{\partial W}{\partial r}\delta r\right\} dx_0 dy_0 dz_0 dt.$$

By virtue of formulas (58), (58′), (59), (59′), we may write:

$$\delta\int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt = \int_{t_1}^{t_2}\iiint_{S_0}\left\{\sum\left[\frac{\partial W}{\partial \xi_i}\left(\eta_i\delta K' - \varsigma_i \delta J' + \frac{\partial \delta' x}{\partial \rho_i} + q_i\delta' z - r_i\delta' y\right)\right.\right.$$
$$+ \frac{\partial W}{\partial \eta_i}\left(\varsigma_i\delta I' - \xi_i\delta K' + \frac{\partial \delta' y}{\partial \rho_i} + r_i\delta' x - p_i\delta' z\right) + \frac{\partial W}{\partial \sigma_i}\left(\xi_i\delta J' - \eta_i\delta I' + \frac{\partial \delta' z}{\partial \rho_i} + p_i\delta' y - q_i\delta' x\right)$$
$$\left. + \frac{\partial W}{\partial p_i}\left(\frac{\partial \delta I'}{\partial \rho_i} + q_i\delta K' - r_i\delta J'\right) + \frac{\partial W}{\partial q_i}\left(\frac{\partial \delta J'}{\partial \rho_i} + r_i\delta I' - p_i\delta K'\right) + \frac{\partial W}{\partial r_i}\left(\frac{\partial \delta K'}{\partial \rho_i} + p_i\delta J' - q_i\delta I'\right)\right]$$
$$+ \frac{\partial W}{\partial \xi}\left(\eta\delta K' - \varsigma\delta J' + \frac{\partial \delta' x}{\partial t} + q\delta' z - r\delta' y\right) + \frac{\partial W}{\partial \eta}\left(\varsigma\delta J' - \xi\delta K' + \frac{\partial \delta' y}{\partial t} + r\delta' x - p\delta' z\right)$$
$$+ \frac{\partial W}{\partial \varsigma}\left(\xi\delta J' - \eta\delta I' + \frac{d\delta' z}{dt} + p\delta' y - q\delta' x\right) + \frac{\partial W}{\partial p}\left(\frac{d\delta I'}{dt} + q\delta K' - r\delta J'\right)$$
$$\left. + \frac{\partial W}{\partial q}\left(\frac{d\delta J'}{dt} + r\delta I' - p\delta K'\right) + \frac{\partial W}{\partial r}\left(\frac{d\delta K'}{dt} + p\delta J' - q\delta I'\right)\right\} dx_0 dy_0 dz_0 dt.$$

We apply GREEN's formula to the terms that explicitly involve a derivative with respect to any of the variables, ρ_1, ρ_2, ρ_3, and perform an integration by parts over the terms that explicitly involve a derivative with respect to time, t. If we let l_0, m_0, n_0, designate the direction cosines with respect to the fixed axes, Ox, Oy, Oz, of the exterior normal to the surface, S_0, that bounds the medium before deformation at the instant, t, and designate the area element of that surface by $d\sigma_0$, then we obtain:

$$\delta\int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt =$$

$$\int_{t_1}^{t_2}\iint_{S_0}\left\{\left(l_0\frac{\partial W}{\partial \xi_1}+m_0\frac{\partial W}{\partial \xi_2}+n_0\frac{\partial W}{\partial \xi_3}\right)\delta' x+\left(l_0\frac{\partial W}{\partial \eta_1}+m_0\frac{\partial W}{\partial \eta_2}+n_0\frac{\partial W}{\partial \eta_3}\right)\delta' y\right.$$

$$+\left(l_0\frac{\partial W}{\partial \varsigma_1}+m_0\frac{\partial W}{\partial \varsigma_2}+n_0\frac{\partial W}{\partial \varsigma_3}\right)\delta' z+\left(l_0\frac{\partial W}{\partial p_1}+m_0\frac{\partial W}{\partial p_2}+n_0\frac{\partial W}{\partial p_3}\right)\delta I'$$

$$\left.+\left(l_0\frac{\partial W}{\partial q_1}+m_0\frac{\partial W}{\partial q_2}+n_0\frac{\partial W}{\partial q_3}\right)\delta J'+\left(l_0\frac{\partial W}{\partial r_1}+m_0\frac{\partial W}{\partial r_2}+n_0\frac{\partial W}{\partial r_3}\right)\delta K'\right\}d\sigma_0\,dt$$

$$+\left\{\iiint_{S_0}\left(\frac{\partial W}{\partial \xi}\delta' x+\frac{\partial W}{\partial \eta}\delta' y+\frac{\partial W}{\partial \varsigma}\delta' z+\frac{\partial W}{\partial p}\delta I'+\frac{\partial W}{\partial q}\delta J'+\frac{\partial W}{\partial r}\delta K'\right)dx_0\,dy_0\,dz_0\right\}_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2}\iiint_{S_0}\left\{\left[\sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial \xi_i}+q_i\frac{\partial W}{\partial \varsigma_i}-r_i\frac{\partial W}{\partial \eta_i}\right)+\frac{\partial}{\partial t}\frac{\partial W}{\partial \xi}+q\frac{\partial W}{\partial \varsigma}-r\frac{\partial W}{\partial \eta}\right]\delta' x\right.$$

$$+\left[\sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial \eta_i}+r_i\frac{\partial W}{\partial \xi_i}-p_i\frac{\partial W}{\partial \varsigma_i}\right)+\frac{\partial}{\partial t}\frac{\partial W}{\partial \eta}+r\frac{\partial W}{\partial \xi}-p\frac{\partial W}{\partial \varsigma}\right]\delta' y$$

$$+\left[\sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial \varsigma_i}+p_i\frac{\partial W}{\partial \eta_i}-q_i\frac{\partial W}{\partial \xi_i}\right)+\frac{\partial}{\partial t}\frac{\partial W}{\partial \varsigma}+\frac{\partial W}{\partial \eta}-q\frac{\partial W}{\partial \xi}\right]\delta' z$$

$$+\left[\sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial p_i}+q_i\frac{\partial W}{\partial r_i}-r_i\frac{\partial W}{\partial q_i}+\eta_i\frac{\partial W}{\partial \varsigma_i}-\varsigma_i\frac{\partial W}{\partial \eta_i}\right)\right.$$

$$\left.+\frac{d}{dt}\frac{\partial W}{\partial p}+q\frac{\partial W}{\partial r}-r\frac{\partial W}{\partial q}+\eta\frac{\partial W}{\partial \varsigma}-\varsigma\frac{\partial W}{\partial \eta}\right]\delta I'$$

$$+\left[\sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial q_i}+r_i\frac{\partial W}{\partial p_i}-p_i\frac{\partial W}{\partial r_i}+\varsigma_i\frac{\partial W}{\partial \xi_i}-\xi_i\frac{\partial W}{\partial \varsigma_i}\right)\right.$$

$$\left.+\frac{d}{dt}\frac{\partial W}{\partial q}+r\frac{\partial W}{\partial p}-p\frac{\partial W}{\partial r}+\varsigma\frac{\partial W}{\partial \xi}-\xi\frac{\partial W}{\partial \varsigma}\right]\delta J'$$

$$+\left[\sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial r_i}+p_i\frac{\partial W}{\partial q_i}-q_i\frac{\partial W}{\partial p_i}+\xi_i\frac{\partial W}{\partial \eta_i}-\eta_i\frac{\partial W}{\partial \xi_i}\right)\right.$$

$$\left.\left.+\frac{d}{dt}\frac{\partial W}{\partial r}+p\frac{\partial W}{\partial q}-q\frac{\partial W}{\partial p}+\xi\frac{\partial W}{\partial \eta}-\eta\frac{\partial W}{\partial \xi}\right]\delta K'\right\}dx_0\,dy_0\,dz_0\,dt.$$

As in sec. **52**, set:

$$F_0'=l_0\frac{\partial W}{\partial \xi_1}+m_0\frac{\partial W}{\partial \xi_2}+n_0\frac{\partial W}{\partial \xi_3},\qquad I_0'=l_0\frac{\partial W}{\partial p_1}+m_0\frac{\partial W}{\partial p_2}+n_0\frac{\partial W}{\partial p_3},$$

$$G_0'=l_0\frac{\partial W}{\partial \eta_1}+m_0\frac{\partial W}{\partial \eta_2}+n_0\frac{\partial W}{\partial \eta_3},\qquad J_0'=l_0\frac{\partial W}{\partial q_1}+m_0\frac{\partial W}{\partial q_2}+n_0\frac{\partial W}{\partial q_3},$$

$$H_0' = l_0 \frac{\partial W}{\partial \varsigma_1} + m_0 \frac{\partial W}{\partial \varsigma_2} + n_0 \frac{\partial W}{\partial \varsigma_3}, \qquad K_0' = l_0 \frac{\partial W}{\partial r_1} + m_0 \frac{\partial W}{\partial r_2} + n_0 \frac{\partial W}{\partial r_3},$$

and, in addition:

$$A' = \frac{\partial W}{\partial \xi}, \qquad B' = \frac{\partial W}{\partial \eta}, \qquad C' = \frac{\partial W}{\partial \varsigma},$$
$$P' = \frac{\partial W}{\partial p}, \qquad Q' = \frac{\partial W}{\partial q}, \qquad R' = \frac{\partial W}{\partial r}.$$

On the other hand, set:

$$X_0' = \sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial \xi_i} + q_i \frac{\partial W}{\partial \varsigma_i} - r_i \frac{\partial W}{\partial \eta_i}\right) + \frac{d}{dt}\frac{\partial W}{\partial \xi} + q\frac{\partial W}{\partial \varsigma} - r\frac{\partial W}{\partial \eta},$$
$$Y_0' = \sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial \eta_i} + r_i \frac{\partial W}{\partial \xi_i} - p_i \frac{\partial W}{\partial \varsigma_i}\right) + \frac{d}{dt}\frac{\partial W}{\partial \eta} + r\frac{\partial W}{\partial \xi} - p\frac{\partial W}{\partial \varsigma},$$
$$Z_0' = \sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial \varsigma_i} + p_i \frac{\partial W}{\partial \eta_i} - q_i \frac{\partial W}{\partial \xi_i}\right) + \frac{d}{dt}\frac{\partial W}{\partial \varsigma} + p\frac{\partial W}{\partial \eta} - q\frac{\partial W}{\partial \xi},$$
$$L_0' = \sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial p_i} + q_i \frac{\partial W}{\partial r_i} - r_i \frac{\partial W}{\partial q_i} + \eta_i \frac{\partial W}{\partial \varsigma_i} - \varsigma_i \frac{\partial W}{\partial \eta_i}\right)$$
$$+ \frac{d}{dt}\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial r} - r\frac{\partial W}{\partial q} + \eta\frac{\partial W}{\partial \varsigma} - \varsigma\frac{\partial W}{\partial \eta},$$
$$M_0' = \sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial q_i} + r_i \frac{\partial W}{\partial p_i} - p_i \frac{\partial W}{\partial r_i} + \varsigma_i \frac{\partial W}{\partial \xi_i} - \xi_i \frac{\partial W}{\partial \varsigma_i}\right)$$
$$+ \frac{d}{dt}\frac{\partial W}{\partial q} + r\frac{\partial W}{\partial p} - p\frac{\partial W}{\partial r} + \varsigma\frac{\partial W}{\partial \xi} - \xi\frac{\partial W}{\partial \varsigma},$$
$$N_0' = \sum\left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial r_i} + p_i \frac{\partial W}{\partial q_i} - q_i \frac{\partial W}{\partial p_i} + \xi_i \frac{\partial W}{\partial \eta_i} - \eta_i \frac{\partial W}{\partial \xi_i}\right)$$
$$+ \frac{d}{dt}\frac{\partial W}{\partial r} + p\frac{\partial W}{\partial q} - q\frac{\partial W}{\partial p} + \xi\frac{\partial W}{\partial \eta} - \eta\frac{\partial W}{\partial \xi}.$$

This makes:

$$\delta \int_{t_1}^{t_2} \iiint_{S_0} W dx_0 dy_0 dz_0 dt$$
$$= \int_{t_1}^{t_2} \iint_{S_0} (F_0'\delta' x + G_0'\delta' y + H_0'\delta' z + I_0'\delta I' + J_0'\delta J' + K_0'\delta K') d\sigma_0 dt$$
$$+ \left\{ \iiint_{S_0} (A'\delta' x + B'\delta' y + C'\delta' z + P'\delta I' + Q'\delta J' + R'\delta K') dx_0 dy_0 dz_0 \right\}_{t_1}^{t_2}$$
$$- \int_{t_1}^{t_2} \iiint_{S_0} (X_0'\delta' x + Y_0'\delta' y + Z_0'\delta' z + L_0'\delta I' + M_0'\delta J' + N_0'\delta K') dx_0 dy_0 dz_0 dt.$$

If we first consider the quadruple integral that figures in the expression for $\delta\int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt$ then we call the segments that have their origin at M and whose projections on the axes Mx', My', Mz' are X_0', Y_0', Z_0' and L_0', M_0', N_0' the *external force and external moment at the point M at the instant t, referred to the unit of volume of the position of the medium at the instant* t_0, respectively.

If we then consider the triple integral that is taken over time and the surface S_0 then we call the segments that issue from the point M whose projections on the axes Mx', My', Mz' are $-F_0', -G_0', -H_0'$ and $-I_0', -J_0', -K_0'$ the *external effort and external moment of deformation at the point M of the surface S that bounds the deformed medium at the instant t.* At a definite point M of *(S)* these last six quantities depend only on the direction of the external normal to the surface *S*. They remain invariant if the region we call (M_0) varies, but the direction of the normal does not change, and they change sign if this direction is replaced by the opposite direction.

Suppose that one traces a surface Σ in the interior of the deformed medium that is bounded by the surface *S*, which, either alone or with a portion of the surface *S* circumscribes a subset (*A*) of the medium, and let (*B*) denote the rest of the medium outside of (*A*). Let Σ_0 be the surface of (M_0) that corresponds to the surface *S* of (*M*), and let (A_0) and (B_0) be the regions of (M_0) that correspond to the regions (*A*) and (*B*) of (*M*). Mentally separate the two subsets *A* and *B*; one may regard the two segments $(-F_0', -G_0', -H_0')$ and $(-I_0', -J_0', -K_0')$ that are determined for the point M and the direction of the normal to Σ_0 that points to the exterior of (A_0) as the external effort and moment of deformation at the point *M* of the frontier Σ of the region (*A*). Similarly, one may regard the two segments (F_0', G_0', H_0') and (I_0', J_0', K_0') to be the external effort and m0ment of deformation at the point *M* of the frontier Σ of the region (*B*). By reason of this remark, we say that $-F_0', -G_0', -H_0'$ and $-I_0', -J_0', -K_0'$ are the components of the *effort and moment of deformation that is exerted on the portion* (*A*) *of the medium* (*M*) *at M* along the axes Mx', My', Mz', and that F_0', G_0', H_0' and I_0', J_0', K_0' are the components of the *effort and moment of deformation that are exerted on the portion* (*B*) *of the medium* (*M*) *at M*, along the axes Mx', My', Mz'.

Finally, if we consider the triple integral over the volume of (*M*) at the instant *t*, whose values are taken at the extreme instants t_1 and t_2, then we call the segments that have their origins at *M* and whose components along the axes Mx', My', Mz' are A', B', C' and P', Q', R' the *quantity of motion and the moment of the quantity of motion at the point M of the deformed medium* (*M*) *at the instant t, respectively.*

63. Diverse specifications for the effort and moment of deformation, the quantity of motion, and the moment of the quantity of motion. – As in sec. **53**, set:

$$A_i' = \frac{\partial W}{\partial \xi_i}, \qquad B_i' = \frac{\partial W}{\partial \eta_i}, \qquad C_i' = \frac{\partial W}{\partial \varsigma_i},$$

$$P_i' = \frac{\partial W}{\partial p_i}, \qquad Q_i' = \frac{\partial W}{\partial q_i}, \qquad R_i' = \frac{\partial W}{\partial r_i};$$

in which A_i', B_i', C_i' and P_i', Q_i', R_i' represent the projections on Mx', My', Mz', respectively, of the effort and moment of deformation that are exerted at the point M of a surface that has a normal that is parallel the axis Ox, Oy, Oz that we describe by the index i before deformation. Indeed, it suffices to recall that we already agreed to replace the letters x_0, y_0, z_0 that correspond to the indices 1, 2, 3 by this convention with ρ_1, ρ_2, ρ_3. Recall that this effort and moment of deformation are referred to the unit of area of the undeformed surface at the instant t.

The new efforts and moments of deformation that we just defined are related the elements that the introduced in the preceding section by the following relations:

$$\begin{aligned}
F_0' &= l_0 A_1' + m_0 A_2' + n_0 A_3', & I_0' &= l_0 P_1' + m_0 P_2' + n_0 P_3', \\
G_0' &= l_0 B_1' + m_0 B_2' + n_0 B_3', & J_0' &= l_0 Q_1' + m_0 Q_2' + n_0 Q_3', \\
H_0' &= l_0 C_1' + m_0 C_2' + n_0 C_3', & K_0' &= l_0 R_1' + m_0 R_2' + n_0 R_3',
\end{aligned}$$

$$\begin{aligned}
&\sum\left(\frac{\partial A_i'}{\partial \rho_i} + q_i C_i' - r_i B_i'\right) + \frac{\partial A'}{\partial t} + qC' - rB' - X_0' = 0, \\
&\sum\left(\frac{\partial B_i'}{\partial \rho_i} + r_i A_i' - p_i C_i'\right) + \frac{\partial B'}{\partial t} + rA' - pC' - Y_0' = 0, \\
&\sum\left(\frac{\partial C_i'}{\partial \rho_i} + p_i B_i' - q_i A_i'\right) + \frac{\partial C'}{\partial t} + pB' - qA' - Z_0' = 0,
\end{aligned}$$

$$\begin{aligned}
&\sum\left(\frac{\partial P_i'}{\partial \rho_i} + q_i R_i' - r_i Q_i' + \eta_i C_i' - \varsigma_i B_i'\right) + \frac{\partial P_i'}{\partial t} + qR' - rQ' + \eta C' - \varsigma B' - L_0' = 0, \\
&\sum\left(\frac{\partial Q_i'}{\partial \rho_i} + r_i P_i' - p_i R_i' + \varsigma_i A_i' - \xi_i C_i'\right) + \frac{\partial Q_i'}{\partial t} + rP' - pQ' + \varsigma A' - \xi C' - M_0' = 0, \\
&\sum\left(\frac{\partial R_i'}{\partial \rho_i} + p_i Q_i' - q_i P_i' + \xi_i B_i' - \eta_i A_i'\right) + \frac{\partial R_i'}{\partial t} + pQ' - qP' + \xi B' - \eta A' - N_0' = 0.
\end{aligned}$$

One may propose to transform the relations we just wrote independently of the values of the quantities that figure in them that are calculated by means of W. Indeed, these relations relate to the segments that are attached to the point M to which we gave the names. Instead of defining these segments by their projections on Mx', My', Mz', we may just as well define them by their projections on other axes; the latter projections will be coupled by relations that are transforms of the preceding ones. Moreover, the transformed relations are obtained immediately if one remarks that the original formulas have simple interpretations ([1]) by the adjunction of axes that are parallel to the moving axes at the point O.

1. As in statics, we confine ourselves to the consideration of the fixed axes Ox, Oy, Oz. Let X_0, Y_0, Z_0 and L_0, M_0, N_0 denote the projections of the external force and the external moment at an arbitrary point M of the deformed medium at an instant t onto these axes, and let F_0, G_0, H_0 and I_0, J_0, K_0 be the projections of the effort and the moment of deformation on a surface whose exterior normal has the direction cosines l_0, m_0, n_0 before deformation at the instant t. Let A_i, B_i, C_i and P_i, Q_i, R_i be the projections of the effort (A_i', B_i', C_i') and the moment of deformation (P_i', Q_i', R_i'), and let A, B, C and P, Q, R be the projections of the quantity of motion (A, B, C) and the moment of the quantity of motion (P, Q, R). The transforms of the preceding relations are obviously:

$$
\begin{aligned}
F_0 &= l_0 A_1 + m_0 A_2 + n_0 A_3, & I_0 &= l_0 P_1 + m_0 P_2 + n_0 P_3,\\
G_0 &= l_0 B_1 + m_0 B_2 + n_0 B_3, & J_0 &= l_0 Q_1 + m_0 Q_2 + n_0 Q_3,\\
H_0 &= l_0 C_1 + m_0 C_2 + n_0 C_3, & K_0 &= l_0 R_1 + m_0 R_2 + n_0 R_3,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial A_1}{\partial x_0} + \frac{\partial A_2}{\partial y_0} + \frac{\partial A_3}{\partial z_0} + \frac{dA}{dt} - X_0 = 0,\\
&\frac{\partial B_1}{\partial x_0} + \frac{\partial B_2}{\partial y_0} + \frac{\partial B_3}{\partial z_0} + \frac{dB}{dt} - Y_0 = 0,\\
&\frac{\partial C_1}{\partial x_0} + \frac{\partial C_2}{\partial y_0} + \frac{\partial C_3}{\partial z_0} + \frac{dC}{dt} - Z_0 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial P_1}{\partial x_0} + \frac{\partial P_2}{\partial y_0} + \frac{\partial P_3}{\partial z_0} + \frac{dP}{dt} + C_1 \frac{\partial y}{\partial x_0} + C_2 \frac{\partial y}{\partial y_0} + C_3 \frac{\partial y}{\partial z_0} + C \frac{dP}{dt}\\
&\qquad - B_1 \frac{\partial z}{\partial x_0} - B_2 \frac{\partial z}{\partial y_0} - B_3 \frac{\partial z}{\partial z_0} - B \frac{dz}{dt} - L_0 = 0,\\
&\frac{\partial Q_1}{\partial x_0} + \frac{\partial Q_2}{\partial y_0} + \frac{\partial Q_3}{\partial z_0} + \frac{dQ}{dt} + A_1 \frac{\partial z}{\partial x_0} + A_2 \frac{\partial z}{\partial y_0} + A_3 \frac{\partial z}{\partial z_0} + A \frac{dz}{dt}\\
&\qquad - C_1 \frac{\partial x}{\partial x_0} - C_2 \frac{\partial x}{\partial y_0} - C_3 \frac{\partial x}{\partial z_0} - C \frac{dx}{dt} - M_0 = 0,\\
&\frac{\partial R_1}{\partial x_0} + \frac{\partial R_2}{\partial y_0} + \frac{\partial R_3}{\partial z_0} + \frac{dR}{dt} + B_1 \frac{\partial x}{\partial x_0} + B_2 \frac{\partial x}{\partial y_0} + B_3 \frac{\partial x}{\partial z_0} + B \frac{dx}{dt}\\
&\qquad - A_1 \frac{\partial y}{\partial x_0} - A_2 \frac{\partial y}{\partial y_0} - A_3 \frac{\partial y}{\partial z_0} - A \frac{dy}{dt} - N_0 = 0.
\end{aligned}
$$

[1] An interesting interpretation to note is the analogue of the one given by P. SAINT-GUILHEM in the context of the dynamics of triads.

2. Now observe that we may express the nine cosines $\alpha, \alpha', \cdots, \gamma''$ by means of the three auxiliary functions λ_1, λ_2, λ_3. Set:

$$\begin{aligned}
\sum \gamma d\beta = -\sum \beta d\gamma &= \varpi_1' d\lambda_1 + \varpi_2' d\lambda_2 + \varpi_3' d\lambda_3, \\
\sum \alpha d\gamma = -\sum \gamma d\alpha &= \chi_1' d\lambda_1 + \chi_2' d\lambda_2 + \chi_3' d\lambda_3, \\
\sum \beta d\alpha = -\sum \alpha d\beta &= \sigma_1' d\lambda_1 + \sigma_2' d\lambda_2 + \sigma_3' d\lambda_3.
\end{aligned}$$

The functions ϖ_i, χ_i, σ_i of λ_1, λ_2, λ_3 so defined satisfy relations that we have written several times already:

$$\begin{aligned}
&\frac{\partial \varpi_j'}{\partial \lambda_i} - \frac{\partial \varpi_i'}{\partial \lambda_j} + \chi_i' \sigma_j' - \chi_j' \sigma_i' = 0, \\
&\frac{\partial \chi_j'}{\partial \lambda_i} - \frac{\partial \chi_i'}{\partial \lambda_j} + \sigma_i' \varpi_j' - \sigma_j' \varpi_i' = 0, \qquad (i, j = 1, 2, 3), \\
&\frac{\partial \sigma_j'}{\partial \lambda_i} - \frac{\partial \sigma_i'}{\partial \lambda_j} + \varpi_i' \chi_j' - \varpi_j' \chi_i' = 0,
\end{aligned}$$

and one has:

$$\begin{aligned}
p_i &= \varpi_1' \frac{\partial \lambda_1}{\partial \rho_i} + \varpi_2' \frac{\partial \lambda_2}{\partial \rho_i} + \varpi_3' \frac{\partial \lambda_3}{\partial \rho_i}, &\qquad p &= \varpi_1' \frac{\partial \lambda_1}{\partial t} + \varpi_2' \frac{\partial \lambda_2}{\partial t} + \varpi_3' \frac{\partial \lambda_3}{\partial t}, \\
q_i &= \chi_1' \frac{\partial \lambda_1}{\partial \rho_i} + \chi_2' \frac{\partial \lambda_2}{\partial \rho_i} + \chi_3' \frac{\partial \lambda_3}{\partial \rho_i}, &\qquad q &= \chi_1' \frac{\partial \lambda_1}{\partial t} + \chi_2' \frac{\partial \lambda_2}{\partial t} + \chi_3' \frac{\partial \lambda_3}{\partial t}, \\
r_i &= \sigma_1' \frac{\partial \lambda_1}{\partial \rho_i} + \sigma_2' \frac{\partial \lambda_2}{\partial \rho_i} + \sigma_3' \frac{\partial \lambda_3}{\partial \rho_i}, &\qquad r &= \sigma_1' \frac{\partial \lambda_1}{\partial t} + \sigma_2' \frac{\partial \lambda_2}{\partial t} + \sigma_3' \frac{\partial \lambda_3}{\partial t},
\end{aligned}$$

in which $x_0 = \rho_1$, $y_0 = \rho_2$, $z_0 = \rho_3$. If we let ϖ_i, χ_i, σ_i denote the projections onto the fixed axes Ox, Oy, Oz of the segment whose projections onto the axes Mx', My', Mz' are $\varpi_i', \chi_i', \sigma_i'$ then we will have:

$$\begin{aligned}
\sum \alpha' d\alpha'' = -\sum \alpha'' d\alpha' &= \varpi_1 d\lambda_1 + \varpi_2 d\lambda_2 + \varpi_3 d\lambda_3, \\
\sum \alpha'' d\alpha = -\sum \alpha d\alpha'' &= \chi_1 d\lambda_1 + \chi_2 d\lambda_2 + \chi_3 d\lambda_3, \\
\sum \alpha d\alpha' = -\sum \alpha' d\alpha &= \sigma_1 d\lambda_1 + \sigma_2 d\lambda_2 + \sigma_3 d\lambda_3,
\end{aligned}$$

by virtue of which [1] the new functions ϖ_i, χ_i, σ_i of λ_1, λ_2, λ_3 satisfy the relations:

$$\frac{\partial \varpi_j}{\partial \lambda_i} - \frac{\partial \varpi_i}{\partial \lambda_j} = \chi_i \sigma_j - \chi_j \sigma_i,$$
$$\frac{\partial \chi_j}{\partial \lambda_i} - \frac{\partial \chi_i}{\partial \lambda_j} = \sigma_i \varpi_j - \sigma_j \varpi_i, \quad (i, j = 1, 2, 3),$$
$$\frac{\partial \sigma_j}{\partial \lambda_i} - \frac{\partial \sigma_i}{\partial \lambda_j} = \varpi_i \chi_j - \varpi_j \chi_i.$$

Once more, we make the remark, which will serve us later on, that if one lets $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$ denote the variations of $\lambda_1, \lambda_2, \lambda_3$ that correspond to the variations $\delta\alpha, \delta\alpha', \cdots, \delta\gamma''$ of $\alpha, \alpha', \cdots, \gamma''$ then one will have:

$$\delta I' = \varpi_1' d\lambda_1 + \varpi_2' d\lambda_2 + \varpi_3' d\lambda_3,$$
$$\delta J' = \chi_1' d\lambda_1 + \chi_2' d\lambda_2 + \chi_3' d\lambda_3,$$
$$\delta K' = \sigma_1' d\lambda_1 + \sigma_2' d\lambda_2 + \sigma_3' d\lambda_3,$$
$$\delta I = \alpha \delta I' + \beta \delta J' + \gamma \delta K' = \varpi_1 \delta\lambda_1 + \varpi_2 \delta\lambda_2 + \varpi_3 \delta\lambda_3,$$
$$\delta J = \alpha' \delta I' + \beta' \delta J' + \gamma'' \delta K' = \chi_1 \delta\lambda_1 + \chi_2 \delta\lambda_2 + \chi_3 \delta\lambda_3,$$
$$\delta K = \alpha'' \delta I' + \beta'' \delta J' + \gamma'' \delta K' = \sigma_1 \delta\lambda_1 + \sigma_2 \delta\lambda_2 + \sigma_3 \delta\lambda_3,$$

in which δI, δJ, δK are the projections onto the fixed axes of the segment whose projections onto Mx', My', Mz' are $\delta I', \delta J', \delta K'$. Now set:

$$\mathcal{I}_0 = \varpi_1' I_0' + \chi_1' J_0' + \sigma_1' K_0' = \varpi_1 I_0 + \chi_1 J_0 + \sigma_1 K_0,$$
$$\mathcal{J}_0 = \varpi_2' I_0' + \chi_2' J_0' + \sigma_2' K_0' = \varpi_2 I_0 + \chi_2 J_0 + \sigma_2 K_0,$$
$$\mathcal{K}_0 = \varpi_3' I_0' + \chi_3' J_0' + \sigma_3' K_0' = \varpi_3 I_0 + \chi_3 J_0 + \sigma_3 K_0,$$

$$\mathcal{L}_0 = \varpi_1' L_0' + \chi_1' M_0' + \sigma_1' N_0' = \varpi_1 L_0 + \chi_1 M_0 + \sigma_1 N_0,$$
$$\mathcal{M}_0 = \varpi_2' L_0' + \chi_2' M_0' + \sigma_2' N_0' = \varpi_2 L_0 + \chi_2 M_0 + \sigma_2 N_0,$$
$$\mathcal{N}_0 = \varpi_3' L_0' + \chi_3' M_0' + \sigma_3' N_0' = \varpi_3 L_0 + \chi_3 M_0 + \sigma_3 N_0.$$

In addition, introduce the following notations:

---

[1] These formulas may serve to define the functions ϖ_i, χ_i, σ_i directly and may be substituted for:

$$\varpi_i = \alpha\varpi_i' + \beta\chi_i' + \gamma\sigma_i',$$
$$\chi_i = \alpha'\varpi_i' + \beta'\chi_i' + \gamma''\sigma_i', \qquad (i, j, = 1, 2, 3),$$
$$\sigma_i = \alpha''\varpi_i' + \beta''\chi_i' + \gamma''\sigma_i'.$$

$$
\begin{aligned}
\Pi_i &= \varpi_1' P_i' + \chi_1' Q_i' + \sigma_1' R_i' = \varpi_1 P_i + \chi_1 Q_i + \sigma_1 R_i, \\
\mathrm{X}_i &= \varpi_2' P_i' + \chi_2' Q_i' + \sigma_2' R_i' = \varpi_2 P_i + \chi_2 Q_i + \sigma_2 R_i, \\
\Sigma_i &= \varpi_3' P_i' + \chi_3' Q_i' + \sigma_3' R_i' = \varpi_3 P_i + \chi_3 Q_i + \sigma_3 R_i, \\
\Pi &= \varpi_1' P' + \chi_1' Q' + \sigma_1' R' = \varpi_1 P + \chi_1 Q + \sigma_1 R, \\
\mathrm{X} &= \varpi_2' P' + \chi_2' Q' + \sigma_2' R' = \varpi_2 P + \chi_2 Q + \sigma_2 R, \\
\Sigma &= \varpi_3' P' + \chi_3' Q' + \sigma_3' R' = \varpi_3 P + \chi_3 Q + \sigma_3 R,
\end{aligned}
$$

and, instead of the latter system, in which either $P_i', Q_i', R_i', P', Q', R'$ or P_i, Q_i, R_i, P, Q, R figure, we have the following:

$$
\begin{aligned}
-\mathcal{L}_0 + \sum_i \Bigg[& \frac{\partial \Pi_i}{\partial \rho_i} - P_i'\left(\frac{\partial \varpi_1'}{\partial \rho_i} + q_i\sigma_1' - r_i\chi_1'\right) - Q_i'\left(\frac{\partial \chi_1'}{\partial \rho_i} + r_i\varpi_1' - p_i\sigma_1'\right) - R_i'\left(\frac{\partial \sigma_1'}{\partial \rho_i} + p_i\chi_1' - q_i\varpi_1'\right) \\
& + A_i'(\chi_1'\varsigma_i - \sigma_1'\eta_i) + B_i'(\sigma_1'\xi_i - \varpi_1'\varsigma_i) + C_i'(\varpi_1'\eta_i - \chi_1'\xi_i)\Bigg] \\
& + \frac{\partial \Pi}{\partial t} - P'\left(\frac{\partial \varpi_1'}{\partial t} + q\sigma_1' - r\chi_1'\right) - Q'\left(\frac{\partial \chi_1'}{\partial t} + r\varpi_1' - p\sigma_1'\right) - R'\left(\frac{\partial \sigma_1'}{\partial t} + p\chi_1' - q\varpi_1'\right) \\
& + A'(\chi_1'\varsigma - \sigma_1'\eta) + B'(\sigma_1'\xi - \varpi_1'\varsigma) + C'(\varpi_1'\eta - \chi_1'\xi) = 0,
\end{aligned}
$$

with two analogous equations. If one remarks that the functions ξ_i, η_i, ζ_i, p_i, q_i, r_i and ξ, η, ζ, p, q, r, and λ_1, λ_2, λ_3, $\frac{\partial \lambda_1}{\partial \rho_i}, \frac{\partial \lambda_2}{\partial \rho_i}, \frac{\partial \lambda_3}{\partial \rho_i}, \frac{d\lambda_1}{d\rho_i}, \frac{d\lambda_2}{d\rho_i}, \frac{d\lambda_3}{d\rho_i}$ give rise to the formulas:

$$
\begin{aligned}
&\frac{\partial \xi_i}{\partial \lambda_j} + \chi_j'\varsigma_i - \sigma_j'\eta_i = 0, &\qquad& \frac{\partial p_i}{\partial \lambda_j} = \frac{\partial \varpi_i'}{\partial \rho_j} + q_j\sigma_i' - r_j\chi_i', \\
&\frac{\partial \eta_i}{\partial \lambda_j} + \sigma_j'\xi_i - \varpi_j'\varsigma_i = 0, && \frac{\partial q_i}{\partial \lambda_j} = \frac{\partial \chi_i'}{\partial \rho_j} + r_j\varpi_i' - p_j\sigma_i', \\
&\frac{\partial \varsigma_i}{\partial \lambda_j} + \varpi_j'\eta_i - \chi_j'\xi_i = 0, && \frac{\partial r_i}{\partial \lambda_j} = \frac{\partial \sigma_i'}{\partial \rho_j} + p_j\chi_i' - q_j\varpi_i', \\
&\frac{\partial \xi}{\partial \lambda_j} + \chi_j'\varsigma - \sigma_j'\eta = 0, && \frac{\partial p}{\partial \lambda_j} = \frac{\partial \varpi_j'}{\partial t} + q\sigma_j' - r\chi_j', \\
&\frac{\partial \eta}{\partial \lambda_j} + \sigma_j'\xi - \varpi_j'\varsigma = 0, && \frac{\partial q}{\partial \lambda_j} = \frac{\partial \chi_j'}{\partial t} + r\varpi_j' - p\sigma_j', \\
&\frac{\partial \varsigma}{\partial \lambda_j} + \varpi_j'\eta - \chi_j'\xi = 0, && \frac{\partial r}{\partial \lambda_j} = \frac{\partial \sigma_j'}{\partial t} + p\chi_j' - q\varpi_j',
\end{aligned}
$$

that result from defining relations for the functions $\varpi_i', \chi_i', \sigma_i'$ and the nine identities they verify, then one may give the preceding system the new form:

$$-\mathcal{L}_0 + \sum_i \left[\frac{\partial \Pi_i}{\partial \rho_i} - A_i' \frac{\partial \xi_i}{\partial \lambda_1} - B_i' \frac{\partial \eta_i}{\partial \lambda_1} - C_i' \frac{\partial \varsigma_i}{\partial \lambda_1} - P_i' \frac{\partial p_i}{\partial \lambda_1} - Q_i' \frac{\partial q_i}{\partial \lambda_1} - R_i' \frac{\partial r_i}{\partial \lambda_1} \right]$$
$$+ \frac{\partial \Pi}{\partial t} - A' \frac{\partial \xi}{\partial \lambda_1} - B' \frac{\partial \eta}{\partial \lambda_1} - C' \frac{\partial \varsigma}{\partial \lambda_1} - P' \frac{\partial p}{\partial \lambda_1} - Q' \frac{\partial q}{\partial \lambda_1} - R' \frac{\partial r}{\partial \lambda_1} = 0,$$

with two analogous equations.

