"Sur la théorie des moments cinétiques propres en Relativité Restreinte," J. math. pures appl. **21** (1942), 267-275.

On the theory of proper kinetic moments in special relativity ([1])

By **OLIVIER COSTA DE BEAUREGARD**

Translated by D. H. Delphenich

---

1. One knows that one of the elements of the success of Dirac's theory is the attribution of a *proper kinetic moment* to the electron, which is a moment whose three components are represented in the quantum manner by the matrices:

$$S_u = -\frac{h}{4\pi i}\,\alpha_v\,\alpha_w\,.$$

(u, v, w denote a circular permutation of the spatial indices 1, 2, 3.) The S_u are found to be the spatial components of a *matrix world-vector* whose temporal component is:

$$S_4 = -\frac{h}{4\pi i}\,i\,\alpha_1\,\alpha_2\,\alpha_3\,.$$

That fact is paradoxical, since a *kinetic world-moment* must have the variance of a *second-order antisymmetric tensor*. Like Louis de Broglie ([2]), it is easy for one to find the origin of the difficulty: *The kinetic moment of an extended body is defined in the classical manner for simultaneous states.* It then follows that the kinetic moments of the same body, as defined in two different Galilean systems *do not belong to the same tensor.* In a general manner, if one considers a continuous medium and a certain finite quantity that is attached to that medium and represented by the integral of a *complete differential form* that refers to the corresponding density then *the definition of the quantity considered will be a function of the individual instants at which the various "molecules"* ([3]) *are taken* ([4]).

Always following the principles of quantum mechanics, Dirac's theory attributed a *proper kinetic moment density* to the *probability fluid* that was represented by a *space-like quadri-vector* σ_i , which was a result that Louis de Broglie likewise justified in the

---

([1]) We have extended and improved our theory since the time when the present article was edited. We intend to present it in its definitive form in a later work.

([2]) "La variance relativiste du moment cinétique d'un corps en rotation," J. de math. **15** (1936), 89.

([3]) In a continuous medium, we use the word "molecule" in the sense of *geometric point of the fluid that follows its motion.*

([4]) There is an exception in the case of electric charge: *Electric charge is a conservative invariant.* However, that situation is due solely to the existence of a *continuity equation.*

case of a rotating solid. In Dirac theory, one can show, in a general fashion, that *the quadri-vector σ_i is orthogonal to the world-current.*

The essential objective of this paper is to discuss and justify those various results *from the classical viewpoint;* we no longer take a rotating solid, like Louis de Broglie, but a continuous medium that we assume, *by hypothesis*, is endowed with a *proper kinetic moment density*. We shall then show that the *quadri-vectorial* character ([5]) of the density σ is a consequence of some very general postulates, and one new postulate that is extremely natural in the relativistic kinematics of continuous media and implies that the latter quadri-vector has the property of being *orthogonal to the world-current.*

One knows, and we shall recall this fact, that classical dynamics denies the existence of a *proper kinetic moment density*. Since the proper kinetic moments (or else the corresponding densities) have been imposed by experiments, it seems that there would be some interest to enlarging the concepts of dynamics on that particular point. Now, the *purely formal* relativistic argument that we shall give will show that this involves some real difficulties. Indeed, it seems that the *negative result* to the question has very deep roots, and that *its true cause is of kinematic order*. In other words, although one cannot assert with full rigor that the problem is insoluble, one can at least conclude that in the study of *proper kinetic moments*, classical continuum physics is very close to its limits of possibility.

In our study, u, v, w will denote a *circular* permutation of the spatial indices 1, 2, 3, and i, j, k, l is an *arbitrary* permutation of the world indices 1, 2, 3, 4.

2. Review of some results from elasticity and pre-relativistic dynamics. – In a stressed elastic medium, let T_{uv} be the nine coefficients that express the vector of *elementary surface tension* δT_u as a function of the corresponding *elementary area* (elastic tensor):

$$\delta T_u = T_{uv}\, ds^v.$$

Now, integrate this over a closed area and transform it into a triple integral. What will appear is the *volumetric density of elastic force* f_u :

$$\text{(1)} \qquad T_u = \iiint \partial^v T_{uv}\, \delta u \qquad \boxed{f_u = \partial^v T_{uv}.}$$

