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THE SPECIAL

THEORY OF RELATIVITY

BY

O. COSTA DE BEAUREGARD

Head of research at the Centre National de la Recherche Scientifique

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PREFACE BY PR. LOUIS DE BROGLIE
Member of the Institute

Translated by D. H. Delphenich

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PREFACE

In the present volume, Olivier Costa de Beauregard gives a summary of the theory of special relativity that especially contains some of the original and important results that he obtained in latter years in the course of his research on the subject, and in particular in his Doctoral Thesis.

The central idea that has guided Costa de Beauregard in his work and in the editing of his book has been that of presenting the special theory of relativity in its most general tensorial form by constantly assuming the Minkowski view of the universe and avoiding "constant-time" slices of that universe as much as possible; i.e., sets of event-points in the universe that are simultaneous in this or that Galilean system. Upon operating in that way, one voluntarily adheres rigorously to the mode of presentation that was adopted originally by Einstein and was then followed by most of his descendents and commentators. It is permissible to think that the elegant methods of Costa de Beauregard cannot replace the usual method for the initiation of the students into the realm of relativistic theories, and likewise for their practical use by physicists. Indeed, it has the undeniable advantage that it arises directly from Einstein's original considerations, which are very deep and fundamental considerations that are attached directly to the facts of experience and correspond to the natural attitude of the physicist who observes the phenomena that those facts manifest. We would not like to completely abandon the consideration of "simultaneous" points in space-time that would lead one to renounce the use of Galilean systems with their proper time that is determined by Einstein's process of synchronizing clocks and then consequently forbid the use of the Lorentz transformation, the notion of Lorentz contraction, the formula for the relativistic composition of velocities, etc. Such renunciations would obscure the bases for a theory that is often difficult and poorly-contrived, and render it far-removed from the intuition of physicists without being necessary in the slightest. However, Costa de Beauregard, who is quite aware of all those questions, has brought considerable finesse to their study in order to adopt an attitude that is also intransigent, and the goal that he pursues is only to prove that upon assuming the ***intrinsic*** *viewpoint on space-time, one will succeed in shedding light upon an entire series of important situations that the usual mode of presentation does not highlight clearly. We cite only this example: The author insists upon the fact that one can define integral tensorial quantities that are attached to a hyper-endcap in space-time and whose value generally varies with the chosen hyper-endcap and which cannot be expressed by considering the space-time slices that are composed of simultaneous event-points for the same Galilean observer. That important remark appreciably clarifies certain problems that have remained very obscure up to now, such as the nature of the proper kinetic moment (i.e.,* ***spin****).*

Upon "foliating" space-time with the aid of an arbitrary family of space-like hypersurfaces, Costa de Beauregard arrived at some concepts that are close to ones that Schwinger recently introduced into the quantum theory of electromagnetic fields, and that alone should suffice to show the fecundity of the viewpoint that he has assumed.

This treatise of Costa de Beauregard, which is very elegant in form and very penetrating at its basis, harmoniously groups for the great pleasure of the informed reader an entire series of questions, some of which are well-known already, while others are known, but all-to-often left in the dark, and finally some questions that he himself has *explained in his personal research. The latter category notably contains his new relativistic definition of force, his considerations regarding fluids that are endowed with spin, and the relationship between spin and the asymmetry of the inertial tensor, his relativistic theory of the barycenter, etc. At the end of this work, one will also find some important complements on the hydrodynamics of Eisenhart, Synge, and Lichnerowicz on the symmetric presentation of analytical mechanics and the manner by which the ideas that are at the basis for wave mechanics relate to those of relativity.*

One of the features that make the book of Costa de Beauregard particular attractive is the number and degree of interest of the remarks that the author presents in the course of his presentation. Indeed, along the way and at the margins of his arguments, he has developed, in a very penetrating way, a whole series of reflections that do not fail to excite interest in all who meditate upon the foundations of the relativistic theories. Along that order of ideas, one should observe, in particular, the paragraphs that are dedicated to the measurement of the velocity of light, the effects of relativistic contraction in rotating bodies, the general problem of the relativistic dynamics of systems.

In summary: *We are in the presence here of a distinguished work of great scientific value that does great honor to its author. By his previous research and the publication of this book, Costa de Beauregard is classified today amongst the school of young theoretical physicists as an entirely eminent and original specialist in the theory of relativity.*

LOUIS DE BROGLIE

AUTHOR'S FOREWORD

In his *Leçons sur les invariants intégraux*, E. Cartan emphasized how interesting that it can be – *even in pre-relativistic mechanics* – to treat the problems of fluid mechanics or the mechanics of systems of points by taking the various fluid molecules or the various points at non-simultaneous time-points. The formulas and arguments thus acquired great generality, and some profound relations exhibit that generality – for example, the one that relates the two expressions $\sum p^k\, dx_k$ and $W\, dt$ for elementary action. In fact, when employed systematically, that way of looking at things will lead one to treat the time variable in the same manner as the spatial coordinates, and to generalize all of the differential formulas that were considered previously by the addition of terms in dt. Those forms, and notably Poincaré's *integral invariants*, appear (to use the words of Cartan himself) to be the *truncated expressions* for differential forms that involve the space and time variables symmetrically.

However, if Cartan's very general viewpoint does a great service to the theoretical presentation of classical mechanics then it goes without saying that it will be in close harmony with the profound spirit of the theory of relativity. One of the guiding principles to which we shall always remain faithful in this book is to write all of the differential forms in the complete and symmetric form that results from the non-simultaneous consideration of an extended system. Whereas that manner of proceeding might seem to be a purely-mathematical luxury in the context of pre-relativistic mechanics, it almost imposes itself in relativity due to precisely the fact that the notion of *simultaneity at a distance* must be essentially *relative*. Hence, *integrations at constant time*, for example, which can seem quite natural in pre-relativistic mechanics, will certainly seem arbitrary in relativity. That is why throughout this book we will systematically replace the families of hyperplanes at constant time that have been employed in classical physics, each of which is defined by a value of t, with *arbitrary* continuous families of hypersurfaces $\mathcal{E}(\theta)$ that are everywhere spacelike. Each of them, which is called a *pseudo-space*, is characterized uniquely by a value of the real parameter θ, which is called *pseudo-time*. The *complete differential forms* thus-introduced present some supplementary terms with respect to the classical *truncated forms* that one can consider to be *corrections for non-simultaneity*.

If, conforming to the classical custom, one considers the totality of space *at constant time* in *any* Galilean frame then in relativity that will imply that two different Galilean observers cannot utilize the same three-dimensional integration hypersurface. Hence, if one is given a certain physical quantity that is represented by a world-tensor as a density, then that will amount to saying that *in practice*, the integral quantity will not have a tensorial character. Indeed, the integral tensor is defined only *relative* to the integration hypersurface, so one will then see that different Galilean observers will utilize different finite tensors to represent that same quantity. That is an important fact to which L. de Broglie has emphatically directed attention in the example of *kinetic moment* [**125**, **126**]. Meanwhile, we remark that the preceding disagreeable situation is found to be avoided for two quantities among all of the important ones, namely, *electric charge* and *mass-impulse*. Since electric charge is *conservative*, its value will be independent of the world hyper-endcap that is used in order to calculate in a given hyper-tube. Moreover, *in an infinitely-thin hyper-tube*, one easily shows that the mass-impulse quadri-vector is independent of the world-orientation of the hypersection considered.

Is it physically obligatory that one must consider all of the points of space to be simultaneous in a given Galilean frame? We do not think so. It is indeed true that, despite the effective existence of simultaneous signals, one can calibrate all clocks easily in the same Galilean frame with the aid of physical operations in such a manner that they will be synchronous in that frame. However, it is obvious that this *possible* operation is an *arbitrary* operation whose only merit is to preserve the customs of the old thinkers in a completely "relative" manner. Conversely, must one believe that the choice of one of our families $\mathcal{E}(\theta)$ will unduly specialize the way that one frames space-time? We expect not. The choice of $\mathcal{E}(\theta)$ is doubly arbitrary, from the form of $\mathcal{E}$, on the one hand, and the manner by which θ is graduated, on the other. We demand only:

1. The everywhere-spatial character of each $\mathcal{E}$.
2. The continual and total sweeping-out of the universe when θ increases from $-\infty$ to $+\infty$.

Other than that, the choice of the family of $\mathcal{E}(\theta)$ is totally free. In fact, we need only the (purely abstract) existence of a family $\mathcal{E}(\theta)$ in order to give a general form to the equations that we write. Finally, our families $\mathcal{E}(\theta)$ are just as *relative* and just as *arbitrary* as a Galilean frame. Their only merit consists of the fact that they give the integral laws a symmetric form in the relativistic sense. A well-defined Galilean observer who has been led to think in terms of families $\mathcal{E}(\theta)$, instead of simultaneity at a distance, will expand his way of thinking, since the various points of an extended system that he considered *at the same time* will no longer be *simultaneous*. That observer can materialize his reference system, as well as before, by making points out of the clocks that are graduated along θ. He will always know how to pass from the general language "in θ" to the classical language "in t," since the relationship between θ and t that is given by the formula $\theta = \theta(x, y, z, t)$ is bijective at each point (x, y, z) in his proper space.

The systematic adoption of the preceding viewpoint will result in a very important consequence: *If it is always understood that the definition of the integral – or "finite" – quantities is a function of the integration hypersurface $\mathcal{E}$ (or, if one prefers, it is "relative" to $\mathcal{E}$) then any density quantity will be associated with by a tensor that is found to be associated with one or more quantities that are each represented by a tensor*. One can even further specify: Since the three-dimensional integration element $[dx^j\, dx^k\, dx^l]$ in the four-dimensional universe is the dual of an infinitesimal quadri-vector that we regularly denote by $ic\, du_i$, we will see that *the finite tensors that are associated with a density tensor of rank n will have rank n* ± 1, *according to whether the definition of* *integral tensor does or does not include a contraction*, resp. ([1]). Two notorious examples of that general situation deal with electric charge, in one case, and mass-impulse, in the other. Charge-current density j^k is a quadri-vector, but charge Q is an invariant. Mass-impulse density is a second-rank tensor T^{ij}, but mass-impulse is a quadri-vector p^j. As another example, we cite the kinetic moment – whether orbital or proper – whose density is a third-rank tensor σ^{ijk} that is antisymmetric in at least i and j, but whose finite form is a second-rank tensor [**127**], if one takes the preceding remarks into account. A new application of that rule is concerned with force: We associate the classical *force-power density* quadri-vector f^i with a second-tensor F^{ij} in order to represent the finite force. We have justified that definition by means of two arguments, one of which was mechanical, and the other of which was electromagnetic [**120**, **108**], and those arguments will be reproduced in the present text, in substance.

Along a neighboring train of thought, one poses the problem of the variance of temperature, but with the prior complication of needing to discuss the variance of the quantity of heat. Temperature is not a quantity that one can associate with a density, but the classical definition of entropy:

$$\delta S = \frac{\delta Q}{T}$$

is a differential form in which the variance of the heat δQ is fixed in advance. Moreover, entropy, as a pure number and the logarithm of an integer number, is certainly an invariant. As a result of some papers on the subject by Einstein and Planck [**137**, **139**], all of the old treatises have refused to give relativistic temperature both invariance and tensorial variance [**4**, §§ 33*a* and *b*; see also **143**]. Those classical rules of variance of temperature are obtained without the least difficulty by calculating the elementary heat δQ in two different Galilean frames *at constant time and by starting with the proper heat density* Q_0 . On the contrary, in an entire series of modern books and papers, the relativistic temperature appears with a covariant tensorial character. Its inverse is defined by either the fourth component of a quadri-vector θ^i or as an invariant θ_0 [**140**, **141**, pp. 676 and pp., 693, **142**, pp. 922, **143**]. Now, that double result will become obvious when one reestablishes the *complete form* of the differential expression, and if one appeals, according to the nature of the problem being treated, to the *quadri-vectorial quantity of heat*, which is homogeneous to an energy-impulse, and which is associated with Van Dantzig and Bergmann's quadri-vector θ^i, or even the *scalar quantity of heat* δQ_0 , which is homogeneous to a proper energy, and which is associated with Tolman and Eckart's scalar θ_0 . If V^i denotes the quadri-velocity of the material medium then the relation:

$$\theta^i = \theta_0 V^i \equiv \frac{1}{T_0} V^i$$

will be imposed in a manner that is not absolutely constricting, but will generally be reasonable [**143**].

([1]) True, if the three-dimensional volume is taken in the form $[dx^j\, dx^k\, dx^l]$ then one will easily see that the ranks $n \pm 3$ of the finite tensor will be added to the indicated ranks $n \pm 1$. However, that remark does not seem to necessarily lead to any interesting physical applications.

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Our presentation of the theory of special relativity and those of kinematics and the optics of the vacuum in relativity, which are coupled together in essence and by tradition, appears in Chapter II. Faithful to our rules of conduct, we establish the formulas for the change of Galilean frame in the four-dimensional form, which is literally immediate once one has acquired certain mathematical facts that are assembled in Chapter I, § B. We make no pretense of originality in that part of the book, so we shall always reason in four-dimensional form and then recover only the expressions for the laws of relativity in their usual forms. Given the spirit that animates our work, in Chapter II, § C, we have very explicitly gathered some general facts of the relativistic kinematics of continuous media and some considerations on differential forms. In particular, one will note that the notion of *proper or scalar volume element* will be defined as a quadri-flux:

$$u_0 = \iiint_{\varepsilon} V_i \, \delta u^i ,$$

which can be of service in regard to some questions, and *in some special cases*, it will permit one to associate any density with a finite tensor of the same rank. For example, it perfectly justifies the notion of a *finite force* quadri-vector that is orthogonal to the world-trajectory, which certain authors have used in the case of a material point [**2**, pp. 115-116].

Chapter III is dedicated to relativistic electromagnetism, although it is very classical to begin with. In § C, we will give our new definition of the *finite* force as a second-rank tensor F^{ij}. That will come about by arguing directly by imitating the argument that establishes the quadri-vectorial character of the force density in a manner that is currently classical. Passing to the case of conduction, we will then give a covariant form to certain known results that make contact with Joule heat in relativity, and infer a general conclusion from them that is of interest in some questions of the creation or annihilation of heat, energy, and mass. In § D, a very brief and direct argument will give us the asymmetric expression for the elastic tensor of the Maxwell-Minkowski field [**99**, **100**, **1**] that we absolutely prefer to the symmetric forms (which differ from each other) that were proposed by Einstein [**105**] and Abraham [**103**, **104**], since it is the one that is suggested naturally by calculation, and it is in harmony with the notion of spin density for the field. We shall establish two electromagnetic spin densities that were discovered by E. Henriot [**107**] with our approach to things by attaching them to the definition of another asymmetric elastic tensor of the field that we considered in [**108**].

Chapter IV, which is the most voluminous of the book, is dedicated to relativistic dynamics. Very special care has been afforded to the deduction of relativistic dynamics in a classical spirit by starting with the laws of force that electromagnetism teaches us. Indeed, we believe that any autonomous, theoretical basis for relativistic dynamics will be arbitrary *a priori*, for the very simple reason that before Einstein mechanics ignored the role that is played by the universal constant c in its proper domain. Once that is known, only two methods for founding relativistic dynamics will remain: The inductive method, which must be founded on any sort of experiments that manifest the role of c in dynamics, such as those of Guye-Lavanchy on the variation of mass [**109**, **110**, **111**, **112**], those of Davisson-Germer on the relationship between mass and frequency [**154**], or even better, for the generality and symmetry of their arguments, the ones on nuclear bookkeeping that the show the universal proportionality between energy and mass [**114**, **115**]. In that way, the laws of relativistic dynamics will be founded upon a method that is entirely analogous to the one that was followed before by Galileo. Following Einstein himself, and with the goal of didactic elegance, we have preferred the deductive method, which we believe that one is obligated to follow in electromagnetism. In the course of that deduction, we shall show that the maximum economy in one's postulates is obtained when one argues in the language of continuous media by starting with a less-known, but very general, form that one can give to the fundamental laws of fluid mechanics [**4**, § 28 *b*]. The rebuttal will be given later on in the deduction of point dynamics. We show that a supplementary postulate will be required if one would like to base relativistic dynamics in terms of a material point. Our § A also contains a generalization of the theory of volume forces, as well as an explicit theory of the force of surface origin that is based upon our covariant definition of the finite force.

§ B will treat the theories of spin and viscosity together, and initially in terms of continuous media. Of course, they are physically quite distinct, since one of them, which is macroscopically unknown, is taken into consideration only as a result of the demands of quantum physics, and the other one appears to have an essentially statistical and microscopically-evanescent sense, but those two questions have an uncontestable mathematical parentage. It is for precisely that reason that we have treated them in the same paragraph. For several years now, we have treated the problem of the relativity of spin when it is posed in terms of continuous media, whereas some authors abroad have generally attacked the same problem in terms of material points [**121**, **122**, **123**, **132**, **134**]. Meanwhile, some work that has many common points of contact with our own – namely, that of Weyssenhoff and Raabe – came about in almost the same epoch and completely independent of our own work [**133**].

§ C is dedicated to the unsolved problem of the relativistic dynamics of systems of *N* interacting points. There, we shall first point out the mean technique for defining the barycenter in a covariant manner that we proposed recently [**135**]. We then show that a relativistic formulation of the general theorems of dynamics will be possible, provided that one takes into account some *potential* phenomena that occur in the field of interaction. As a result of those simple calculations, we think that we should then conclude that the true problem of the relativistic dynamics of systems is a field problem [**136**]; i.e., a problem of partial differential equations.

§ D is concerned with relativistic thermodynamics. We shall treat only some questions of principle, and above all, the question of the variance of temperature. Two very simple applications will be given by way of illustration, one of which bears upon Fourier's law of conduction, while the other one bears upon the law of adiabatic compression of a perfect gas [**128**, pp. 199-200].

