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TO

PROFESSOR LOUIS DE BROGLIE

MEMBER OF THE INSTITUTE

FIRST THESIS.

CONTRIBUTION TO THE STUDY

OF

DIRAC'S THEORY OF THE ELECTRON

FOREWORD

The various studies that constitute this work are, so to speak, placed under the double sign of relativity and quanta. However, the first chapter is the only one in which the very arduous problem that is posed by the relationships between those two great theories is partially begun – only once moreover, and only in the context of Dirac's theory. The mechanism by which Dirac's theory arrives at a reconciliation, if not a harmonization, of the two formalisms is truly paradoxical, and seems to hide an enigma.

However, having accepted that point (and some suitably-adopted conventions), the questions that we shall address in the following chapters will remain "intrinsic" to concepts that are either relativistic or quantum in their present form. We will be dealing with an extension of relativistic dynamics that is intended to include the notion of proper kinetic moment – or *spin* (Chap. II), and then a study of and an attempt to interpret some quantities and relations from Dirac's theory that are of interest to the statistical fluid (Chap. III), a comparison of the classical electromagnetism of polarized media with Dirac's theory, and a return to several questions that remain pending (Chap. IV).

When we need to refer to certain results from classical special relativity, we shall cite our earlier book on that subject, for the very simple reason that we shall preserve its notations here, and that to our way of thinking that first book gave us a way of approaching the present topic. We refer to the classic work of R. Becker for the questions in the electromagnetism of polarized media that we have, unfortunately, left aside.

Notations used. – Throughout the entire work, u, v, w will denote a *circular* permutation of the spatial indices 1, 2, 3, and the temporal index will then be equal to 4 explicitly. The two sets i, j, k, l and p, q, r, s = 1, 2, 3, 4 will be the set of *world-tensor indices* and the set of *matrix* or *spinor* indices that belong to Dirac's theory, resp. In general, we shall use the summation convention over dummy indices from tensor calculus, except that in some of the calculations in Chapter III, paragraph I, in which we shall introduce some special conventions and use the tensor indices λ, μ, … *without summation* when they are repeated. Finally, according to the usual convention in Dirac's theory, the upper-case Roman indices that are attributed to γ will vary from 1 to 16, and we shall possibly apply the convention of summing over dummy indices to them.

Along with several authors, we consider Dirac's four-component ψ to be a matrix with four rows and one column, while Gordon-Pauli's $\psi^* = \psi^+$ and $\psi^\times = i\,\psi^+\gamma^4$ will then be associated with matrices with one row and four columns. In a manner that is analogous to that of W. Franz ([1]), for example, we introduce, along with the usual partial differential operator that acts on the right, which we denote by $\underset{\rightarrow}{\partial}^i$, the analogous operator $\underset{\leftarrow}{\partial}^i$ that acts on the left, and we shall define the two operators:

$$[\partial^i] = \underset{\rightarrow}{\partial}^i - \underset{\leftarrow}{\partial}^i, \qquad (\partial^i) = \underset{\rightarrow}{\partial}^i + \underset{\leftarrow}{\partial}^i,$$

which, like the Dirac γ matrices, act on both the right and the left. We shall use a dot to *stop* the action of one the preceding operators to the right or left. Finally, we preserve the usual notation ∂^i in order to denote the *un-notated operator*, which acts only to its immediate right ([2]).

Along with the five classical density tensors of Dirac and Darwin,

$$\rho_D = \psi^\times \gamma \psi,$$

we shall also systematically introduce five tensors of the type:

$$\rho_S = \psi^\times [\partial^i]\, \gamma \psi,$$

that we call *Schrödingerian*, due to the fact that they pertain to the current trivector in Schrödinger's original theory. *By definition*, the $\gamma = \gamma^{ij}$ … in these tensors is a product $\gamma^i\,\gamma^j$… of Dirac matrices, in which the *tensor indices i, j, … are essentially assumed to all be distinct.* Indeed, one knows that these γ^{ij} … behave like the components of a completely-antisymmetric *matrix tensor*, so it would be appropriate to *define* the components with two or more equal indices to be zero. A bar over a γ or a completely-antisymmetric tensor will denote the dual of that quantity. An exception to that is in Chapter III, paragraph I, where we shall use a *partial double bar over two indices* to overbar third-rank, completely-antisymmetric tensors.

All of our notations are in accord with those of our book on special relativity, *up to a change in sign in the definition of the quadri-potential A^i for the field H^{ij}* ([3]); for example, we shall often use the duals $ic\,\delta u^i$ and $ic\,\delta s^{kl}$ of the integration elements $[dx^i\,dx^j\,dx^k]$ and $[dx^i\,dx^j]$, respectively. Moreover, we set:

$$\delta u = ic\,\delta u^4 = [dx^1\,dx^2\,dx^3] \quad \text{and} \quad \delta u^w = ic\,\delta u^{w4} = [dx^u\,dx^v]$$

([1]) "Zur Methodik der Dirac-Gleichung," See, pp. 404.

([2]) With t denoting time, we shall generally write ∂^t or ∂_t for $\partial / \partial t$.

([3]) And up to a change in sign in the definition of the *spatial* vector product, with no repercussions in the four-dimensional formulas.

in order to denote the usual volume element and the three components of the area element, respectively. We shall make use of the general formula for the transformation of multiple integrals:

$$\int_\mu A_{\alpha\beta\ldots\rho\sigma\ldots}\ [dx^\alpha\, dx^\beta \ldots\, dx^\rho\, dx^\sigma \ldots] = \int_{\mu+1} \partial_\omega A_{\alpha\beta\ldots\rho\sigma\ldots}\ [dx^\alpha\, dx^\beta \ldots\, dx^\rho\, dx^\sigma \ldots dx^\omega]\ ,$$

with several reprises. The integration elements of rank p $[dx^\lambda\ dx^\mu \ldots]$ are completely-antisymmetric tensors that are defined by the (signed) determinants that are extracted from the matrix with p rows and n columns:

$$\| dx_i^j \|,$$

in which n is the number of dimensions of the space considered ([1]).

In the present work, as in our cited book, we shall use what one can call the *Heaviside e. m. units*, which are units in which the electromagnetic mass-impulse of a point charge will have the expression QA^i ([2]). For the electron, one will have:

$$Q = -\frac{e}{c},$$

in such a way that its electromagnetic mass-impulse can be written $- e / c\, A^i$.

The interesting equations or relations in the rest of this monograph will be numbered in brackets; on the contrary, the ones that serve only as intermediate calculations shall be denoted by a symbol in brackets, when necessary.

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([1]) *La Relativité restreinte*, pp. 6-7 and 31.
([2]) *Op. cit.*, pp. 34-35, 48, and 62.

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CHAPTER I

ON THE RELATIONSHIP OF RELATIVITY AND QUANTA TO DIRAC'S THEORY.

1. The problem of the relationship of special relativity and wave mechanics to Dirac's theory (if one is to "restrict" it with respect to the combined relativistic and quantum theory) is already sufficiently difficult to merit a deeper examination. The present chapter does not actually aim to study the restricted problem in its entirety, but only to examine certain special aspects of it in detail.

A first remark, which is classical, is the following one: Whereas non-relativistic wave mechanics arises without difficulty from the work of Schrödinger to create a mechanics of systems of *n* interacting points, up to now, the relativistic wave mechanics of a point that is endowed with spin knows only how to treat the problem of a single point that is embedded in a pre-established field (whose formal basis remains the Dirac equation) with no approximations. That grave situation does not properly belong to wave mechanics, since it is encountered already in pre-quantum dynamics. It is therefore relativity that has failed in that case, but one can remark that the theory of quanta has not contributed to a clarification of the problem.

It seems that the constitution of a relativistic dynamics of *n* interacting points encounters two principal difficulties, which are connected, moreover. The first one results from the replacement of the classical *instantaneous potentials* with potentials that propagate with a finite velocity, which is equal to *c* in the case of electromagnetism. It follows from the fact that each of the *points* will be influenced by the states of the other *points* "of the same wave as it," which will be states that are located on the *light hypercone* that has that point for its summit. Therefore, if one would like to treat the problem of the dynamics of *n* points then one would have to consider, at the same time, *n* hypersurfaces of that type instead of the single *simultaneous hyperplane* of classical dynamics. As hard as it is, the problem is physically determined on that basis, and therefore, it is certainly capable of being formulated. The difficulties in its solution, which are perhaps currently insurmountable, are of only a mathematical nature.

One of those difficulties obviously consists of the fact that the world-positions of the *n* points are, in principle, independent of each other, while in the old dynamics, they were all taken to be in the same *simultaneity hyperplane*. Each of the relativistic *points* indeed possesses its own *proper time*, but one does not see *a priori* how to introduce a global evolution parameter for the collective "cloud." One can demand that one would not have any reason to establish such a parameter, either for physical reasons that presently elude us or for mathematical reasons that the effective study of the problem might cause to appear. Perhaps one can then confirm that the *world-positions* that the *n* points take not only belong to the neighborhood of the same space-like hypersurface that displaces in the direction of increasing time, but are also characterized by the increasing values of an *action* function. Without wanting to prejudice the results of such a study (which would be interesting to undertake), it seems to us that these simple considerations will clarify the nature of the problem that was posed, and that the solution that we suggested will, *in* *principle*, respect both the relativistic symmetry and the classical custom of having an *evolution parameter* and a *three-dimensional configuration hypersurface* that is valid for the system collectively.

It is clearly that same *objective evolution parameter* and *objective configuration hypersurface* that one must establish in wave mechanics in order to harmonize the relativistic symmetries and quantum principles completely. Now, it is remarkable that Dirac's theory seems to not only lend itself poorly to such an operation, but even provides a very clear counter-indication. Indeed, one can be tempted to "measure" an objective evolution parameter along the Dirac streamlines, which are time-like, but those lines will not generally admit orthogonal hypersurfaces. Conversely, the quadri-vector density that we call $U^i_{(1)}$ is found to be a world-gradient [*see*, much later, equations (61′ B) and (62 B II)], but it is not necessarily time-like. To our knowledge, Dirac's theory does not provide any quadri-vector field that enjoys both of those properties at once, and both of them seem necessary if one is to be able to succeed in the indicated way.

Another important reason seems to us to doom the progress of any attempt of that kind: *The set of the four matrices γ^i does not possess any relativistic symmetry*, in the sense that the four γ^i can be chosen to be all Hermitian, but not three Hermitian ones and one anti-Hermitian one. One can guarantee that, and we shall show that it is impossible to modify Dirac's theory in such a manner as to remedy that state of affairs. The desired condition is not compatible with the well-known Dirac condition:

$$\tfrac{1}{2}(\gamma^i\,\gamma^j + \gamma^j\,\gamma^i) = \delta^{ij},$$

which is indispensable for Gordon's second-order equation to be a consequence of the theory in the absence of a field. These remarks are important. If it is proved in a definitive manner that it is impossible to introduce an objective evolution parameter into wave mechanics that is analogous to the *proper time* of relativity (or the *cosmic time* of the theories that treat the universe collectively) then one must conclude, for example, that *the Dirac electron, when taken by itself, will ignore time.* The passing of time is manifested only in a macroscopic reference system of observation. On the contrary, one recalls that the only evolution parameter that is endowed with any clear significance in non-cosmic relativity is the *proper time* of material points, or of systems that are sufficiently small that they can be associated with material points.

Not only will the asymmetry of the set of γ^i prevent Dirac's theory from giving relativistic symmetry to the general quantum principles, but *it is precisely a special intervention of the matrix γ^4 that will permit one to reconcile the formalism of these principles with the demands of relativity (in connection with the integration at constant time).* After recalling the principal manifestations of the lack of relativistic symmetry in the general principles of wave mechanics in paragraph I, we shall analyze the very paradoxical mechanism by which Dirac's theory manages to arrive at that reconciliation in paragraph II. The question is closely connected with what we have called the *second principle* of relativity ([1]). We shall essentially show that the condition:

$$S^{+}\gamma^4 S = \gamma^4$$

([1]) *La Relativité restreinte*, pp. 15.

that Von Neumann imposed upon the matrix S of the change of Galilean frame, and which is a condition that makes the special role of γ^4 appear, is nothing but the *matrix expression for the second principle of relativity* (no. **7**). After having shown how those considerations contribute notably to the definition of Tetrode's inertia tensor, we shall use them to specify the tensorial and physical classification of the sixteen γ operators of Dirac's theory (no. **8**).

As for the *matrix expression of the first principle of relativity* ([1]), we shall show that it is provided by the set of two known conditions: The first one, which is very immediate and was stated by Von Neumann and Pauli, is the commutation of S with the *second matrix invariant* $\bar{\gamma} = \gamma^{uvw4}$. The second one, which is due to Pauli, can be written:

$$\tilde{S}BS = B.$$

$\tilde{S}$ denotes the *transpose* of the matrix S, and B is a certain matrix that is introduced by Pauli whose definition we shall recall (no. **4**).

Can we hope that the matrix formulation of the *first* and *second principles* of relativity will give us the means to comprehend the properties of space and time more than is permitted by Minkowski's tensorial laws? Unfortunately, that does not seem to be true. One knows that the direct attachment of two ways of changing Galilean frames to each other in Dirac's theory is quite laborious, which is a situation that constitutes a serious obstacle to the deeper comprehension of what Dirac's theory presents in that topic that is new ([2]). All of that study leaves the impression that despite (or even, one might say, due to) its great ingenuity, Dirac's theory does not actually constitute the last word in what the relativistic quantum theory of the electron must be. In summary, one can say that the profound conflict between relativity and quanta persists even in Dirac's theory, but also that Dirac's theory realizes a *modus vivendi* that is so clever that the conflict, which is already latent, never erupts ([3]). The experimental fact that it fails to be both

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([1]) *Op. cit.*, pp. 10.

([2]) To our knowledge, the reciprocal calculation of the elements of the matrices S and o_i^j relative to the two ways of changing Galilean frames has never been given explicitly in its general form. Along that train of ideas, we cite Pauli's calculation of S as functions of the ε_i^j of the infinitesimal transformation (*Handb. Phys.*, $XXIV_2$, pp. 222), and Möglich's calculation of the S that correspond to three special Lorentz transformations (and also a rotation of the spatial axes) in the case where one adopts a particular representation for the γ^i (Zeit. Phys. **48**, pp. 852).

The general theory of changing Galilean frames in Dirac's theory that was given to us by Dirac, Von Neumann, and Pauli rests entirely upon the *existence theorem* for S that was proved by Pauli; the elegant restricting conditions that were imposed upon S by those authors are then proved by a very indirect method.

([3]) Meanwhile, there is a very delicate point at which the problem demands a deeper study than we have undertaken. From the principles of wave mechanics, a measurement that is made at the instant t will *determine* the wave function ψ_i (x^1, x^2, x^3) at a future time when another measurement fixes that function again. In the non-relativistic universe, there is no problem presented by making the various *measurement hyperplanes* parallel to each other. Things are no longer the same in the relativistic universe for two measurements that are made on the *same system* in *two different Galilean frames*. The measurement hyperplanes will then intersect, and will determine two regions that are the "past" for one measurement and the "future" for the other one, which will be exchanged when one changes the region.

quantum and relativistic proves that the two theories are both "true," in a sense. The theoretical fact of the "conflict" shows that at least one of the two formalisms (and probably both of them) is imperfectly adequate in the eyes of physical reality. In that case, one must not hope to discover a means of "reconciling" relativity and quanta, properly speaking, but the advent of new and more powerful conceptions: The problem will not be of a logical nature, but a physical one.

I. – ON THE LACK OF RELATIVISTIC SYMMETRY IN THE GENERAL PRINCIPLES OF WAVE MECHANICS.

2. Wave mechanics, whether non-relativistic or relativistic, makes any physical quantity r that is attached to the system (viz., a system of interacting points in the non-relativistic case, a single point in the relativistic case) correspond to a linear Hermitian operator R that operates on all of part of the *spatial* coordinates x^u. The time t, which is never a variable that is "operated" upon by an operator R, properly speaking, can figure as a parameter in the definition of R. As for Hermiticity, it is defined at each instant t in a domain of pure space U by the condition:

$$\int_U \varphi^* \cdot R\psi \cdot \delta u = \varepsilon \int_U (R\varphi^*) \cdot \psi \cdot \delta u \qquad \text{with} \qquad \varepsilon = ? + 1.$$

In the non-relativistic case of a system of interacting points, δu denotes the volume element $[dx^1, dx^2, \ldots, dx^{3n}]$ of configuration space. The integrals are taken over the entire domain of interest for the x^u. It is that same purely-spatial domain, when considered at a well-defined "instant t" that enters into the definition of the values of the proper functions of the operator R, as we shall recall in the following number. One then sees that the *evolution variable t* and the *configuration domain U* are *relative* to the Galilean frame of the observer. It seems that this first fundamental asymmetry in the quantum principles is related closely to everything that we shall encounter in the rest of this chapter.

Wave mechanics makes the *spatial coordinates* x^u correspond to operators:

(1) $$\boxed{X^u = x^u \times,}$$

which are obviously Hermitian, and the operators:

(2) $$\boxed{P^u = -\frac{h}{2\pi i}\partial^u,}$$

to the homologous components of the *impulse* (Lagrange's *conjugate momenta*). The Hermiticity of the latter operators results from the classical calculation ([1]):

The difficulties that are raised by this *quantum problem of changing Galilean frames* are certainly considerable. One sees that there is a certain interference between the notions of *relativity* and *quantum subjectivity* that is present.

([1]) In the second expression, it is intended that there should be *no summation* over the index u.

$$\frac{h}{2\pi i}\int_{U_{3n}}(\varphi^*\partial^u\psi+\partial^u\varphi^*\psi)\,\delta u$$
$$=\frac{h}{2\pi i}\int_{U_{3n}}\partial^u(\varphi^*\psi)[dx^1dx^2\cdots dx^{u-1}dx^u dx^{u+1}\cdots dx^{3u}]$$
$$=\frac{h}{2\pi i}\int_{U_{3n-1}}(\varphi^*\psi)[dx^1dx^2\cdots dx^{u-1}dx^{u+1}\cdots dx^{3u}]=0,$$

since the last integral is necessarily zero. Indeed, if that were not true then the integral:

$$\frac{h}{2\pi i}\int_{U_{3n}}(\varphi^*\psi)[dx^1dx^2\cdots dx^{u-1}dx^u dx^{u+1}\cdots dx^{3u}]$$

would diverge, which is a case that is by the very definition of functions in Hilbert space ([1]).

Wave mechanics, whether relativistic or not, makes the *energy* of a system correspond to a certain Hermitian operator H that is a function of the preceding operators ([2]), and consequently operates upon the *spatial* coordinates x^u. Finally, it defines the *wave equation* of the system as being the partial differential equation:

(3) $$\boxed{-\frac{h}{2\pi i}\partial^t\psi=H\psi.}$$

In Schrödinger's pre-relativistic wave mechanics, the interpretation of the operator H is a natural consequence of that of the operators X^u and P^u, in the sense that the operator function $H(P^u, X^u)$ is the exact transposition of the Hamiltonian expression for the energy $\mathcal{H}(p^u, x^u)$ of the pre-relativistic analytical mechanics of the system. That will no longer be true in the relativistic theory of a point that is endowed with spin: Since it will cease to be deduced by simple transposition of a pre-quantum mechanical expression, the *Hamiltonian* operator must be defined especially.

As has been pointed many a time, it is natural to seek to give relativistic symmetry to the preceding definitions by making the "temporal coordinates" $x^4 = ict$ correspond to the operator:

(1′) $$X^4 = x^4\times,$$

which is clearly anti-Hermitian, and the energy, to the operator:

(2′) $$P^4=-\frac{h}{2\pi i}\partial^4=-\frac{1}{ic}\frac{h}{2\pi i}\partial^t.$$

The question of knowing whether the operator X^4 can or cannot *physically* represent time is closely linked with the one that was raised in the preceding number that touched upon

([1]) J. Von NEUMANN, *Mathematische Grundlagen der Quantenmechanik*, pp. 49 and 50.
([2]) For the notion of operators that are functions of operators, *see*, for example, Von Neumann, *op. cit.*, pp. 46 *et seq.*

the search for an *objective* evolution parameter in wave mechanics; we shall return to that problem here. On the subject of the operator P^4, it poses two questions that are, in fact, characteristic of the ambiguous state of the relationship between relativity and quanta:

1. Does the operator P^4 correspond physically to the energy of a point that is endowed with spin?

That idea does not seem to be impossible to justify in the case of a free point. However, in the problems of the quantization of the atom, into which a potential energy of interaction enters, it is the operator H that represents the energy, and which permits the effective calculation of its discontinuous spectrum. Nevertheless, although they are not *equivalent*, the operators icP^4 and H will yield the same result when one applies them to the wave function ψ; that will be true, by virtue of the fundamental wave equation (3).

2. Is the operator P^4 anti-Hermitian?

One cannot confirm that. The calculation that was given in the context of P^u fails to show that, since the differential dx^4 does not figure in δu; however, the operator $\frac{1}{ic}H$ is, in fact, anti-Hermitian.

3. Having recalled these preliminaries, the most general statement that one can make of the *principles of the interpretation* of wave mechanics is the following one: Let $E(r)$ be the projector that yields the decomposition of the Hermitian operator R at the instant t ([1]).

1. The probability that a measurement that is made at the instant t will yield a value for the quantity r that is found in a given interval Δr is ([2]):

$$\Delta W_t = \int_U \psi_t^* \cdot \Delta E\, \psi_t \cdot \delta u \,. \tag{4}$$

2. If the measurement shows that the quantity r effectively has a value that is found in the interval Δr then one can affirm that at the instant t, the new wave function ψ that is "created" by the measurement will be a *mixture* of the linearly-independent functions that are contained in the Hilbertian subspace ΔE ([3]).

Upon applying the formula (4) to the operator X^u that represents a coordinate, one can show ([4]) that the probability of "finding" the point (in the mechanics of systems, the figurative point of the system) in the volume element δu at the instant t is $(\psi^*\psi)\ \delta u$, so it will follow that the *probable mean point* that is provided by a large number of

([1]) Von NEUMANN, *op. cit.*¸ pp. 61.
([2]) *Op. cit.*, pp. 105.
([3]) *Op. cit.*, pp. 105.
([4]) *Op. cit.*, pp. 117 *et seq.*

measurements that are made at the instant t of the system that is described by the same ψ will have the coordinates ([1]):

$$[5x] \qquad x^u = \int x^u (\psi^* \psi)\, \delta u \ .$$

In non-relativistic wave mechanics, and in the case of the single corpuscle of proper mass m_0, that mean point will then appear as the barycenter of a fictitious statistical fluid whose mass density will be defined by the *real* quantity:

$$\rho = m_0\, (\psi^* \psi).$$

A consequence of the preceding general statement is that the probable mean value $\overline{r}$ of the quantity r, which is the result of a large number of measurements that are made at the instant t on the system that is described by that same ψ, will be ([2]):

$$(5) \qquad \overline{r}_t = \int_t \psi^* \cdot R\psi \cdot \delta u \ .$$

That probable mean value thus appears as the integral of a *fictitious statistical density* $\psi^* R \psi$ over the entire domain U at the instant t. More exactly, we remark that the asymmetry in the formulas (4) and (5) has the consequence that the Hermiticity of R (or ΔE) does not generally permit one to confirm the reality of the statistical density that was just defined. Under these conditions, it is advantageous to symmetrize those formulas *by appealing to the Hermiticity property* of the operators that they contain; for example, formula (5) will then become:

$$(5') \qquad \overline{r}_t = \int \psi^* \cdot R\psi \cdot \delta u = \int (R\psi)^* \cdot \psi \cdot \delta u = \tfrac{1}{2} \int \{\psi^* \cdot R\psi + (R\psi)^* \cdot \psi\}\, \delta u \ ,$$

and one will see that the newly-defined statistical density { } is indeed real upon taking the complex conjugate ([3]).

We apply the principle that is expressed by formula (5′) to the operator P^u that represents a component of the impulse; we get:

$$\overline{p}_u = -\frac{h}{4\pi i} \int \{\psi^* \cdot \partial^u \psi - \partial^u \psi^* \cdot \psi\}\, \delta u = -\frac{h}{4\pi i} \int \psi^* [\partial^u] \psi \cdot \delta u \ .$$

Conforming to our general conventions of the Foreword, we have defined the *antisymmetric partial differential operator:*

([1]) The general formula (5), which will be given in a moment, leads directly to that result, but without affording it the detailed description in the text.

([2]) Von NEUMANN, *op. cit.*, pp. 105.

([3]) Things will still be the same for an arbitrary non-Hermitian operator; however, the last expression (5′) will no longer be equal to the first two then, and the general principles of wave mechanics will cease to apply.

$$[\partial^u] = \underset{\rightarrow}{\partial}^u - \underset{\leftarrow}{\partial}^u ,$$

which is a formula in which $\underset{\rightarrow}{\partial}^u$ represents the usual operator that acts on the right, and the analogous operator $\underset{\leftarrow}{\partial}^u$ acts on the left. The operator $[\partial^u]$ acts on both the right and the left, and its introduction will completely symmetrize the expression for the statistical impulse density:

(6) $$\rho\, v^u = \psi^* [\partial^u]\, \psi.$$

The classical statements and their applications (which are likewise classical) that we just recalled raise a problem *a priori* that is important from the relativistic viewpoint. From these statements, it seems that the "probable mean" quantity and the associated density quantity will have the same tensorial variance. That is inadmissible in relativity, where the element δu is, up to the factor *ic*, the fourth component of the quadri-vector δu^i that is dual to the trilinear element $[dx^j\, dx^k\, dx^l]$. The fact that wave mechanics takes all of its integrals "at constant time" [or even, to recall a terminology that we have employed moreover ([1]), that it is always subordinate to the "simultaneity hypothesis"] does not raise that objection: In particular, from the fact that the element δu^i, it specializes the "probable mean value" of the tensor integral, *but it cannot change the rank of that tensor*. We shall recall that problem in detail in the context of Dirac's theory.

In Schrödinger's non-relativistic wave mechanics, the three densities ρ^u define a *current density* vector. A question that poses itself is to know whether, in fact, that vector can be associated with the mass density ρ in such a manner that the classical continuity equation:

$$\partial_u (\rho\, v^u) + \partial_t\, \rho = 0$$

is satisfied. This condition is obviously necessary for the validity of the notion of a *fictitious fluid* that is statistically equivalent to that of "material point" m_0. One knows that the response is affirmative: *The preceding equation of continuity is a consequence of the wave equation* (3) *in the case where that equation is of non-relativistic, Schrödinger type* ([2]). One can say that Schrödinger theory refers to it here in the manner that is necessary *a priori*.

Things are much less simple in Dirac's relativistic theory. As we shall discuss in detail in Chapter III, paragraph II, inductive reasoning must play an important role. First of all, in a relativistic theory, the definition of the barycenter demands certain precautions. One no longer has the right to conclude from formula [5*x*] (which is always valid as an application of general quantum principles) that the quantity $m_0\ (\psi^* \psi)$ represents a mass density. A simple induction will be of some help: Since mass is equivalent to energy in relativity, we are entitled to associate the three operators $[\partial^u]$, which correspond to impulsion *densities* according to Schrödinger, to the operator $[\partial^4]$, which we think must correspond to the energy *density*. What follows will show that this induction is indeed the one that is suitable in Dirac's theory (nos. **8** and **18**). We remark

([1]) *La Relativité restreinte*; *see* especially, pp. 29.
([2]) *See*, for example, L. DE BROGLIE, *L'Électron magnétique*, pp. 81.

here that the difficulties that relate to the anti-Hermiticity of the operator $\frac{h}{2\pi i}\partial^4$ will vanish in the case of the density operator $\frac{h}{4\pi i}[\partial^4]$. Upon taking its conjugate, one sees that the expression:

$$ic\,\rho = \frac{h}{4\pi i}\psi^*[\partial^4]\psi$$

is indeed pure imaginary ([1]).

In Dirac's theory, and up to a factor *ic*, the four density quantities that were just in question will belong to Tetrode's asymmetric inertia tensor, which is conservative in the absence of a field. The other twelve components of that tensor are "suppressed" by the quantum hypothesis "of simultaneity." Moreover, the quantity $(\psi^*\psi)$ remains the temporal component of a conservative current quadri-vector that maintains a certain relationship with Tetrode's inertia tensor (Chap. IV, no. **26**).

One last well-known consequence of the general statement that is summarized in formula (4) is the following one ([2]): *The only values that a measurement that is made at the instant t can yield for a certain quantity r are the proper values of its operator R.* First of all, we see the same problem reappear that related to the probable mean value: How does one arrange in relativistic wave mechanics that the finite quantity and the density quantity should have convenient variances, respectively? That double problem will be treated in the following paragraph in the context of Dirac's theory.

Another problem that is much more serious, and that we shall only mention, is the following one: In wave mechanics, the finite quantity that is effectively provided by a measurement no longer relates *explicitly* to the (statistical) density quantity that corresponds to it by the intermediary of the volume element δu. Indeed, that finite quantity is a proper value of the operator R, so the corresponding density quantity will be $\psi^*\cdot R\psi$ [or rather $\frac{1}{2}\{\psi^*\cdot R\psi + (R\psi)^*\cdot\psi\}$]. The element δu then intervenes only *implicitly* in the definition of the proper values of R. This fact is completely revolutionary in comparison to the old mechanics, and its importance must be considerable, since it touches upon the elementary properties of space ([3]).

([1]) The fact that this property is established independently of any Hermitian or anti-Hermitian character of the operator ∂^t emphasizes the difference between the treatment of space coordinates and time in wave mechanics, as well as the inductive character of our argument.

([2]) One refers to the cited passages in Von NEUMANN's book.

([3]) See the closely-analogous remarks by L. de Broglie in Arch. Sci. Phys. and Nat., **15**, Geneva 1933, pp. 479.

II. – THE MECHANISM OF GALILEAN FRAME CHANGE ([1]) AND THE RECONCILIATION OF RELATIVISTIC AND QUANTUM DEMANDS IN DIRAC'S THEORY.

4. The first way of changing the Galilean frame. Matrix expressions for the "first principle" of relativity. – We say *the first principle* of relativity, or principle of the reciprocal partial transformation of space into time when a Galilean frame changes, to mean the Lorentz-Minkowski transformation ([2]):

$$(7) \qquad v'^i = o^i_j v^j, \qquad \sum_{i=1}^{4} o^j_i o^k_i = \delta^{jk},$$

with $i, j, k = 1, 2, 3, 4$, and $x^4 = ict$. It results from these laws that the square $\mathcal{S}^2$ of the *world-length* of a quadri-vector X^i will be an invariant. One can consider the formula:

$$\sum_{u=1}^{3} X_u^2 - c^2\, T^2 = \mathcal{S}^2$$

to be another expression of the *law of the equivalence of space and time.* When one is dealing with only the *first principle*, the real or pure imaginary character of quantities such as x^i, X^i, and o^i_j is not an issue; for the moment, those quantities will be assumed to be represented by arbitrary complex numbers.