Similarly, take the surface force moment with respect to the origin, integrate it, and transform it. What will appear is a *proper ponderomotive moment density of elastic origin* μ_{uv} :

$$\iint (T_{uw}\, x_v - T_{vw}\, x_u)\, \delta s^w$$
$$= \iiint (x_v\, \partial^w T_{uw} - x_u\, \partial^u T_{vw})\, \delta u + \iiint (T_{uw} \partial^w x_v - T_{vw}\, \partial^w x_u)\, \delta u$$

([5]) In orthogonal Cartesian axes of equal measure (up to an arbitrary sign), there is no difference between a quadri-vector and a third-order completely-antisymmetric tensor. In reality, the Dirac spin density is a third-order antisymmetric tensor.

$$= \iiint (f_u\, x_v - f_v\, x_u)\, \delta u + \iiint (T_{uw} - T_{vw})\, \delta u\,,$$

$$\boxed{\mu_{uv} = T_{uv} - T_{vu}\,.}$$

Therefore:

(A) *Elasticity establishes the possibility of the existence of a proper ponderomotive moment density, which is represented by a second-order antisymmetric tensor and which will be annulled when the elastic tensor is symmetric.*

Now take the equations of the dynamics of continuous media; f_u denotes the total inertial force density, v_u is the fluid velocity, and ρ is its density. One can write:

(3) $$f_u = \partial^v (\rho\, v_u\, v_v) + \partial^4 (\rho\, v_u).$$

Hence, the inertial force density is the sum of a term of elastic form that derives from a *symmetric* tensor $\rho\, v_u\, v_v$, and a term $\partial^4 (\rho\, v_u)$ that is reducible to that – i.e., it is properly volumetric. Consequently, dynamics asserts that *the proper inertial force moment density is identically zero,* and by virtue of d'Alembert's principle, when it is applied to moments:

(B) *The same thing is true for the proper ponderomotor moment density.*

Moreover, it is easy to confirm that result in the following manner: Take a spherical droplet of radius r in a continuous medium and follow its motion; its moment of inertia is $\frac{8}{15}\pi\rho\, r^3$, and its angular velocity is $\frac{1}{2}\overrightarrow{\mathrm{rot}\cdot v}$. Its *kinetic moment* is then a fifth-order infinitesimal, *which does not allow us to define proper kinetic moment density.* In other words:

(C) *Dynamics denies the existence of proper kinetic moment density.*

Now, the existence of proper kinetic moments at the atomic scale is manifested by gyromagnetic experiments, for example. It would seem interesting to enlarge the classical concepts of continuum physics in such a manner as to attribute a density of proper kinetic moment to matter. Similarly, the existence of a proper kinetic moment for the electron is manifested by spectroscopy, while quantum mechanics utilizes a proper kinetic moment density as an intermediary in space-time calculations, and it would be interesting from the purely relativistic viewpoint to justify its properties.

Indeed, we shall be able to show that *the properties of Dirac's density σ are imposed by the relativistic formalism as necessary consequences of very general postulates that almost impose themselves* (no. **5**). Moreover, one sees that *the negative results* (B) *and* (C) *of pre-relativistic theory have a very deep origin, and that expanding upon that point will not be simple in the old dynamics* (no. **4**).

5. Relativistic determination of the properties of the proper kinetic moment density. – In finite form, the relativistic variance of a kinetic moment C will result unambiguously from the considerations of a material point whose coordinates are x^i and whose mass-impulse is p^i ; indeed, from the old dynamics, one can only be dealing with the antisymmetric tensor:

(4) $$C^{ij} = p^i x^j - p^j x^i \qquad (i, j = 1, 2, 3, 4),$$

whose three components C^{uv} represent the kinetic moment, properly speaking, with respect to the spatial origin (u, v = 1, 2, 3).

As for the three C^{4u}, their interpretation is simple if one replaces x^4 and the p^i by their values ict and $p^u = mv^u$, $p^4 = imc$, resp. ; indeed, one will have:

(4′) $$C^{4u} = icm\,(x^u - v^u t),$$

and one will see that this amounts to the *barycentric moment with respect to the origin, generalized by the hypothesis of non-simultaneity,* up to a factor. In particular, for infinitely-small t, one recovers the usual barycentric moment at time zero ($-\,v^u\,dt$ then represents a "correction from non-simultaneity").