3. Finally, we shall subject the preceding two equations that we introduced to a transformation that is analogous to the one that led us, in sec. **53**, to the generalization of the equations of the theory of elasticity that relate to effort.

To abbreviate the notation, let $\mathcal{X}_0', \mathcal{Y}_0', \mathcal{Z}_0', \mathcal{L}_0', \mathcal{M}_0', \mathcal{N}_0'$ denote – for the moment – the left-hand sides of the transformation relation that refers to X_0, Y_0, Z_0, L_0, M_0, N_0, respectively, and observe that one may summarize the twelve equations we have established by the following:

$$\int_{t_1}^{t_2} \iiint_{S_0} (\mathcal{X}_0'\lambda_1 + \mathcal{Y}_0'\lambda_2 + \mathcal{Z}_0'\lambda_3 + \mathcal{L}_0'\mu_1 + \mathcal{M}_0'\mu_2 + \mathcal{N}_0'\mu_3)\, dx_0\, dy_0\, dz_0\, dt$$
$$+ \int_{t_1}^{t_2} \iint_{S_0} \{(F_0 - l_0 A_1 - m_0 A_2 - n_0 A_3)\lambda_1 + (G_0 - l_0 B_1 - m_0 B_2 - n_0 B_3)\lambda_2$$
$$+ (H_0 - l_0 C_1 - m_0 C_2 - n_0 C_3)\lambda_3 + (I_0 - l_0 P_1 - m_0 P_2 - n_0 P_3)\mu_1$$
$$+ (J_0 - l_0 Q_1 - m_0 Q_2 - n_0 Q_3)\mu_2 + (K_0 - l_0 R_1 - m_0 R_2 - n_0 R_3)\mu_3\}\, d\sigma_0\, dt = 0,$$

in which λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary functions, and the integrals are taken over, on the one hand, the time interval between the instants t_1 and t_2, and, on the other hand, the surface S_0, of the medium (M_0) and the domain it bounds. If we apply GREEN'S theorem and integrate by parts then the relation that we just wrote becomes the following one:

$$- \int_{t_1}^{t_2} \iiint_{S_0} (X_0\lambda_1 + Y_0\lambda_2 + Z_0\lambda_3 + L_0\mu_1 + M_0\mu_2 + N_0\mu_3)\, dx_0\, dy_0\, dz_0\, dt$$
$$+ \int_{t_1}^{t_2} \iint_{S_0} (F_0\lambda_1 + G_0\lambda_2 + H_0\lambda_3 + I_0\mu_1 + J_0\mu_2 + K_0\mu_3)\, d\sigma_0\, dt$$
$$+ \left\{ \iiint_{S_0} (A\lambda_1 + B\lambda_2 + C\lambda_3 + P\mu_1 + Q\mu_2 + R\mu_3)\, dx_0\, dy_0\, dz_0 \right\}_{t_1}^{t_2}$$
$$- \int_{t_1}^{t_2} \iiint_{S_0} \left(A_1 \frac{\partial \lambda_1}{\partial x_0} + A_2 \frac{\partial \lambda_1}{\partial y_0} + A_3 \frac{\partial \lambda_1}{\partial z_0} + A \frac{d\lambda_1}{dt} + B_1 \frac{\partial \lambda_2}{\partial x_0} + B_2 \frac{\partial \lambda_2}{\partial y_0} + B_3 \frac{\partial \lambda_2}{\partial z_0} + B \frac{d\lambda_2}{dt} \right.$$
$$\left. + C_1 \frac{\partial \lambda_3}{\partial x_0} + C_2 \frac{\partial \lambda_3}{\partial y_0} + C_3 \frac{\partial \lambda_3}{\partial z_0} + C \frac{d\lambda_3}{dt} \right) dx_0\, dy_0\, dz_0\, dt$$
$$- \int_{t_1}^{t_2} \iiint_{S_0} \left(P_1 \frac{\partial \mu_1}{\partial x_0} + P_2 \frac{\partial \mu_1}{\partial y_0} + P_3 \frac{\partial \mu_1}{\partial z_0} + P \frac{\partial \mu_1}{\partial t} + Q_1 \frac{\partial \mu_2}{\partial x_0} + Q_2 \frac{\partial \mu_2}{\partial y_0} + Q_3 \frac{\partial \mu_2}{\partial z_0} + Q \frac{d\mu_2}{dt} \right.$$

$$+ R_1 \frac{\partial \mu_3}{\partial x_0} + R_2 \frac{\partial \mu_3}{\partial y_0} + R_3 \frac{\partial \mu_3}{\partial z_0} + R \frac{d\mu_3}{dt}\Bigg) dx_0 dy_0 dz_0 dt$$

$$+ \int_{t_1}^{t_2} \iiint_{S_0} \Bigg(C_1 \frac{\partial y}{\partial x_0} + C_2 \frac{\partial y_1}{\partial y_0} + C_3 \frac{\partial y}{\partial z_0} + C \frac{dy}{dt}$$

$$- B_1 \frac{\partial z}{\partial x_0} - B_2 \frac{\partial z}{\partial y_0} - B_3 \frac{\partial z}{\partial z_0} - B \frac{dz}{dt}\Bigg) \mu_1 dx_0 dy_0 dz_0 dt$$

$$+ \int_{t_1}^{t_2} \iiint_{S_0} \Bigg(A_1 \frac{\partial z}{\partial x_0} + A_2 \frac{\partial z_1}{\partial y_0} + A_3 \frac{\partial z}{\partial z_0} + A \frac{dz}{dt}$$

$$- C_1 \frac{\partial x}{\partial x_0} - C_2 \frac{\partial x}{\partial y_0} - C_3 \frac{\partial x}{\partial z_0} - C \frac{dx}{dt}\Bigg) \mu_2 dx_0 dy_0 dz_0 dt$$

$$+ \int_{t_1}^{t_2} \iiint_{S_0} \Bigg(B_1 \frac{\partial x}{\partial x_0} + B_2 \frac{\partial x}{\partial y_0} + B_3 \frac{\partial x}{\partial z_0} + B \frac{dx}{dt}$$

$$- A_1 \frac{\partial y}{\partial x_0} - A_2 \frac{\partial y}{\partial y_0} - A_3 \frac{\partial y}{\partial z_0} - A \frac{dy}{dt}\Bigg) \mu_3 dx_0 dy_0 dz_0 dt = 0.$$

We seek to transform this last relation when one takes the functions x, y, z for other new variables, while preserving t. We apply the elementary formulas for the change of variables that we recalled in sec. **53** to the functions λ_1, λ_2, λ_3, μ_1, μ_2, μ_3. With S always indicating the surface of the medium (M) at the instant t that corresponds to the surface S_0 of (M_0). Moreover, let X, Y, Z, L, M, N be the projections on Ox, Oy, Oz of the external force and external moment that are applied to the point M at the instant t, and referred to the unit of volume of the deformed medium (M), and let F, G, H, I, J, L denote the projections on Ox, Oy, Oz of the effort and moment of deformation that are exerted at the point M on S, referred to the unit of area of S. Finally introduce, as in sec. **53**, eighteen new auxiliary functions p_{xx}, …, q_{xx}, … by the formulas:

$$\Delta p_{xx} = A_1 \frac{\partial x}{\partial x_0} + A_2 \frac{\partial x}{\partial y_0} + A_3 \frac{\partial x}{\partial z_0}, \qquad \Delta q_{xx} = P_1 \frac{\partial x}{\partial x_0} + P_2 \frac{\partial x}{\partial y_0} + P_3 \frac{\partial x}{\partial z_0},$$
$$\Delta p_{yx} = A_1 \frac{\partial y}{\partial x_0} + A_2 \frac{\partial y}{\partial y_0} + A_3 \frac{\partial y}{\partial z_0}, \qquad \Delta q_{yx} = P_1 \frac{\partial y}{\partial x_0} + P_2 \frac{\partial y}{\partial y_0} + P_3 \frac{\partial y}{\partial z_0},$$
$$\Delta p_{zx} = A_1 \frac{\partial z}{\partial x_0} + A_2 \frac{\partial z}{\partial y_0} + A_3 \frac{\partial z}{\partial z_0}, \qquad \Delta q_{zx} = P_1 \frac{\partial z}{\partial x_0} + P_2 \frac{\partial z}{\partial y_0} + P_3 \frac{\partial z}{\partial z_0},$$

and the analogous one that is obtained by replacing:

$$A_1, A_2, A_3, p_{xx}, p_{yx}, p_{zx}, P_1, P_2, P_3, q_{xx}, q_{yx}, q_{zx}$$

by

$$B_1, B_2, B_3, p_{xy}, p_{yy}, p_{zy}, Q_1, Q_2, Q_3, q_{xy}, q_{yy}, q_{zy},$$

and then by

$$C_1, C_2, C_3, p_{xz}, p_{yz}, p_{zz}, R_1, R_2, R_3, q_{xz}, q_{yz}, q_{zz},$$

respectively, with the quantity Δ having the same expression as it did in sec. **53**. We obtain the transformed relation:

$$
\begin{aligned}
&-\int_{t_1}^{t_2}\iiint_{S_0}(X\lambda_1+Y\lambda_2+Z\lambda_3+L\mu_1+M\mu_2+N\mu_3)dxdydzdt \\
&+\int_{t_1}^{t_2}\iint_{S_0}(F\lambda_1+G\lambda_2+H\lambda_3+I\mu_1+J\mu_2+K\mu_3)d\sigma dt \\
&+\left\{\iiint_{S_0}(\frac{A}{\Delta}\lambda_1+\frac{B}{\Delta}\lambda_2+\frac{C}{\Delta}\lambda_3+\frac{P}{\Delta}\mu_1+\frac{Q}{\Delta}\mu_2+\frac{R}{\Delta}\mu_3)dxdydz\right\}_{t_1}^{t_2} \\
&-\int_{t_1}^{t_2}\iiint_{S_0}\left(p_{xx}\frac{\partial\lambda_1}{\partial x}+p_{yx}\frac{\partial\lambda_1}{\partial y}+p_{zx}\frac{\partial\lambda_1}{\partial z}+p_{xy}\frac{\partial\lambda_2}{\partial x}+\cdots+p_{zz}\frac{\partial\lambda_3}{\partial y}\right. \\
&\qquad\qquad\left.+\frac{A}{\Delta}\frac{d\lambda_1}{dt}+\frac{B}{\Delta}\frac{d\lambda_2}{dt}+\frac{C}{\Delta}\frac{d\lambda_3}{dt}\right)dxdydzdt \\
&-\int_{t_1}^{t_2}\iiint_{S_0}\left(q_{xx}\frac{\partial\mu_1}{\partial x}+q_{yx}\frac{\partial\mu_1}{\partial y}+q_{zx}\frac{\partial\mu_1}{\partial z}+q_{xy}\frac{\partial\mu_2}{\partial x}+\cdots+q_{zz}\frac{\partial\mu_3}{\partial z}\right. \\
&\qquad\qquad\left.+\frac{P}{\Delta}\frac{d\mu_1}{dx}+\frac{Q}{\Delta}\frac{d\mu_2}{dx}+\frac{R}{\Delta}\frac{d\mu_3}{dx}\right)dxdydzdt \\
&+\int_{t_1}^{t_2}\iiint_{S_0}\left\{\left(p_{yz}-p_{zy}+\frac{C}{\Delta}\frac{dy}{dt}-\frac{B}{\Delta}\frac{dz}{dt}\right)\mu_1+\left(p_{zx}-p_{xz}+\frac{A}{\Delta}\frac{dz}{dt}-\frac{C}{\Delta}\frac{dx}{dt}\right)\mu_2\right. \\
&\qquad\qquad\left.+\left(p_{xy}-p_{yx}+\frac{B}{\Delta}\frac{dx}{dt}-\frac{A}{\Delta}\frac{dy}{dt}\right)\mu_3\right\}dxdydzdt=0,
\end{aligned}
$$

in which the integrals are taken over, on the one hand, the time interval between the instants t_1 and t_2, and, on the other hand, the surface S of the medium (M) at the instant t, and the domain it bounds, with $d\sigma$ designating the area element of S.

Once again, we apply the GREEN formula to the terms that refer to the derivatives of λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 with respect to x, y, z, and an integration by parts ([1]) of the terms that involve the derivatives of λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 with respect t, and let l, m, n denote the direction cosines of the exterior normal to the surface S at the instant t with respect to the fixed axes. Since λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary, they become:

$$
\begin{aligned}
F &= lp_{xx}+mp_{yx}+np_{zx}, & I &= lq_{xx}+mq_{yx}+nq_{zx}, \\
G &= lp_{xy}+mp_{yy}+np_{zy}, & J &= lq_{xy}+mq_{yy}+nq_{zy}, \\
H &= lp_{xz}+mp_{yy}+np_{zz}, & K &= lq_{xz}+mq_{yz}+nq_{zz},
\end{aligned}
$$

$$\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}+\frac{1}{\Delta}\frac{dA}{dt}-X=0,$$

$$\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}+\frac{1}{\Delta}\frac{dB}{dt}-Y=0,$$

[1] Since the field of variation actually varies with t, we perform that integration by parts by the intermediary of passing to the variables x_0, y_0, z_0. We suppose that Δ is positive and equal to $|\Delta|$.

$$\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}+\frac{1}{\Delta}\frac{dC}{dt}-Z=0,$$
$$\frac{\partial q_{xx}}{\partial x}+\frac{\partial q_{yx}}{\partial y}+\frac{\partial q_{zx}}{\partial z}+p_{yz}-p_{zy}+\frac{1}{\Delta}\frac{dP}{dt}+\frac{C}{\Delta}\frac{dy}{dt}-\frac{B}{\Delta}\frac{dz}{dt}-L=0,$$
$$\frac{\partial q_{xy}}{\partial x}+\frac{\partial q_{yy}}{\partial y}+\frac{\partial q_{zy}}{\partial z}+p_{yx}-p_{zx}+\frac{1}{\Delta}\frac{dQ}{dt}+\frac{A}{\Delta}\frac{dz}{dt}-\frac{C}{\Delta}\frac{dx}{dt}-M=0,$$
$$\frac{\partial q_{xz}}{\partial x}+\frac{\partial q_{yz}}{\partial y}+\frac{\partial q_{zz}}{\partial z}+p_{xy}-p_{yx}+\frac{1}{\Delta}\frac{dR}{dt}+\frac{B}{\Delta}\frac{dx}{dt}-\frac{A}{\Delta}\frac{dy}{dt}-N=0.$$

The significance of the eighteen new auxiliary functions p_{xx}, …, q_{xx}, … result immediately from the relations that we just wrote. Indeed, it is clear that the coefficients, p_{xx}, p_{xy}, p_{xz} of l in the expressions of F, G, H represent the projections onto Ox, Oy, Oz of the effort that is exerted at the point M on a surface whose exterior normal is parallel to Ox, and that the coefficients q_{xx}, q_{xy}, q_{xz} of l in the expressions for I, J, K are the projections onto Ox, Oy, Oz of the moment of deformation at M relative to the same surface.

64. Exterior virtual work; theorems analogous to those of Varignon and Saint-Guilhem. Remarks on the auxiliary functions that were introduced in the preceding paragraphs. – On a deformed medium (M) between the instants t_1 and t_2 in an arbitrary state of virtual deformation, we give the name of *external virtual work* to the expression:

$$\begin{aligned}\delta\mathcal{T}_e=&-\left\{\iiint_{S_0}(A'\delta'x+B'\delta'y+C'\delta'z+P'\delta I'+Q'\delta J'+R'\delta K')dx_0dy_0dz_0\right\}_{t_1}^{t_2}\\&-\int_{t_1}^{t_2}\iint_{S_0}(F_0'\delta'x+G_0'\delta'y+H_0'\delta'z+I_0'\delta I'+J_0'\delta J'+K_0'\delta K')d\sigma_0dt\\&+\int_{t_1}^{t_2}\iiint_{S_0}(F_0'\delta'x+G_0'\delta'y+H_0'\delta'z+I_0'\delta I'+J_0'\delta J'+K_0'\delta K')dx_0dy_0dz_0dt.\end{aligned}$$

We refer to the notations of sec. **60**, and, moreover, let δI, δJ, δK be denote the projections onto the fixed axes of the segment whose projections onto Mx', My', Mz' are $\delta I', \delta J', \delta K'$ in such a way that one has, for example:

$$-\delta I=\alpha''\delta\alpha'+\beta''\delta\beta'+\gamma''\delta\gamma'=-(\alpha'\delta\alpha''+\beta'\delta\beta''+\gamma'''),$$

in which we are always supposing that the axes in question have the same disposition.

This being the case, suppose, as in sec. **63**, that one has given the arbitrary functions λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 the significance that is defined by the formulas:

$$\lambda_1=\delta x,\quad\lambda_2=\delta y,\quad\lambda_3=\delta z,\qquad\mu_1=\delta I,\quad\mu_2=\delta J,\quad\mu_3=\delta K.$$

We then see that the preceding relations we obtained between the new auxiliary functions express only the following condition:

If a trajectory of the deformed medium is given any of the virtual displacements of sec. **60** *then the external virtual work* $\delta\mathcal{T}_e$ *is given by either the relation:*

$$
\begin{aligned}
-\delta\mathcal{T}_e = \int_{t_1}^{t_2}\iiint_{S_0} \Bigg(& p_{xx}\frac{\partial \delta x}{\partial x} + p_{yx}\frac{\partial \delta x}{\partial y} + p_{zx}\frac{\partial \delta x}{\partial z} + p_{xy}\frac{\partial \delta y}{\partial x} + \cdots + p_{zz}\frac{\partial \delta z}{\partial y} \\
& + \frac{A}{\Delta}\frac{d\delta x}{dt} + \frac{B}{\Delta}\frac{d\delta y}{dt} + \frac{C}{\Delta}\frac{d\delta z}{dt}\Bigg) dxdydzdt \\
+ \int_{t_1}^{t_2}\iiint_{S_0} \Bigg(& q_{xx}\frac{\partial \delta I}{\partial x} + q_{yx}\frac{\partial \delta I}{\partial y} + q_{zx}\frac{\partial \delta I}{\partial z} + q_{xy}\frac{\partial \delta J}{\partial x} + \cdots + q_{zz}\frac{\partial \delta K}{\partial z} \\
& + \frac{P}{\Delta}\frac{d\delta I}{dx} + \frac{Q}{\Delta}\frac{d\delta J}{dx} + \frac{R}{\Delta}\frac{d\delta K}{dx}\Bigg) dxdydzdt \\
- \int_{t_1}^{t_2}\iiint_{S_0} \Bigg\{ & \left(p_{yz} - p_{zy} + \frac{C}{\Delta}\frac{dy}{dt} - \frac{B}{\Delta}\frac{dz}{dt}\right)\delta I + \left(p_{zx} - p_{xz} + \frac{A}{\Delta}\frac{dz}{dt} - \frac{C}{\Delta}\frac{dx}{dt}\right)\delta J \\
& + \left(p_{xy} - p_{yx} + \frac{B}{\Delta}\frac{dx}{dt} - \frac{A}{\Delta}\frac{dy}{dt}\right)\delta K \Bigg\} dxdydzdt,
\end{aligned}
$$

in which the integrals are taken over the time interval between the instants t_1 *and* t_2 *and the deformed medium, or by the relation:*

$$
\begin{aligned}
-\delta\mathcal{T}_e = \int_{t_1}^{t_2}\iiint_{S_0} \Bigg(& A_1\frac{\partial \delta x}{\partial x_0} + A_2\frac{\partial \delta x}{\partial y_0} + A_3\frac{\partial \delta x}{\partial z_0} + B_1\frac{\partial \delta y}{\partial x_0} + \cdots + C_3\frac{\partial \delta z}{\partial z_0} \\
& + A\frac{d\delta x}{dt} + B\frac{d\delta y}{dt} + C\frac{d\delta z}{dt}\Bigg) dx_0dy_0dz_0dt \\
+ \int_{t_1}^{t_2}\iiint_{S_0} \Bigg(& P_1\frac{\partial \delta I}{\partial x_0} + P_2\frac{\partial \delta I}{\partial y_0} + P_3\frac{\partial \delta I}{\partial z_0} + Q_1\frac{\partial \delta J}{\partial x_0} + \cdots + R_3\frac{\partial \delta K}{\partial z_0} \\
& + P\frac{d\delta I}{dt} + Q\frac{d\delta J}{dt} + R\frac{d\delta K}{dt}\Bigg) dx_0dy_0dz_0dt \\
- \int_{t_1}^{t_2}\iiint_{S_0} \Bigg(& C_1\frac{\partial y}{\partial x_0} + C_2\frac{\partial y_1}{\partial y_0} + C_3\frac{\partial y}{\partial z_0} + C\frac{dy}{dt} \\
& - B_1\frac{\partial z}{\partial x_0} - B_2\frac{\partial z}{\partial y_0} - B_3\frac{\partial z}{\partial z_0} - B\frac{dz}{dt}\Bigg)\delta I dx_0dy_0dz_0dt \\
- \int_{t_1}^{t_2}\iiint_{S_0} \Bigg(& A_1\frac{\partial z}{\partial x_0} + A_2\frac{\partial z_1}{\partial y_0} + A_3\frac{\partial z}{\partial z_0} + A\frac{dz}{dt} \\
& - C_1\frac{\partial x}{\partial x_0} - C_2\frac{\partial x}{\partial y_0} - C_3\frac{\partial x}{\partial z_0} - C\frac{dx}{dt}\Bigg)\delta J dx_0dy_0dz_0dt
\end{aligned}
$$

$$-\int_{t_1}^{t_2}\iiint_{S_0}\left(B_1\frac{\partial x}{\partial x_0}+B_2\frac{\partial x}{\partial y_0}+B_3\frac{\partial x}{\partial z_0}+B\frac{dx}{dt}\right.$$
$$\left.-A_1\frac{\partial y}{\partial x_0}-A_2\frac{\partial y}{\partial y_0}-A_3\frac{\partial y}{\partial z_0}-A\frac{dy}{dt}\right)\delta K dx_0dy_0dz_0dt=0,$$

in which the integrals are taken over the time interval between the instants t_1 and t_2 and the undeformed medium at the instant t, because the formula that we gave above:

$$\begin{aligned}\delta\mathcal{T}_e=&-\left\{\iiint_{S_0}(A'\delta'x+B'\delta'y+C'\delta'z+P'\delta I'+Q'\delta J'+R'\delta K')dx_0dy_0dz_0\right\}_{t_1}^{t_2}\\&-\int_{t_1}^{t_2}\iint_{S_0}(F_0'\delta'x+G_0'\delta'y+H_0'\delta'z+I_0'\delta I'+J_0'\delta J'+K_0'\delta K')d\sigma_0dt\\&+\int_{t_1}^{t_2}\iiint_{S_0}(F_0'\delta'x+G_0'\delta'y+H_0'\delta'z+I_0'\delta I'+J_0'\delta J'+K_0'\delta K')dx_0dy_0dz_0dt,\end{aligned}$$

which serves to define the external virtual work, may also be written:

$$\begin{aligned}\delta\mathcal{T}_e=&-\left\{\iiint_{S_0}(A\delta x+B\delta y+C\delta z+P\delta I+Q\delta J+R\delta K)dx_0dy_0dz_0\right\}_{t_1}^{t_2}\\&-\int_{t_1}^{t_2}\iint_{S_0}(F_0\delta x+G_0\delta y+H_0\delta z+I_0\delta I+J_0\delta J+K_0\delta K)d\sigma_0dt\\&+\int_{t_1}^{t_2}\iiint_{S_0}(X_0\delta x+Y_0\delta y+Z_0\delta z+L_0\delta I+M_0\delta J+N_0\delta K)dx_0dy_0dz_0dt,\end{aligned}$$

by virtue of the significance of X_0, Y_0, Z_0, L_0, M_0, N_0, F_0, G_0, H_0, I_0, J_0, K_0, A, B, C, P, Q, R, and likewise:

$$\begin{aligned}\delta\mathcal{T}_e=&-\left\{\iiint_{S}\left(\frac{A}{\Delta}\delta x+\frac{B}{\Delta}\delta y+\frac{C}{\Delta}\delta z+\frac{P}{\Delta}\delta I+\frac{Q}{\Delta}\delta J+\frac{R}{\Delta}\delta K\right)dxdydz\right\}_{t_1}^{t_2}\\&-\int_{t_1}^{t_2}\iint_{S}(F\delta x+G\delta y+H\delta z+I\delta I+J\delta J+K\delta K)d\sigma dt\\&+\int_{t_1}^{t_2}\iiint_{S}(X\delta x+Y\delta y+Z\delta z+L\delta I+M\delta J+N\delta K)dxdydzdt,\end{aligned}$$

by virtue of the significance of $X, Y, \ldots, N, F, G, \ldots, K$.

Start with the formula:

$$\delta\int_{t_1}^{t_2}\iiint_{S_0}\delta W dx_0dy_0dz_0dt+\delta\mathcal{T}_e=0,$$

applied to an arbitrary part of the medium that is bounded by a surface S_0 and the time interval between the instants t_1 and t_2. Since δW must be identically null when the variations δx, δy, δz are given by the formulas (60) of sec. **61**, namely:

$$\delta x = (a_1 + \omega_2 z - \omega_3 y)\delta t,$$
$$\delta y = (a_2 + \omega_3 x - \omega_1 z)\delta t,$$
$$\delta z = (a_3 + \omega_1 y - \omega_2 x)\delta t,$$

by virtue of the invariance of W under the group of Euclidean displacements, and δI, δJ, δK are given by:

$$\delta I = \omega_1 \delta t, \qquad \delta J = \omega_2 \delta t, \qquad \delta K = \omega_3 \delta t,$$

and that this is true for any values of the constants a_1, a_2, a_3, ω_1, ω_2, ω_3 we conclude from the expressions for δT_e that just insisted on ([1]) that one has:

$$\left\{\iiint_{S_0} A\,dx_0\,dy_0\,dz_0\right\}_{t_1}^{t_2} + \int_{t_1}^{t_2}\iint_{S_0} F_0\,d\sigma_0\,dt - \int_{t_1}^{t_2}\iiint_{S_0} X_0\,dx_0\,dy_0\,dz_0\,dt = 0,$$
$$\left\{\iiint_{S_0} (P + Cy - Bz)\,dx_0\,dy_0\,dz_0\right\}_{t_1}^{t_2} + \int_{t_1}^{t_2}\iint_{S_0} (I_0 + H_0 y - G_0 z)\,d\sigma_0\,dt$$
$$- \int_{t_1}^{t_2}\iiint_{S_0} (L_0 + Z_0 y - Y_0 z)\,dx_0\,dy_0\,dz_0\,dt = 0,$$

and four analogous equations. *In these formulas, one may imagine that the frontier S_0 is variable.*

The auxiliary functions that were introduced in the preceding paragraphs are not the only ones that one may imagine. Upon confining ourselves to their consideration, we add the same simple remarks as in sec. **54**.

By definition, we have introduced two systems of efforts and moments of deformation relative to a point M of the deformed medium at the instant t. The first of them are the ones that are exerted on surfaces that have their normal parallel to one of the fixed axes Ox, Oy, Oz before deformation. The second are the ones that are exerted on surfaces that have their normal parallel to one of the same fixed axes Ox, Oy, Oz after deformation. The formulas that we indicated give the latter elements in terms of the former; however, by an immediate solution, which we will not elaborate upon, one inversely obtains the former elements in terms of the latter.

Now suppose that one introduces the function W. The first efforts and moments of deformation have the expressions we already indicated, and one immediately deduces the expressions for the second ones. However, in these calculations, one may specify the functions that one must introduce according to the nature of the problem, and which are, *for example, x, y, z,* and three parameters ([2]) λ_1, λ_2, λ_3, by means of which one expresses $\alpha, \alpha', \cdots, \gamma''$.

---

[1] The passage from the elements that are referred to the unit of volume of the undeformed medium and the area of the frontier S_0 to the elements that refer to the unit of volume of the deformed medium and the area of the frontier S at the instant t is sufficiently immediate that it suffices to confine oneself, as we have done, to the first, for example.

[2] For such auxiliary functions λ_1, λ_2, λ_3 one may take, for example, the components of the rotation, which makes the axes Ox, Oy, Oz parallel to Mx', My', Mz', respectively.