In an analogous manner, if one neglects the Hermitian or non-Hermitian character of the Dirac γ^i, for the moment, then one must seek the restrictive conditions that must be imposed upon the matrix S in order for the transformation:

$$(8) \qquad \gamma'^i = S^{-1} \gamma^i S$$

to be equivalent to the Minkowskian transformation:

$$(9) \qquad \gamma'^i = o^i_j \gamma^j, \qquad \sum_{i=1}^{4} o^j_i o^k_i = \delta^{jk}.$$

One knows that the equivalent transformations (8) and (9) characterize the *change of Galilean frame in the first manner*; i.e., with invariance of the wave function ψ. Indeed, it is clear that the expressions $P_i\, \psi$ will then transform like the i^{th} component of a quadri-vector, the postulate of invariance of the Dirac equations:

([1]) For the theory of the change of Galilean frame in Dirac's theory, we refer to Dirac's fundamental papers ["The Quantum Theory of the Electron," Proc. Roy. Soc. **117** (1928), § 3, pp. 615], Von Neumann ["Einige Bemerkungen zur Diracschen Theorie," Zeit. Phys. **48** (1928), pp. 871], and Pauli ["Contributions Mathématiques à la Théorie de Dirac," Ann. Inst. H. Poincaré **6** (1936), § 5, pp. 123], as well as the classic book by L. de Broglie, *l'Électron magnétique,* pp. 149.

([2]) O. COSTA de BEAUREGARD, *La Relativité restreinte,* pp. 10 *et seq.*

(10) $$(P_i \gamma^i - i\, m_0\, c)\, \psi = 0$$

demands that the γ^i must transform according to the law (9), in which the o^i_j are the Minkowski coefficients.

The main point is that, not only equations (10), but also Dirac's fundamental conditions:

(11) $$\boxed{\tfrac{1}{2}(\gamma^i\gamma^j + \gamma^i\gamma^j) = \delta^{ij},}$$

are invariant under the transformation (9). Therefore, if one agrees to say *Dirac's theory* when one means the set of consequences of equations (10) and the conditions (11) (for the moment, we are abstracting from any hypothesis on the possible Hermiticity of the γ^i) then one will agree to say that *Dirac's theory is invariant under a change of Galilean frame in the first manner* [viz. formulas (8) or (9)].

It is interesting to show rigorously how the tensorial transformation law of the $\gamma^{ij}\ldots$ results from (9) and (11). (From our general conventions in the Foreword, $\gamma^{ij}\ldots$ denotes the product of the matrices $\gamma^i\gamma^j\ldots$, *with essentially* $i \neq j \neq \ldots$) For the second rank γ^{ij}, one will have:

(9′) $$\gamma^{ij} = \underbrace{o^i_k o^j_l \gamma^k \gamma^l}_{\text{for any } k \text{ and any } l} = \underbrace{o^i_k o^j_l \gamma^k \gamma^l}_{\text{for any } k \neq l} + \sum_{l=1}^{4} o^i_l o^j_l = o^i_k o^j_l \gamma^{kl}. \qquad \text{Q. E. D}$$

The same argument applies by recurrence to the matrices $\gamma^{ijk} = \gamma^{ij}\gamma^k$ and $\gamma^{ijkl} = \gamma^{ijk}\gamma^l$.

In order to conclude the law (8) from the law (9), one must appeal to the invariance of (11) and a main theorem whose general direct proof was given by W. Pauli:

If one is given two distinct sets of four square matrices γ^i *of rank* 4, *whether Hermitian or not, that both satisfy the conditions* (11) *then there will exist one and only one square matrix S of rank* 4 *(that is defined up to a complex factor) that admits an inverse* S^{-1} *and is such that one has formula* (8) ([1]).

Conversely, one sees immediately that the transformation (8) preserves the conditions (11). The proof of the preceding theorem involves an important lemma that is quite useful for us, and whose proof was given already by Pauli: *The sixteen matrices* γ^A *of Dirac's theory form a complete system*, i.e., any square matrix of rank 4 can be developed in one and only one manner in the form $c_A\, \gamma^A$, where the c_A denote complex constants ([2]).

The invariance of conditions (11) under a transformation (9) is established by the following well-known calculation:

$$\tfrac{1}{2}(\gamma'^i\gamma'^j + \gamma'^i\gamma'^j) = \tfrac{1}{2} o^i_k o^j_l (\gamma^k\gamma^l + \gamma^l\gamma^k) = o^i_k\, o^j_l \delta^{kl} = \sum_{i=1}^{4} o^i_l\, o^j_l = \delta^{ij},$$

([1]) W. PAULI, *op. cit.*, § III, pp. 115.

([2]) *Op. cit.*, § II, pp. 111.

in which the second equality results from the Dirac conditions (11), and the fourth one, from the Minkowski conditions (9_2). Pauli's theorem then shows that there exists one and only one transformation (8) that is equivalent to (9). Conversely, *if, by hypothesis, a transformation of the type* (8) *is such that the four transformed* γ'^i *are congruent to the original four* γ^i, i.e., if one has the relations (9_1), then the preceding calculation will show that *the o are the Minkowski coefficients*, which satisfy the conditions (9_2).

Indeed, it is reasonable *a priori* that in general the γ'^i that are transforms of the γ^i by way of a law (8) will not be congruent to the initial γ^i, but they can still be developed in terms of the system of sixteen γ^A. A simple example will convince us that this is, in fact, the case. If one introduces the four $\bar{\gamma}^i$ that are dual to the γ^{jkl}, which are defined as follows ([1]):

$$\bar{\gamma}^u = \gamma^{vw4}, \qquad \bar{\gamma}^4 = \gamma^{uvw},$$

then one will effortlessly verify that they satisfy the same conditions as the γ^i :

$$\tfrac{1}{2}(\bar{\gamma}^i\bar{\gamma}^j + \bar{\gamma}^i\bar{\gamma}^j) = \delta^{ij}.$$

If one considers the transforms:

$$\gamma'^i = o^i_j \gamma^j, \qquad \sum_{i=1}^{4} o^j_i o^k_i = \delta^{jk}$$

then the preceding calculation will show that the γ'^i, like the γ^i, satisfy the conditions (11), and consequently, that one can again pass from the γ^i to the γ'^i by a transformation (8). However, by reason of the *complete* character of the sixteen γ^A, the present transformation will be irreducible to the Minkowski transformation.

More generally, if γ^A denotes any of the sixteen γ, and γ'^A denotes its homologue in the system γ' then the matrix expression for the transformation (9) and (9′) will obviously be:

(8′) $$\boxed{\gamma'^A = S^{-1}\gamma^A S.}$$

However, an arbitrary transformation of that type is not a tensorial transformation of the five matrix tensors γ^A, in the sense that the γ of a given tensorial rank will not remain congruent to themselves under the transformation; *the matrix I is an exception*, since it obviously always transforms into itself.

Therefore, if one desires that the transformation (8) should be an equivalence of the Minkowskian transformation (9) then certain restrictive conditions must be imposed upon the matrix *S*. When combined with (8) or (8′), they constitute what one can call the *matrix expression of the first principle of relativity;* we now seek those conditions.

A first necessary condition, which is quite clear, was stated by Von Neumann ([2]) and Pauli ([1]); we give it a general statement. Conforming to our conventions in the Foreword, we set:

([1]) This definition differs in sign from the one that we shall adopt later on [eq. (45)].
([2]) *Op. cit.*, pp. 877.

$$\gamma = \gamma^{\mu\nu w4},$$

and verify effortlessly that:

$$\gamma^2 = I.$$

If, by hypothesis, we demand that the matrix $\bar{\gamma}$ should remain congruent to itself:

$$\bar{\gamma}' = c\,\gamma$$

under a transformation (8) then a simple calculation will show that the coefficient c must be ± 1:

$$c^2\,\bar{\gamma}^2 = S^{-1}\gamma S\, S^{-1}\bar{\gamma}\, S, \qquad c^2 = 1, \qquad c = \pm 1. \qquad \text{Q.E.D.}$$

Therefore, we find that the first necessary condition is that S and S^{-1} *must both commute or anti-commute with* $\bar{\gamma}$. Now, the sixteen γ^A divide into two classes of eight. On the one hand, the 1, γ^i, $\bar{\gamma}$ commute with $\bar{\gamma}$, and on the other, the γ^i and $\bar{\gamma}^i$ anti-commute with $\bar{\gamma}$. One then effortlessly verifies that *in either case, the* $S^{-1}\gamma' S$ *can be developed in the system of eight* γ^i and $\bar{\gamma}^i$. One can say nothing more. Even when one introduces the numerical relations that exist between the coefficients of S^{-1} and S from the fact that $S^{-1}S = 1$, one will confirm that neither the set of γ^i, nor that of $\bar{\gamma}^i$, can be eliminated from the result. That is, *the necessary condition* (12) (although it is already clearly restrictive, since it reduces the number of basic matrices by half) *is not, by itself, a sufficient condition for the* γ'^i *to be congruent to the* γ^i.

However, one can associate the condition (12) with another necessary condition (13) that was discovered by Pauli ([2]), such that the set of (12) and (13) collectively constitutes a sufficient condition for the result that we have in mind. In order to establish that new condition, we remark, with Pauli, that the matrices $\tilde{\gamma}^i$ that are the *transposes* of the γ^i will obviously satisfy the conditions (11):

$$\tfrac{1}{2}(\tilde{\gamma}^i\tilde{\gamma}^j + \tilde{\gamma}^j\tilde{\gamma}^i) = \delta^{ij}.$$

From the fundamental theorem, there will then exist a matrix B that is defined up to a complex constant and is such that:

$$\tilde{\gamma}^i = B^{-1}\gamma^i B.$$

Now consider the transforms γ'^i of the γ^i by (8); one will define the matrix B' that transforms γ'^i into $\tilde{\gamma}'^i$ similarly. Since one obviously has:

$$B'^{-1}\gamma'^i B' = \tilde{S}\,\tilde{\gamma}^i\tilde{S}^{-1}, \qquad B'^{-1}S^{-1}\gamma^i SB' = \tilde{S}\,B^{-1}\gamma^i B\tilde{S}^{-1},$$
$$(SB')^{-1}\gamma^i(SB') = (B\tilde{S}^{-1})\gamma^i(B\tilde{S}^{-1}),$$

([1]) *Op. cit.*, pp. 126, equation (28).
([2]) *Op. cit.*, § IV, pp. 119 and 126.

one will define the following relation between B, B', and S (a denotes a complex constant):

$$[p] \qquad B\tilde{S}^{-1} = a\, SB' \quad \text{or} \quad B = a\, SB'\tilde{S}\,.$$

More generally, if one seeks the transforms of the sixteen γ^A under B then one will find, with no difficulty, that for the five tensorial ranks of 0, 1, 2, 3, 4:

$$[q] \qquad +\tilde{1} = 1,\ +\tilde{\gamma}^i,\ -\tilde{\gamma}^{ij},\ -\tilde{\gamma}^{ijk},\ +\tilde{\gamma}^{uvw4}\,.$$

That being the case, develop the γ'^i, which are the transforms of the γ^i by S, in the system of sixteen γ^A, and transform the matrices on the two sides by B. From what was just said, the transforms of the γ^A on the right-hand side will be the $\tilde{\gamma}^A$, up to sign, and the sign will be given by the table [q]. One can then write, upon distinguishing the coefficients c_M and c_N that relate to the matrices of each sign:

$$B^{-1}\gamma'^i B = c_M\tilde{\gamma}^M - c_N\tilde{\gamma}^N\,.$$

It is clear that in this formula, the necessary and sufficient condition for the coefficients c_N to be identically zero is that one must have:

$$[r] \qquad B^{-1}\gamma'^i B = \tilde{\gamma}'^i = B'^{-1}\gamma'^i B', \qquad \text{in which } B = b\, B'$$

(b denotes a complex coefficient) ([1]).

Finally, approach the results [p_2], [r_2], and [q], we see that *necessary and sufficient condition for the matrices that are the transforms* γ'^i *of the* γ^i *by S to be congruent in the system I,* γ^i, $\bar{\gamma}$ *is that one must have:*

$$(13) \qquad \boxed{B = c\, SB\tilde{S},}$$

in which c denotes a complex constant, and that is the Pauli condition that we stated.

Now take *both* of the conditions (12) and (13); they demand that the γ'^i should be congruent:

to " γ^i " γ'^i "
and to I γ^i " " $\bar{\gamma}$,

respectively, in such a way that ultimately the four γ'^i must be congruent to the four γ^i, and consequently, from a previous remark, they can be deduced by a Minkowski transformation. *Conditions* (12) *and* (13), *when taken together, therefore indeed constitute a matrix expression for the first principle of relativity.*

([1]) By reason of the complex character of the c_M, that conclusion will not still be valid if one takes the γ^{+i}, which are adjoints, in place of the $\tilde{\gamma}^i$, which are transposes.

5. The second way of changing the Galilean frame. Necessity of defining a "buffer matrix" γ^0. – One knows that Dirac's theory utilizes two *associated* equations of the type:

$$\zeta\left(-\underset{\leftarrow}{P}^i\gamma_i + im_0c\right) = 0, \qquad \left(\underset{\rightarrow}{P}^i\gamma_i + im_0c\right)\psi = 0, \tag{14}$$

with, by definition, the quadri-operators P^i:

$$\underset{\leftarrow}{P}^i = -\frac{h}{2\pi i}\underset{\leftarrow}{\partial}^i - \frac{e}{c}A^i, \qquad \underset{\rightarrow}{P}^i = -\frac{h}{2\pi i}\underset{\rightarrow}{\partial}^i + \frac{e}{c}A^i. \tag{15}$$

For what follows, we further define the two quadri-operators:

$$[P^i] = \tfrac{1}{2}[\underset{\rightarrow}{P}^i - \underset{\leftarrow}{P}^i] = -\frac{h}{2\pi i}[\partial^i] + \frac{e}{c}A^i, \qquad i(P^i) = \tfrac{1}{2}(\underset{\rightarrow}{P}^i + \underset{\leftarrow}{P}^i) = -\frac{h}{4\pi i}(\partial^i), \tag{15′}$$

in which $[\partial^i]$ and (∂^i) are the "anti-symmetric" and "symmetric" differential operators, respectively, that act on both the right and the left, and that we defined in the Foreword. From what we said in number **3**, the operator $[P^i]$ must be regarded as something that corresponds to *inertial mass-impulse* (viz., total – electromagnetic) as a density; that remark will find its application later on [no. **8**, eq. (26)]. Along with several authors, we consider the *associated wave functions* with four components ζ and ψ to be two matrices, the first of which has one row and four columns, while the second one has four rows and one column. Under those conditions, equations (14) will both be differential matrix equations.

In Dirac's theory, one introduces five well-known statistical density tensors ρ_D, and also five density tensors ρ_S, whose definitions we shall introduce systematically in Chapter III, and that have the types:

$$\rho_D = \zeta\,\gamma^A\psi, \qquad \rho_S = \zeta\,[\partial^i]\,\gamma^A\psi, \tag{16}$$

respectively. From our conventions, the 32 expressions (16) will be matrices with one row and one column; i.e., simple numbers. As far as the tensorial components are concerned, their variance is, *by hypothesis*, the one that is indicated by the indices $i, j, \ldots$, of the *significant matrix* γ^A.

Under a *change of Galilean frame in the first manner*, the y and z are transformed invariantly, the $\underset{\rightarrow}{P}^i$ and the $\underset{\leftarrow}{P}^i$, according to the quadri-vectorial law (7) (which, it would be appropriate to remark, leaves the symbols $\underset{\rightarrow}{\partial}^i$ and $\underset{\leftarrow}{\partial}^i$ invariant), and finally the γ^A transform according to the law (9), or – what amounts to the same thing – according to the law (8′), when "restricted" by the conditions (12) and (13). Equations (14) and the definitions (16) will then become:

$$\zeta\left(-\underset{\leftarrow}{P}'^i S^{-1}\gamma_i + im_0c\,S^{-1}\right) = 0, \qquad \left(\underset{\rightarrow}{P}^i S\gamma_i + im_0c\,S\right)\psi = 0,$$
$$\rho'_D = \zeta\,(S^{-1}\gamma^A S)\,\psi, \qquad \rho'_S = \zeta\,[\partial^i]\,(S^{-1}\gamma^A S)\,\psi.$$

As is well-known, one immediately deduces the laws of a *change of Galilean frame in the second manner* from these; i.e., *with an invariant transformation of the sixteen* γ^A. Indeed, the preceding formulas can be written:

$$\zeta S^{-1}\left(-\underset{\leftarrow}{P}'^{i}\gamma_i + im_0 c\right) = 0, \qquad \left(\underset{\rightarrow}{P}^{i}\gamma_i + im_0 c\right) S\psi = 0,$$
$$\rho'_D = \zeta S^{-1}(\gamma^A)\, S\psi, \qquad \rho'_S = \zeta S^{-1}([\partial^i]\,\gamma^A)\, S\psi.$$

The desired transformations laws for ζ and ψ are then:

(17) $$\boxed{\zeta' = \zeta S^{-1},} \qquad \boxed{\psi' = S\psi.}$$

The essential remark is that, *in principle, the transformations laws* (17) *forbid us to consider the matrices* ζ *and* ψ *to be adjoint.* There might be an exception when the matrix of the transformation S can be taken to be unitary, but we shall confirm in the following number that this is simply not possible. Now, according to the tradition that originated in Schrödinger's theory, Dirac's theory proposes to write the statistical densities – $\rho_{D'}$, for example – in the form:

$$\psi^+\gamma^B\psi \qquad \text{or} \qquad \zeta\,\gamma^B\zeta^+ .$$

The matrix γ^B, which is different from γ^A, no longer directly exhibits the tensorial variance of the ρ. However, if we select the first of the two forms above, for example, then it will be certain that the matrices ζ and ψ^+ can be deduced from each other by a square matrix γ^0 ; by convention ([1]), we set:

(18) $$\boxed{\gamma^B = i\gamma^0\gamma^A} \qquad \text{and} \qquad \boxed{\zeta = \psi^\times = i\psi^+\gamma^0.}$$

It is important for what follows to establish the laws of transformation of the *buffer matrix* γ^0 that we just introduced. Under a *change of Galilean frame in the first manner*, ψ, and therefore ψ^+ and $\zeta = \psi^\times$ will be transformed invariantly, so the same thing will be true for γ^0. *In the second manner,* ψ^+ and $\psi^\times$ will be transformed according to the different laws:

(17′) $$\psi'^\times = \psi^\times S^{-1}, \qquad \psi'^+ = \psi^+ S^{-1},$$

so the following transformation law for γ^0 will result:

(19) $$\gamma^0 = (S^{-1})^+\gamma^0\,(S^{-1}),$$

([1]) The factor i is introduced arbitrarily in order to simplify the expressions for the important *current-density of presence* quadri-vector $\psi^\times\gamma^i\psi$. Our notations are the ones that are generally adopted in the course of the survey article by W. Pauli ["Allgemeinen Prinzipien der Wellenmechanik, Relativistischen Theorien," Handb. d. Phys. XXIV$_3$ (1933), pp. 220]. The principle of these notations is due to W. Gordon ["Der Strom in der Diracschen Elektronentheorie," Zeit. Phys. **50** (1928), pp. 620, eqs. (3) and (4).]

which is completely different from the one that is true *in the first manner* for the γ^A [eq. (8)].

6. Lack of relativistic symmetry in the Dirac equations with respect to the second principle. Special role of the matrix γ^4. – The essential novelty in relativity is comprised of the law (7) for the change of Galilean frame, which is a law that expresses an *equivalence* between space and time in the sense that was discussed. That being the case, it is obviously paramount to formulate the principles that distinguish time from space, which are principles that are remarkable for the fact that they are ultimately expressed by inequalities, and notably imply the emergence of a unique time direction.

The classical formulation of the *second principle* of relativity is naturally performed in two stages ([1]). One can state the *second principle in its broad form* in the following manner: *Any objective event is represented by three real space coordinates x^u and one pure imaginary time coordinate x^4.* It follows that *the coefficients o^i_j of a change of objective Galilean frame will be real if they contain the index* 4 *zero or two times, and pure imaginary if they contain it once.* One consequence of the last statement and the Minkowskian conditions (7_2) is the inequality:

$$(o^4_4)^2 \geq 1. \tag{20′}$$

Recall the existence of two inequalities that are equivalent to the preceding one:

$$v^2 \leq c^2, \qquad ds^2 \leq 0;$$

v denotes the relative velocity of the spatial origin of the two objective Galilean frames, and *ds* denotes the element of length of the temporal axis of an arbitrary objective Galilean frame.

An important consequence of the inequality (20′) is that the *square $\mathcal{S}^2$ of the length of a world-quadri-vector will have the same sign in all objective Galilean frames,* which will give an *objective* significance to the classification of quadri-vectors into *time-like* and *space-like* according to the sign of their $\mathcal{S}^2$. In its *broad form*, the *second principle* thus already succeeds in making a strong, clear distinction between space and time.

The inequality (20′) can then be written:

$$\boxed{o^4_4 \leq -1} \qquad \text{or} \qquad \boxed{+1 \leq o^4_4.} \tag{20}$$

The *narrow form* of the *second principle* is then obtained by postulating, along with the conditions that were pointed out previously, that *the transformations that remain*

([1]) O. COSTA DE BEAUREGARD, *La Relativité restreinte*, pp. 15 *et seq.* The "second principle of relativity" is, in reality, only one part of what would be a true *second principle* for time. Relativity does not oblige time to elapse any more than classical physics (or quantum physics); it limits itself to asserting that time elapses, and starting from that, it shows that it elapses in the same sense for all objective observers.

permissible under (20′) or (20) *must form a continuous group that notably contains the identity transformation*, like the initial (7). It is then obvious that the inequality (20_1) must be discarded, and one effortlessly verifies that the inequality (20_2) can be preserved. Under these conditions, the properties of a time-like quadri-vector are found to be specified in the following form: *The sign of the fourth component of a time-like quadri-vector will be the same in all objective Galilean frames.* Conversely, *if one knows a certain quadri-vector whose temporal component possesses a well-defined sign then one can assert that:*

1. *The second principal of relativity is satisfied in the narrow form.*
2. *The quadri-vector considered is time-like.*

Since the temporal axes of objective Galilean frames are all time-like, by virtue of the inequality (20′), one can say that the more restrictive inequality (20_2) implies that *the sense in which time elapses will be the same in all objective Galilean frames.* One can pass from the case that was predicted by the inequality (20_2) to the one that is predicted by the rejected inequality (20_1) by means of a reversal of the time axis.

Having recalled these preliminaries, a main point for what follows is that, despite how they first appear, *the Dirac equations* (14) *do not have relativistic symmetry that corresponds to the "second principle."* In that regard, and up to certain difficulties that were mentioned before, we saw in no. **5** that the *total mass-impulse* quadri-operator:

$$-\frac{h}{2\pi i}\partial^i$$

can be considered to have relativistic symmetry, since its first three components are Hermitian, and the fourth one behaves in the wave equations as if it were anti-Hermitian. Moreover, the *electromagnetic mass-impulse* quadri-vector:

$$-\frac{e}{c}A^i\times$$

possesses relativistic symmetry, since the three A^u are real, and A^4 is pure imaginary. Finally, except for certain difficulties, the operators $\underset{\rightarrow}{P}^i$ and $\underset{\leftarrow}{P}^i$ that are defined by (15) must be considered to possess relativistic symmetry of the second principle. Moreover, the difficulties in question disappear completely for both operators $[P^i]$ and (P^i) that are defined by (15′), which are operators that will be quite useful in what follows. *Those operators possess relativistic symmetry of the second principle.*

On the contrary, *the set of four matrices γ^i does not possess relativistic symmetry of the second principle.* In order for that to be true, it is necessary that the three γ^u can be chosen to be Hermitian and γ^4, to be anti-Hermitian. Now, that is excluded by the Dirac conditions (11): A Hermitian matrix can indeed have a square of I, but not an anti-Hermitian matrix.

Is it possible to modify Dirac's theory in such a way as to give relativistic symmetry to the set of γ^i by replacing the conditions (11) with the analogous conditions:

$$\tfrac{1}{2}(\gamma^{+i}\,\gamma^{j} + \gamma^{j}\,\gamma^{+i}) = \delta^{ij}\ ?$$

First of all, contrary to (11), and taking into account the relativistic symmetry of the o_i^j with respect to the second principle, these conditions will not be invariant under a Minkowski transformation. Moreover, the conditions considered will not permit one to recover the Gordon equation as a consequence of the theory in the absence of a field. Indeed, instead of multiplying the Dirac equation (14_2) on the left by the operator $\left(\gamma_i \underset{\rightarrow}{P}^i - im_0 c\right)$ ([1]), one must multiply it by one or the other of the operators:

$$\left(\gamma_i^+ \underset{\rightarrow}{P}^{+i} - im_0 c\right) \qquad \text{or} \qquad \left(\gamma_i^+ \underset{\rightarrow}{P}^i - im_0 c\right).$$

The first of them is not suitable to provide the square term:

$$P_i^+ P^{+i} = \sum_{u=1}^{3} P_u^2 - P_4^2\,,$$

instead of the $P_i\,P^{\,i}$ that should be found; the second one is not suitable to provide the term:

$$i\,m_0\,c\,(\gamma_i\,P^{\,i} - \gamma_i^+ P^{+i}),$$

which is no longer cancelled.

Finally, the relativistic symmetry of the γ_i with respect to the second principle (Hermiticity of the γ^u and anti-Hermiticity γ^4) and the relativistic symmetry of the same operators with respect to the first principle that is expressed by the conditions (11) will be conserved separately by a Minkowski transformation (9), *but they are mutually incompatible.* If one would like to give a name to the particular aspect that the conflict in relativity and quanta is characterized by here then one must call it *the conflict of the two i symbols* that are germane to each theory. Among its other consequences, that conflict implies that the two operators $\left(\gamma_i \underset{\rightarrow}{P}^i + im_0 c\right)$ and $\left(\gamma_i \underset{\rightarrow}{P}^i - im_0 c\right)$, whose product is the Gordon operator, are not adjoint operators. Since it is absolutely necessary that the Gordon equation should be a consequence of Dirac's theory, we conclude that *it is impossible to arrange in some way that the set of Dirac matrices γ^i possesses the relativistic symmetry of second principle without destroying the very fundamentals of that theory.*

In order to measure the importance of the problem that results from that statement, we make some further remarks. It is obvious that the necessary and sufficient condition for the expressions $\psi^+\gamma^B\psi$ to be real or pure imaginary is that the matrix γ^B should be Hermitian or anti-Hermitian, resp. Now, if the four γ^i, and consequently the table of sixteen γ^A, possesses the relativistic symmetry of the second principle, then by setting simply:

$$\gamma^\times = \gamma^+, \qquad \gamma^A = \gamma^B, \qquad \gamma^0 = 1,$$

([1]) P. A. M. DIRAC, "The Quantum Theory of the Electron." Proc. Roy. Soc. **117** (1928), pp. 613; or L. DE BROGLIE, *L'Électron magnétique*, pp. 137, eq. (15).

the statistical densities ρ_D and ρ_S will be assured to have relativistic symmetry, in the sense that the ρ of the same rank will be real or pure imaginary according to the absence or presence of the index 4 in γ^B, resp. Since that property is obviously preserved under a change of Galilean frame in the first manner, it must also be preserved under a change of Galilean frame in the second manner. The matrix of the transformation (8) must then be unitary, in such a way that one will always have:

$$S^{-1} = S^+, \qquad \gamma'^0 = \gamma^0 = I, \quad \psi^\times = \psi^+.$$

In fact, we just saw that all of this effortless harmony is only an illusion that must then be rejected. If the four γ^i cannot be chosen to be three Hermitian ones and one anti-Hermitian one the Dirac has shown that all four of them can be taken to be Hermitian ([1]), which is a very paradoxical situation from the relativistic standpoint, and in it we will see the first manifestation of the very asymmetric special role that the matrix plays in Dirac's theory ([2]).

It is obvious (and well-known) that when one takes into account the symmetry of the o^i_j with respect to the second principle, *the Hermiticity condition of the four γ^i will not be preserved under a change of Galilean frame of the first kind* [eq. (9)], and it will follow from this that *the transformation matrix S cannot be unitary* [eq. (8)]. On the contrary, *that same condition will be invariant under a change of Galilean frame of the second manner*, since the γ^i (and consequently, all of the *significant matrices* γ^A) will then be transformed invariantly. Moreover, if the *buffer matrix* γ^0 has itself been chosen to be Hermitian then formula (19) will show that *this condition is conserved under a change of Galilean frame in the second manner.* It is by appealing to these remarks that we shall now establish the matrix expression of the *second principle* of relativity, which is an expression that clearly exhibits the asymmetric special role of the matrix γ^4.

7. Matrix expression for the second principle of relativity. – Lemma. – *If the four matrices γ^i, as well as the buffer matrix γ^0, are chosen to be Hermitian, as in Dirac* (which is a condition for it to be invariant under a change of Galilean frame in the second manner) *then the necessary and sufficient condition for the statistical densities ρ to enjoy relativistic symmetry of the second principle is that one must have:*

([1]) *Op. cit.*, pp. 614.

([2]) The Hermiticity of the four γ^i that Dirac postulated does not seem to be an essential element of the theory, moreover, and one can probably remove it at the expense of a certain complication in the calculations and formulas. For example, one knows that the fact that the values ± 1, or ± *i*, are the proper values of the γ^A does not result from any possible Hermiticity of the γ^i at all, but only from the conditions (11). Indeed, since the γ^i, for example, have squares equal to *I*, if *g* denotes any of their proper values, and *f* denotes a corresponding proper function then one must have:

$$\gamma f = gf, \qquad \gamma^2 f = g\,\gamma f = g^2 f, \qquad g^2 = \pm 1, \qquad \text{Q. E. D.}$$

[G. PETIAU, "Sur les fonctions propres des opérateurs fondamentales de la Théorie de l'Électron de Dirac," Acad. Roy. de Belgique, Cl. des Sci. **24** (1938), pp. 488.

(21′) $$\gamma^0 = (a\,I + ib\,\bar{\gamma})\,\gamma^4,$$

in which a and b denote two real numbers that are not simultaneously zero.

Indeed, with the premises that were posed, the necessary and sufficient condition for the desired result to be true is that γ^0 must commute or anti-commute with the significant γ^A of the same rank according to the absence or presence of the index 4, resp. If one then develops γ^0 in the system of sixteen γ^A then the matrices I, γ^{u4}, γ^{uv4}, and γ^{uvw4} will be eliminated since they commute or anti-commute with *both* γ^4 and at least one of the γ^u, and the γ^u and γ^{uv} will be eliminated since they anti-commute with both the γ^{u4} and γ^{uw}, for example. Only the matrices γ^4 and $\gamma^{uvw} = -\bar{\gamma}^4$ are not eliminated by criteria of that kind, and one thus has the statement (21′). Conversely, it is obvious that the statement (21′) is a sufficient condition for the desired result. As for the coefficients a and b, since γ^0 and γ^4 are Hermitian, by hypothesis, and γ^4 anti-commutes with $\bar{\gamma} = \gamma^{1234}$, which is Hermitian, they will necessarily be real; moreover, they are not simultaneously zero, because all of the densities ρ would be zero then.