Our Chapter V is further dedicated to dynamics, and expressly to our presentation of three special topics. First, to a very ingenious relativistic generalization of the theory of vortices in inviscid fluids that is due, in principle, to Synge [**149**, **150**], and into which A. Lichnerowicz has introduced the beautiful neatness of the theory of integral invariants [**151**, **152**]. It is obvious that we should present that theory in terms of special relativity. To conclude, we will show that Lichnerowicz's "definition B" of an incompressible fluid

[**153**], which is more restrictive than Synge's hypothesis, constitutes the true relativistic extension of the notion of *perfect fluid*, in which the pseudo-velocity field is derived from a double potential of source/sinks and vortex filaments, as well as a potential of the potential that generalizes that of Poincaré [**128**, pp. 203-204]. In § B, we once more present our completely-symmetric presentation of the analytical mechanics of the material point, in which the *world-force* is assumed to be derived from a quadri-potential, as in electromagnetism [**8**, IV].

Finally, § C presents Louis de Broglie's famous initial *wave mechanics* [**163**] systematically in four-dimensional terms, which is a treatment of light in microphysics, and in order to begin it, one must recall everything that is concerned with the relationship between the theories of relativity and quanta. We cannot think of a better way to conclude a treatise on *special relativity* then to embark upon one of the major problems in physics today.

We do not pretend to have exhausted the true or virtual content of the theory of special relativity with the preceding subjects. We have voluntarily limited ourselves to the questions that we have personally pondered. For example, we see quite well that the developments of elasticity or thermodynamics would have their place, and it seems to us that our covariant definition of the finite force must permit one to revive the former subject. Despite everything, we think that we have said what is essential, and we hope that we have provided a tool that will be useful in everything that is of interest in relativity, whether in its own right or as an intermediary in the study of the problems of either astrophysics or microphysics.

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We have directed the preceding remarks to readers that are already familiar with the theory so that we could present the reasons that have encouraged us to add several excellent treatises or synthetic articles to our modest book. Now, we would like to say a few words to the novice reader, for whom our book is also intended, to our way of thinking.

We have constantly sought to be very clear, and at the same time, to present the questions in a truly relativistic context, which is that of a systematic symmetrization of space and time. Rather than making long speeches, we have profoundly desired to know the *spirit* of relativity, and at the same time, to even show the unique means for seeing things absolutely clearly in its questions, which is the use of four-dimensional thinking.

Today, more than 40 years after its formulation, special relativity is no longer a difficult theory, technically speaking. Nonetheless, one must think that certain points are still misunderstood, judging by the nature of certain objections that are occasionally formulated. Along with the aforementioned caveats, we have then had to precede our exposition with a long and detailed introduction, while providing the reader with some pure contingencies, moreover; we hope that we have thus given them some useful clarifications.

§ B of the first chapter contains a simple presentation of the required mathematical notions. Principally, they are the definition of tensors in oblique, rectilinear axes in pseudo-Euclidian spaces and the establishment of the general formula for the transformation of multiple integrals.

In the entire course of this book, with rare exceptions, the same letters will denote the same physical quantities. The bold numbers between brackets refer to the bibliographic index. The chapters and the equations inside of each of the five chapters are numbered. The chapter number will be a Roman numeral. In general, we shall omit that numeral prefix in the references that are internal to a given chapter, and enclose references to equations in other chapters in square brackets, along with the chapter number.

In the entire course of the book, i, j, k, l will denote an *arbitrary* permutation of the world-tensorial indices 1, 2, 3, 4. u, v, w will denote a *circular* permutation of the spatial indices 1, 2, 3, so the world-indices will then be u, v, w, 4.

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FIRST CHAPTER

INTRODUCTION

I.1. – Sub-chapter A of the present chapter will present the epistemological history of Newton's and Einstein's *principles of relativity*. We have sought to give it brevity and clarity. However, we have insisted upon some points whose importance has been revealed to us in our discussion with certain adversaries of relativity.

At the heads of the last chapters, the novice reader will find some indications regarding the manner by which the conversion of the branches of physics in those chapters into four-dimensional geometry will be treated.

In sub-chapter B, we have compiled some of the mathematical ideas that are necessary for understanding the rest of the book. Since we intend to always refer everything to Minkowski space with orthogonal axes of equal measure, we shall have no need to summarize either the theory or the rules of the general tensor calculus. We shall give a very simplified, but quite complete, presentation of the notion of tensor and the rules of tensorial calculus in planar spaces that are referred to rectilinear axes of constant measure. In order to do that, we shall not at all seek to argue in an axiomatic manner, but we shall largely appeal to the reader's intuition in order to extrapolate the results of ordinary geometry to the *n*-dimensional case. The knowledge that is required in that paragraph is summarized in combinatorial analysis, the theorem of determinants and linear equations, and the elementary theory of multiple integrals.

A. – THE SUCCESSIVE STATEMENTS OF NEWTON'S AND EINSTEIN'S PRINCIPLES OF RELATIVITY.

I.2. – **The "principle of relativity" in Galilean-Newtonian kinematics and dynamics.** – One knows that Newton thought that he could base the kinematics and dynamics that are called *classical* or *Newtonian* today upon the principle of *absolute space;* that was a purely verbal statement. The *principles of relativity* that are valid in those two respective branches of *rational mechanics* are, in fact, much broader.

Classical kinematics is subject to what one calls the *principle of relative motion.* In order to understand that principle, one must remember that classical kinematics utilizes the notions of a (three-dimensional) Euclidean space and a "universal" time. On the one hand, the theorems of Euclidian geometry are invariant with respect to the group of rigid displacements. On the other hand, the *postulate of a universal time* signifies that the framing of an event in time can be done independently of the motion of the (rigid) system of reference. Finally, the changes of the spatial variables x^u (u = 1, 2, 3) and the time t that respect the *principle of relative motion* have the form:

$$x'^u = x'^u (x^u, x^v, x^w, t) \qquad t' = t'(t), \tag{I.1}$$

in which it is assumed that the transformation $x \to x'$ essentially leaves the expressions for the spatial distances or their squares invariant:

(I.2) $$\sum \Delta x_u'^2 = \sum \Delta x_u^2 .$$

One sometimes says that this essential autonomy of space that is respected by the *principle of relative motion* forms a *principle of absolute space.* One should be careful, since this new sense is quite different from the original sense, which postulated the existence of a *privileged frame* in space. Be that as it may, in the original sense, as well as in the derived sense that was just mentioned, the *principle of absolute space* will be rejected by relativity, as well as that of *universal time.*

Before Einstein, one considered the *principle of relative motion* in kinematics as having been verified quite well by experiments. Einstein revealed that this was only an approximate truth. The new kinematics that he promoted assumed only a *principle of relativity* in the large, namely, the one that one knows already from Newtonian mechanics.

The *principle of relativity* is presented in detail in all treatises. It will result from this that the fundamental formula $\mathbf{F} = m\,\boldsymbol{\gamma}$ of point dynamics is a characteristic invariant of the group of uniform translations (extended in the classical sense) of the reference x^u, t; i.e., transformations of the form:

(I.3) $$x'^u = x^u - v^u\, t, \qquad t' = t - \text{const.}$$

Those transformations leave the accelerations invariant, and one will assume, as a postulate, that the masses and the forces are also transformed as invariants.

Physically, Newtonian dynamics must then restrict considerably the *principle of relative motion* that Newtonian kinematics assumes. The accelerated transformations or rotations (which are both uniform) of reference frames (which are assumed to be rigid) become *absolute.* As one knows, experiments have verified that deduction by exhibiting the appearance of ordinary or Coriolis forces of inertia, and consequently permitting the effective determination of the group of *Galilean frames*. The laws of dynamics will be formulated in an *equivalently privileged* manner relative to that group of frames, which are all in uniform translation with respect to each other.

I.3. – **Absolute space and the ether from Arago (1818) to Michelson-Morley (1887). The "ether wind" is hidden from optical experimentation. –** The only domain in classical theoretical science in which Newton's *absolute space* is anything but a pipe dream is in optical (or electromagnetic) kinematics. Indeed, consider a monochromatic point source that emits isotropic waves of speed c in a certain Galilean frame $\mathcal{G}_0$. From classical ideas, in any other Galilean frame $\mathcal{G}$, the speed of the waves will no longer be isotropic, but will have values that are found between $c \pm v = c\,(1 \pm \beta)$; v denotes the relative velocity of $\mathcal{G}$ and $\mathcal{G}_0$, which is assumed to be less than c, and we have set $\beta \equiv v / c$. In a less elementary manner, one can insure that the d'Alembert equation or those of Maxwell or Lorentz are not invariant under Galilean transformations. It follows naturally from this that in the classical theory, optical or electromagnetic experimentation seems to be capable of characterizing the hypothetical absolute framing of space that is further called *absolute space.*

One knows that the great legislators of optics and electromagnetism Fresnel, Maxwell, and Lorentz supposed the existence of a medium of propagation for waves that they more or less (and in fact, less and less) assumed to be elastic waves. From the definition itself of a *medium of propagation*, the speed of the waves will be isotropic. That hypothetical *ether* will then be quite naturally identified with the hypothetical *absolute space.* One then sees that classical science, by the very force of things, had to identify a kinematical notion with an optical one, while Einstein's relativity would condemn both of them together. That was like a contradictory premonition of the close kinship that relativity would establish between kinematics and optics.

In optics, the most direct effects of the motion of sources and receivers are aberration (1728) and the Doppler effect (1842). They are effects of first order in the velocities, and the classical theory, which assumes the ether, explains them easily. The important point is that *the first-order terms involve only the relative velocity of the source-observer.* The absolute velocities, which exhibit what they called the *ether wind* in the time of Michelson, appear only to second order in $\beta \equiv v / c$. Up to a very recent date (Ives and Stilwell, 1941), the second order in β remained beyond the reach of experiments in that domain.

From the classical ideas, the *ether wind* further intervened to first order by altering the velocity of light that progresses along a rigid base. However, that amounts to a totally abstract viewpoint, so no apparatus could allow one to test that experimentally. For example, one might wish to measure the time duration of the light trajectory along a rigid base *AB*. In order to synchronize the chronometers at *A* and *B*, no procedure will be any better than an exchange of optical (or Hertzian) signals between *A* and *B*. That is ultimately equivalent to measuring the time duration of the trajectory along a round trip of the base, and that is practically what one does in the classical methods of Fizeau, Foucault, and Karolus-Mittelstaedt. Any effect of the *ether wind* will then disappear to first order in β, which is the only one that can be experimentally attained.

In order to measure the speed of light on a one-way trajectory, one can appeal to the two closely-related phenomena of the Doppler effect and aberration, and in fact that is how the first evaluations of the constant *c* were obtained. Indeed, Römer's observation (1676) was mathematically equivalent to the observation of the Doppler effect, in which the frequency of emission of a wave train was replaced with the frequency of occultation of the satellites of Jupiter. The second evaluation of *c* was that of Bradley (1728). In the two cases, any effect of a hypothetical ether wind that blows parallel to the plane of the ecliptic would disappear to first order in β, in such a way that *the two phenomena considered will indeed provide evaluations of the constant c, and none of the evaluations of c augmented or diminished the hypothetical ether wind.* We also remark that the preceding methods of evaluation of *c* along a one-way trajectory of light succeed only because of the fact that the receiver – namely, Earth – successively occupies different Galilean frames (in fact, an infinitude of them, but two will suffice, in principle). The relative velocities of those various Galilean frames must be known directly.

However, there is a domain in which the *ether wind* seems to be manifested to first order *a priori*, from classical ideas: It is the domain of experiments that involve transmission in a transparent medium, such as glass or water. Indeed, if the hypothetical *ether* is not carried along by the material medium in question at all, or completely, then a first-order effect will manifest itself – for example, in the observed aberration that is observed with a lunette that is filled with water, and also in the deviation that is due to a prism. Nothing of the sort has been observed. Notably, the second experiment was attempted by Arago in 1818, and repeated in 1872 in a refined way by Mascart, and with a perfectly "negative" result. In 1818, Fresnel showed that the necessary and sufficient condition for annulling the *ether wind* to first order (which was Arago's experimental result) was that one must assume a *partial dragging of the ether by the matter in the prism* according to the well-known formula:

$$c' = \frac{c}{n} \pm v\left(1 - \frac{1}{n^2}\right). \tag{I.4}$$

Throughout the XIX$^{\text{th}}$ Century, Babinet, Fizeau, Angström, Hoek, and Mascart accumulated experiments that were equivalent to the preceding ones, in principle. The results were consistently "negative," but with the exception of a series of experiments by Fizeau on the rotation of the plane of polarization in a block of ice. However, that same experiment, which was repeated in a much more precise manner in 1905 by Brace and in 1907 by Strasser, gave perfectly "negative" results both times. Among the experiments in question, special mention must be made of the ones by which Fizeau (1851), Michelson (1886), and Zeeman (1914) verified Fresnel's dragging formula in a completely direct way.

As a result of all those experiments, and notably the ones that he himself had performed, Mascart was convinced that the optical search for *absolute space* must be in vain. The group of Galilean frames must enjoy the same *privileged equivalence* in optics that one knows from dynamics. In purely qualitative terms, that is what one would anticipate from Einstein's thesis. In the same era (1873-1874), Veltmann, and then Potier, brought into plain view the general result that was implied by Fresnel's formula, and which Stokes has pointed out already: *Fresnel's law of dragging of the ether is equivalent to the unconditional annulling of any effect of an "ether wind" to first order.* A little bit later, Lorentz extended that theoretical result to all of electromagnetism. The first campaign in the search for an "ether wind" that was inaugurated by Arago then arrived at a totally "negative" result.

The second campaign opened with the celebrated second-order experiments of Michelson (1881), and then Michelson-Morley (1887). Once again, the result was absolutely "negative." This time, the theory countered with an appropriate formula that translated into a convenient and universal effect of the supposed "ether wind." It was the formula for the *contraction of length by the ether wind* according to law:

$$l = l_0\sqrt{1-\beta^2}\ , \tag{I.5}$$

which was proposed by Fitzgerald (1893) and Lorentz (1895). Epistemologically, the Fitzgerald-Lorentz hypothesis has the same very grave defect as the older hypothesis of Fresnel: *At the same time, it asserts the existence of the ether and the impossibility of proving that fact experimentally.*

Moreover, the theory does not stop with that: With Lorentz and Poincaré, it adds the hypothesis of a *proper* time for each Galilean frame to the preceding one. Finally, it writes the formula for the change of Galilean frame with the aid of the group of formulas that is becoming celebrated:

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}}, \qquad t' = \frac{t - vx/c^2}{\sqrt{1-\beta^2}}, \tag{I.6}$$

which agrees with the Galilei formulas, to an approximation that is first-order in $1 / c$. Einstein's theory of relativity consists essentially of the adequate interpretation of the preceding formulas.

Michelson's experiment was repeated by various authors, and generally with its totally "negative" result. Meanwhile, Miller's experiments seemed to reveal an "ether wind" of 4 or 5 km/s, which is much smaller than the one that was initially sought, and varies according to different laws, as well. However, it does not seem that those experiments should be retained, since newer and more precise experiments, one of which was due to Kennedy and Illingsworth (1927), and the other of which, to Joos (1930), gave totally negative results. From the latter experiments, the "ether wind" will be less than 1 or 1.5 km/s (the value of the experimental imprecision), while the speed of the Earth in its orbit is of order 30 km/s.

Along with the second-order experiments of Michelson type, one must cite the electromagnetic experiments of Trouton-Noble type (1903) and Trouton-Rankine type (1908). Those experiments have also been renovated – notably by Chase (1927) and Tomashek (1925-1927). Their results, whose precision could reach 4 or 5 km/s, were always negative.

Naturally, the concept was calling into question the experiments on the optical effects of rotations. There as well, the effect that was predicted by the classical theory was of first order in β, and experiments verified it precisely (Harress, 1912, Sagnac, 1913). It then asserted that rotations have an *absolute* character, in optics, as well as dynamics. The Michelson-Gale experiment (1925, which was less precise, moreover), in which the Earth was taken to be a rotating rigid body, is the true optical analogue of the Foucault pendulum experiment.

I.4. – **Einstein's relativity postulate.** –

A. – For Einstein, as well as for Mascart and Poincaré before him, the qualitative lesson of all the experimentation that was carried out in kinematical optics from Arago to Michelson-Morley was this: *The law of privileged equivalence of Galilean frames is not just a* ***principle of relativity*** *that is intrinsic to dynamics, it is a principle of relativity that is universal to all physics.* In particular, the *principle of relative motion* of classical mechanical kinematics and the *postulate of absolute space* or *ether* of classical optical kinematics are both false; the one, in the very large and the other, in the very small.

Here, we must respond to a group of objections against relativity *a priori* that are often posed by the practitioners of *rational mechanics.*

For them, true kinematics must be an integral part of rational mechanics. They stress that one claims to be overhauling kinematics in the name of the laws that were discovered by optics. But, in reality, kinematics is not a special case of mechanics; on the contrary, it is the general context in which all physics is inscribed. By virtue of an uncontestable epistemological view of Duhem, it is not at all excluded *a priori* then that experimental progress in any chapter of physics should retroactively impose a refining of the most "primary" postulates of science: namely, those of kinematic type.

The objectors in question then ask: "Why give preference to optics in a conflict between traditional mechanics and optics?" First, by virtue of its uncontestable precision. The proof of a universal acknowledgement of the superior precision in optics – or, what amounts to the same thing, electromagnetism – is provided implicitly by the two facts that as of today, all high-precision metrology is subordinate to the spatial standard that is defined by light, and all high-precision chronometry is subordinate to the Hertzian time standard. However, there is more: It is not just by its unequaled precision that the science of waves naturally dominates geometry and kinematics, but because of the fact that it *is* really the most geometrical and the most kinematical of the physical sciences. If one leaves aside the ancillary questions of intensity then optics essentially treats length (viz., *wave length*), time (viz., *period*), and pure numbers (viz., *phase*).