If one introduces $x, y, z, \lambda_1, \lambda_2, \lambda_3$, and if one continues to let W denote the function that depends on x_0, y_0, z_0, the first derivatives of x, y, z with respect to x_0, y_0, z_0, t on $\lambda_1, \lambda_2, \lambda_3$, and their first derivatives with respect to x_0, y_0, z_0, t that are obtained by replacing the various quantities $\xi_i, \eta_i, \zeta_i, p_i, q_i, r_i, \xi, \eta, \zeta, p, q, r$ in the function $W(x_0, y_0, z_0, t, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i, \xi, \eta, \zeta, p, q, r)$ by the values they are given by formulas (54), (55), (54′), and (55′), then one will have:

$$
\begin{array}{llll}
A_1 = \dfrac{\partial W}{\partial \dfrac{\partial x}{\partial x_0}}, & A_2 = \dfrac{\partial W}{\partial \dfrac{\partial x}{\partial y_0}}, & A_3 = \dfrac{\partial W}{\partial \dfrac{\partial x}{\partial z_0}}, & A = \dfrac{\partial W}{\partial \dfrac{dx}{dt}}, \\
B_1 = \dfrac{\partial W}{\partial \dfrac{\partial y}{\partial x_0}}, & B_2 = \dfrac{\partial W}{\partial \dfrac{\partial y}{\partial y_0}}, & B_3 = \dfrac{\partial W}{\partial \dfrac{\partial y}{\partial z_0}}, & B = \dfrac{\partial W}{\partial \dfrac{dy}{dt}}, \\
C_1 = \dfrac{\partial W}{\partial \dfrac{\partial z}{\partial x_0}}, & C_2 = \dfrac{\partial W}{\partial \dfrac{\partial z}{\partial y_0}}, & C_3 = \dfrac{\partial W}{\partial \dfrac{\partial z}{\partial z_0}}, & C = \dfrac{\partial W}{\partial \dfrac{dz}{dt}}, \\
\Pi_i = \dfrac{\partial W}{\partial \dfrac{\partial \lambda_1}{\partial \rho_i}}, & \mathrm{X}_i = \dfrac{\partial W}{\partial \dfrac{\partial \lambda_2}{\partial \rho_i}}, & \Sigma_i = \dfrac{\partial W}{\partial \dfrac{\partial \lambda_3}{\partial \rho_i}}, & \\
\Pi_i = \dfrac{\partial W}{\partial \dfrac{\partial \lambda_1}{\partial \rho_i}}, & \mathrm{X}_i = \dfrac{\partial W}{\partial \dfrac{\partial \lambda_2}{\partial \rho_i}}, & \Sigma_i = \dfrac{\partial W}{\partial \dfrac{\partial \lambda_3}{\partial \rho_i}}, & \\
\Pi = \dfrac{\partial W}{\partial \dfrac{d\lambda_1}{dt}}, & \mathrm{X} = \dfrac{\partial W}{\partial \dfrac{d\lambda_2}{dt}}, & \Sigma = \dfrac{\partial W}{\partial \dfrac{d\lambda_3}{dt}}. &
\end{array}
$$

65. Notion of energy of deformation and motion. – We must remark that our present exposition contains the statics of deformable media as a special case. Indeed, it suffices to consider a *reversible virtual modification,* in the sense of DUHEM, instead of envisioning a *realizable virtual deformation,* as we have done.

This observation leads us to consider the notion of the energy of deformation and motion. We propose to determine the work done by external forces and moments, as well as external efforts and moments, of deformation that depend on an arbitrary time interval for a *real modification.* For this, it suffices to calculate the elementary work relative to time dt. The latter is:

$$\left\{\iiint_{S_0} (\xi X_0' + \eta Y_0' + \cdots)\, dx_0\, dy_0\, dz_0 - \iint_{S_0} (\xi F_0' + \eta G_0' + \cdots)\, d\sigma \right\} dt.$$

If one replaces $X_0', Y_0', \cdots, F_0', G_0', \cdots,$ by their expression as a function of the action, and if one performs an inverse calculation to the one that led us to their definition, then one immediately obtains, by virtue of the CODAZZI equations:

$$\left\{\iiint_{S_0}\left(\frac{dE}{dt}+\frac{\partial W}{\partial t}\right)dx_0\,dy_0\,dz_0\right\}dt,$$

in which we have set:

$$E = \xi\frac{\partial W}{\partial \xi}+\eta\frac{\partial W}{\partial \eta}+\varsigma\frac{\partial W}{\partial \varsigma}+p\frac{\partial W}{\partial p}+q\frac{\partial W}{\partial q}+r\frac{\partial W}{\partial r}-W.$$

In particular, if one considers the case in which W does not contain t explicitly, in such a way that $\frac{\partial W}{\partial t}$ is null, then the preceding value becomes the differential with respect to time of the expression:

$$\iiint_{S_0} E\,dx_0\,dy_0\,dz_0,$$

which may be called the *energy of deformation and movement at the instant t.*

At this point in the discussion, we need to make several important general remarks that will find further application in what follows in the theory of Euclidean action.

The only notion of Euclidean action of deformation and motion that *suffices* for us furnishes, in a very extended case, a *constructive* definition of the quantity of motion and the moment of the quantity of motion, the effort and moment of deformation, and the force and external moment. One may distinguish a dynamical part and a static part in the force and the external moment by grouping, on the one had, the terms that contain only the dynamical acceleration, and, on the other hand, the terms that contain only what one may call the *kinematical acceleration;* this distinction obviously expresses an extension of d'ALEMBERT's *principle.* Similarly, suppose that external work is null, and that the energy of deformation and motion remains invariant in time. We thus obtain the notion of *conservation of energy,* which simply translates into the hypothesis that the medium is *isolated* from the external world. In turn, we recover all of the fundamental ideas of classical mechanics, and it is manifest that the particular form that they take in the latter context must be what one envisions for the state of motion and deformation *in an infinitesimal neighborhood of the natural state,* in which one supposes that W and its derivatives are null.

66. Initial state and natural states. General indications on the problem that led us to the consideration of deformable media. – In the foregoing, we considered the trajectory of the deformed state, and, after describing the *initial position* (M_0) of that deformed state at a definite instant t_0 we referred it to the position (M) at an arbitrary instant t. Considerations that are analogous to the ones we developed in sec. **56**, and in which the parameter that was thus introduced is now replaced by time t may be repeated here if we make one of the deformed states play the role that we attributed to the initial state (M_0).

However, one may also imagine that the functions x, y, z that determine the trajectory of the deformed state depend on one parameter, and that one distinguishes a particular value of this parameter. One thus defines a sequence of states that one may call *natural states,* and their trajectory may be called the *trajectory of natural states.* One may use the new parameter as we did in our *Note sur la dynamique du point et du corps invariable* and study, in particular, the trajectory of the deformed states that infinitely close to the trajectory of the natural states.

Conforming to the previous indications, suppose, to fix ideas, that the external force and moment are given by means of simple functions of x_0, y_0, z_0, t, the elements that fix the position of the triad $Mx'y'z'$. We may consider the equations of sec. **62** that relate to the external force and moment as partial differential equations that relate to x, y, z and three parameters λ_1, λ_2, λ_3, by means of which one expresses $\alpha, \alpha', \cdots, \gamma''$. This viewpoint is the one that presents itself most naturally. The expressions ξ_i, η_i, ζ_i, p_i, q_i, r_i, ξ, η, ζ, p, q, r will be functions of $\frac{\partial x}{\partial \rho_i}, \frac{\partial y}{\partial \rho_i}, \frac{\partial z}{\partial \rho_i}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \lambda_1, \cdots, \frac{\partial \lambda_1}{\partial \rho_i}, \cdots, \frac{d\lambda_1}{dt}, \cdots$ (setting $\rho_1 = x_0$, $\rho_2 = y_0$, $\rho_3 = z_0$, as always) that we may calculate by means of formulas (54), (55), (54′) and (55′).

Suppose that $X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0$, or, what amounts to the same thing, X_0, Y_0, Z_0, L_0, M_0, N_0 are given functions of x_0, y_0, z_0, t, x, y, z, λ_1, λ_2, λ_3. After substituting the values of ξ_i, …, r_i, ξ, …, r that one deduces from formulas (54), (55), (54′) and (55′), the expression W is a definite function of:

$$x_0, y_0, z_0, t, \frac{\partial x}{\partial x_0}, \cdots, \frac{\partial z}{\partial z_0}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \lambda_1, \lambda_2, \lambda_3, \frac{\partial \lambda_1}{\partial z_0}, \cdots, \frac{\partial \lambda_3}{\partial z_0}, \frac{d\lambda_1}{dt}, \frac{d\lambda_2}{dt}, \frac{d\lambda_3}{dt}$$

that we continue to denote by W, and the equations of the problem may be written:

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial x}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial x}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial x}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial \frac{dx}{dt}} = X_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial y}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial y}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial y}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial \frac{dy}{dt}} = Y_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial z}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial z}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial z}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial \frac{dz}{dt}} = Z_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial \frac{d\lambda_1}{dt}} - \frac{\partial W}{\partial \lambda_1} = \mathcal{L}_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial\frac{\partial\lambda_2}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial\frac{\partial\lambda_2}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial\frac{\partial\lambda_2}{\partial z_0}}+\frac{d}{dt}\frac{\partial W}{\partial\frac{d\lambda_2}{dt}}-\frac{\partial W}{\partial\lambda_2}=\mathcal{M}_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial\frac{\partial\lambda_3}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial\frac{\partial\lambda_3}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial\frac{\partial\lambda_3}{\partial z_0}}+\frac{d}{dt}\frac{\partial W}{\partial\frac{d\lambda_3}{dt}}-\frac{\partial W}{\partial\lambda_3}=\mathcal{N}_0,$$

in which $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are functions of x_0, y_0, z_0, t, x, y, z, λ_1, λ_2, λ_3 that result from the definitions of sec. **63**. This pertains to the formulas of the preceding paragraphs directly, in a way that is more immediate than the definition of the X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ may be summarized in the relation:

$$\delta\int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt+\delta\mathcal{T}_e=0,$$

i.e., in:

$$\begin{aligned}&\delta\int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt\\
&=\left\{\iiint_{S_0}(A\delta x+B\delta y+C\delta z+P\delta\lambda_1+Q\delta\lambda_2+R\delta\lambda_3)dx_0dy_0dz_0\right\}_{t_1}^{t_2}\\
&+\int_{t_1}^{t_2}\iint_{S_0}(F_0\delta x+G_0\delta y+H_0\delta z+\mathcal{I}_0\delta\lambda_1+\mathcal{J}_0\delta\lambda_2+\mathcal{K}_0\delta\lambda_3)d\sigma_0dt\\
&-\int_{t_1}^{t_2}\iiint_{S_0}(X_0\delta x+Y_0\delta y+Z_0\delta z+\mathcal{L}_0\delta\lambda_1+\mathcal{M}_0\delta\lambda_2+\mathcal{N}_0\delta\lambda_3)dx_0dy_0dz_0dt.\end{aligned}$$

67. Notions of hidden triad and hidden *W*. Case in which *W* depends only on x_0, y_0, z_0, t, ξ_i, η_i, ζ_i, ξ, η, ζ, and is independent of p_i, q_i, r_i, p, q, r. Extension of the classical dynamics of deformable bodies. The gyrostatic medium and kinetic anisotropy. – The considerations that we exposed previously in regard to the hidden triad and hidden *W* are also applicable to the deformable medium in motion. It suffices to simply add that a hidden *W* will correspond to a hidden motion.

In particular, we shall examine the case in which *W* depends only on the quantities x_0, y_0, z_0, t, ξ_i, η_i, ζ_i, ξ, η, ζ but not on the p_i, q_i, r_i, p, q, r. The equations of sec. **66** then reduce to the following:

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial\frac{\partial x}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial\frac{\partial x}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial\frac{\partial x}{\partial z_0}}+\frac{d}{dt}\frac{\partial W}{\partial\frac{dx}{dt}}=X_0,\qquad \frac{\partial W}{\partial\lambda_1}+\mathcal{L}_0=0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial\frac{\partial y}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial\frac{\partial y}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial\frac{\partial y}{\partial z_0}}+\frac{d}{dt}\frac{\partial W}{\partial\frac{dy}{dt}}=Y_0,\qquad \frac{\partial W}{\partial\lambda_2}+\mathcal{M}_0=0$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial z}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial z}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial z}{\partial z_0}}+\frac{d}{dt}\frac{\partial W}{\partial \frac{dz}{dt}}=Z_0, \qquad \frac{\partial W}{\partial \lambda_3}+\mathcal{N}_0=0,$$

in which W depends only x_0, y_0, z_0, t, $\frac{\partial x}{\partial x_0},\cdots,\frac{\partial x}{\partial x_0},\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt},\lambda_1,\lambda_2,\lambda_3,$ and they show us that if we take the simple case in which X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are given functions ([1]) of x_0, y_0, z_0, t, $\frac{\partial x}{\partial x_0},\cdots,\frac{\partial x}{\partial x_0},\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt},\lambda_1,\lambda_2,\lambda_3$ then the three equations on the right may be solved for λ_1, λ_2, λ_3. One thereby finally obtains three partial differential equations that, by our hypotheses, refer only to x_0, y_0, z_0, t, and to x, y, z, and their first and second derivatives.

Imagine the particular case in which the given functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are null; the same will be true for the corresponding values of the functions in any of the systems: (L_0', M_0', N_0'), (L_0, M_0, N_0), (L, M, N). From this, it results that the equations:

$$\frac{\partial W}{\partial \lambda_1}=0, \qquad \frac{\partial W}{\partial \lambda_2}=0, \qquad \frac{\partial W}{\partial \lambda_3}=0,$$

amounts to:

$$\begin{aligned}
&C_1\frac{\partial y}{\partial x_0}+C_2\frac{\partial y}{\partial y_0}+C_3\frac{\partial y}{\partial z_0}-B_1\frac{\partial z}{\partial x_0}-B_2\frac{\partial z}{\partial y_0}-B_3\frac{\partial z}{\partial z_0}=B\frac{dz}{dt}-C\frac{dy}{dt},\\
&A_1\frac{\partial z}{\partial x_0}+A_2\frac{\partial z}{\partial y_0}+A_3\frac{\partial z}{\partial z_0}-C_1\frac{\partial x}{\partial x_0}-C_2\frac{\partial x}{\partial y_0}-C_3\frac{\partial x}{\partial z_0}=C\frac{dx}{dt}-A\frac{dz}{dt},\\
&B_1\frac{\partial x}{\partial x_0}+B_2\frac{\partial x}{\partial y_0}+B_3\frac{\partial x}{\partial z_0}-A_1\frac{\partial y}{\partial x_0}-A_2\frac{\partial y}{\partial y_0}-A_3\frac{\partial y}{\partial z_0}=A\frac{dy}{dt}-B\frac{dx}{dt},
\end{aligned}$$

i.e., to:

$$p_{yz}-p_{zy}=\frac{1}{\Delta}\left(B\frac{dz}{dt}-C\frac{dy}{dt}\right), \qquad p_{zx}-p_{xz}=\frac{1}{\Delta}\left(C\frac{dx}{dt}-A\frac{dz}{dt}\right),$$

$$p_{xy}-p_{yx}=\frac{1}{\Delta}\left(A\frac{dy}{dt}-B\frac{dx}{dt}\right),$$

which one may interpret as saying that the motion of the deformable body in question, which constitutes the classical theory of elasticity as a special case, gives rise to a *moment* whose three components are:

[1] To simplify the exposition and to indicate more easily what we are alluding to, we suppose that X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ do not refer to the derivatives of λ_1, λ_2, λ_3.

$$\frac{1}{\Delta}\left(B\frac{dz}{dt}-C\frac{dy}{dt}\right),\qquad \frac{1}{\Delta}\left(C\frac{dx}{dt}-A\frac{dz}{dt}\right),\qquad \frac{1}{\Delta}\left(A\frac{dy}{dt}-B\frac{dx}{dt}\right),$$

and thus has the effect of *destroying* the equalities:

$$p_{yz} = p_{zy},\qquad p_{zx} = p_{xz},\qquad p_{xy} = p_{yz}\,.$$

Having said this, we observe that if one starts with a trajectory that is supposed to be *given* and deduces the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$, as in sec. **63**, then, in the case in which these three functions are null one may arrive at the result that accidentally presents itself, i.e., for a certain set of particular trajectories; however, one may arrive at this for any trajectory (M) as a consequence of the nature of the medium (M), and its motions, i.e., from the form of W.

Imagine the latter case, which is particularly interesting; W is then a simple function ([1]) of x_0, y_0, z_0, t, and ten expressions ε_1, ε_2, ε_3, γ_1, γ_2, γ_3, φ_1, φ_2, φ_3, v^2 that is defined by the following formulas:

$$\varepsilon_1 = \frac{1}{2}\left\{\left(\frac{\partial x}{\partial x_0}\right)^2+\left(\frac{\partial y}{\partial x_0}\right)^2+\left(\frac{\partial z}{\partial x_0}\right)^2-1\right\}=\frac{1}{2}(\xi_1^2+\eta_1^2+\varsigma_1^2-1),$$

$$\varepsilon_2 = \frac{1}{2}\left\{\left(\frac{\partial x}{\partial y_0}\right)^2+\left(\frac{\partial y}{\partial y_0}\right)^2+\left(\frac{\partial z}{\partial y_0}\right)^2-1\right\}=\frac{1}{2}(\xi_2^2+\eta_2^2+\varsigma_2^2-1),$$

$$\varepsilon_3 = \frac{1}{2}\left\{\left(\frac{\partial x}{\partial z_0}\right)^2+\left(\frac{\partial y}{\partial z_0}\right)^2+\left(\frac{\partial z}{\partial z_0}\right)^2-1\right\}=\frac{1}{2}(\xi_3^2+\eta_3^2+\varsigma_3^2-1),$$

$$\gamma_1 = \frac{\partial x}{\partial y_0}\frac{\partial x}{\partial z_0}+\frac{\partial y}{\partial y_0}\frac{\partial y}{\partial z_0}+\frac{\partial z}{\partial y_0}\frac{\partial z}{\partial z_0}=\xi_2\xi_3+\eta_2\eta_3+\varsigma_2\varsigma_3,$$

$$\gamma_2 = \frac{\partial x}{\partial z_0}\frac{\partial x}{\partial x_0}+\frac{\partial y}{\partial z_0}\frac{\partial y}{\partial x_0}+\frac{\partial z}{\partial z_0}\frac{\partial z}{\partial x_0}=\xi_3\xi_1+\eta_3\eta_1+\varsigma_3\varsigma_1,$$

$$\gamma_3 = \frac{\partial x}{\partial x_0}\frac{\partial x}{\partial y_0}+\frac{\partial y}{\partial x_0}\frac{\partial y}{\partial y_0}+\frac{\partial z}{\partial x_0}\frac{\partial z}{\partial y_0}=\xi_1\xi_2+\eta_1\eta_2+\varsigma_1\varsigma_2,$$

$$\varphi_1 = \frac{dx}{dt}\frac{\partial x}{\partial x_0}+\frac{dy}{dt}\frac{\partial y}{\partial x_0}+\frac{dz}{dt}\frac{\partial z}{\partial x_0}=\xi\xi_1+\eta\eta_1+\varsigma\varsigma_1,$$

$$\varphi_2 = \frac{dx}{dt}\frac{\partial x}{\partial y_0}+\frac{dy}{dt}\frac{\partial y}{\partial y_0}+\frac{dz}{dt}\frac{\partial z}{\partial y_0}=\xi\xi_2+\eta\eta_2+\varsigma\varsigma_2,$$

$$\varphi_3 = \frac{dx}{dt}\frac{\partial x}{\partial z_0}+\frac{dy}{dt}\frac{\partial y}{\partial z_0}+\frac{dz}{dt}\frac{\partial z}{\partial z_0}=\xi\xi_3+\eta\eta_3+\varsigma\varsigma_3,$$

$$v^2 = \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=\xi^2+\eta^2+\varsigma^2.$$

[1] The triad is completely hidden; thus, we may also imagine that we have a simply pointlike medium.

The equations deduced in sec. **62** and **63** reduce to either:

$$\sum\left(\frac{\partial A_i'}{\partial\rho_i}+q_iC_i'-r_iB_i'\right)+\frac{dA'}{dt}+qC'-rB'=X_0', \qquad F_0'=l_0A_1'+m_0A_2'+n_0A_3',$$
$$\sum\left(\frac{\partial B_i'}{\partial\rho_i}+r_iA_i'-p_iC_i'\right)+\frac{dB'}{dt}+rA'-pC'=Y_0', \qquad G_0'=l_0B_1'+m_0B_2'+n_0B_3',$$
$$\sum\left(\frac{\partial C_i'}{\partial\rho_i}+p_iB_i'-q_iA_i'\right)+\frac{dC'}{dt}+pB'-qA'=Z_0', \qquad H_0'=l_0C_1'+m_0C_2'+n_0C_3',$$

in which one has:

$$A_i'=\xi_i\frac{\partial W}{\partial\varepsilon_i}+\xi_k\frac{\partial W}{\partial\gamma_j}+\xi_j\frac{\partial W}{\partial\gamma_k}+\xi\frac{\partial W}{\partial\varphi_i},$$
$$B_i'=\eta_i\frac{\partial W}{\partial\varepsilon_i}+\eta_k\frac{\partial W}{\partial\gamma_j}+\eta_j\frac{\partial W}{\partial\gamma_k}+\eta\frac{\partial W}{\partial\varphi_i}, \qquad (i, j, k = 1, 2, 3),$$
$$C_i'=\varsigma_i\frac{\partial W}{\partial\varepsilon_i}+\varsigma_k\frac{\partial W}{\partial\gamma_j}+\varsigma_j\frac{\partial W}{\partial\gamma_k}+\varsigma\frac{\partial W}{\partial\varphi_i},$$
$$A'=\frac{1}{v}\frac{\partial W}{\partial v}\xi+\sum\xi_i\frac{\partial W}{\partial\varphi_i},$$
$$B'=\frac{1}{v}\frac{\partial W}{\partial v}\eta+\sum\eta_i\frac{\partial W}{\partial\varphi_i},$$
$$C'=\frac{1}{v}\frac{\partial W}{\partial v}\varsigma+\sum\varsigma_i\frac{\partial W}{\partial\varphi_i},$$

or to:

$$\frac{\partial A_1}{\partial x_0}+\frac{\partial A_2}{\partial y_0}+\frac{\partial A_3}{\partial z_0}+\frac{dA}{dt}=X_0, \qquad F_0=l_0A_1+m_0A_2+n_0A_3,$$
$$\frac{\partial B_1}{\partial x_0}+\frac{\partial B_2}{\partial y_0}+\frac{\partial B_3}{\partial z_0}+\frac{dB}{dt}=Y_0, \qquad G_0=l_0B_1+m_0B_2+n_0B_3,$$
$$\frac{\partial C_1}{\partial x_0}+\frac{\partial C_2}{\partial y_0}+\frac{\partial C_3}{\partial z_0}+\frac{dC}{dt}=Z_0, \qquad H_0=l_0C_1+m_0C_2+n_0C_3,$$

in which one has:

$$A_1=\Omega_1\frac{\partial x}{\partial x_0}+\Xi_3\frac{\partial x}{\partial y_0}+\Xi_2\frac{\partial x}{\partial z_0}+\Phi_1\frac{dx}{dt},$$
$$B_1=\Omega_1\frac{\partial y}{\partial x_0}+\Xi_3\frac{\partial y}{\partial y_0}+\Xi_2\frac{\partial y}{\partial z_0}+\Phi_1\frac{dy}{dt},$$
$$C_1=\Omega_1\frac{\partial z}{\partial x_0}+\Xi_3\frac{\partial z}{\partial y_0}+\Xi_2\frac{\partial z}{\partial z_0}+\Phi_1\frac{dz}{dt},$$

with analogous expressions for A_2, B_2, C_2, A_3, B_3, C_3 and

$$A=\Phi_1\frac{\partial x}{\partial x_0}+\Phi_2\frac{\partial x}{\partial y_0}+\Phi_3\frac{\partial x}{\partial z_0}+\frac{1}{v}\frac{\partial W}{\partial v}\frac{dx}{dt},$$
$$B=\Phi_1\frac{\partial y}{\partial x_0}+\Phi_2\frac{\partial y}{\partial y_0}+\Phi_3\frac{\partial y}{\partial z_0}+\frac{1}{v}\frac{\partial W}{\partial v}\frac{dy}{dt},$$
$$C=\Phi_1\frac{\partial z}{\partial x_0}+\Phi_2\frac{\partial z}{\partial y_0}+\Phi_3\frac{\partial z}{\partial z_0}+\frac{1}{v}\frac{\partial W}{\partial v}\frac{dz}{dt},$$

upon setting:

$$\Omega_i=\frac{\partial W}{\partial\varepsilon_i},\qquad \Xi_i=\frac{\partial W}{\partial\gamma_i},\qquad \Phi_i=\frac{\partial W}{\partial\varphi_i},$$

or again to:

$$\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}+\frac{1}{\Delta}\frac{dA}{dt}=X,\qquad F=lp_{xx}+mp_{yx}+np_{zx},$$
$$\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}+\frac{1}{\Delta}\frac{dB}{dt}=Y,\qquad G=lp_{xy}+mp_{yy}+np_{zy},$$
$$\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}+\frac{1}{\Delta}\frac{dC}{dt}=Z,\qquad H=lp_{xz}+mp_{yz}+np_{zz},$$

in which one has:

$$p_{xx}=\frac{1}{\Delta}\left\{\Omega_1\left(\frac{\partial x}{\partial x_0}\right)^2+\Omega_2\left(\frac{\partial x}{\partial y_0}\right)^2+\Omega_3\left(\frac{\partial x}{\partial z_0}\right)^2+2\Xi_1\frac{\partial x}{\partial z_0}\frac{\partial x}{\partial x_0}++2\Xi_3\frac{\partial x}{\partial x_0}\frac{\partial x}{\partial y_0}\right.$$
$$\left.+\left(\Phi_1\frac{\partial x}{\partial x_0}+\Phi_2\frac{\partial x}{\partial y_0}+\Phi_3\frac{\partial x}{\partial z_0}\right)\frac{dx}{dt}\right\},$$

$$p_{yx}=\frac{1}{\Delta}\left\{\Omega_1\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial x_0}+\Omega_2\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial y_0}+\Omega_3\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial z_0}\right.$$
$$+\Xi_1\left(\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial z_0}+\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial y_0}\right)+\Xi_2\left(\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial x_0}+\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial z_0}\right)+\Xi_3\left(\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0}+\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right)$$
$$\left.+\left(\Phi_1\frac{\partial y}{\partial x_0}+\Phi_2\frac{\partial y}{\partial y_0}+\Phi_3\frac{\partial y}{\partial z_0}\right)\frac{dx}{dt}\right\},$$

$$p_{zx}=\frac{1}{\Delta}\left\{\Omega_1\frac{\partial z}{\partial x_0}\frac{\partial y}{\partial x_0}+\Omega_2\frac{\partial z}{\partial y_0}\frac{\partial y}{\partial y_0}+\Omega_3\frac{\partial z}{\partial z_0}\frac{\partial y}{\partial z_0}\right.$$
$$+\Xi_1\left(\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial z_0}+\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial y_0}\right)+\Xi_2\left(\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial x_0}+\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial z_0}\right)+\Xi_3\left(\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial y_0}+\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial x_0}\right)$$
$$\left.+\left(\Phi_1\frac{\partial z}{\partial x_0}+\Phi_2\frac{\partial z}{\partial y_0}+\Phi_3\frac{\partial z}{\partial z_0}\right)\frac{dx}{dt}\right\},$$

with analogous expressions for p_{xy}, p_{yy}, p_{zy}, p_{xz}, p_{yz}, p_{zz} . We thus obtain the most general equations of motion for the classical deformable body.

In order for the effort to satisfy the relations:

$$p_{yz} = p_{zy}, \qquad p_{zx} = p_{xz}, \qquad p_{xy} = p_{yx},$$

it is sufficient that one has:

$$\varphi_1 = 0, \qquad \varphi_2 = 0, \qquad \varphi_3 = 0,$$

i.e., that W is independent of the arguments φ_1, φ_2, φ_3. More particularly, if one must have:

$$p_{yz} = p_{zy} = 0, \qquad p_{zx} = p_{xz} = 0, \qquad p_{xy} = p_{yx} = 0,$$

then W must be a simple function of Δ and v, and one finds that:

$$p_{xx} = p_{yy} = p_{zz} = \frac{\partial W}{\partial \Delta};$$

one then finds the motion of a *perfect* fluid in this case.

When the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are not null, W will have the twelve translations ξ_i, η_i, ζ_i, ξ, η, ζ for its arguments. On the one hand, the medium may be regarded as *gyrostatic*, by giving a justifiable extension to this word, which was coined by LORD KELVIN, and, on the other hand, the medium is endowed with *kinetic anisotropy*, in the sense envisioned by RANKINE and then by LORD RAYLEIGH. For example, one therefore makes the theory of the double refraction of light, such as was exposed by LORD RAYLEIGH and GLAZEBROOK, rest on a purely mechanical basis.

V. – EUCLIDEAN ACTION AT A DISTANCE, ACTION OF CONSTRAINT, AND DISSIPATIVE ACTION

68. – Euclidean action of deformation and motion in a discontinuous medium. – Consider a discrete system of n triads in which each triad is distinguished by an index i that consequently takes the values 1, 2, …, n. Let $M_i x_i' y_i' z_i'$ be the triad whose index is i, with an origin M_i that has the coordinates x_i, y_i, z_i, and axes $M_i x_i'$, $M_i y_i'$, $M_i x_i'$ that have the direction cosines $\alpha_i, \alpha_i', \alpha_i''; \beta_i, \beta_i', \beta_i''; \gamma_i, \gamma_i', \gamma_i''$ with respect to three fixed rectangular axes Ox, Oy, Oz. We suppose that the quantities x_i, y_i, z_i, $\alpha_i, \alpha_i', \cdots, \gamma_i''$ are functions of time t, and we introduce the six arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i that are defined by formulas $(54')$ and $(55')$ of sec. **60** with the index i.

Envision a function W of two infinitely close positions of the system of triads $M_i x_i' y_i' z_i'$, i.e., a function of t, of x_i, y_i, z_i, $\alpha_i, \alpha_i', \cdots, \gamma_i''$, and their first derivatives with respect to t (i takes the values 1, 2, …, n). We propose to determine what sort of form W must take in order for that function to remain invariant under any infinitesimal transformation of the group of Euclidean displacements such as (60). Observe that the relations $(54')$ and $(55')$ of sec. **60**, with the index i, permit us to express the first derivatives of the nine direction cosines $\alpha_i, \alpha_i', \cdots, \gamma_i''$ with respect to t by means of well-known formulas that involve these cosines and p_i, q_i, r_i, and, on the other hand, to express these nine cosines $\alpha_i, \alpha_i', \cdots, \gamma_i''$ by means of ξ_i, η_i, ζ_i, and the first derivatives of x_i, y_i, z_i with respect to t. We may therefore finally express the function W that we seek as a function of t, of x_i, y_i, z_i, and their first derivatives, and finally, of ξ_i, η_i, ζ_i, p_i, q_i, r_i, which we indicate by writing:

$$W = W\left(t, x_i, y_i, z_i, \frac{dx_i}{dt}, \frac{dx_i}{dt}, \frac{dx_i}{dt}, \xi_i, \eta_i, \varsigma_i, p_i, q_i, r_i\right).$$

Since the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$, δp_i, δq_i, δr_i are null in the present case, as a result of the well-known theory of moving frames, we must write the new form for W that one obtains by virtue of formulas (60), when taken with the index i, and for any a_1, a_2, a_3, ω_1, ω_2, ω_3 :

$$\sum_i \left(\frac{\partial W}{\partial x_i}\delta x_i + \frac{\partial W}{\partial y_i}\delta y_i + \frac{\partial W}{\partial z_i}\delta z_i + \frac{\partial W}{\partial \frac{dx_i}{dt}}\delta\frac{dx_i}{dt} + \frac{\partial W}{\partial \frac{dy_i}{dt}}\delta\frac{dy_i}{dt} + \frac{\partial W}{\partial \frac{dz_i}{dt}}\delta\frac{dz_i}{dt} \right) = 0.$$

Replace δx_i, δy_i, δz_i with their values in (60) and $\delta\frac{dx_i}{dt}, \delta\frac{dy_i}{dt}, \delta\frac{dz_i}{dt}$ with the values one obtains by differentiating them. Equate the coefficients of a_1, a_2, a_3, ω_1, ω_2, ω_3; we obtain the following six conditions:

(63) $$\sum_i \frac{\partial W}{\partial x_i} = 0, \qquad \sum_i \frac{\partial W}{\partial y_i} = 0, \qquad \sum_i \frac{\partial W}{\partial z_i} = 0,$$

and

(64) $$\sum \left(y_i \frac{\partial W}{\partial z_i} - z_i \frac{\partial W}{\partial y_i} + \frac{dy_i}{dt} \frac{\partial W}{\partial \frac{dz_i}{dt}} - \frac{dz_i}{dt} \frac{\partial W}{\partial \frac{dy_i}{dt}} \right) = 0,$$

with analogous relations.