The two terms in (21′) are not actually distinct, because $\bar{\gamma}\gamma^A$ is nothing but $\bar{\gamma}^A$, up to sign, in such a way that the ρ of a given rank are expressed as sums of components of the two tensors $\rho^{(1)}$ and $\rho^{(2)}$ that are dual to each other. Under these conditions, it is clear that we can assign the coefficients a and b arbitrarily *in a well-defined Galilean frame that one calls "initial."* We shall do that by taking:

(22^0) $$a_0 = 1, \qquad b_0 = 0,$$

in such a manner as to recover the formula, which is becoming classical:

(21) $$\boxed{\gamma^0 = \gamma^4}$$

in that initial frame.

It is well-known that the double presence of the γ^4 in the ρ that are defined by (16) and (18), which result from the equality (21), when combined with the fundamental Dirac conditions (10), will reconcile the Hermiticity of the four γ^i with the symmetry of the ρ with respect to the second principle, and even, more precisely, that it will uphold the second condition by means of the first one. One can say that the asymmetric role of the matrix γ^4 is, from the relativistic viewpoint, "compensated" by that of the buffer matrix γ^0, and one will see quite well that the mechanism of that compensation is closely linked with the formulation of the *second principle* in Dirac's theory.

In order to establish the *matrix expression for the second principle*, we remark that the result (21′) is valid in every Galilean frame that satisfies the second principle in its broad form, since, on the one hand, its premises are preserved for any change of Galilean frame in the second manner, and on the other hand, the statistical densities enjoy symmetry of the second principle, by hypothesis. We then seek to find out what the coefficients a and b will become in a new frame when it is intended that they have been defined "initially" according to (22^0).

If b^0 were chosen to be zero initially then all of the components of the previously-defined tensor $\rho^{(2)}$ would be zero, in such a way that *all of the transformed b would be* *zero.* Moreover, *no transformed a can be zero,* since all of the components of the tensor $\rho = \rho^{(1)}$ would be annulled at once then. Since, as we recall, *a is real*, we must consider the *two cases that are possible a priori:*

(22′) $$a < 0, \qquad a > 0,$$

which are cases that are completely distinct, since one cannot pass from one to the other by continuity, as the value $a = 0$ has been excluded.

I shall now say that upon disposing of the arbitrary constant that appears in the matrix S, one can arrange that it should make $a = -1$ in the case of $a < 0$, and $a = +1$ in the case of $a > 0$. For that to be true, it is sufficient to replace S^{-1} with $\sqrt{a}\,S^{-1}$ in (19), and consequently, in the first case, one replaces $(S^{-1})^+$ with $-\sqrt{a}\,(S^{-1})^+$, and in the second case, $(S^{-1})^+$ with $+\sqrt{a}\,(S^{-1})^+$. Under these conditions, the expression for the possible cases will become:

(22) $$\boxed{a = -1,} \qquad \boxed{a = +1,}$$

in such a way that when one takes into account the convention (22^0), the transformed expression for the *buffer matrix* can only be:

$$\gamma'^0 = \pm\,\gamma^0 = \pm\,\gamma^4 .$$

From the argument that we just presented, that is a necessary condition for the second principle in its broad form to be respected; conversely, that condition is obviously sufficient. Finally, if we refer to the transformation law (19) of γ^0 then we will find that the *matrix expression for the second principle in its broad form is:*

(23′) $$\boxed{\gamma^0 = \pm S^+\gamma^0 S;}$$

this is a restrictive condition that is imposed upon the transformation matrix S, in which the asymmetric special role of the matrix $\gamma^4 = \gamma^0$ appears clearly.

As for the expression of the second principle in its narrow form, in order to find it, it suffices to remark that:

1. One obviously has $a = 1$ for the identity transformation, and
2. a is certainly a continuous function of the Minkowski o_j^i.

It is clear that under these conditions *the two conditions* (22) *will correspond bijectively with the two classical conditions* (20). Consequently, in the set of conditions:

(23) $$\gamma^0 = -\,S^+\gamma^0\,S, \qquad \boxed{\gamma^0 = S^+\gamma^0 S,}$$

the second of these will correspond to the *rotations* of a world-tetrad, and the first of them to the *reflections.* It is Von Neumann who gets the credit for the discovery of the important condition (23_2) ([1]) in an equivalent form. We think that it is interesting to accentuate the fact that this condition is nothing but *the matrix expression for the second principle in its narrow form.*

Finally, *the validity of the second principle in its narrow form will imply the invariance of the definition* (21) *of the buffer matrix for the changes of Galilean frame in the second manner.* The proof of that result that we gave, which was very paradoxical on first glance, rests upon a later analysis of the *existence theorem* for the matrix *S* of the transformation. It is instructive to recover the converse of that result in the following manner: If, by hypothesis, the definition (21) is invariant under changes of Galilean frame in the second manner then the temporal component of the current-density of presence quadri-vector can be written:

$$(j^4) = \psi^+ \gamma^4 \psi = i\,(\psi^+ \psi), \tag{24}$$

where the last expression is invariant. Now, the parenthesis is the *positive-definitive form* that Dirac imposed in order to recover an expression for the probability density of presence. Therefore, *by means of the hypothesis* (21), *the sign of the fourth component of the Dirac current must be the same in all Galilean frames that are embraced by the theory. That is:*

1. *The second principle is respected in its narrow form.*
2. *The Dirac quadri-current is time-like.*

8. On the calculation of finite quantities in Dirac's theory. – We now show that *it is once more the asymmetric, special role of the matrix γ^4 that permits Dirac's theory to reconcile the general principles of wave mechanics with the relativistic demands of the statistical densities.* Here, as one sees, the presence of the matrix γ^4 as a "buffer" serves to compensate for the relativistic asymmetry that is caused by the integration "at constant time" of wave mechanics.

From the general principles of wave mechanics, the probable mean value $\bar{a}$ of the quantity that is represented by a certain operator *A* is:

$$\bar{a} = \iiint_U \psi^+ \cdot A\psi \cdot u\,, \tag{25}$$

in which the integral is taken over a world-hypersurface at constant time. In relativistic wave mechanics, two variances of 4 are concealed in the right-hand side of that formula. The first one comes from the fact that $\delta u = ic\ \delta u^4$ is the fourth component of the quadri-vector $ic\ \delta u^i$, and is dual to the trilinear expression $[dx^i\ dx^j\ dx^k]$ ([2]) and the second one is that, from the preceding, one will have $\psi^+ = -\,i\ \psi^\times \gamma^4$. It is, moreover, clear that *formula* (25) *contains a* virtual *summation over an index that takes only the value* 4 *here, from the*

([1]) "Einige Bemerkungen zur Diracschen Theorie," pp. 878.
([2]) These notations are the ones that we used in our *Relativité restreinte.*

fact that one is integrating at constant time. As a corollary, one sees that *the symbolic variance of the operator A that is the quantum representative of a certain finite quantity will necessarily be the same as that of the tensor that corresponds to classical relativity.* That remark will be useful to us in what follows.

We first apply these considerations to two important examples. From Schrödinger theory, we know that *operator of presence* is the operator 1. The general formula (25) will then give the expression (24_2) above for the statistical density of presence, which is an expression that will appear like the fourth component of a quadri-vector when one takes into account the *invariant* definition (21) of the *buffer matrix.*

Along the same lines, we know that the quadri-vector:

$$P^i = -\frac{h}{4\pi i}[\partial^i] + \frac{e}{c}A^i \tag{15′}$$

(which is an operator that is endowed with relativistic symmetry of the second principle) corresponds to the *proper* or *kinetic energy-impulse* of the electron as a density. From the general quantum principle (25), the four statistical densities that will appear under an integration at constant time are:

$$-\frac{ich}{4\pi i}\,\psi^+\,[\partial^i]\,\psi + ie\,A^i\psi^+\psi = \frac{ich}{4\pi}\;\psi^\times\,[\partial^i]\,\gamma^4\psi + ie\,A^i\psi^\times\,\gamma^4\psi\,.$$

One recognizes the four components T^{i4} of Tetrode's *inertia tensor* in this, namely

$$\boxed{T^{ij} = \frac{ich}{4\pi}\psi^\times[\partial^i]\,\gamma^j\psi + A^i\cdot e\psi^\times\,\gamma^j\psi,} \tag{26}$$

whose second term involves Dirac's *charge-current density:*

$$\boxed{j^k = -e\psi^\times\gamma^k\psi.} \tag{24′}$$

The formula:

$$\bar{p}^i = \frac{1}{ic}\iiint T^{i4}\delta u \;=\; \iiint T^{i4}\delta u_4\,, \tag{27′}$$

which is imposed by the quantum principle (25), is the truncated expression for the relativistic formula ([1]):

$$\bar{p}^i = \iiint T^{ij}\delta u_j \tag{27}$$

that corresponds to an integration *at constant time.* It is important to remark that *quantum principles impose the summation over the second index of* T^{ij} in formula (27). Now, it is in precisely this manner that:

([1]) *La Relativité restreinte*, pp. 50.

1. The significant index of the first term that is provided by T^{ij} (viz., the total mass-impulse) is the index of the differential operator $[\partial^i]$ (and not that of the matrix operator γ^j), and

2. If $\bar{A}^i$ denotes the mean value of the quadri-potential on the hypersurface of integration then the second term that is provided by T^{ij} (viz., the *electromagnetic* mass-impulse) will rejoin the following classical expression ([1]):

$$(27') \qquad -\iiint A^i j^k \delta u_k = \frac{e}{c}\bar{A}^i .$$

In summary, one sees that:

1. It is precisely the special intervention of the matrix γ^4 that permits Dirac's theory to reconcile the quadri-vectorial variance of the mass-impulse operator $(15'_1)$ with the variance of the second-rank tensor that is Tetrode's inertia density tensor (26).

2. The expression for the inertia tensor density that conforms to the general principles of wave mechanics is Tetrode's *asymmetric* expression T^{ij} ([2]) , and not Pauli's symmetrized expression $\Theta^{ij} = \frac{1}{2}(T^{ij} + T^{ji})$ ([3]).

3. In the calculation of the probable mean mass-impulse, the "virtual" dummy index that is imposed by the general principles of wave mechanics is the index of the matrix γ^j, and the significant index is then the index that is common to the operators $[\partial^i]$ and A^i.

In the preceding two examples, the variance and physical significance of the operator was known *a priori*, and we deduced the definition of the corresponding density tensor by the intermediary of the general rule (25). We shall now apply the same rule in the opposite sense, in such a manner as to slightly sharpen what we said in general about the classification and physical interpretation of the sixteen matrices γ^A upon starting with the known interpretation of the five density tensors $\psi^\times \gamma\, \psi$. First of all, a series of important remarks must be made.

All of those remarks proceed from the following one: Contrary to what happens in pre-relativistic physics, *there are several integral tensors (i.e., finite tensors) that are associated with the same tensor density in relativistic physics.* Under these conditions, the brute-force application of the quantum formula (25) to *all* of the components of a given tensor density will generally provide components that belong to not just one, but in fact *several, distinct finite tensors.* Moreover, each of those tensors is "realized" by formula (25) only in a "truncated" manner," since some of its components will be "suppressed" by the integration at constant time ($\delta u^u = 0$ for $u = 1, 2, 3$). It follows from this that *in order to apply the quantum formula* (25) *wisely, one must be in possession of a relativistic theory of the quantity under study.*

([1]) *Op. cit.*, pp. 47 and 62.
([2]) "Der Impuls-Energiesatz in der Diracschen Quantentheorie," Zeit. Phys. **49** (1928), pp. 858.
([3]) "Die allgemeinen Prinzipien der Wellenmechanik, B : Relativistischen Theorie," Handb. d. Phys. $XXIV_3$, 1933, pp. 235.

An example might make what we just said more comprehensible. The four finite quantities that one obtains upon applying formula (25) to the components of Dirac's charge-current density $j^{k} = -e\ \psi^{\times}\gamma^{k}\psi$ do not constitute a geometric entity. On the contrary, the invariant *finite electric charge* is provided solely by the fourth integral:

$$[a] \qquad Q = \iiint j^4 \delta u_4 = \iiint q\,\delta u \qquad \left(q = \frac{1}{ic} j^4,\ \delta u = ic\,\delta u^4 \right),$$

in which the three terms $j^{v}\,du_v$ are "suppressed" by the integration at constant time ([1]). The other three integrals have no relationship to the electric charge.

However, from pre-relativistic physics, we know that these three integrals represent the *finite electric current.* In relativity, we must then seek to define a world-integral tensor whose three components will give back the classical pre-relativistic expressions for an integration at constant time. The simplest definition of the *finite electric world-current* will obviously be:

$$[b] \qquad \delta\Gamma^{kl} = j^{k}\,\delta u^l - j^{l}\,\delta u^k,$$

in which the three $\delta\Gamma^{u4}$ represent the current, properly speaking, while the three $\delta\Gamma^{uv}$ will be "suppressed" by a constant-time integration.

We shall pursue that analysis for the other four tensors $\psi^{\times}\gamma^{A}\psi$. As far as the spin density σ^i is concerned, in Chapter II, we shall show that the finite spin must be calculated by a formula that is analogous to [*b*]:

$$[c] \qquad \delta B^{kl} = \sigma^k\,\delta u^l - \sigma^l\,\delta u^k,$$

from which, it will result that under constant-time integration, the three $ie\ B^{u4}$ will be equal to $\iiint \sigma^u \delta u$ (i.e., spin, properly speaking), while the three B^{uv} will be zero. From the relativistic viewpoint, it is therefore not precise to consider the quantity:

$$[d] \qquad X = \iiint \sigma^4 \delta u = ic \iiint \sigma^4 \delta u_4$$

to be the fourth component of a finite spin, as one is occasionally tempted to do. What one must say is that [*d*] is the truncated expression for an invariant whose physical interpretation is not better known, and which is to the density σ^k what the electric charge is to the density j^k.

An interesting question is that of the finite magnetic moment $\mathcal{M}$ and electric moment $\mathcal{E}$. The simplest relativistic definitions that one can give for those quantities are ([2]):

([1]) *La Relativité restreinte*, pp. 44.
([2]) O. COSTA DE BEAUREGARD, "Sur deux questions de Relativité," C. R. Acad. Sci. **213** (1941), pp. 822. Another system of equivalent definitions is obviously:

$$[e^2] \qquad \delta\mathcal{M}^i = \tfrac{1}{2}\, m_{jk}\, [dx^i\, dx^j\, dx^k], \qquad \delta\mathcal{E}^i = \tfrac{1}{2}\, \bar{m}_{jk}\, [dx^i\, dx^j\, dx^k].$$

$$[e^1] \qquad \delta\mathcal{M}^i = \bar{m}^{ij}\,\delta u_j\,, \qquad \delta\mathcal{E}^{\,i} = m^{ij}\delta u_j\,,$$

in which m^{ij} denotes the antisymmetric tensor of the proper magneto-electric moment density, and $\bar{m}^{ij}$ is its dual. Among these formulas, the finite moments are defined to be two quadri-vectors that are orthogonal to the quadri-vector δu_j. Their temporal components will then be "suppressed" by a constant-time integration. As for the other three components, they will then take on (up to a factor of *ic*) the classical truncated form:

$$[e^0] \qquad \delta\mathcal{M}^w = m^{uv}\,\delta u\,, \qquad \delta\mathcal{E}^{\,u} = m^{u4}\delta u\,.$$

To conclude, we shall study two density invariants. The relation (62 B I) of Chapter III equates (up to a factor) the invariant $(\omega^1) = \psi^\times\,\psi$ and the trace of Tetrode's inertial tensor, which permits one to interpret it as the *proper mass density* of the statistical fluid (*see* Chapter II, § II). The quadri-vector:

$$[f] \qquad p^i_{(0)} \approx \iiint (\omega^1)\,\delta u^i \approx \iiint T^j_j\,\delta u^i\,,$$

which is collinear with the quadri-vector δu^i (and therefore has its three components $p^u_{(0)}$ "suppressed" by a constant-time integration), is physically homogeneous to the mass-impulse quadri-vector:

$$p^i = \iiint T^{ij}\,\delta u_j\,,$$

but is obviously distinct from it; for that reason, we say that the integral:

$$\iiint (\omega^1)\,\delta u$$

represents the *proper pseudo-mass* of the electron, up to a factor.

In an analogous manner, one defines a quadri-vector integral, whose physical interpretation is still unknown, of the invariant (ω^2):

$$[g] \qquad Y^i = \iiint (\omega^2)\,\delta u^i\,,$$

Contrary to what we said in the cited Note, it is even possible to define $\mathcal{M}$ and $\mathcal{E}$ to be completely-antisymmetric tensors of rank 3 whose duals will enjoy all of the properties that were indicated in the text, by means of the formulas:

$$[e^3] \qquad \delta\mathcal{M}^{ijk} = \sum m^{ij}\,\delta u^{\,k}, \qquad \delta\mathcal{E}^{ijk} = \sum \bar{m}^{\,ij}\,\delta u^{\,k},$$

$$[e^4] \qquad \delta\mathcal{M}^{ijk} = \sum m_l^{\,i}\,[dx^l\,dx^j\,dx^k], \qquad \delta\mathcal{M}^{ijk} = \sum \bar{m}_l^{\,i}\,[dx^l\,dx^j\,dx^k],$$

in which the various signs $\sum$ are intended to mean *over all circular permutations of i, j, k.* The same considerations are obviously valid for the electromagnetic field, which is homogenous to an electromagnetic moment density; they will find an application in L. de Broglie's Theory of the Photon.

whose three components Y^u are "suppressed" by a constant-time integration.

Endowed with these results, we are now in a position to establish the stated interpretation table for the sixteen γ^A. From a remark that was made in the context of the general quantum formula (25), we know that the set of operators that represent the same finite quantity can have the same *symbolic* variance as the corresponding tensor of classical relativity. We then suppose that we have calculated the components of a certain classical finite tensor that is attached to a tensor density with a known interpretation *by a constant-time integration.* Certain components can then be zero (viz., the *suppressed components*), in which case, the rule (25) will give *zero* for the corresponding symbolic operator component. The other components involve only one term in their expression (*truncated expression*), and after transforming $\psi^\times$ into ψ^+ according to $\psi^\times = i\,\psi^+\gamma^4$, the rule (25) will provide the corresponding non-zero operator components.

For these non-zero components, and when one is dealing with the five tensor densities of the type $\psi^\times\gamma^A\psi$, the desired operator γ^B will be then given by the formula:

$$\gamma^B = i\ \gamma^4\gamma^A,$$

whose systematic application will lead to the following table:

Symbolic variance of the operator that is imposed by relativity		
Invariant…………………………	$\underbrace{\text{Electric charge}}_{I}$	
Quadri-vector……………………	$\underbrace{\text{Proper electric moment}}_{\gamma^1\ \gamma^2\ \gamma^3}$	$\underbrace{\text{Proper pseudo-mass}}_{\gamma^4}$
Antisymmetric tensor of rank 2…	$\underbrace{\text{Proper kinetic moment (spin)}}_{\gamma^{12}\ \gamma^{23}\ \gamma^{31}}$	$\underbrace{\text{Finite electric current}}_{\gamma^{14}\ \gamma^{24}\ \gamma^{34}}$
Pseudo-quadri-vector……………	$\underbrace{\text{Unknown quantity } Y^4}_{\gamma^{123}}$	$\underbrace{\text{Finite electric current}}_{\gamma^{14}\ \gamma^{24}\ \gamma^{34}}$
Pseudo-invariant……………….		$\underbrace{\text{Unknown quantity } X}_{\gamma^{1234}}$

Therefore, *from the standpoint of physical interpretation, each of the five matrix tensors* I, γ^i, γ^{ij}, γ^{ijk}, γ^{ijkl} *will split into two matrix tensors of the same rank*, and we shall say, by convention, that one of them is *time-like* and the other one is *space-like*. In order to exhibit these two *matrix tensors* completely, it will suffice to establish the *zero* operators that were previously predicted. For example, the quadri-vector γ^i of the second row will generate the two quadri-operators:

$$\underbrace{\text{Proper electric moment}}_{\gamma^1\,\gamma^2\,\gamma^3\,0} \qquad \underbrace{\text{Proper pseudo-mass}}_{0\,0\,0\,\gamma^4}\ .$$

In order to facilitate the comparison with the classical table that was constructed by L. de Broglie ([1]), we give the transcription of these results in terms of the α matrices. If the physical coefficients are systematically neglected then the table will be as follows:

Electric charge………………..				I			
Proper electric moment…….....		α^{14}	α^{24}		α^{34}	0	
Finite electric current…………	0	0	0		α^1	α^2	α^4
Proper magnetic moment……..		α^{234}	α^{314}		α^{124}	0	
Proper pseudo-mass…………..		0	0		0	α^4	
Proper kinetic moment (spin)…	α^{23}	α^{21}	α^{12}		0	0	0
Unknown quantity *Y*………....		0	0		0	α^{1234}	
Unknown quantity *X*………….				α^{123}			

Obviously, nothing will change in the *physical* interpretation of the sixteen α. The difference that this yields is that here the α are grouped as if they were the components of a *finite* tensor of classical relativity, whereas in the book by L. de Broglie, they are grouped as if they were the components of the tensor density $\psi^+\alpha\psi$.

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([1]) *L'Électron magnétique*, pp. 225-226.

CHAPTER II

SPECIAL RELATIVISTIC THEORY OF CONTINUOUS MEDIA THAT ARE ENDOWED WITH A PROPER KINETIC MOMENT DENSITY.

9. The present chapter contains the latest state of a study of the dynamics of continuous media that are endowed with a proper kinetic moment or *spin* ([1]) that we began to develop some time ago ([2]). Our initial project, which was quite modest, was simply to show that the properties of spin in Dirac's theory conform to relativistic demands, and that one can deduce them from general postulates that one will encounter in most relativistic questions. For example, that is how we have justified *a posteriori* the representation of spin density by a space-like pseudo-quadri-vector σ^i and established the known formula:

$$\sigma^4 = \frac{1}{c}(\boldsymbol{\sigma} \cdot \boldsymbol{v})$$

by a purely-relativistic argument.

Since then, our theory has continued to develop – driven by its internal potential, so to speak. The new results that are obtained are found to conform to some consequences of Dirac's theory that are not explicit (or at least insufficiently explicit), and to whose development and interpretation they can then contribute. In particular, we have indicated that the inertia tensor T^{ij} of a medium that is endowed with spin is not symmetric, and gave an interpretation to the relation:

$$\overline{T^{ji} - T^{ij}} = -ic\,[\partial^j \sigma^i - \partial^i \sigma^j]$$

that is satisfied by the asymmetric tensor T^{ij} that was originally defined by Tetrode ([3]) in Dirac's theory. That remark, combined with some other considerations that we shall present in the following chapters, leads us to conclude that *the true inertia tensor in Dirac's theory* (and in the theory of spinning particles that are obtained by fusion) *is not Pauli's symmetrized tensor with four terms, but, in fact, Tetrode's asymmetric tensor with two terms.*

In a general manner, one will see that the accord between our pre-quantum, relativistic dynamics of media that are endowed with spin and Dirac's theory (or, more generally, with the theory of spinning particles) is as perfect as one can hope for.

([1]) Although, in principle, the term *spin* is reserved for the *quantum* representation of a *finite* proper kinetic moment, here, we shall use it like a simple synonym in order to abbreviate the discussion of the notion of proper kinetic moment, whether finite or density.

([2]) O. COSTA DE BEAUREGARD, C. R. Acad. Sc. **211** (1940), pp. 228 and 499, and Jour. de Math. **21**, fasc. 3 (1942), pp. 267. We now regard the "pessimistic" conclusions of the latter work to be unjustified, as sub-paragraphs 8 and 9, which now follow 7 (no. **12**, below), will show.

([3]) That relation was given in an equivalent form by Tetrode, and by several other authors since then. However, its true significance in the absence of any dynamics of media that are endowed with spin seems to have escaped them.

Unfortunately, the same thing is not true for the kinematics that we shall give, in turn, which nevertheless seems to be satisfactory in its own right. For example, we think that, by analogy with the classical relativistic case, the asymmetric inertia tensor T^{ij} must be the general product of two non-collinear quadri-vectors, one of which represents the "true" world-current, and the other of which represents a "false" current. Now, although, as one knows, Dirac's theory introduces two current quadri-vectors:

$$\psi^\times \gamma^i \psi \qquad \text{and} \qquad \psi^\times [\partial^i]\, \psi - 2i\varepsilon A^i\, \psi^\times \psi,$$

one can show, with the aid of some quadratic identities of Kofink that we shall discuss in Chapter IV, that the Tetrode tensor:

$$\psi^\times [\partial^i]\, \gamma^j \psi - 2i\varepsilon A^i\, \psi^\times \gamma^j \psi$$

is not homothetic to the general product of those two currents. It is nonetheless remarkable that its expression "resembles" the product in question, and we shall see in Chapter IV that one can appeal to that "resemblance" in order to infer some valid conclusions. The kinematics that we shall propose, although it is insufficient to lead back to Dirac's theory, can then be considered to be a first approximation of what that would constitute.

Among the various notions that enter into the dynamics and kinematics that we shall present, the ones that figure notably are that of a *mass-impulse that is oblique to the world-trajectory of a material point that is endowed with spin* and correspondingly that of a *transverse mass-impulse*. These notations have been encountered already by numerous authors that have treated relativity ([1]), but it seems to us that it still remains for them to be integrated into the bosom of a coherent theory.

I. – DYNAMICS OF MEDIA THAT ARE ENDOWED WITH SPIN.

10. The fundamental remark is that *the origin of a proper kinetic moment density σ will not be found in the inertial forces of traditional dynamics.* Indeed, isolate a spherical droplet of radius r in a material medium of density ρ and animate it with a velocity $\mathbf{v}$. The moment of inertia of that droplet will be $\frac{8}{15}\pi\rho\, r^5$, and its angular velocity will be $\frac{1}{2}\text{rot}\,\mathbf{v}$, so its kinetic moment will be an infinitesimal of fifth order in r, which is an order that is too high by two units in order for one to be able to define a corresponding density. That is, in order to establish the dynamics of media that are endowed with spin, we shall be forced to proceed axiomatically by imposing the tensorial rules of variance and homogeneity, and we shall appeal to some reasonable postulates that are suggested by the classical theories of relativity.

The point of departure of all our theory will be this: *A finite kinetic moment is represented by the three components C^{uv} of an antisymmetric second-rank tensor C^{ij}.* That fundamental fact will result unambiguously from the consideration of a material

([1]) Von LAUE, *Relativité* (trad. G. Létang). t. I, pp. 126 and 255; PROCA, *Thesis*, pp. 145.

point with world-coordinates x^i and mass-impulse p^i. Indeed, from classical dynamics, the *kinetic moment at the origin* of that material point is represented by the three components C^{uv} of the antisymmetric tensor:

$$C^{ij} = x^i p^j - x^j p^i. \tag{31}$$

We seek the significance of the three components C^{u4} of that tensor. If one replaces x^4 and the p^i with their values ict and $p^u = mv^u$, $p^4 = icm$ (m denotes the relativistic mass of the point, and v^u is its ordinary velocity), respectively, then one will get:

$$C^{u4} = icm\,(x^u - v^u t). \tag{31′}$$

Therefore, the three C^{u4} are nothing but the components of the *generalized barycentric moment* ([1]); here, we denote the dual of the tensor C^{ij} by $ic\,B^{kl}$.

Having established our point of departure as we just said, we shall proceed to the axiomatic establishment of the dynamics of continuous media that are endowed with a proper kinetic moment density σ; the successive postulates that we will be led to formulate will, in turn, be denoted by Roman capitals A, B, …

11. –

1. *If the quantity* σ *exists* (existence postulate A) *then it will be represented by a tensor.*

Indeed, by the definition itself of a density, a certain product of σ with the world-volume element tensor $[dx^i\,dx^j\,dx^k]$ must yield a kinetic moment, which is represented by a tensor.

2. *The fact that the finite kinetic moment has rank* 2 *implies that the rank of the corresponding density* σ *is* 1 *or* 3.

Indeed, let n be the unknown rank of the tensor σ, let m be the number of dummy indices in the tensor product of σ by $[dx^i\,dx^j\,dx^k]$, and let $s = n - m$ be the number of significant indices of σ. One has the two homogeneity relations:

$$\begin{aligned} 2 &= (3 - m) + s \qquad &\text{or}\qquad m &= 1 + s,\\ n &= m + s \qquad &\text{so}\qquad n &= 1 + 2s. \end{aligned}$$

Since the integers n, m, and s must be positive or zero, one can write the inequalities:

([1]) When the material point is considered simultaneously with the origin of the moment C^{ij}, the three $\frac{1}{ic}C^{u4}$ will represent the usual barycentric moment; for t that is infinitely small ($t = dt$), the additive term will be interpreted as a *correction for non-simultaneity*.

$$s < m \leq n,$$

and since the same integers must be equal to at most 4, the only admissible hypotheses will ultimately be:

$$[H] \qquad \begin{cases} s = 0, & \text{which gives} \quad m = 1 \quad \text{and} \quad \boxed{n = 1,} \\ s = 1, & \quad\text{"} \qquad\qquad m = 2 \quad \text{and} \quad \boxed{n = 3.} \end{cases}$$

The hypothesis $s = 2$ is not appropriate, since it will give $n = 5$.

Finally, as we have asserted, the rank n of σ is necessarily 1 or 3. Moreover, the preceding argument fixes the number m of dummy indices under each of the two hypotheses $s = 0$, $s = 1$, which are hypotheses that we denote by $[H_1]$ and $[H_2]$, respectively.

3. It remains for us to now address the antisymmetry property of the finite tensor C^{ij}, which is a property that must necessarily be satisfied.

Up to an unimportant factor, the "natural" way to write the hypothesis $[H_1]$ is:

$$[H_1] \qquad \delta C^{kl} = \sigma_i \, [dx^i \, dx^k \, dx^l] \, .$$

One sees that this notation will "automatically" insure the antisymmetry of the finite tensor C^{kl}.

On the contrary, the "brute-force" writing of the hypothesis $[H_2]$:

$$[H_2'] \qquad \delta B^{ij} = \tfrac{1}{2} \sigma^i{}_{kl} \, [dx^j \, dx^k \, dx^l] \qquad \text{or} \qquad \delta B^{ij} = \tfrac{1}{2} \sigma^i{}_{kl} \, [dx^j \, dx^k \, dx^l]$$

will not "automatically" insure the antisymmetry of the finite tensor δB^{ij} ([1]). That antisymmetry will be insured only by means of a particular choice of trilinear integration element, which is inadmissible. If we desire that the hypothesis $n = 3$ might be suitable then it will be necessary to satisfy the postulate (B) for *an arbitrary trilinear integration element.*

Here, postulate $[B]$ will oblige us to replace the expressions $[H_2']$ with their antisymmetric combination:

$$[H_2] \qquad \delta B^{ij} = \tfrac{1}{2} \{ \sigma^i{}_{kl} \, [dx^j \, dx^k \, dx^l] - \sigma^j{}_{kl} \, [dx^i \, dx^k \, dx^l] \}.$$

The factor 1/2 was introduced for a reason that will become apparent in a moment.