Moreover, the final deciding point in this controversy is provided by the formulation of *wave mechanics*. It is no longer just kinematics, but dynamics, that is subordinate to the science of waves. A synthesis of the kinematics of points and the kinematics of waves is found within the context of relativity. Optics, as is the will of *wave mechanics*, is nothing but the wave mechanics of the *photon*, which is a corpuscle whose very small physical mass is physically indiscernible. It is solely by virtue of that particular fact that the physical speed of light – viz., the group velocity of the *photon* – is indiscernible from the universal constant *c*. In truth, the constant *c* is not at all special to optics; it belongs to all physics. One simply arrives at the fact that from the fact of the vanishing mass of the *photon*, the dynamics of the photon sublimates, so to speak, to the state of pure kinematics.

In anticipation of what follows to some extent, those are the reasons of an epistemological order that oppose the objection of traditional mechanicians, and which justify Einstein for having identified kinematics and optics, in a way.

B. – For Einstein, the law of privileged equivalence of Galilean frames in the optics of the vacuum translates thus: *The speed of light not only **seems** isotropic in every Galilean frame, it **is** isotropic.* In opposition to the epistemological frailty of the original hypotheses of Fresnel and Fitzgerald-Lorentz, Einstein's new hypothesis adequately translates the homogeneity that stands out in the experiments into a theoretical statement.

At this point in the presentation, it is important to examine closely the significance of the Michelson-Morley experiment, in order to see and distinguish the conclusions that it *imposes* more clearly, along with the postulates that it *permits.*

The *net result* of the Michelson-Morley experiment is this: *The round-trip speed of the optical phase is isotropic*, and consequently in independent of the *direction* of the so-called *ether wind.* Two questions seem to have been left in suspense then, and one must demand to know the circumstances under which it is permissible to ask:

1. Might the round-trip phase velocity be a function of the *absolute value* of the *ether wind?*
2. Might the phase velocity of a *one-way trip* be a function of the *direction* of the *ether wind?*

We remark that it is easy to translate those two questions into the language of wave length. First of all, the *net result* of the Michelson-Morley experiment can be formulated equivalently in the following form: *The number of stationary waves that are carried by a given material yardstick and emitted from a monochromatic source that is at rest with respect to that yardstick is isotropic.* Under those conditions, is it permissible to demand to know (and then, under what conditions):

1. Whether that number can be a function of the *absolute value* of the hypothetical *ether wind?*
2. Whether the *number of waves that propagate* along the yardstick and issue from a source that is coupled with it can be a function of the *direction* of the hypothetical *ether wind?*

We first address the former question. When posed in terms of the *ether wind*, one can hardly see how it could make any intelligent sense, because the Michelson-Morley experiment showed precisely that if there is an *ether wind* then its physical behavior will differ radically from the properties that a true *wind* of true *ether* would have. Under those conditions, it is not indicated that one should continue to appeal to words that evoke inexact images, and one agrees to pose the first question in one or the other of the following direct forms: *Is the number of stationary waves of a well-defined physical radiation that are carried by a given yardstick subject to secular variations?* Or even: *Is the round-trip speed of the optical phase subject to secular variations when it is evaluated with the aid of a material standard of length and a standard of time that is the period of the source utilized?*

Before returning to that interesting first question a little bit later, we shall pass on to an examination of the second one.

Experimentally speaking, it would seem that there is *no sense* in demanding that the speed of light must have a given value along a one-way trip.

It results clearly from what was said in the preceding no. that the process of measuring the speed of light along a one-way trajectory (which one can, in principle, base upon the Bradley effect, or what is equivalent to it, the Römer and Doppler effects) can no longer be provided by processes that utilize hypothetical first-order round-trip effects of the *ether wind*. Similarly, no process can permit one to enumerate the number of propagating waves that are carried by a given material ruler by taking into account the classical hypothetical effect of the ether wind. For example, if one places a source at the extremity A of the ruler and receives its waves with a grid that is placed at the extremity B then the Doppler effect for emission will be compensated by a second Doppler effect for reception, in such a way that the apparatus will provide the number of waves that are carried by the ruler in the absence of an ether wind, and nothing more. That experiment with a source-grid that is fixed in the laboratory is the one that Angström and Mascart performed. Finally, to address the Michelson-Morley experiment, one cannot learn more from the consideration of one-way trajectories than one can from real round-trip trajectories.

C. – As a result of the *negative result* of the Michelson-Morley experiment, Einstein formulated the *fundamental postulate* of his new kinematics in the following form: *The* *speed of light in vacuo c is an absolute constant.* We shall show (as an application of the Duhem's epistemological viewpoint) that *that postulate is not properly imposed by the Michelson-Morley experiment, it is the Michelson-Morley experiment that makes it possible and suggests it.* We will then show (and this will be an application of Poincaré's epistemological viewpoint) that *Einstein's postulate is equivalent to a definition, namely, the "universal" attachment of a standard of time to a standard of length.*

We refer to the interferential experiments by which Michelson-Benoît, Benoît-Pêrot-Fabry, and then some other authors, have referred the standard meter to the wave length of the red line in cadmium. It is clear that if the material lengths and optical wave lengths are not unconditionally common to each other in comparison to the assumed *ether wind* – which is the net result of experiments of Michelson-Morley type – then the metrological comparisons in question can have no sense at all. They must be accompanied by a determination of the direction and speed of the *ether wind*, which is not the case, in fact ([1]). *The negative result of the experiments of Michelson-Morley type then appears to be a necessary condition for the substitution of the optical standard of length for the material standard.*

Now, the red line of cadmium (or any other well-defined line) ([2]) was chosen to be the standard of length (by way of its wave length), but its period can just as well be a standard of time, in principle. It is clear that *to make an optical line the standard of length, as well as time, is equivalent to assuming that the speed is an absolute constant* ([3]). We have then established the two points that we asserted, and they are closely connected.

What would happen if the experimental evaluations of the speed of light were to reveal a secular variation of that so-called absolute constant? That is a question that is not absolutely Platonic, and several authors – the most eminent of whom is Esclangon – have raised it effectively ([4]). From the preceding argument, and contrary to what some have suggested, it is clear that the eventuality considered will affect nothing in the theory of relativity. Einstein's postulate rests upon the experimental fact of *isotropy*, and upon it alone. It then makes c an absolute constant, *by definition.* Let ρ_1 denote the ratio of the standard meter to the wave length of a certain well-defined monochromatic line then, and let ρ_2 denote the ratio of the sidereal second to the period of the same line. Assuming that the experimental value of c is subject to secular variations is equivalent to assuming that at least one of the ratios ρ_1 or ρ_2 is subject to secular variations. That question is entirely independent of the question of the foundations of special relativity.

([1]) The optical method permits one to detect relative variations of the material lengths of order 10^{-8}. From classical ideas, it will then permit one to detect values of the *ether wind* of order 50 km/s, which is a degree precision that is quite inferior to that of experiments of Michelson-Morley type. The question of the principle that was formulated in the text will then continue to exist.

([2]) The use of the green line of an isotope of mercury with zero nuclear spin has recently been proposed. One knows that such lines are particularly fine, all other things being equal, moreover.

([3]) In the present state of science, one does not know how to get the period of an optical line directly. By contrast, in the centimeter and millimeter Hertzian domain, certain molecular lines can be compared to the astronomical second directly. It has already been proposed that the standards of time that are defined down to 10^{-8} to 10^{-9} in the inversion spectrum of ammonia can be made finer and more stable [**77′**].

([4]) E. ESCLANGON, *La notion du temps*, Paris, 1938, pp. 16-18. – GHEURY DE BRAY, Nature **133** (1934), pp. 464 and 948. – EDMONDSON, *ibid.*, pp. 759.

In fact, the most recent measurements of c by Anderson did not confirm the laws of variation that were proposed, and it seems that the best interpretation of the set of terrestrial measurements that were made in the last hundred years is that of the constancy of c [**70**, **71**, **74**] ([1]). Moreover, nine comparisons of the standard meter with the wave length of the red line in cadmium that were spread over fifty years are mutually coherent to almost 10^{-8} [**77**] ([2]).

D. – It results from Einstein's postulate that *the formulas for the change of Galilean frame* x^u, t (u = 1, 2, 3) *are no longer the formulas for the Galilei group, but those of the Lorentz-Poincaré group.* The proof is found in Einstein's work, and it was already implicit in the work of Lorentz and Poincaré. We shall give that proof again, in our own manner, in no. II.2, where we will link it immediately to the notion of Minkowski's four-dimensional space-time.

B. – MATHEMATICAL INTRODUCTION: TENSORS IN PSEUDO-EUCLIDEAN SPACES.

I.5 – **Covariant and contravariant components of an *n*-vector. Metric tensor.** – Consider an n-dimensional space that is referred to a system of n rectilinear axes with constant, but not necessarily equal, magnitudes, and which are generally oblique to each other. We shall put ourselves in a very general case that contains the particular case of the pseudo-Euclidian space-time – or *universe* – of the theory of special relativity by assuming that the n coordinates n_i are complex (i = 1, 2, 3, …, n).

Each of the n axes can be defined by its unit director n-vector $\mathbf{u}_i$, and the lengths of those vectors are not equal, in general [*sic*]. The most general spatial vector **s** considered can be written uniquely as the sum:

([1]) Birge [**70**, **71**] held the best measurements of c to be:

1. Rosa-Dorsey (1906, report on the bases for E. S. U. and E. M. U., which was a correction to the report on the bases for absolute and international electric units).
2. Mercier (1923, stationary radio-electric waves).
3. Karolus-Mittelstaedt (1928, beam chopped at very high frequency by a Kerr cell and having a very short base).
4. Michelson-Pease-Parson (1932, rotating mirror, base of 1 mile *in vacuo*).

To them, it is suitable to add:

5. Anderson (1941) [**74**], who ingeniously refined the method in 3.

Here is a table of those results:

1.	(1906)	c =	299,781 ± 10 km/s
2.	(1923)		299,782 ± 30 "
3.	(1928)		299,778 ± 10 "
4.	(1932)		299,774 ± 4 "
5.	(1941)		299,776 ± 14 "

This table does not at all create the impression that there is an ample and rapid secular variation of c, as some authors have assumed. On the contrary, the preceding five results, which were obtained by four very different methods and over dates that were spread over 35 years, are remarkably coherent.

([2]) It was quite recently that a measurement of c was made in the centimeter Hertzian domain by a procedure that was equivalent to the simultaneous measurement of the period and wave length of a stationary wave. The precision obtained was equal to or greater than that of the previous methods.

(I.7) $$\mathbf{s} = \sum_{i=1}^{n} x^i \, \mathbf{u}_i \, ,$$

in which the *n coordinates* x^i are, as we said, assumed to be complex. We now introduce and utilize the *summation over dummy indices convention*, which is well-known in tensor calculus and consists of this: *Whenever the same index appears twice in a monomial – once above and once below – one must sum over all values of that index.* The monomial in question will then be a polynomial, in reality, and in what follows in this book we shall speak indifferently of *its term* or *its terms*, according to whether our attention is directed spontaneously to its formal aspect or the reality that it implies; we shall see that this will introduce no confusion. We then agree to write the expression (7) in the form:

(I.8) $$\mathbf{s} = x^i \, \mathbf{u}_i \, .$$

The repeated index i is called *dummy* because since the summation over i is automatically required by the form itself of the right-hand side of (8), that index will not figure essentially in the expression for **s**.

Upon scalar-multiplying the vector **s** by itself, we will get the square of its length, which is a number (that is complex, from our general hypothesis), and whose expression will be ([1]):

(I.9) $$\mathbf{s}^2 = \mathbf{u}_i \, \mathbf{u}_j \, x^i \, x^j = \tfrac{1}{2}(\mathbf{u}_i \, \mathbf{u}_j + \mathbf{u}_j \, \mathbf{u}_i) \, x^i \, x^j.$$

Of course, we shall continue to utilize the *convention of summation over dummy indices*, and $\mathbf{u}_i \, \mathbf{u}_j \, x^i \, x^j$, for example, is intended to mean:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{u}_i \mathbf{u}_j x^i x^j \, .$$

The n complex numbers x^i are called *contravariant coordinates* of the vector **s**, and the n^2 numbers:

(I.10) $$g_{ij} = \tfrac{1}{2}(\mathbf{u}_i \, \mathbf{u}_j + \mathbf{u}_j \, \mathbf{u}_i),$$

which are also complex and symmetric in i, j, are called *covariant coordinates of the metric tensor.* By definition, the n numbers:

(I.11) $$\boxed{x_i = g_{ij} x^j}$$

are *covariant coordinates* of the vector **s**.

([1]) In the Hilbert space that is used by quantum theory, the scalar product of two vectors is defined to be what we will call the scalar product of one vector with the conjugate of the other one.

In order to pass from first expression in (9) to the second one, one must postulate that the scalar product of the two director vectors $\mathbf{u}_i$ is commutative; some geometries that reject that postulate have been proposed.

Equations (11), in which one sums over the dummy index j, are n in number, since i can take all of the values from 1 to n. Depending upon whether we direct our attention spontaneously to the vectorial significance or the analytical explanation for a formula such as (11), we shall speak indifferently of the equation or equations of the formula that is represented; we shall see that no confusion should follow from that.

Equations (11) can be regarded as a system of n linear equations in n unknowns x^i. One then essentially supposes that the determinant of that system:

$$\Delta = | g^{ij} | \neq 0 \tag{I.12}$$

is non-zero.

Equations (11) can then be inverted into the form:

$$\boxed{x^i = g^{ij} x_j ,} \tag{I.13}$$

in which the g^{ij} are the *normalized minors* of the elements g_{ij} of Δ. By definition, the n^2 numbers g^{ij}, which are complex and symmetric in i and j, are called the *contravariant coordinates* of the metric tensor. Their determinant is found to be equal to Δ, by definition, and it will then be non-zero:

$$| g_{ij} | = \Delta \neq 0. \tag{I.14}$$

The square s^2 of the length can successively take on the three forms:

$$\boxed{s^2 = g^{ij} x_i \, x_j = x^i x_i = g_{ij} x^i x^j .} \tag{I.15}$$

If δ^i_j denotes the well-known Kronecker symbol then the second of those forms can be written:

$$s^2 = \delta^j_i \, x^i \, x_j , \qquad \delta^j_i = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases} \tag{I.16}$$

By definition, the n^2 real numbers:

$$g^j_i \equiv \delta^j_i \tag{I.17}$$

are called the *mixed components* of the metric tensor.

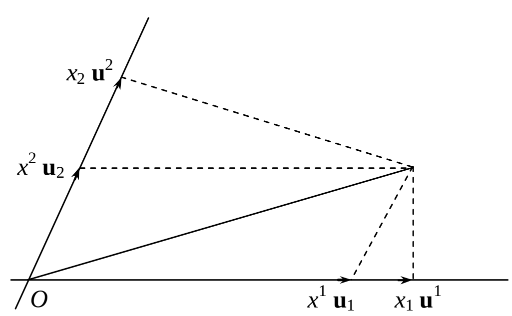

Figure 1.

Formulas (11), (13), and:

$$x^i = \delta^i_j x^j \qquad \text{or} \qquad x_j = \delta^i_j x_i ,$$

show that *the tensors* g^{ij}, g_{ij} , *and* $g^i_j \equiv \delta^i_j$ *deserve the names of raising and lowering operators, and the substitution of the index* i, *resp.*

We address the special case of a two-dimensional space with real coordinates and *seek the geometric interpretation of the covariant coordinates* x_i *of the vector* **s**, *it being intended that the contravariant coordinates are nothing but the oblique coordinates in the usual sense of the term;* the reader can pass to the general case effortlessly. If α denotes the angle between the director vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ then we will have:

$$g_{11} = (u_1)^2, \qquad g_{22} = (u_2)^2, \qquad g_{12} = g_{21} = u_1\, u_2 \cos \alpha,$$

and consequently:

$$x_1 = u_1\, (u_1\, x^1 + u_2\, x^2 \cos \alpha),$$
$$x_2 = u_2\, (u_2\, x^2 + u_1\, x^1 \cos \alpha).$$

We then introduce two vectors $\mathbf{u}^1$ and $\mathbf{u}^2$ that are collinear to $\mathbf{u}_1$ and $\mathbf{u}_2$, resp., and are such that the two scalar products $\mathbf{u}_1\, \mathbf{u}^1$ and $\mathbf{u}_2\, \mathbf{u}^2$ are equal to 1. We will see that *the extremities* $x_1\, \mathbf{u}^1$ *and* $x_2\, \mathbf{u}^2$ *are nothing but the orthogonal projections of the extremities of the vector* **s** *onto the coordinate axes.*

I.6 – **Formulas for a change of axes. General definition of a tensor. Tensorial rules of homogeneity.** – Take n linearly-independent vectors $\bar{\mathbf{u}}_i$ in the space considered, and let $\bar{o}_i^j$ denote the j^{th} contravariant component of the i^{th} vector. By definition, one has the system of relations:

(I.18) $$\bar{\mathbf{u}}_i = \bar{o}_i^j \mathbf{u}_j$$

between the system of vectors $\bar{\mathbf{u}}_i$ and the basis vectors $\mathbf{u}_i$. Consider the vector that is defined by (8) again, and look for its contravariant components in the new system of axes that are defined by the n vectors $\bar{\mathbf{u}}_i$; by definition, one has:

$$\mathbf{s} = \underline{x}^i\, \bar{\mathbf{u}}_i = \bar{o}_i^j\, \underline{x}^i \mathbf{u}_j ,$$

from which, one concludes the formulas:

(I.19) $$\boxed{x^j = \bar{o}_i^j\, \underline{x}^i ,}$$

which express the old contravariant coordinates as functions of the new ones. Since, by hypothesis, the n vectors $\bar{\mathbf{u}}_i$ are linearly independent, the determinant $\left|\bar{o}^i_j\right|$ will be non-zero. Since the $\bar{o}^i_j$ denote the normalized minors, (19) will invert to:

(I.20) $$\boxed{\underline{x}^j = \underline{o}_i^j x^i} \qquad (\left|\bar{o}_j^i\right| = \left|\underline{o}_j^i\right| \neq 0),$$

and one will then see that the $\underline{o}_j^i$ represent the j^{th} contravariant components of the vector $\mathbf{u}_i$ in the system $\bar{\mathbf{u}}_i$.