If we suppose that *the points* (x_i, y_i, z_i) *describe all possible trajectories* then we arrive at identities that verified by the function W of the $6n$ arguments of x_i, y_i, z_i, $\frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}$, and the last arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i, which we leave aside for the moment. We seek to discover the resulting form for W.

We commence by treating the case of the system of three equations:

(65) $$\begin{cases} \sum_{i=1}^{i=p} \left(y_i \frac{\partial W}{\partial z_i} - z_i \frac{\partial W}{\partial y_i} \right) = 0, \\ \sum_{i=1}^{i=p} \left(z_i \frac{\partial W}{\partial x_i} - x_i \frac{\partial W}{\partial z_i} \right) = 0, \\ \sum_{i=1}^{i=p} \left(x_i \frac{\partial W}{\partial y_i} - y_i \frac{\partial W}{\partial x_i} \right) = 0, \end{cases}$$

that determine a function W of the $3n$ arguments x_i, y_i, z_i. We have already encountered this system in the context of the statics of the line, surface, and continuous three-dimensional medium, in the case where $p = 1$, $p = 2$, $p = 3$. We leave aside the case $p = 1$, in which the three equations reduce to two. For $p = 2$ and $p = 3$, we have three equations that form a complete system. For $p = 2$, we have three equations, six variables, and three independent solutions:

$$x_i^2 + y_i^2 + z_i^2 \qquad (i = 1, 2), \qquad x_1x_2 + y_1y_2 + z_1z_2;$$

for $p = 3$, we have three equations, nine variables, and six independent solutions:

$$x_i^2 + y_i^2 + z_i^2 \qquad (i = 1, 2, 3), \qquad x_ix_i + y_iy_i + z_iz_i \qquad (i = 1, 2, 3).$$

For $p > 3$, the system is still complete. To prove this it suffices to show that they admit $3p - 3$ independent solutions, in which the number of equations is 3 and the number of variables is $3p$. We effectively have first, the p solutions:

$$x_i^2 + y_i^2 + z_i^2 \qquad (i = 1, 2, \ldots, p),$$

then the solution:

$$x_1x_2 + y_1y_2 + z_1z_2,$$

and finally, the $2(p-2)$ solutions:

$$x_1x_i + y_1y_i + z_1z_i, \qquad x_2x_i + y_2y_i + z_2z_i \qquad (i = 3, 4, 5, .., p),$$

which are independent. W is thus a function of the $3(p-1)$ independent arguments that we just enumerated.

Now return to the proposed system that is formed from conditions (63) and (64). The conditions (63) prove that W depends on $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$ only by the intermediary of the expressions:

$$\begin{array}{llll} X_2 = x_2 - x_1, & X_3 = x_3 - x_1, & \ldots, & X_n = x_n - x_1, \\ Y_2 = y_2 - y_1, & Y_3 = y_3 - y_1, & \ldots, & Y_n = y_n - y_1, \\ Z_2 = z_2 - z_1, & Z_3 = z_3 - z_1, & \ldots, & Z_n = z_n - z_1. \end{array}$$

On the other hand, set:

$$\frac{dx_i}{dt} = X_{n+i}, \qquad \frac{dy_i}{dt} = Y_{n+i}, \qquad \frac{dz_i}{dt} = Z_{n+i},$$

and demand that equations (64) be verified by the function W of the arguments $X_2, X_3, \ldots, X_{2n}$; $Y_2, Y_3, \ldots, Y_{2n}$; $Z_2, Z_3, \ldots, Z_{2n}$. For example, consider the first of equations (64); they become:

$$-y_1\left(\frac{\partial W}{\partial Z_2} + \frac{\partial W}{\partial Z_3} + \cdots + \frac{\partial W}{\partial Z_n}\right) + z_1\left(\frac{\partial W}{\partial Y_2} + \frac{\partial W}{\partial Y_3} + \cdots + \frac{\partial W}{\partial Y_n}\right)$$
$$+ (y_1 - Y_2)\frac{\partial W}{\partial Z_2} - (z_1 - Z_2)\frac{\partial W}{\partial Y_2} + \cdots = 0.$$

y_1 and z_1 disappear, and what remains are the first of the equations:

$$\sum_{i=1}^{i=2n}\left(y_i\frac{\partial W}{\partial z_i} - z_i\frac{\partial W}{\partial y_i}\right) = 0,$$
$$\sum_{i=1}^{i=2n}\left(z_i\frac{\partial W}{\partial x_i} - x_i\frac{\partial W}{\partial z_i}\right) = 0,$$
$$\sum_{i=1}^{i=2n}\left(x_i\frac{\partial W}{\partial y_i} - y_i\frac{\partial W}{\partial x_i}\right) = 0.$$

We thus come down to the system (65), in which x_i, y_i, z_i are replaced by $X_{i+1}, Y_{i+1}, Z_{i+1}$, and p by $2n-1$.

If we first suppose that $n = 2$, then we see that W is abstractly given in terms of the arguments $\xi_i, \eta_i, \zeta_i, p_i, q_i, r_i$ as a function of the independent expressions:

$$X_2^2 + Y_2^2 + Z_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

$$X_3^2+Y_3^2+Z_3^2=(\frac{dx_1}{dt})^2+(\frac{dy_1}{dt})^2+(\frac{dz_1}{dt})^2=\xi_1^2+\eta_1^2+\varsigma_1^2,$$

$$X_4^2+Y_4^2+Z_4^2=(\frac{dx_2}{dt})^2+(\frac{dy_2}{dt})^2+(\frac{dz_2}{dt})^2=\xi_2^2+\eta_2^2+\varsigma_2^2,$$

$$X_2X_3+Y_2Y_3+Z_2Z_3=(x_2-x_1)\frac{dx_1}{dt}+(y_2-y_1)\frac{dy_1}{dt}+(z_2-z_1)\frac{dz_1}{dt},$$

$$X_2X_4+Y_2Y_4+Z_2Z_4=(x_2-x_1)\frac{dx_2}{dt}+(y_2-y_1)\frac{dy_2}{dt}+(z_2-z_1)\frac{dz_2}{dt},$$

$$X_3X_4+Y_3Y_4+Z_3Z_4=\frac{dx_1}{dt}\frac{dx_2}{dt}+\frac{dy_1}{dt}\frac{dy_2}{dt}+\frac{dz_1}{dt}\frac{dz_2}{dt}.$$

Therefore, we finally have that W is a function of ξ_i, η_i, ζ_i, p_i, q_i, r_i, and the *four arguments:*

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2,$$

$$(x_2-x_1)\frac{dx_1}{dt}+(y_2-y_1)\frac{dy_1}{dt}+(z_2-z_1)\frac{dz_1}{dt},$$

$$(x_2-x_1)\frac{dx_2}{dt}+(y_2-y_1)\frac{dy_2}{dt}+(z_2-z_1)\frac{dz_2}{dt},$$

$$\frac{dx_1}{dt}\frac{dx_2}{dt}+\frac{dy_1}{dt}\frac{dy_2}{dt}+\frac{dz_1}{dt}\frac{dz_2}{dt}.$$

If we suppose that $n > 2$ then we see that W is abstractly given in terms of the arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i as a function of $6(n-1)$ independent arguments:

$$X_i^2+Y_i^2+Z_i^2=\begin{cases}(x_i-x_1)^2+(y_i-y_1)^2+(z_i-z_1)^2 \quad (i=1,2,\cdots,n),\\ \left(\dfrac{dx_k}{dt}\right)^2+\left(\dfrac{dy_k}{dt}\right)^2+\left(\dfrac{dz_k}{dt}\right)^2=\xi_k^2+\eta_k^2+\varsigma_k^2,\end{cases}$$

$$X_2X_3+Y_2Y_3+Z_2Z_3=(x_2-x_1)(x_3-x_1)+(y_2-y_1)(y_3-y_1)+(z_2-z_1)(z_3-z_1),$$

$$X_2X_i+Y_2Y_i+Z_2Z_i=\begin{cases}(x_2-x_1)(x_3-x_1)+(y_2-y_1)(y_3-y_1)+(z_2-z_1)(z_3-z_1),\\ (x_2-x_1)\dfrac{dx_k}{dt}+(y_2-y_1)\dfrac{dy_k}{dt}+(z_2-z_1)\dfrac{dz_k}{dt},\end{cases}$$

$$X_3X_i+Y_3Y_i+Z_3Z_i=\begin{cases}(x_3-x_1)(x_i-x_1)+(y_3-y_1)(y_i-y_1)+(z_3-z_1)(z_i-z_1),\\ (x_3-x_1)\dfrac{dx_k}{dt}+(y_3-y_1)\dfrac{dy_k}{dt}+(z_3-z_1)\dfrac{dz_k}{dt}.\end{cases}$$

We remark that one has:

$$(x_i-x_j)(x_i-x_j)+(y_i-y_j)(y_i-y_j)+(z_i-z_j)(z_i-z_j)=\tfrac{1}{2}(r_{ij}^2+r_{ik}^2-r_{kj}^2),$$

in which r is the distance between two points of the system. From symmetry reasons, one may have to involve arguments in W that are *not independent*, in which case, one may take, independently of the ξ_i, η_i, ζ_i, p_i, q_i, r_i, the following arguments:

$$r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2,$$
$$\psi_{ij} = \frac{dx_i}{dt}\frac{dx_j}{dt} + \frac{dy_i}{dt}\frac{dy_j}{dt} + \frac{dz_i}{dt}\frac{dz_j}{dt},$$
$$\lambda_{ijk} = (x_i - x_j)\frac{dx_k}{dt} + (y_i - y_j)\frac{dy_k}{dt} + (z_i - z_j)\frac{dz_k}{dt};$$

the latter subsume the arguments with three indices λ_{iji} and arguments with four indices λ_{ijk}. They figure only when there are more than two points, and one sees that the action on two points is influenced by all of the other points in this case. It is easy to establish the relations that exist between these dependent arguments in a form that is sufficiently complex; they are analogous to the relations between the distances r_{ij} when the number of points is ≥ 5.

If we know the expression for the Euclidean action W in a the system of triads in question, then, by a calculation that repeats the ones we made before, one may easily find the expression for the external force and moment on an arbitrary triad. Since the action W is a function of x_i, y_i, z_i, $\frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}$, by the intermediary of r_{ij}, ψ_{ij}, λ_{ijk}, it is easy to regard W as primarily a function of x_i, y_i, z_i, $\frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}$, and of ξ_i, η_i, ζ_i, p_i, q_i, r_i. We have:

$$\delta\int_{t_1}^{t_2} W dt$$
$$= \left[\sum_i (A_i\delta x_i + B_i\delta y_i + C_i\delta z_i + P_i\delta I_i + Q_i\delta J_i + R_i\delta K_i)\right]_{t_1}^{t_2}$$
$$- \int_{t_1}^{t_2} \sum_i (X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i + L_i\delta I_i + M_i\delta J_i + N_i\delta K_i)dt,$$

in which we have set:

$$A_i = \alpha_i\frac{\partial W}{\partial \xi_i} + \beta_i\frac{\partial W}{\partial \eta_i} + \gamma_i\frac{\partial W}{\partial \varsigma_i}, \qquad P_i = \alpha_i\frac{\partial W}{\partial p_i} + \beta_i\frac{\partial W}{\partial q_i} + \gamma_i\frac{\partial W}{\partial r_i},$$
$$B_i = \alpha_i'\frac{\partial W}{\partial \xi_i} + \beta_i'\frac{\partial W}{\partial \eta_i} + \gamma_i'\frac{\partial W}{\partial \varsigma_i}, \qquad Q_i = \alpha_i'\frac{\partial W}{\partial p_i} + \beta_i'\frac{\partial W}{\partial q_i} + \gamma_i'\frac{\partial W}{\partial r_i},$$
$$C_i = \alpha_i''\frac{\partial W}{\partial \xi_i} + \beta_i''\frac{\partial W}{\partial \eta_i} + \gamma_i''\frac{\partial W}{\partial \varsigma_i}, \qquad R_i = \alpha_i''\frac{\partial W}{\partial p_i} + \beta_i''\frac{\partial W}{\partial q_i} + \gamma_i''\frac{\partial W}{\partial r_i},$$

in which (A_i, B_i, C_i) and (P_i, Q_i, R_i) are the quantity of motion and the moment of the quantity of motion, respectively, for the triad of index i, and:

$$X_i = \frac{dA_i}{dt} + \frac{d}{dt}\left(\frac{\partial W}{\partial \frac{dx_i}{dt}}\right) - \frac{\partial W}{\partial x_i}, \qquad L_i = \frac{dP_i}{dt} + C_i \frac{dy_i}{dt} - B_i \frac{dz_i}{dt},$$

$$Y_i = \frac{dB_i}{dt} + \frac{d}{dt}\left(\frac{\partial W}{\partial \frac{dy_i}{dt}}\right) - \frac{\partial W}{\partial y_i}, \qquad M_i = \frac{dQ_i}{dt} + A_i \frac{dz_i}{dt} - C_i \frac{dx_i}{dt},$$

$$Z_i = \frac{dC_i}{dt} + \frac{d}{dt}\left(\frac{\partial W}{\partial \frac{dz_i}{dt}}\right) - \frac{\partial W}{\partial z_i}, \qquad N_i = \frac{dR_i}{dt} + B_i \frac{dx_i}{dt} - A_i \frac{dy_i}{dt},$$

in which (X_i, Y_i, Z_i) and (L_i, M_i, N_i) are the external force and external moment of the triad of index i; what remains in these calculations is to exhibit the arguments r_{ij}, ψ_{ij}, λ_{ijk}, but this is easy.

We remark that the expression for the external force may be decomposed into two parts. The first, which depends on the segments (A_i, B_i, C_i), (P_i, Q_i, R_i) and their derivatives, is the properly dynamical part. The second, which results from the presence of the arguments r_{ij}, ψ_{ij}, λ_{ijk} in W corresponds to the force that the triad of index i is subject to on the part of the other triads of the system. Consider the expression:

$$\sum_i \Big[X_i \frac{dx_i}{dt} + Y_i \frac{dy_i}{dt} + Y_i \frac{dz_i}{dt} + L_i(\alpha_i p_i + \beta_i q_i + \gamma_i r_i) \\ + M_i(\alpha'_i p_i + \beta'_i q_i + \gamma'_i r_i) + N_i(\alpha''_i p_i + \beta''_i q_i + \gamma''_i r_i)\Big] dt,$$

which represent the sum of the elementary works of the forces applied to the different triads. If we calculate them upon replacing X_i, Y_i, Z_i, L_i, M_i, N_i, with the preceding values then we find the following expression for the elementary work relative to the dynamical part of the external force and the external moment:

$$\sum_i \left[\frac{d}{dt}\left(\xi_i \frac{\partial W}{\partial \xi_i} + \eta_i \frac{\partial W}{\partial \eta_i} + \varsigma_i \frac{\partial W}{\partial \varsigma_i} + p_i \frac{\partial W}{\partial p_i} + q_i \frac{\partial W}{\partial q_i} + r_i \frac{\partial W}{\partial r_i} \right) \\ - \left(\frac{\partial W}{\partial \xi_i}\frac{d\xi_i}{dt} + \frac{\partial W}{\partial \eta_i}\frac{d\eta_i}{dt} + \cdots + \frac{\partial W}{\partial r_i}\frac{dr_i}{dt} \right) \right] dt,$$

and, for the elementary work due to the forces that are exerted between the triads of the system, we have:

$$\sum_i \left[\frac{d}{dt}\left(\frac{dx_i}{dt}\frac{\partial W}{\partial \frac{dx_i}{dt}} + \frac{dy_i}{dt}\frac{\partial W}{\partial \frac{dy_i}{dt}} + \frac{dz_i}{dt}\frac{\partial W}{\partial \frac{dz_i}{dt}} \right) \right.$$

$$\left. - \left(\frac{\partial W}{\partial \frac{dx_i}{dt}}\frac{d^2 x_i}{dt^2} + \frac{\partial W}{\partial \frac{dy_i}{dt}}\frac{d^2 y_i}{dt^2} + \frac{\partial W}{\partial \frac{dz_i}{dt}}\frac{d^2 z_i}{dt^2} + \frac{\partial W}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial W}{\partial y_i}\frac{dy_i}{dt} + \frac{\partial W}{\partial z_i}\frac{dz_i}{dt} \right) \right] dt.$$

If we add these two expressions, and set:

$$E = \sum_i \left(\xi_i \frac{\partial W}{\partial \xi_i} + \eta_i \frac{\partial W}{\partial \eta_i} + \varsigma_i \frac{\partial W}{\partial \varsigma_i} + p_i \frac{\partial W}{\partial p_i} + q_i \frac{\partial W}{\partial q_i} + r_i \frac{\partial W}{\partial r_i} \right.$$
$$\left. + \frac{dx_i}{dt}\frac{\partial W}{\partial \frac{dx_i}{dt}} + \frac{dy_i}{dt}\frac{\partial W}{\partial \frac{dy_i}{dt}} + \frac{dz_i}{dt}\frac{\partial W}{\partial \frac{dz_i}{dt}} - W \right).$$

then we see that the sum of the elementary works is:

$$dE + \frac{\partial W}{\partial t} dt,$$

in which we suppose that W is independent of t, and when we give E the name of *energy of motion and position* for the system of triads in question, we obtain a proposition that is entirely analogous to that of sec. **65**.

From the foregoing, it is easy to deduce a system dynamic that is established on the same basis as the classical theory, but without limiting ourselves to central forces, as in the latter case. Moreover, the actual exposition presents the advantage of associating the diverse laws of force at a distance that were studied by GAUSS, RIEMANN, WEBER, and CLAUSIUS ([1]), who uniquely introduced the arguments r_{ij}, ψ_{ij}, γ_{ijk} to their true origin.

69. The Euclidian action of constraint and the dissipative Euclidian action. – The considerations that we must develop in regard to the Euclidian action at a distance lead to the notion of *constraint* in a natural manner, a notion that was due to GAUSS and, as one knows, was applied by HERTZ to the study of the foundations of mechanics by

[1] See R. REIFF and A. SOMMERFELD, *Encyclopädie der Math. Wissenschaften*, **52**, pp. 3-62.

following a path already explored by BELTRAMI, R. LIPSCHITZ, and G. DARBOUX ([1]).

To simplify, let there be a point that describes a definite trajectory by the three functions x_0, y_0, z_0, and time t when its movement is *free*. On the other hand, denote the functions of time t that describe its trajectory when it is subject to constraints by x, y, z. We may envision the two points (X, Y, Z), (X_0, Y_0, Z_0), whose coordinates are obtained, for example, by the formulas:

$$X = x + \frac{dx}{dt}dt + \frac{1}{2}\frac{d^2x}{dt^2}dt^2, \qquad X_0 = x_0 + \frac{dx_0}{dt}dt + \frac{1}{2}\frac{d^2x_0}{dt^2}dt^2,$$
$$Y = y + \frac{dy}{dt}dt + \frac{1}{2}\frac{d^2y}{dt^2}dt^2, \qquad Y_0 = y_0 + \frac{dy_0}{dt}dt + \frac{1}{2}\frac{d^2y_0}{dt^2}dt^2,$$
$$Z = z + \frac{dz}{dt}dt + \frac{1}{2}\frac{d^2z}{dt^2}dt^2, \qquad Z_0 = z_0 + \frac{dz_0}{dt}dt + \frac{1}{2}\frac{d^2z_0}{dt^2}dt^2,$$

which provide the TAYLOR development up to the first three terms. If we assume that the constraints are *frictionless* then we may demand that at the instant t in question one has:

$$x = x_0, \qquad y = y_0, \quad z = z_0, \quad \frac{dx}{dt} = \frac{dx_0}{dt}, \quad \frac{dy}{dt} = \frac{dy_0}{dt}, \qquad \frac{dz}{dt} = \frac{dz_0}{dt}.$$

Having said this, the introduction of the notion of constraint due to GAUSS amounts to replacing r by its value, where r denotes the distance, after having considered the *Euclidean action at a distance* $U_1(r)$ in such a way that one is led to the function U of the argument γ that is defined by the formula:

$$\gamma^2 = \left(\frac{d^2x}{dt^2} - \frac{d^2x_0}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2} - \frac{d^2y_0}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2} - \frac{d^2z_0}{dt^2}\right)^2.$$

If we then apply the method of variable action, we have:

$$\delta U = \mathcal{X}\left(\delta\frac{d^2x}{dt^2} - \delta\frac{d^2x_0}{dt^2}\right) + \mathcal{Y}\left(\delta\frac{d^2y}{dt^2} - \delta\frac{d^2y_0}{dt^2}\right) + \mathcal{Z}\left(\delta\frac{d^2z}{dt^2} - \delta\frac{d^2z_0}{dt^2}\right),$$

in which we have set:

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[1] BELTRAMI, *Sulla teoria generale dei parametric differenziali, Mem. Della R. Accad. Di Bologna,* Feb. 25, 1869.

R. LIPSCHITZ, *Untersuchungen eines Problemes der Variationsrechnung, in welchem das Problem der Mechanik enthalten ist, Journ. fhr die reine und angewandte Mathmematik,* **74**, pp. 116-149, 1872; *Bemerkung zu dem Princip des kleinsten Zwanges, ibid.*, **82**, pp. 311-342, 1877.

G. DARBOUX, *Leçons sur la théorie générale des surfaces,* 2nd Part, Book V, Chap. VI, VII, VIII, Paris, 1889.

$$\mathcal{X} = \frac{1}{\gamma}\frac{dU}{d\gamma}\left(\frac{d^2x}{dt^2} - \frac{d^2x_0}{dt^2}\right), \qquad \mathcal{Y} = \frac{1}{\gamma}\frac{dU}{d\gamma}\left(\frac{d^2y}{dt^2} - \frac{d^2y_0}{dt^2}\right), \qquad \mathcal{Z} = \frac{1}{\gamma}\frac{dU}{d\gamma}\left(\frac{d^2z}{dt^2} - \frac{d^2z_0}{dt^2}\right).$$

If, with GAUSS, we call the argument γ the *constraint* then the force $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ may be called the *force of constraint* that is applied to the point (*x, y, z*), and may be regarded as having the effect of impeding the free motion of the point; on the contrary, the force $-\mathcal{X}$, $-\mathcal{Y}$, $-\mathcal{Z}$ has the effect of changing the free motion into the constrained motion.

The essential difference between the present conception of force and the one that results from NEWTON's laws of motion is the following: in the latter form, one considers the action relative to two infinitely close positions – one present, one future – *on the same trajectory;* in the conception of GAUSS and HERTZ, the action is referred to two future positions: one on the trajectory we called *free,* the other on the trajectory we called *constrained.* In the two cases, one obviously has a theory that permits us to *predict* the future motion, which is the object of point dynamics. However, in addition, and this is the point that we would particularly like to clarify, the action is *Euclidean.*

On the subject, it is interesting to remark that GAUSS has explicitly established an agreement between the action of constraint and the *law of errors,* which has the same form in effect. One therefore sees that the fundamental character of the law of errors is *the Euclidean invariance* of that law, and that the new branch of mechanics, which was created by MAXWELL, BOLTZMANN, and W. GIBBS in the name of *statistical mechanics,* may likewise receive the deductive form that we propose to give ordinary mechanics here.

We may further observe that the forces of constraint translate into an *indeterminacy* that is the product of the definition of the force, and which leads to the introduction of LAGRANGE multipliers, just as in the mechanics that one derives from NEWTON's ideas as in what one deduced from the notion of GAUSS constraint.

GAUSS's idea may also be applied to friction by envisioning a Euclidean action on the two points:

$$\begin{aligned} X &= x + \frac{dx}{dt}dt, & X_0 &= x_0 + \frac{dx_0}{dt}dt, \\ Y &= y + \frac{dy}{dt}dt, & Y_0 &= y_0 + \frac{dy_0}{dt}dt, \\ Z &= z + \frac{dz}{dt}dt, & Z_0 &= z_0 + \frac{dz_0}{dt}dt, \end{aligned}$$

in which the point x_0, y_0, z_0 refers to a free trajectory, and the point *x, y, z* refers to a trajectory that is traversed with friction. As we are dealing with sliding friction here, we must set :$x = x_0$, $y = y_0$, $z = z_0$, $\frac{dx}{dt} = \mu\frac{dx_0}{dt}$, $\frac{dy}{dt} = \mu\frac{dy_0}{dt}$, $\frac{dz}{dt} = \mu\frac{dz_0}{dt}$. We are then led to a function of the velocity $v_0 = \sqrt{\left(\frac{dx_0}{dt}\right)^2 + \left(\frac{dy_0}{dt}\right)^2 + \left(\frac{dz_0}{dt}\right)^2}$ for the action, affected with a factor $1 - \mu$, which corresponds precisely to the notion of the *dissipation of the free action at a point* x_0, y_0, z_0.

The arguments r_{ij}, ψ_{ij}, λ_{ijk} that we considered in sec. **68**, translate, by definition, into an analogous idea with regard to a triad we take to be isolated in the system of n triads in question. One may, if one prefers, distinguish between these arguments, and say that r_{ij} is a *potential* argument, and that ψ_{ij}, λ_{ijk} are *dissipative* arguments. The central force hypothesis thus amounts to considering only the dynamics of systems without *friction at a distance* in mechanics. From the arguments r_{ij}, ψ_{ij}, λ_{ijk}, one may, on the other hand, derive the particular argument of WEBER $\frac{dr_{ij}}{dt}$, and if one passes from the discontinuous medium to the continuous medium, in which the concept rests on the consideration of ds^2 for the space, then one finds oneself led to introduce the *viscosity arguments* $\frac{d\varepsilon_1}{dt}, \frac{d\varepsilon_2}{dt}, \frac{d\varepsilon_3}{dt}, \frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt}, \frac{d\gamma_3}{dt}$ in the action W. Beside such arguments, which were envisioned for the first time by NAVIER and POISSON, one must obviously also place arguments such as the argument $\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2$, which was considered in sec. **47**, and arguments such as $\varphi_1, \varphi_2, \varphi_3$ from sec. **67**. We confine ourselves to these summary indications on viscosity, which has not been given further study in a sufficiently systematic manner up till now.

VI. – THE EUCLIDEAN ACTION FROM THE EULERIAN VIEWPOINT

70. The independent variables of Lagrange and Euler. The auxiliary functions considered from the hydrodynamical viewpoint. – In the statics and dynamics of deformable media, we took x_0, y_0, z_0, and x_0, y_0, z_0, t, respectively, to be the independent variables. In the former case (statics), one lets x_0, y_0, z_0 denote the coordinates of the point M_0 of the natural state (M_0) by imaging that this natural state is deformed in an infinitely slow fashion so that its points do not acquire any velocity, and passes from the position (M_0) to the position (M) in a continuous fashion ([1]). In the second case (dynamic), one lets x_0, y_0, z_0 denote the coordinates of the position M_0 at a definite instant t_0 of the point that is at M at the instant t. The position (M_0) of the medium *plays a particular role.*

The deformable medium (M) has been considered to be generated by a triad $Mx'y'z'$, whose origin M has the coordinates x, y, z, and whose vectors have the direction cosines $\alpha, \alpha', \alpha''; \beta, \beta', \beta''; \gamma, \gamma', \gamma''$ with respect to the fixed axes Ox, Oy, Oz. In the static case x, y, $z, \alpha, \alpha', \cdots, \gamma''$ are considered to be functions of the independent variables x_0, y_0, z_0, and, in the dynamics case, as functions of the four independent variables x_0, y_0, z_0, t. In either case, we say that the independent variables imagined are the LAGRANGE *variables,* and if we would like to make this concept specific we demand that:

(66) $\qquad x = x(x_0, y_0, z_0), \qquad y = y(x_0, y_0, z_0), \qquad z = z(x_0, y_0, z_0),$

or:

(66′) $\qquad x = x(x_0, y_0, z_0, t), \qquad y = y(x_0, y_0, z_0, t), \qquad z = z(x_0, y_0, z_0, t),$

and, similarly, we have either:

(67) $\qquad \alpha = \alpha(x_0, y_0, z_0), \qquad \alpha' = \alpha'(x_0, y_0, z_0), \qquad \alpha'' = \alpha''(x_0, y_0, z_0),$

or

(67′) $\qquad \alpha = \alpha(x_0, y_0, z_0, t), \qquad \alpha' = \alpha'(x_0, y_0, z_0, t), \qquad \alpha'' = \alpha''(x_0, y_0, z_0, t),$

with analogous formulas for $\beta, \beta', \beta'', \gamma, \gamma', \gamma''$.

However, we may now imagine that one performs a change of variables on the independent variables. In particular, by analogy with what one does in hydrodynamics, we may imagine that one takes x, y, z, or x, y, z, t to be the independent variables. We then say that we are imagining the EULER *variables.*

What is the fundamental question we must ask? In the theory that we just developed, where one considered that question to be either the question of defining the elements of force, etc., or, conversely, that of determining the position (M), we encountered the

[1] In this conception of the infinitely slow deformation of a medium, which is analogous to the reversible transformations of thermodynamics, we have defined the external force and moment, the effort and moment of deformation that one may qualify as *static,* and then the work done in passing from (M_0) to (M), and, consequently, we obtain the notion of the *energy of deformation,* which is placed beside that of *action,* which we started with.

functions $x, y, z, \alpha, \alpha', \cdots, \gamma''$ of x_0, y_0, z_0, or of x_0, y_0, z_0, t that are defined by (66), (67), or by $(66'), (67')$. Imagine that one solves equations (66) or $(66')$ with respect to x_0, y_0, z_0; one has:

(68) $\quad x_0 = x_0(x, y, z), \qquad y_0 = y_0(x, y, z), \qquad z_0 = z_0(x, y, z),$

or

(68′) $\quad x_0 = x_0(x, y, z, t), \qquad y_0 = y_0(x, y, z, t), \qquad z_0 = z_0(x, y, z, t),$

and, substituting these in (67) or $(67')$, we have:

(69) $\quad \alpha = \alpha(x, y, z), \qquad \alpha' = \alpha'(x, y, z), \qquad \alpha'' = \alpha''(x, y, z),$

or

(69′) $\quad \alpha = \alpha(x, y, z, t), \qquad \alpha' = \alpha'(x, y, z, t), \qquad \alpha'' = \alpha''(x, y, z, t).$

We presently know the functions x_0, y_0, z_0, $\alpha, \alpha', \cdots, \gamma''$ of x, y, z, or of x, y, z, t, and, conversely, by solving (68), (69) or $(68'), (69')$ one will then pass to (66), (67) or to $(66'), (67')$.

However, one must complete the statement that must be made by observing that in either case it may be convenient to introduce the auxiliary functions.

If we imagine the case of LAGRANGE variables, it may happen that the functions x, y, z do not figure in the question explicitly ([1]); it may therefore be convenient to introduce the first derivatives of x, y, z with respect to x_0, y_0, z_0, or with respect to x_0, y_0, z_0, t as auxiliary variables ([2]). In this case, by imagining $x, y, z, \alpha, \alpha', \cdots, \gamma''$, one may also introduce the translations and rotations ξ_i, …, r_i, ξ, …, r as auxiliary functions if only x_0, y_0, z_0 or x_0, y_0, z_0, t figure in the givens.

If we imagine the case of the EULER variables then we may indicate analogous circumstances in which the use of the auxiliary variables may offer advantages. First, suppose that the hypotheses that we must consider for the LAGRANGE variables are realized. We may preserve the indicated auxiliary functions. The only essential difference from the preceding case resides in the *ultimate* determination of formulas (66), (67) or the analogous ones, if one performs them. If we suppose, furthermore, that x_0, y_0, z_0 do not figure in the question then we may introduce the derivatives of x_0, y_0, z_0 with respect to x, y, z or with respect to x, y, z, t as the auxiliary variables.

Following these indications, one sees that there may be some use for the equations that served as the point of departure since they were presented in a convenient form from the standpoint of the auxiliary functions. One observes that this goal is already attained by the equations that we previously obtained, in which the auxiliary functions ξ_i, …, r_i, ξ, …, r already figure.

[1] This is what normally happens if one starts with results like the ones given in our exposition and if one does not modify the expressions of force, etc., by virtue of the formulas (66), (67) or $(66'), (67')$; indeed, the letters x, y, z do not figure explicitly in W.

[2] These auxiliary functions are actually coupled by relations that are easy to form; the same remark applies in general. They are not introduced in hydrodynamics, where the auxiliary functions are derivatives with respect to just the variable t (and where the use of these auxiliary functions is often limited to the case of introducing the EULER variables).