4. The hypothesis $[H_1]$ presents another remarkable property that does not generally pertain to $[H_2]$. When one integrates *at constant time* – i.e., when the three $[dx^u \, dx^v \, dx^4]$

([1]) In what follows, we will show that the tensors C^{kl} and $ic \, B^{ij}$, which are defined by $[H_1]$ and $[H_2]$, respectively, are in fact duals of each other.

are zero (which is a hypothesis that is called "simultaneity") – the expression $[H_1]$ will reduce to:

$$[H_1^0] \qquad \delta C^{vw} = \sigma_u\, \delta u, \qquad \delta C^{u4} = 0.$$

δu denotes the "pure volume" element $[dx^u\, dx^v\, dx^w]$. Therefore, with the notations $[H_1]$, the three components of the proper barycentric moment will be zero under the simultaneity hypothesis, and the three components of the proper kinetic moment will each be linked to their densities σ_u by the well-known pre-relativistic formula. These are circumstances that one constantly encounters in the usual theory of special relativity, and it is very natural to seek to insure theM in a general manner in a physical theory like the one that we are establishing. We therefore have postulate (*C*), namely, that *under the simultaneity hypothesis, the barycentric moment must be zero and the kinetic moment must contain only one term in its expression.*

It is indeed clear that if the tensor $\sigma^i{}_{kl}$ of hypothesis $[H_2]$ is arbitrary then that postulate will not be satisfied. Indeed, we first see that for a well-defined pair k, l of dummy indices, each of the terms that are written down will yield two generally different terms by permutation of k and l. Our postulate (*C*), which, by virtue of (*B*), must be satisfied for no particular Galilean frame, will then already impose antisymmetry on σ_{ikl} with respect to the pair of indices k, l. By means of that, the two terms that are obtained by permutation of k and l will be identically equal, and we can agree to group them in such a way as to neglect the factor 1 / 2.

We then write $[H_2]$ under the simultaneity hypothesis. If we take $j = v$ then the only non-zero term in the first group will be provided by $k = w$ and $l = u$. Similarly, if we take $i = u$ then the only non-zero term in the second group will be provided by $k = v$ and $l = w$, while if we take $i = 4$ then all of the terms in the second group will be zero. Finally, under the hypothesis of simultaneity, $[H_2]$ will be specified as follows ([1]):

$$\delta B^{uv} = (\sigma^{uwu} - \sigma^{vvw})\,[dx^1\, dx^2\, dx^3] = -\,(\sigma^{uuw} + \sigma^{vvw})\,[dx^1\, dx^2\, dx^3],$$
$$\delta B^{4v} = \sigma^{4wu}\,[dx^1\, dx^2\, dx^3].$$

By virtue of postulate (*C*), it is necessary that one of these groups of components (which represents the barycentric moment) should be zero, and that the other one (which represents the kinetic moment) should involve only one term in its expression. Moreover, by virtue of postulate (*B*), that result must be obtained in any Galilean frame. Now, with σ that are not identically zero, that will be possible only if the quantity $(\sigma^{uuw} + \sigma^{vvw})$ is identically zero, which will imply the antisymmetry of the tensor σ^{ijk} with respect to the first two indices.

5. Finally, since the set of postulates (*B*) and (*C*) is intended to provide the case $n = 3$ with two remarkable properties that are inherent to the case $n = 1$, we impose the complete antisymmetry of the tensor σ^{ijk} upon the expression $[H_2]$. I then say that *the two cases* $n = 1$ *and* $n = 3$ *are completely coincident;* indeed, they are written:

(1) Of course, that notation is intended to mean that there is *no summation* over repeated indices.

$$[H] \qquad \begin{cases} \delta C^{kl} = \sigma_i [dx^i dx^k dx^l], \\ \delta B^{ij} = \frac{1}{2}\{\sigma^i_{\ kl}[dx^j dx^k dx^l] - \sigma^j_{\ kl}[dx^i dx^k dx^l]\}. \end{cases}$$

If one passes to the dual quantities then the last two statements will coincide with:

$$[K] \qquad \delta B^{ij} = \sigma^i \delta u^j - \sigma^j \delta u^i .$$

Indeed, first take the formula $[H_2]$. i differs from j due to the antisymmetry of δB^{ij}, k differs from l due to the antisymmetry of σ or of $[dx^i\, dx^j\, dx^k]$, and k and l will be different from i and j for the same reason. If one abstracts from the permutation of k and l then each of the groups of terms that are written down will yield only one term, and upon introducing the dual quadri-vectors σ^j and $ic \cdot \delta u^i$ ([1]) of the two third-rank tensors, one will in fact arrive at formula $[K]$. Now, take formula $[H_1]$. Since k differs from l, and i differs from k and l, there will be only two non-zero terms in the right-hand side. If one introduces the duals $ic \cdot \delta B^{ij}$ and $ic \cdot \delta u^i$ ([1]) of δC^{kl} and $[dx^i\, dx^j\, dx^k]$, resp., then one will arrive at precisely the formula $[K]$. Conforming to what we said, the two tensors C^{ij} and $ic \cdot B^{kl}$ seem to be duals of each other.

Before we go on, we summarize the conclusions that were obtained already: The set of postulates (A) (*existence of a density* σ), (B) (*arbitrary integration hypersurface*), (C) (*vanishing of the barycentric moment and the writing of just one kinetic moment term under the simultaneity hypothesis*) lead us to couple the finite moment δC or δB to the corresponding density that is represented by a quadri-vector σ^i by one or the other of the following equivalent expressions:

$$(32) \qquad \boxed{\delta C^{kl}_{(p)} = \sigma_i [dx^i dx^k dx^l],} \qquad \boxed{\delta B^{ij}_{(p)} = \sigma^i \delta u^j - \sigma^j \delta u^i,}$$

in which $ic \cdot \delta u^i$ denotes quadri-vector that is dual to the trilinear element $[dx^i\, dx^j\, dx^k]$, and the component $ic \cdot \delta u^4 = [dx^u\, dx^v\, dx^w] = \delta u$ represents the usual *pure volume element*. The three $\delta C^{uv} = ic \cdot \delta B^{w4}$ are the components of the proper kinetic moment $\delta\mathbf{C}$, and the three $\frac{1}{ic}\delta C^{w4} = \delta B^{uv}$ are those of the proper barycentric moment $\delta\mathbf{B}$ of a fluid droplet δu^i. Under the hypothesis of simultaneity, (32) will reduce to:

$$(32') \qquad \delta C^u = \sigma^u\, \delta u, \quad \delta B^u = 0,$$

which are formulas that are nothing but the ones that are used "spontaneously" by Dirac's theory.

6. The usual theories of special relativity suggest that we must formulate a new postulate that is found to imply another known property of the Dirac density σ. We know that the degenerate forms of the tensorial equations under the simultaneity hypothesis and

([1]) These notations are the ones that we have utilized in our *Relativité restreinte*; see especially pp. 19 and 31.

under the hypothesis in which the Galilean frame is the *co-moving* or *proper* frame are generally quite analogous. We then postulate (*D*) that *one must recover the density formula* ($32'_1$) *for the kinetic moment* δC *in the co-moving Galilean frame.* By virtue of (32), that demands that σ^i must be annulled in that system, or equivalently, that the quadri-vector σ^i must be orthogonal to the world-current quadri-vector $v^i = dx^i : ds$:

(33) $$\boxed{\sigma^i v_i = 0,}$$

or furthermore, if **v** denotes the velocity of the fluid in the usual sense, and $\boldsymbol{\sigma}$ denotes the spatial vector that has the three σ^u for its components:

(33′) $$\sigma^4 = \frac{1}{c}(\boldsymbol{\sigma} \cdot \mathbf{v}).$$

One knows that the property that is expressed by formula (33) is effectively satisfied in Dirac's theory ([1]).

One sees that the last postulate (*D*) is introduced quite independently of the postulates (*A*), (*B*), (*C*). Despite its "natural" character, one can demand that it must be somewhat arbitrary. The following number will show that (33) has a whole series of interesting simplifications in the formulas as a consequence, in such a way that we shall make it an essential element of our theory, even though it is considered to be independent of Dirac's theory.

12. –

7. *The ponderomotive moments that are coupled with the proper kinetic moment.* – Take the integral of the expression (32_2) over a closed tri-dimensional domain and transform it into a quadruple integral; one will get:

[*p*] $$\iiint \delta B^{ij} = \frac{1}{ic}\int\iiint (\partial^j\sigma^i - \partial^i\sigma^j)\,[dx^1\,dx^2\,dx^3\,dx^4].$$

Choose that closed tri-dimensional domain to be the one that is determined by the lateral hyper-wall of a current world-tube, and two generally curvilinear, everywhere-space-like hypersurfaces, in which figure two distinct "non-simultaneous states" of the same finite "finite drop." On the left-hand side of formula [*p*], the portion of the integral that corresponds to the two hypersurfaces represents the variation dB^{ij} of the *proper kinetic moment-barycentric moment* of the drop considered. One can then say that formula [*p*] provides a decomposition of the variation of the *proper kinetic moment-barycentric moment* of the fluid drop, in which one of the terms of that decomposition is the portion

([1]) The general proof of that formula in Dirac's theory was give by W. Pauli; see below, formula (48_1). It goes without saying that in Dirac's theory, the current quadri-vector of the statistical fluid does not define a *co-moving Galilean frame.*

of the triple integral that corresponds to the hypersurface (with the sign changed), and the other, to the quadruple integral that appears in the right-hand side. In order to find the interpretation of those two terms, we specialize the two hypersurfaces into planes that are orthogonal to the time axis in the Galilean frame used, and infinitely-close in time; let dt be their temporal interval.

Recall a property of the trilinear element of a hypersurface. It is natural to define that element with the aid of:

1. Two small space-like vectors $\delta x_1'$ and $\delta x_2'$ that are tangent to the surface contour of the fluid drop (which is "not simultaneous," in general).

2. The element of the time-like world-current dx^i.

Under these conditions, the four components of the trilinear element of the hyper-wall are the determinants that are extracted from the matrix:

$$\begin{Vmatrix} \delta x_1^1 & \delta x_1^2 & \delta x_1^3 & \delta x_1^4 \\ \delta x_2^1 & \delta x_2^2 & \delta x_2^3 & \delta x_2^4 \\ dx^1 & dx^2 & dx^3 & dx^4 \end{Vmatrix}.$$

The six minors that are extracted from the first two rows are the components of the *generalized area element*, while the three $[\delta x^u\ \delta x^v]$ represent the area, properly speaking. It is then obvious that the trilinear element of the hyper-wall is a certain exterior product of that element with the area by the quadri-vector element of the trajectory. In a more precise manner, if we introduce the duals $ic\ \delta u^i$ and $ic\ \delta s^{kl}$ of the trilinear element of the hyper-wall and the bilinear element of the contour of the drop, resp., then we will effortlessly verify the relation ([1]):

$$\delta u^j = \delta s^{ij}\, dx_i\ ;$$

the three $ic\ \delta s^{w4} = \delta s^w$ are the components of the area element, properly speaking.

Under the special hypothesis of simultaneity, which we adopt as was said, the three δs^{uv} will be zero. If one recalls that $x^4 = ict$, and if one always lets v^u denote the three components of the ordinary velocity $\mathbf{v}$ of the fluid then the formula considered will be specified as follows:

$$\delta u^w = -\,\delta s^w\, dt, \qquad ic\ \delta u^4 = \delta s^u\, dx_u = \delta s^u \cdot v_u\, dt = (\delta \mathbf{s} \cdot \mathbf{v})\, dt.$$

Finally, if one substitutes these expressions in (32_2) then one will get:

$$\delta B^{uv} = -\,(s^u\ \delta s^v - s^v\ \delta s^u)\, dt = dt\,(\boldsymbol{\sigma} \wedge \delta \mathbf{s})^w,$$
$$ic\ \delta B^{w4} = \{(\delta \mathbf{s} \cdot \mathbf{v}) + \sigma^4 \cdot ic\ \delta s^w\}\, dt$$

for the triple integral element of a hyper-wall.

([1]) O. COSTA DE BEAUREGARD, *La Relativité restreinte*, pp. 31.

Conforming to what we said, the last expression will be "arranged" in a remarkable manner if we take into account the consequence (33′) of postulate (*D*): Indeed, we will see the following double exterior product appear:

$$ic\,\delta B^{w4} = \sigma^{w}(\delta\mathbf{s}\cdot\mathbf{v}) - \delta s^{w}(\boldsymbol{\sigma}\cdot\mathbf{v})\,dt = \{\mathbf{v}\wedge(\boldsymbol{\sigma}\cdot\delta\mathbf{s})\}^{w}\,dt.$$

We then remember that on a hypersurface, the three δB^{uv} represent the elementary proper barycentric moment $\delta\mathbf{B}$, and the three $ic\delta B^{w4}$ represent the elementary proper kinetic moment $\delta\mathbf{C}$. If we set δB^{uv} equal to $-\,d_{(s)}\,B^{w}$ and $ic\delta B^{w4}$ equal to $-\,d_{(s)}\,C^{w}$, by definition, then the preceding formulas can be written:

$$[p'] \qquad d_{(s)}\mathbf{B} = -\,dt\iint \boldsymbol{\sigma}\wedge\delta\mathbf{s}, \qquad d_{(s)}\mathbf{C} = -\,dt\iint \mathbf{v}\wedge(\boldsymbol{\sigma}\wedge\delta\mathbf{s}).$$

From classical dynamics, we see clearly then that the quantities $-\,\boldsymbol{\sigma}\wedge\delta\mathbf{s}$ and $-\,\mathbf{v}\wedge(\boldsymbol{\sigma}\wedge\delta\mathbf{s})$ are interpreted as the elementary surface ponderomotive moments that correspond to the barycentric moment and the proper kinetic moment.

After studying the triple integral over the hyper-wall, we then study the quadruple integral that appears in the right-hand side of [*p*]. Always using the particular tri-dimensional contour that we have specified, it can be written:

$$d_{w}\,B^{ij} = dt\iiint [\partial^{j}\sigma^{i} - \partial^{i}\sigma^{j}]\,\delta u,$$

in which δu denotes the pure volume element $[dx^1\,dx^2\,dx^3]$, which is formula in which we read, from classical dynamics, that the quantity in brackets is (up to a factor *ic*) a volume density of world-ponderomotive moment. We then set:

$$\mu^{ij}_{(1)} = ic\,\overline{[\partial^{j}\sigma^{i} - \partial^{i}\sigma^{j}]},$$

in which the three μ^{uv} represent the density of ponderomotive moment in the usual sense.

One will immediately get:

$$d_{(v)}\,B^{uv} = dt\iiint [\text{rot}\,\boldsymbol{\sigma}]^{w}\,\delta u$$

for the barycentric moment B^{uv} and $C^{vw} = ic\,B^{u4}$ for the kinetic moment. If one takes into account the consequence (33_2) of postulate (*D*) then the bracket will be "arranged" in such a manner that it contains only the three spatial components of σ^{i}:

$$d_{(v)}\,C^{uv} = dt\iiint \{\partial^{4}\sigma^{w} - \partial^{w}(\boldsymbol{\sigma}\cdot\mathbf{v})\}\,\delta u.$$

Finally, we can write:

$$[p''] \qquad d_{(v)}\,\mathbf{B} = dt\iiint \text{rot}\,\boldsymbol{\sigma}\,\delta u, \qquad d_{(v)}\,\mathbf{C} = dt\iiint \{\text{grad}\,(\boldsymbol{\sigma}\cdot\mathbf{v}) + \frac{\partial}{\partial t}\boldsymbol{\sigma}\}\,\delta u,$$

in such a way that the quantities rot $\boldsymbol{\sigma}$ and {grad ($\boldsymbol{\sigma} \cdot \mathbf{v}$) + $\frac{\partial}{\partial t}\boldsymbol{\sigma}$} are interpreted as volume ponderomotive densities that correspond to the barycentric and proper kinetic moments, resp. One has the relation:

$$d = d_{(s)} + d_{(v)}$$

between the differentials $d_{(s)}$ and $d_{(v)}$ that were defined by formulas [p'] and [p''] and the d that was defined initially (p. o. ?). Our analysis will then permit us to distinguish the contribution of the surface forces from that of the volume forces in the global variation of the *proper kinetic moment-barycentric moment*.

8. *The ponderomotive moments that are coupled to the orbital kinetic moment. Expression for the total volume density of proper ponderomotive moment.* – One knows that if T^{ij} denotes the inertia tensor of a continuous material medium then *the mass-impulse* of a finite portion of the medium will be given by the integral:

$$p^i = \iiint T^{ij}\delta u_j \,, \tag{34}$$

which is extended over a space-like hypersurface. Correspondingly, one shows that *ponderomotive world-force density* that is applied to medium considered is derived from T^{ij} by the formula of generalized elasticity ([1]):

$$f^i = \partial_j T^{ij} \,. \tag{35}$$

In classical relativistic dynamics, the tensor T^{ij} is symmetric by its very definition, in such a way that the dummy index in the summation in the preceding formulas will be arbitrary. However, as in pre-relativistic elasticity, we shall consider an *asymmetric* tensor T^{ij} ([2]) that is intended to account for the proper ponderomotive moments. Under those conditions, the summation index in formulas (34), (35), and some formulas that are consequences of them must be fixed by an initial definition. We then agree that for all of what follows in this work, the *summation index will be the second index in* T^{ij}, or equivalently, that the significant index of the quadri-vectors p^i and f^i will be the first index of T^{ij}.

From the classical general definition (31), the moment δu_k of the mass-impulse of a droplet at the origin – or *orbital kinetic moment* of that droplet – is:

$$\boxed{\delta C_0^{ij} = [x^i T^{jk} - x^j T^{ik}]\,\delta u_k ,} \tag{36}$$

([1]) *La Relativité restreinte*, pp. 50. That proof utilizes the fact that the hyper-wall integral is identically zero; we shall return to that point in the following paragraph.

([2]) In general relativity, asymmetry in the inertia tensor would imply asymmetry in the tensor $R^{ij} - \frac{1}{2} R\, g^{ij}$; i.e., an asymmetry in the curvature tensor, since one cannot image what asymmetry in the metric tensor would correspond to. That asymmetry can be obtained either by the use of a Weyl gauge or, more simply, by torsion in the universe. We specify that this torsion is not intended to be interpreted as electromagnetism, as in some unitary theories, but that is corresponds to a *gravitational effect of spin*.

in such a way that the *orbital kinetic moment* of a finite portion of the medium will be given by the integral:

$$\delta C_0^{ij} = \iiint [x^i T^{jk} - x^j T^{ik}]\, \delta u_k$$

that is extended over a space-like hypersurface. The tensor in brackets, which has rank three and is antisymmetric in *i, j*, is then interpreted as the *orbital kinetic moment density* of the medium with respect to the origin of the space-time coordinates ([1]).

Take the integral of the expression (36) over the closed tri-dimensional domain that is defined by two space-like hypersurfaces 1 and 2 and the lateral hyper-wall of a current world-tube. Upon assuming that the hyper-wall integral is identically zero ([2]), the expression that will be obtained will obviously represent the *variation of the orbital kinetic moment* of the fluid drop considered between state 1 and state 2. Moreover, when one transforms that integral into a quadruple integral and takes into account the relation (35), as well as the fact that $\partial_k x^i = \delta_k^i$, one will get the expression:

$$\frac{1}{ic}\int\iiint \{[x^j f^i - x^j f^i] + [T^{ji} - T^{ij}]\}[dx^1 dx^2 dx^3 dx^4],$$

from which, one reads, upon repeating the argument that was made before, that the action of the *orbital ponderomotive moment density:*

$$\mu_0^{ij} = [x^j f^i - x^i f^j]$$

is added to that of a *proper ponderomotive moment density* of "elastic" type:

$$\mu_p^{ij} = [T^{ji} - T^{ij}].$$

Now consider the *total barycentric moment-kinetic moment* (orbital + proper) of the finite fluid drop:

$$\boxed{C^{ij} = C_0^{ij} + C_p^{ij}.}$$

It evolves under the action of the set of ponderomotive moments that we studied, which are:

1. *The orbital ponderomotive moment*, which is derived from the volume density μ_0^{ij}.

([1]) Some authors who treated the kinetic moment in Dirac's theory seemed to assume that the [ ] considered represented the *total kinetic moment density* (orbital + proper). The origin of that way of looking at things – which, to us, is not compatible with the primitive definitions (31) and (34) – is the presence of the second [ ] in the quadruple integral that will be given in a moment, and the fact that the expression for [ ] transforms by virtue of formula (37′), which is a consequence of Dirac's theory. One will see how our theory interprets that second [ ] *directly*, and in a manner that is the only correct one, to our understanding.

([2]) The necessary and sufficient condition for that result is that the tensor T^{ij} must have the expression (39) above.

2. *The proper ponderomotive moment of volume origin*, whose total density μ^{ij} is the sum of two terms μ_2^{ij} and μ_1^{ij} that are attached to the *orbital kinetic moment* and the *proper kinetic moment*, respectively, and has the expression:

$$\boxed{\mu^{ij} = [T^{ji} - T^{ij}] + ic\overline{[\partial^j\sigma^i - \partial^i\sigma^j]}.} \tag{37}$$

3. Finally, the *proper ponderomotive moment of surface origin*, which is defined by [*p′*], pp. 38, and obviously corresponds to a contact action of the rest of the fluid on the bounded drop by its motion.

In the following chapter, we will see that in Dirac's theory, one will have:

$$\overline{[T^{ji} - T^{ij}]} + ic[\partial^j\sigma^i - \partial^i\sigma^j] = 0 \tag{37′}$$

identically, in which T^{ij} denotes Tetrode's asymmetric inertia tensor. From what we just said, that relation signifies that the *total volume density of the proper ponderomotive moment is identically zero* ([1]).

([1]) Upon posing the last result as a *postulate*, one can consider formula (37) to be a consequence of our theory, but that postulate is not imposed out of necessity, and we prefer to preserve our general formula (37).

An equivalent expression for the postulate in question is that the total kinetic moment must be conserved in the absence of forces f^i (since, in fact, the action of the surface ponderomotive moment will disappear when one stretches the wall of the world-hyper-tube indefinitely). When Pauli applied that criterion to the expression:

$$\iiint [x^i T^{jk} - x^j T^{ik}]\delta u_k ,$$

which is regarded as representing the *total* kinetic moment (orbital + proper), that author was led to set:

$$\Theta^{ji} - \Theta^{ij} = 0,$$

instead of our formula (37′).

For us, since the expression considered is formally that of an *orbital* moment, the tensor Θ^{ij} that it involves will not be the *true* inertia tensor of a medium that is endowed with spin. Nevertheless, in Dirac's theory, Pauli's symmetrized tensor:

$$\Theta^{ij} = \tfrac{1}{2}(T^{ij} + T^{ji})$$

permits one to calculate the quantities of the *mass-impulse integral* and the *total kinetic moment* by means of formulas that are valid for classical media without spin; as such, it can be regarded as the inertia tensor of a fictitious fluid *without spin* that is integrally equivalent to the Dirac statistical fluid.

Indeed, as far as the kinetic moment is concerned, if one assumes (as will be verified *a posteriori*) that the tensors T^{ij} and Θ^{ij} of Dirac's theory satisfy the criterion that $\mu^{ij} \equiv 0$ then the integral equivalence in question will be attained by means of the condition:

$$\partial_j \Theta^{ij} \equiv \partial_j \Theta^{ji} = \partial_j T^{ij},$$

which is effectively realized thanks to the continual equality of the two divergences of the Tetrode tensor:

$$\partial_j T^{ij} \equiv \partial_j T^{ji}$$

9. *Transverse mass-impulse.* We agree to call the two quadri-vectors:

$$(34') \qquad p^i_{(1)} = \iiint T^{ji}\,\delta u_j\,, \qquad p^i_{(2)} = \iiint [T^{ij} - T^{ji}]\,\delta u_j\,, \qquad (p^i = p^i_{(1)} + p^i_{(2)})$$

the *false mass-impulse* $p^i_{(1)}$ and the *transverse mass-impulse* $p^i_{(2)}$, respectively, and we propose to study the *transverse mass-impulse* $p_{(2)}$. To commence, we remark that this quadri-vector is orthogonal to the quadri-vector world-volume element δu_i , since the expression $[T^{ij} - T^{ji}]\ \delta u_j\ \delta u_i$ is identically zero by virtue of the antisymmetry of $[T^{ij} - T^{ji}]$. That being the case, the relation (37′) will permit us to write:

$$p^i_{(2)} = -\iiint \mu^{ji}\,\delta u_j - ic\iiint \overline{[\partial^i\sigma^j - \partial^j\sigma^i]}\,\delta u_j$$

$$= -\iiint \mu^{ji}\,\delta u_j - \tfrac{1}{2}\iiint [\partial_k\sigma_l - \partial_l\sigma_k][du^i dx^k dx^l]\,.$$

One passes from the first expression to the second one by taking the duals of the two antisymmetric tensors in the last term. It is easy to verify that these terms are the same, and that an equal number of them will enter into each of the two expressions. One then transforms the last integral into a double integral by:

$$\tfrac{1}{2}\iint \sigma_l[dx^i dx^l] + \sigma_k[dx^i dx^k] = \iint \sigma_k[dx^i dx^k]\,.$$

Under the hypothesis of simultaneity, the three $[\delta x^u\ \delta x^v] = \delta s^w$ that represent the area element, properly speaking, will only be non-zero, and if one sets $v^u = \mu^{u4}\, ic$ then one will get simply:

$$p^u_{(2)} = -\frac{1}{ic}\iiint \mu^{u4}\delta u - \iint \sigma^w\delta s^v - \sigma^v\delta s^w = -\iiint v^u\,\delta u - \iint [\boldsymbol{\sigma}\wedge\delta\mathbf{s}]^u\,, \qquad p^4_{(2)} = 0\,.$$

In other words, and conforming to the remark that we just made, the transverse mass will then be zero, while the transverse impulse will be given by the formula:

$$(34'') \qquad \mathbf{p}_{(2)} = -\iiint \mathbf{v}\,\delta u - \iint \boldsymbol{\sigma}\wedge\delta\mathbf{s}\,.$$

These two integrals were already taken under consideration in 7. They can then be interpreted as volume and surface *ponderomotive moments* that correspond to the *proper barycentric moment.* That coincidence, which is fortuitous and correct from the

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(*see* below, no. **23**). An analogous argument will be true for the problem of mass-impulse.

We insist upon the fact that the equivalence of T^{ij} and Θ^{ij} is only *integral*, and that one must take into account the contribution from the hyper-wall if one does not integrate over all space. For us, the *true* inertial tensor of Dirac's theory remains Tetrode's asymmetric tensor T^{ij}.

[W. PAULI, Handb. Phys. $XXIV_3$, pp. 235] standpoint of dimensional analysis, moreover, appears only under constant-time integration. More generally, one sees that the quantities of *proper kinetic moment* and *transverse mass-impulse* of a finite material drop are two consequences of the existence of the density σ^i, but that they are not coupled directly by any algebraic formula. Finally, we remark that one will recover the generate formula (37‴) in the Galilean frame that moves with the fluid.

Remark. – Besides the quantity $p^i_{(1)}$ that was just defined, another "*false mass-impulse*" that appears in certain problems is:

$$p^i_{(0)} = T^j_j \delta u^i .$$

That quadri-vector is collinear with the world-volume element δu^i. Since we just saw that the *transverse mass-impulse* is orthogonal to δu^i, a seductive idea is present in spirit: Is it possible to choose the world-orientation of the quadri-vector δu^i in such a manner that the two false *mass-impulses* are everywhere equal to each other? In that manner, the true mass-impulse would be found to decompose into two mutually-orthogonal quadri-vectors, one of which is time-like, and the other of which is space-like.

Solving that problem amounts to solving the system of four homogeneous linear equations:

$$T^{ji}\,\delta u_j = T^j_j\,\delta u^i \qquad \text{or} \qquad \boxed{\{T^{ik} - T^j_j \delta^{ik}\}\delta u_i = 0.}$$

The necessary and sufficient condition for the world-direction of the quadri-vector δu_i to be determined uniquely is that the determinant of:

$$T^{ik} - T^j_j \delta^{ik}$$

must be zero, while one of its minors of rank three is not. Moreover, by reason of the obvious symmetry, there then will exist a second world-orientation of the quadri-vector δu^i that will make the infinitesimal *true mass-impulse* $T^{ij}\,\delta u_j$ collinear with $T^j_j \delta u^i$. This situation is encountered in the definition of the tensor T^{ij} that we shall now give.

II. – ON THE KINEMATICS OF MEDIA THAT ARE ENDOWED WITH SPIN.

13. The arguments in the preceding paragraph are not independent of all kinematical considerations. We had to introduce the notion of current world-line, or – what amounts to the same thing – that of the velocity of the material points in the ordinary sense:

1. In order to define the generalized barycentric moment [eq. (31′)].
2. In order to establish the relation (33) by starting from the postulate (*D*).
3. In order to interpret the hyper-wall trilinear element (eqs. [*q*] and [*q*′]).

The consequences of relations (33) and [q'] are found in the expressions [p'] and [p''] for the surface and volume ponderomotive densities.

In classical relativistic theory, the notions of *velocity* **v** of material particles and *mass density* ρ of continuous medium are sufficient to define the inertia tensor, which is obviously symmetric, by the formulas:

$$T^{uv} = \rho\, v^u\, v^w, \qquad T^{u4} = T^{4u} = ic \cdot \rho\, v^u, \qquad T^{44} = -\rho\, c^2 .$$

Equivalently, if ρ_0 denotes the *proper mass density* of the medium (i.e., the value of ρ in the co-moving Galilean frame), and v^i is the *unit quadri-vector that is tangent to the world-trajectory* then one will have the condensed definition ([1]):

$$(39') \qquad T^{ij} = -c^2 \rho_0\, v^i\, v^j.$$

As for the trace of that tensor, one will have two interesting expressions for it:

$$(40') \qquad T_i^i = \rho\,(v^2 - c^2) = -c^2 \rho_0 .$$

The well-known Lorentz contraction factor enters into the first one; the second one shows that, up to the factor $-c^2$, the trace of the inertia tensor is equal to the proper mass density ρ_0 .

The classical tensor T^{ij}, which is the general product of the quadri-vector $ic\sqrt{\rho_0}\,v^i$ with itself, is obviously not the most general symmetric second-rank tensor; the most general one would have ten components that are distinct in modulus, and would thus depend upon ten arbitrary constants.

The "minimal generalization" of the classical definition will permit us to obtain an asymmetric tensor T^{ij}, as the theory of the preceding paragraph would demand, which obviously consists of setting T^{ij} equal to *the general product of two non-collinear quadri-vectors.* If v^i denotes a unitary quadri-vector, u^i, a quadri-vector that is not collinear with v^i such that scalar product $u^i\, v_i$ has the value 1, and finally, $-c^2\rho_0$ is the product of the corresponding scalar factors then one will finally have:

$$(39) \qquad \boxed{T^{ij} = -c^2 \rho_0 \cdot u^i v^j .}$$

The tensor T^{ij} thus-defined will then depend upon seven arbitrary constants, instead of 16 for the most general second-rank tensor. The value of its trace will remain, as before:

$$(40) \qquad \boxed{T_i^i = -c^2 \rho_0 ,}$$

and one can say that, *by definition*, ρ_0 represents the proper mass density of the medium that is endowed with spin.

([1]) *La Relativité restreinte*, pp. 50.