That being the case, we know that we must obtain a second-order antisymmetric tensor δC^{ij} by suitably multiplying the unknown components of the density σ by the *generalized volume element* $[dx^i\, dx^j\, dx^k]$; the component $[dx^u\, dx^v\, dx^w]$ represents the usual *pure volume,* while the other three can be considered to be generated by a change of Galilean frame. Conversely, one can then give an *arbitrary non-simultaneous state* of a set of fluid molecules by taking an arbitrary hypercap to a current world-tube that is restricted only by the demand that it must be everywhere *space-like*. Indeed, it is obviously necessary that *local simultaneity* can be insured in a suitable Galilean frame, which is, in fact, true thanks to the preceding condition: The *local simultaneity system* is then one whose temporal axis is orthogonal to the hypercap at the world-point considered.

One then sees that the unknown quantity σ is necessarily a tensor, whose order n must be determined, along with any possible symmetries. Let m be the number of its dummy indices, which much *saturate* certain indices of $[dx^i\, dx^j\, dx^k]$, and let s be the number of its significant indices. The three integers n, m, and s are essentially positive and equal to at most 4, and one can write down the homogeneity relations:

$$2 = (3 - m) + s \qquad \text{or} \qquad m = 1 + s,$$
$$m + s = n \qquad \text{or} \qquad n = 1 + 2s,$$

For	$s = 0$,	one will have	$m = 1$, $n = 1$,
"	$s = 1$,	"	$m = 2$, $n = 3$;

One discards the hypothesis $s = 2$, since it will give $n = 5$.

Therefore, the density σ – *if it exists* (Postulate I) – is necessarily a *tensor of order 1 or 3*.

We first study the first hypothesis. It is written:

(5) $$\delta C^{ij} = \sigma_k\,[dx^i\,dx^j\,dx^k],$$

and thanks to that formula, *the antisymmetry of the tensor product δC is guaranteed for any choice of trilinear integration element*, which is obviously necessary *a priori*. Moreover, one sees that *under the hypothesis of simultaneity*, by which, only the component $[dx^1\,dx^2\,dx^3]$ is non-zero, *one will recover the usual definition of a vector density for the density σ*, in the sense that:

1. The three components δC^{uv} of the kinetic moment, properly speaking, involve only one term in their expression, namely, σ^w.

2. The three components of the barycentric moment δC^{4v} are zero.

If one introduces the complements $ic\ \delta B^{ij}$ and $ic\ \delta u^i$ of the two antisymmetric tensors that enter into (5) then one will have the new notation:

(5′) $$\boxed{\delta B^{kl} = \sigma^k\,\delta u^l - \sigma^l\,\delta u^k.}$$

Now, take the hypothesis $n = 3$ and postulate that:

(II) *The antisymmetry of the tensor product must remain insured for any choice of trilinear integration element.*

We are then obliged to adopt, not just the simple contracted product over two indices, but the classical combination:

(5″) $$\delta B_i^j = \tfrac{1}{2}\{\sigma_{jkl}\,[dx^i\,dx^k\,dx^l] - \sigma_{ikl}\,[dx^j\,dx^k\,dx^l]\}.$$

We then postulate that:

(III) *Under the hypothesis of simultaneity, the character of being a spatial vector must be recovered for the density σ.*

We remark, in turn, that for a given pair of dummy indices, each of the non-zero terms in the preceding expression is, in reality, the sum of two terms that are generally different and correspond to the permutation of those indices in σ. Consequently, our postulates already impose the antisymmetry of the tensor σ in the two dummy indices, which is a necessary and sufficient condition for the two terms under consideration to always be equal. One can then group them together and neglect the coefficient 1/2 . Always under the simultaneity hypothesis, that will permit us to write (the summation convention is not used, and u, v, w denotes a circular permutation of the spatial indices 1, 2, 3):

$$\delta B_{uv} = \tfrac{1}{2}(\sigma_{vvw} + \sigma_{uuw})\,[dx^1\,dx^2\,dx^3], \qquad \delta B_{w4} = \sigma_{4uv}\,[dx^1\,dx^2\,dx^3].$$