71. Expressions for $\xi_i, \ldots, r_i$ (or for $\xi_i, \ldots, r_i, \xi, \ldots, r$) by means of the functions $x_0, y_0, z_0, \alpha, \alpha', \cdots, \gamma''$ of x, y, z (or of x, y, z, t) and their derivatives; introduction of the Eulerian arguments. – From the explanations that must be given, it results that it may be useful to have expressions for $\xi_i, \ldots, r_i$ or for $\xi_i, \ldots, r_i, \xi, \ldots, r$, which are evaluated, no longer in accord with formulas (66), (67) or $(66'), (67')$, which suppose that x_0, y_0, z_0 or x_0, y_0, z_0, t are independent variables, but in accord with formulas (68), (69) or $(68'), (69')$, which introduce the functions $x_0, y_0, z_0, \alpha, \alpha', \cdots, \gamma''$ of x, y, z or of x, y, z, t.

We think about the case in which t figures in a general manner. The formulas obtained give, in particular, the case in which $x, y, z, \alpha, \alpha', \cdots, \gamma''$ are independent of t. By virtue of $(66'), (67')$, the quantities $\xi_i, \ldots$ are calculated by the formulas ([1]):

$$(70) \qquad \begin{cases} \xi_i = \alpha \dfrac{\partial x}{\partial \rho_i} + \alpha' \dfrac{\partial y}{\partial \rho_i} + \alpha'' \dfrac{\partial z}{\partial \rho_i}, & \xi = \alpha \dfrac{dx}{dt} + \alpha' \dfrac{dy}{dt} + \alpha'' \dfrac{dz}{dt}, \\ \eta_i = \beta \dfrac{\partial x}{\partial \rho_i} + \beta' \dfrac{\partial y}{\partial \rho_i} + \beta'' \dfrac{\partial z}{\partial \rho_i}, & \eta = \beta \dfrac{dx}{dt} + \beta' \dfrac{dy}{dt} + \beta'' \dfrac{dz}{dt}, \\ \varsigma_i = \gamma \dfrac{\partial x}{\partial \rho_i} + \gamma' \dfrac{\partial y}{\partial \rho_i} + \gamma'' \dfrac{\partial z}{\partial \rho_i}, & \varsigma = \gamma \dfrac{dx}{dt} + \gamma' \dfrac{dy}{dt} + \gamma'' \dfrac{dz}{dt}, \end{cases}$$

$$(71) \qquad \begin{cases} p_i = \sum \gamma \dfrac{\partial \beta}{\partial \rho_i} = -\sum \beta \dfrac{\partial \gamma}{\partial \rho_i}, & p = \sum \gamma \dfrac{d\beta}{dt} = -\sum \beta \dfrac{d\gamma}{dt}, \\ q_i = \sum \alpha \dfrac{\partial \gamma}{\partial \rho_i} = -\sum \gamma \dfrac{\partial \beta}{\partial \rho_i}, & q = \sum \alpha \dfrac{d\gamma}{dt} = -\sum \gamma \dfrac{d\alpha}{dt}, \\ r_i = \sum \beta \dfrac{\partial \alpha}{\partial \rho_i} = -\sum \alpha \dfrac{\partial \beta}{\partial \rho_i}, & r = \sum \beta \dfrac{d\alpha}{dt} = -\sum \alpha \dfrac{d\beta}{dt}, \end{cases}$$

(in which $\rho_1 = x_0, \rho_2 = y_0, \rho_3 = z_0$), and these are calculated by means of $x_0, y_0, z_0, \alpha, \alpha', \cdots, \gamma''$ and their derivatives with respect to x, y, z using formulas $(68'), (69')$.

To that effect, we shall show that the quantities $\xi_i, \ldots, r_i, \xi, \ldots, r$, which will henceforth be called *Lagrangian arguments*, are simply expressed by means of the following auxiliary functions, which we call Eulerian arguments:

$$(72) \qquad \begin{cases} (\xi_i) = \alpha[\xi_i] + \alpha'[\eta_i] + \alpha''[\varsigma_i], & (\xi) = \dfrac{\partial \rho_1}{\partial t}, \\ (\eta_i) = \beta[\xi_i] + \beta'[\eta_i] + \beta''[\varsigma_i], & (\eta) = \dfrac{\partial \rho_2}{\partial t}, \\ (\varsigma_i) = \gamma[\xi_i] + \gamma'[\eta_i] + \gamma'[\varsigma_i], & (\varsigma) = \dfrac{\partial \rho_3}{\partial t}, \end{cases}$$

[1] We use the habitual notations for the derivatives with respect to t. (See e.g., APPELL, *Traité de Mécanique,* T. III, 1st ed., pp. 277).

$$(73)\quad \begin{cases} (p_i) = \alpha[p_i] + \alpha'[q_i] + \alpha''[r_i], & (p) = \sum \gamma \dfrac{\partial \beta}{\partial t} = -\sum \beta \dfrac{\partial \gamma}{\partial t}, \\ (q_i) = \beta[p_i] + \beta'[q_i] + \beta''[r_i], & (q) = \sum \alpha \dfrac{\partial \gamma}{\partial t} = -\sum \gamma \dfrac{\partial \alpha}{\partial t}, \\ (r_i) = \gamma[p_i] + \gamma'[q_i] + \gamma'[r_i], & (r) = \sum \beta \dfrac{\partial \alpha}{\partial t} = -\sum \alpha \dfrac{\partial \beta}{\partial t}, \end{cases}$$

in which we have set:

$$(74)\begin{cases} [\xi_i] = \dfrac{\partial \rho_i}{\partial t}, & [\eta_i] = \dfrac{\partial \rho_i}{\partial t}, & [\varsigma_i] = \dfrac{\partial \rho_i}{\partial t}, \\ [p_1] = \sum \gamma \dfrac{\partial \beta}{\partial x} = -\sum \beta \dfrac{\partial \gamma}{\partial x}, & [q_1] = \sum \gamma \dfrac{\partial \beta}{\partial y} = -\sum \beta \dfrac{\partial \gamma}{\partial y}, & [r_1] = \sum \gamma \dfrac{\partial \beta}{\partial z} = -\sum \beta \dfrac{\partial \gamma}{\partial z}, \end{cases}$$

with analogous formulas for $[p_2]$, $[q_2]$, $[r_2]$, and for $[p_3]$, $[q_3]$, $[r_3]$ that are obtained by first changing γ, β into α, γ, and then into β, α, and we employ the well-known notations ([1]) $\dfrac{\partial \alpha}{\partial t}, \dfrac{\partial \beta}{\partial t}, \dfrac{\partial \gamma}{\partial t}, \cdots$.

We differentiate relations (68′) successively with respect to the LAGRANGE variables; they become four systems of three equations that, by virtue of notations (70) and (72), one may write:

$$(75)\qquad \xi_i(\xi_i) + \eta_i(\eta_i) + \zeta_i(\zeta_i) = 1, \qquad \xi_j(\xi_k) + \eta_j(\eta_k) + \zeta_j(\zeta_k) = 0, \qquad (j\ \gamma k),$$

$$(76)\qquad \begin{cases} (\xi) + \xi(\xi_1) + \eta(\eta_1) + \varsigma(\varsigma_1) = 0, \\ (\eta) + \xi(\xi_2) + \eta(\eta_2) + \varsigma(\varsigma_2) = 0, \\ (\varsigma) + \xi(\xi_3) + \eta(\eta_3) + \varsigma(\varsigma_3) = 0. \end{cases}$$

By virtue of the preceding relations (75) (as well as things that result from formulas (78) given before), the last three relations (76) may be written:

$$(76')\qquad \begin{cases} (\xi) + \xi_1(\xi) + \xi_2(\eta) + \xi_3(\varsigma) = 0, \\ (\eta) + \eta_1(\xi) + \eta_2(\eta) + \eta_3(\varsigma) = 0, \\ (\varsigma) + \varsigma_1(\xi) + \varsigma_2(\eta) + \varsigma_3(\varsigma) = 0. \end{cases}$$

Once we solve equations (75) and (76), we observe that we may replace these systems with equivalent systems that are obtained by differentiating relations (66′) with respect to the EULER variables *x*, *y*, *z*, *t* successively, and which, by virtue of notations (72), may be written (upon multiplying by $\alpha, \alpha', \alpha''$ and adding, etc.).

[1] See APPELL, *Traité de Mécanique*, T. III, 1st ed., pp. 277.

$$(75'')\qquad \begin{cases} \alpha = \sum (\xi_i)\dfrac{\partial x}{\partial \rho_i}, & \beta = \sum (\eta_i)\dfrac{\partial x}{\partial \rho_i}, & \gamma = \sum (\varsigma_i)\dfrac{\partial x}{\partial \rho_i}, \\ \alpha' = \sum (\xi_i)\dfrac{\partial y}{\partial \rho_i}, & \beta' = \sum (\eta_i)\dfrac{\partial y}{\partial \rho_i}, & \beta'' = \sum (\varsigma_i)\dfrac{\partial y}{\partial \rho_i}, \\ \alpha'' = \sum (\xi_i)\dfrac{\partial z}{\partial \rho_i}, & \gamma' = \sum (\eta_i)\dfrac{\partial z}{\partial \rho_i}, & \gamma'' = \sum (\varsigma_i)\dfrac{\partial z}{\partial \rho_i}, \end{cases}$$

to which we adjoin (76′). By multiplying system (75″) by $\alpha, \alpha', \alpha''$ and adding, etc., it may also be written:

$$(75')\qquad \begin{cases} \sum \xi_i(\xi_i) = 1, & \sum \xi_i(\eta_i) = 0, & \sum \xi_i(\varsigma_i) = 0, \\ \sum \eta_i(\xi_i) = 0, & \sum \eta_i(\eta_i) = 1, & \sum \eta_i(\varsigma_i) = 0, \\ \sum \varsigma_i(\xi_i) = 0, & \sum \varsigma_i(\eta_i) = 1, & \sum \varsigma_i(\varsigma_i) = 1. \end{cases}$$

Once again, observe that the following form, which implies (75), is intermediate between (75″) and (75), and ultimately results from formulas (70) combined with (75) and formulas (74):

$$(75''')\qquad \begin{cases} \alpha = \sum \xi_i[\xi_i], & \beta = \sum \eta_i[\xi_i], & \gamma = \sum \varsigma_i[\xi_i], \\ \alpha' = \sum \xi_i[\eta_i], & \beta' = \sum \eta_i[\eta_i], & \beta'' = \sum \varsigma_i[\eta_i], \\ \alpha'' = \sum \xi_i[\varsigma_i], & \gamma' = \sum \eta_i[\varsigma_i], & \gamma'' = \sum \varsigma_i[\varsigma_i]. \end{cases}$$

One sees that the Lagrangian arguments are functions of only the Eulerian arguments and conversely (at least as far as translations are concerned).

First determine the Lagrangian arguments by means of the Eulerian arguments. Let Δ denote the determinant:

$$\Delta = \begin{vmatrix} \xi_1 & \eta_1 & \varsigma_1 \\ \xi_2 & \eta_2 & \varsigma_2 \\ \xi_3 & \eta_3 & \varsigma_3 \end{vmatrix}, \qquad \text{which is } \frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)}, \text{ if } \begin{vmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{vmatrix} = 1.$$

Let $\xi_1', \eta_1', \varsigma_1', \xi_2', \eta_2', \varsigma_2', \xi_3', \eta_3', \varsigma_3'$ be the coefficients of the elements of the determinant Δ, i.e., the minors given a convenient sign, which therefore amounts to setting:

$$\xi_1' = \eta_2\varsigma_3 - \eta_3\varsigma_2, \qquad \eta_1' = \varsigma_2\xi_3 - \varsigma_3\xi_2, \qquad \varsigma_1' = \xi_2\eta_3 - \xi_3\eta_2, \ \ldots$$

Upon solving equations (75) with respect to (ξ_i), (η_i), (ζ_i), (ξ), (η), (ζ), and then substituting in (76), one obtains:

(77)
$$\begin{cases}(\xi_i)=\dfrac{\xi_i'}{\Delta}, & (\xi)=-\dfrac{\xi\xi_1'+\eta\eta_1'+\varsigma\varsigma_1'}{\Delta},\\ (\eta_i)=\dfrac{\eta_i'}{\Delta}, & (\eta)=-\dfrac{\xi\xi_2'+\eta\eta_2'+\varsigma\varsigma_2'}{\Delta},\\ (\varsigma_i)=\dfrac{\varsigma_i'}{\Delta}, & (\varsigma)=-\dfrac{\xi\xi_3'+\eta\eta_3'+\varsigma\varsigma_3'}{\Delta},\end{cases}$$

Conversely, determine ξ_i, η_i, ζ_i, ξ, η, ζ as a function (ξ_i), (η_i), (ζ_i), (ξ), (η), (ζ). We observe that the determinant whose elements are $\Delta(\xi_i)$, $\Delta(\eta_i)$, $\Delta(\zeta_i)$ is the *adjoint determinant* ([1]) of Δ, in such a way that we must let $\dfrac{1}{\Delta}$ designate the determinant:

(78)
$$\frac{1}{\Delta}=\begin{vmatrix}(\xi_1) & (\eta_1) & (\varsigma_1)\\ (\xi_2) & (\eta_2) & (\varsigma_2)\\ (\xi_3) & (\eta_3) & (\varsigma_3)\end{vmatrix}.$$

Solve formulas (75) and (76) with respect to ξ_i, η_i, ζ_i, ξ, η, ζ. Upon designating the coefficients of the elements of the determinant (78) by $(\xi_i'),(\eta_i'),(\varsigma_i')$, they become ([2]):

(79)
$$\begin{cases}\xi_i=\Delta(\xi_i'), & \xi=-\Delta\{(\xi)(\xi_1')+(\eta)(\xi_2')+(\varsigma)(\xi_3')\},\\ \eta_i=\Delta(\eta_i'), & \eta=-\Delta\{(\xi)(\eta_1')+(\eta)(\eta_2')+(\varsigma)(\eta_3')\},\\ \varsigma_i=\Delta(\varsigma_i'), & \varsigma=-\Delta\{(\xi)(\varsigma_1')+(\eta)(\varsigma_2')+(\varsigma)(\varsigma_3')\}.\end{cases}$$

We now propose to determine the rotations.

Differentiate relations $(67')$ with respect to *x, y, z, t*. While always employing the well-known notation for derivatives with respect to time, we have ([3]):

$$\frac{\partial\alpha}{\partial x}=\frac{\partial\alpha}{\partial x_0}\frac{\partial x_0}{\partial x}+\frac{\partial\alpha}{\partial y_0}\frac{\partial y_0}{\partial x}+\frac{\partial\alpha}{\partial z_0}\frac{\partial z_0}{\partial x},$$

[1] This adjoint determinant is the square of Δ.

[2] The first nine formulas of (79) (I = 1, 2, 3) are true if one considers the known consequences of the theory of adjoint determinants. It is clear that all of the present calculations may be attached to the theory of forms and to that of linear substitutions.

[3] We distinguish $\dfrac{d\alpha}{dt}$ from $\dfrac{\partial\alpha}{\partial t}$, …, consistent with the notation employed by APPELL, *Traité de Mécanique,* T. III., pp. 277. As for x_0, y_0, z_0, we do not need to introduce $\dfrac{dx_0}{dt},\dfrac{dy_0}{dt},\dfrac{dz_0}{dt}$, since they are zero. One observes that the present x_0, y_0, z_0, t are functions of *x, y, z, t,* which, when equated to the old x_0, y_0, z_0, define functions *x, y, z* that are thus implicit functions. We shall return to this point later.

$$\frac{\partial\alpha}{\partial y} = \frac{\partial\alpha}{\partial x_0}\frac{\partial x_0}{\partial y} + \frac{\partial\alpha}{\partial y_0}\frac{\partial y_0}{\partial y} + \frac{\partial\alpha}{\partial z_0}\frac{\partial z_0}{\partial z},$$
$$\frac{\partial\alpha}{\partial z} = \frac{\partial\alpha}{\partial x_0}\frac{\partial x_0}{\partial z} + \frac{\partial\alpha}{\partial y_0}\frac{\partial y_0}{\partial z} + \frac{\partial\alpha}{\partial z_0}\frac{\partial z_0}{\partial z},$$
$$\frac{\partial\alpha}{\partial t} = \frac{\partial\alpha}{\partial x_0}\frac{\partial x_0}{\partial t} + \frac{\partial\alpha}{\partial y_0}\frac{\partial y_0}{\partial t} + \frac{\partial\alpha}{\partial z_0}\frac{\partial z_0}{\partial t} + \frac{d\alpha}{dt},$$

with analogous formulas for the cosines $\beta, \gamma, \cdots, \gamma''$.

The formulas (74) then give:

$$\begin{aligned}
&[p_1] = \sum p_i[\xi_i], && [p_2] = \sum q_i[\xi_i], && [p_3] = \sum r_i[\xi_i],\\
&[q_1] = \sum p_i[\eta_i], && [q_2] = \sum q_i[\eta_i], && [q_3] = \sum r_i[\eta_i],\\
&[r_1] = \sum p_i[\varsigma_i], && [r_2] = \sum q_i[\varsigma_i], && [r_3] = \sum r_i[\varsigma_i],
\end{aligned}$$

and, using formulas (72), formulas (73) give:

$$(80)\qquad \begin{cases}
(\mathrm{p}_1) = \sum p_i(\xi_i), \quad (\mathrm{p}_2) = \sum q_i(\xi_i), \quad (\mathrm{p}_3) = \sum r_i(\xi_i),\\
(\mathrm{q}_1) = \sum p_i(\eta_i), \quad (\mathrm{q}_2) = \sum q_i(\eta_i), \quad (\mathrm{q}_3) = \sum r_i(\eta_i),\\
(\mathrm{r}_1) = \sum p_i(\varsigma_i), \quad (\mathrm{r}_2) = \sum p_i(\varsigma_i), \quad (\mathrm{r}_3) = \sum p_i(\varsigma_i),\\
\qquad (p) = p_1(\xi) + p_2(\eta) + p_3(\varsigma) + p,\\
\qquad (q) = q_1(\xi) + q_2(\eta) + q_3(\varsigma) + q,\\
\qquad (r) = r_1(\xi) + r_2(\eta) + r_3(\varsigma) + r,
\end{cases}$$

which give us the latter Eulerian arguments (p_i), (q_i), (r_i), (p), (q), (r) by means of the Lagrangian arguments (it suffices to replace (ξ_1), … with their values).

Conversely, to obtain the latter Lagrangian arguments p_1, …, we may solve the system (80), but one may also directly differentiate the relations with respect to x_0, y_0, z_0, t successively; we have:

$$\frac{\partial\alpha}{\partial x_0} = \frac{\partial\alpha}{\partial x}\frac{\partial x}{\partial x_0} + \frac{\partial\alpha}{\partial y}\frac{\partial y}{\partial x_0} + \frac{\partial\alpha}{\partial z}\frac{\partial z}{\partial x_0},$$
$$\frac{\partial\alpha}{\partial y_0} = \frac{\partial\alpha}{\partial x}\frac{\partial x}{\partial y_0} + \frac{\partial\alpha}{\partial y}\frac{\partial y}{\partial y_0} + \frac{\partial\alpha}{\partial z}\frac{\partial z}{\partial z_0},$$
$$\frac{\partial\alpha}{\partial z_0} = \frac{\partial\alpha}{\partial x}\frac{\partial x}{\partial z_0} + \frac{\partial\alpha}{\partial y}\frac{\partial y}{\partial z_0} + \frac{\partial\alpha}{\partial z}\frac{\partial z}{\partial z_0},$$
$$\frac{d\alpha}{dt} = \frac{\partial\alpha}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial\alpha}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\alpha}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial\alpha}{\partial t}.$$

After taking (70) into account, relations (71) then give us:

$$(81)\qquad \begin{cases} p_1 = (p_1)\xi_1 + (q_1)\eta_1 + (r_1)\varsigma_1, \\ q_1 = (p_2)\xi_1 + (q_2)\eta_1 + (r_2)\varsigma_1, \\ r_1 = (p_3)\xi_1 + (q_3)\eta_1 + (r_3)\varsigma_1, \end{cases}$$

which one may write in the intermediate form:

$$\begin{aligned} p_1 &= [p_1]\frac{\partial x}{\partial x_0} + [q_1]\frac{\partial y}{\partial x_0} + [r_1]\frac{\partial z}{\partial x_0}, \\ q_1 &= [p_2]\frac{\partial x}{\partial x_0} + [q_2]\frac{\partial y}{\partial x_0} + [r_2]\frac{\partial z}{\partial x_0}, \\ r_1 &= [p_3]\frac{\partial x}{\partial x_0} + [q_3]\frac{\partial y}{\partial x_0} + [r_3]\frac{\partial z}{\partial x_0}, \end{aligned}$$

with analogous formulas for p_2, q_2, r_2; p_3, q_3, r_3 that one obtains upon changing ξ_1, η_1, ζ_1, into ξ_2, η_2, ζ_2, and then into ξ_3, η_3, ζ_3, or upon changing x_0 into y_0, and then into z_0; one has, moreover:

$$(81')\qquad \begin{cases} p = (p_1)\xi + (q_1)\eta + (r_1)\varsigma + (p), \\ q = (p_2)\xi + (q_2)\eta + (r_2)\varsigma + (p), \\ r = (p_3)\xi + (q_3)\eta + (r_3)\varsigma + (p). \end{cases}$$

72. Static equations of a deformable medium relative to the Euler variables as deduced from the equations obtained from the Lagrange variables. We have already performed the passage from the LAGRANGE variables to the EULER variables in the context of the statics of deformable media. It will suffice for us to complete the results so obtained ([1]).

We found formulas such as the following in sec. 53:

$$\begin{aligned} \Delta p_{xx} &= \frac{\partial x}{\partial x_0}A_1 + \frac{\partial x}{\partial y_0}A_2 + \frac{\partial x}{\partial z_0}A_3, & \Delta q_{xx} &= \frac{\partial x}{\partial x_0}P_1 + \frac{\partial x}{\partial y_0}P_2 + \frac{\partial x}{\partial z_0}P_3, \\ \Delta p_{yx} &= \frac{\partial y}{\partial x_0}A_1 + \frac{\partial y}{\partial y_0}A_2 + \frac{\partial y}{\partial z_0}A_3, & \Delta q_{yx} &= \frac{\partial y}{\partial x_0}P_1 + \frac{\partial y}{\partial y_0}P_2 + \frac{\partial y}{\partial z_0}P_3, \\ \Delta p_{zx} &= \frac{\partial z}{\partial x_0}A_1 + \frac{\partial z}{\partial y_0}A_2 + \frac{\partial z}{\partial z_0}A_3, & \Delta q_{zx} &= \frac{\partial z}{\partial x_0}P_1 + \frac{\partial z}{\partial y_0}P_2 + \frac{\partial z}{\partial z_0}P_3, \end{aligned}$$

in which one has:

$$A_i = \alpha\frac{\partial W}{\partial \xi_i} + \beta\frac{\partial W}{\partial \eta_i} + \gamma\frac{\partial W}{\partial \varsigma_i}, \qquad P_i = \alpha\frac{\partial W}{\partial p_i} + \beta\frac{\partial W}{\partial q_i} + \gamma\frac{\partial W}{\partial r_i}.$$

[1] We then seek to obtain the definitive results directly.

Suppose that W is expressed by means of the arguments (ξ_i), (η_i), (ζ_i), (p_i), (q_i), (r_i), and set:

$$W = \Delta\Omega.$$

By virtue of the formulas (77) of the preceding paragraph, one will have:

$$\frac{\partial W}{\partial \xi_i} = \Delta \frac{\partial \Omega}{\partial \xi_i} + \Omega \xi_i' = \Delta\left\{\frac{\partial \Omega}{\partial \xi_i} + \Omega(\xi_i)\right\},$$
$$\frac{\partial W}{\partial \eta_i} = \Delta \frac{\partial \Omega}{\partial \eta_i} + \Omega \eta_i' = \Delta\left\{\frac{\partial \Omega}{\partial \eta_i} + \Omega(\eta_i)\right\},$$
$$\frac{\partial W}{\partial \varsigma_i} = \Delta \frac{\partial \Omega}{\partial \varsigma_i} + \Omega \varsigma_i' = \Delta\left\{\frac{\partial \Omega}{\partial \varsigma_i} + \Omega(\varsigma_i)\right\},$$

and, as a result, since Δ does not depend on p_i, q_i, r_i:

$$A_i = \Delta\left\{\alpha \frac{\partial \Omega}{\partial \xi_i} + \beta \frac{\partial \Omega}{\partial \eta_i} + \gamma \frac{\partial \Omega}{\partial \varsigma_i} + \Omega[\xi_i]\right\},$$
$$P_i = \Delta\left\{\alpha \frac{\partial \Omega}{\partial p_i} + \beta \frac{\partial \Omega}{\partial q_i} + \gamma \frac{\partial \Omega}{\partial r_i}\right\}.$$

Upon differentiating relations (75) with respect to ξ_i, one gets:

$$\xi_i \frac{\partial (\xi_j)}{\partial \xi_i} + \eta_i \frac{\partial (\eta_j)}{\partial \xi_i} + \varsigma_i \frac{\partial (\varsigma_j)}{\partial \xi_i} = -(\xi_j), \qquad \xi_j \frac{\partial (\xi_k)}{\partial \xi_j} + \eta_j \frac{\partial (\eta_k)}{\partial \xi_j} + \varsigma_j \frac{\partial (\varsigma_k)}{\partial \xi_j} = 0 \quad (i \neq j),$$

from which, one deduces:

$$\frac{\partial (\xi_j)}{\partial \xi_i} = -(\xi_j)\frac{\xi_i'}{\Delta} = -(\xi_i)(\xi_j),$$
$$\frac{\partial (\eta_j)}{\partial \xi_i} = -(\xi_j)\frac{\eta_i'}{\Delta} = -(\eta_i)(\xi_j),$$
$$\frac{\partial (\varsigma_j)}{\partial \xi_i} = -(\xi_j)\frac{\varsigma_i}{\Delta} = -(\varsigma_i)(\xi_j);$$

and then, by the relations (80):

$$\frac{\partial (p_j)}{\partial \xi_i} = -(p_i)(\xi_j),$$
$$\frac{\partial (q_j)}{\partial \xi_i} = -(p_i)(\eta_j),$$

$$\frac{\partial(r_j)}{\partial\xi_i} = -(p_i)(\varsigma_j),$$

with analogous formula for the derivatives with respect to η_i, ζ_i. If one sets:

$$(A'_i) = \frac{\partial\Omega}{\partial(\xi_i)}, \qquad (B'_i) = \frac{\partial\Omega}{\partial(\eta_i)}, \qquad (C'_i) = \frac{\partial\Omega}{\partial(\varsigma_i)},$$
$$(P'_i) = \frac{\partial\Omega}{\partial(p_i)}, \qquad (Q'_i) = \frac{\partial\Omega}{\partial(q_i)}, \qquad (R'_i) = \frac{\partial\Omega}{\partial(r_i)},$$

then one has:

$$\begin{aligned}\frac{1}{\Delta}A_i &= \Omega[\xi_i] \\ &\quad -[\{(\xi_i)(A'_1)+(\eta_i)(B'_1)+(\varsigma_i)(C'_1)\}[\xi_1]+\{(\xi_i)(P'_1)+(\eta_i)(Q'_1)+(\varsigma_i)(R'_1)\}[p_1] \\ &\quad +\{(\xi_i)(A'_2)+(\eta_i)(B'_2)+(\varsigma_i)(C'_2)\}[\xi_2]+\{(\xi_i)(P'_2)+(\eta_i)(Q'_2)+(\varsigma_i)(R'_2)\}[p_2] \\ &\quad +\{(\xi_i)(A'_3)+(\eta_i)(B'_3)+(\varsigma_i)(C'_3)\}[\xi_3]+\{(\xi_i)(P'_3)+(\eta_i)(Q'_3)+(\varsigma_i)(R'_3)\}[p_3]].\end{aligned}$$

By virtue of the formulas (72), (73), (74), $(75'')$, and upon letting $[A_i]$, $[B_i]$, $[C_i]$; $[P_i]$, $[Q_i]$, $[R_i]$ denote the components relative to the axes Ox, Oy, Oz of the two vectors whose components with respect to the axes Mx', My', Mz' are $(A'_i), (B'_i), (C'_i)$; $(P'_i), (Q'_i), (R'_i)$, one deduces the following three formulas:

$$\begin{aligned} p_{xx} &= \Omega - \sum [A_i][\xi_i] - \sum [P_i][p_i], \\ p_{yx} &= \quad - \sum [B_i][\xi_i] - \sum [Q_i][p_i], \\ p_{zx} &= \quad - \sum [C_i][\xi_i] - \sum [R_i][p_i], \end{aligned}$$

with analogous formulas for B_i, C_i, and p_{xy}, p_{yy}, p_{zy}, p_{xz}, p_{xz}, p_{xz}. One then has:

$$\begin{aligned}\frac{1}{\Delta}P_i &= \alpha\left\{(\xi_i)\frac{\partial\Omega}{\partial(p_1)}+(\eta_i)\frac{\partial\Omega}{\partial(q_1)}+(\varsigma_i)\frac{\partial\Omega}{\partial(r_1)}\right\} \\ &\quad +\beta\left\{(\xi_i)\frac{\partial\Omega}{\partial(p_2)}+(\eta_i)\frac{\partial\Omega}{\partial(q_2)}+(\varsigma_i)\frac{\partial\Omega}{\partial(r_2)}\right\} \\ &\quad +\gamma\left\{(\xi_i)\frac{\partial\Omega}{\partial(p_3)}+(\eta_i)\frac{\partial\Omega}{\partial(q_3)}+(\varsigma_i)\frac{\partial\Omega}{\partial(r_3)}\right\},\end{aligned}$$

and, again taking $(75'')$, into account, we obtain the following three formulas:

$$\begin{aligned} q_{xx} &= \alpha[P_1] + \beta[P_2] + \gamma[P_3], \\ q_{yx} &= \alpha[Q_1] + \beta[Q_2] + \gamma[Q_3], \end{aligned}$$

$$q_{zx} = \alpha[R_1] + \beta[R_2] + \gamma[R_3],$$

with analogous formulas for Q_i, R_i, and q_{xy}, q_{yy}, q_{zy}, q_{xz}, q_{xz}, q_{xz}.

73. Dynamical equations of the deformable medium relative to the Euler variables as deduced from the equations obtained for the Lagrange variables. – We have also performed the passage from the LAGRANGE variables to the EULER variables in the context of the dynamics of the deformable medium. We shall first complete the results so obtained.