By hypothesis, we suppose that one of the two quadri-vectors u^i or v^i represents the material world-current in the usual sense. From the general principles of relativity, that quadri-vector of the *true current* must be time-like ([1]). Moreover, from a result of the preceding paragraph, the spin density quadri-vector must be orthogonal to it. As for the other quadri-vector, we say that, by definition, it represents a *false current* that we shall seek to interpret.

We first have to demand to know *which of the two quadri-vectors* u^i *and* v^i *represents the true current*. Conforming to the method that was followed up to now, in order to do that, we shall analyze classical theory in detail and generalize it in the "minimal" manner that respects the essential results. We recall that, by virtue of a fundamental convention (35), *the finite mass-impulse that corresponds to an asymmetric tensor* T^{ij} *must be calculated with by summing over the second index; i.e., from the formula:*

$$p^i = \iiint T^{ij}\,\delta u_j\,. \tag{38}$$

In classical theory, in which the tensor T^{ij} is symmetric, since it is defined by (39′), *the preceding integral over the hyper-wall of a current world-tube will be identically zero.* Indeed, if one replaces T^{ij} using (39′) and the hyper-wall element δu_j using formula [*q*] of no. **12** then one will make the doubly-contracted product $\delta s_{jk}\, dx^j\, dx^k$ of an antisymmetric tensor and a symmetric tensor appear under the $\iiint$ sign. Now, *that result is essential for the notion of ponderomotive force density that is defined by* (35) *to have any meaning* ([2]). The necessary and sufficient condition for it to be conserved with the new definition (39) is obviously that *the quadri-vector of true current must be* v^j.

Another interesting situation is produced in classical theory: *For an infinitely-thin current world-tube, the mass-impulse quadri-vector will be defined intrinsically, independently of the orientation of the hypersection* δu_j. Indeed, the scalar product $dx^j\,\delta u_j$ will appear under the $\iiint$ sign, which will obviously be invariant under changes in the orientation of the hyperplane section of the same infinitely-thin tube. The mass-impulse of the material point without classical spin will then be defined intrinsically; as a special postulate, one might demand that it must have a constant length $ic\mu$ ([2]). The necessary and sufficient condition for the result in question to be preserved with the new definition (39) is that v_j must be the true current. That being the case, the new fact with respect to the classical theory is that the mass-impulse of the *material point that is endowed with spin* will no longer be tangent to the true streamline v^j (mean line of the hyper-tube), but, in fact, *to the false streamline; the mass-impulse of the material point that is endowed with spin will be oblique to the world-trajectory.* It is obviously quite natural to postulate that *its projection* $ic\mu$ *on the tangent to the trajectory must be constant* ([3]).

Finally, an examination of the situation that prevails in the Galilean frame that moves with the true current will allow us to confirm that v^j is the true current. If v^j is the false

([1]) *La Relativité restreinte*, pp. 18 and 19.
([2]) *Op. cit.*, pp. 51.
([3]) These last results agree perfectly with the ones that certain considerations of the analytical mechanics of a point would suggest (*La Relativité retreinte*, pp. 62).

current then the mass-impulse will be collinear to the true current, in such a way that the three components of the *proper impulse* will be zero. As for the *proper mass*, its expression will consist of four terms. On the contrary, if one assumes that v^j is the true current then not all of the four p^i will be zero, in general, in the co-moving Galilean frame, but each of them will consist of only one term in its expression, namely, $\iiint T^{ik}\,\delta u_k$. Now, we know that in all classical theories of relativity, the things that take place in the co-moving Galilean frame are very close to the ones that realize the *simultaneity hypothesis.* Here, the simultaneity hypothesis realizes the second situation precisely, so we further conclude that the true current is v^j.

Remark. – According to the idea in the remark that concluded the preceding paragraph, we demand to know what the direction of the hyperplane section of the infinitely-thin tube would have to be in order for the two *false mass-impulse* $T^{ji}\,\delta u_j$ and $T_j^j\,\delta u^i$ to coincide; that direction will obviously be the world-normal to the hypertube.

CHAPTER III

STUDY OF THE FICTITIOUS STATISTICAL FLUID IN DIRAC'S THEORY.

14. It is truly a new chapter in Dirac theory that results from the matrix formula:

$$[a] \qquad \sum_{A=1}^{16} \gamma^A_{pr}\, \gamma^A_{qs} = 4\, \delta_{ps}\, \delta_{qr}$$

that Pauli showed to be a consequence of the Dirac conditions ([1]):

$$[b] \qquad \tfrac{1}{2}(\gamma^i\, \gamma^j + \gamma^j\, \gamma^i) = \delta^{ij}\, ;$$

by definition, the γ^A are the sixteen *Hermitian* matrices 1, γ^i, $i\gamma^{ij}$, $i\gamma^{ijk}$, γ^{1234}. The quadratic formula [*a*] contains two distinct groups of terms, which are written on the left-hand and right-hand sides, respectively; the former are the terms "in *pr*, *qs*," while the latter are the terms "in *ps*, *qr*." One sees that the *first indices p* and *q* and the *second indices r* and *s* are the same in these two groups of terms, which are characterized by the different *coupling* of the *first indices* with the *second indices.*

Pauli perceived from the beginning that formula [*a*], and some other analogous formulas that one can deduce, permit one to establish the following relations between the five Dirac-Darwin statistical tensors *in the general case*, when they were known up to now only in the case of the monochromatic plane wave ([2]):

$$(j^k)\,(j_k) = -\,(\sigma^k)\,(\sigma_k) = (i\ \omega_1)^2 + (\omega_2)^2, \qquad \tfrac{1}{2}(m^{kl})\,(m_{kl}) = (i\ \omega_1)^2 - (\omega_2)^2,$$
$$(j^k)\,(\sigma_k) = 0, \qquad \tfrac{1}{2}(m^{kl})\,(i\,\bar{m}_{kl}) = 2\,(i\ \omega_1)\,(\omega_2);$$

We have written the five *Dirac tensors* in parentheses in order to indicate that one is dealing with *abstract tensors* that are not provided with their physical coefficients; by contrast, we have taken care to reinstate the factor *i*, in according with the demands of relativity.

W. Kofink addressed the question in a paper in 1937, and then in the first of a series of four papers in 1940) ([3]), and completed the first series of *quadratic identities between the statistical densities* with the following ones:

$$(m^{kl})\,(j_k) = -\,(\omega_2)\,(\sigma^l), \qquad (i\,\bar{m}^{kl})\,(j_k) = (i\omega_1)\,(\sigma^l),$$
$$(m^{kl})\,(\sigma_k) = -\,(\omega_2)\,(j^l), \qquad (i\,\bar{m}^{kl})\,(\sigma_k) = (i\omega_1)\,(j^l),$$

([1]) *Pieter Zeeman Verhandelingen*, 1935, pp. 31; "Contributions mathématiques à la Théorie de Dirac," Ann. Inst. H. Poincaré **6** (1936), pp. 118.

([2]) *Op. cit.*, see Ann. Inst. H. Poincaré **4**, § 6, pp. 131.

([3]) "Über das magnetische und elektrische Moment des Elektrons, Ann. Phys. (Leipzig) **30** (1937), pp. 91; "Zur Diracschen Theorie, I: Algebraische Identitäten zwischen den Wahrscheinlichkeitsdichten," Ann. Phys. (Leipzig) **38** (1940), pp. 421.

$$(j^k)(\sigma^l) - (j^l)(\sigma^k) = (\omega_2)(m^{kl}) - (i\omega_1)(i\,\bar{m}^{kl}) .$$

These *identities* and the preceding ones constitute the complete set of relations that are consequences of the Dirac matrix conditions [*b*] that exist between the tensors of type ρ_D. In order to establish them, Pauli and Kofink employed the matrix *B* that we spoke of in Chapter I, which is a matrix that canonically transforms the four γ^i into their transposes. G. Petiau, in some work that has been published incompletely, moreover, avoided introducing the matrix *B* in dealing with that question, which we will likewise do in the following pages.

Pauli remarked from the beginning that this type of calculation can be generalized in such a manner as to form some *identities* that relate to density quantities other than those of type ρ_D. For example, one can make the densities that we called *Schrödingerian* ones appear, whose definition involved our operator $[\partial_i]$, and which will figure in some differential relations that we shall discuss later on. That is precisely the objective of the second of the 1940 papers by Kofink ([1]), in which the author established 52 *quadratic identities with backward differentiation* (i.e., ones that involve our symbol $\underset{\leftarrow}{\partial}^i$) in spatial vector form and gave the rules for deducing the *identities with forward differentiation* $\underset{\rightarrow}{\partial}_i$ ([2]); one obtains some *identities* by adding and subtracting that are consequences that involve our symbols $[\partial^i]$ and (∂^i), i.e., ones that are concerned with the five Schrödingerian tensors ρ_S and the partial derivatives of the five Diracian tensors ρ_D.

Among these identities, one family that we shall study in detail contains ten *tensorial identities* whose left-hand sides have the form ([3]):

$$\psi^\times \gamma^A \psi \cdot \psi^\times \gamma^B \psi - \psi^\times [\partial^i]\, \gamma^A \psi \cdot \psi^\times [\partial^i] \gamma^B \psi,$$

and in general, two different left-hand sides that are sums of terms of the form:

$$\sum \pm \psi^\times \gamma^M \psi \cdot \partial^i (\psi^\times \gamma^N \psi) .$$

We establish these *identities* by the same method of Kofink, but while being careful to not destroy the tensorial symmetry of the universe in the calculations and the results. The spatial vector notation is not, moreover, the only reason that Kofink's formulas are deprived of a general tensorial validity. For example, it happens that certain formulas are established only for the values $i \neq j$ of the tensorial indices, and that one must perform certain manipulations in order to extend to the case of $i = j$. Under these conditions, we believe that it is useful to prolong the Kofink calculations in order to arrive at formulas that will have a general tensorial validity.

The third of the 1940 Kofink papers ([4]) began by reestablishing some differential relations that were consequences of the Dirac equation and conditions that were given

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([1]) "…II: Algebraische Identitäten die Differentialquotienten enthalten," Ann. Phys. (Leipzig) **38**, pp. 435.

([2]) *Op. cit.*, pp. 437 and 442.

([3]) *Op. cit.*, § 11, pp. 454.

([4]) "…III: Folgen der Realität der Potentiale," Ann. Phys. (Leipzig) **38**, pp. 565.

already by W. Franz ([1]), and that Al. Proca had proposed in order to systematically take into consideration the case of the free electron ([2]). From that family of relations, one already knows Dirac's *equation of continuity*, an uninterpreted relation of Uhlenbeck and Laporte, and Gordon's *formula for the decomposition of current*. Finally, a relation that was given by Tetrode relates the *lack of symmetry* in its inertia tensor to the rotation of the spin density, which our theory in Chapter II permits us to interpret. As we have said already, that circumstance seems to us to constitute a serious argument in favor of believing that the true inertia tensor of Dirac's theory is the asymmetric tensor that was originally defined by Tetrode.

The relations in question, when given in vectorial form, appear to be sixteen in number in the papers of Franz and Kofink; when one writes them in world-tensor form, they will be *ten* in number. Those are the ten relations that we have established in our own right in complete ignorance of the work of Franz and Kofink, by introducing the systematic definition of a "Schrödingerian" tensor of the type $\psi^{\times}\, [\partial^{\,i}]\; \gamma\, \psi$, which is a type of tensor that includes the Gordon current $\psi^{\times}[\partial^i]\psi$ and Tetrode's asymmetric inertia tensor $\psi^{\times}[\partial^i]\gamma^j\psi$, as well as Proca's magnetic current $\psi^{\times}[\partial^i]\bar{\gamma}\,\psi$ ([3]).

Having established the relations in question, neither Franz nor Kofink devoted any attention to their physical significance. On the contrary, in the course of his third 1940 paper, Kofink eliminated our five *Schrödingerian tensors*, which were "uninterpretable quantities" to him ([4]), from the ten differential relations that were considered and the ten *identities with differentials* that he had established for that in his second paper. He then obtained, along with Dirac's continuity equation and the Uhlenbeck-Laporte formula (which does not involve Schrödingerian tensor), 6 + 2 spatial vector relations that were quadratic. After starting with that, Kofink's attention, like that of Franz before, diverged completely from ours.

Indeed, we pose the problem of the physical interpretation of the ten differential relations considered, as well as that of the Diracian and Schrödingerian tensors that they involve. A detailed examination will permit us to extend and slightly sharpen the results that were obtained. Finally, five of the ten relations in question, one of the five Diracian tensors (the invariant ω_2), and three of the six Schrödingerian tensors ([5]) will remain uninterpreted.

Collectively, Franz's ten differential relations split into two families of five that each possesses its own Diracian and Schrödingerian tensors. Since all of the relations and all of the tensors that are presently interpreted are of an electromagnetic nature in one of the families and of a dynamical nature in the other, one is led to say that *by definition* the relations and tensors of the first family are *electromagnetic*, and those of the second one

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([1]) "Zur Methodik der Dirac Gleichung," Sitz. Bay. Akad. Wiss, Math. Abt. **3** (1935), pp. 404; § 10.

([2]) "Théorie de l'Électron de Dirac dans un champ nul," no. 18, Ann. de Physique **10** (1933), pp. 401.

([3]) "Sur dix relations conséquences des équations de Dirac," C. R. Acad. Sci. **214** (1942), pp. 818.

([4]) That term (Ger. *undeutbare Grossen*) is somewhat surprising, due to the fact that two of those quantities were interpreted in 1928, one, by Tetrode, and the other by Gordon. Kofink, who referred to the Gordon paper (but not to that of Tetrode) specified on that occasion that the "*undeutbare Grossen*" were density quantities that were possibly capable of interpretation, but which did not present themselves as derivatives of the Diracian tensors (*Op. cit.*, pp.569-570).

([5]) One of the five tensors $\psi^{\times}\, [\partial^{\,i}]\; \gamma\,\psi$ intervenes only by two of its *contractions* on the index *i*, in such a way that one has indeed six tensors of that type to interpret physically.

are *dynamic*. According to that *definition*, the relations and tensors in each family, which are still uninterpreted, will be said to belong to a theory of *electromagnetism* and a theory of *dynamics*, respectively, that have been expanded with respect to the classical theory. Moreover, according to a well-known property that generalizes one that relates to the Gordon current and the Tetrode tensor, the presence of an external quadri-potential A^i will add an *interaction term* of the type $A^i\,(\psi^\times\gamma\,\psi)$ to each of the Schrödingerian tensors, since the Diracian tensor ( ) has a physical nature that is "opposite" to that of the Schrödingerian tensor that contains it. The presence of the quadri-potential A^i then manifests a *coupling whose form is perfectly symmetric with respect to electromagnetism and dynamics*. In Dirac's theory, one will then no longer see any reason to qualify the potential A^i as being *electromagnetic*, rather than *dynamical*. In the absence of a prevailing potential, the coupling in question will disappear, and the two families of five relations will become independent of each other. One can think that these various remarks are not without some bearing on the problem of the unitary theory of electromagnetism and inertia-gravity.

In order to conclude the present chapter, we shall indicate how one can extend the general definition of the Diracian and Schrödingerian tensors in the theory of particles with integer spin, and we will show that Franz's ten differential relations will remain valid in that theory.

I. – THE PAULI-KOFINK QUADRATIC IDENTITIES.

15. Establishing the starting formulas. Pauli-Kofink identities that relate to the Diracian tensors. – Consider the Pauli formula ([1]):

$$\sum_{A=1}^{16} \gamma^A_{pr}\gamma^A_{qs} = 4\,\delta_{ps}\,\delta_{qr}\,, \tag{41}$$

in which one has, as was said before:

$$\gamma^A = 1,\ \gamma^i,\ i\,\gamma^{ij},\ i\,\gamma^{ijk} = i\,\bar{\gamma}^l,\ \gamma^{uvw4} = \bar{\gamma}\,, \tag{42}$$

in which the notations are the ones that were specified in the Foreword. Recall that the γ or $\bar{\gamma}$ of a given tensor rank behave like the components of a completely antisymmetric tensor, in such a way that:

1. *In all of the calculations that follow, the tensor indices i, j, … in the same γ or $\bar{\gamma}$ are essentially supposed to be all distinct.*

We complete that convention with some others that are especially destined to simplify the calculation and formulas that follow:

([1]) For the proof of that formula, *see* W. Pauli, "Contributions mathématiques à la Théorie de Dirac," Ann. Inst. H. Poincaré **6** (1936), pp. 115; § 3.

2. *In all of the formulas that are derived from* (41) *that we shall establish and utilize*, in formulas that will be denoted by Roman capitals between brackets, *we shall neglect to write the lower matrix indices, since it will be intended that the left-hand side will always be "in pr, qs" and the second one "in ps, qr,"* and the sense of that terminology will be the one that is evident in (41).

3. *When tensor indices i, j, … are repeated* (upper indices), *that will always mean that they are summed and then divided by a suitable number in order that each term in the result should be provided only once* (1).

4. *A tensor index that is denoted by a Greek letter* λ, μ, … *will always be intended to mean one that is not summed, even if it is repeated.*

With these conventions, formula (41) can be transcribed as follows:

$$[A] \qquad \delta\delta + \gamma^i\,\gamma^i - \gamma^{ij}\,\gamma^{ij} - \bar{\gamma}\,\bar{\gamma} + \gamma\gamma = 4\,\delta\delta.$$

Upon multiplying (for example, on the left) *all* of the matrices in [*A*] by *the same* matrix γ that is chosen to have each of the four ranks 1, 2, 3, 4, in turn, one can form four new formulas that are analogous to [*A*]. For $\gamma = \bar{\gamma}$, after reversing the sense of the notation in the left-hand side:

$$[E] \qquad \delta\delta - \gamma^i\,\gamma^i - \gamma^{ij}\,\gamma^{ij} + \bar{\gamma}^i\,\gamma^i + \bar{\gamma}\,\bar{\gamma} = 4\,\bar{\gamma}\,\bar{\gamma}.$$

Contrary to [*A*] and [*E*], the three unwritten formulas [*B*], [*C*], [*D*] are not symmetric in the upper indices. However, upon writing those formulas once and only once for each combination C_4^n of the upper indices in play (*n* denotes the tensorial rank of the multiplying matrix) and adding them, one can form some symmetrized formulas. For what follows, we shall need simply the symmetrized formula [*BS*], which we shall define.

Construct a table of rank five with double entries, whose columns correspond to the successive groups of terms in the left-hand side of [*A*], and whose rows correspond to the tensor rank of the matrices γ in the product term that is obtained. Above each column, we give the sign of the first term in [*A*]. It is clear that each group of *first terms* of rank *n* generally provides two groups of terms of rank $n \pm 1$, which will be written in the same column, and in the appropriate row (2). Under these conditions, and with the conventions that we have adopted, the table will take the following form when it has been filled:

(1) It goes without saying that this particular *summation convention* is not the usual one in tensor calculus.

(2) It goes without saying that [*E*] will yield formula [*D*] by that same process. The same table will then suffice, with the signs taken being the ones that we indicated *below*, for reference.

+	+	–	–	+
	$\delta\delta$			
$\gamma^i\,\gamma^\lambda$		$\gamma^i\gamma^i$ $(i \neq \lambda)$		
	$\gamma^{\lambda i}\,\gamma^{\lambda i}$		$\overline{\gamma}^{\lambda i}\,\overline{\gamma}^{\lambda i}$	
		$\overline{\gamma}^i\overline{\gamma}^i$ $(i \neq \lambda)$		$\overline{\gamma}^\lambda\,\overline{\gamma}^\lambda$
			$\overline{\gamma}\,\overline{\gamma}$	
+	–		+	+

We now inscribe the number of *different* terms that each box in the table contains, and take the *algebraic sum* of the numbers in question for each row, when they have been given the sign at the head of the column; multiply by 4, and write the final result to the right of the row; it is clearly the total number of terms of tensorial rank that were considered in the symmetrized formula [*BS*].

	+	+	–	–	+	
1		1				4
4	1		3			– 8
6		3		3		0
4			3		1	– 8
1				1		4

Moreover, let C_4^n = 1, 4, 6, 4, 1 be the *theoretical number* of terms contained in each tensorial group $\gamma^A\,\gamma^A$, which is a number that we indicate to the left of the rows. Upon dividing the numbers on the right by the numbers on the left, one will get the coefficients of these groups of terms in [*BS*], which will then be written:

[*BS*] $$4\,(\delta\delta - \overline{\gamma}\,\overline{\gamma}) - 2\,(\gamma^i\,\gamma^i + \overline{\gamma}^i\overline{\gamma}^i) = 4\,\gamma^i\,\gamma^i.$$

If one now adds and subtracts corresponding sides of [*A*] and [*E*] and then subtracts corresponding sides of the second formula that is obtained from [*BS*] then one will get three formulas that will be quite useful to us:

$$\begin{array}{|ll|}\hline [X] & -\gamma^{ij}\gamma^{ij}+(\delta\delta+\bar{\gamma}\,\bar{\gamma})=2(\delta\delta+\bar{\gamma}\,\bar{\gamma}), \\ [Y] & \gamma^{i}\gamma^{i}-\bar{\gamma}^{i}\bar{\gamma}^{i}=2(\delta\delta-\bar{\gamma}\,\bar{\gamma}), \\ [Z] & \gamma^{i}\gamma^{i}-(\delta\delta-\bar{\gamma}\,\bar{\gamma})=2\{\gamma^{i}\gamma^{i}-(\delta\delta-\bar{\gamma}\,\bar{\gamma})\}. \\ \hline\end{array}$$

Upon matrix-multiplying ([1]) the terms of these three formulas by $\psi_p^\times$, $\psi_q^\times$, ψ_r, ψ_s, and referring to the usual definitions and notations for the five Diracian tensors [for that, *see* eq. (60)], one will find that the parentheses contain, by definition, the *squares* of the five Diracian tensors (viz., sums of squares of the components, taken once and only once), respectively:

$$(m)^2 = (i\ \omega_1)^2 - (\omega_2)^2, \qquad (j)^2 = -(\sigma)^2, \qquad (j)^2 = (i\ \omega_1)^2 + (\omega_2)^2.$$

(In the present number, we introduce the factor i whenever it is necessary in order for the components of the Diracian tensors to be, according to a known rule of relativity, pure imaginary or real according to whether they do or do not contain the index 4, resp.). One will then have:

$$(43) \qquad \boxed{(m)^2 = (i\,\omega_1)^2 - (\omega_2)^2,} \qquad \boxed{(j)^2 = -(\sigma)^2 = (i\,\omega_1)^2 + (\omega_2)^2,}$$

which is a group of formulas that permits one to express the *square* of each Diracian tensor as a function of the *squares* of the other ones; (43) shows that the quadri-vector (j) is *time-like*, and the quadri-vector (σ) is *space-like.*

We now give the *final formulas*, properly speaking, that will permit us to establish the aforementioned set of relations (viz., the Pauli-Kofink *identities*). If P and Q denote two square matrices of rank 4, and on the one hand, ψ, ζ_1, and ξ_1, $\psi^\times$, $\zeta_2^\times$, $\xi_2^\times$, on the other, are *wave functions* of the *associated* type then upon matrix-multiplying all of the terms of a formula of the type (41) by $(\xi_2^\times P)_p$, $(\zeta_2^\times Q)_q$, ψ_r, and ψ_s, the "coupling difference" of the lower indices will be erased from the result because:

1. The "first two" indices p and q are the same on both sides, and
2. The difference between the "second indices" r and s is "erased" by the common multiplier ψ.

Finally, all of the terms obtained will have the same type, as far as the lower indices p, q, r, s are concerned, and one can make them pass from one side of the equation to the other one, and add them algebraically if they have the same upper indices. The same conclusions are obviously valid for a multiplication by the matrices $\psi_p^\times$, $\psi_q^\times$, $(P\ \xi_1)_r$, $(Q\ \zeta_1)_s$. Applying these considerations to the preceding formulas [X] and [Y], one will get two groups of *final formulas:*

([1]) We always consider $\psi^\times$ to be a matrix with one row and four columns and ψ to be a matrix with four rows and one column.

$$(44)\qquad \boxed{\begin{array}{ll} (X)\left\{\begin{array}{ll} (1) & \tfrac{1}{2}\psi^{\times}\gamma^{ij}P\,\xi_1\cdot\psi^{\times}\gamma_{ij}Q\,\zeta_1 = -(\psi^{\times}P\xi_1\cdot\psi^{\times}Q\,\zeta_1+\psi^{\times}P\overline{\gamma}\,\xi_1\cdot\psi^{\times}\overline{\gamma}\,Q\,\zeta_1), \\ (2) & \tfrac{1}{2}\xi_2^{\times}P\,\gamma^{ij}\psi\cdot\zeta_2^{\times}\,Q\,\gamma_{ij}\zeta_1 = -(\xi_2^{\times}P\psi^{\times}\cdot\zeta_2^{\times}Q\psi+\xi_2^{\times}P\overline{\gamma}\,\psi\cdot\zeta_2^{\times}Q\,\overline{\gamma}\,\psi), \end{array}\right. \\ (Z)\left\{\begin{array}{ll} (1) & \tfrac{1}{2}\psi^{\times}\gamma^{i}P\,\xi_1\cdot\psi^{\times}\gamma_{i}Q\,\zeta_1 = \psi^{\times}P\xi_1\cdot\psi^{\times}Q\,\zeta_1-\psi^{\times}P\overline{\gamma}\,\xi_1\cdot\psi^{\times}\overline{\gamma}\,Q\,\zeta_1, \\ (2) & \tfrac{1}{2}\xi_2^{\times}P\,\gamma^{ij}\psi\cdot\zeta_2^{\times}\,Q\,\gamma_{ij}\zeta_1 = \xi_2^{\times}P\psi^{\times}\cdot\zeta_2^{\times}Q\psi-\xi_2^{\times}P\overline{\gamma}\,\psi\cdot\zeta_2^{\times}Q\,\overline{\gamma}\,\psi. \end{array}\right. \end{array}}$$

In these formulas, we have reverted to the usual summation convention of tensor calculus; (44 *Z*) are the starting formulas that were utilized by Kofink in his second 1940 paper ([1]).

For the applications that we have in mind, the matrices P and Q will be chosen from the table of the sixteen γ: $P = \gamma^P$, $Q = \gamma^Q$. In this and the following paragraph, we will often have to pass to dual quantities, for which, it will be advantageous to distinguish the index 4 from the indices u, v, w = 1, 2, 3; we can then construct the following conversion table for the sixteen γ^A ([2]) in which, the conventions were explained in the Foreword:

(45)

$1 = \overline{\gamma}^{uvw4}$	$\gamma^{uw} = \overline{\gamma}^{w4}$,	$\gamma^{w4} = \overline{\gamma}^{uv}$,	$\gamma^{uvw4} = \overline{\gamma}$
$\gamma^{w} = -\overline{\gamma}^{uv4}$,	$\gamma^{4} = \overline{\gamma}^{uvw}$,	$\gamma^{uv4} = \overline{\gamma}^{w}$,	$\gamma^{uvw} = \overline{\gamma}^{4}$

Along the same order of ideas, and in view of the calculations that must follow, it will be useful to construct the following multiplication table, in which we group the commuting matrices on the left and the anti-commuting ones on the right:

(46)

$\overline{\gamma}\,\gamma^{lj} = \gamma^{lj}\,\overline{\gamma} = -\overline{\gamma}^{lj}$, $\overline{\gamma}\,\overline{\gamma}^{lj} = \overline{\gamma}^{lj}\,\overline{\gamma} = -\gamma^{lj}$, $\qquad\overline{\gamma}^2 = 1$,	$\overline{\gamma}\,\gamma^{l} = -\gamma^{l}\,\overline{\gamma} = \overline{\gamma}^{l}$, $\overline{\gamma}\,\overline{\gamma}^{l} = -\overline{\gamma}^{l}\,\overline{\gamma} = \gamma^{l}$,
$\gamma^{l}\,\overline{\gamma}^{j} = \overline{\gamma}^{j}\,\gamma^{l} = \overline{\gamma}^{lj}$,	$\gamma^{\lambda}\,\overline{\gamma}^{\lambda} = -\overline{\gamma}^{\lambda}\,\gamma^{\lambda} = -\overline{\gamma}$,
$\gamma^{\lambda}\,\overline{\gamma}^{\lambda i} = \overline{\gamma}^{\lambda i}\,\gamma^{\lambda} = \overline{\gamma}^{i}$, $\overline{\gamma}^{\lambda}\,\overline{\gamma}^{\lambda i} = \overline{\gamma}^{\lambda i}\,\overline{\gamma}^{\lambda} = \overline{\gamma}^{lj}$,	$\gamma^{\lambda}\,\overline{\gamma}^{\lambda} = -\overline{\gamma}^{\lambda}\,\gamma^{\lambda} = -\overline{\gamma}$ $\overline{\gamma}^{(?)}\,\overline{\gamma}^{(?)} = -\overline{\gamma}^{(?)}\,\gamma^{(?)} = -\overline{\gamma}^{(?)}$
$\gamma^{ij}\,\overline{\gamma}^{k} = \overline{\gamma}^{k}\,\gamma^{ij} = \overline{\gamma}^{ijk}$,	$\gamma^{i\lambda}\,\overline{\gamma}^{\lambda} = -\overline{\gamma}^{\lambda}\,\gamma^{i\lambda} = \gamma^{i}$.

Upon setting $\xi = \zeta = \psi$ in (44) and methodically taking P and Q to be matrices with differing tensor ranks in the table of sixteen γ, we shall define what one can call the Pauli-Kofink *rectangle identities*. For $P = 1$, $Q = \overline{\gamma}$, formulas (44 *Z*) and (44 *X*) provide two tensorial *identities*, respectively, that are well-known in the particular case of monochromatic plane waves ([3]):

$$(47)\qquad \boxed{(j^k)(\sigma_k) = 0,} \qquad \boxed{\tfrac{1}{2}(m^{lj})(i\,\overline{m}_{lj}) = 2(i\,\omega_1)(\omega_1).}$$

([1]) *Op. cit.*, II, pp. 438, eq. (*), and pp. 441, eq. (**).

([2]) N. B. $\overline{\gamma}^{ij}\cdots$ is not equal to the product $\overline{\gamma}^{i}\,\overline{\gamma}^{j}\cdots$, as one can verify in some examples.

([3]) Formula (47_1) is equivalent to our formula (33) of Chapter II, from which the quadri-vector of spin density must be orthogonal to the world-current.