By virtue of the definition itself of the inverse coefficients $\underline{o}_j^i$, if one starts with the direct coefficients $\bar{o}_j^i$, and if δ_k^i always denotes the Kronecker symbol then one will have the double system of identities:

(I.21) $$\boxed{\bar{o}_j^i\, \underline{o}_k^j \equiv \bar{o}_k^j\, \underline{o}_j^i = \delta_k^i.}$$

Let $\underline{o}_j^{*i}$ denote the j^{th} covariant component of the i^{th} vector $\bar{\mathbf{u}}_k$, and let $\bar{x}_i$ denote the new covariant components, so formula (19) will permit one to write:

$$x_j = \underline{o}_j^{*i}\, \bar{x}_i .$$

The latter formula, when combined with (19), permits one to write:

$$x_i\, x^i \equiv \underline{x}_i\, \bar{x}^i = \bar{o}_k^{\,j} \underline{o}_j^{*i}\, \underline{x}^k\, \bar{x}_l ,$$

and one concludes from this that:

$$\bar{o}_k^{\,j} \underline{o}_j^{*i} = \delta_k^l .$$

Recalling (21), the latter formula will show that:

$$\underline{o}_i^{*j} = \bar{o}_i^{\,j} , \qquad \bar{o}_i^{*j} = \underline{o}_i^{\,j} ;$$

i.e., $\underline{o}_i^{\,j}$, which is the normalized minor of $\bar{o}_i^{\,j}$, is nothing but the i^{th} covariant component of the j^{th} vector $\bar{\mathbf{u}}_j$ in the system $\mathbf{u}_i$ .When one compares this result with the preceding result, one will see that the j^{th} covariant component of the vector $\bar{\mathbf{u}}_j$ in terms of $\mathbf{u}_i$ is equal to the i^{th} contravariant components of the vector $\mathbf{u}_i$ in terms of $\bar{\mathbf{u}}_j$. Finally, (19) and (20) imply the consequences:

(I.22) $$\boxed{x_j = \underline{o}_j^i\, \bar{x}_i, \quad \bar{x}_j = \bar{o}_j^i\, x_i;}$$

the reciprocal consequences are established with no difficulty.

Let **a**, **b**, **c**, … be a certain number of n-vectors, and consider, for example, an expression such as $a^i\, b^j\, c_k$, which is doubly-contravariant and simply-covariant. It transforms according to the law:

$$\bar{a}^i\,\bar{b}^j\,\bar{c}_k = \bar{o}^i_{i'}\,\bar{o}^j_{j'}\,\bar{o}^k_{k'}\,a^{i'}\,b^{j'}\,c_{k'}.$$

By definition, a tensor is a geometric object whose *components* transform like such products, and its *rank* is the number of n vectors that occur in the product considered. For example, the doubly-contravariant, simply-covariant components of a tensor of rank 3 transform according to the law:

$$\boxed{\bar{\bar{T}}{}^{ij}_{..k} = \bar{o}^i_{i'}\,\bar{o}^j_{j'}\,\bar{o}^k_{k'}\,T^{i'j'}_{..k'}.} \tag{I.23}$$

As was explained in the preceding no., the operators g_{il}, g^{kl}, $g^k_i = \delta^k_i$ can be applied to the two sides of such equations in order to lower, raise, or substitute an index, resp.; i.e., in the first two cases, one can turn a covariant index into a contravariant one, and *vice versa*. Notably, one will then see that a second-rank tensor will have contravariant components T_{ij}, covariant components T^{ij}, and "mixed" components $T_j^{\ i}$. By definition, one calls the operation that consists of setting $i = j$ and summing over all i from 1 to n the *contraction* of a tensor over a pair of indices i, j, one of which is raised and one of which is lowered (if they are not that way already). For example, the contraction over i, j of the preceding tensor T is written:

$$T_{ik}^{\ i},$$

and that expression is intended to mean:

$$\sum_{i=1}^{n} T_{ik}^{\ i}.$$

The contraction $T_i^{\ i}$ of a second-rank tensor, which is a scalar, is often called its *trace*. As examples of tensors, we have already encountered *scalar magnitudes*, which are *tensorial invariants*, and the *n-vectors*, which are tensors of rank 1. The *square of a quadri-vector* – namely, $x_i\,x^i$ – is the contraction of the *general vectorial product* $x^i\,x^j$, and is a scalar, moreover.

Formally, any tensorial expression presents itself as a polynomial. Tensorial homogeneity demands that all of the indices that are not dummy indices – viz., the *significant indices* – must occur once and only once in each formal monomial, and with the same upper or lower position each time. The reader will have noticed that we have appealed to a rule for the placement of the bar that is similar to the one that is utilized in the formulas for the change of a system of axes, which one calls a *frame*, moreover.

If all of the expressions with indices in a formal monomial that has tensorial validity are tensors except for one of them then one can assert that the latter is a tensor that has obvious variances, as one can effortlessly verify thanks to the formula for the change of a frame. In order to abbreviate the discussion, it is in current usage to say *covariant, contravariant, or mixed tensor,* even though rigor would demand that one must speak of the set of covariant, contravariant, or mixed components of a certain tensor, resp.

I.7. – **Symmetric and antisymmetric tensors. Dual tensor of a completely-antisymmetric tensor.** – A tensor is called symmetric or antisymmetric with respect to a pair of indices i, j (that are both upper or both lower) if one has:

$$T^{ij} = \pm T^{ji} \tag{I.24}$$

identically. Under the hypothesis that the tensor is antisymmetric, one will have $T^{ij} = 0$ identically for $i = j$.

Quite often, from the fact that certain symmetries or antisymmetries are known in a certain frame, one can conclude a more complete symmetry or antisymmetry of the tensor from that. The following chapters will provide examples of the arguments of that type, where at least one of them will imply a conclusion of the greatest physical importance ([1]).

As examples of completely-antisymmetric tensors (i.e., ones that are antisymmetric in all of their indices), one has the exterior products of vectors, whose rank is defined in the following manner: Suppose that one has p linearly-independent vcctors, and that x_j^i denotes the j^{th} covariant component of the i^{th} vector ($i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$). The C_n^p non-zero covariant components of the exterior product of rank p of those vectors are, by definition, the determinants of rank p that are extracted from the matrix:

$$\left\| x_j^i \right\|,$$

and are each affected with a sign. By definition, we denote the components in question in the form:

$$[x_i\, x_j \ldots x_k], \tag{I.25}$$

in which the indices $i, j, \ldots, k$ are all different, and there are p of them. As always by definition, the tensor considered will be said to represent the *hyper-volume of order p* in n-dimensional space of the hyper-parallelepiped that is constructed from the p vectors x_j^i, and its components will measure the *projections* of that geometric entity onto the C_n^p linear varieties of order p that are defined by the coordinate axes. The exterior product of rank n, which has a sign that is defined by the order of the indices, involves just one non-zero component and is said to be a *pseudo-scalar*, which is an expression that will be justified in a moment. The well-known equality from combinatorial analysis:

$$C_n^p = C_n^{n-p} \tag{I.26}$$

shows that two completely-antisymmetric tensors of ranks p and $n - p$ will have the same number of a non-zero components.

All of the non-zero components (contravariant, for example) of a completely-antisymmetric tensor of rank n have the same modulus, and their signs will be + or – according to whether the permutation of indices has even or odd class, resp.; call the

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([1]) Namely: the universal proportionality between energy and mass that was discovered by special relativity.

common modulus of those components Ω. In the case of contravariant components, it will clearly result from what was said about the definition of the $[x^i\, x^j \ldots]$ that under a change of frame, Ω will vary like the evaluation of a hyper-volume of order n with the aid of a hyper-volume ω of the n-parallelepiped that is constructed from the director vectors of the axes, which will be called the *gauge* of the frame used. If $A^{12\ldots n}$ denotes a completely-antisymmetric tensor of rank n then the expression with just one component:

$$\mathcal{A} = \omega A^{12\ldots n} \tag{I.27}$$

will then be a *tensorial invariant* – or *scalar quantity* – that is called the *dual* of the completely-antisymmetric tensor $A^{ij\ldots n}$ of rank n.

More generally, if $A^{mn\ldots z}$ are the contravariant components of a completely-antisymmetric tensor of rank p then the covariant components $\mathcal{A}_{ab\ldots l}$ of its dual, which is completely-antisymmetric and of rank $n - p$, will be defined by:

$$\mathcal{A}_{ab\ldots l} = \varepsilon\, \omega A^{mn\ldots z}. \tag{I.28}$$

There are n indices $a, b, \ldots, l$ and $m, n, \ldots, z$, and they are all different. ε will equal + 1 or – 1 according to whether the *total* permutation $a, b, \ldots, z$ has even or odd class, respectively.

In order to show that the $\mathcal{A}_{ab\ldots l}$ are indeed the covariant components of a tensor, we introduce a completely-antisymmetric auxiliary tensor $B^{ab\ldots l}$ of rank $n - p$ and form the contracted product:

$$B^{ab\ldots l}\, \mathcal{A}_{ab\ldots l} = \omega \sum \varepsilon B^{ab\cdots l} A^{mn\cdots z}\,. \tag{I.29}$$

The sum $\sum$ extends over all permutations of the n indices $a, b, \ldots, z$. The tensor $\sum$ of rank n is completely antisymmetric. Indeed, let i, j be an arbitrary pair of indices. The sum $\sum$ is composed of terms for which i and j occur together in one of the two permutations $a, b, \ldots, l$ and $m, n, \ldots, z$ and terms for which i and j occur in one and the other of those permutations. Each of the terms of the first kind is antisymmetric in i, j and terms of the second kind can be grouped into pairs whose sum is antisymmetric, so the two terms in each pair will differ by only the exchange of the indices i and j. In regard to those pairs of terms, each of them will be reproduced in modulus when one exchanges i and j, and its sign will change because the total permutation $a, b, \ldots, z$ will change class. The tensor $\sum$ will then be completely antisymmetric. Upon comparing formula (29) with (27), one will see that the dual of $\sum$, which is a scalar, is nothing but:

$$B^{ab\ldots l}\, \mathcal{A}_{ab\ldots l}\,,$$

and since $B^{ab\ldots l}$ is a contravariant tensor, the antisymmetric expressions $\mathcal{A}_{ab\ldots l}$ will indeed be the components of a covariant tensor. Q. E. D.

As a particular example of dual tensors, other than the example of a tensor of rank n and an invariant, one has the example of a tensor of rank $n - 1$ and an n-vector. For example, in three-dimensional space, the dual of an exterior product of two vectors is a vector that is defined with an arbitrariness in its sign. That is the origin of the distinction between "polar vectors" and "axial vectors" that is well-known to physicists. In three-dimensional space, the fact that all circular permutations u, v, w of the numbers 1, 2, 3 have even class will make the passage to duals very simple.

In four-dimensional space, which will be that of the theory of relativity, it is easy in practice to distinguish the three indices 1, 2, 3 (u, v, w will denote a circular permutation of them) from the time index 4. Passing to duals then comes about according to the schema, which refers to those notations:

$$\boxed{\begin{array}{ccc} u, v, w \cdots 4 & \text{and} & u, v, \cdots, -w \\ u, v, \cdots w, 4 & \text{and} & w, 4, \cdots, u, v. \end{array}} \tag{I.30}$$

I.8. – **Tensorial derivation in Cartesian axes.** – Let T be a tensor of arbitrary rank that is defined in a spatial domain and is an analytic function of the n-point x_i . When one passes from the point x_i to the point $x_i + dx_i$, T will submit to a certain increase dT, and one will have:

$$dT \equiv \sum_{i=1}^{n} dx_i \frac{\partial}{\partial x_i} T \equiv \sum_{i=1}^{n} dx^i \frac{\partial}{\partial x^i} T$$

identically in any coordinate system, whether rectilinear or curvilinear. If we set, by definition:

$$\boxed{\partial^i \equiv \frac{\partial}{\partial x_i}, \quad \partial_i \equiv \frac{\partial}{\partial x^i},} \tag{I.31}$$

then when we take the summation over dummy indices convention into account, the preceding differential identity can be written:

$$dT \equiv dx_i \, \partial^i \, T \equiv dx^i \, \partial_i \, T, \tag{I.32}$$

or further, in symbolic or operator form:

$$d \equiv dx_i \, \partial^i \equiv dx^i \, \partial_i \, . \tag{I.33}$$

Now, make a change of coordinates (or variables) $x^i - \underline{x}^{i'}$, and let $(\overline{x}^i_{i'})$ denote the partial derivative of x^i with respect to $\underline{x}^{i'}$:

$$(\overline{x}^i_{i'}) \equiv \frac{\partial x^i}{\partial \underline{x}^{i'}} .$$

From elementary analysis, the operator $dx^j \, \partial_i$, for example, transforms by invariance, since one will have:

$$\overline{\partial}_{i'} = (\overline{x}^i_{i'})\,\partial_i$$

and

$$dx^i = (\overline{x}^i_{i'})\,d\underline{x}^{i'}.$$

How, one must once more take into account the expression for T as a function of the new variables. In rectilinear axes of constant measures, one will have simply the expressions that were deduced in (19), in which the coefficients o are constants [with $\overline{o}^i_{i'} \equiv (\overline{x}^i_{i'})$, moreover]. *In our particular case of rectilinear axes of constant measure,* it will then follow that the operators ∂_i will not act upon the o, in such a way that the preceding analysis will exhibit their *variance as a* ***covariant*** *n-vector symbolically;* the same argument will apply to the contravariant symbols ∂^i. That will not be true at all for general curvilinear coordinates. The analysis that one will then carry out, and which will provide the covariant and contravariant definitions of the first-order tensorial derivatives, will constitute one of the essential pieces of the general tensorial calculus [**18**, **19**, **20**, **22**]. Recall, in passing, that the symbolic vector ∂^u is known by the name of *nabla* in three-dimensional space with Cartesian coordinates.

Always in Cartesian axes, one will then see by recursion that the operators $\partial^i \partial^j$ or $\partial_i \partial_j$ have the symbolic variance of tensors of second rank. By virtue of the property of the commutability of partial derivatives, that symbolic tensor will be symmetric. By definition, we set:

$$\partial^{ij} = \partial^{ji} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}, \qquad \partial_{ij} = \partial_{ji} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}. \tag{I.34}$$

The contraction of that second-rank tensorial operator:

$$\partial^i_i \equiv \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x^i} \tag{I.35}$$

is well-known in three-dimensional space by the name of the *Laplacian*, and in the four-dimensional space with $x_4 = ict$ by the name of *d'Alembertian.*

The covariant components, for example, of an arbitrary partial derivative of order p are written:

$$\partial_{ijk\cdots} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}\cdots \tag{I.36}$$

I.9 – **The general formula for the transformation of multiple integrals** ([1]). – Let T be a certain tensor that is defined in a spatial domain, and consider the definite integral that is taken along a curved arc $\mathcal{L}$ between an n-point M_1 and an n-point M_2 :

([1]) In the argument that follows, we assume that all of the variables are essentially real. Meanwhile, upon reading nos. II.3 and II.4, the reader will understand that the formula that is obtained will be valid in pseudo-Euclidian Minkowski space.

(I.37) $$\int_{M_1}^{M_2} \partial_i \, T\, dx^i = T(M_2) - T(M_1).$$

Now, suppose that the tensor T is interpreted physically as a density of order p; i.e., that the integral:

(I.38) $$\iiint \cdots \int_{\mathcal{V}} T[dx_j dx_k \cdots dx_r]$$

that is taken over a variety $\mathcal{V}$ of order $p < n$ that is generally curvilinear is meaningful. The [ ] denotes the general covariant components of the tensor that was defined in no. 8 whose dual represents the hyper-volume of an infinitesimal hyper-parallelepiped of order p that is tangent to $\mathcal{V}$ at the n-point x_i . We finally recall that the hyper-volume is "automatically" referred to its natural gauge; i.e., to the system of volumes of the p-parallelepipeds that are defined by the director vectors of the axes.

Take a simply-connected curvilinear manifold $\mathcal{V}$ of order $p < n$ in the spatial domain considered, and let $\mathcal{C}$ denote the closed contour of order $p - 1$ that surrounds it. Pass a curve $\mathcal{L}$ that is not contained in $\mathcal{V}$ through each of the points of $\mathcal{V}$, and take two points M_1 and M_2 on $\mathcal{L}$ that are distinct from the point of intersection P of $\mathcal{L}$ and $\mathcal{V}$. We assume that the curve $\mathcal{L}$ and the two points M_1 and M_2 will vary continuously when P describes the manifold $\mathcal{V}$, and that M_1 and M_2 will coincide with each other, as well as P, when P varies along $\mathcal{C}$. In that way, the two points M_1 and M_2 will describe a closed manifold of order p, namely, $\mathcal{W}$. Finally, let $\mathcal{D}$ denote a manifold of order $p + 1$ that is bounded by $\mathcal{W}$.