A_i is augmented with:

$$\Delta\left[\left\{\alpha\frac{\partial(\xi)}{\partial\xi_i}+\beta\frac{\partial(\xi)}{\partial\eta_i}+\gamma\frac{\partial(\xi)}{\partial\varsigma_i}\right\}\frac{\partial\Omega}{\partial(\xi)}+\left\{\alpha\frac{\partial(p)}{\partial\xi_i}+\beta\frac{\partial(p)}{\partial\eta_i}+\gamma\frac{\partial(p)}{\partial\varsigma_i}\right\}\frac{\partial\Omega}{\partial(p)}\right.$$
$$+\left\{\alpha\frac{\partial(\eta)}{\partial\xi_i}+\beta\frac{\partial(\eta)}{\partial\eta_i}+\gamma\frac{\partial(\eta)}{\partial\varsigma_i}\right\}\frac{\partial\Omega}{\partial(\eta)}+\left\{\alpha\frac{\partial(q)}{\partial\xi_i}+\beta\frac{\partial(q)}{\partial\eta_i}+\gamma\frac{\partial(q)}{\partial\varsigma_i}\right\}\frac{\partial\Omega}{\partial(q)}$$
$$\left.+\left\{\alpha\frac{\partial(\varsigma)}{\partial\xi_i}+\beta\frac{\partial(\varsigma)}{\partial\eta_i}+\gamma\frac{\partial(\varsigma)}{\partial\varsigma_i}\right\}\frac{\partial\Omega}{\partial(\varsigma)}+\left\{\alpha\frac{\partial(r)}{\partial\xi_i}+\beta\frac{\partial(r)}{\partial\eta_i}+\gamma\frac{\partial(r)}{\partial\varsigma_i}\right\}\frac{\partial\Omega}{\partial(r)}\right];$$

however, from (76) and (80):

$$\frac{\partial(\xi)}{\partial\xi_1}=-(\xi_1)(\xi),\qquad \frac{\partial(\xi)}{\partial\xi_2}=-(\xi_1)(\eta),\qquad \frac{\partial(\xi)}{\partial\xi_3}=-(\xi_1)(\varsigma),$$
$$\frac{\partial(p)}{\partial\xi_1}=-(p_1)(\xi),\qquad \frac{\partial(p)}{\partial\xi_2}=-(p_1)(\eta),\qquad \frac{\partial(p)}{\partial\xi_3}=-(p_1)(\varsigma),$$

with analogous formulas, in such a way that if we set:

$$(A')=\frac{\partial\Omega}{\partial(\xi)},\quad (B')=\frac{\partial\Omega}{\partial(\eta)},\quad (C')=\frac{\partial\Omega}{\partial(\varsigma)},$$
$$(P')=\frac{\partial\Omega}{\partial(p)},\quad (Q')=\frac{\partial\Omega}{\partial(q)},\quad (R')=\frac{\partial\Omega}{\partial(r)},$$

then we must add

$$A(\xi),\quad A(\eta),\quad A(\zeta),$$

respectively, to the given values of A_i , i = 1, 2, 3, that were given in the last paragraph, where we have set:

$$-\frac{A}{\Delta}=(A')[\xi_1]+(B')[\xi_2]+(C')[\xi_3]+(P')[p_1]+(Q')[p_2]+(R')[p_3].$$

The expressions that we add to the values of p_{xx}, p_{xy}, p_{xz}, of the preceding paragraph are therefore:

$$\frac{A}{\Delta}\left\{(\xi)\frac{\partial x}{\partial x_0}+(\eta)\frac{\partial x}{\partial y_0}+(\varsigma)\frac{\partial x}{\partial z_0}\right\},\qquad \frac{A}{\Delta}\left\{(\xi)\frac{\partial y}{\partial x_0}+(\eta)\frac{\partial y}{\partial y_0}+(\varsigma)\frac{\partial y}{\partial z_0}\right\},$$
$$\frac{A}{\Delta}\left\{(\xi)\frac{\partial z}{\partial x_0}+(\eta)\frac{\partial z}{\partial y_0}+(\varsigma)\frac{\partial z}{\partial z_0}\right\};$$

however, from the values (76) of (ξ),(η),(ζ), one has:

$$(\xi)\frac{\partial x}{\partial x_0}+(\eta)\frac{\partial x}{\partial y_0}+(\varsigma)\frac{\partial x}{\partial z_0}=-\xi\sum(\xi_i)\frac{\partial x}{\partial\rho_i}-\eta\sum(\eta_i)\frac{\partial x}{\partial\rho_i}-\varsigma\sum(\varsigma_i)\frac{\partial x}{\partial\rho_i},$$
$$(\xi)\frac{\partial y}{\partial x_0}+(\eta)\frac{\partial y}{\partial y_0}+(\varsigma)\frac{\partial y}{\partial z_0}=-\xi\sum(\xi_i)\frac{\partial y}{\partial\rho_i}-\eta\sum(\eta_i)\frac{\partial y}{\partial\rho_i}-\varsigma\sum(\varsigma_i)\frac{\partial y}{\partial\rho_i},$$
$$(\xi)\frac{\partial z}{\partial x_0}+(\eta)\frac{\partial z}{\partial y_0}+(\varsigma)\frac{\partial z}{\partial z_0}=-\xi\sum(\xi_i)\frac{\partial z}{\partial\rho_i}-\eta\sum(\eta_i)\frac{\partial z}{\partial\rho_i}-\varsigma\sum(\varsigma_i)\frac{\partial z}{\partial\rho_i},$$

i.e., by virtue of formulas (75″) :

$$(\xi)\frac{\partial x}{\partial x_0}+(\eta)\frac{\partial x}{\partial y_0}+(\varsigma)\frac{\partial x}{\partial z_0}=-(\alpha\xi+\beta\eta+\gamma\varsigma),$$
$$(\xi)\frac{\partial y}{\partial x_0}+(\eta)\frac{\partial y}{\partial y_0}+(\varsigma)\frac{\partial y}{\partial z_0}=-(\alpha'\xi+\beta'\eta+\gamma'\varsigma),$$
$$(\xi)\frac{\partial z}{\partial x_0}+(\eta)\frac{\partial z}{\partial y_0}+(\varsigma)\frac{\partial z}{\partial z_0}=-(\alpha''\xi+\beta''\eta+\gamma''\varsigma),$$

in such a way that the expressions that we must add to the p_{xx}, p_{xy}, p_{xz} of the preceding paragraph are:

$$-\frac{A}{\Delta}\frac{dx}{dt},\qquad -\frac{A}{\Delta}\frac{dy}{dt},\qquad -\frac{A}{\Delta}\frac{dz}{dt}.$$

One will have analogous expressions for p_{yx}, …, p_{zx},… by the obvious change of A into two analogous expressions B and C that are deduced by reducing the $[\xi_i]$, $[p_i]$ by the corresponding quantities $[\eta_i]$, $[q_i]$ and $[\zeta_i]$, $[r_i]$.

We now introduce the notations A, B, C; we show that they are identical to the notations introduced in the Lagrangian theory:

$$A=\alpha\frac{\partial W}{\partial\xi}+\beta\frac{\partial W}{\partial\beta}+\gamma\frac{\partial W}{\partial\gamma},\cdots.$$

Indeed, one has:

$$\frac{A}{\Delta} = \alpha\left[(A')\frac{\partial(\xi)}{\partial\xi} + (B')\frac{\partial(\eta)}{\partial\xi} + \cdots + (R')\frac{\partial(r)}{\partial\xi}\right] + \beta\left[(A')\frac{\partial(\xi)}{\partial\eta} + \cdots\right] + \gamma\left[(A')\frac{\partial(\xi)}{\partial\varsigma} + \cdots\right].$$

However, from formulas (76) and (80), one has:

$$\frac{\partial(\xi)}{\partial\xi} = -(\xi_1), \qquad \frac{\partial(\eta)}{\partial\xi} = -(\xi_2), \qquad \frac{\partial(\varsigma)}{\partial\xi} = -(\xi_3),$$
$$\frac{\partial(p)}{\partial\xi} = -(p_1), \qquad \frac{\partial(q)}{\partial\xi} = -(p_2), \qquad \frac{\partial(r)}{\partial\xi} = -(p_3),$$

and analogous relations for η, ζ. By virtue of relations (72), we obtain:

$$-\frac{A}{\Delta} = (A')[\xi_1] + (B')[\xi_2] + (C')[\xi_3] + (P')[p_1] + (Q')[p_2] + (R')[p_3].$$

Similarly, for the P, Q, R of the Lagrangian theory, namely:

$$P = \alpha\frac{\partial W}{\partial p} + \beta\frac{\partial W}{\partial q} + \gamma\frac{\partial W}{\partial r}, \cdots,$$

one has, by virtue of the relations (80):

$$\frac{P}{\Delta} = \alpha(P') + \beta(Q') + \gamma(R'), \cdots.$$

Finally, consider the modification that must be made to the formulas of the preceding paragraph in order to have the q_{xx}, … relate to the actual case of dynamics.

The quantities that we have called P_i are augmented for $i = 1, 2, 3$, either by:

$$\Delta\left[(P')\left\{\alpha\frac{\partial(p)}{\partial p_1} + \beta\frac{\partial(p)}{\partial q_1} + \gamma\frac{\partial(p)}{\partial r_1}\right\} + (Q')\left\{\alpha\frac{\partial(q)}{\partial p_1} + \cdots\right\} + (R')\left\{\alpha\frac{\partial(r)}{\partial p_1} + \cdots\right\}\right]$$
$$\Delta\left[(P')\left\{\alpha\frac{\partial(p)}{\partial p_2} + \beta\frac{\partial(p)}{\partial q_2} + \gamma\frac{\partial(p)}{\partial r_2}\right\} + (Q')\left\{\alpha\frac{\partial(q)}{\partial p_2} + \cdots\right\} + (R')\left\{\alpha\frac{\partial(r)}{\partial p_2} + \cdots\right\}\right]$$
$$\Delta\left[(P')\left\{\alpha\frac{\partial(p)}{\partial p_3} + \beta\frac{\partial(p)}{\partial q_3} + \gamma\frac{\partial(p)}{\partial r_3}\right\} + (Q')\left\{\alpha\frac{\partial(q)}{\partial p_3} + \cdots\right\} + (R')\left\{\alpha\frac{\partial(r)}{\partial p_3} + \cdots\right\}\right]$$

or by

$$\Delta(\xi)\{\alpha(P') + \beta(Q') + \gamma(R')\}$$
$$\Delta(\eta)\{\alpha(P') + \beta(Q') + \gamma(R')\}$$

$$\Delta(\varsigma)\{\alpha(P') + \beta(Q') + \gamma(R')\},$$

by virtue of formulas (80). One sees that these increases are:

$$P(\xi), \quad P(\eta), \quad P(\zeta).$$

The expressions that must be added to the values of q_{xx}, q_{xy}, q_{xz} of the preceding section are thus:

$$\frac{P}{\Delta}\left\{(\xi)\frac{\partial x}{\partial x_0} + (\eta)\frac{\partial x}{\partial y_0} + (\varsigma)\frac{\partial x}{\partial z_0}\right\},$$

$$\frac{P}{\Delta}\left\{(\xi)\frac{\partial y}{\partial x_0} + (\eta)\frac{\partial y}{\partial y_0} + (\varsigma)\frac{\partial y}{\partial z_0}\right\}, \qquad \frac{P}{\Delta}\left\{(\xi)\frac{\partial z}{\partial x_0} + (\eta)\frac{\partial z}{\partial y_0} + (\varsigma)\frac{\partial z}{\partial z_0}\right\},$$

i.e.,

$$-\frac{P}{\Delta}(\alpha\xi + \beta\eta + \gamma\varsigma), \qquad -\frac{P}{\Delta}(\alpha\xi' + \beta'\eta + \gamma'\varsigma), \qquad -\frac{P}{\Delta}(\alpha\xi'' + \beta''\eta + \gamma''\varsigma),$$

or finally

$$-\frac{P}{\Delta}\frac{dx}{dt}, \qquad -\frac{P}{\Delta}\frac{dy}{dt}, \qquad -\frac{P}{\Delta}\frac{dz}{dt}.$$

One will have analogous expressions for q_{yz}, …; q_{zx}, … by changing P into Q, and then into R.

74. Variations of the Eulerian arguments deduced from those of the Lagrangian arguments. – With the aim of directly formulating the Eulerian equations that relate to the deformable medium, we shall calculate the variations of the Eulerian arguments. We commence by deducing the variations from the Lagrangian arguments in order to verify them, and then we calculate them directly.

If we apply δ to equations (75) then they become three systems like the following one:

$$\begin{aligned}
\xi_1\delta(\xi_1) + \eta_1\delta(\eta_1) + \zeta_1\delta(\zeta_1) &= -(\xi_1)\delta\xi_1 - (\eta_1)\delta\eta_1 - (\zeta_1)\delta\zeta_1,\\
\xi_2\delta(\xi_1) + \eta_2\delta(\eta_1) + \zeta_2\delta(\zeta_1) &= -(\xi_1)\delta\xi_2 - (\eta_1)\delta\eta_2 - (\zeta_1)\delta\zeta_2,\\
\xi_3\delta(\xi_1) + \eta_3\delta(\eta_1) + \zeta_3\delta(\zeta_1) &= -(\xi_1)\delta\xi_3 - (\eta_1)\delta\eta_3 - (\zeta_1)\delta\zeta_3\,.
\end{aligned}$$

Hence, keeping relations (77) in mind:

$$\begin{aligned}
-\delta(\xi_1) &= (\xi_1)\{(\xi_1)\delta\xi_1 + (\eta_1)\delta\eta_1 + (\zeta_1)\delta\zeta_1\} + (\xi_2)\{(\xi_1)\delta\xi_1 + \dots\} + (\xi_3)\{(\xi_1)\delta\xi_1 + \dots\}\\
&= (\xi_1)\sum(\xi_i)\delta\xi_i + (\eta_1)\sum(\xi_i)\delta\eta_i + (\varsigma_1)\sum(\xi_i)\delta\varsigma_i\,,
\end{aligned}$$

or, upon replacing $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$ with their values, and taking relations (75′) and (80) into account:

$$\delta(\xi_1) = (\eta_1)\delta K' - (\varsigma_1)\delta J' - (\xi_1)\left\{(\xi_1)\frac{\partial\delta' x}{\partial x_0} + (\xi_2)\frac{\partial\delta' x}{\partial y_0} + (\xi_3)\frac{\partial\delta' x}{\partial z_0} + (p_2)\delta' z - (p_3)\delta' y\right\}$$
$$- (\eta_1)\left\{(\xi_1)\frac{\partial\delta' y}{\partial x_0} + (\xi_2)\frac{\partial\delta' y}{\partial y_0} + (\xi_3)\frac{\partial\delta' y}{\partial z_0} + (p_2)\delta' x - (p_3)\delta' z\right\}$$
$$- (\varsigma_1)\left\{(\xi_1)\frac{\partial\delta' z}{\partial x_0} + (\xi_2)\frac{\partial\delta' z}{\partial y_0} + (\xi_3)\frac{\partial\delta' z}{\partial z_0} + (p_2)\delta' y - (p_3)\delta xz\right\};$$

however, by virtue of equations $(75'')$ one has:

$$\sum(\xi_i)\frac{\partial\delta' x}{\partial\rho_i} = \frac{\partial\delta' x}{\partial x}\sum(\xi_i)\frac{\partial x}{\partial\rho_i} + \frac{\partial\delta' x}{\partial y}\sum(\xi_i)\frac{\partial y}{\partial\rho_i} + \frac{\partial\delta' x}{\partial z}\sum(\xi_i)\frac{\partial z}{\partial\rho_i}$$
$$= \alpha\frac{\partial\delta' x}{\partial x} + \alpha'\frac{\partial\delta' x}{\partial y} + \alpha''\frac{\partial\delta' x}{\partial z},$$

for example. We therefore obtain the following relation:

$$\delta(\xi_1) = (\eta_1)\delta K' - (\varsigma_1)\delta J' - (\xi_1)\left\{\alpha\frac{\partial\delta' x}{\partial x} + \alpha'\frac{\partial\delta' x}{\partial y} + \alpha''\frac{\partial\delta' x}{\partial z} + (p_2)\delta' z - (p_3)\delta' y\right\}$$
$$- (\eta_1)\left\{\alpha\frac{\partial\delta' y}{\partial x} + \alpha'\frac{\partial\delta' y}{\partial y} + \alpha''\frac{\partial\delta' y}{\partial z} + (p_2)\delta' x - (p_3)\delta' z\right\}$$
$$- (\varsigma_1)\left\{\alpha\frac{\partial\delta' z}{\partial x} + \alpha'\frac{\partial\delta' z}{\partial y} + \alpha''\frac{\partial\delta' z}{\partial z} + (p_2)\delta' y - (p_3)\delta xz\right\};$$

in order to find $\delta(\eta_1)$, $\delta(\zeta_1)$, it suffices to make a circular permutation of (ξ_1), (η_1), (ζ_1) to replace $\alpha, \alpha', \alpha''$ with β, β', β'', and then with $\gamma, \gamma', \gamma''$, and to replace the p_i with q_i and then with r_i. One has analogous systems of formulas for $\delta(\xi_2)$, $\delta(\eta_2)$, $\delta(\zeta_2)$; $\delta(\xi_3)$, $\delta(\eta_3)$, $\delta(\zeta_3)$.

By means of (76) and the values for $\delta\xi$, $\delta\eta$, $\delta\zeta$, one has, in turn:

$$\delta(\xi) = -\{\xi\delta(\xi_1) + \eta\delta(\eta_1) + \varsigma\delta(\varsigma_1)\} - \{(\xi_1)\delta\xi + (\eta_1)\delta\eta + (\varsigma_1)\delta\varsigma\}$$
$$= -(\xi_1)\Big[\frac{d\delta' x}{dt} - (\alpha\xi + \beta\eta + \gamma\varsigma)\frac{\partial\delta' x}{\partial x} - (\alpha'\xi + \beta'\eta + \gamma\varsigma)\frac{\partial\delta' x}{\partial y} - (\alpha''\xi + \beta''\eta + \gamma''\varsigma)\frac{\partial\delta' x}{\partial z}$$
$$+ \{q - (p_2)\xi - (q_2)\eta - (r_2)\varsigma\}\delta' z - \{r - (p_3)\xi - (q_3)\eta - (r_3)\varsigma\}\delta' y\Big]$$
$$- (\eta_1)\Big[\frac{d\delta' y}{dt} - (\alpha\xi + \beta\eta + \gamma\varsigma)\frac{\partial\delta' y}{\partial x} - (\alpha'\xi + \beta'\eta + \gamma\varsigma)\frac{\partial\delta' y}{\partial y} - (\alpha''\xi + \beta''\eta + \gamma''\varsigma)\frac{\partial\delta' y}{\partial z}$$
$$+ \{q - (p_3)\xi - (q_3)\eta - (r_3)\varsigma\}\delta' x - \{p - (p_1)\xi - (q_1)\eta - (r_1)\varsigma\}\delta' z\Big]$$
$$- (\varsigma_1)\Big[\frac{d\delta' z}{dt} - (\alpha\xi + \beta\eta + \gamma\varsigma)\frac{\partial\delta' z}{\partial x} - (\alpha'\xi + \beta'\eta + \gamma\varsigma)\frac{\partial\delta' z}{\partial y} - (\alpha''\xi + \beta''\eta + \gamma''\varsigma)\frac{\partial\delta' z}{\partial z}$$

$$+\{p-(p_1)\xi-(q_1)\eta-(r_1)\varsigma\}\delta'y-\{q-(p_2)\xi-(q_2)\eta-(r_2)\varsigma\}\delta'x];$$

however, by virtue of (76), relations (80) give:

$$\begin{aligned}(p_1)\xi+(q_1)\eta+(r_1)\zeta&=-\{p_1(\xi)+p_2(\eta)+p_3(\zeta)\},\\(p_2)\xi+(q_2)\eta+(r_2)\zeta&=-\{q_1(\xi)+q_2(\eta)+q_3(\zeta)\},\\(p_3)\xi+(q_3)\eta+(r_3)\zeta&=-\{r_1(\xi)+r_2(\eta)+r_3(\zeta)\},\end{aligned}$$

from which, we finally have:

$$\begin{aligned}\delta(\xi)=-(\xi_1)&\left\{\frac{d\delta'x}{dt}-\frac{dx}{dt}\frac{\partial\delta'x}{\partial x}-\frac{dy}{dt}\frac{\partial\delta'x}{\partial y}-\frac{dz}{dt}\frac{\partial\delta'x}{\partial z}+(q)\delta'z-(r)\delta'y\right\}\\-(\eta_1)&\left\{\frac{d\delta'y}{dt}-\frac{dx}{dt}\frac{\partial\delta'y}{\partial x}-\frac{dy}{dt}\frac{\partial\delta'y}{\partial y}-\frac{dz}{dt}\frac{\partial\delta'y}{\partial z}+(r)\delta'x-(p)\delta'z\right\}\\-(\varsigma_1)&\left\{\frac{d\delta'z}{dt}-\frac{dx}{dt}\frac{\partial\delta'z}{\partial x}-\frac{dy}{dt}\frac{\partial\delta'z}{\partial y}-\frac{dz}{dt}\frac{\partial\delta'z}{\partial z}+(p)\delta'y-(q)\delta'x\right\}.\end{aligned}$$

One will get analogous values for $\delta(\eta)$, $\delta(\zeta)$ upon changing (ξ_1), (η_1), (ζ_1) into (ξ_2), (η_2), (ζ_2), and then into (ξ_3), (η_3), (ζ_3).

From (80), we now have:

$$\delta(p_1)=(\xi_1)\delta p_1+(\xi_2)\delta p_2+(\xi_3)\delta p_3+p_1\delta(\xi_1)+p_2\delta(\xi_2)+p_3\delta(\xi_3),$$

i.e., by virtue of formulas $(75'')$:

$$\begin{aligned}\delta(p_1)&=(q_1)\delta K'-(r_1)\delta J'\\&+\alpha\frac{\partial\delta I'}{\partial x}+\alpha'\frac{\partial\delta I'}{\partial y}+\alpha''\frac{\partial\delta I'}{\partial z}+(p_2)\delta K'-(p_3)\delta J'\\&-(p_1)\left\{\alpha\frac{\partial\delta'x}{\partial x}+\alpha'\frac{\partial\delta'x}{\partial y}+\alpha''\frac{\partial\delta'x}{\partial z}+(p_2)\delta'z-(p_3)\delta'y\right\}\\&-(q_1)\left\{\alpha\frac{\partial\delta'y}{\partial x}+\alpha'\frac{\partial\delta'y}{\partial y}+\alpha''\frac{\partial\delta'y}{\partial z}+(p_3)\delta'x-(p_1)\delta'z\right\}\\&-(r_1)\left\{\alpha\frac{\partial\delta'z}{\partial x}+\alpha'\frac{\partial\delta'z}{\partial y}+\alpha''\frac{\partial\delta'z}{\partial z}+(p_1)\delta'y-(p_2)\delta'z\right\}\end{aligned}$$

with analogous formulas for $\delta(q_1)$, $\delta(r_1)$, and for $\delta(p_2)$, $\delta(q_2)$, $\delta(r_2)$; $\delta(p_3)$, $\delta(q_3)$, $\delta(r_3)$.

We have have:

$$\delta(p)=\delta p+(\xi)\delta p_1+(\eta)\delta p_2+(\zeta)\delta p_3+p_1\delta(\xi)+p_2\delta(\eta)+p_3\delta(\zeta),$$

i.e., by virtue of formulas $(75'')$, (76), and (80):

$$\delta(p) = \frac{d\delta I'}{dt} - \frac{\partial I'}{\partial x}\frac{dx}{dt} - \frac{\partial I'}{\partial y}\frac{dy}{dt} - \frac{\partial I'}{\partial z}\frac{dz}{dt} + (q)\delta K' - (r)\delta J'$$
$$- (p_1)\left\{\frac{d\delta' x}{dt} - \frac{\partial \delta' x}{\partial x}\frac{dx}{dt} - \frac{\partial \delta' x}{\partial y}\frac{dy}{dt} - \frac{\partial \delta' x}{\partial z}\frac{dz}{dt} + (q)\delta' z - (r)\delta' y\right\}$$
$$- (q_1)\left\{\frac{d\delta' y}{dt} - \frac{\partial \delta' y}{\partial x}\frac{dx}{dt} - \frac{\partial \delta' y}{\partial y}\frac{dy}{dt} - \frac{\partial \delta' y}{\partial z}\frac{dz}{dt} + (r)\delta' x - (p)\delta' z\right\}$$
$$- (r_1)\left\{\frac{d\delta' z}{dt} - \frac{\partial \delta' z}{\partial x}\frac{dx}{dt} - \frac{\partial \delta' z}{\partial y}\frac{dy}{dt} - \frac{\partial \delta' z}{\partial z}\frac{dz}{dt} + (p)\delta' y - (q)\delta' x\right\}$$

with analogous formulas for $\delta(q)$, $\delta(r)$.

Now, we seek to find the formulas that must be established when one introduces the auxiliary functions δx, δy, δz, δI, δJ, δK, which are defined as before. For example, one has:

$$\frac{\partial \delta x}{\partial x} = \alpha\frac{\partial \delta' x}{\partial x} + \beta\frac{\partial \delta' y}{\partial x} + \gamma\frac{\partial \delta' z}{\partial x} + \frac{\partial \alpha}{\partial x}\delta' x + \frac{\partial \beta}{\partial x}\delta' y + \frac{\partial \gamma}{\partial x}\delta' z,$$

and analogous expressions for $\frac{\partial \delta y}{\partial x}, \frac{\partial \delta z}{\partial x}$, from which, we have the system:

$$\alpha\frac{\partial \delta x}{\partial x} + \alpha'\frac{\partial \delta y}{\partial x} + \alpha''\frac{\partial \delta z}{\partial x} = \frac{\partial \delta' x}{\partial x} + [p_2]\delta' z - [p_3]\delta' y,$$
$$\beta\frac{\partial \delta x}{\partial x} + \beta'\frac{\partial \delta y}{\partial x} + \beta''\frac{\partial \delta z}{\partial x} = \frac{\partial \delta' y}{\partial x} + [p_3]\delta' x - [p_1]\delta' z,$$
$$\gamma\frac{\partial \delta x}{\partial x} + \gamma'\frac{\partial \delta y}{\partial x} + \gamma''\frac{\partial \delta z}{\partial x} = \frac{\partial \delta' z}{\partial x} + [p_1]\delta' y - [p_2]\delta' x,$$

and analogous systems for the derivatives with respect to y and z. One has similar formulas that relate to $\delta I', \delta J', \delta K'$ and δI, δJ, δK. By virtue of formulas (72), and upon supposing that the determinant $|\alpha'\beta'\gamma''| = 1$, one then has:

$$\delta(\xi_1) = -[\xi_1]\left(\alpha\frac{\partial \delta x}{\partial x} + \alpha'\frac{\partial \delta x}{\partial y} + \alpha''\frac{\partial \delta x}{\partial z}\right) + (\alpha'[\varsigma_1] - \alpha''[\eta_1])\delta I \tag{82}$$
$$-[\eta_1]\left(\alpha\frac{\partial \delta y}{\partial x} + \alpha'\frac{\partial \delta y}{\partial y} + \alpha''\frac{\partial \delta y}{\partial z}\right) + (\alpha'[\xi_1] - \alpha''[\varsigma_1])\delta J$$
$$-[\varsigma_1]\left(\alpha\frac{\partial \delta z}{\partial x} + \alpha'\frac{\partial \delta z}{\partial y} + \alpha''\frac{\partial \delta z}{\partial z}\right) + (\alpha'[\eta_1] - \alpha''[\xi_1])\delta K,$$

with analogous formulas.

The value of $\delta(\xi)$ that was written on page (?) may be put into the form:

$$\delta(\xi) = -(\xi_1)\left\{\frac{\partial \delta' x}{\partial t} + (q)\delta' z - (r)\delta' y\right\}$$
$$- (\eta_1)\left\{\frac{\partial \delta' y}{\partial t} + (r)\delta' x - (p)\delta' z\right\}$$
$$- (\varsigma_1)\left\{\frac{\partial \delta' z}{\partial t} + (p)\delta' y - (q)\delta' x\right\};$$

however, by virtue of formulas (73) that define (p), (q), (r), one has formulas like the following ones:

$$\frac{\partial \delta' x}{\partial t} + (q)\delta' z - (r)\delta' y = \alpha \frac{\partial \delta x}{\partial t} + \alpha' \frac{\partial \delta y}{\partial t} + \alpha'' \frac{\partial \delta z}{\partial t},$$

and, as result, by virtue of formulas (72), one has:

(83) $$\delta(\xi) = -\left([\xi_1]\frac{\partial \delta x}{\partial t} + [\eta_1]\frac{\partial \delta y}{\partial t} + [\varsigma_1]\frac{\partial \delta z}{\partial t}\right),$$

a formula in which one may revert to the derivatives $\frac{d}{dt}$, as we shall see in detail later on.

By virtue of the formulas that define δx, δy, δz, δI, δJ, δK, one has:

$$\delta(p_1) = \alpha\left(\alpha \frac{\partial \delta I}{\partial x} + \alpha' \frac{\partial \delta J}{\partial x} + \alpha'' \frac{\partial \delta K}{\partial x}\right) + [\gamma(q_1) - \beta(r_1)]\delta I$$
$$+ \alpha'\left(\alpha \frac{\partial \delta I}{\partial y} + \alpha' \frac{\partial \delta J}{\partial y} + \alpha'' \frac{\partial \delta K}{\partial y}\right) + [\gamma'(q_1) - \beta'(r_1)]\delta J$$
$$+ \alpha''\left(\alpha \frac{\partial \delta I}{\partial z} + \alpha' \frac{\partial \delta J}{\partial z} + \alpha'' \frac{\partial \delta K}{\partial z}\right) + [\gamma''(q_1) - \beta''(r_1)]\delta K$$
$$- (p_1)\left[\alpha\left(\alpha \frac{\partial \delta x}{\partial x} + \alpha' \frac{\partial \delta y}{\partial x} + \alpha'' \frac{\partial \delta z}{\partial x}\right) + \alpha'\left(\alpha \frac{\partial \delta x}{\partial y} + \alpha' \frac{\partial \delta y}{\partial y} + \alpha'' \frac{\partial \delta z}{\partial y}\right) + \alpha''\left(\alpha \frac{\partial \delta x}{\partial z} + \cdots\right)\right]$$
$$- (q_1)\left[\alpha\left(\beta \frac{\partial \delta x}{\partial x} + \beta' \frac{\partial \delta y}{\partial x} + \beta'' \frac{\partial \delta z}{\partial x}\right) + \alpha'\left(\beta \frac{\partial \delta x}{\partial y} + \beta' \frac{\partial \delta y}{\partial y} + \beta'' \frac{\partial \delta z}{\partial y}\right) + \alpha''\left(\beta \frac{\partial \delta x}{\partial z} + \cdots\right)\right]$$
$$- (r_1)\left[\alpha\left(\gamma \frac{\partial \delta x}{\partial x} + \gamma' \frac{\partial \delta y}{\partial x} + \gamma'' \frac{\partial \delta z}{\partial x}\right) + \alpha'\left(\gamma \frac{\partial \delta x}{\partial y} + \gamma' \frac{\partial \delta y}{\partial y} + \gamma'' \frac{\partial \delta z}{\partial y}\right) + \alpha''\left(\gamma \frac{\partial \delta x}{\partial z} + \cdots\right)\right],$$

which, by virtue of formulas (73), may be written:

(84) $$\delta(p_1) = \alpha\left(\alpha \frac{\partial \delta I}{\partial x} + \alpha' \frac{\partial \delta I}{\partial y} + \alpha'' \frac{\partial \delta I}{\partial z}\right) + (\alpha'[r_1] - \alpha''[q_1])\delta I$$

$$
\begin{aligned}
&+\alpha'\left(\alpha\frac{\partial\delta J}{\partial x}+\alpha'\frac{\partial\delta J}{\partial y}+\alpha''\frac{\partial\delta J}{\partial z}\right)+(\alpha''[p_1]-\alpha[r_1])\delta J\\
&+\alpha''\left(\alpha\frac{\partial\delta K}{\partial x}+\alpha'\frac{\partial\delta K}{\partial y}+\alpha''\frac{\partial\delta K}{\partial z}\right)+(\alpha[q_1]-\alpha'[p_1])\delta K\\
&-[p_1]\left(\alpha\frac{\partial\delta x}{\partial x}+\alpha'\frac{\partial\delta x}{\partial y}+\alpha''\frac{\partial\delta x}{\partial z}\right)\\
&-[q_1]\left(\alpha\frac{\partial\delta y}{\partial x}+\alpha'\frac{\partial\delta y}{\partial y}+\alpha''\frac{\partial\delta y}{\partial z}\right)\\
&-[r_1]\left(\alpha\frac{\partial\delta z}{\partial x}+\alpha'\frac{\partial\delta z}{\partial y}+\alpha''\frac{\partial\delta z}{\partial z}\right),
\end{aligned}
$$

and one has analogous results for $\delta(q_1)$, …

Finally, observe that one may write:

$$
\begin{aligned}
\delta(p)=\frac{\partial\delta I}{\partial t}&+(q)\delta K'-(r)\delta J'\\
&-(p_1)\left[\frac{\partial\delta' x}{\partial t}+(q)\delta' z-(r)\delta' y\right]\\
&-(q_1)\left[\frac{\partial\delta' y}{\partial t}+(r)\delta' x-(p)\delta' z\right]\\
&-(r_1)\left[\frac{\partial\delta' z}{\partial t}+(p)\delta' y-(q)\delta' x\right],
\end{aligned}
$$

or:

$$
\begin{aligned}
\delta(p)=\alpha\frac{\partial\delta I}{\partial t}&+\alpha'\frac{\partial\delta J}{\partial t}+\alpha''\frac{\partial\delta K}{\partial t}\\
&-(p_1)\left[\frac{\partial\delta' x}{\partial t}+(q)\delta' z-(r)\delta' y\right]\\
&-(q_1)\left[\frac{\partial\delta' y}{\partial t}+(r)\delta' x-(p)\delta' z\right]\\
&-(r_1)\left[\frac{\partial\delta' z}{\partial t}+(p)\delta' y-(q)\delta' x\right],
\end{aligned}
$$

or finally:

(85) $$\delta(p)=\alpha\frac{\partial\delta I}{\partial t}+\alpha'\frac{\partial\delta J}{\partial t}+\alpha''\frac{\partial\delta K}{\partial t}-[p_1]\frac{\partial\delta x}{\partial t}-[q_1]\frac{\partial\delta y}{\partial t}-[r_1]\frac{\partial\delta z}{\partial t},$$

a formula in which one may also revert to the derivatives $\frac{d}{dt}$. One has two analogous formulas for $\delta(q)$, $\delta(r)$.