Then, four interesting tensor identities that are due to Kofink are provided by starting with formula (44 *Z*), for two different systems of "values" of *P* and *Q* that we indicate on the left:

$$(48) \quad \left\{ \begin{array}{l} (1, \gamma^i, \bar{\gamma}^i, \gamma^{ik}) \\ (1, \bar{\gamma}^i, \gamma^i, \gamma^{ik}) \\ (\bar{\gamma}, \gamma^i, \bar{\gamma}^i, \bar{\gamma}^{ik}) \\ (\bar{\gamma}, \bar{\gamma}^i, \gamma^i, \bar{\gamma}^{ik}) \end{array} \right. \qquad \boxed{\begin{array}{c} (m^{ki})(j_k) = -\,(\omega_2)(\sigma^i), \\ (i\,\bar{m}^{ki})(j_k) = +(i\,\omega_1)(\sigma^i), \\ (m^{ki})(\sigma_k) = -\,(\omega_2)(j^i), \\ (i\,\bar{m}^{ki})(\sigma_k) = +(i\,\omega_2)(j^i). \end{array}}$$

When the same values of *P* and *Q* are substituted in (44 *X*), that will lead to some *identities* that are consequences of these ones ([1]). Finally, one last interesting formula, which is likewise due to Kofink, is provided by starting from (44 *Z*), by three different systems of "values" for *P* and *Q*, which we indicate on the left ([2]):

$$(49) \quad (1,\ \bar{\gamma}^{ij}\ ;\ \bar{\gamma},\ \gamma^{ij}\ ;\ \gamma^i,\ \gamma^j) \qquad \boxed{[(j^k)(\sigma^l) - (j^l)(\sigma^k)] = (\omega_2)(m^{kl}) - (i\,\omega_1)(i\,\bar{m}^{kl});}$$

the same "values," when substituted into (44 *X*) will lead to the identity 0 = 0.

16. Kofink identities that involve the Schrödingerian tensor $\psi^\times [\partial^{\,i}]\ \gamma\,\psi$. – Among these new identities, some of them can be considered to be direct generalizations of the preceding *square* and *rectangle* identities. For example, if one replaces the multiplication by the matrix $\psi_p^\times$ with multiplication by the matrix $\psi_p^\times \underset{\leftarrow}{\partial}{}^i$ in the symmetric formulas [*X*], [*Y*], [*Z*] that allowed us to establish the *square identities* (43) then one will obtain what Kofink called a series of relations "with backward derivation"; similarly, replacing ψ_r with $\underset{\rightarrow}{\partial}{}^i \psi_r$ and subtracting the relations "with forward-derivation" thus-obtained ([3]) from the preceding, one will easily form three *identities* that correspond

([1]) One should note the "resemblance" between (48_1) and the formula $f^i = \Pi^{ki} j_k$ from Lorentz's theory of electromagnetism. Here, the polarization tensor m^{ki} replaces the field tensor, and the spin density replaces the world-force.

([2]) With the third system of *values* for *P* and *Q*, one must take into account the identity $m^{ik}\bar{m}^j_{\ k} + \bar{m}^{jk} m^i_{\ k} = 0$, which is valid for $i \neq j$; the formula is then established only for $i \neq j$, but its validity for $i = j$ is obvious, due to the antisymmetry of the three groups of terms. – Upon taking the duals of these three groups of terms and combining the result thus-obtained with (49), one can form the expression for the tensor (m^{kl}) as a function of $(j^k)(\sigma^l) - (j^l)(\sigma^k)$ and the dual of that exterior product. That relation (which Kofink gave explicitly) permits one to answer a physically-interesting question: In the case of monochromatic plane waves, one knows that the three components (m^{uv}) of the tensor (m^{ij}) (viz., the *electric* part of the tensor) are annulled in the co-moving Galilean frame. Is the same thing true in the general case for the time-like quadri-vector (j^k) in the "co-moving" frame at each point and each instant? The answer is *no*, since the "second invariant" (ω_2) will not be zero in the general case.

([3]) Upon adding, one would form the *derivative* of (43).

bijectively to (43) ([1]), and we shall write only the third one, in view of a physical question that we shall pose in the last chapter:

$$(43'_2) \qquad \boxed{\psi^\times[\partial^i]\gamma^k\psi\cdot\psi^\times\gamma_k\psi = \psi^\times\psi\cdot\psi^\times[\partial^i]\psi - \psi^\times\bar{\gamma}\psi\cdot\psi^\times[\partial^i]\bar{\gamma}\psi.}$$

One can obviously "generalize" this in a manner that is analogous to the various *rectangle identities* that were given in the preceding number ([2]).

In this number, we propose to establish (in the manner of Kofink) a complete collection of ten *quadratic identities* whose left-hand sides have the form ([3]):

$$K_1 \equiv \psi^\times\,[\partial^i]\,\gamma^A\psi\cdot\,\psi^\times\,\gamma^B\psi - \psi^\times\,\gamma^A\psi\cdot\,\psi^\times\,[\partial^i]\,\gamma^B\psi,$$

in which γ^A and γ^B denote two well-defined arbitrary matrices from the table of sixteen γ, and whose right-hand sides have the form:

$$K_2 \equiv \sum \pm\,\psi^\times\gamma^K\psi\cdot\partial^i\,(\psi^\times\gamma^L\psi) = \sum \mp\,\partial^i\,(\psi^\times\gamma^K\psi)\cdot\psi^\times\,\gamma^L\psi.$$

The *identities* are all obtained by adding or subtracting from the *starting formulas* (44 Z), where one sets $P = \gamma^P$, $Q = \gamma^Q$, $\xi_1 = \underrightarrow{\partial}^i\psi$, $\xi_2^\times = \psi^\times\underleftarrow{\partial}^i$, $\zeta_1 = \psi$, $\zeta_2^\times = \psi^\times$.

Thus, the left-hand sides are differences of the two products of a Schrödingerian tensor with a Diracian tensor, where the Diracian tensor in each product has the same matrix significance that the Schrödingerian tensor has in the other one, and the right-hand sides, which are generally susceptible to being written in two ways, are sums of products of a Diracian tensor with the partial derivative of a Diracian tensor. In order to pass from one notation for the right-hand side to the other one, one must:

1. Change all of the signs.
2. "Shift" the partial differential operator ∂^i of one Diracian tensor to the other one in all of the terms ([4]).

The rules that relate to the right-hand sides are consequences of the ones that relate to the left-hand sides. Indeed, if, for example, the formula considered is obtained by adding (44 Z) when P and Q both commute or anti-commute with the γ in a certain term of (44 Z) then the term that will be generated will obviously be "of type K_2." In the contrary case, it will be "of type K" – viz., the product of a Schrödingerian tensor with a Diracian tensor. Now, switch the roles of the matrices P and Q; i.e., set $P = \gamma^Q$ and $Q = \gamma^P$. The new "K_2 terms" are obviously deduced from the preceding ones by a simple transfer of the symbol ∂^i. As for the new "K_1 terms," if, *by hypothesis*, they define a difference of

---

([1]) In Kofink's 1940 second paper, the *identity* that corresponds to (43_1) was obtained by subtracting either (25, 26), or (48, 49); the identity that corresponds to (43_2) is (1), and the one that corresponds to (43_3) is (9). ["…II…," Ann. Phys. (Leipzig) **38** (1940), pp. 436.]

([2]) There are two possible "generalizations" for the *identity* (49) that Kofink gave in (3, 4) and (10, 11).

([3]) That family of *identities* is represented by 35 vectorial relations between the 52 identities that are given in Kofink's second paper.

([4]) *Op. cit.*, pp. 441 and 442.

the indicated type then the roles of γ^A and γ^B will be simply inverted [due to the symmetry of the formulas (44 Z)], which will obviously have the effect of changing the sign of the "left-hand side of type K_1 ." Q. E. D.

We shall now introduce some special conventions, which are intended to simplify the writing of the calculations and the results of the present number. Since the index of partial differentiation is unique, and the same in all of the terms, we shall neglect it, in such a way that [ ] will mean $[\partial^i]$. Furthermore, we shall denote the partial differentiation of a certain quantity by a simple underline; for example, $\underline{\psi^\times\gamma^L\psi}$ will be intended to mean $\partial^i(\psi^\times\gamma^L\psi)$. We shall continue to denote the five Diracian tensors according to the usual conventions [on this, *see*, eq. (60)], *but we shall not neglect the parentheses* by which we generally specify that we are dealing with an "abstract tensor" that is devoid of any physical significance. Similarly, we neglect to reestablish the symbol *i* that gives the density tensors the real or pure imaginary character that is required by relativity.

With these various conventions, the general symbolic writing of the ten desired formulas will be:

$$[K] \qquad \boxed{\{\psi^\times[\;]\gamma^A\psi\cdot\psi^\times\gamma^B\psi-\psi^\times\gamma^A\psi\cdot\psi^\times[\;]\gamma^B\psi\}=\sum\pm\psi^\times\gamma^K\psi\cdot\underline{\psi^\times\gamma^L\psi}=\sum\mp\underline{\psi^\times\gamma^K\psi}\cdot\psi^\times\gamma^L\psi.}$$

As we have said before, these formulas are deduced from (44 *Z*), as in Kofink, when one sets:

$$\xi=\underline{\psi}, \qquad \zeta=\psi; \qquad P=\gamma^P, \qquad Q=\gamma^Q.$$

Finally, just as in the preceding number, we shall indicate the *values* that are given to the matrices *P* and *Q* on the left, as well as whether one must proceed by addition or subtraction.

One first gets:

$$(50) \qquad (1,\ \bar\gamma\,;-) \qquad \boxed{\{\psi^\times[\;]\psi\cdot\psi^\times\bar\gamma\psi-\cdots\}=-\underline{j^k}\sigma_k=j_k\underline{\sigma^k},}$$

$$(51) \qquad \left\{\begin{array}{l}(1,\gamma^i;-)\\(1,\bar\gamma^i;+)\\(\bar\gamma,\gamma^i;+)\\(\bar\gamma,\gamma^i;-)\end{array}\right. \qquad \boxed{\begin{array}{c}\{\psi^\times[\;]\psi\cdot\psi^\times\gamma^i\,\psi-\cdots\}=+\underline{j_k}m^{kl}+\underline{\omega_2}\,\sigma^l=\cdots,\\ \{\psi^\times[\;]\bar\gamma\,\psi\cdot\psi^\times\gamma^i\,\psi-\cdots\}=-\underline{j_k}\bar m^{kl}+\underline{\omega_1}\,\sigma^l=\cdots,\\ \{\psi^\times[\;]\psi\cdot\psi^\times\gamma^i\,\psi-\cdots\}=+\underline{\sigma_k}m^{kl}+\underline{\omega_2}\,\sigma^l=\cdots,\\ \{\psi^\times[\;]\bar\gamma\,\psi\cdot\psi^\times\gamma^i\,\psi-\cdots\}=-\underline{\sigma_k}\bar m^{kl}+\underline{\omega_1}\,\sigma^l=\cdots,\end{array}}$$

which are five formulas that directly have the tensorial form ([1]).

One similarly obtains:

([1]) The correspondence between our formulas and the vectorial formulas of Kofink's second paper is established as follows: (50) → (2); (51_1) → (7, 8); (51_2) → (5, 6); (51_3) → (14, 15); (51_4) → (12, 13).

$$
(52')\quad
\begin{cases}
(\bar\gamma^i,\bar\gamma^{jk};+) & \{\psi^\times[\;]\gamma^i\psi\cdot\psi^\times\gamma^{jk}\,\psi-\cdots\}=-\underline{\omega_2}\,\bar j^{ijk}-[\underline{\bar m_{ij}}\,\sigma^k-\underline{\bar m^{ik}}\sigma^j]-\underline{\sigma}^i\bar m^{jk}=\cdots,\\
(\gamma^l,\bar\gamma^{ik};-) & \{\psi^\times[\;]\gamma^i\,\psi\cdot\psi^\times\bar\gamma^{jk}\,\psi-\cdots\}=-\underline{\omega_1}\,\bar j^{ijk}-[\underline{m^{ij}}\,\sigma^k-\underline{m^{ik}}\sigma^j]-\underline{\sigma}^i m^{jk}=\cdots,\\
(\bar\gamma^l,\gamma^{jk};-) & \{\psi^\times[\;]\bar\gamma^i\psi\cdot\psi^\times\gamma^{jk}\,\psi-\cdots\}=-\underline{\omega_2}\,\bar\sigma^{ijk}-[\underline{\bar m_{ij}}\,j^k-\underline{\bar m^{ik}}\,j^j]-\underline{j}^i\bar m^{jk}=\cdots,\\
(\gamma^l,\gamma^{jk};+) & \{\psi^\times[\;]\gamma^i\,\psi\cdot\psi^\times\bar\gamma^{jk}\,\psi-\cdots\}=+\underline{\omega_1}\,\bar\sigma^{ijk}-[\underline{m^{ij}}\,j^k-\underline{m^{ik}}\,j^j]-\underline{j}^i m^{jk}=\cdots,
\end{cases}
$$

which are formulas that will not have a general tensorial validity ([1]), since they were established for $i \neq j, k$ ($j \neq k$, by hypothesis). However, it is obvious that upon writing the left-hand sides, when one groups these formulas into pairs, as was indicated, each of the pairs will specify a general tensorial formula. It is easy to write those two general formulas; indeed, upon adding and subtracting a term from the right-hand sides, the first pair – for example – can be written:

$$\{\psi^\times[\;]\gamma^i\psi\cdot\psi^\times\gamma^{jk}\psi-\ldots\}=-\left[\underline{\omega_2}\,\bar j^{ijk}+\underline{\bar m^{ij}}\,\sigma^k+\underline{\bar m^{ki}}\,\sigma^j+\underline{\bar m^{jk}}\,\sigma^i\right]+\left[\underline{\bar m^{jk}}\,\sigma^i-\underline{\sigma}^i\,\bar m^{jk}\right]=\ldots$$

$$\{\psi^\times[\;]\gamma^i\psi\cdot\psi^\times\gamma^{jk}\psi-\ldots\}=-\left[-\underline{\omega_1}\,\bar j^{ijk}+\underline{\bar m^{ij}}\,\sigma^k+\underline{\bar m^{ki}}\,\sigma^j+\underline{\bar m^{jk}}\,\sigma^i\right]+\left[\underline{\bar m^{jk}}\,\sigma^i-\underline{\sigma}^i\,\bar m^{jk}\right]=\ldots$$

The first [ ] *in the right-hand side are completely anti-symmetric in i, j, k,* in such a way that they must be considered to be zero for $i = j, k$. Consequently, if one gives i all possible values that are independent of the values of j and k ($j \neq k$) then there will always be one and only one of the first [ ] that is not zero. One can say that when one passes from the case $i = j, k$ to the case $i \neq j, k$, one of the first [ ] will replace the other. As for the second [ ], just like the left-hand sides [ ], they are not distinct from each other; as far as that is concerned, one passes from one equation to the other by taking dual quantities in j, k. Finally, the two desired general formulas are then:

$$
(52)\quad
\boxed{
\begin{aligned}
\{\psi^\times[\;]\gamma^i\psi\cdot\psi^\times\gamma^{jk}\psi-\cdots\} &= [\underline{\bar m^{jk}}\sigma^i-\underline{\sigma}^i m^{jk}]-\overline{\left[\;\omega_1\bar j^{ijk}+\sum\underline{m^{ij}}\sigma^k\right]}^{[ik]}\\
&\qquad\qquad\left[-\underline{\omega_2}\bar j^{ijk}+\sum\underline{m^{ij}}\sigma^k\right]\quad=\cdots\\
\{\psi^\times[\;]\gamma^i\psi\cdot\psi^\times\gamma^{jk}\psi-\cdots\} &= [\underline{\bar m^{jk}}\,j^i-\underline{j}^i m^{jk}]-\overline{\left[\;\omega_1\,\sigma^{ijk}+\sum\underline{m^{ij}}\,j^k\right]}^{[ik]}\\
&\qquad\qquad\left[+\underline{\omega_2}\,\sigma^{ijk}+\sum\underline{m^{ij}}\,j^k\right]\quad=\cdots
\end{aligned}}
$$

in which the summations $\sum$ are done over all circular permutations of i, j, k, and the “bar with indices” over the [ ] is intended to mean the *dual over j, k.*

Here is a pair of *identities* that are proved essentially for $k \neq l$ ([2]):

([1]) For a “rough calculation” of (52′), one must pass to dual quantities on the right-hand sides in order to give { } the form $[K_1]$. The correspondence with the Kofink formulas is then established by: $(52'_1) \to$ (17, 20, 33, 34); $(52'_2) \to$ (19, 28, 37, 38); $(52'_3) \to$ (16, 29, 31, 32); $(52'_4) \to$ (18, 27, 35, 36).

([2]) These *identities* correspond to (20, 43) and (21, 44), respectively, of the cited paper by Kofink.

$$(53') \quad \begin{cases} (\gamma^k, \gamma^l; +) & \{\psi^\times[\ ]\psi \cdot \psi^\times \gamma^{kl} \psi \cdots\} = \underline{j^k}\, j^l - \underline{\sigma^k}\, \sigma^l - \underline{m^{ki}}\, m^l_{\ i} = \cdots \\ (\bar\gamma^k, \bar\gamma^l; -) & \{\psi^\times[\ ]\gamma\psi \cdot \psi^\times \gamma^{kl} \psi \cdots\} = \underline{j^k}\, j^l - \underline{\sigma^k}\, \sigma^l + \underline{\bar m^{ki}}\, \bar m^l_{\ i} = \cdots \end{cases}$$

We remark that the third group of terms in the right-hand sides of $(53'_2)$, for example, can be transformed by means of the identity:

$$(53^*) \qquad \underline{\bar m^{ki}}\, \bar m^l_{\ i} = -\underline{m^{ki}}\, m^l_{\ i},$$

which is valid for $k \neq l$. *Formulas* (53′) *have no general tensorial validity;* indeed, from our conventions in the Foreword, the left-hand sides must be considered to be the components of a tensor that is anti-symmetric in *k*, *l*, so it must be annulled for $k = l$. Now, the right-hand sides of (53′) are not annulled for $k = l$. However, upon taking one-half the sum of the two right-hand sides, one will get:

$$(53) \quad \begin{aligned} &\{\psi^\times[\ ]\psi \cdot \psi^\times \gamma^{kl} \psi - \cdots\} &&= \tfrac{1}{2}\{[\underline{j^k}\, j^l - \underline{j^l}\, j^k] - [\underline{\sigma^k}\, \sigma^l - \underline{\sigma^l}\, \sigma^k] - [\underline{m^{ki}}\, m^l_{\ i} - \underline{m^{li}}\, m^l_{\ i}]\}, \\ &\{\psi^\times[\ ]\bar\gamma\psi \cdot \psi^\times \gamma^{kl} \psi - \cdots\} &&= \tfrac{1}{2}\{[\qquad\qquad] - [\qquad\qquad] + [\qquad\qquad]\}, \end{aligned}$$

and since the three [ ] in the new right-hand sides are anti-symmetric in *k*, *l*, *the identities* (53) *will have a general tensorial validity for all values of k and l.* We remark that the right-hand sides of those two *identities* contain the same groups of terms, while the sign is different for just the third one.

Finally, the last of the *tensorial identities* of the family considered, whose left-hand side is:

$$\{\psi^\times[\ ]\gamma^i\psi \cdot \psi^\times \bar\gamma^j\,\psi - \psi^\times\gamma^i\psi \cdot \psi^\times[\ ]\bar\gamma^j\,\psi\},$$

must be established separately for the values $i \neq j$ and $i = j$ of the indices. One first gets, with essentially $i \neq j$ ([1]):

$$(54''_1) \quad (\gamma^{ki}, \gamma^{kj}; +) \quad \{\psi^\times[\ ]\gamma^\lambda\psi \cdot \psi^\times \gamma^{\lambda ij} \psi - \ldots\} = \underline{m^{\lambda i}}\, m^{\lambda j} - \underline{\bar m^{\lambda i}}\, \bar m^{\lambda j} - \underline{j^i}\, j^j - \underline{\sigma^{\lambda ik}}\, \sigma^{\lambda j}_{\ \ldots k} = \ldots$$

We shall transform the writing, which is not correct from the tensorial viewpoint, due to the fact that, notably, there is an *unsummed index* λ present. For example, take $l = 4$, $i = u$, $j = v$; upon passing to dual quantities, one will get:

$$\begin{aligned} \{\psi^\times[\ ]\gamma^4\psi \cdot \psi^\times \bar\gamma^w\,\psi - \ldots\} &= \underline{\bar m^{\mu 4}}\, m^{\mu w} + \underline{\bar m^{\nu w}}\, m^{\nu 4} - \underline{j^\mu}\, \bar j^{\mu w 4} - \underline{\sigma^\nu}\, \bar\sigma^{\nu w 4} \\ &= \underline{\bar m^{\nu 4}}\, m^{\nu w} + \underline{\bar m^{\mu w}}\, m^{\mu 4} - \underline{j^\nu}\, \bar j^{\nu w 4} - \underline{\sigma^\mu}\, \bar\sigma^{\mu w 4}. \end{aligned}$$

([1]) Recall that, according to an earlier convention, our indices *l*, *m*, … are intended to *not be summed.* Our $(54''_1)$ corresponds to Kofink's (39, 40, 45, 46).

In order to free ourselves from the *unsummed indices* μ, ν, we take one-half the sum of the right-hand sides; it will be written:

$$\{\psi^{\times}[\ ]\gamma^4\psi\cdot\psi^{\times}\bar{\gamma}^w\psi - \dots\} = \tfrac{1}{2}(\underline{\bar{m}^{i4}}\,m_i^{\ w} + \underline{\bar{m}^{iw}}\,m_i^{\ 4}) - \tfrac{1}{2}[j_k\,\bar{j}^{w4k} + \sigma_k\,\bar{\sigma}^{w4k}],$$

so, upon reestablishing the arbitrary values for the indices, while the notation *is valid only for* $i \neq j$:

$$(54_1') \qquad \boxed{\{\psi^{\times}[\ ]\gamma^j\psi\cdot\psi^{\times}\bar{\gamma}^i\psi - \cdots\} = \tfrac{1}{2}(\underline{\bar{m}^{i4}}\,m_i^{\ w} + \underline{\bar{m}^{iw}}\,m_i^{\ 4}) - \tfrac{1}{2}[j_k\,\bar{j}^{w4k} + \sigma_k\,\bar{\sigma}^{w4k}],}$$

which is a notation in which the right-hand side is the sum of a symmetric tensor and an anti-symmetric tensor in i, j. We remark that, by virtue of an identity that was invoked before, the symmetric tensor can take on two other forms:

$$(54^*) \qquad \boxed{\tfrac{1}{2}(\underline{\bar{m}^{ik}}\,m^j_{\ k} + \underline{\bar{m}^{jk}}\,m^i_{\ k}) = -\tfrac{1}{2}(\underline{m^{ik}}\,\bar{m}^j_{\ k} - \underline{\bar{m}^{ik}}\,m^j_{\ k}) = -\tfrac{1}{2}(\underline{m^{ik}}\,\bar{m}^j_{\ k} + \underline{m^{jk}}\,\bar{m}^i_{\ k}).}$$

It remains for us to establish the *identity* considered for $i = j$; one has [1]:

$$(54_2'') \qquad (\psi^{\lambda}, ;\ -) \qquad \boxed{\{\psi^{\times}[\ ]\gamma^{\lambda}\psi\cdot\psi^{\times}\bar{\gamma}^{\lambda}\psi - \cdots\} = -\underline{\omega_1}\omega_2 + \underline{m^{\lambda k}}\,\bar{m}^{\lambda}_{\ k} = \cdots,}$$

or, upon taking one-half the sum of the right-hand sides, as before:

$$(54_2') \qquad \{\psi^{\times}[\ ]\gamma^{\lambda}\psi\cdot\psi^{\times}\bar{\gamma}^{\lambda}\psi - \cdots\} = \tfrac{1}{2}(\underline{\omega_2}\omega_1 - \underline{\omega_1}\omega_2 + \underline{m^{\lambda k}}\,\bar{m}^{\lambda}_{\ k} - \underline{\bar{m}^{\lambda k}}\,m^{\lambda}_{\ k}).$$

Finally, upon comparing the two formulas $(54_1')$ and $(54_2')$, we see that the general tensorial expression for the desired *identity* is:

$$(54) \quad \boxed{\{\psi^{\times}[\ ]\gamma^j\psi\cdot\psi^{\times}\bar{\gamma}^j\psi - \cdots\} = \tfrac{1}{2}(\underline{m^{ik}}\,\bar{m}^j_{\ k} - \underline{\bar{m}^{ik}}\,m^j_{\ k} + (\underline{\omega_2}\omega_1 - \underline{\omega_1}\omega_2)\,\delta^{ij}) + \tfrac{1}{2}\left[\underline{j_k}\,\bar{j}^{ijk} + \underline{\sigma_k}\bar{\sigma}^{ijk}\right],}$$

in which δ_{ij} denotes the Kronecker symbol; the right-hand side takes the form of the sum of a tensor that is symmetric in i, j and an anti-symmetric one [2].

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[1] This *identity* corresponds to (24) and (47) of Kofink.

[2] The cited paper of Kofink contains some further *identities* "with backward derivation" that do not fit into any of the categories that were considered in this number, and which seem to be some "specimens" of much vaster families; tensorially, one agrees to group them thus: 25, 48; 26, 49; 22, 23, 41, 42; 50; 51; 52. The tensorial variance of these identities is not always directly evident in Kofink. For example, the last three [*sic*] cited ones then have the variance 1, 2, 3, 4. (*Op. cit.*, "…II…," § 1, pp. 438-441.

II. – ESTABLISHEMENT AND PHYSICAL STUDY OF THE FRANZ-KOFINK DIFFERENTIAL RELATIONS.

17. While always observing our general conventions in the Foreword, we shall write the symbolic Dirac equation here and its associated Gordon-Pauli equation in the form ([1]):

(56) $$\boxed{\left\{\gamma_i\left(\underset{\rightarrow}{\partial}^i - i\varepsilon A^i\right) + \mu_0\right\}\psi = 0,} \qquad \boxed{\psi^\times\left\{\left(-\underset{\leftarrow}{\partial}^i - i\varepsilon A^i\right)\gamma_i + \mu_0\right\} = 0,}$$

upon setting:

(57) $$\boxed{\varepsilon = \frac{2\pi}{h}\cdot\frac{e}{c} = \frac{\nu}{e}, \quad \nu = \frac{2\pi e^2}{hc};} \qquad \boxed{\mu_0 = \frac{2\pi}{h}cm_0.}$$

ν denotes the *fine-structure constant*, which is a pure number; the operators $\underset{\rightarrow}{\partial}^i$ and $\underset{\leftarrow}{\partial}^i$, as well as the constant μ_0, have the dimension of inverse length.

The way that we shall define the ten differential relations that we have in mind is the following: Let γ_A be any of the sixteen γ of Dirac's theory. Multiply (56_1) on the left by $\psi^\times\gamma_A$, (56_2), on the right by $\gamma_A\psi$, and then add and subtract. Upon successively operating on the five tensorial ranks 0, 1, 2, 3, 4 with γ_A, we will cause $2 \times 5 = 10$ relations to appear that will have a tensorial character, by virtue of what was said in Chapter I.

In the general case where the rank n of the multiplying matrix γ_A is not 0 or 4, the terms in γ_i in (56) will generate two groups of terms in which the rank of the matrix product γ is $n \pm 1$. In one of these groups of terms, the γ_A commute with the γ_i, while in the other, they anti-commute. Consequently, by the aforementioned addition or subtraction, the terms in γ_i in (56) will generate the following two types ([2]):

(58) $$\begin{cases} \{\ \} \equiv \psi^\times([\partial^i] - 2i\varepsilon A^i)\gamma^B\psi = \{\psi^\times([\partial^i]\gamma^B\psi - 2i\varepsilon A^i\cdot\psi^\times\gamma^B\psi\}, \\ \partial^i(\ ) \equiv \psi^\times(\partial^i)\gamma^C\psi = \partial^i(\psi^\times\gamma^C\psi). \end{cases}$$

One sees that the tensors { } are sums of two tensors: The first one, which we call *Schrödingerian*, because its definition involves the operator $[\partial^i]$ of the current in Schrödinger's original theory, is independent of the prevailing quadri-potential, and must therefore be considered to belong to statistical electronic fluid. Up to a factor, the second one is a product of the prevailing quadri-potential with one of the five *Diracian* tensors () = $\psi^\times\gamma\,\psi$, and must then be considered to be an *interaction* term between the field and the electronic fluid. As for the tensors (58_2), they are the derivatives of the Diracian tensors (). Finally, the term in μ_0 in (56) will give:

(58′) $$2\mu_0\,(\psi^\times\gamma^A\psi) \quad \text{and zero,}$$

([1]) We shall always consider ψ to be a matrix with four rows and one column, and $\psi^\times$, to be a matrix with one row and four columns.

([2]) *See* AL. PROCA, "Sur la Théorie de Dirac dans un champ nul," Ann. de Physique **20** (1933), pp. 401, 404.

resp., by addition and subtraction, resp.

In order to effectively establish the ten relations in question, it is often advantageous to pass to dual quantities, which the table (45) permits one to do with no difficulty. The results obtained are the following ones, in which the symbol [ ] corresponds to the *value* that one attributes to the multiplying matrix γ:

$$[I]\quad \begin{cases} \partial^i(\psi^\times\gamma_i\psi) = 0, \\ \{\psi^\times[\partial^i]\gamma_i\psi - 2i\varepsilon A^i\cdot\psi^\times\gamma_i\psi\} + 2\mu_0(\psi^\times\psi) = 0; \end{cases}$$

$$[\gamma_i]\quad \begin{cases} \partial^i(\psi^\times\psi) + \{\psi^\times[\partial^i]\gamma^{ij}\psi - 2i\varepsilon A^i\cdot\psi^\times\gamma^{ij}\psi\} = 0, \\ \{\psi^\times[\partial^i]\psi - 2i\varepsilon A^i\cdot\psi^\times\psi\} + \partial_j(\psi^\times\gamma^{ij}\psi) + 2\mu_0(\psi^\times\gamma^i\psi) = 0; \end{cases}$$

$$[\gamma_m]\quad \begin{cases} \{\psi^\times([\partial^v]\gamma^u - [\partial^u]\gamma^v)\psi - 2i\varepsilon(A^v\cdot\psi^\times\gamma^u\psi - A^u\cdot\psi^\times\gamma^v\psi)\} + \partial^w(\psi^\times\overline{\gamma}^4\psi) - \partial^4(\psi^\times\overline{\gamma}^w\psi) = 0, \\ \partial^v(\psi^\times\gamma^u\psi) - \partial^u(\psi^\times\gamma^v\psi) + \{\psi^\times([\partial^w]\overline{\gamma}^4 - [\partial^4]\overline{\gamma}^w)\psi - 2i\varepsilon(A^w\cdot\psi^\times\overline{\gamma}^4\psi - A^4\cdot\psi^\times\overline{\gamma}^w\psi)\} + 2\mu_0(\psi^\times\gamma^{uv}\psi) = 0; \end{cases}$$

$$[\gamma_{uvw}]\quad \begin{cases} \{\psi^\times[\partial^4]\overline{\gamma}\psi) - 2i\varepsilon A^4\cdot\psi^\times\overline{\gamma}\psi\} + \partial_u(\psi^\times\overline{\gamma}^{u4}\psi) = 0, \\ \partial^4(\psi^\times\overline{\gamma}\psi) + \{\psi^\times[\partial_u]\overline{\gamma}^{u4}\psi) - 2i\varepsilon A_u\cdot\psi^\times\overline{\gamma}^{u4}\psi\} + 2\mu_0(\psi^\times\overline{\gamma}^4\psi) = 0; \end{cases}$$

$$[\gamma_{uvw4}]\quad \begin{cases} \{\psi^\times[\partial^i]\overline{\gamma}_i\psi - 2i\varepsilon A^i\cdot\psi^\times\overline{\gamma}_i\psi\} = 0, \\ \partial^i(\psi^\times\overline{\gamma}_4\psi) + 2\mu_0(\psi^\times\overline{\gamma}\psi) = 0. \end{cases}$$

In order to simplify the calculation of the dual quantities, we have taken $\gamma_{ij} = \gamma_{uv}$ and $\gamma_{ijk} = \gamma_{uvw}$, so the ultimate establishment of the general indices can be accomplished with less difficulty. One sees the *equation of continuity for the Dirac current* ([1]) in $[I^4]$, the *decomposition formula for the Gordon current* ([2]) in $[I^2]$, a relation that was given by H. Tetrode in a somewhat different form ([3]) in $[\gamma^1_{uv}]$, a formula from classical magnetism in $[\gamma^1_{uvw}]$, as Al. Proca recognized in the particular case where the prevailing potential is zero ([4]), and finally, the formula that was already interpreted by Uhlenbeck and Laporte ([5]) in $[\gamma^2_{uvw4}]$. Among the Schrödingerian tensors { } that appear in these ten relations, the following three have be taken into consideration: The *Gordon current* $\psi^\times[\partial^i]\psi - 2i\varepsilon A^i\psi^\times\psi$, the *asymmetric Tetrode tensor* $\psi^\times[\partial^i]\gamma^j\psi - 2i\varepsilon A^i\psi^\times\gamma^j\psi$, and finally, in the case of the free electron, the *Proca magnetic current*, whose general expression is $\psi^\times[\partial^i]\overline{\gamma}\psi - 2i\varepsilon A^i\cdot\psi^\times\overline{\gamma}\psi$.