First, argue in orthogonal axes, and take $\mathcal{L}$ to be a line that is parallel to one of the axes (say, the first one i), take $\mathcal{V}$ to be a plane that is parallel to p of the other axes, and finally, take $\mathcal{D}$ to be contained in the planar subspace that is defined by the $p + 1$ axes in question. In order to take into account the order in which those axes are enumerated, we affect their permutations with convenient signs, which are defined up to an arbitrary initial permutation, moreover. We then consider the equality that was deduced in (37) and (38):

(I.39) $$\int_{\mathcal{V}} [dx_j dx_k \cdots] \int_{M_1(P)}^{M_2(P)} \partial^i T \, dx_i = \int_{\mathcal{V}} \{T[M_2(P)] - T[M_1(P)]\} [dx_i dx_j \cdots].$$

By the definition itself, we have the following integral of order $p + 1$ on the left-hand side:

$$\int_{\mathcal{D}} \partial^i T [dx_i dx_j \ldots dx_k].$$

On the right-hand side, we have the difference of two integrals of order p that are taken over $\mathcal{V}$, which is equivalent to a single integral that is taken over $\mathcal{W}$. Since the hyper-surface $\mathcal{W}$ is met by the curves of the congruence $\mathcal{L}$ at only two points, it is natural to orient the "upper sheet" M_2 in the same sense as $\mathcal{V}$ and the "lower sheet" M_1 in the opposite sense, which would imply replacing the – sign with the + sign. Finally, formula (39) is written:

$$\boxed{\int_{(p+1)} \partial^i T[dx_i dx_j \dots dx_k] = \int_{(p)} T[dx_i dx_j \dots dx_k].} \tag{I.40}$$

The integral of order p is taken over a closed domain that is met by the parallel to the i-axis at two points, and the integral of order $p + 1$ is taken over the domain that is enclosed by the preceding one. As in ordinary differential geometry, one is effortlessly liberated from the first restriction, and one extends that formula to more complicated domains. Naturally, nothing will prevent one from raising the indices in [ ] and performing contractions of the indices in the [ ] with the ones in the tensor T. Note, as well, that that $\mathcal{V}$ and the $\mathcal{L}$ have disappeared from the result, so the arbitrariness that has seemed to prevail in their choice has had no consequences, at least as long as they remain contained in the planar subspace that is defined by the $p + 1$ axes that were considered previously.

In order to free ourselves from that restriction, as well as that of the choice of a planar $\mathcal{D}$, we begin by making an arbitrary change of rectilinear axes of constant measure. Those of the components of [ ] that are zero (namely, the ones that contain indices other than those of the $p + 1$ axes that are considered) will cease to be so. Formula (40), which is formally correct from the tensorial viewpoint, will then be automatically completed by terms that are zero. That being the case, one will easily reduce the case of an arbitrary curvilinear $\mathcal{D}$ to that of a planar $\mathcal{D}$ by dividing that domain into sufficiently small pieces and summing them in the well-known manner.

In the course of this book, we will indeed often have the dual of [ ] under the $\int_{(p)}$ sign, instead of [ ]. In that case, the rule to apply will be to *differentiate with respect to the indices that occur in the dual to* [ ], where each term is, of course, affected with a convenient sign.

As an application of the general formula (40), we shall show that *in three-dimensional space*, one in fact recovers the known formulas that are called the Stokes, gradient, divergence, and rotation formulas ([1]) that are classically written as:

$$\begin{aligned}
&\int \mathbf{A}\, d\boldsymbol{l} = \iint \text{rot}\, \mathbf{A} \cdot d\mathbf{s}, \\
&\iint V\, d\mathbf{s} = \iiint \text{grad}\, V \cdot du, \\
&\iint \mathbf{A} \wedge \mathbf{B} \cdot d\mathbf{s} = \iiint \text{div}(\mathbf{A} \wedge \mathbf{B})\, du, \\
&\iint \mathbf{A} \wedge d\mathbf{s} = -\iiint \text{rot}\, \mathbf{A}\, du.
\end{aligned} \tag{I.41}$$

Formula (40) "automatically" gives (u, v, w = 1, 2, 3):

([1]) See, for example, R. BRICARD, *Calcul vectoriel*, Paris, 1932, pp. 140.

(I.42)
$$\int A^v\, dx_v = \iint \partial^u A^v [dx_u dx_v],$$
$$\iint V [dx_v dx_w] = \iiint \partial^u V [dx_u dx_v dx_w],$$
$$\iint A^v B^w [dx_v\, dx_w] = \iiint \partial^u A^v B^w [dx_u dx_v dx_w],$$
$$\iint A^u [dx_v dx_w] = \iiint \partial^u A^v [dx_u dx_v dx_w].$$

The right-hand sides of (42_2) and (42_4) consist of only one non-zero term. The left-hand side of (42_4) consists of only two, and upon introducing the dual ds^w to $[dx_u\, dx_v]$, one can interpret it as an exterior product. In the right-hand sides of (42_1), (42_4), and the left-hand side of (42_3), the antisymmetry of [ ] makes a rotation or an exterior product appear. Finally, that same antisymmetry will appear on the right-hand side of (42_3) as a sum over circular permutations of *u*, *v*, *w* and an exterior product, in succession. Taking that remark into account, (42) will be written:

(I.43)
$$\int A^u\, dx_u = \tfrac{1}{2} \iint (\partial^u A^v - \partial^v A^u) [dx_u dx_v],$$
$$\iint V\, ds^u = \iiint \partial^u V\, du,$$
$$\tfrac{1}{2} \iint (A^v B^w - A^w B^v) [dx_v\, dx_w] = \tfrac{1}{6} \iiint \sum \partial^u (A^v B^w - A^w B^v)\, du,$$
$$\iint A^v ds^u - A^u ds^v = \iiint (\partial^u A^v - \partial^v A^u)\, du.$$

(41) and (43) are, in fact, equivalent, by passing to dual tensors, if necessary.

The simple verification that we just carried can initiate the novice reader to some of calculations that they will find the rest of the book.

I.10 – **The very important particular case of orthogonal axes of equal measure. Orthogonal linear substitutions.** – To conclude with the general considerations, we refer our pseudo-Euclidian space to orthogonal axes of equal measure, or *Cartesian axes in the narrow sense.* The square s^2 of the length of a vector will take one or the other equivalent form:

(I.44)
$$s^2 = g\, \delta^{ij} x_i\, x_j \equiv g\, \delta_{ij}\, x^i\, x^j,$$

in such a way that the values of the covariant and contravariant components of a vector or a tensor will always become equal to each other ([1]). There is then no longer any reason to distinguish between covariant or contravariant components of a tensor. Nonetheless,

([1]) That property is also obvious from Figure 1 and the arguments at the end of no. 6.

for clarity in our formulas, we shall continue to write the dummy summation indices with one above and one below.

Nothing will prevent us from setting:

(I.45) $$g = 1$$

in (44), which amounts to taking the standard of length to be the common measure of all the axes. Under those conditions, the metric tensor (whose general mixed components will be δ_j^i) will admit the Kronecker symbols:

(I.46) $$g_{ij} = \delta_{ij}\,, \qquad g^{ij} = \delta^{ij}$$

for its covariant and contravariant components.

The expression:

(I.47) $$s^2 = \delta^{ij}\, x_i\, x_j$$

can then be written as a sum of squares:

(I.48) $$s^2 = \sum_{i=1}^{n} x_i^2\ .$$

One calls the linear substitutions that preserve that form for s^2 (i.e., the orthogonality of the axes) when one starts with a system of variables such as one has in (47) *orthogonal linear substitutions.* By hypothesis, the coefficients of such a substitution will satisfy the relations:

(I.49) $$\sum_{i=1}^{n} \overline{o}_k^l\, \overline{o}_l^i = \delta_{kl}\,,$$

which, when compared with (21), will show that *an orthogonal linear substitution is characterized by the system of relations* ([1]):

(I.50) $$\boxed{\overline{o}_j^i = \underline{o}_i^j.}$$

It is clear from this that the square of the two determinants $\left|\overline{o}_j^i\right|$ and $\left|\underline{o}_i^j\right|$ has the same value 1, in such a way that those two determinants will simultaneously have the value + 1 or – 1. If they equal + 1 then one will say that the two systems of axes (or the initial and final *n-hedra*) have *the same sense*, and that one passes from one to the other by a *rotation*. If they equal – 1 then one will say that the two *n-hedra* have *opposite senses*, and that one passes from one to the other by a *reflection.*

([1]) These relations are obvious if one refers to the geometrical interpretation of the coefficients $\overline{o}_j^i$ that was given previously.

If one takes $g = 1$, as was said, then the gauge ω of the system of axes (i.e., the hyper-volume of the n-cube that is defined by the director vectors of the axes) will be 1 in modulus, by definition, and if it equals + 1 then the sense of reference *n-hedron* will be *direct*, and one can always enumerate the axes in an order such that the permutation of the indices will have even class. If one eliminates the reflections, in order to keep only the rotations of the reference *n-hedron*, then one can always preserve the gauge $\omega = + 1$.

CHAPTER II

RELATIVISTIC KINEMATICS AND OPTICS

II.1. – As one knows, Minkowski was the author of a remark that is essential to both the comprehension and the development of the theory of relativity. Let x^u, t (u = 1, 2, 3) be a system of four Galilean variables, in the sense that is implied by the Lorentz-Poincaré formulas (no. I.4). If c denotes Einstein's absolute constant then set:

$$x^4 = ict$$

and complete the tri-rectangular trihedron Ox^u to a quadri-rectangular quadrihedron Ox^i (i = 1, 2, 3, 4). Any *event* x^u, t will then be referred to a four-dimensional *diagram*. One is easily assured that the Lorentz-Poincaré formulas express nothing more than a *rotation of the quadrihedron* Ox^i. Thanks to Minkowski, the *kinematic* law of the privileged equivalence of Galilean frames will then take on a perfectly clear interpretation: It is identified with the *privileged equivalence of Cartesian frames in the narrow sense*, which is well-known in Euclidian (or pseudo-Euclidian) geometry. One will also see that the expression:

$$\Delta s^2 = \sum \Delta x_n^2 - c^2 \Delta t^2 ,$$

which is called the *square of the world-interval,* is an invariant of the group in question. That is not the case for either the square of the spatial distance or the square of the temporal interval.

In summary, in the new kinematics, any Galilean frame is a frame that is *no longer just spatial, but also temporal* (Lorentz's notion of *local* or *proper time*). The solidarity between the spatial standard and the time standard that was introduced by Einstein's postulate translates into an effective *physical equivalence* of space and time, and the change of Galilean frame will partially transform one into the other.

Furthermore, the whole time that relativity introduces that novel *equivalence* (which is also quite paradoxical from the classical viewpoint), it also mathematically accounts for the complete disparity that experiments reveal between space and time, such as the characteristic irreversibility of time. Notably, thanks to the distinction that is established between *spacelike quadri-vectors* and *timelike quadri-vectors*, the latter can be divided into two classes according to the sign of the fourth component. The well-known distinction between *past, future,* and *unphysical* (†) forms a body of notions that are corollary to the preceding ones. Those statements, along with some other statements about inequalities such as the one that is implied by the bounded character of the speed c, define various expressions for what one can call the "second principle" of relativity.

True to our rule of always arguing in four-dimensional geometry, we shall deduce Minkowski's kinematical laws directly from Einstein's postulate. It will only be in sub-chapter B, as a prelude to other applications, that we shall show their equivalence with the Lorentz-Poincaré form. The other applications will be concerned with kinematics and

(†) Translator's note: The original French was *d'ailleurs* = "elsewhere."

optics indifferently. For the most part, they will be the ones that are given in the classical treatises. We shall strive to present them in an essentially geometric light.

The ends of sub-chapter B, and above all, sub-chapter C, are dedicated to some general questions that are indispensible if one is to embark upon the various chapters of relativistic physics. We shall insist upon the benefit that can be derived from treating fluids or systems of points in a *non-simultaneous* manner, which was pointed out by E. Cartan; in relativity, that benefit becomes almost a necessity. It is only then that one can insure that the *integral* or *finite* quantities have a tensorial character. Another remark that is closely connected with the preceding one, and also quite important, is that *in relativity, the tensorial ranks of a density quantity and its homologous finite quantity will differ by one unit.* That is due to the quadri-vectorial character of the three-dimensional volume element.

A. – THE NEW EINSTEIN-MINKOWSKI KINEMATICS

II.2. – **The relativistic equivalence of space and time.** – Consider a region of the vacuum that is traversed by light waves, and let x_1, x_2, x_3, t be a system of four Galilean coordinates, in the sense that was specified in no. I.4. Let dx_u (u = 1, 2, 3) be an elementary vector that, from Huygens principle, is traversed by light, and in a time interval dt. If c denotes the velocity of light waves then one will have:

$$\sum_{u=1}^{3} dx_u^2 - c^2\,dt^2 = 0.$$

From Einstein's postulate, c is an absolute constant, and the preceding formula is universally true in any Galilean frame. In order to make the expression on the left-hand side symmetric, with Minkowski, we set:

(II.1) $$\boxed{x^4 = ict,}$$

and when we take the summation convention into account (no. I.5), that will give the left-hand side the form:

(II.2) $$\boxed{ds^2 \equiv dx_i dx^i = 0} \qquad (i = 1, 2, 3, 4).$$

Also with Minkowski, we refer *events* to a *system of four Galilean axes* that are orthogonal and of equal measure $Ox_1\ x_2\ x_3\ x_4$ and call the "continuum" thus-defined *space-time* or the *universe.* The left-hand side of formula (2) represents the square of a world-distance, and by hypothesis, that expression will remain invariant under all changes of Galilean frame. It will then follow that: *Galilean frames are represented in the Minkowski universe by systems of four orthogonal rectilinear axes of equal measure – or Cartesian quadrihedra, in the narrow sense – and only by those systems.* The notion of *privileged equivalence of Galilean frames,* which is well-known in classical dynamics and extended universally to physics by means of relativity, will then take on a simple and clear interpretation that is attached to the privileged equivalence of Cartesian frames in the narrow sense that is well-known in geometry.

Finally, if we take into account what we recalled in Chapter I, § B then the formulas for the change of Galilean frame in the Minkowski universe will be those of an orthogonal linear substitution:

$$\text{(II.3)} \qquad \boxed{\bar{x}^i = \bar{o}^i_j\, x^j, \quad x^i = \underline{o}^i_j\, \bar{x}^j,} \qquad \boxed{\bar{o}^i_j\, \underline{o}^j_k = \underline{o}^i_j\, \bar{o}^j_k = \delta^i_k,} \qquad \boxed{\bar{o}^i_j = \underline{o}^j_i.}$$

With such substitutions, the covariant or contravariant coordinates of a vector – or, more generally, a tensor – will always be equal to each other, and there will be no reason to be preoccupied with distinguishing them. Nevertheless, in the name of elegance in the formulas, as well as in view of their possible extension to general relativity, we shall be obliged to respect the rules of tensorial homogeneity rigorously.

It is obvious that the system of transformations (3) preserves the form of the d'Alembertian operator ∂^i_i, whose classical expression is:

$$\square \equiv \sum_{u=1}^{3} \frac{\partial^2}{\partial x_u^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2},$$

and that, reciprocally, we can conclude formulas (3) by arguing with the d'Alembertian as we did with $ds^2 \equiv dx_i\, dx^i$.

The preceding set of formulas express various aspects of the *equivalence* of the notions of space and time that relativity established. That *equivalence*, which is implied from the outset by the universal coupling of the standards of space and time [formula (1)], translates into a partial reciprocal transformation of space and time into each other under a change of Galilean frame [formula (3)], and also by the introduction of a synthetic notion that encompasses the notions of spatial interval and temporal interval in a quadratic form, namely:

$$\text{(II.4)} \qquad \Delta s^2 \equiv \Delta x^i\, \Delta x_i\,,$$

whose expression in classical terms will be:

$$\Delta s^2 = \sum_{u=1}^{3} \Delta x_u^2 - c^2\, \Delta t^2.$$

Note that if the Δx_u and Δt are real then Δs^2 will be real, and it can be positive, negative, or zero. By definition, the quadri-vector Δx_i will be called *spacelike*, *timelike*, or *isotropic*, respectively, in those three cases. The profound significance of those distinctions will appear in the following no.

For the readers who are not familiar with the theory of relativity, we shall insist a bit on the new situations that will appear in Einstein-Minkowski kinematics that are consequences of rejecting the notions of *absolute space* and *absolute time*. Two *events* that happen simultaneously at two distant points $x_{(2)u}$ and $x_{(1)u}$ in a certain Galilean frame are represented in the universe by two instant-points $x_{(2)i}$ and $x_{(1)i}$ in such that the quadri-vector $x_{(2)i} - x_{(1)i}$ is orthogonal to the axis Ox_4. That property is not *absolute*, but *relative* to the spatio-temporal frame that is utilized, so it is not preserved under a change of Galilean frame, in general. That amounts to saying that, in general, two *events* will no longer be simultaneous in any Galilean frame besides the initial one, which is a situation that was ignored completely by classical kinematics. Similarly, two *events* that happen at the same point x_u in space at two successive instants t_1 and t_2 in a certain Galilean frame will be represented in the universe by the extremities $x_{(1)i}$ and $x_{(2)i}$ of a quadri-vector that is parallel to the axis Ox_4. Since that property is *relative*, two successive events will not take place at the same point in the other Galilean frames, which is a situation that already happens in classical kinematics.

More generally, the spatial distance, as well as the temporal interval, is a *relative* notion in relativity. By contrast, the *tensorially-invariant spatio-temporal interval*, whose expression is (II.4), will be absolute. The spatial distance between two points that was first considered in the preceding paragraph will increase, in general, when one changes the Galilean frame, and under the same conditions, the temporal interval of the two events considered after that will also increase.

II.3. – **Mathematical expressions for the distinction between space and time. Interpretation of the** o_j^i**.** – The relativistic discovery of an *equivalence* of space and time is certainly quite troubling. One must demand to know if – and how – the radical distinction between space and time that our perception establishes will be preserved in the formulas, as well as the irreversibility of things that is present in time, but not space.

As a result of Einstein, most authors have written that the manner in which relativity accounts for the difference in the physical behavior of space and time is by way of the negative sign on the temporal square in the Δs^2. We shall show that, in fact, the special character of time follow from that, as well as several corollaries.

If the constant c remains essentially real then formulas (1), (3), and (4) will just as well be true in complex variables. We shall now infer the detailed consequences of the fact that the four physical variables x_u and t are real, and consequently, that the three x_u of Minkowski are essentially real, and x_4 is pure imaginary.