75. Direct determination of the variations of the Eulerian arguments. – We suppose that one subjects the functions x, y, z of x_0, y_0, z_0, t to the variations δx, δy, δz. Consider the relations that one obtains by differentiating relations (68′) successively with respect to the LAGRANGE variables; from this, we deduce:

$$\frac{\partial x}{\partial \rho_i}\delta[\xi_i]+\frac{\partial y}{\partial \rho_i}\delta[\eta_i]+\frac{\partial z}{\partial \rho_i}\delta[\varsigma_i]+[\xi_i]\frac{\partial \delta x}{\partial \rho_i}+[\eta_i]\frac{\partial \delta y}{\partial \rho_i}+[\varsigma_i]\frac{\partial \delta z}{\partial \rho_i}=0;$$

however, one has:

$$\begin{aligned}
\frac{\partial \delta x}{\partial \rho_i}&=\frac{\partial \delta x}{\partial x}\frac{\partial x}{\partial \rho_i}+\frac{\partial \delta x}{\partial y}\frac{\partial y}{\partial \rho_i}+\frac{\partial \delta x}{\partial z}\frac{\partial z}{\partial \rho_i},\\
\frac{\partial \delta y}{\partial \rho_i}&=\frac{\partial \delta y}{\partial x}\frac{\partial x}{\partial \rho_i}+\frac{\partial \delta y}{\partial y}\frac{\partial y}{\partial \rho_i}+\frac{\partial \delta x}{\partial z}\frac{\partial y}{\partial \rho_i},\\
\frac{\partial \delta z}{\partial \rho_i}&=\frac{\partial \delta z}{\partial x}\frac{\partial x}{\partial \rho_i}+\frac{\partial \delta z}{\partial y}\frac{\partial y}{\partial \rho_i}+\frac{\partial \delta z}{\partial z}\frac{\partial z}{\partial \rho_i};
\end{aligned}$$

if one substitutes the values of these derivatives into the preceding expression then one has:

$$\begin{aligned}
&\frac{\partial x}{\partial \rho_i}\left\{\delta[\xi_i]+[\xi_i]\frac{\partial \delta x}{\partial x}+[\eta_i]\frac{\partial \delta y}{\partial y}+[\varsigma_i]\frac{\partial \delta z}{\partial z}\right\}\\
&+\frac{\partial y}{\partial \rho_i}\left\{\delta[\eta_i]+[\xi_i]\frac{\partial \delta x}{\partial y}+[\eta_i]\frac{\partial \delta y}{\partial y}+[\varsigma_i]\frac{\partial \delta z}{\partial y}\right\}\\
&+\frac{\partial z}{\partial \rho_i}\left\{\delta[\varsigma_i]+[\xi_i]\frac{\partial \delta x}{\partial z}+[\eta_i]\frac{\partial \delta y}{\partial z}+[\varsigma_i]\frac{\partial \delta z}{\partial z}\right\}=0;
\end{aligned}$$

the parentheses in this latter equality are thus null, and one has:

$$\begin{aligned}
\delta[\xi_i]&=-\left\{[\xi_i]\frac{\partial \delta x}{\partial x}+[\eta_i]\frac{\partial \delta y}{\partial y}+[\varsigma_i]\frac{\partial \delta z}{\partial z}\right\},\\
\delta[\eta_i]&=-\left\{[\xi_i]\frac{\partial \delta x}{\partial y}+[\eta_i]\frac{\partial \delta y}{\partial y}+[\varsigma_i]\frac{\partial \delta z}{\partial y}\right\},\\
\delta[\varsigma_i]&=-\left\{[\xi_i]\frac{\partial \delta x}{\partial z}+[\eta_i]\frac{\partial \delta y}{\partial z}+[\varsigma_i]\frac{\partial \delta z}{\partial z}\right\}.
\end{aligned}$$

Similarly, we have:

$$\delta(\xi)=-\frac{dx}{dt}\delta[\xi_1]-\frac{dy}{dt}\delta[\eta_1]-\frac{dz}{dt}\delta[\varsigma_1]-[\xi_1]\frac{d\delta x}{dt}-[\eta_1]\frac{d\delta y}{dt}-[\varsigma_1]\frac{d\delta z}{dt};$$

upon replacing $\delta[\xi_1]$, $\delta[\eta_1]$, $\delta[\zeta_1]$ with the values that we must obtain they become:

$$\delta(\xi) = \frac{dx}{dt}\left\{[\xi_i]\frac{\partial\delta x}{\partial x} + [\eta_i]\frac{\partial\delta y}{\partial y} + [\varsigma_i]\frac{\partial\delta z}{\partial z}\right\},$$
$$+\frac{dy}{dt}\left\{[\xi_i]\frac{\partial\delta x}{\partial y} + [\eta_i]\frac{\partial\delta y}{\partial y} + [\varsigma_i]\frac{\partial\delta z}{\partial y}\right\},$$
$$+\frac{dz}{dt}\left\{[\xi_i]\frac{\partial\delta x}{\partial z} + [\eta_i]\frac{\partial\delta y}{\partial z} + [\varsigma_i]\frac{\partial\delta z}{\partial z}\right\}.$$
$$-[\xi_1]\frac{d\delta x}{dt} - [\eta_1]\frac{d\delta y}{dt} - [\varsigma_1]\frac{d\delta z}{dt};$$

with analogous formulas for $\delta(\eta)$, $\delta(\zeta)$. To retrieve the formula that we obtained in sec. 74, it suffices to remark that one has:

$$\frac{d\delta x}{dt} = \frac{\partial\delta x}{\partial x}\frac{dx}{dt} + \frac{\partial\delta x}{\partial y}\frac{dy}{dt} + \frac{\partial\delta x}{\partial z}\frac{dz}{dt} + \frac{\partial\delta x}{\partial t},$$
$$\frac{dyx}{dt} = \frac{\partial\delta y}{\partial x}\frac{dx}{dt} + \frac{\partial\delta y}{\partial y}\frac{dy}{dt} + \frac{\partial\delta y}{\partial z}\frac{dz}{dt} + \frac{\partial\delta y}{\partial t},$$
$$\frac{d\delta z}{dt} = \frac{\partial\delta z}{\partial x}\frac{dx}{dt} + \frac{\partial\delta z}{\partial y}\frac{dy}{dt} + \frac{\partial\delta z}{\partial z}\frac{dz}{dt} + \frac{\partial\delta z}{\partial t};$$

but we will not use the formula on page (?) and its analogues in what follows. Indeed, it is convenient to observe only the domain of integration of the integrals over *x, y, z,* which we consider to *depend* on *t*, in the case in which *x, t, z, t* are the *independent variables,* and not revert to the integrations over *x, y, z,* and *t*, as is the habitual custom (as with x_0, y_0, z_0). If one must integrate by parts with respect to *t* then one must introduce the auxiliary variables x_0, y_0, z_0, and use only derivatives with respect to *t* that take the form $\frac{d}{dt}$, which will necessitate the use of formulas such as the one that wrote above for $\delta(\xi)$.

The calculations that must be done in order to obtain $\delta(p_i)$, $\delta(q_i)$, $\delta(r_i)$, $\delta(p)$, $\delta(q)$, $\delta(r)$, like the ones that lead to expressions for $\delta(\xi_i)$, $\delta(\eta_i)$, $\delta(\zeta_i)$, $\delta(\xi)$, $\delta(\eta)$, $\delta(\zeta)$, presently rest upon formulas that we just obtained for $\delta[\xi_i]$, $\delta[\eta_i]$, $\delta[\zeta_i]$. The transformation that the expressions $\delta(p)$, $\delta(q)$, $\delta(r)$, which were given in sec. 74, must be subjected to in order to put the derivatives with respect to *t* into the form $\frac{d}{dt}$, is the same as the one that we indicated for $\delta(\xi)$, $\delta(\eta)$, $\delta(\zeta)$.

76. The action of deformation and motion in terms of Euler variables. Invariance of the Eulerian arguments. Application to the method of variable action. – The action of deformation and motion becomes:

$$\int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt,$$

in which W is a function of x_0, y_0, z_0, t; ξ_i, η_i, ζ_i, p_i, q_i, r_i; ξ, η, ζ, p, q, r.

From formulas (79) and (81), (81′), one may also say that W is a function of x_0, y_0, z_0, t; (ξ_i), (η_i), (ζ_i), (p_i), (q_i), (r_i); (ξ), (η), (ζ), (p), (q), (r), and, if one sets ([1]):

$$\Omega = \frac{W}{\Delta}$$

then the preceding action may be written:

$$\int_{t_1}^{t_2}\iiint_{S} \Omega dx dy dz dt.$$

The integration over x, y, z is taken over the medium S, i.e., *over a domain that varies with time.*

One may also see how one can arrived at this latter action independently of the former. Indeed, the Lagrangian arguments are, as we saw before, *Euclidian invariants;* however, since the Eulerian arguments are uniquely functions of the Lagrangian arguments, from formulas (77) and (80), it results from this that they are also *Euclidian invariants;* furthermore, one may establish this *in a direct manner* by means of formulas (82), (83) and (84), (85), by setting:

$$\delta x = (a_1 + \omega_2 z - \omega_3 y)dt,$$
$$\delta y = (b_1 + \omega_3 x - \omega_1 z)dt,$$
$$\delta z = (c_1 + \omega_1 y - \omega_2 x)dt,$$
$$\delta I = \omega_1 \delta t, \qquad \delta J = \omega_2 \delta t, \qquad \delta K = \omega_3 \delta t.$$

From this, it results that one is directly led to give the following form to the *action of deformation and movement in terms of the* EULER *variables* taken over the interior of the surface S, and during the time interval between instants t_1 and t_2:

$$\int_{t_1}^{t_2}\iiint_{S} \Omega dx dy dz dt,$$

in which *the function* Ω *has the following remarkable:*

$$\Omega(x_0, y_0, z_0, t; (\xi_i), (\eta_i), (\zeta_i), (p_i), (q_i), (r_i); (\xi), (\eta), (\zeta), (p), (q), (r)).$$

Consider an *arbitrary* variation of the action of deformation and motion in the interior of a surface (S) in the medium (M), and the time interval between the instants t_1 and t_2, and, to that effect, give the x, … the variations δx, …

---

[1] We suppose that Δ is positive and therefore equal to $|\Delta|$.

For the moment, write the integral in the form:

$$\int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt;$$

its variation is:

$$\int_{t_1}^{t_2}\iiint_{S_0} (\Delta\delta\Omega + \Omega\delta\Delta) dx_0 dy_0 dz_0 dt,$$

or:

$$\int_{t_1}^{t_2}\iiint_{S_0} (\delta\delta\Omega + \Omega\frac{\delta\Delta}{\Delta}) dx_0 dy_0 dz_0 dt.$$

However:

$$\begin{aligned}
\Delta &= \frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)},\\
\delta\Delta &= \frac{\partial(y,z)}{\partial(y_0,z_0)}\frac{\partial\delta x}{\partial x_0} + \frac{\partial(y,z)}{\partial(z_0,x_0)}\frac{\partial\delta x}{\partial y_0} + \frac{\partial(y,z)}{\partial(x_0,y_0)}\frac{\partial\delta x}{\partial z_0} + \cdots\\
&= \left\{\frac{\partial(y,z)}{\partial(y_0,z_0)}\frac{\partial x}{\partial x_0} + \frac{\partial(y,z)}{\partial(z_0,x_0)}\frac{\partial x}{\partial y_0} + \frac{\partial(y,z)}{\partial(x_0,y_0)}\frac{\partial x}{\partial z_0}\right\}\frac{\partial\delta x}{\partial x} + \cdots\\
&= \frac{\partial\delta x}{\partial x}\Delta + \frac{\partial\delta y}{\partial y}\Delta + \frac{\partial\delta z}{\partial z}\Delta,
\end{aligned}$$

i.e.,

$$\frac{\delta\Delta}{\Delta} = \frac{\partial\delta x}{\partial x} + \frac{\partial\delta y}{\partial y} + \frac{\partial\delta z}{\partial z},$$

and, as a result, the variation of the integral is:

$$\int_{t_1}^{t_2}\iiint_{S} \left\{\Omega\left(\frac{\partial\delta x}{\partial x} + \frac{\partial\delta y}{\partial y} + \frac{\partial\delta z}{\partial z}\right) + \delta\Omega\right\} dx dy dz dt.$$

The variation $\delta\Omega$ of Ω is:

$$\delta\Omega = \sum\left\{\frac{\partial\Omega}{\partial(\xi_i)}\delta(\xi_i) + \frac{\partial\Omega}{\partial(\eta_i)}\delta(\eta_i) + \cdots\right\} + \frac{\partial\Omega}{\partial(p)}\delta(p) + \cdots + \frac{\partial\Omega}{\partial(q)}\delta(q),$$

in which $\delta(\xi_i)$, $\delta(\eta_i)$, …, $\delta(r)$ are determined by the formulas of sec. 74 and 75, in such a way that only the derivatives with respect to t in the form $\frac{d}{dt}$ are involved. We may apply GREEN'S formula to the terms that explicitly refer to a derivative with respect to one of the variables x, y, z. As far as the terms that explicitly refer to a derivative with respect to time are concerned, here is how we deal with them (the domain of integration over x, y, z *varies with time*): let:

$$\int_{t_1}^{t_2}\iiint_S g\frac{dh}{dt}dxdydzdt,$$

be a typical term; if we pass to the intermediary of the variables x_0, y_0, z_0 then it becomes:

$$\int_{t_1}^{t_2}\iiint_{S_0} g\Delta\frac{dh}{dt}dx_0dy_0dz_0dt,$$

or, on integrating by parts:

$$\iiint_{S_0}[g\Delta h]_{t_1}^{t_2}dx_0dy_0dz_0 - \int_{t_1}^{t_2}\iiint_{S_0} h\frac{d(g\Delta)}{dt}dx_0dy_0dz_0dt$$

$$= \left[\iiint_{S_0} g\Delta h dx_0dy_0dz_0\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\iiint_{S_0} h\frac{d(g\Delta)}{dt}dx_0dy_0dz_0dt$$

$$= \left[\iiint_S ghdxdydz\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\iiint_S \frac{h}{\Delta}\frac{d(g\Delta)}{dt}dx_0dy_0dz_0dt,$$

when we revert to the variables x, y, z ([1]).

If we let l, m, n denote the direction cosines of the exterior normal to the surface S that bounds the medium after deformation at the instant t with respect to the fixed axes Ox, Oy, Oz, and let $d\sigma$ be the area element of that surface:

$$\delta\int_{t_1}^{t_2}\iiint\Omega dxdydzdt$$

$$= \int_{t_1}^{t_2}\iint_S\{(lp_{xx}+mp_{yx}+np_{zx})\delta x+(lp_{xy}+mp_{yy}+np_{zy})\delta y+(lp_{xz}+mp_{yz}+np_{zz})\delta z$$

$$+(lq_{xx}+mq_{yx}+nq_{zx})\delta I++(lq_{xy}+mq_{yy}+nq_{zy})\delta J+(lq_{xz}+mq_{yz}+nq_{zz})\delta K\}d\sigma dt$$

$$+\left\{\iiint_S\left(\frac{A}{\Delta}\delta x+\frac{B}{\Delta}\delta y+\frac{C}{\Delta}\delta z+\frac{P}{\Delta}\delta I+\frac{Q}{\Delta}\delta J+\frac{R}{\Delta}\delta K\right)dxdydz\right\}_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2}\iiint_S\left\{\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}+\frac{1}{\Delta}\frac{dA}{dt}\right)\delta x\right.$$

$$+\left(\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}+\frac{1}{\Delta}\frac{dB}{dt}\right)\delta y$$

$$+\left(\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}+\frac{1}{\Delta}\frac{dC}{dt}\right)\delta z$$

---

[1] Here one may replace $\frac{d\Delta}{dt}$ by the value it derives from:

$$\frac{1}{\Delta}\frac{d\Delta}{dt}=\frac{\partial}{\partial x}\left(\frac{dx}{dt}\right)+\frac{\partial}{\partial y}\left(\frac{dy}{dt}\right)+\frac{\partial}{\partial z}\left(\frac{dz}{dt}\right).$$

$$+\left(\frac{\partial q_{xx}}{\partial x}+\frac{\partial q_{yx}}{\partial y}+\frac{\partial q_{zx}}{\partial z}+\frac{1}{\Delta}\frac{dP}{dt}+p_{yz}-p_{zy}+\frac{C}{\Delta}\frac{dy}{dt}-\frac{B}{\Delta}\frac{dz}{dt}\right)\delta I$$
$$+\left(\frac{\partial q_{xy}}{\partial x}+\frac{\partial q_{yy}}{\partial y}+\frac{\partial q_{zy}}{\partial z}+\frac{1}{\Delta}\frac{dQ}{dt}+p_{zx}-p_{xz}+\frac{A}{\Delta}\frac{dz}{dt}-\frac{C}{\Delta}\frac{dx}{dt}\right)\delta J$$
$$\left.+\left(\frac{\partial q_{xz}}{\partial x}+\frac{\partial q_{yz}}{\partial y}+\frac{\partial q_{zz}}{\partial z}+\frac{1}{\Delta}\frac{dR}{dt}+p_{xy}-p_{yx}+\frac{B}{\Delta}\frac{dx}{dt}-\frac{A}{\Delta}\frac{dy}{dt}\right)\delta K\right\}dxdydzdt,$$

in which we have set, following the notations of sec. 73:

$$\frac{A}{\Delta}=-(A')[\xi_1]-(B')[\xi_2]-(C')[\xi_3]-(P')[p_1]-(Q')[p_2]-(R')[p_3],$$
$$\frac{B}{\Delta}=-(A')[\eta_1]-(B')[\eta_2]-(C')[\eta_3]-(P')[q_1]-(Q')[q_2]-(R')[q_3],$$
$$\frac{C}{\Delta}=-(A')[\varsigma_1]-(B')[\varsigma_2]-(C')[\varsigma_3]-(P')[r_1]-(Q')[r_2]-(R')[r_3],$$
$$\frac{P}{\Delta}=[P]=\alpha(P')+\beta(Q')+\gamma(R'),$$
$$\frac{Q}{\Delta}=[Q]=\alpha'(P')+\beta'(Q')+\gamma'(R'),$$
$$\frac{R}{\Delta}=[P]=\alpha''(P')+\beta''(Q')+\gamma''(R'),$$
$$p_{xx}=\Omega-\sum[A_i][\xi_i]-\sum[P_i][p_i]-\frac{A}{\Delta}\frac{dx}{dt},$$
$$p_{yx}=\quad-\sum[B_i][\xi_i]-\sum[Q_i][p_i]-\frac{A}{\Delta}\frac{dy}{dt},$$
$$p_{zx}=\quad-\sum[C_i][\xi_i]-\sum[R_i][p_i]-\frac{A}{\Delta}\frac{dz}{dt},$$
$$p_{xy}=\quad-\sum[A_i][\eta_i]-\sum[P_i][q_i]-\frac{A}{\Delta}\frac{dx}{dt},$$
$$p_{yy}=\Omega-\sum[B_i][\eta_i]-\sum[Q_i][q_i]-\frac{B}{\Delta}\frac{dy}{dt},$$
$$p_{zy}=\quad-\sum[C_i][\eta_i]-\sum[R_i][q_i]-\frac{B}{\Delta}\frac{dz}{dt},$$
$$p_{xz}=\quad-\sum[A_i][\varsigma_i]-\sum[P_i][r_i]-\frac{C}{\Delta}\frac{dx}{dt},$$
$$p_{yz}=\quad-\sum[B_i][\varsigma_i]-\sum[Q_i][r_i]-\frac{C}{\Delta}\frac{dy}{dt},$$
$$p_{zz}=\Omega-\sum[C_i][\varsigma_i]-\sum[R_i][r_i]-\frac{C}{\Delta}\frac{dz}{dt},$$

and, in addition:

$$q_{xx} = \alpha[P_1] + \beta[P_2] + \gamma[P_3] - \frac{P}{\Delta}\frac{dx}{dt},$$
$$q_{yx} = \alpha[Q_1] + \beta[Q_2] + \gamma[Q_3] - \frac{P}{\Delta}\frac{dy}{dt},$$
$$q_{zx} = \alpha[R_1] + \beta[R_2] + \gamma[R_3] - \frac{P}{\Delta}\frac{dz}{dt},$$

with analogous formulas for q_{xy}, q_{yy}, q_{zy}, q_{xz}, q_{yz}, q_{zz}.

77. Remarks on the variations introduced in the preceding sections. Application of the method of variable action as in the usual calculus of variations. – We used the calculus of variations in the preceding section; it is useful to elaborate on the significance of those formulas according to the approach of JORDAN ([1]).

For the sake of completeness, recall the exposition of JORDAN. JORDAN sought the variation of

$$S\phi \, dxdydz$$

when one supposes, on the one hand, that x, y, z are subject to variations, and, on the other hand, that the functions that figure in ϕ are also subject to variation. From this fact, ϕ is subject to *two* variations whose effects are added together. JORDAN successively considered the variation due to the variation of the functions that figure in ϕ, and then the variation due to the variation of x, y, z that is juxtaposed with the preceding.

One may just as well search for the complete effect of juxtaposing the two variations on the letters $u, \ldots, u_{\alpha\beta\gamma}, \ldots$ that figure in ϕ. If we call these complete variations $\delta u, \ldots$ then one will have:

$$\delta\varphi = \frac{\partial \varphi}{\partial u}\delta u + \cdots$$

for the *complete* variation $\delta\varphi$ of φ.

Having said this, one remarks that the previously calculated variations are what we must call the *complete* variations and that the calculations in the preceding section were carried out from this latter viewpoint.

If one prefers to present things in a form that is *identical* to that of JORDAN then here is what one must do. In what follows, we introduce the functions x_0, y_0, $z_0, \alpha, \alpha', \cdots, \gamma''$, of x, y, z, which figure explicitly and by their derivatives, at least in part. The functions x_0, y_0, z_0 of x, y, z, t are the ones that must be used in the left-hand side of (68′) in order to derive x, y, z as functions of x_0, y_0, z_0, t. From this, and the fact that x, y, z are subjected to variations $\delta x, \delta y, \delta z$, it results that *these functions* x_0, y_0, z_0 of x, y, z, t

[1] JORDAN, *Cours d' Analyse de l'Ecole polytechnique*, 1st ed., T. III, no. 339, pp. 533-535; 2nd ed., T. III, no. 396, pp. 528-530.

are also subjected to variations, which we designate [1] by (δx_0), …, and one has the formulas:

(86) $$\begin{cases} 0 = (\delta x_0) + \dfrac{\partial x_0}{\partial x}\delta x + \dfrac{\partial x_0}{\partial y}\delta y + \dfrac{\partial x_0}{\partial z}\delta z, \\ 0 = (\delta y_0) + \dfrac{\partial y_0}{\partial x}\delta x + \dfrac{\partial y_0}{\partial y}\delta y + \dfrac{\partial y_0}{\partial z}\delta z, \\ 0 = (\delta z_0) + \dfrac{\partial z_0}{\partial x}\delta x + \dfrac{\partial z_0}{\partial y}\delta y + \dfrac{\partial z_0}{\partial z}\delta z, \end{cases}$$

which express that the *complete* variations of these function are null. The variations (δx_0), (δy_0), (δz_0) that figure in the last three formulas are *copied from* the variations that figure in the exposition of JORDAN, as we shall see. This remark seems to seems to have been discussed in the considerations that were developed by C. NEUMANN in his research [2] on the MAXWELL and HERTZ equations; it conforms, on the one hand, to the rules of calculus that were adopted by H. POINCARÉ, in his memoir *on the dynamics of the electron* [3], which we shall discuss later on.

As far as $\alpha, \alpha', \cdots, \gamma''$ are concerned, we have the variations $(\delta\alpha)$, …, in the sense of JORDAN; however, the variations that were introduced in the preceding sections, and which we continue to denote by $\delta\alpha$, …, will be the complete variations, in such a way that one will have:

$$\delta\alpha = (\delta\alpha) + \frac{\partial\alpha}{\partial x}\delta x + \frac{\partial\alpha}{\partial y}\delta y + \frac{\partial\alpha}{\partial z}\delta z.$$

This amounts to saying that when we introduce the variations $(\delta\alpha)$, …, *in the sense of* JORDAN, we introduce, in addition, the auxiliary functions $\delta I', \delta J', \delta K'$, which we *define* in terms of $(\delta\alpha)$, δx, … by way of:

[1] In general, in order to avoid confusion we denote the variations that areobtained by leaving x, y, z fixed by (δ).

[2] C. NEUMANN. – *Die elektrischen Kräfte,* T. II, Leipzig, 1898; *Über die Maxwell-Hertz'sche Theorie (Abhandl. der k. Sächs Gesells. der Wiss. zu Leipzig; Math.-phys. Klasses,* T. XXVII, nos. 2 and 8, 1901-1902).

[3] H. POINCARÉ, *Rend. di Palermo,* Tome XXI, pp. 129 et seq. (1905), 1906. H. POINCARÉ uses different notations from ours, in particular, as far as derivatives with respect to t are concerned; our notation, d, ∂, which is that of APPELL (*Traité de Mecanique,* Tome II, 1st ed., pp. 277), is the opposite of POINCARÉ. He distinguishes the ordinary variation $(\delta\varphi)$ of a function φ in the sense of JORDAN, which he denotes by $\frac{d\varphi}{d\varepsilon}d\varepsilon$, from its variation $\delta\varphi$ (which we call *complete*), which he denotes by $\frac{\partial\varphi}{\partial\varepsilon}\delta\varepsilon$ [in particular, see the formula (11 bis), page 140].

$$(87)\qquad \begin{cases} \delta I' = \sum \gamma\delta\beta = \sum \gamma(\delta\beta) + [p_1]\delta x + [q_1]\delta y + [r_1]\delta z, \\ \delta J' = \sum \alpha\delta\gamma = \sum \alpha(\delta\gamma) + [p_2]\delta x + [q_2]\delta y + [r_2]\delta z, \\ \delta K' = \sum \beta\delta\alpha = \sum \beta(\delta\alpha) + [p_3]\delta x + [q_3]\delta y + [r_3]\delta z. \end{cases}$$

The fundamental convention is expressed by the relations (86), as one sees. It will be found, in an eventual work on the theory of *temperature,* for the functions that figure by way of their differential parameters – for example, in the case that amounts to a pointlike medium – if one abstracts from the formulas in which the complete variations of these functions are presented.

One will observe that *presently* the simplest way to perform these calculations is not the one that was followed in the aforementioned exposition of JORDAN, but consists of determining, as we did before, the *complete* variation of the function under the integration sign. Nevertheless, in view of the comparisons that are to be performed when one develops the two viewpoints that are suggested by the notion of *temperature* later on, it will be useful to likewise follow the path of JORDAN.

We have:

$$(88)\qquad \delta\int_{t_1}^{t_2} \iiint_S \Omega\,dx\,dy\,dz\,dt = \int_{t_1}^{t_2} \iiint_S \left[\frac{\partial\Omega}{\partial x_0}(\delta x_0) + \frac{\partial\Omega}{\partial y_0}(\delta y_0) + \frac{\partial\Omega}{\partial z_0}(\delta z_0)\right.$$
$$+ \sum\left\{\frac{\partial\Omega}{\partial(\xi_i)}\big(\delta(\xi_i)\big) + \cdots + \frac{\partial\Omega}{\partial(r_i)}\big(\delta(r_i)\big)\right\} + \frac{\partial\Omega}{\partial(\xi)}\big(\delta(\xi)\big) + \cdots + \frac{\partial\Omega}{\partial(r)}\big(\delta(r)\big)$$
$$\left. + \frac{d}{dx}(\Omega\delta x) + \frac{d}{dy}(\Omega\delta y) + \frac{d}{dz}(\Omega\delta z)\right] dx\,dy\,dz\,dt,$$

in which the (δ) sign corresponds to the variation that is obtained by leaving x, y, z fixed, in such a way that one has, in a general fashion:

$$(89)\qquad (\delta\mathcal{F}) = \delta\mathcal{F} - \frac{d\mathcal{F}}{dx}\delta x - \frac{d\mathcal{F}}{dy}\delta y - \frac{d\mathcal{F}}{dz}\delta z\,.$$

We substitute the auxiliary functions δx, δy, $\delta z, \delta I', \delta J', \delta K'$ that are defined by the formulas (86), (87) for the variations (δx_0), … *In regard to the integration over t,* we must also recall that the domain of integration over x, y, z varies with t, and that one may not switch the order of integrating over t and the system of integrations over x, y, z *in the habitual fashion that is employed for the variables* x_0, y_0, z_0.

If we replace (δx_0), (δy_0), (δz_0), $(\delta(\xi_i))$, … by their values from (89), which subsumes (86), we obtain:

$$(90)\qquad \delta\int_{t_1}^{t_2} \iiint_S \Omega\,dx\,dy\,dz\,dt = \int_{t_1}^{t_2} \iiint_S \left[-\frac{d\Omega}{dx}\delta x - \frac{d\Omega}{dy}\delta y - \frac{d\Omega}{dz}\delta z\right.$$

$$+\sum\left\{\frac{\partial\Omega}{\partial(\xi_i)}(\delta(\xi_i))+\cdots+\frac{\partial\Omega}{\partial(r_i)}(\delta(r_i))\right\}+\frac{\partial\Omega}{\partial(\xi)}(\delta(\xi))+\cdots+\frac{\partial\Omega}{\partial(r)}(\delta(r))$$
$$\left.+\frac{d}{dx}(\Omega\delta x)+\frac{d}{dy}(\Omega\delta y)+\frac{d}{dz}(\Omega\delta z)\right]dx\,dy\,dz\,dt.$$

If we consider first

$$(91)\qquad \int_{t_1}^{t_2}\iiint_S\left[-\frac{d\Omega}{dx}\delta x-\frac{d\Omega}{dy}\delta y-\frac{d\Omega}{dz}\delta z\right.$$
$$\left.+\frac{d}{dx}(\Omega\delta x)+\frac{d}{dy}(\Omega\delta y)+\frac{d}{dz}(\Omega\delta z)\right]dx\,dy\,dz\,dt,$$

and then:

$$(92)\ \int_{t_1}^{t_2}\iiint_S\left[\sum\left\{\frac{\partial\Omega}{\partial(\xi_i)}(\delta(\xi_i))+\cdots\right\}+\frac{\partial\Omega}{\partial(\xi)}(\delta(\xi))+\cdots+\frac{\partial\Omega}{\partial(r)}(\delta(r))\right]dx\,dy\,dz\,dt,$$

just as, in the preceding section, we divided the sum into:

$$(91')\qquad \int_{t_1}^{t_2}\iiint_S\Omega\left(\frac{\partial\delta x}{\partial x}+\frac{\partial\delta y}{\partial y}+\frac{\partial\delta z}{\partial z}\right)dx\,dy\,dz\,dt,$$

and (92), one sees that the calculation is identical to the one that we did earlier.

78. – The Lagrangian and Eulerian conceptions of action. The method of variable action applied to the Eulerian conception of action as expressed by the Euler variables. – In his work *sur la dynamique de l'électron,* which was presented at the July 23, 1905 session of the Cercle de Palerme, H. POINCARÉ presented a conception of the action *for an infinite domain* that was different from the one that we envisioned up till now. If one clarifies the idea of H. POINCARÉ when considering a *finite domain* then one is led to distinguish the following two conceptions of action, the one being *Lagrangian,* and the other, *Eulerian.*

We may integrate the general function W or Ω over the independent variables ([1]) x_0, y_0, z_0, or the independent variables ([2]) *x, y, z in a fixed domain,* and then integrate over t.

1. Start with the space (M_0), and imagine that an observer attached to the reference axes directs his attention to a portion (S_0) of that space and to the different positions that it ultimately takes, namely: (S) at an arbitrary instant t, (S_1) and (S_2) at the times t_1 and t_2.

We imagine the integral:

[1] In this case, we denote the function by W.

[2] In this case, we denote the function by Ω.

$$\int_{t_1}^{t_2} \iiint_S \Omega dxdydzdt,$$

in which the domain of integration (S) with respect to x, y, z *varies with t,* and which takes the form:

$$\int_{t_1}^{t_2} \iiint_{S_0} W dx_0 dy_0 dz_0 dt,$$

upon effecting the change of variables that is defined by $(66')$ or $(68')$, in which W denotes the expression that is obtained by replacing the letters x, y, z in $\Omega\Delta$ by their expressions in $(66')$, and the domain of integration over x_0, y_0, z_0, (S_0) is *independent of t.* We then have the *Lagrangian* conception of the action.

2. While always envisioning an observer that is *fixed with respect to the reference axes,* imagine that he constantly directs his attention to *fixed and definite* portion of space (M); let x_0, y_0, z_0 denote the coordinates that are calculated by means of formulas $(68')$ at the point M_0 of (M_0), and becomes the point M of (M), with coordinates, x, y, z at the instant t, and let (S_0) be the region contained in M_0 that becomes (S) at the instant, t; we may then let (S_{01}), (S_{02}) denote the regions that (S_0), which varies with t, becomes for the values t_1 and t_2 of t.