([1]) "The quantum theory of the electron," Proc. Roy. Soc. London **118** (1928), pp. 35. *See* also J. VON NEUMANN, "Einige Bemerkungen zur Diracschen Theorie," Zeit. Phys. **48** (1928), pp. 868 and 880.

([2]) "Der Strom der Diracschen Elektronentheorie," Zeit. Phys. **50** (1928), pp. 630.

([3]) "Der Impulse-Energiesatz in der Diracschen Quantentheorie," Zeit. Phys. **49** (1928), eq. (16), pp. 861. The same formula was given, but without interpretation by J. Géhéniau, *Mécanique ondulatoire de l'électron et du photon*, Brussels, 1938, eqs. (54) and (58), pp. 59-60.

([4]) "Sur la Théorie de Dirac dans un champ nul," Ann. de Physique **20** (1933), pp. 429.

([5]) "New covariant relations following from the Dirac Equations," Phys. Rev. **37** (1931), pp. 1553, eq. (2).

We now group and systematize these results. The ten relations that were obtained involve *abstract density tensors* that are:

1. The five classical Dirac and Darwin tensors of the type $\psi^{\times}\gamma\psi$, which we shall denote by the symbol ( ), to abbreviate.

(59)

$$\text{(A)}\quad \boxed{\begin{aligned}(j^l) &= \psi^{\times}\gamma^l\psi,\\ (m^{ij}) &= \psi^{\times}\gamma^{ij}\psi,\end{aligned}} \qquad \text{(B)}\quad \boxed{\begin{aligned}(\omega_1) &= \psi^{\times}\psi,\\ (\sigma^i) &= \psi^{\times}\overline{\gamma}^i\psi,\\ (\omega_2) &= \psi^{\times}\overline{\gamma}\,\psi.\end{aligned}}$$

2. Five other tensors, which we denote by the symbol { }, namely:

(60)

$$\text{(A)}\quad \boxed{\begin{aligned}\{k^i\} &= \psi^{\times}[\partial^i]\psi \quad - 2i\varepsilon A^i(\omega_1),\\ \{l^i\} &= \psi^{\times}[\partial^i]\overline{\gamma}^i\psi \ - 2i\varepsilon A^i(\omega_2),\\ \{S^{kl}\} &= \psi^{\times}[\partial^k]\overline{\gamma}^l\,\psi - 2i\varepsilon A^k(\sigma^l),\end{aligned}} \qquad \text{(B)}\quad \boxed{\begin{aligned}\{U^{ijk}\} &= \psi^{\times}[\partial^i]\gamma^{jk}\psi - 2i\varepsilon A^i(m^{jk}),\\ \{T^{kl}\} &= \psi^{\times}[\partial^k]\gamma^l\psi - 2i\varepsilon A^k(j^l).\end{aligned}}$$

As was said, the latter tensors present themselves as sums of a *Schrödingerian* tensor $\psi^{\times}$ [ ] $\gamma\psi$, which is independent of the prevailing quadri-potential A^i, and which we denote by the symbol { }′, and a tensor $-2i\varepsilon A^i$ ( ) that is the product of the prevailing quadri-potential with one of the Diracian tensors, and which we denote by the symbol { }″ ([1]). All of these tensors are *abstract* density tensors; i.e., ones that are devoid of coefficients that would give them physical dimensions and a convenient real or pure imaginary character. Finally, we remark that the third-rank tensor $\{U^{ijk}\}$ enters into the preceding relations only by its two contractions:

(61′ B)
$$\boxed{\{U^i\} = \psi^{\times}[\partial_j]\gamma^{ij}\psi - 2i\varepsilon A_j(m^{ij}), \quad \{U_2^i\} = \psi^{\times}[\partial_j]\overline{\gamma}^{ij}\psi - 2i\varepsilon A_j(\overline{m}^{ij}).}$$

Once these definitions have been recalled or introduced, the ten relations in question can be written:

(A)

$$\boxed{\begin{array}{ll}\text{I} & \partial_l\,(j^l) = 0\\ \text{II} & \{k^l\} + \partial_l\,(m^{lj}) = -\,2\mu_0\,(j^l),\\ \text{III} & [\partial^l\,(j^k) - \partial^k\,(j^l)] - \overline{[\{S^{lk}\} - \{S^{kl}\}]} = -\,2\mu_0\,(m^{kl}),\\ \text{IV} & \{U_2^i\} - \partial_j(\overline{m}^{ij}) = 0,\\ \text{V} & \{S_i^i\} = 0;\end{array}}$$

(61)

([1]) The tensorial character of the quantities (59) and (60) is obvious for a change of Galilean frame that is performed "in the first manner" (no. **4**).

(B)

I	$\{T_l^l\} = -2\mu_0\,(\omega_1)$,
II	$\partial^l\,(\omega_1) + \{U_1^l\} = 0$,
III	$[\partial^k\,(\sigma^l) - \partial^l\,(\sigma^k)] - \overline{[\{T^{lk}\} - \{T^{kl}\}]} = 0$,
IV	$\partial^l\,(\omega_2) - \{U_1^l\} = -2\mu_0\,(\sigma^l)$,
V	$\partial_l\,(\sigma^l) = -2\mu_0\,(\omega_2)$.

18. Physical study of the ten relations (61). – A first fundamental remark is that the tensors (j^l), (m^{kl}), $\{k^i\}$, $\{l^i\}$, $\{S^{ij}\}$, on the one hand, properly belong to the sub-system of five equations (61 A), and the tensors (ω_1), (ω_2), (σ^i), $\{U^{ijk}\}$, and $\{T^{ij}\}$, on the other, to the sub-system of five equations (61 B). Now, those of these tensors that are physically well-defined are, on the one hand, (j^l) (the Dirac charge-current density), $\{k^i\}$ (the Gordon charge-current density), (m^{kl}) (magneto-electric moment density), and $\{l^i\}$ (magnetic charge-current density). On the other hand, one has (σ^i) (spin density), $\{T^{ij}\}$ (asymmetric Tetrode inertial tensor), and (ω_1) (proper mass density). Thus, *all of the tensors that were identified in* (61 A) *have an electromagnetic significance, and all of the tensors that were identified in* (61 B) *have a dynamical significance.* Correspondingly, those of the relations (61) that have presently been interpreted are, in the one hand, (A I) (conservation of Dirac current), (A II) (decomposition of the Dirac current) and (A IV) (expression for the magnetic current). On the other hand, (B III) [our relation (37) between the inertia tensor and the spin density] and (B I) (expression for the proper mass density). Therefore, among the relations (61) that are presently being interpreted, all of the (A) are electromagnetic relations, and all of the (B) are dynamical relations. All of this empowers us to say that, *by definition, the five relations* (61 A) *and the five tensors that they belong to characterize the electromagnetic behavior of the statistical fluid of Dirac's theory, and the five relations* (61 B) *and the five tensors that they belong to, its dynamical behavior.* From that definition, the relations (A III) and (A IV), as well as the tensor $\{S^{ij}\}$, which does not belong to classical electromagnetism, belongs to an extended electromagnetism; similarly, the relations (B II), (B III), and (B IV) (Uhlenbeck and Laporte), as well as the tensors (ω_2) and $\{U^{ijk}\}$, belong to an extended dynamics ([1]).

In the absence of an external quadri-potential A^i, the five tensors { } reduce to their *Schrödingerian part* $\{\ \}' = \psi^\times[\partial^i]\gamma\psi$. Since the five tensorial operators γ define *Diracian tensors* ( ) bijectively and the five operators $[\partial^i]\,\gamma$ define *Schrödingerian tensors* $\{\ \}'$, we see that *in the absence of an external quadri-potential* A^i, *the two sub-systems* (61 A) *and* (61 B) *are completely independent;* the electromagnetic and dynamic properties of the statistical fluid each evolve by themselves without interacting with each other.

([1]) One can doubt whether the results that are acquired from classical theories and also provide a sufficient basis for formal arguments that are analogous to the ones in Chapter II permit one to justify the five formulas (61) that remain to be interpreted. In that case, it is Dirac's theory that one must start with in order to "enlarge" electromagnetism and dynamics in the indicated sense.

On the contrary, a non-null external quadri-potential A^i will add *an interaction term* $\{\}'' = -2i\varepsilon A^i$ ( ) to each *Schrödingerian* tensor { }′. It is remarkable that the Diracian tensor ( ) that enters into { }″ has the *opposite* physical nature to that of the corresponding { }′ (electrical for mechanical, and *vice versa*). Therefore, *the external quadri-potential* A^i *will produce an electro-mechanical coupling between the two sub-systems* (61 A) *and* (61 B), *which is a coupling that is completely symmetric with respect to electromagnetism and dynamics.*

The *ponderomotive effect of the field* is then manifested in a perfectly symmetric manner. In Dirac's theory, there is no reason to qualify the prevailing potential A^i as being "electrical," rather than "mechanical," ([1]). That is an entirely new situation in comparison to the classical theory: In the analytical mechanics of the electrically-charged point, the total mass-impulse indeed appears to be the sum of a *proper* term p^i and an *electromagnetic* term QA^i ([2]), but that fact seems to be isolated and has no *symmetric* counterpart in the sense that was just discussed.

Detailed examination of the "electromagnetic" relations (61 A). The relation (A I) is nothing but Dirac's fundamental continuity equation. Dirac's inductions, although they have succeeded brilliantly, are no less audacious, as one sees here notably: Indeed, if the fourth component $\psi^\times\gamma^4\psi = i\,\psi^*\psi$ of the Dirac world-current density is, in fact, the exact transposition of the Schrödinger charge density $\psi^*\psi$ ([3]) then the three $\psi^\times\gamma^u\psi$ are by no means analogues of the Schrödinger current densities $\psi^*[\partial^u]\,\psi$. One finds that the Gordon formula (A II) permits one to reduce the "amplitude" of the corresponding induction: The expression for the Gordon current $\psi^\times[\partial^u]\,\psi$ is clearly apparent in the Schrödinger expression $\psi^*[\partial^u]\,\psi$, and one will confirm later on in this number and in Chapter IV, paragraph II, that the situation is further ameliorated when one takes into consideration the Tetrode inertia tensor $\psi^\times[\partial^i]\,\gamma^j\psi$, whose components in ($i$, 4) are written $i\psi^*[\partial^i]\,\psi$.

The relation (A II), which has the same form as a well-known relation from the theory of electromagnetism in polarized media ([4]) was given by Gordon as providing a decomposition of the *total charge-current density* (j^l) into a *convection current* $\{k^i\}$ and a *polarization current* ∂_i (m^{ik}). In Chapter IV, we will confirm that this terminology is indeed the one that is imposed from the electromagnetic viewpoint, but that it raises some difficulties from the standpoints of kinematics and dynamics. As for the expression ∂_k (m^{lk}), it obviously represents the polarization current, and one can consider that the relation (A II) is justified, upon starting with the interpretation of the quadri-vector (j^l) as a *charge-current density*, and that of the antisymmetric tensor (m^{kl}) as *the magneto-electric moment density.*

([1]) An analogous remark is true for the *proper mass of the electron* m_0 . Recall that the *proper mass of the photon* enters into the equations and definition of the theory of the photon. (L. DE BROGLIE, *Mécanique ondulatoire du Photon*, pp. 156 and 158.)

([2]) O. COSTA DE BEAUREGARD, *la Relativité restreinte*, pp. 48 and 62. – In number **8** of the present work, we showed that by integrating the two terms of the Tetrode tensor over a hypersurface, one will recover the classical expression for the proper or *kinetic* mass-impulse *in the mean.*

([3]) *See* above, Chapter I, no. **8**.

([4]) *See*, for example, R. BECKER, *Théorie des Électrons*, pp. 124 and 365.

In an analogous manner, formula (A IV), which was studied by Proca in the case of the free electron, permits one to interpret that quadri-vector $\{l^i\}$ as a *magnetic charge-current density*; the vanishing of the right-hand side expresses the *absence of true magnetism* that one expected *a priori*.

Before we go on, we make the definitions of the four tensors that were just in question more precise by introducing suitable physical factors. One knows that in Heaviside's e.s.u., the Dirac charge-current density is $j^k = -e\ \psi^\times \gamma^k \psi$; starting from that, one proceeds step-by-step:

(62 A)

Dirac electric charge-current density	$j^k = -e(j^k),$
Gordon " "	$k^i = +\dfrac{e}{2\mu_0}\{k^l\},$
Magneto-electric moment density	$m^{ij} = +\dfrac{e}{2\mu_0}\{m^{ij}\},$
Magnetic charge-current of polarization	$l^i = -\dfrac{ec}{2\mu_0}\{l^i\}.$

Now, take equation (61 A III). One can consider that it provides a decomposition of the magneto-electric moment density m^{kl} into two terms that (up to suitable factors) are the rotation of the quadri-current j^k and the "lack of symmetry" in a certain asymmetric tensor $\{S^{kl}\}$. On first glance, the first term seems to conform to what intuition would suggest: Since the rotation of an electrified droplet produces a magnetic moment, it would seems that a vorticial electric world-current must manifest a magneto-electric moment density, such that the magnetic moment would correspond to the rotation of the spatial tri-current, which is precisely what happens in formula (A III). In fact, that way of seeing things falls apart on the basis of an objection that was encountered before in the context of kinetic moments (Chapter II, no. **10**): The magnetic moment of a uniformly-charged sphere of radius r with a density of q and animated with an angular velocity of ω will be $4\pi q\, r^5 / 15$; it is a fifth-order infinitesimal in r, which is an order that is too high by two units in order for it to define a *density*. We are then certain *a priori* that the first term in formula (A III) cannot be interpreted in terms of *classical* electromagnetism; that is what the introduction of physical coefficients confirms for us.

Recalling the physical definitions of the quantities j^k and m^{kl} that were just given, we find that *the first physical component* of m^{kl} has the value:

$$m^{kl}_{(1)} = \left(\frac{1}{2\mu_0}\right)^2 (\partial^l j^k - \partial^k j^l).$$

This intervention of the square of the proper mass of the electron in a formula that, by definition, we have said was a formula from electromagnetism, shows clearly that the electromagnetism that we are dealing with here is not classical electromagnetism ([1]). As for the *second component* $m_{(2)}^{kl}$ of m^{kl}, it is equal (up to a factor) to the *lack of symmetry* of a certain tensor $\{S^{kl}\}$. In Chapter IV, no. **24**, we calculate the two divergences of the tensor $\{S^{kl}\}$ whose physical interpretation has eluded us. Moreover, we can no longer give any *a priori* justification for the fact that the trace of that tensor must be zero, as the relation (61 A V) requires.

Detailed examination of the "dynamical" relations (61 B). The relation (B III) is identical to our formula (37′) of Chapter II; in order to verify this, we introduce convenient physical factors into the expressions for the spin density σ^k and Tetrode's asymmetric inertia tensor T^{kl}, which are, as one knows:

(62 B′)

Proper kinetic moment density.........	$\sigma^k = -\dfrac{h}{4\pi}(\sigma^k)$,
Tetrode's asymmetric inertia tensor...	$T^{kl} = +\dfrac{ich}{4\pi}\{T^{kl}\}$;

conforming to what we said, one will get:

$$\overline{[T^{kl} - T^{lk}]} = -ic[\partial^k \sigma^l - \partial^l \sigma^k]. \tag{63}$$

Note that *the coincidence of the relations* (37′) *and* (63) *is not only true in modulus, but also in sign;* indeed, in the two cases, and under the hypothesis of simultaneity, the finite mass-impulse can be calculated from the formula:

$$P^i = \iiint T^{i4}\,\delta u_4 = \frac{1}{ic}\iiint T^{i4}\,\delta u_4\,,$$

in which the significant index is the *first index* of T^{ij}. For (37′), that will result from what we said in Chapter II, no. **12**, 8, and for (63), from what we said in Chapter I, no. **8**.

Finally, our theory of pre-quantum relativistic dynamics in Chapter II permits us to interpret formula (B III) (which was given initially by Tetrode in an equivalent form) as having the following significance: *The volumetric density of fictitious proper ponderomotive moment, when applied to the polarized statistical fluid by the field, will be identically zero.* Later on, we shall recall in number **21** that this situation differs from the one that we encountered in the classical electromagnetic theory of polarized media.

Now, take the relation (B I). In a "classical medium" without spin, the trace of the inertia tensor T^{kl} will be nothing but the proper mass density m_0 (up to a factor of $-c^2$). *By definition*, that result can be preserved in the theory of media that are endowed with

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([1]) Recall that the square of the proper mass *of the photon* enters into the modified equations of the first group of Maxwell-Lorentz equations in L. de Broglie's theory of the photon (*Méc. ond. photon*, t. I, pp. 158.)

spin (Chap. II, no. **13**); the relation considered then gives the following expression for the proper mass density in Dirac's theory:

(63 B) Proper mass density.......... $\rho_0 = m_0 i(\omega_1) = -m_0(\psi^* \gamma^4 \psi);$

this well-known result is then found to be justified in an elegant manner.

Unfortunately, the three relations (B II), (B IV), and (B V) (Uhlenbeck and Laporte) remain lacking in an interpretation, just like the tensors (ω_2), $\{U_1^i\}$, and $\{U_2^i\}$ ([1]).

19. Extension of the definitions and relations (59), (60), (61) **to the theories of the photon and the graviton.** – L. de Broglie's theory of the photon and M. A. Tonnelat's theory of the graviton involve two distinct categories of tensorial density quantities. The first of them refers to the original proper elements of those theories, namely, the creation of the electromagnetic or gravitational field by the transition of the corpuscle from the state $\Phi(x^1, x^2, x^3, x^4)$ and an "annihilation state" Φ^0 ; these quantities are not the ones that we shall study here. The density quantities of the second category, only some of which have ever been given any physical consideration, are attached to the propagation of the statistical corpuscular fluid. For example, one is dealing with the presence-current density quadri-vector (j^l), the spin density quadri-vector σ^l, and "corpuscular" inertia tensor T^{kl} ; these quantities, which are completely analogous to the ones that one considers in Dirac's theory, are the ones that we would now like to say a few words about.

The *photon* and the *graviton* are particular cases of the corpuscles that are obtained by the fusion of n Dirac corpuscles, corresponding to the values $n = 2$ and $n = 4$. In a general manner, the fundamental equations of the "corpuscle n," which are called "ones of type I" by L. de Broglie, are composed of n systems of 4^n equations of Diracian type. Each of these systems utilizes a set of four matrices $\mathcal{A}_\nu^i$ of rank 4^n ($i = 1, 2, 3, 4$; $\nu = 1, 2, \ldots, n$) that satisfy Dirac's fundamental relations (11); moreover, for $\mu \neq \nu$, any matrix $\mathcal{A}_\mu^i$ will commute with any matrix $\mathcal{A}_\nu^j$ ([2]). Under these conditions, it is clear that if one sets:

$$\mathcal{A}_\nu^u = -i\, a_\nu^u\, a_\nu^4, \qquad \mathcal{A}_\nu^4 = a_\nu^4, \tag{64$_1$}$$

([1]) The quadri-vector $\iiint (\omega_1)\,\delta u^1 \approx \iiint (T_l^{\,j})\,\delta u^l$, when calculated over a space-like hypersurface, is homogeneous in the mass-impulse $\iiint T^{ij}\,\delta u_j$. We call it the *false mass-impulse* and take the integral $\iiint\int \{U_{(1)}^i\}[dx^1dx^2dx^3dx^4]$ over the world-volume that is bounded by two infinitely-close "constant time" hypersurfaces 1 and 2 and by the hyper-wall of a current tube; from (B II), that integral is equal (up to a factor) to the integral $\iiint \rho_0\,\delta u^1$, when taken over the contour of the preceding volume. Since the portion of the triple integral that corresponds to the hypersurfaces represents the *variation of the false mass-impulse* when one passes from the state 1 to the state 2, we can interpret (up to a factor) the quadri-vector $\{U_{(1)}^i\}$ as a *volumetric density of false ponderomotive force;* in an analogous manner, the portion of the triple integral that corresponds to the hypersurface will permit us to introduce a *false surface ponderomotive force.*

([2]) L. DE BROGLIE, *Théorie générale des particules à spin*, pp. 138 *et seq.*

so ([1]):

(64₂)
$$\Phi^{\times} = i\,\Phi^{+} a_1^4\, a_2^4 \cdots a_n^4,$$

then one will find the same advantages of relativistic symmetry in the writing of "system 1" and the "associated system" as the Gordon-Pauli nomenclature in the theory of the electron. The generalized definition of the Diracian tensors (59) and the Schrödingerian tensors (60) ([2]) must be the following one: One replaces $\psi^{\times}$ in each expression (59) or (60) with the $\Phi^{\times}$ that was just defined, and each γ^i with the *normalized sum:*

(64₂)
$$a^i - \tfrac{1}{2}\sum_{\nu=1}^{n} a_\nu^i;$$

the physical coefficient will remain the same as in Dirac's theory ([3]). One effortlessly verifies that this definition is, in fact, the one that led to the particular expressions that were given by L. de Broglie in the theory of the photon ([4]) and by M. A. Tonnelat in the theory of the graviton ([5]). That being the case, it is clear that the system of ten relations (61) will remain valid. It suffices to recall the calculations of no. **17** verbatim, upon operating on each isolated sub-corpuscle with the aid of "equations I" and their associated ones, and then adding the results.

In light of the foregoing, it is interesting to examine the problem that is posed by the definitions of the various *energy-impulse tensor densities* that are considered by these theories. First of all, it results from what we have said collectively that, according to us, it is not convenient to symmetrize the expression for the inertia tensor that is called "corpuscular," which belongs to the preceding family. It seems to us that the expression for that tensor must be given in the form:

(65)
$$T^{ij} = \frac{ich}{4\pi}\,\Phi^{\times}\left\{\frac{1}{n}[\partial^i]\sum_{\nu=1}^{n} a_\nu^i\right\}\Phi,$$

in which the operator { }, which is part differential and part matrix, acts on both the right and the left. An essential remark is that *the expression for the tensor T^{ij} is symmetric with respect to the index v; i.e., with respect to the constituent sub-corpuscles.*

Aside from the corpuscular tensor, whose definition involves differential operators, the general theory of fusion introduces other energy-impulse tensor densities, whose number increases with that of the fusing corpuscles, into the definition, which will not

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([1]) We will always consider the components of Φ and Φ^* to be the elements of two adjoint matrices.

([2]) The introduction of the *potential terms* into the equations for the basic corpuscle is not always exempt from complications. Be that as it may, the statements that we shall make will be true for the *free corpuscle*, which is a case in which the Schrödingerian tensors reduce to their first term.

([3]) Except for the *presence-current density quadri-vector* (j^i), it is unclear what physical significance of the tensors (59 A) or (60 A) is case of the uncharged corpuscle; In Dirac's theory, the charge *e* is a factor in the physical expressions for all of these tensors (eq. 62 A).

([4]) *Mécanique ondulatoire du Photon*, pp. 173, 185, 187, eqs. (2), (46), (52).

([5]) "Étude de la Particule de Spin 2," pp. 197 and 200, Ann. de Physique **17** (1942).

involve differential operators, but only matrices a ([1]). For example, one defines a "Maxwellian" tensor in the theory of the photon whose expression is:

$$\mathcal{M}^{ij} = m_0 c^2\, \Phi^{\times} \left\{ \frac{1}{(?)^{l}_{2n}} \sum_{\mu \neq \nu} a^i_\mu a^j_\nu \right\} \Phi, \tag{65′}$$

which is, one sees, an expression that is symmetric with respect to the set of indices i and μ; that double symmetry is recovered in all of the inertia tensors "of type M" that are defined by the general theory of fusion. It is clear that the symmetry of those tensors in μ, ν, … – i.e., their symmetry with respect to the constituent corpuscles, which is necessary *a priori* – automatically implies their symmetry in i, j, … It cannot be a question of "de-symmetrizing" the expression for these "M tensors," which can be one good reason to think that their physical interpretation is less direct than that of the "corpuscular" tensor T^{ij}.

In the case of a superposition of monochromatic plane waves, one knows that the corpuscular tensor is *integrally equivalent* to the tensors "of type M" ([2]). That result will not be altered when one replaces the symmetrized corpuscular tensor with the asymmetric tensor (65), since the latter tensor will once more become symmetric in the case considered ([3]).

([1]) *Mécanique ondulatoire du Photon*, pp. 189, eq. (60); "Étude de la Particule de Spin 2," pp. 200; *Théorie générale des particules à spin*, pp. 154.

([2]) *Mécanique ondulatoire du Photon*, pp. 190; "Étude de la Particule de Spin 2," pp. 201; *Théorie générale des particules à spin*, pp. 155.

([3]) *See* above, no. **26**.

CHAPTER IV

STUDY OF THE FICTITIOUS STATISTICAL FLUID IN DIRAC'S THEORY (cont.)

20. In the present chapter, we shall examine some particular aspects of the agreement between the properties of the fictitious statistical fluid of Dirac's theory and those of a classical continuous medium that is endowed with electromagnetic and dynamical polarization, in the sense of Chapter II. The results that will be obtained will be perfectly ambiguous, or even contradictory; however, for certain reasons that we shall point out in no. **21**, that fact should not be surprising, and one must expect it *a priori*.

For example, some considerations of a purely electromagnetic order lead one to clearly assign the Dirac current to the *total electromagnetic current*, which conforms to the terminology of Darwin and Gordon. One knows that in the example of the *Darwin globule*, the Gordon current seems to be translation current (no. **22**); however, as we shall see in a moment, that concept does not seem susceptible to extension in the most general case, in such a way that the qualifier *"of convection"* that Gordon applied to its current raises some difficulties.

Indeed, some converging kinematical and dynamical arguments (the latter ones are drawn from our Chapter II), in their own right, lead one to associate the Dirac current with the *kinematical current* of a customary fluid (nos. **25** and **26**). That seems to be a paradox to us, which cannot be lacking in relationships to other paradoxes that were pointed out by various authors in regard to the study of magnetism, even pre-quantum.

In no. **23**, we shall calculate the two divergences of Tetrode's inertia tensor, according to a method of this author, and at the same time, the two divergences of our own asymmetric electromagnetic tensor S^{kl}. One knows that the double result of Tetrode converges to Lorentz's electrodynamical formula, and that is precisely how Tetrode justified the interpretation of his tensor T^{kl} as *inertial*. In reality, the discussion of the question shows that the agreement between the classical ideas is not complete, in such a way that the convergence in question seems very formal (no. **23**). This latent disaccord of Dirac's theory with classical electrodynamics seems much clearer in the question of the proper ponderomotive moments (no. **24**).

One knows that Pauli had profited from the fact that the two divergences of the tensor T^{kl} are equal in order to symmetrize that tensor *a posteriori*, which is an operation that, as we have said before, seems contestable from the viewpoint of general quantum principles (no. **8**), and also that of the theory of media that are endowed with spin (no. **12**, 8). In any case, one can say that the fact that was invoked by Pauli can just as well be interpreted as the lifting of the need to symmetrize T^{kl}. At the end of no. **26**, we shall recapitulate the entire set of arguments that were encountered in the course of this work, and from which the "true" inertial tensor of Dirac's theory seems to us to be, not Pauli's symmetrized tensor, but, in fact, Tetrode's original asymmetric tensor.

I. – ON THE RELATIONSHIPS BETWEEN DIRAC'S THEORY AND THE CLASSICAL ELECTROMAGNETISM OF POLARIZED MEDIA.

21. Summary of classical electromagnetism. Definition of electromagnetism according to Dirac. – One knows that all of electromagnetism and all of the classical electrodynamics of polarized media can be derived from the following three groups of basically independent equations:

$$\text{[I]} \qquad \partial_k E^{ik} = 0, \qquad \text{[II]} \qquad \partial_k F^{ik} = j^i, \qquad \text{[III]} \qquad f^i = H^{ik} j_k .$$

The antisymmetric E^{ik}, which is sometimes called the *final field*, H^{ik} is its dual, which contains the *electric field* E^{uv} and the *magnetic induction* E^{u4}, while the antisymmetric tensor F^{ik} contains the *magnetic field* H^{uv} and the *electric induction* H^{u4}. It is regrettable that the terminology that has been consecrated by its use makes it difficult to define the tensors E^{ik} (or H^{ik}) and F^{ik} globally. Finally, j^i is the *total charge-current density* quadri-vector, and f^i is the *total force-power density* that is applied to the latter by the *final field* H^{ik}.

Correspondingly, the tensor F^{ik} and the quadri-vector j^i decompose according to the formulas:

$$\text{[IV]} \qquad F^{ik} = H^{ik} + m^{ik}, \qquad\qquad \text{[V]} \qquad j^i = k^i + \partial_k m^{ik}.$$

The antisymmetric tensor m^{ik} is the *magneto-electric moment density* of the medium considered, $\partial_k m^{ik}$ is the *polarization charge-current density*, and k^i is the *convection charge-current density* ([1]). In the case of a *truly continuous* medium, it seems natural to assume the well-known notation for the latter quadri-vector:

$$k^u = q\, v^u, \qquad k^4 = ic\, q,$$

from which, it will be *time-like* ([2]). On the contrary, the quadri-vector j^i has an arbitrary type *a priori*.

Equations [I], which are independent of the properties of the material medium, are condition equations for the field. Under very broad conditions, they are equivalent to the following equations, which translate into the existence of a vector-potential:

$$\text{[I′]} \qquad H^{ij} = \partial^i A^j - \partial^j A^i.$$

Equations [II] are the expression of a *magneto-electric correlation* between the field and the medium. Physically, one imagines that this correlation translates into the creation of a field by a distribution of current and polarization that is given *a priori*. If one imposes Lorentz's supplementary condition $\partial_i A^i = 0$ on the field then equations [II] can be put into the equivalent form $\partial_i^i A^l = j^l$. Finally, equations [III] express an *electrodynamical*

([1]) For all of this, *see*, for example, R. BECKER, *Théorie des Électrons*, pp. 121, 124, 359, 365.