Under those conditions, formulas (3) show that *the coefficients of the change of Galilean frame will be essentially real if they contain the index* 4 *either zero or two times, and imaginary if they contain it once.* That being the case, one will have, in particular:

(II.5) $$(o_4^4)^2 = 1 - \overline{o}_4^u \, \underline{o}_u^4 , \qquad (\overline{o}_4^4 = \underline{o}_4^4 = o_4^4 , \; \overline{o}_4^u = \underline{o}_u^4),$$

and since the sum $\overline{o}_4^u \, \underline{o}_u^4$ is essentially negative, one will conclude that:

$$(o_4^4)^2 \geq 1,$$

and consequently:

$$\text{either} \qquad o_4^4 \leq -1 \qquad \text{or} \qquad o_4^4 \geq +1.$$

Hence, as in thermodynamics, a *second principle* is expressed by the inequalities that restrict the possibilities that are offered by the *principle of equivalence* in relativity. The initial group of transformations (3) has now been split into two completely distinct families.

If one then postulates (as would seem natural) that *the transformations that remain permissible must again form a group* then the first family, which does not contain the identity transformation, will be found to have been eliminated. On the contrary, the second family, which does contain the identity transformation, does indeed form a group, since, as one can effortlessly see, the transformation (o^i_j) will produce two transformations (o'^i_j) and (o''^i_j), such that $o'^4_4 \geq 1$ and $o''^4_4 \geq 1$, so $o^4_4 \geq 1$.

Finally, in addition to the conditions that were formulated already in (3), the o^i_j of a change of Galilean frame must satisfy the inequality:

$$\text{(II.6)} \qquad \boxed{o^4_4 \geq 1.}$$

First, take the hypothesis that $o^4_4 = 1$. From (5), the $\bar{o}^u_4 = \underline{o}^4_u$ will then be zero, and the transformation will reduce to:

$$\text{(II.7)} \qquad t' = t, \qquad \bar{x}^u = \bar{o}^u_v x^v \qquad (u, v = 1, 2, 3),$$

which is interpreted as a simple rotation of a spatial trihedron, and not as a change of Galilean frame, properly speaking. In one blow, the interpretation of the nine coefficients $\bar{o}^u_v = \underline{o}^v_u$ is found to be given in the particular case, as well as the general one (at least, to a first approximation).

In order to study a true change of Galilean frame, suppose that $o^4_4 > 1$. In order to "follow the motion" of the origin o of the spatial axes of the frame (x_i), set $x_u \equiv 0$ (u = 1, 2, 3) in (3). One will then have:

$$\text{(II.8)} \qquad \bar{x}^u = \bar{o}^u_4 x^4, \qquad \bar{x}^4 = \bar{o}^4_4 x^4.$$

If one substitutes x^4 from (8_2) into (8_1) then one will get the *equations of motion of the spatial origin o with respect to the Galilean frame* $\bar{x}^i$:

$$\text{(II.8}') \qquad \bar{x}^u = \frac{1}{o^4_4}\bar{o}^u_4 \bar{x}^4 \equiv \frac{ic}{o^4_4}\bar{o}^u_4 \bar{t}\,.$$

One will then see that the three real quantities $\frac{ic}{o^4_4}\bar{o}^u_4$ (or $\frac{ic}{o^4_4}\underline{o}^u_4$, for that matter, due to the equality of the covariant and contravariant components) are nothing but the components of the velocity of the spatial origin o of the "moving" Galilean frame (x^u, t) in the "fixed" fame $(\bar{x}^u, \bar{t})$. In relativity, it is customary to set:

(II.9) $$\boxed{\beta^u \equiv \frac{1}{c} v^u} \qquad (u = 1, 2, 3),$$

in such a way that one will finally write:

(II.10) $$\bar{v}^u = \frac{ic}{o_4^4}\bar{o}_4^u, \qquad \bar{\beta}^u = \frac{i}{o_4^4}\bar{o}_4^u.$$

Hence, *the three components* $\boldsymbol{\beta}$ *of the* ***reduced*** *velocity are interpreted as the three* ***tangent*** *directors or* ***angular coefficients*** *of the axis* Ox_4 *in the quadrihedron* $O\bar{x}^i$.

Now suppose that the matrix of the nine $\bar{o}_v^u$ is diagonal, which amounts to taking the spatial axes to be "parallel" to each other, in the common language of space and time ([1]). Formulas (3_2) will then show that:

(II.11) $$\bar{o}_4^u = -\bar{o}_u^4$$

(the + sign is excluded, due to the fact that the matrix of sixteen o_j^i for a rotation cannot be symmetric). Since $\bar{o}_4^u \equiv \underline{o}_4^u$, one will conclude that:

(II.12) $$\bar{o}_4^u = -\underline{o}_4^u, \qquad \bar{v}^u = -v^u.$$

The speed of the origin $\bar{o}$ *of the spatial axes of the system* $O\bar{x}^i$ *with respect to the system* Ox^i *is a vector that is "equal to"* – $\mathbf{v}$, in the usual language of space.

We finally show that *the coefficient* o_4^4 *relates to the absolute value of the reduced velocity*. If we form the spatial scalar product $\beta_u\,\beta^u \equiv \boldsymbol{\beta}^2$ and take (8) into account once more then we will get:

(II.13) $$\beta^2 = \frac{(o_4^4)^2 - 1}{(o_4^4)^2} \qquad \text{or} \qquad \boxed{o_4^4 = \frac{1}{\sqrt{1-\beta^2}}.}$$

Since o_4^4 is real, one can conclude the following important inequality from that:

(II.14) $$\boxed{\beta^2 \leq 1} \qquad \text{or} \qquad \boxed{v^2 \leq c^2.}$$

The relative velocity of the spatial origins of two Galilean frames that are "physical" or "real" is always less than the universal constant c in modulus. We already see the

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([1]) None of the axes Ox^u in the universe is parallel to any of the $O\bar{x}^u$. However, the axes ox^u and $\bar{o}\,\bar{x}^u$ can be called "parallel," in the common sense of the term, because if the Galilean observer (x^i), for example, considers all of the material points that are found along the axis $\bar{o}\,\bar{x}^u$ then they will be aligned along the axis ox^u for him.

constant c appear in kinematics here by way of its role as an upper limit on physical velocities.

We then write the ds^2 of the axis Ox^2 of the "moving" Galilean frame Ox^i in the "fixed" quadrihedron $O\overline{x}^4$. We have:

$$ds^2 = dx^u\, dx_u - c^2\, dt = (v^2 - c^2)\, dt^2 \leq 0, \tag{II.15}$$

which shows that *the temporal axes of physical Galilean frames are all timelike*, and consequently, *their spatial axes will all be spacelike.* We then see the appearance of a more concrete interpretation of the distinction between a *timelike quadri-vector and a spacelike quadri-vector* that was pointed out in the preceding no. A characteristic property of the former kind of quadri-vector that exists a certain "proper" Galilean frame in which its three spatial components are zero, and a characteristic property of the latter kind is that there exist certain Galilean frames in which the temporal component is zero. Three linearly-independent spacelike quadri-vectors define a unique proper Galilean frame (up to a rotation of the spatial axes), namely, the one in which the temporal axis is orthogonal to the other three, and the temporal components of the latter are all zero. Two instant-points – or events – that can be localized at the same spatial point will define a timelike quadri-vector, and two events can appear to be simultaneous will define a spacelike quadri-vector; the converses are obvious.

As a result of the restricting hypothesis that translates into the inequality (6), one sees that *the sign of the temporal component* ([1]) *of a timelike quadri-vector is the same in every physical Galilean frame, and that property is characteristic.* If one is given an instant-point O as an origin and considers the two sheets of the cone that has that point for its summit and isotropic generators (in the sense that was defined no. 3) then the universe will be found to have been divided into three regions: The interior of the upper sheet contains the extremities of timelike quadri-vectors whose temporal component is positive, and is called the *future region*. The interior of the lower sheet contains the extremities of the timelike quadri-vectors with negative temporal components, and is called the *past region*. The exterior of the two sheets contains the extremities of the spacelike quadri-vectors, and is called the *unphysical region.*

In particular, the temporal axes of all physical Galilean frames are oriented into the future. That will permit one to say that *time flows in the same sense in any real Galilean frame*. Thus, as was said before, in relativity, one finds, at the same time, a very neat distinction between the concepts of *space* and *time*, which are now depicted as *equivalent*, and a mathematical formulation of the irreversibility of time. The preceding statements, as well as the inequalities (6) and (14), constitute various consequences of something that deserves the name of the *second principle of relativity*. That principle was initially stated with the aid of the formal demands of physical reality and the fact that the transformations must define a group.

Remark: The quantities o_4^4 and β are related to the angle θ between two temporal axes by the formulas:

$$\cos\theta = o_4^4 \geq 1, \qquad \beta = -\,i\tan\theta. \tag{II.16}$$

([1]) When that expression is applied to a pure imaginary quantity, its sense will be self-referent.

(16_2) results from (13_2). The *angle* θ *is pure imaginary.*

II.4. – **Some other relativistic principles.** – We shall now state some principles that will prepare us for the passage from kinematics to relativistic physics.

1. Certain phenomena that are ideally capable of being perfectly localized as points in space and time have the property that they can be followed in the course of time. Their other properties evolve in a sufficiently continuous manner that one can recognize the "same" phenomenon at successive instants. As very important examples of those "persistent phenomena," one has the classical *material point* that is endowed with mass, as well as the classical fluid *molecule*, or a geometrical material point of that fluid in its motion.

In the universe, a *persistent point-like phenomenon* describes a *trajectory* $\mathcal{T}$ that is curvilinear in the general case of an accelerated motion. Now, it is an experimental fact that has been made into a *fundamental postulate* by the theory of relativity that there is always an objective Galilean frame in which the persistent phenomenon is at rest at a given instant. It is the *proper Galilean frame,* which is also called *comoving* or *tangent*, because in ordinary space, its motion will be tangent to that of the point-like phenomenon in the sense of rational mechanics, and its temporal world-axis will be geometrically tangent to the trajectory $\mathcal{T}$ of the phenomenon.

Hence, *any persistent point-like phenomenon is, at each instant, the virtual origin of a Galilean trihedron in objective space.* It follows that all of the properties that are proved for the origins of Galilean trihedra will extend to persistent phenomena, and notably that:

α) *The speed of propagation of any objective phenomenon cannot exceed the value c.*

β) *The world-trajectories* $\mathcal{T}$ *of objective phenomena are timelike, and their curvilinear abscissas are constantly-increasing functions.*

2. Now imagine a point-like observer. It is a "persistent phenomenon" that is endowed with the preceding properties and which possess a conscience, moreover. Relativity poses the *fundamental postulate* that the framework of space and time to which the *point-like observer* spontaneously refers the phenomena that he is aware of must be the *tangents* and the *normal three-dimensional hyperplanes*, respectively, to his world trajectory. Any material point is considered to be occupied by an observer, and one says *proper time* and *proper space* of the material point to refer to the preceding two linear varieties, which are subtended by the temporal axis Ox^4 and the three spatial axes Ox^u, respectively, of the instantaneously-comoving Galilean frame. The *duration of the accelerated material point* is obtained by integrating over proper time $d\mathcal{T}$, or what amounts to the same thing, the curvilinear abscissa:

$$ds = ic\, d\tau. \tag{II.17}$$

3. Relativistic physics can be interpreted in terms of four-dimensional geometry, and its equations will refer to space-time tensors, vectors, and scalars. Of course, the usual

relations must be recovered from the new relations that can sometimes be predicted, at least in the first approximation. In any event, the various algebraic components of a tensorial relation must all be physically interpreted in such a manner that the quantities implied are presented with the desired real character. The latter result will be achieved as long as one respects the following *rule: Any component of a world-tensor will be real if it contains the index* 4 *zero, two, or four times, and pure imaginary if it contains it one or three times.* In particular, *the homologous components of two dual antisymmetric tensors will be real in one case and pure imaginary, in the other.*

B. – VARIOUS APPLICATIONS OF AND EXPLANATIONS FOR THE NEW KINEMATICS

II.5. – **The Lorentz-Poincaré formulas. Minkowski's hyperbolic universe. –** For the sake of completeness, we return to the considerations that permitted us to write formulas (8), (10), (13). Upon assuming that the spatial axes are mutually "parallel," in the language of ordinary space, and taking the diagonal matrix of nine o_v^u into consideration, we can write (3), *without the summation convention*, in the form:

$$\bar{x}^u = \bar{o}_u{}^u x^u + \bar{o}_4{}^u x^4 = \sqrt{1-(\bar{o}_u{}^u)^2}\, x^u + \bar{o}_4{}^u x^4 ,$$
$$\bar{x}^4 = -\sum \bar{o}_4{}^u x^u + \bar{o}_4{}^4 x^4 .$$

The sign in front of the $\sqrt{\;}$ will be + if the homologous axes have the same sense. Taking (10) and (13) into account, those equations can be written:

$$\text{(II.18)} \qquad \bar{x}^u = \frac{x^u\sqrt{1+\beta^{u2}-\beta^2}+v^u t}{\sqrt{1-\beta^2}}, \qquad \overline{ct} = \frac{ct+\boldsymbol{\beta}\cdot\mathbf{x}}{\sqrt{1-\beta^2}};$$

the inverse equations are obtained by changing the $\mathbf{v}$ (or $\boldsymbol{\beta}$) into $-\,\mathbf{v}$ (or $-\,\boldsymbol{\beta}$).

In the case where one takes the spatial axes ox^u and $\bar{o}\,\bar{x}^u$ to be "parallel" to the relative velocity, in addition to the preceding hypotheses, one will have:

$$\beta^v = -\,\bar{\beta}^v = 0, \qquad \beta^w = -\,\bar{\beta}^w = 0,$$

so (18) and their inverses will take on the form that has become quite familiar today:

(II.19)
$$\begin{cases} \bar{x}^u = \dfrac{x^u + v^u t}{\sqrt{1-\beta^2}}, & x^u = \dfrac{\bar{x}^u - \bar{v}^u t}{\sqrt{1-\beta^2}}, \\ \bar{x}^v = x^v, & x^v = \bar{x}^v, \\ \bar{x}^w = x^w, & x^w = \bar{x}^w, \\ \bar{t} = \dfrac{-t + v\, x^u / c^2}{\sqrt{1-\beta^2}} & t = \dfrac{\bar{t} - v\,\bar{x}^u / c^2}{\sqrt{1-\beta^2}}. \end{cases}$$

As we said, those equations were obtained from Lorentz and Poincaré in their interpretation of Michelson's experiment.

In some applications of the preceding "planar problem," it is advantageous to utilize some formulas that make the relativistic symmetry between space and time more apparent than (19) do, and meanwhile refer to nothing but real variables. To that end, it is natural to set:

(II.20)
$$x = x^u, \qquad y = ct$$

and

(II.21)
$$o_1^1 = o_4^4 = \cosh\varphi, \qquad \bar{o}_1^1 = \bar{o}_4^4 = -\underline{o}_1^1 = -\underline{o}_4^4 = \sinh\varphi,$$

which will lead to the formulas:

(II.22)
$$\begin{cases} \bar{x} = x\cosh\varphi + y\sinh\varphi, \\ \bar{y} = x\sinh\varphi + y\cosh\varphi, \end{cases} \qquad \begin{cases} x = \quad \bar{x}\cosh\varphi - \bar{y}\sinh\varphi, \\ y = -\bar{x}\sinh\varphi + \bar{y}\cosh\varphi. \end{cases}$$

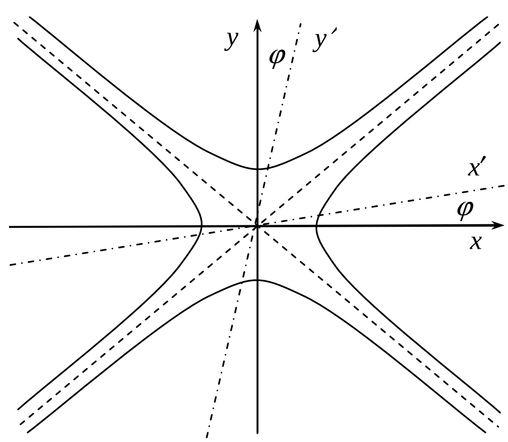

Figure 2.

If one then considers two equilateral hyperbolas that have the same asymptotes and the same moduli (which is taken to be 1) for their real semi-axes then each pair of Galilean axes will be represented by a pair of conjugate diameters, and the director vectors of each axis will describe the corresponding hyperbola. A double mesh of such hyperbolas will provide the standard of length for the spacelike and timelike world-vectors, and a double mesh of their conjugate diameters will likewise provide a standard for the measurement of the angle between two world-vectors.

That hyperbolic model for the universe is advantageous whenever one must effectively draw a figure. One explicitly recovers the three *past, future, and unphysical* regions that were defined previously. One has the relations:

(II.23) $$\beta = \tanh \varphi, \qquad \theta = i\,\varphi$$

between the real parameter φ in (21) and (22) and the parameter θ that is defined by (16).

II.6. – **The slowing of clocks. Lorentz contraction. An expression for the four-dimensional volume element. –** Consider two successive events that happen at the same material point along its motion; for example, two successive beats of a tiny point-like *clock*. In the proper Galilean frame $O\bar{x}^i$, they will be separated by a certain proper time interval $\Delta t_0 = \Delta\bar{t} = \Delta\bar{x}^4 / ic$. One assumes that the time interval Δt_0 is sufficiently brief that the corresponding arc of the world-trajectory can be considered to be a small line segment. In any Galilean frame Ox^i, with respect to which the material point has the reduced velocity β at the mean proper instant considered, the projection of the proper interval Δt_0 onto Ox^4 will have the value:

(II.24) $$\Delta t = o_4^4\ \Delta t_0 = \frac{\Delta t_0}{\sqrt{1-\beta^2}}.$$

Δt is greater than Δt_0, because $o_4^4 = \cos\theta$ is greater than unity.

That phenomenon of *the slowing-down of clocks by their motion*, which is a phenomenon that is *relative* to the Galilean reference system, is one of the most direct consequences of the new kinematics. It manifests itself quite neatly in the elongation of the mean lifetime of the *meson* in cosmic rays [**85**, **86**].