We then once more bring postulate (III) into play. By virtue of an intrinsic property of the tensor σ, it is necessary that one of the two groups of components that must be written should be zero, and that the other one should include only one term in its expression. The second group cannot be zero, since the components σ_{uuw} and σ_{vvw} would always have to be equal to each other then, which is possible only if they are zero: The tensor σ will then be identically zero, which is an unacceptable hypothesis. One must then have that the σ_{4uv} are non-zero, but the σ_{vvw} are always zero: That will be possible only if the tensor σ is antisymmetric with respect to the first two indices. Finally, the tensor σ must be completely antisymmetric, in such a way that the notation (5″) must agree with the notation (5′), with the corollary that *the barycentric moment is zero under the hypothesis of simultaneity.*

Now, add to the preceding postulates:

(IV) *The necessity of recovering the character of the quantity σ being a spatial vector in the comoving Galilean frame.*

We see that the component σ_4 must be annulled in that system; i.e., the *quadri-vector σ must be orthogonal to the world-current:*

(6) $$\sigma_i\, dx^i = 0 \qquad \text{or} \qquad \boxed{\sigma_4 = \frac{i}{c}(\boldsymbol{\sigma}\cdot\mathbf{v}).}$$

($\boldsymbol{\sigma}$ denotes the spatial vector that has the three σ^u for its components, and $\mathbf{v}$ denotes the fluid velocity in its usual sense.)

One knows that in Dirac's theory, the density σ is a third-order antisymmetric tensor, and that the relation (6) is effectively verified. Finally, we have indeed recovered the entire set of properties of the Dirac density σ by means of the four general postulates: *Existence* (I). *The arbitrariness of the trilinear integration element* (II). *The vector density recovered in the simultaneity system* (III), *and in the comoving system* (IV).

4. Study of the hypercap integral. The proper ponderomotive moment density. – It is necessary that we complete our study in the following manner: We take the triple integral of the expression (5′) over the particular domain that is composed of two different hypercaps that relate to the same molecules and the hypercap of the corresponding current world-tube. Indeed, in order to assert that the set of two hypercap integrals indeed represents the *variation of the kinetic moment-barycentric moment* of the same fluid drop, it is necessary that we know how to interpret the triple integral over the hypercap, and also the quadruple integral that is obtained in the right-hand side by transformation.

By hypothesis, the trilinear element of the hypercap contains the quadri-vector element of the trajectory dx^i ; it is therefore everywhere *time-like.* In order to arrive at the definition, one takes two elementary *space-like* quadri-vectors δu_1^i and δu_2^i, in such a way that its various components will be the determinants that one extracts from the table:

$$\left\| \begin{matrix} \delta x_1^1 & \delta x_1^2 & \delta x_1^3 & \delta x_1^4 \\ \delta x_2^1 & \delta x_2^2 & \delta x_2^3 & \delta x_2^4 \\ dx^1 & dx^2 & dx^3 & dx^4 \end{matrix} \right\| .$$

It is even clear that the exterior product of the quadri-vectors δx_1^i and δx_2^i represents the *generalized area element* of the fluid drop along its motion, where the three components in (u, v) correspond to the area, properly speaking, and the three components in $(w, 4)$ can be considered to be generated when one makes a change of Galilean frame ([6]).

If one introduces the tensor $ic\ \delta s^{ij}$ that is complementary to the preceding exterior product then one can confirm that the trilinear element of the hypercap will take the contracted form:

$$\delta u^{\,j} = \delta s^{\,ij}\, dx_i \,,$$

in which $ic\ \delta s^{u4}$ represents the usual area. *On the hypercap*, one then has:

$$\delta B^{kl} = (\sigma^k\, \delta s^{il} - \sigma^l\, \delta s^{ik})\, dx_l \,, \tag{7}$$

by virtue of (5′), and thanks to that formula, the antisymmetry of δB^{kl} will remain insured automatically.

The most essential fact upon which we must insist is that *the vanishing of the hypercap integral is not insured automatically* when one takes into account the relation between the density σ and the element of the world-trajectory [viz., formula (6)] ([7]). Under those conditions, we shall successively envision several hypotheses.

Hypothesis 1: *The hypercap integral is zero for any hypercap.*

That amounts to saying that things must take place in our present problem as they do in the classical problems of *electric charge* and *mass-impulse.*

Take two planar hypercaps that are perpendicular to the time axis and infinitely close in time, integrate over them, and transform the integral into a triple integral. On the left-hand side, one has the *variation of the kinetic moment-barycentric moment*, and one can write:

$$dB^{kl} = dt \iiint (\partial^l \sigma^k - \partial^k \sigma^l)\, \delta u \tag{8}$$

after introducing the product $\delta u \cdot dt$ of a pure volume with a pure time on the right-hand side, so *the three components μ^{uv} of the complement μ^{ij} to the world-rotation of the density σ represents the proper ponderomotor moment density*, which is clearly consistent with the result in elasticity (A) [formula (2)]:

([6]) By a formula that is identical to the one that is known by the name of the *Frenkel formula* in relativistic electromagnetism.