If Ω refers to both x, y, z, and the functions expressed by the formulas $(66')$ then we envision:

$$\int_{t_1}^{t_2} \iiint_S \Omega dxdydzdt,$$

in which the domain of integration over x, y, z – namely, (S) – is *independent of t* this time, and which takes the form:

$$\int_{t_1}^{t_2} \iiint_{S_0} W dx_0 dy_0 dz_0 dt,$$

upon effecting the change of variables that is defined by $(66')$ or $(68')$, in which the domain of integration over x, y, z – namely, (S) – *varies with t.* We then have the *eulerian* conception of action.

We have considered the first case in the earlier paragraphs; we shall now occupy ourselves with the second one. Formula (88) is then replaced with the following ([1]):

$$(88') \quad \left(\delta \int_{t_1}^{t_2} \iiint_S \Omega dxdydzdt\right) = \int_{t_1}^{t_2} \iiint_S \left[\frac{\partial \Omega}{\partial x_0}(\delta x_0) + \frac{\partial \Omega}{\partial y_0}(\delta y_0) + \frac{\partial \Omega}{\partial z_0}(\delta z_0) \right.$$

[1] Upon referring to the exposition of JORDAN, one will observe that the terms $\frac{d}{dx}(\Omega\delta x) + \frac{d}{dy}(\Omega\delta y) + \frac{d}{dz}(\Omega\delta z)$ come from the fact that the domain is moving, and correspond to the variation of the letters x, y, z, as well as the independent variables.

$$+\sum\left\{\frac{\partial\Omega}{\partial(\xi_i)}(\delta(\xi_i))+\cdots+\frac{\partial\Omega}{\partial(r_i)}(\delta(r_i))\right\}+\frac{\partial\Omega}{\partial(\xi)}(\delta(\xi))+\cdots+\frac{\partial\Omega}{\partial(r)}(\delta(r))\Bigg]dxdydzdt;$$

and, by virtue of (89), formula (90) is replaced by the following one:

$$(90')\qquad \left(\delta\int_{t_1}^{t_2}\iiint_S \Omega dxdydzdt\right)=\int_{t_1}^{t_2}\iiint_S\Bigg[-\frac{d\Omega}{dx}\delta x-\frac{d\Omega}{dy}\delta y-\frac{d\Omega}{dz}\delta z$$
$$+\sum\left\{\frac{\partial\Omega}{\partial(\xi_i)}\delta(\xi_i)+\cdots+\right\}+\frac{\partial\Omega}{\partial(\xi)}\delta(\xi)+\cdots+\frac{\partial\Omega}{\partial(r)}\delta(r)\Bigg]dxdydzdt.$$

This sequence of calculations resembles the ones in sec. **77**. At the same time, a difference was introduced as far as the derivatives with respect to time are concerned. At the moment, one may exchange the integration over t and the integration over the domain of the variables x, y, z, and, that exchange having been performed, the integration over time must be done by imagining that x, y, z are constant. The integration by parts over time must be done by making them depend on the derivatives $\frac{\partial}{\partial t}$, and not on $\frac{d}{dt}$, as we did in sec. **76** and **77**, and conforming to the remark made in sec. **75** and **76**.

The integration by parts now gives:

$$\left(\delta\int_{t_1}^{t_2}\iiint_S \Omega dxdydzdt\right)$$
$$=\int_{t_1}^{t_2}\iint_S\{(l'p'_{xx}+m'p'_{yx}+n'p'_{zx})\delta x+(l'p'_{xy}+m'p'_{yy}+n'p'_{zy})\delta y+(l'p'_{xz}+m'p'_{yz}+n'p'_{zz})\delta z$$
$$+(l'q'_{xx}+m'q'_{yx}+n'q'_{zx})\delta I+(l'q'_{xy}+m'q'_{yy}+n'q'_{zy})\delta J+(l'q'_{xz}+m'q'_{yz}+n'q'_{zz})\delta K\}d\sigma'dt$$
$$+\left\{\iiint_S\left(\frac{A'}{\Delta}\delta x+\frac{B'}{\Delta}\delta y+\frac{C'}{\Delta}\delta z+\frac{P'}{\Delta}\delta' I+\frac{Q'}{\Delta}\delta' J+\frac{R'}{\Delta}\delta' K\right)dxdydz\right\}_{t_1}^{t_2}$$
$$-\int_{t_1}^{t_2}\iiint_S\Bigg\{\left(\frac{\partial p'_{xx}}{\partial x}+\frac{\partial p'_{yx}}{\partial y}+\frac{\partial p'_{zx}}{\partial z}+\frac{\partial}{\partial t}\frac{A'}{\Delta}+\frac{d\Omega}{dx}\right)\delta x$$
$$+\left(\frac{\partial p'_{xy}}{\partial x}+\frac{\partial p'_{yy}}{\partial y}+\frac{\partial p'_{zy}}{\partial z}+\frac{\partial}{\partial t}\frac{B'}{\Delta}+\frac{d\Omega}{dy}\right)\delta y$$
$$+\left(\frac{\partial p'_{xz}}{\partial x}+\frac{\partial p'_{yz}}{\partial y}+\frac{\partial p'_{zz}}{\partial z}+\frac{\partial}{\partial t}\frac{C'}{\Delta}+\frac{d\Omega}{dz}\right)\delta z$$
$$+\left(\frac{\partial q'_{xx}}{\partial x}+\frac{\partial q'_{yx}}{\partial y}+\frac{\partial q'_{zx}}{\partial z}+\frac{\partial}{\partial t}\frac{P'}{\Delta}+p'_{yz}-p'_{zy}\right)\delta I$$
$$+\left(\frac{\partial q'_{xy}}{\partial x}+\frac{\partial q'_{yy}}{\partial y}+\frac{\partial q'_{zy}}{\partial z}+\frac{\partial}{\partial t}\frac{Q'}{\Delta}+p'_{yx}-p'_{xz}\right)\delta J$$

$$+\left(\frac{\partial q'_{xz}}{\partial x}+\frac{\partial q'_{yz}}{\partial y}+\frac{\partial q'_{zz}}{\partial z}+\frac{\partial}{\partial t}\frac{R'}{\Delta}+p'_{xy}-p'_{yx}\right)\delta K\Bigg\}dxdydzdt,$$

in which we have set, with the notations of sec. **72** and **73:**

$$\begin{aligned}
&\frac{A'}{\Delta}=\frac{A}{\Delta}=-(A')[\xi_1]-(B')[\xi_2]-(C')[\xi_3]-(P')[p_1]-(Q')[p_2]-(R')[p_3],\\
&\frac{B'}{\Delta}=\frac{B}{\Delta}=-(A')[\eta_1]-(B')[\eta_2]-(C')[\eta_3]-(P')[q_1]-(Q')[q_2]-(R')[q_3],\\
&\frac{C'}{\Delta}=\frac{C}{\Delta}=-(A')[\varsigma_1]-(B')[\varsigma_2]-(C')[\varsigma_3]-(P')[r_1]-(Q')[r_2]-(R')[r_3],\\
&\frac{P'}{\Delta}=\frac{P}{\Delta}=[P]=\alpha(P')+\beta(Q')+\gamma(R'),\\
&\frac{Q'}{\Delta}=\frac{Q}{\Delta}=[Q]=\alpha'(P')+\beta'(Q')+\gamma'(R'),\\
&\frac{R'}{\Delta}=\frac{R}{\Delta}=[R]=\alpha''(P')+\beta''(Q')+\gamma''(R'),\\
&p'_{xx}=-\sum\{[A_i][\xi_i]+[P_i][p_i]\}\\
&p'_{yx}=-\sum\{[B_i][\xi_i]+[Q_i][p_i]\}\\
&p'_{zx}=-\sum\{[C_i][\xi_i]+[R_i][p_i]\}
\end{aligned}$$

with analogous formulas for $p'_{xy}, p'_{yy}, p'_{zy}; p'_{xz}, p'_{yz}, p'_{zz}$ that are obtained by changing $[\xi_i]$, $[p_i]$ into $[\eta_i]$, $[q_i]$, and then into $[\zeta_i]$, $[r_i]$, respectively, and, in addition:

$$\begin{aligned}
q'_{xx}&=\alpha[P_1]+\beta[P_2]+\gamma[P_3],\\
q'_{yx}&=\alpha[Q_1]+\beta[Q_2]+\gamma[Q_3],\\
q'_{zx}&=\alpha[R_1]+\beta[R_2]+\gamma[R_3],
\end{aligned}$$

with analogous formulas for $q'_{xy}, q'_{yy}, q'_{zy}; q'_{xz}, q'_{yz}, q'_{zz}$ that are obtained by changing α, β, γ, into α', β', γ', and then into $\alpha'', \beta'', \gamma''$, respectively.

Observe that:

$$\frac{\partial}{\partial t}\frac{A'}{\Delta}=\frac{d}{dt}\frac{A'}{\Delta}-\frac{dx}{dt}\frac{\partial}{\partial x}\frac{A'}{\Delta}-\frac{dy}{dt}\frac{\partial}{\partial y}\frac{A'}{\Delta}-\frac{dz}{dt}\frac{\partial}{\partial z}\frac{A'}{\Delta}$$

may, by virtue of the relation:

$$\frac{1}{\Delta}\frac{d\Delta}{dt}=\frac{\partial}{\partial x}\frac{dx}{dt}+\frac{\partial}{\partial y}\frac{dy}{dt}+\frac{\partial}{\partial z}\frac{dz}{dt},$$

be written:

$$\frac{\partial}{\partial t}\frac{A'}{\Delta} = \frac{1}{\Delta}\frac{dA'}{dt} - \frac{\partial}{\partial x}\left(\frac{A'}{\Delta}\frac{dx}{dt}\right) - \frac{\partial}{\partial y}\left(\frac{A'}{\Delta}\frac{dy}{dt}\right) - \frac{\partial}{\partial z}\left(\frac{A'}{\Delta}\frac{dz}{dt}\right);$$

similarly:

$$\frac{\partial}{\partial t}\frac{P'}{\Delta} = \frac{1}{\Delta}\frac{dP'}{dt} - \frac{\partial}{\partial x}\left(\frac{P'}{\Delta}\frac{dx}{dt}\right) - \frac{\partial}{\partial y}\left(\frac{P'}{\Delta}\frac{dy}{dt}\right) - \frac{\partial}{\partial z}\left(\frac{P'}{\Delta}\frac{dz}{dt}\right).$$

On the other hand, $A' = A, P' = P;$ from this it results that one has:

$$\frac{\partial p'_{xx}}{\partial x} + \frac{\partial p'_{yx}}{\partial y} + \frac{\partial p'_{zx}}{\partial z} + \frac{\partial}{\partial t}\frac{A'}{\Delta} + \frac{d\Omega}{dt} = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} + \frac{1}{\Delta}\frac{dA}{dt},$$

and:

$$\frac{\partial q'_{xx}}{\partial x} + \frac{\partial q'_{yx}}{\partial y} + \frac{\partial q'_{zx}}{\partial z} + \frac{\partial}{\partial t}\frac{P'}{\Delta} + p'_{yz} - p'_{zy}$$
$$= \frac{\partial q_{xx}}{\partial x} + \frac{\partial q_{yx}}{\partial y} + \frac{\partial q_{zx}}{\partial z} + \frac{1}{\Delta}\frac{dP}{dt} + p_{yz} - p_{zy} + \frac{C}{\Delta}\frac{dy}{dt} - \frac{B}{\Delta}\frac{dz}{dt},$$

with analogous relations.

The force and exterior moment thus have the same definition as in sec. **62**, **63**. However, *the same is not the case for the effort and the moment of deformation;* from sec. **72**, **76**, we have:

$$(93)\qquad \begin{cases} p_{xx} - p'_{xx} = \pi_{xx} = \Omega - \dfrac{A}{\Delta}\dfrac{dx}{dt}, \\ p_{yx} - p'_{yx} = \pi_{yx} = \quad -\dfrac{A}{\Delta}\dfrac{dy}{dt}, \\ p_{zx} - p'_{zx} = \pi_{zx} = \quad -\dfrac{A}{\Delta}\dfrac{dz}{dt}, \end{cases}$$

with analogous expressions for $\pi_{xy}, \pi_{yy}, \pi_{zy}$; $\pi_{xz}, \pi_{yz}, \pi_{zz}$ that are obtained by cyclic permutation of *A*, *B*, *C*, and *x, y, z;* in addition:

$$(93')\qquad \begin{cases} q_{xx} - q'_{xx} = \chi_{xx} = -\dfrac{P}{\Delta}\dfrac{dx}{dt}, \\ q_{yx} - q'_{yx} = \chi_{yx} = -\dfrac{P}{\Delta}\dfrac{dy}{dt}, \\ q_{zx} - q'_{zx} = \chi_{zx} = -\dfrac{P}{\Delta}\dfrac{dz}{dt}, \end{cases}$$

with analogous expressions for $\chi_{xy}, \chi_{yy}, \chi_{zy}$; $\chi_{xz}, \chi_{yz}, \chi_{zz}$ that are obtained by cyclic permutation of *A*, *B*, *C*, and *x*, *y*, *z*.

79. The method of variable action applied to the Eulerian conception of action as expressed by the Lagrange variables. – We shall once more develop the Eulerian concept of action with the Lagrange variables. We begin with the integral:

$$\int_{t_1}^{t_2} \iiint_{S_0} W dx_0 dy_0 dz_0 dt,$$

in which the domain of integration over x_0, y_0, z_0 now varies with time t, and corresponds to the fixed integration domain that is described by the point (x, y, z).

Following the exposition of JORDAN, we have:

$$\begin{aligned}
&\delta \int_{t_1}^{t_2} \iiint_{S_0} W dx_0 dy_0 dz_0 dt \\
&= \int_{t_1}^{t_2} \iiint_{S_0} \Bigg[\sum \left(\frac{\partial W}{\partial \xi_i} \delta\xi_i + \cdots + \frac{\partial W}{\partial r_i} \delta r_i \right) + \frac{\partial W}{\partial \xi} \delta\xi + \cdots + \frac{\partial W}{\partial r} \delta r \\
&\qquad + \frac{d}{dx_0} \big(W(\delta x_0) \big) + \frac{d}{dy_0} \big(W(\delta y_0) \big) + \frac{d}{dz_0} \big(W(\delta z_0) \big) \Bigg] dx_0 dy_0 dz_0 dt,
\end{aligned}$$

in which (δx_0), (δy_0), (δz_0) are defined by formulas (86) by means of the auxiliary variables δx, δy, δz.

The sequence of calculations resembles those that we encountered in the dynamics of deformable media; at the same time, a difference was introduced, insofar as differentiation with respect to time is concerned. This time, one may not change the order of integrating over time and integration over the domain of variables x_0, y_0, z_0. One will therefore apply reasoning analogous to that of sec. **76**. One first introduces only the derivatives with respect to time in the form $\frac{\partial}{\partial t}$ by using the formula:

$$\frac{\partial \mathcal{F}}{\partial t} = \frac{d\mathcal{F}}{dt} + \frac{\partial \mathcal{F}}{\partial x_0}\frac{\partial x_0}{\partial t} + \frac{\partial \mathcal{F}}{\partial y_0}\frac{\partial y_0}{\partial t} + \frac{\partial \mathcal{F}}{\partial z_0}\frac{\partial z_0}{\partial t}$$

in which $\frac{\partial x_0}{\partial t}, \frac{\partial y_0}{\partial t}, \frac{\partial z_0}{\partial t}$ denote the derivatives with respect to t of the functions x_0, y_0, z_0, of x, y, z, t that one infers from formulas (66′). Upon using the notations we introduced before, the preceding formulas may be written:

$$\frac{\partial \mathcal{F}}{\partial t} = \frac{d\mathcal{F}}{dt} - (\xi)\frac{\partial \mathcal{F}}{\partial x_0} - (\eta)\frac{\partial \mathcal{F}}{\partial y_0} - (\varsigma)\frac{\partial \mathcal{F}}{\partial z_0}. \tag{94}$$

If one has a term of the form:

$$\int_{t_1}^{t_2} \iiint_{S_0} g \frac{\partial h}{\partial t} dx_0 dy_0 dz_0 dt$$

then one writes:

$$\int_{t_1}^{t_2}\iiint_S \frac{g}{\Delta}\frac{\partial h}{\partial t}dxdydzdt,$$

and, upon integrating by parts:

$$\iiint_S \left\{\frac{g}{\Delta}h\right\}_{t_1}^{t_2} dxdydz - \int_{t_1}^{t_2}\iiint_S h\frac{\partial}{\partial t}\left(\frac{g}{\Delta}\right)dxdydzdt,$$

$$= \left\{\iiint_S \frac{g}{\Delta}hdxdydz\right\}_{t_1}^{t_2} - \int_{t_1}^{t_2}\iiint_S h\frac{\partial}{\partial t}\left(\frac{g}{\Delta}\right)dxdydzdt,$$

i.e., reverting to the variables x_0, y_0, z_0:

$$= \left\{\iiint_{S_0} ghdx_0dy_0dz_0\right\}_{t_1}^{t_2} - \int_{t_1}^{t_2}\iiint_S h\Delta\frac{\partial}{\partial t}\left(\frac{g}{\Delta}\right)dx_0dy_0dz_0dt.$$

Having said this, from the previous formulas for the dynamics of deformable media and from (94), we obtain, upon integrating by parts:

$$\begin{aligned}
&\delta\int_{t_1}^{t_2}\iiint_{S_0} Wdx_0dy_0dz_0dt \\
&= \int_{t_1}^{t_2}\iint_{S_0}(F_0'\delta' x + G_0'\delta' y + +H_0'\delta' z + I_0'\delta I' + J_0'\delta J' + K_0'\delta K')d\sigma_0 dt \\
&+ \left\{\iiint_{S_0}(A'\delta' x + B'\delta' y + C'\delta' z + P'\delta I' + Q'\delta J' + R'\delta K')dx_0dy_0dz_0\right\}_{t_1}^{t_2} \\
&- \int_{t_1}^{t_2}\iiint_{S_0}(X_0'\delta' x + Y_0'\delta' y + Z_0'\delta' z + L_0'\delta I' + M_0'\delta J' + N_0'\delta K')dx_0dy_0dz_0dt,
\end{aligned}$$

upon setting:

$$\begin{aligned}
F_0' &= l_0\left\{\frac{\partial W}{\partial \xi_1} - (\xi_1)W - (\xi)\frac{\partial W}{\partial \xi}\right\} + m_0\left\{\frac{\partial W}{\partial \xi_2} - (\xi_2)W - (\eta)\frac{\partial W}{\partial \xi}\right\} + n_0\left\{\frac{\partial W}{\partial \xi_3} - (\xi_3)W - (\varsigma)\frac{\partial W}{\partial \xi}\right\}, \\
G_0' &= l_0\left\{\frac{\partial W}{\partial \eta_1} - (\eta_1)W - (\xi)\frac{\partial W}{\partial \eta}\right\} + m_0\left\{\frac{\partial W}{\partial \eta_2} - (\eta_2)W - (\eta)\frac{\partial W}{\partial \eta}\right\} + n_0\left\{\frac{\partial W}{\partial \eta_3} - (\eta_3)W - (\varsigma)\frac{\partial W}{\partial \eta}\right\}, \\
H_0' &= l_0\left\{\frac{\partial W}{\partial \varsigma_1} - (\varsigma_1)W - (\xi)\frac{\partial W}{\partial \varsigma}\right\} + m_0\left\{\frac{\partial W}{\partial \varsigma_2} - (\varsigma_2)W - (\eta)\frac{\partial W}{\partial \varsigma}\right\} + n_0\left\{\frac{\partial W}{\partial \varsigma_3} - (\varsigma_3)W - (\varsigma)\frac{\partial W}{\partial \varsigma}\right\},
\end{aligned}$$

$$\begin{aligned}
I_0' &= l_0\left\{\frac{\partial W}{\partial p_1} - (\xi)\frac{\partial W}{\partial p}\right\} + m_0\left\{\frac{\partial W}{\partial p_2} - (\eta)\frac{\partial W}{\partial p}\right\} + n_0\left\{\frac{\partial W}{\partial p_3} - (\varsigma)\frac{\partial W}{\partial p}\right\}, \\
J_0' &= l_0\left\{\frac{\partial W}{\partial q_1} - (\xi)\frac{\partial W}{\partial q}\right\} + m_0\left\{\frac{\partial W}{\partial q_2} - (\eta)\frac{\partial W}{\partial q}\right\} + n_0\left\{\frac{\partial W}{\partial q_3} - (\varsigma)\frac{\partial W}{\partial q}\right\},
\end{aligned}$$

$$K_0' = l_0\left\{\frac{\partial W}{\partial r_1} - (\xi)\frac{\partial W}{\partial r}\right\} + m_0\left\{\frac{\partial W}{\partial r_2} - (\eta)\frac{\partial W}{\partial r}\right\} + n_0\left\{\frac{\partial W}{\partial r_3} - (\varsigma)\frac{\partial W}{\partial r}\right\},$$

$$X_0' = \frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial \xi_1} - (\xi)\frac{\partial W}{\partial \xi}\right) + \frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial \xi_2} - (\eta)\frac{\partial W}{\partial \xi}\right) + \frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial \xi_3} - (\varsigma)\frac{\partial W}{\partial \xi}\right)$$
$$+ \sum \left(q_i\frac{\partial W}{\partial \varsigma_i} - r_i\frac{\partial W}{\partial \eta_i}\right) + \Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial \xi}\right) + q\frac{\partial W}{\partial \varsigma} - r\frac{\partial W}{\partial \eta},$$

$$Y_0' = \frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial \eta_1} - (\xi)\frac{\partial W}{\partial \eta}\right) + \frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial \eta_2} - (\eta)\frac{\partial W}{\partial \eta}\right) + \frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial \eta_3} - (\varsigma)\frac{\partial W}{\partial \eta}\right)$$
$$+ \sum \left(r_i\frac{\partial W}{\partial \xi_i} - p_i\frac{\partial W}{\partial \varsigma_i}\right) + \Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial \eta}\right) + r\frac{\partial W}{\partial \xi} - p\frac{\partial W}{\partial \varsigma},$$

$$Z_0' = \frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial \varsigma_1} - (\xi)\frac{\partial W}{\partial \varsigma}\right) + \frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial \varsigma_2} - (\eta)\frac{\partial W}{\partial \varsigma}\right) + \frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial \varsigma_3} - (\varsigma)\frac{\partial W}{\partial \varsigma}\right)$$
$$+ \sum \left(p_i\frac{\partial W}{\partial \eta_i} - q_i\frac{\partial W}{\partial \xi_i}\right) + \Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial \varsigma}\right) + p\frac{\partial W}{\partial \eta} - q\frac{\partial W}{\partial \xi},$$

$$L_0' = \frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial p_1} - (\xi)\frac{\partial W}{\partial p}\right) + \frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial p_2} - (\eta)\frac{\partial W}{\partial p}\right) + \frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial p_3} - (\varsigma)\frac{\partial W}{\partial p}\right)$$
$$+ \sum \left(q_i\frac{\partial W}{\partial r_i} - r_i\frac{\partial W}{\partial q_i} + \eta_i\frac{\partial W}{\partial \varsigma_i} - \varsigma_i\frac{\partial W}{\partial \eta_i}\right) + \Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial p}\right) + q\frac{\partial W}{\partial r} - r\frac{\partial W}{\partial q} + \eta\frac{\partial W}{\partial \varsigma} - \varsigma\frac{\partial W}{\partial \eta},$$

$$M_0' = \frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial q_1} - (\xi)\frac{\partial W}{\partial q}\right) + \frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial q_2} - (\eta)\frac{\partial W}{\partial q}\right) + \frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial q_3} - (\varsigma)\frac{\partial W}{\partial q}\right)$$
$$+ \sum \left(r_i\frac{\partial W}{\partial p_i} - p_i\frac{\partial W}{\partial r_i} + \varsigma_i\frac{\partial W}{\partial \xi_i} - \xi_i\frac{\partial W}{\partial \varsigma_i}\right) + \Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial q}\right) + r\frac{\partial W}{\partial p} - p\frac{\partial W}{\partial r} + \varsigma\frac{\partial W}{\partial \xi} - \xi\frac{\partial W}{\partial \varsigma},$$

$$N_0' = \frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial r_1} - (\xi)\frac{\partial W}{\partial r}\right) + \frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial r_2} - (\eta)\frac{\partial W}{\partial r}\right) + \frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial r_3} - (\varsigma)\frac{\partial W}{\partial r}\right)$$
$$+ \sum \left(p_i\frac{\partial W}{\partial q_i} - q_i\frac{\partial W}{\partial p_i} + \xi_i\frac{\partial W}{\partial \eta_i} - \eta_i\frac{\partial W}{\partial \xi_i}\right) + \Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial r}\right) + p\frac{\partial W}{\partial q} - q\frac{\partial W}{\partial p} + \xi\frac{\partial W}{\partial \eta} - \eta\frac{\partial W}{\partial \xi}.$$

We may observe that by virtue of (94) X_0', for example, may be written:

$$X_0' = \sum \left(\frac{\partial}{\partial \rho_i}\frac{\partial W}{\partial \xi_i} + q_i\frac{\partial W}{\partial \varsigma_i} - r_i\frac{\partial W}{\partial \eta_i}\right) + \frac{d}{dt}\frac{\partial W}{\partial \xi} + q\frac{\partial W}{\partial \varsigma} - r\frac{\partial W}{\partial \eta}$$
$$-\left(\frac{1}{\Delta}\frac{\partial \Delta}{\partial t} + \frac{\partial(\xi)}{\partial x_0} + \frac{\partial(\eta)}{\partial y_0} + \frac{\partial(\varsigma)}{\partial z_0}\right)\frac{\partial W}{\partial \xi};$$

however, one has:

$$\frac{1}{\Delta}\frac{\partial\Delta}{\partial t} = -\left(\frac{\partial(\xi)}{\partial x_0} + \frac{\partial(\eta)}{\partial y_0} + \frac{\partial(\varsigma)}{\partial z_0}\right),$$

and, as a result, X_0' *has the same value:*

$$X_0' = \sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\xi_i} + q_i\frac{\partial W}{\partial\varsigma_i} - r_i\frac{\partial W}{\partial\eta_i}\right) + \frac{d}{dt}\frac{\partial W}{\partial\xi} + q\frac{\partial W}{\partial\varsigma} - r\frac{\partial W}{\partial\eta},$$

as in sec. **62**; *the same remarks apply to* $Y_0', Z_0', L_0', M_0', N_0'$. However, *the same is not true for the effort and moment of deformation;* by simple transformations, one once more recovers relations (93) and $(93')$ of sec. **78**.

80. The notion of radiation of the energy of deformation and motion. – We have seen that the density of energy of deformation and motion, when expressed as a function of the Lagrangian arguments and referred to the space of (x_0, y_0, z_0), is:

(95) $$E = \xi\frac{\partial W}{\partial\xi} + \eta\frac{\partial W}{\partial\eta} + \varsigma\frac{\partial W}{\partial\varsigma} + p\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial q} + r\frac{\partial W}{\partial r} - W;$$

this same density, *when referred to the space of* (x, y, z) and expressed by means of the function Ω of the Eulerian arguments (ξ_i), (η_i), (ζ_i), (p_i), (q_i), (r_i); (ξ), (η), (ζ), (p), (q), (r) is:

(96) $$\frac{E}{\Delta} = (\xi)\frac{\partial W}{\partial(\xi)} + (\eta)\frac{\partial W}{\partial(\eta)} + (\varsigma)\frac{\partial W}{\partial(\varsigma)} + (p)\frac{\partial W}{\partial(p)} + (q)\frac{\partial W}{\partial(q)} + (r)\frac{\partial W}{\partial(r)} - \Omega.$$

This result is obtained either by transforming expression (95) by means of the relations that we indicated before that exist between the Lagrangian arguments and the Eulerian arguments, or by directly repeating the reasoning of sec. **65** on the elementary work:

$$dt\left\{\iiint_{S_0}(\xi X_0' + \eta Y_0' + \varsigma Z_0' + pL_0' + qM_0' + rN_0')\,dx_0\,dy_0\,dz_0 - \iint_{S_0}(\xi F_0' + \eta G_0' + \varsigma H_0' + pI_0' + qJ_0' + rK_0')\,d\sigma_0\right\},$$

that the forces and external moments and the efforts and external moments of deformation exert on the portion (M) of the medium that the portion (M_0) of the natural state occupies at the instant t. By this latter path, we recover the expression:

$$dt\left\{\iiint_{S_0}\frac{dE}{dt}\,dx_0\,dy_0\,dz_0\right\}$$

for the elementary work, in which Ω is assumed to be independent of t.

If we observe that we has the following identity:

$$\frac{1}{\Delta}\frac{dE}{dt} = \frac{\partial}{\partial t}\left(\frac{E}{\Delta}\right) + \frac{\partial}{\partial x}\left(\frac{E}{\Delta}\frac{dx}{dt}\right) + \frac{\partial}{\partial y}\left(\frac{E}{\Delta}\frac{dy}{dt}\right) + \frac{\partial}{\partial z}\left(\frac{E}{\Delta}\frac{dz}{dt}\right),$$

which was employed by POINCARÉ in the memoir that was cited in sec. **77**, and which we apply to an arbitrary function, then we arrive at the following new expression:

$$dt\left\{\frac{\partial}{\partial t}\iiint_S \frac{E}{\Delta}\,dxdydz \right.$$
$$\left. + \iiint_S \left[\frac{\partial}{\partial x}\left(\frac{E}{\Delta}\frac{dx}{dt}\right) + \frac{\partial}{\partial y}\left(\frac{E}{\Delta}\frac{dy}{dt}\right) + \frac{\partial}{\partial z}\left(\frac{E}{\Delta}\frac{dz}{dt}\right)\right] dxdydz\right\},$$

or:

$$dt\left\{\frac{\partial}{\partial t}\iiint_S \frac{E}{\Delta}\,dxdydz + \iint_S \frac{E}{\Delta}\left(l\frac{dx}{dt} + m\frac{dy}{dt} + n\frac{dz}{dt}\right)d\sigma\right\}, \tag{97}$$

for the elementary work.

The second integral in (97) expresses *the flux of energy of deformation and motion* across *a fixed surface S* in the deformed body.

Now consider the Eulerian conception of action. In the preceding sections we confirmed that the values of the forces and external moments remain the same, but that the following terms disappear from the expressions for the efforts p_{xx}, p_{xy}, p_{xz}:

$$\pi_{xx} = \Omega - \frac{A}{\Delta}\frac{dx}{dt},$$
$$\pi_{xy} = \quad -\frac{B}{\Delta}\frac{dx}{dt},$$
$$\pi_{xz} = \quad -\frac{C}{\Delta}\frac{dx}{dt},$$

and the following terms disappear from the expressions for the moments of deformation q_{xx}, q_{xy}, q_{xz}:

$$\chi_{xx} = -\frac{P}{\Delta}\frac{dx}{dt},$$
$$\chi_{xy} = -\frac{Q}{\Delta}\frac{dx}{dt},$$
$$\chi_{xz} = -\frac{R}{\Delta}\frac{dx}{dt},$$

with analogous expressions for the quantities π_{yz}, π_{yy}, π_{yz}, π_{zx}, π_{zy}, π_{zz}, and χ_{yz}, χ_{yy}, χ_{yz}, χ_{zx}, χ_{zy}, χ_{zz} . From this, it results that the elementary work that is obtained in the preceding must be added to a new surface integral that has the expression:

$$dt\left\{\iint_s\left(l\frac{dx}{dt}+m\frac{dy}{dt}+n\frac{dz}{dt}\right)\left[\Omega-\frac{1}{\Delta}\left(A\frac{dx}{dt}+B\frac{dy}{dt}+C\frac{dz}{dt}\right)\right.\right.$$
$$\left.\left.-\{p(P')+q(Q')+r(R')\right]d\sigma\right\}.$$

One may call this new integral *the flux of radiant energy crossing the boundary* S *of the deformed body.*

The reasoning made in sec. **64**, which was based on the *Euclidean invariance* of the action density, no longer leads to the same conclusions for the forces and external moments as it does for the *new* efforts and external moments of deformation. This may be interpreted by saying that the new efforts and moments of deformation no longer satisfy what POINCARÉ called the *principle of reaction.* This latter conclusion is likewise recovered, as one knows, in the electric theory of LORENTZ. However, the existence of radiation that we just showed permits us to approach the efforts and moments of deformation $\pi_{xx}, \pi_{yx}, \ldots, \chi_{xx}, \chi_{yx}, \ldots$ as being what MAXWELL, from considerations deduced from the electromagnetic theory of light, and BARTOLI, from those of thermodynamics, called the *pressure of radiant energy,* and one may therefore retrieve the *principle of reaction.*