Now consider two material points that are simultaneously at rest at two distant points $\bar{p}$ and $\bar{q}$ in a certain Galilean frame $O\bar{x}^i$; for example, they might be the extremities of a *yardstick*. Those two world-points will generate two rectilinear trajectories that are parallel to the axis $O\bar{x}^4$, and their simultaneous spatial manifestations p and q in an arbitrary Galilean frame Ox^i will be the traces of those lines in a hyperplane x^4 = const. Since $o_4^4 = \cos\theta$ is greater than 1, the vector $\overrightarrow{q-p}$ will have a length that is less than or equal to that of $\overrightarrow{\bar{q}-\bar{p}}$.

Decompose the vector $\overrightarrow{q-p}$ into components that are "parallel" and "perpendicular" to the translation velocity **v**. By arguing four-dimensionally, one will effortlessly see that the transverse component has the same length as it did at rest, and the longitudinal component will be contracted by the ratio:

(II.25) $$\Delta x = \Delta\bar{x}\sqrt{1-\beta^2}.$$

For example, a body that presents itself as a motionless rigid sphere in a certain *proper* Galilean frame will appear to be an ellipsoid that has flattened in the direction of motion by the ratio of $\sqrt{1-\beta^2}$ in any other frame. If the velocity of the body can attain the value c then it will reduce to a disc that is normal to the displacement.

That is the contraction of material lengths that Lorentz and Michelson quite justifiably imposed upon the elements that comprised the Michelson interferometer, when it is explained by a purely kinematical theory, and it can be considered to be verified directly by the experiment in question. In Lorentz's original way of thinking, that contraction, which was assumed to be absolute, was attributed to a physical effect of the ether wind. However, due to the fact that the Lorentz formulas, when completed by Poincaré, will become those of group, that way of thinking is not admissible, because the Lorentz contraction will become *reciprocal*, and consequently, *relative*. Two Galilean observers that are animated with respect to each other with a velocity **v** and each make use of two equal rules of measurement when one juxtaposes them at rest will each find that the other's ruler has been shortened by the ratio $\sqrt{1-\beta^2}$ when they compare the lengths in motion when they have been placed "parallel" to **v**.

Now consider a material droplet inside a moving fluid, and let δu denote the evaluation of a volume that is made at a given instant in a certain Galilean frame. Let δu_0 be the "proper" value of δu; i.e., the value that δu takes in the Galilean frame that follows the mean point of the droplet. Moreover, let dt be the evaluation of a certain "proper time duration" dt_0 that is attached to the mean molecule of the droplet. It results from the preceding that if δu and dt vary according to the law:

$$dt = \frac{dt_0}{\sqrt{1-\beta^2}}, \qquad \delta u = \delta u_0 \sqrt{1-\beta^2} \tag{II.26}$$

then the product $\delta u\, dt$ will be a relativistic invariant. Obviously, $ic\ \delta u\, dt$ is the evaluation of the elementary four-dimensional volume element $[dx_1\, dx_2\, dx_3\, dx_4]$, or more precisely, its dual $ic\ \delta\omega$:

$$\boxed{\delta u\, dt = \delta\omega \equiv \frac{1}{ic}[dx_1\, dx_2\, dx_3\, dx_4].} \tag{II.27}$$

In the course of this book, we will often appeal to that formula.

II.7. – **The composition of velocities in relativity. The Fresnel-Fizeau law of dragging.** – Recall that *the laws of classical kinematics are integrally conserved by special relativity inside a given Galilean frame.*

The present problem is the following one: If one is given three spatial Galilean trihedra (I), (II), (III), and one knows the velocities $\mathbf{v}_2 = \mathbf{v}$ (III, II) and $\mathbf{v}_1 = \mathbf{v}$ (II, I), then find the velocity $\mathbf{v} = \mathbf{v}$ (III, I). Formulas (3) or (19) permit one to treat that problem completely, and one confirms that the relativistic correction has second order in $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ in comparison to classical kinematics. Here, we shall treat the case of parallel velocities

$\mathbf{v}_1$ and $\mathbf{v}_2$, which leads to a very interesting physical application that was pointed out by Von Laue [**91**].

The composition of two world-velocities relative to a Galilean trihedron translates into the "product" of two quadrihedral rotations whose magnitudes are defined by the angles θ_1 and θ_2, which are coplanar here. One will then have:

$$\tan\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2},$$

and thus, the *relativistic law for the composition of parallel velocities:*

(II.28) $$v = \frac{v_1 + v_2}{1 + \beta_1\beta_2} = (v_1 + v_2)(1 - \beta_1\beta_2 + \ldots).$$

Naturally, that law is symmetric in v_1 and v_2, and it will coincide with the classical law up to second order in β_1 and β_2.

If one of the velocities – for example, v_1 – equals c then one will find that $v = c$, as would be necessary *a priori*. *Hence, any velocity, when composed with the limiting speed c, will once again give the limiting speed c.*

Now suppose that the velocity v_2 is the speed of propagation of light in an isotropic material medium of index n that is at rest in a certain Galilean frame and is naturally taken to be a reference rigid body for the evaluation of that velocity. From classical optics, one has:

$$v_2 = \frac{c}{n}$$

in every direction.

Now, if $v_1 = v$ denotes the speed of translation of the medium with respect to an arbitrary Galilean frame then the longitudinal values of the speed of light relative to the latter frame will be, from (28):

$$c' = \frac{c/n \pm v}{1 \pm v/nc} = \left(\frac{c}{n} \pm v\right)\left(1 \mp \frac{v}{nc} + \cdots\right).$$

Upon retaining only the first two terms, one will recover the celebrated law that was stated by Fresnel on the basis of Arago's experiment and verified directly by Fizeau later on:

(II.29) $$\boxed{c' = \frac{c}{n} \pm v\left(1 - \frac{1}{n^2}\right).}$$

According to Fresnel, the term v / n^2 accounts for the *dragging of the ether* by the transparent body. Basically, *the old experiments of Arago and Fizeau, even before that of Michelson, were experiments that virtually revealed the laws of relativistic kinematics, and were justly entitled to a purely kinematical explanation.* Fresnel's idea of the dragging of the ether by transparent bodies, like Lorentz's idea of the contraction of length by the ether wind, were only provisional expedients, and not very satisfying in their own right *a priori*. Each of them, while asserting the existence of the *ether*, had the goal and the effect of rendering it experimentally inaccessible.

II.8. – **Some words about the kinematics of rotating rigid bodies** ([1]). –

1. To begin with, consider a continuous set of material points that define a rigid body $\mathcal{P}$ that rotates with a uniform angular velocity ω around a fixed axis when it is referred to a certain Galilean system $\mathcal{G}$. To abbreviate the discussion, we shall call that rotating rigid body a *planet*. We demand to know what kind of geometry (or, if one prefers, *geodesy*) that the *inhabitants* of that planet (which are assumed to be point-like) will construct for themselves.

First of all, the *planet* $\mathcal{P}$ will be considered to be a rigid body by its *inhabitants*; indeed, let p be a point that follows it. In the spatial domain of the inhabitant that is stationed at p, take three local axes $p\xi_1$, which is *radial*, $p\,\xi_2$, which is *tangential*, and $p\xi_3$, which is parallel to the axis of rotation, or *vertical*. The standards of length $p\xi_1$ and $p\xi_3$ for that inhabitant will be the same as they are for the Galilean observer that regards the planet as turning. By contrast, from the fact of Lorentz contraction, the standard $p\xi_2$ will be $1 / \sqrt{1-\beta^2}$ times greater for the inhabitant of the planet than it is for the Galilean observer. One has:

$$\beta = \frac{\omega r}{c}, \tag{II.30}$$

in which r denotes the radius of the *parallel* that is described by p when it is evaluated in the Galilean frame $\mathcal{G}$. Since, on the one hand, the three spatial axes $p\xi_1\xi_2\xi_3$ move with $\mathcal{P}$, and on the other hand, the Lorentz contraction constantly affects only the axis $p\xi_2$, in the same way, the matter of the planet will indeed seem to be indeformable to the inhabitant p. He cannot exhibit the rotation of his planet by purely local geodesic operations. Nonetheless, from the principles of relativity, that rotation will certainly have some absolute kinematical consequences.

Now suppose that our *inhabitant* sets about the step-by-step measurement of the length and radius of a parallel on his planet, while provisionally assuming that he has characterized that parallel, moreover. He will find:

$$l = \frac{l_0}{\sqrt{1-\beta^2}} \tag{II.31}$$

for the length of that parallel, in which l_0 denotes the length that is measured by the Galilean observer. Similarly, he will find:

([1]) For a theory that is phrased in terms of general relativity, the reader can refer to P. LANGEVIN, C. R. Acad. Sci. **173** (1921), pp. 831; *ibid.* **200** (1935), pp. 48; *ibid.* **205** (1937), pp. 304.

$$r = r_0 \tag{II.32}$$

for the radius. Since one will certainly have:

$$l_0 = 2\pi r_0$$

in the Galilean frame $\mathcal{G}_0$, the inhabitant of $\mathcal{P}$ will find the relation:

$$l = \frac{2\pi r}{\sqrt{1-\omega^2 r^2 / c^2}} \tag{II.33}$$

between the length and the radius of a parallel. Hence, *the geometry that is valid on the planet is not Euclidian* (Ehrenfest and Pfaff), and one will see that *some extended geodesic operations that are performed on the planet will permit its inhabitants (who are assumed to be familiar with the theory of relativity) to determine the axis and the angular velocity of the absolute rotation.*

But that is not all: Complete the trihedron $p\xi_u$ with the temporal axis $p\xi_4$ that is tangent to the world-helix that is described by p. *Since the congruence of helices p does not admit an orthogonal trajectory, there exists no system of world-coordinates that are coupled with the planet and realize the separation of space and time in an extended way.* It is obvious that a coordinate system ξ_i will be called *coupled with the planet* if the three equations ξ_u = const (u = 1, 2, 3) define a world-trajectory for a point of the planet. It follows from this that *there is no time that is valid for the entire planet collectively*, but only *a local time* at each point p. As a corollary, one sees that *relativity rejects the classical notion of a rotating body as an absurdity*.

We leave to the reader the task of verifying that if two inhabitants of the planet traverse the same parallel in the opposite senses with the same speed w, which is measured with respect to the Sun with the aid of their respective chronometers, then when those inhabitants return to their common starting point p_0, they will disagree with each other by:

$$\Delta l = 2 l_0 \frac{\omega r w}{c^2}$$

in regard to the length of the parallel, such that the excess will be found by the one that travels to the east, and their returns to p_0 will no longer be simultaneous, but will differ by:

$$\Delta t = \frac{\Delta l}{w} \qquad \text{or} \qquad \Delta t = \frac{4\mathcal{A}w}{c^2},$$

in which $\mathcal{A}$ denotes the area of the parallel that is evaluated by the Galilean observer.

2. Another important problem is concerned with the kinematics of a rotating rigid body: What will happen when one takes a rigid body $\mathcal{P}$ that is initially at rest in a certain

Galilean frame $\mathcal{G}$, and one progressively puts it into uniform rotation in that same frame? Let l^0 and r^0 denote the length and radius, respectively, of a future parallel of $\mathcal{P}$ when evaluated in $\mathcal{G}$, so one will have:

$$l^0 = 2\pi r^0.$$

Now put the body $\mathcal{P}$ into rotation. *When viewed in* $\mathcal{G}$, each element of the parallel will be subject to the Lorentz contraction *if no tension is applied to its extremities*, and the circumference will then collectively suffer a dislocation at any arbitrarily fine scale. However, one must naturally think that the cohesion of matter would be opposed to such a phenomenon. *Putting* $\mathcal{P}$ *into rotation must then cause a tension to appear along each parallel,* which is a tension that will be easy to calculate as a function of the corresponding elastic modulus of the matter, as well as the formula for the Lorentz contraction. Those tensions of relativistic origin, whose value will increase with β, and therefore with r, will tend to shorten the circumference of the parallels, and thus, to contract their radii r. *They will then induce radial, centripetal forces* according to the process that is well-known in artillery by the name of *shrinking* (*frettage*). If the matter of $\mathcal{P}$ possesses neither radial cohesion nor mass density (which is an entirely schematic case) then no force could oppose the tensions considered, which will cancel themselves after having transformed the initial length l^0 into:

$$l_0 = l^0\sqrt{1-\beta^2}\,, \tag{II.34}$$

and consequently, the initial radius r^0 into:

$$r_0 = r^0\sqrt{1-\beta^2}\,. \tag{II.35}$$

With that schematic hypothesis, one will then have:

$$l = l^0, \qquad r = r^0\sqrt{1-\beta^2}\,, \tag{II.36}$$

in which l and r always denote the results of *surveys* that are made by an *inhabitant* of $\mathcal{P}$. We also remark that if one takes the expression (30) for β into account then one will have:

$$|\,\omega r\,| = \frac{c}{\sqrt{1+\dfrac{c^2}{\omega^2 r_0^{\,2}}}} < c, \tag{II.37}$$

which shows that *upon communicating an arbitrarily large angular velocity to a rigid body* $\mathcal{P}$ *that is initially arbitrarily large, the linear speed will remain less than c at every point of* $\mathcal{P}$.

In reality, neither the forces of cohesion nor the centrifugal forces of inertia of matter on $\mathcal{P}$ will be zero. They will largely surpass the very weak centripetal forces of relativistic origin, which remain far below the scale of any possible experiments.

II.9. – **Theory of the Haress and Sagnac experiments.** – The almost contemporaneous experiments of Harress (1912) and Sagnac (1913) are experiments that have first-order in β, and therefore do not permit one to distinguish between classical and relativistic kinematics, so they will answer to the same theory. There is some interest to proving the absolute character of rigid rotations by optics, just as the Foucault pendulum experiment did by dynamics ([1]).

In those experiments, one causes two optical rays that traverse the same material circuit in opposite directions to interfere when they both follow the uniform rotation of a disc, and the circuit is composed of a broken line that is defined by planar mirrors. In the Harress experiment, the light traveled through prisms of index n that followed the rotation, and in that of Sagnac, through air, which corresponds to a vacuum. When the disc turned with an angular velocity ω, the two experiments showed that the *optical delay* Δt was given by the same formula, independently of the index n:

$$\boxed{\Delta t = \frac{4\mathcal{A}\omega}{c^2}.} \tag{II.38}$$

$\mathcal{A}$ denotes the area of the projection of the mean circuit $\mathcal{L}$ onto a plane that is normal to the axis of rotation, and that mean circuit will be well-defined if the interfering rays are thin. We shall justify that result.

Take an *arbitrary* Galilean frame $\mathcal{G}$, and neglect the component of the velocity that is normal to the disc. Since classical and relativistic kinematics are equivalent to first order, everything will happen *at that order* as if the disc were turning as a unit with an angular velocity ω around an *instantaneous center I*, which is the instantaneously-fixed point in $\mathcal{G}$. Set $\mathbf{r} = P - I$, where P denotes an arbitrary point of the circuit $\mathcal{L}$.

By virtue of Einstein's postulate, the speed of light will be isotropic (i.e., the ether wind will be absent) in the Galilean frame that instantaneously follows P and at the point P. In the Harress experiment, it equaled c / n, and in the Sagnac experiment, it was c. One can devise a simultaneous theory of the two experiments by setting $n = 1$ in the Sagnac case.

Now let v be the component of the velocity of the material point P that is tangent to $\mathcal{L}$ in the Galilean frame $\mathcal{G}$ that instantaneously follows an arbitrary point I of the disc. If $\mathbf{t}$ denotes the unit vector that is tangent to $\mathcal{L}$ then it will have the expression:

$$v = (\boldsymbol{\omega} \wedge \mathbf{r})\, \mathbf{t} \equiv \boldsymbol{\omega}(\mathbf{r} \wedge \mathbf{t}).$$

([1]) One can read a study of the optical effects of a rectilinear acceleration of the reference system from the pen of E. DURAND. Annales de Physique, **20** (1945), 535-544 and *ibid*. **1** (1946), 216-231.

Along that same axis, and in the same sense, the speed of light will be given by the Fresnel-Fizeau law (29), which is a direct consequence of the relativistic law for the composition of velocities, and which will give c in the Sagnac experiment. *Upon subtracting* $\pm c$ *from that formula, one will get two speeds of light with respect to the disc, when evaluated in the Galilean frame that instantaneously follows I:*

$$c_1 = \frac{c}{n}\left(1 - \frac{\beta}{n}\right), \qquad c_2 = \frac{c}{n}\left(1 + \frac{\beta}{n}\right).$$

Consequently, for a small element of the circuit dl that is placed at P, *one will get the following two unequal time durations for the traversal* (always evaluated in $\mathcal{G}$):

$$dl_1 = \frac{n\,dl}{c}\left(1 + \frac{\beta}{n} + \cdots\right), \qquad dl_2 = \frac{n\,dl}{c}\left(1 - \frac{\beta}{n} + \cdots\right).$$

Upon taking their difference, the index n will be eliminated to first order, and if $d\mathcal{A}$ denotes the area of the sector I, $d\boldsymbol{l}$ then one will get the following partial expression for the optical delay:

$$\delta t = 2\frac{v\,dl}{c^2} = 2\frac{\boldsymbol{\omega}(\mathbf{r} \wedge d\mathbf{t})}{c^2} = \frac{4\omega}{c^2}\,d\mathcal{A}\,.$$

One will indeed obtain the stated formula (38) when one integrates this along $\mathcal{L}$.

It goes without saying that, physically, one does not observe a difference in time Δt, but a displacement of the interference fringes that corresponds to an *optical delay* Δt. Obviously, since the optical frequency is the same for the two rays at any point that follows $\mathcal{L}$, the preceding theory will agree verbatim with an argument that makes the number of stationary waves occur explicitly.