([7]) Indeed, one knows that in the problem of electric charge, the vanishing of the hypercap integral will result from the collinearity of the charge-current quadri-vector density with the element of the world-trajectory dx^i, and that the same thing will be true in the problem of mass-impulse by virtue of the proportionality of the material tensor with the symmetric tensor $dx^i\, dx^j$.

(9) $$\mu^{uv} = \frac{1}{ic}(\partial^4\sigma^w - \partial^w\sigma^4).$$

Formula (9) is completely analogous to the classical formula (1): In both cases, the *ponderomotor density* is a derivative of the *inertial density*.

Unfortunately, not all of these satisfying results can be realized in fact. Indeed, to say that the integral (7) must be identically zero is to say that *the density* σ *must be identically zero.* We then get back to the *negative results* (B) and (C) of dynamics, but special relativity has the advantage of showing us that *these results have a basically kinematical origin:* The hypothesis of the *vanishing of the hypercap integral* has only reduced us to the classical case, since it is sufficient to imply the positive result (A) and the negative results (B) and (C).

Hypothesis 2: *The hypercap integral is non-zero, but the quadruple integral is identically zero.*

This makes the hypothesis that is expressed by:

(9′) $$\partial^j\sigma^i - \partial^i\sigma^j = 0$$

seductive *a priori*, since it is the one that permits us, in principle, to replace the defective notion of *volumetric density of ponderomotor moment* with that of *surface density*. Indeed, if we introduce the fluid velocity into (7) then we can include the time interval dt as a factor, in such a way that the coefficients of the $\delta s^{ij} \cdot dt$ will be the components of the *surface density of the proper ponderomotor moment.*

Unfortunately, one again bumps into a difficulty that is entirely analogous to the preceding one. Indeed, if one specifies the expressions for our density under the hypothesis of simultaneity (viz., $\delta s^{uv} \equiv 0$) then one will have

(10) $$\left\{\begin{aligned} \delta B^{uv} &= (\sigma^u\delta s^v - \sigma^v\delta s^u)\,dt = -[\boldsymbol{\sigma}\wedge\delta\mathbf{s}]\,dt, \\ ic\,\delta B^{u4} &= \sigma^u\delta s^v dx_v + \sigma^4\delta s^u dx_4, \\ &= \{\sigma^u(\delta\mathbf{s}\cdot\mathbf{v}) - (\boldsymbol{\sigma}\cdot\mathbf{v})\delta s^v\}dt = [\mathbf{v}\wedge(\boldsymbol{\sigma}\wedge\delta\mathbf{s})]^u\,dt \end{aligned}\right.$$

for the hypercap integral, and since the δB^{uv} *of a hypercap* are identically zero (always under the hypothesis of simultaneity), one can conclude that:

(10′) $$\boldsymbol{\sigma}\wedge\delta\mathbf{s} = 0,$$

and as a result, that the σ^i must be zero. The present hypothesis, just like the preceding one, thus serves to *deny the existence of proper moments.*

Hypothesis 3: *The hypercap integral and the quadruple integral are non-zero.*

It results from the preceding that this hypothesis is necessary in order to preserve the existence of proper moments. Moreover, it agrees with Dirac's theory, in which the quadri-vector σ is not irrotational. *However, one cannot interpret it from the classical standpoint.* Indeed, the disappearance of one or the other of the preceding restrictions will convert the transformation of the triple integral into a quadruple integral into a pure tautology, *which will not permit us to define a proper ponderomotor density.* Having done that, the notion of *proper kinetic moment density* would make no sense *from the classical viewpoint:* In order to justify it, we must appeal to a hypothesis that is foreign to both pure kinematics and traditional dynamics; i.e., a hypothesis that is completely artificial in our present state of understanding.

Since progress in that understanding takes place very neatly in the quantum sense, the wisest thing to do is probably to conclude that the preceding study makes one feel that the possibility of explaining things by means of the old continuum mechanics is indeed limited, at least for one particular point.