([2]) *La Relativité restreinte*, pp. 36. – R. Becker, like H. A. Lorentz, considered the case of a cloud of electrified corpuscles (viz., classical point-like corpuscles without spin). In that case, the *mean* convection current is not time-like, due to the existence of charges of the two signs. In the present work, we shall systematically limit ourselves to the case of *truly continuous* medium (*see*, notably, pp. 31 and pp. 67).

correlation between the field and the medium that translates into the action of the given field upon the medium *a priori*.

Among the consequences of the basic equations that are of interest to us, we cite the *continuity equations for the currents* j^i *and* k^i :

$$\text{[VI]} \qquad \partial_i\, j^i = 0, \qquad \partial_i\, k^i = 0,$$

as well as the expression for the *proper ponderomotive moment density of the universe*, which applies to the medium when it has been polarized by the prevailing field ([1]):

$$\text{[VII]} \qquad m^{ij} = H^{ik}\, m^{j}{}_{k} - H^{jk}\, m^{i}{}_{k} \,.$$

That being the case, consider the set of equations (11) and (14) from Dirac's theory, properly speaking, and denote them by [II D], here. Those equations, like [II], translate into the existence of an electromagnetic correlation between the ambient field and the electron (or, for us, between the field and the statistical electronic fluid). However, since one neglects the reaction of the electron on the field here, one will be dealing with the action of a field that is given *a priori* on the electron that it embedded in it. It is easy to verify directly that *equations* [II] *and* [II D] *are incompatible*, which should not be surprising if one recalls that [I] and [II], on the one hand, and [II D], on the other, correspond to some distinct limiting cases of L. de Broglie's general equations of interaction for the photon-electron ([2]).

By themselves, the equations of Dirac's theory do not suffice to constitute a complete theory of electromagnetism, but rather they constitute a theory of electrodynamics. However, one knows that from the beginning that Dirac's theory appealed to the classical formula [I′] in order to establish the existence of a proper magnetism of the electron ([3]), and that Tetrode invoked that same formula in the calculation of the two divergences of his inertia tensor, which was a calculation that led him to recover the electrodynamical formula [III] in Dirac's theory ([4]). One can then say that *electromagnetism according to Dirac* and classical electromagnetism both use equations [I] as a basis, and differ by the incompatible basic equations [II] and [II D]. Equations [III], which constitute a basic element that is independent of classical theory, are recovered as *consequences of the set of equations* [I] *and* [II D], which is a truly remarkable result, and we say in passing that it is compatible with the fact that [II D] translate into the action of the field that was given *a priori* on the electron. As for the juxtaposition of [I] and [II D] to form a theory of electromagnetism, one can even say that it seems arbitrary *a priori*, since it is legitimate only because it is not contradictory.

([1]) In spatial-vector notation, one will have, in classical notation $\boldsymbol{\mu} = -\mathbf{H} \wedge \boldsymbol{\mathcal{H}} - \mathbf{E} \wedge \boldsymbol{\mathcal{E}}$. That expression, and that of the *energy density* $w = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})$, appear as consequences of the *asymmetric* expression for the Maxwell tensor when it is extended to the case of polarizable media:

$$M^{ij} = -\frac{1}{2}(H^{ik}\, F^{j}{}_{k} + \bar{F}^{ik}\, \bar{H}^{j}{}_{k}).$$

([2]) *La Mécanique ondulatoire du Photon*, t. II, pp. 132-136.

([3]) *See*, for example, *l'Électron magnétique*, pp. 241, eq. (28).

([4]) "Der Impuls-Energiesatz in der Diraschen Quantentheorie," Zeit. Phys. **49** (1928), pp. 860.

Another fact that is remarkable *a posteriori* is this one: Although they are quite different from each other, and even incompatible, *equations* [II] *and* [II D] *imply the same continuity equation* [VI_1] *for the current* j^i. Since the decomposition formula [V] is recovered in Dirac's theory as a consequence of [II D] (eq. 61 A III), moreover, *the Gordon current* k^i *is itself also conservative* [eq. VI_2] ([1]).

By contrast, it results from what we said in Chapter III that the expression for the fictitious ponderomotive moment density that is applied to the statistical fluid by the field is zero, which is a result that differs from the one that is expressed by the classical formula [VII].

Finally, one sees that classical electromagnetism and what we call *electromagnetism according to Dirac* are two theories that are incompatible *a priori*. Meanwhile, they have in common, as an independent basis element, the *first group* of Maxwell-Lorentz equations, as they are called. Some of the subsequent relations are, as one must expect, quite different from each other. However, one will find that due to a very surprising fact some of the more important subsequent relations are, on the contrary, and at least formally ([2]), identities in both theories, so we shall continue to pursue the comparison.

22. On the total electric current in Dirac's theory. – In anticipation of paragraph II, we say here that whether kinematically or dynamically, Dirac's current j^i, which is time-like, must be associated with the *kinematical current* or *true current* of the statistical fluid. Therefore, if Dirac's theory must agree with classical theory on that particular point then it would seem that the Dirac current j^i must coincide with the *electric convention current*, and the Gordon current k^i, with the *total electric current* of the statistical fluid. We shall now see that, on the contrary, several important arguments lead one to associate the *Dirac current* $j^i = \psi^\times \gamma^i \psi$ with the *total electric current.*

First of all, one knows that the charge – e (e.s.u., C.G.S.) of the electron must be calculated by integrating the Dirac current, by virtue of the normalization conditions ([3]):

$$\iiint \psi^+ \psi \cdot \delta u = 1 \qquad \text{or} \qquad -e \iiint \psi^+ \psi \cdot \delta u = -e.$$

Now, it is quite clear that the measured charge – e is the total charge (true charge + polarization charge), which shows that j^i must be considered to be the total current ([4]). In the second place, Tetrode's formula:

$$f^i = \partial_k T^{ik} = \partial_k T^{ki} = H^{ik} j_k ,$$

which we shall establish in the following number, shows clearly that j^i must be associated with the total current when one compares it with the classical formula [III]. Finally, it is

([1]) When the same argument is applied to the relation (61 A IV), it will show that the magnetic current quadri-vector l^i is conservative.

([2]) *See* below, end of no. **25**.

([3]) The "constant time" hyperplane of integration cuts the Dirac streamlines, which are time-like, once and only once.

([4]) Since the quantum – e is a universal constant, it is convenient to say, in classical terminology, that the possible variations of the *true charge* and the *polarization charge* compensate for each other.

not until one addresses Darwin's spherical globule theory that it too will behave the same way.

One knows that the equations of the Darwin globule are a solution of the Dirac equation that is valid in the absence of a field and in the non-relativistic approximation. One confirms that the Dirac current can then be decomposed into a first term that is orthogonal to the phase hyperplane, which is time-like here, and a second term that is none other than the polarization current. One effortlessly verifies that, as the general formula [62 (A) II] would demand, the first term in question is the Gordon current ([1]). Furthermore, upon dividing the three terms of that relation by $\psi^\times\psi$, one will cause three corresponding *fictitious* velocities to appear. One finds that the velocity $\mathbf{u}$ that is associated with the Dirac current is the sum of the group velocity $\mathbf{v}$ of the phase plane and a velocity $\boldsymbol{\omega} \wedge \mathbf{r}$ that corresponds to a collective rotation of the globule ([2]). Under these conditions, it is completely natural to say, with Darwin, that the *total current* j^i (or $\mathbf{u}$) is the sum of a *translation current* k^i (or $\mathbf{v}$) and a *complementary current* that corresponds to the polarization current from the electromagnetic viewpoint and to the vorticity of the globule from the kinematical viewpoint. In sum, in that example, one associates the notation of *total kinematical current* to that of *total electromagnetic current*. However, in the general case, the Gordon current k^i is not necessarily time-like, so it would seem difficult to associate it to a "translation current." On the contrary, the "total kinematical current" j^i is necessarily time-like, which permits one to always consider it to be the *final kinematical convection current* (and here, especially).

The final conclusion from the preceding seems to us to be the following one: *In Dirac's theory, the total electromagnetic current coincides with the (fictitious) kinematical current of the statistical fluid*, which is a situation that seems "revolutionary" to us in comparison to the classical theory ([3]). The Gordon current will then have no strict equivalent in classical electromagnetic theory. Moreover, from what was said in the preceding number, the brutal fact of a "conflict" between classical electromagnetism and "electromagnetism according to Dirac" is not surprising, and one must expect it *a priori*.

25. Calculating the two divergences of the asymmetric tensors $\{T^{ik}\}$ and $\{S^{ik}\}$. – The differences between the two divergences considered are provided in a very simple manner by the relations [(61) B III] and [(61) A III]. Indeed, taking into account the fact that the divergences of the dual of a rotation are identically zero, as well as the definition of the quadri-vector $\{l^i\}$ in [(61) A IV], in the second case, the relations that were invoked will permit one to write:

([1]) *See*, for example, *l'Électron magnétique*, pp. 170. In the non-relativistic approximation, and with Dirac's particular α^i, the Gordon current will have the expression $-\psi_4^*[\partial^u]\psi_4 - \psi_4^*[\partial^u]\psi_4$, and consequently, with the notations (6) and (10) of the cited passage, $\rho\,\mathbf{v}$.

([2]) *Op. cit.*, pp. 178. The fact that one divides by $\psi^*\psi = -\psi^\times\gamma^4\psi$ in order to make the velocity $\mathbf{u}$ appear amounts to *postulating* that the Dirac quadri-current can be put into the form $\mathbf{j} = \rho\,\mathbf{u}$, $j^4 = ic\rho$. As for the velocities $\mathbf{v}$ and $\boldsymbol{\omega} \wedge \mathbf{r}$, their introduction by the indicated process seems a bit artificial.

([3]) This paradox cannot fail to have some relationship to the other paradoxes that were pointed out by several authors. For example, we cite the absence of mutual energy between currents and permanent magnets. (P. JANET, *Leçons d'Électrotechnique générale*, t. 1, pp. 84)

(66) $$\boxed{\partial_j\{T^{ji}\} - \partial_j\{T^{ij}\} = 0,} \qquad \boxed{\partial_j\{S^{ji}\} - \partial_j\{S^{ij}\} = 2\mu_0\{l^i\}.}$$

It will then suffice to calculate the simpler of the two divergences of each tensor, which is found to be the one that relates to the *second index* (matrix index). It was precisely in that manner that Tetrode calculated the two divergences of T^{ij} in his cited paper ([1]).

For the *Schrödingerian part* { }′ of the two tensors considered, one can write:

[T] $$\begin{aligned}\partial_j\{T^{ij}\}' &= \partial_j\{\psi^\times[\partial^i]\gamma^j\psi\} \\ &= \left(\psi^\times\underset{\to}{\partial}^i\cdot\gamma^j\underset{\to}{\partial}_j\psi + \psi^\times\underset{\leftarrow}{\partial}_j\gamma^j\cdot\underset{\to}{\partial}^i\psi\right) - \left(\psi^\times\underset{\leftarrow}{\partial}^i\cdot\gamma^j\underset{\to}{\partial}_j\psi + \psi^\times\underset{\leftarrow}{\partial}_j\gamma^j\cdot\underset{\leftarrow}{\partial}^i\psi\right),\end{aligned}$$

[S] $$\begin{aligned}\partial_j\{S^{ij}\}' &= \partial_j\{\psi^\times[\partial^i]\bar{\gamma}^j\psi\} \\ &= \left(\psi^\times\underset{\to}{\partial}^i\cdot\bar{\gamma}^j\underset{\to}{\partial}_j\psi + \psi^\times\underset{\leftarrow}{\partial}_j\bar{\gamma}^j\cdot\underset{\to}{\partial}^i\psi\right) - \left(\psi^\times\underset{\leftarrow}{\partial}^i\cdot\bar{\gamma}^j\underset{\to}{\partial}_j\psi + \psi^\times\underset{\leftarrow}{\partial}_j\bar{\gamma}^j\cdot\underset{\leftarrow}{\partial}^i\psi\right).\end{aligned}$$

The four expressions $\gamma^j\underset{\to}{\partial}_j\psi$, … in [T] are provided by the Dirac equations (56); similarly, the four expressions $\bar{\gamma}^j\underset{\to}{\partial}_j\psi$, … in [S] are provided by the transformations:

$$\left\{\bar{\gamma}^i(\underset{\to}{\partial}^i - i\varepsilon A^i) + \mu_0\,\bar{\gamma}\right\}\psi = 0 \quad \text{and} \quad \psi^\times\left\{-(\underset{\leftarrow}{\partial}^i + i\varepsilon A^i)\,\bar{\gamma}^i - \mu_0\,\bar{\gamma}\right\} = 0$$

of the Dirac equations, which served for us to establish the [γ_{uvw4}] (pp. 62). *In the absence of the external quadri-potential*, one also has:

$$\partial_j\{T^{ij}\}' = (0) - (0) = 0,$$
$$\partial_j\{S^{ij}\}' = \left(-2\mu_0\psi^\times\bar{\gamma}\underset{\to}{\partial}\psi\right) - \left(-2\mu_0\psi^\times\underset{\leftarrow}{\partial}\,\bar{\gamma}\psi\right) = -2\mu_0\{l^i\}';$$

i.e.:

$$\partial_j\{T^{ij}\}' = 0, \qquad \partial_j\{S^{ij}\}' = -2\mu_0\{l^i\}.$$

In the presence of an external quadri-potential A^i, the principle of the calculation is the same, but one must take into account the commutation law for the operators $\underset{\to}{\partial}^i$, $\underset{\leftarrow}{\partial}^i$, and A^j. From a classical remark in wave mechanics, one will have:

$$\underset{\to}{\partial}^i A^j - A^j\,\underset{\to}{\partial}^i = \partial^i A^j - A^j\left(\underset{\to}{\partial}^i - \underset{\to}{\partial}^i\right) = \partial^i A^j,$$

and similarly

$$A^j\underset{\leftarrow}{\partial}^i - \underset{\leftarrow}{\partial}^i A^j = \partial^i A^j,$$

in which ∂^i denotes the *un-notated operator*, which acts only to its immediate right.

That being the case, the first parenthesis in [T^{ij}] gives:

([1]) "Der Impuls-Energiesatz in der Diracschen Quantentheorie des Elektrons," Zeit. Phys. **49** (1928), pp. 858. Tetrode's formula (16) is equivalent to our [62 (B) III].

$$i\varepsilon\,\psi^{\times}\left(\underset{\rightarrow}{\partial}^{i}A^{j}-A^{j}\underset{\rightarrow}{\partial}^{i}\right)\gamma_{j}\,\psi=i\varepsilon\,(\psi^{\times}\gamma_{j}\,\psi)\,\partial^{\,i}A^{\,j}.$$

The second parentheses will give the same result in modulus, as well as in sign, by virtue of a double change of sign in (67_1) and in (56). The calculation that relates to $\{S^{\,ij}\}$ is analogous, and one will finally have:

$$[67'] \qquad \partial_k\{T^{\,ik}\}'=2i\varepsilon\,(j_k)\,\partial^{\,i}A^k, \qquad \partial_k\{S^{\,ik}\}'=-\,2\mu_0\,\{l^{\,i}\}'+2i\varepsilon\,(\sigma_k)\,\partial^{\,i}A^k.$$

We similarly calculate the divergences of the *interaction terms* $\{\ \}''$ of $\{T^{ik}\}$ and $\{S^{ik}\}$. If we take into account, on the one hand, the Dirac continuity equation [(61) A I], and on the other, the Uhlenbeck-Laporte relation [(61) B V], then we will get:

$$[67''] \qquad \begin{cases} \partial_k\{T^{ik}\}''=-2i\varepsilon\,\partial_k\{A^i(j^k)\}=-2i\varepsilon(j_k)\,\partial^kA^i, \\ \partial_k\{S^{ik}\}''=-2i\varepsilon\,\partial_k\{A^i(\sigma^k)\}=-2\mu_0\{l^i\}''-2i\varepsilon\,(\sigma_k)\,\partial^kA^i. \end{cases}$$

Finally, if one adds corresponding sides of [67′] and [67″] then one will see the rotation of A^i appear; if one then takes into account the *definition:*

$$(68) \qquad \boxed{H^{ij}=\partial^iA^j-\partial^jA^i}$$

of the prevailing field when one starts with the potential, as well as (66), which was proved to begin with, then one can write:

$$(67) \qquad \begin{cases} \boxed{\partial_k\{T^{ki}\}=\partial_k\{T^{ik}\}=-2i\varepsilon\,H^{ik}(j_k),} \\ \boxed{\partial_k\{S^{ki}\}=\partial_k\{S^{ik}\}+2\mu_0\{l^i\}=-2i\varepsilon\,H^{ik}(\sigma_k).} \end{cases}$$

In these very analogous formulas, the tensors $\{S^{ik}\}$ and (j_k) have an electromagnetic interpretation, while the tensors $\{T^{\,ik}\}$ and (σ_k) have a mechanical one. We then see that, conforming to what was said before on the subject of the quadri-potential A^i, the field $H^{\,ik}$ plays a role that is perfectly symmetric with respect to the electromagnetic and mechanical properties, in such a way that there is no reason to qualify it with "electromagnetic" more especially.

Remark. – We now give some indications about the manner in which one directly calculates the divergences of $\{T^{ik}\}$ and $\{S^{ik}\}$ on the first index (viz., the differential index). For the *Schrödingerian part* $\{\ \}'$, for example, one will have:

$$\partial_i\{T^{ij}\}'=\partial_i\,\psi^{\times}[\partial^{\,i}]\,\gamma^j\,\psi=\psi^{\times}\underset{\leftarrow}{\partial}_i\,\gamma^j\,\underset{\rightarrow}{\partial}^i\psi-\psi^{\times}\underset{\leftarrow}{\partial}^i\,\gamma^j\underset{\rightarrow}{\partial}_i\,\psi+\psi^{\times}\gamma^j\underset{\rightarrow}{\partial}{}^i_i\,\psi-\psi^{\times}\underset{\leftarrow}{\partial}{}^i_i\,\gamma^j\psi\,,$$

in such a way that, since the first two terms cancel, what will be left is:

$$\partial_i\{T^{\,ij}\}'=\psi^{\times}\gamma^j\underset{\rightarrow}{\partial}{}^i_i\,\psi-\psi^{\times}\underset{\leftarrow}{\partial}{}^i_i\,\gamma^j\psi \qquad \text{and} \qquad \partial_i\{S^{\,ij}\}'=\psi^{\times}\overline{\gamma}^{\,j}\underset{\rightarrow}{\partial}{}^i_i\,\psi-\psi^{\times}\underset{\leftarrow}{\partial}{}^i_i\,\overline{\gamma}^{\,j}\psi\,.$$

The symbols $\underset{\rightarrow}{\partial}{}_i^i$ and $\underset{\leftarrow}{\partial}{}_i^i$ denote the Laplacians that act on the right and on the left, respectively. Since the direct calculation of the divergences considered involves some second derivatives, in order to complete them, one must resort to the second-order equation that is a consequence of the Dirac equations. In the general case where the quadri-potential A^i is not zero, that equation will contain a factor that is, as one knows, the field H^{ik} that was defined by (68), as well as the quadri-potential A^i ([1]), moreover. If one expresses $\underset{\rightarrow}{\partial}{}_i^i \psi$ and $\psi^\times \underset{\leftarrow}{\partial}{}_i^i$ with the aid of that equation and its *Gordon-Pauli transform* then one will see the right-hand sides of (67) appear, as well as the expressions $\partial_i\{^{ij}\}''$.

24. Relationships between Dirac's theory and classical electrodynamics. – Formula (67_2), into which the tensor $\{S^{ij}\}$ enters, whose significance is still unknown, does not seem to be interpretable in the present state of our knowledge. On the contrary, formula (67_1) seems to be identical with Lorentz's formula of classical electrodynamics. In order to verify that, it will suffice to replace ε with its value in (57), and to reestablish the physical coefficient $\frac{ich}{4\pi}$ of the tensor $\{T^{ij}\}$ and ec of the quadri-vector j_k [eqs. (62_A) and (62_B)]. One will get:

(69)
$$\partial_k T^{ki} = \partial_k T^{ik} = -H^{ik} j_k ,$$

which is, in fact, the Lorentz formula ([2]), by virtue of the dynamical formula (35). One knows that in the cited paper Tetrode appealed to formula (69) in order to justify the interpretation of T^{ik} as an inertia tensor and to fix the physical coefficient $\frac{ich}{4\pi}$ of that tensor. With the line of reasoning that we have adopted in this work, the electrodynamical formula (69) seems, on the contrary, to be a consequence of Dirac's theory, so the interpretation of T^{ik} and the value of its physical coefficient will result from the general principles of wave mechanics (no. **8**). Then again, if one prefers, one can attach it to the interpretation of the quadri-vector σ thanks to our theory of Chapter II [eqs. (37′) and (63)].

One knows that Pauli had profited from the fact that the two divergences of the tensor T^{ik} were equal in order to symmetrize that tensor by setting ([3]):

$$\Theta^{ik} = \tfrac{1}{2}(T^{ik} + T^{ki}),$$

which is a definition that allows the relations (69) to be preserved by Θ^{ik}. We pointed out the significance of that operation in a note in Chapter II, and in the context of our theory of media endowed with spin, and to us, it was somewhat arbitrary. Here, we remark

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([1]) L. de Broglie, *l'Électron magnétique*, Chap. X, eqs. (6) and (30), pp. 132 and 141.
([2]) *La Relativité restreinte*, pp. 40.
([3]) Die allgemeinen Prinzipien der Wellenmechanik, B: Relativistische Theorien," Handb. d. Phys. **24** (1933), pp. 235.

simply that the definition of Θ^{ik} is more complicated than that of T^{ik}, since it contains four terms instead of two.

Finally, from our dynamical theory of Chapter II, and by virtue of formula (63), the fictitious proper ponderomotive moment density that is applied to the statistical fluid by the field is identically zero, which is a result that differs from the one that was expressed by the classical formula [VII]. That example of the *proper ponderomotive moment density* seems to us to illustrate what we said about the divergence that would be expected by the properties of the statistical fluid in Dirac's theory and those of a polarized medium in classical electromagnetism.

II. – ON THE DIRAC AND GORDON CURRENT QUADRI-VECTORS AND TETRODE'S ASYMMETRIC INERTIA TENSOR.

25. To begin with, we wish to see how the pseudo-classical theory of a continuous medium that is endowed with not only a mass density and an electric charge density, but also a *proper kinetic moment density* σ^i and a *magneto-electric moment density* m^{ij} presents itself. The notion of *kinematic velocity*, or – what amounts to the same thing – that of *world-trajectories* of a fluid is perfectly clear, and we know from a general principle of relativity that the trajectories in question must be *time-like* at each of their points ([1]).

From the dynamical viewpoint, and for a medium that is endowed with *dynamical polarization* in the sense of Chapter II, we were led to introduce, in addition to the preceding congruence, which is called the *true current*, a second congruence that is called the *false current*, which is not necessarily time-like, and to define the *asymmetric inertia tensor* of the medium that is endowed with spin as the general product of the two current quadri-vectors [eq. (39)]. We then showed that in the calculation of the finite mass-impulse according to the formula:

$$p^i = \iiint T^{ij}\, \delta u_j\,,$$

in which the significant index i must be that of the *false current* (no. **15**). Moreover, we have previously shown that for reasons of a kinematic nature, the *spin density* quadri-vector σ^i must be orthogonal to the true current [eq. (33)].

The classical electromagnetism of a polarized medium takes into consideration two quadri-vectors of current density that are both conservative, in addition to the latter ones. One of them, which corresponds to the true charge, is called the *electric convection current* and is tangent to the current of the kinematical streamlines ([2]), and thus, the *true current* of our Chapter II. The other one, which is not necessarily time-like and is called the *total electric current*, is the sum of the preceding one and a *fictitious polarization current* $\partial_j m^{ij}$. It is obviously natural, but by no means necessary *a priori*, to *postulate* that the latter current must be tangent to our congruence of the *false current* in Chapter II, paragraph II.

([1]) *La Relativité restreinte*, pp. 18.

([2]) See above, no. **21**.

We shall now examine the extent to which these various properties are recovered *mutatis mutandis* in Dirac theory; it is, of course, only Dirac theory that we shall investigate, independently of any appeal to classical relativity.

From the standpoint of electromagnetism, Dirac's theory introduces two conservative current quadri-vectors – viz., those of Dirac and Gordon – which are defined (up to a factor) by:

$$(j^i) = \psi^\times \gamma^i \psi, \qquad \{k^i\} = \psi^\times [\partial^i]\, \psi - 2i\varepsilon A^i \cdot \psi^\times \psi,$$

respectively. From the standpoint of dynamics, from Chapter II, paragraph II, and from what we just said, we expect to recover these two current quadri-vectors in the expression for the Tetrode asymmetric inertia tensor:

$$\{T^{ij}\} = \psi^\times [\partial^i]\, \gamma^j \psi - 2i\varepsilon A^i \cdot \psi^\times \gamma^j \psi.$$

We immediately see that the operators that enter into the definition of T^{ij} are indeed the ones that we hoped for, and even that the second term in the Tetrode tensor is, in fact, the general product of the Dirac current with the second term in the Gordon current, in the ratio $(\omega_1) = \psi^\times \psi$.

In order to see whether our formula (39) is satisfied or not, we must examine whether the expression:

$$\psi^\times [\partial^i]\, \gamma^j \psi \cdot \psi^\times \psi - \psi^\times [\partial^i]\, \psi \cdot \psi^\times \gamma^j \psi$$

is or is not zero, respectively. The response, which is negative, is provided by the Kofink identity (51_1), which gives the values of that expression as (1):

$$(j_k)\, \partial^i (m^{kj}) + (\omega_2)\, \partial^i (\sigma^j) = -\, \partial^i (j_k)\, (m^{kj}) - \partial^i (\omega_2)\, (\sigma^j).$$

Therefore, the relationship of the Tetrode tensor to the two Dirac and Gordon currents is apparently the one that we predicted qualitatively, but quantitatively, it is less rigorous: *The Tetrode inertia tensor is not a general product of the two quadri-vectors.* In Dirac's theory, there is a new situation that is quite "revolutionary" with respect to the classical theories: It results from what was just said that the integral $\iiint T^{ij}\, \delta u_j$ will no longer be zero when it is taken over the hyper-wall of the kinematical world-current, and that there exists no hyper-wall that will enjoy that property, either. Now, the vanishing of the integral in question is absolutely necessary for the *classical* interpretation of the quantity f_j (ponderomotive force density), p^i (finite mass-impulse), and T^{ij} (mass-impulse tensor (2). It follows that the interpretation of Tetrode's formula (71) is less clear than it first seemed to be, and that one can hardly infer anything that is better than a formal argument.

26. However, abstracting from the latter group of difficulties, it still remains permissible for us to demand to know to what degree the properties of the two quadri-

(1) The same conclusion can be inferred from the identity $(43'_2)$, from which, the contracted product $T^{ik} j_k$ is congruent to not only the Gordon current k^i, but also to the *magnetic current* l^i (pp. 56).

(2) *La Relativité restreinte*, no. **23**, pp. 50.

vectors j^i and k^i conform to the ones that the classical considerations of the preceding number predicted.

From the kinematical standpoint, the Dirac current j^i is time-like, as the positive-definite expression:

$$(j^4) = \psi^\times \gamma^4 \psi = i \cdot \psi^\times \psi$$

shows, or even the Pauli identity (43_2). As one can say nothing about the Gordon current k^i, we see that the Dirac current plays the fictitious role of *kinematical current*, or *ordinary current* of the statistical fluid.

From the dynamical standpoint, the Pauli identity (47_1) shows that the spin density quadri-vector σ^i is orthogonal to the Dirac current without being able to say anything about the Gordon current. Moreover, from the general principles of wave mechanics, the "virtual" dummy index in the calculation of the probable mean mass-impulse is that of the operator γ^j of the Dirac current, or – what amounts to the same thing – the significant index is that of the operators $[\partial^i]$ and A^i of the Gordon current [eqs. (26) and (27)]. By virtue of what we said in Chapter II, these two criteria converge to each other, and converge with the preceding kinematical criteria in such a way as to associate the Dirac current with our *true current,* or world-current in the usual sense. So far, everything points to the agreement with pseudo-classical theory that we have expected.

From the standpoint of electromagnetism, we saw in no. **21** that it is appropriate to associate the Dirac current with the *total electromagnetic current*, whereas from the preceding, one would expect to associate it with the *electromagnetic convection current.* We have already remarked how paradoxical that result is, and suggested that cannot be lacking in some relationship to certain curious remarks that are due to several authors. Under those conditions, we would see incorrectly what the classical equivalent of the Gordon current would be in a coherent density theory. It does not seem to us that a general conclusion could be drawn from the fact that it manifests itself like a *translation current* in the theory of the Darwin globule.

General conclusion that relates to the inertia tensor. – From the entire collection of facts and properties that were encountered in the course of this work, we believe that we can conclude formally that *the true inertia tensor of Dirac's theory is not Pauli's symmetrized tensor Θ^{ik}, but Tetrode's original asymmetric tensor T^{ik} that was defined* in equation (26) of Chapter I. We shall now recapitulate those facts and properties:

1. The definition in question is the one that the general principles of wave mechanics impose when one starts with the definition (15) of the inertial mass-impulse quadri-operator (no. **8**).

2. The probable mean value of the *total* kinetic moment, when expressed as a function of the Pauli tensor Θ^{ik} is of *orbital type* formally; in order to decompose the *total* momentum into an *orbital* momentum and a *proper* momentum, one must utilize the Tetrode tensor T^{ik} (no. **12**, 8).

3. Although the relationship between the Tetrode tensor T^{ik} and the current quadri-vectors k^i and j^k is not as close as the one that we predicted in number **13**, the qualitative

resemblance between the two definitions (39) and [(60) B II] is real, and we have can use it in an argument.

4. In the course of the calculation, it is always the tensor T^{ik}, and never the symmetrized tensor $\Theta^{ik} = \frac{1}{2}(T^{ik} + T^{ki})$, that appears "spontaneously": One sees that in the context of the Kofink identities (51_1), (51_2), (52_1) and (54), the Franz relations [(61) B I] and [(61) B III], and finally, in the calculations that arrive at the double result (71).

Remark. – In the absence of an external potential, the Dirac equations admit monochromatic plane waves as solutions. Since the Dirac and Gordon quadri-vectors are then collinear with each other and collinear with the wave rays, they will both be time-like. The proper mass-impulse quadri-vector of the electron, which is therefore well-defined and collinear with the rays, is likewise time-like ([1]). Finally, the Tetrode tensor T^{ij} will be symmetric in that particular case.

One can read off that double group of results from formulas [(61), A II] and [(61) B III], since the density tensors will be constant in all of space-time in the case of a monochromatic plane wave ([2]).

([1]) L. DE BROGLIE, *l'Électron magnétique*, pp. 162 to 166.

([2]) In the same manner, one can read off from (61), for example, that the invariant (ω_2), as well as the quadri-vectors $\{l^i\}$ (magnetic charge-current density) and $\{U^i_{(1)}\}$, will be annulled in the case of a monochromatic plane wave.

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