One must remark that *to first order, the relativistic theory coincides exactly with the classical theory that one devises by supposing that the ether is at rest in the Galilean frame that is used.* That confirms, in a particular example, the general assertion that *everything happens in relativity as if the classical ether were at rest in* ***every*** *Galilean frame.*

The preceding theory permits one confirm that *there exists an optical anisotropy on the rotating rigid body that is proportional to:*

α) *The distance from the point considered to the point that is occupied by the observer.*

β) *The angular velocity* ω.

That is no contradiction with Einstein's fundamental postulate, since that anisotropy will disappear when:

α) $r \to 0$; i.e., at the point that is occupied by the observer.

β) $\omega \to 0$; i.e., at all points of a "Galilean rigid body."

Remark: The experiments of Dufour and Prunier. – Dufour and Prunier have complicated the experiment of Harress-Sagnac type by constituting their closed optical circuit from an element that is fixed in the laboratory and an element that follows the rotation of the disc [**82**] ([1]). The theory of that experiment that we proposed several years ago [**92**] will no longer be suitable when, as the aforementioned authors have since realized, the "fixed" part of the circuit has a complicated form that possibly involves some prisms of index *n*.

It goes without saying that the theory of the experiment in question can be given in an arbitrary Galilean frame, and by virtue of the general theory of Veltmann-Potier (or, if one prefers, by some properties of the Lorentz-Poincaré transformation), that theory will not differ at all (to first order) from the classical theory that one constructs with an "ether" at rest. Naturally, the Fresnel-Fizeau formula for the dragging of the frames must be used for the portions of the circuit that involve prisms of index *n*.

Here, we shall give a theory of those experiments from the viewpoint of an observer that is coupled with the disc.

Let $\overset{\frown}{ASB}$ be the portion of the circuit that is dragged by the disc, in which *S* denotes the point of the disc that is occupied by a window that separates the two interfering rays at the start and superimposes them when they return. Let $\mathcal{G}$ be the Galilean frame that is materialized by the laboratory. Two events that happen simultaneously at the points *A* and *B* relative to $\mathcal{G}$ will no longer be simultaneous for an observer that follows the disc. Moreover, their temporal shift Δt will be a function of the *comoving path* around which the synchronizing signals circulate. Along the arc $\overset{\frown}{ASB}$, one will have (to first order):

$$c^2\,\Delta t = \int_A^B \mathbf{v}\,d\mathbf{s} = \int_A^B (\boldsymbol{\omega}\,\mathbf{r}\,d\mathbf{s}) = 2\,\omega\,\mathcal{A},$$

by virtue of the Lorentz formulas, and with the same notations as before. $\mathcal{A}$ denotes the area in which one sees the comoving portion $\overset{\frown}{ASB}$ of the circuit from the center of rotation of the disc. Now, if the events considered *A* and *B* are no longer simultaneous in $\mathcal{G}$ then the preceding Δt will remain the expression for the supplementary shift that is due to the rotation of the disc. Furthermore, if the events in question are caused by a signal that is emitted by a point that follows *S* then the preceding Δt must be doubled in order to give the total delay that is caused by the rotation of the disc. One will then indeed predict the:

$$\Delta t = \frac{4\mathcal{A}\omega}{c^2}$$

that was found experimentally, from the viewpoint of the comoving observer.

([1]) See also Arch. Sci. Phys. Nat. (5) **28** (1946), pp. 73, *et seq.*

II.10. – **The spatio-temporal frequency quadri-vector. Aberration and Doppler Effect. Reflection from a moving mirror.** – Consider a planar, monochromatic light wave. The vectorial quantities that characterize it depend upon the spatial variables x^u and time t by the *phase* function, which is expressed by:

(II.39) $$\varphi = \frac{\alpha_u x^u}{L} - \frac{1}{T}.$$

The α^u are the three direction cosines of the *light ray* – viz., the normal to the *wave planes* $\varphi(x^u, t)$ = const. The constants L and T – namely, the *period* and *wave length* – are related to each other by:

(II.40) $$L = c\,T.$$

The phase, which is a pure number, enters into formulas by way of the exponent of the number e. It can only be a relativistic invariant, in such a way that one sets:

(II.41) $$\boxed{\lambda_u = \frac{1}{L}\alpha_u, \quad \lambda_4 = \frac{i}{cT}.}$$

Equation (39) transcribes into:

(II.42) $$\boxed{\varphi = \lambda_i\, x^i,}$$

and one sees that the four λ_i are the components of a quadri-vector that is *isotropic*, moreover, by virtue of (40):

(II.43) $$\boxed{\lambda_i\, \lambda^i = 0.}$$

That quadri-vector, which was systematically considered by L. de Broglie in his thesis [**163**], obviously deserves the name of the *spatio-temporal frequency* (each of its spatial components is nothing but the number of waves per unit length in that direction). By definition, the quadri-vector λ_i has the direction of the isotropic *light world-ray*. The hyperplanes φ = const. are the *world-hyperwaves*, which are likewise isotropic and are both orthogonal to λ_i and contain λ_i .

2. We shall now give the relativistic theory of aberration and the Doppler Effect by appealing to the notion of the quadri-vector λ_i .

Take two Galilean frames whose spatial axes are "parallel," with their relative velocity being directed along ox_1 and $\overline{ox_1}$, and apply the Lorentz-Poincaré formulas (19) to the quadri-vector λ_i :

(II.44) $$\bar{\lambda}_1 = \frac{\lambda_1 - i\beta\lambda_4}{\sqrt{1-\beta^2}}, \qquad \bar{\lambda}_4 = \frac{\lambda_4 + i\beta\lambda_1}{\sqrt{1-\beta^2}}, \qquad \bar{\lambda}_2 = \lambda_2, \qquad \bar{\lambda}_3 = \lambda_3 .$$

If we let α_1 denote the cosine of the angle between the ray and the relative velocity then, by virtue of (40) and (41), we can replace λ_4 with $i\,\lambda_1 / \alpha_1$ in (441), and then divide both sides by:

$$+\sqrt{\bar{\lambda}_2^2 + \bar{\lambda}_3^2} = +\sqrt{\lambda_2^2 + \lambda_3^2}\,.$$

One brings about the cotangents $\bar{\chi}_1$ and χ_1 of the angles between the ray and the relative velocity, and one will get the *relativistic law of aberration:*

(II.45) $$\bar{\chi}_1 = \frac{1 + \beta / \alpha_1}{\sqrt{1-\beta^2}}\,\chi_1\,.$$

If the plane waves emanate from a distant source that is fixed in the unprimed system then α_1 / χ_1 will be the sine of the angle between the ray and the apparent velocity of the source, in such a way that $\alpha_1 c / \chi_1$ will be the transverse component of the velocity along the ray, namely, c_t. The preceding formula will then be further written:

(II.46) $$\boxed{\bar{\chi}_1 = \frac{\chi_1 + v / c_t}{\sqrt{1-\beta^2}},}$$

and this will coincide with the well-known classical formula when one lets $\sqrt{1-\beta^2}$ go to 1.

If one takes (40), (41) into account, as well as the usual definition for the frequency:

(II.47) $$\nu = \frac{1}{T},$$

and if α_1 always denotes the cosine of the angle between the ray and the relative velocity then formula (44) can be written:

(II.48) $$\bar{\nu} = \frac{1 + \alpha_1 \beta}{\sqrt{1-\beta^2}}\,\nu\,.$$

That is the *relativistic law of the transformation of frequencies;* i.e., the Doppler Effect. Always assuming that the plane waves emanate from a distant source that is at rest in the unprimed system, $\alpha_1 \beta$ will be nothing but the radial component of the reduced translational velocity – namely, β_r – and the preceding formula can be further written:

(II.49) $$\boxed{\bar{\nu} = \frac{1 + \beta_r}{\sqrt{1-\beta^2}}\,\nu.}$$

This will agree with the well-known classical formula when one lets $\sqrt{1-\beta^2}$ go to 1.

In the early days of relativity, one could find that the new explanation for aberration and the Doppler Effect was less simple and less direct than the old one and invoke the analogy with sound in which the old explanation was phrased. Indeed, that analogy was not correct, because sound propagates in a medium that, by the very fact of its existence, defines a privileged reference system. If one then remarks that *the so-called absolute velocities occur in the classical formulas only to first order* then one must concede that *aberration and the Doppler Effect speak in favor of relativity, and not against it, at least to first order.*

But there is more: *Some recent experiments have permitted us to exhibit the intervention of the relativistic factor* $\sqrt{1-\beta^2}$, which expresses, in short, the *slowing-down of clocks.* Recall formula (49). Set:

$$\beta_r = 0$$

in it; i.e., suppose that the velocity of the source is normal to the direction of observation. One will get:

$$\bar{\nu} = \frac{\nu}{\sqrt{1-\beta^2}}, \tag{II.49}$$

which is the formula for the transverse Doppler Effect, which is a relativistic effect that was not predicted by the classical theory; in the latter, one simply sets:

$$\bar{\nu} = \nu.$$

Now, the recent experiments of Ives and Stilwell [**83**, **84**, **86**] have permitted us to exhibit the transverse Doppler Effect, which is a second-order effect that is in perfect accord with the relativistic formulas.

If one sets:

$$\beta_r = \pm\,\beta$$

in formula (49′) – i.e., if one considers a source velocity that is purely radial – then one will get the relativistic law for the purely-longitudinal Doppler Effect:

$$\bar{\nu} = \nu\sqrt{\frac{1\pm\beta}{1\mp\beta}}, \tag{II.49″}$$

which differs from the classical law by a factor of $\sqrt{1-\beta^2}$.

3. We shall now study the aberration and Doppler Effect that are obtained by reflection from a moving mirror. In the classical books on relativity, that question, like the preceding one, was generally treated by means of electromagnetic theory, which has the inconvenience that it masks the purely-kinematical nature of the problem to some

extent [1]. As before, we shall argue directly on the basis of the law of transformation for the quadri-vector λ^i.

In classical optics, the set that is composed of a point-like source that is fixed in the reference system and a plane mirror that translates with a component of the velocity $\mathbf{v}_r$ that is normal to that mirror is equivalent to a point-like source s' that is symmetric to s at each instant, and thus animated with a velocity of $2\mathbf{v}_r$. We shall show that those properties persist in relativity to first order.

First, take a Galilean frame $p\ x_1\ x_2\ x_3,\ t$ that is fixed in the mirror, with px_1 being directed along the normal. From Descartes' first law, the planes of incidence and reflection will coincide; let $x_3 = 0$ be that plane, to simplify matters. From Descartes' second law, the angles of incidence and reflection will be equal. Finally, from wave optics, there will be conservation of frequency. In the proper system of the mirror, the two quadri-vectors λ^i and λ'^i, which relate to the incident and reflected wave, respectively, will then satisfy the *proper laws of reflection:*

$$\text{(II.50)} \qquad \lambda'_i = -\lambda_i\,, \qquad \lambda'_2 = \lambda_2 \neq 0, \quad \lambda'_3 = \lambda_3 = 0, \quad \lambda'_4 = \lambda_4\,.$$

We seek to know what those laws will become in a Galilean frame that is not proper. First of all, we verify effortlessly that a translation of the mirror parallel to itself that does not move with the source will leave the classical laws unaltered. The only second-order differences between the present reference system and the one that is coupled with the mirror relate to a simultaneous alteration of the Descartes angles and the frequency.

We shall now study the more interesting problem of a translation of the mirror normal to itself. If we take (50) into account then formulas (44) will permit us to write:

$$\text{(II.51)} \qquad \bar{\lambda}_1 = \frac{\lambda_1 - i\beta\lambda_4}{\sqrt{1-\beta^2}}, \qquad \bar{\lambda}'_1 = \frac{-\lambda_1 - i\beta\lambda_4}{\sqrt{1-\beta^2}},$$

$$\text{(II.52)} \qquad \bar{\lambda}_4 = \frac{\lambda_4 + i\beta\lambda_1}{\sqrt{1-\beta^2}}, \qquad \bar{\lambda}'_4 = \frac{\lambda_4 - i\beta\lambda_1}{\sqrt{1-\beta^2}},$$

$$\text{(II.53)} \qquad \bar{\lambda}_2 = \bar{\lambda}'_4 = \lambda_2 = \lambda'_2\,, \qquad \bar{\lambda}_3 = \bar{\lambda}'_3 = 0.$$

If one divides the corresponding sides of the equations in (51), replaces λ_4 with $i\,\lambda_1 / \alpha_1$, as before, and introduces the cotangents $\bar{\chi}_1$ and $\bar{\chi}'_1$ of the angles of incidence and reflection, resp., then one will get the *relativistic law of aberration by reflection:*

$$\text{(II.54)} \qquad -\frac{\bar{\lambda}'_1}{\bar{\lambda}_1} = -\frac{\bar{\chi}'_1}{\bar{\chi}_1} = \frac{1 + i\beta\,\lambda_4/\lambda_1}{1 - i\beta\,\lambda_4/\lambda_1} = \frac{1 - \beta/\alpha_1}{1 + \beta/\alpha_1},$$

[1] In his cited thesis, L. DE BROGLIE used purely-kinematical arguments that involved the mass-impulse of the photon.

$$\boxed{-\bar{\chi}_1' = \frac{1-\beta/\alpha_1}{1+\beta/\alpha_1}\,\bar{\chi}_1 = (1 - 2\frac{\beta}{\alpha_1} + \cdots)\bar{\chi}_1.}$$

Similarly, if one divides the corresponding sides of the equations in (52) then one will get the *relativistic law of the Doppler Effect by reflection:*

(II.56) $$\frac{\bar{\lambda}_4}{\bar{\lambda}_4'} = \frac{1+i\beta\lambda_1/\lambda_4}{1-i\beta\lambda_1/\lambda_4} = \frac{1+\beta\alpha_1}{1-\beta\alpha_1},$$

$$\boxed{\bar{\nu}' = \frac{1-\beta\alpha_1}{1+\beta\alpha_1}\,\bar{\nu} = (1-2\beta\alpha_1 + \cdots)\,\bar{\nu}.}$$

If one compares (46) and (49) then formulas (54) and (55) will show that:

1. In full relativistic rigor, there is no equivalence between the relations that couple the directions and frequencies of incidence and reflection with the relations that couple the "proper" and "relative" directions and frequencies.

2. One recovers the classical equivalence to first order, so translation of the mirror normal to itself with velocity $\mathbf{v}_r$ can be replaced by the fictitious translation of a source with velocity $2\mathbf{v}_r$.

II.11. – **The stationary world-wave. The formula for the retarded potential.** – In this number, we shall make two brief series of remarks that are independent of each other.

1. The well-known equation:

$$\psi = a \sin 2\pi \frac{t}{T} \sin 2\pi \frac{\alpha_1}{L}$$

is that of a stationary wave that results from the superposition of two sinusoidal waves with the same amplitude, period T, and wave lengths $\pm L$ that propagate in the opposite sense along the same Galilean axis $x_1' x_1$. Those quantities and T are coupled by (40).

If one adopts Minkowski's hyperbolic diagram, along with the customary definitions (20), then one can write the perfectly symmetric expression:

(II.56) $$\psi = a \sin 2\pi \frac{x}{L} \sin 2\pi \frac{y}{L}.$$

One sees that the *stationary world-wave* will admit two series of nodal lines that form a grid that is orthogonal to the square mesh. The timelike ones are the world-trajectories of the ordinary nodes, and the spacelike ones correspond to the instants $t = y / c$, where the entire axis $x_1 = x$ is at rest. In that way, one can consider the stationary world-wave as defining its proper Galilean frame in an extended manner.

2. If ∂_i^i always denotes the d'Alembertian operator then consider the equation with the well-known form:

$$\boxed{\partial_i^i A = a,} \tag{II.57}$$

which presents itself in the course of studying numerous equations, and in which A and a are two tensors with the same rank. By definition, the tensor a is the *source density,* which is generally non-zero only in the interiors of certain space-time domains. It *generates* the *field* tensor A at any instant-point.

As one knows, Kirchhoff gave the general equation (57) the solution:

$$A = \frac{1}{4\pi}\iiint \{a\}\frac{\partial u}{r},$$

in which, if the instant-point at which one calculates A is taken to be the origin, to simplify, then $\{a\}$ will denote the value of the source at the current instant-point whose distance r and temporal precedence t will be related by (58_1). One will then obtain the solution that is called the *retarded potential.* Mathematically, the solution (58_2), which is called the *advanced potential,* would be just as suitable, but one cannot know its physical manifestations:

$$r = -\,c\,t, \qquad r = +\,c\,t. \tag{II.58}$$

Physically, if the source a is the *cause* of the tensor A then the *retarded potential* will translate into the existence of a causality of the type that is usually invoked in order to pass from the past to the future, while the *advanced potential* represents a causality that will point from the future into the past, which is entirely unconventional.

That being the case, one must verify that formula (57) is covariant from the relativistic viewpoint, and consequently that the expression $\delta u / r$ is an invariant. Always let α^u denote the three direction cosines of the direction that links the origin $x^i = 0$, where one calculates A, to the current source point, and introduce the pair of isotropic quadri-vectors:

$$r^u = \alpha^\mu\, r, \qquad r_4 = \pm\, ir \qquad (\alpha^\mu\, \alpha_u = 1). \tag{II.59}$$

The – sign corresponds to the retarded potential, and the + sign, to the advanced potential. Similarly, if δu denotes the ordinary volume element $[dx_1\, dx_2\, dx_3]$ then introduce the new pair of isotropic quadri-vectors:

$$c\,\delta u^u = \alpha^\mu\,\delta u, \qquad c\,\delta u^4 = \pm\, i\,\delta u, \tag{II.60}$$

in which the – sign must be taken in the case of the retarded potential, and the + sign, in the case of the advanced potential. The two homologous quadri-vectors r^i and δu^i are, by definition, collinear and, at the same time, orthogonal, from the fact of their isotropy. In world geometry, the quadri-vector r^i represents the segment of the generator of the isotropic hypercone that has its summit at the instant-point where the one calculates A and passes through the instant-point where one considers a. The quadri-vector $ic\ \delta u^i$ is the dual of the three-dimensional volume element that is carried by the sheet in question of the hypercone at the instant-point (a). Therefore, all of the notions that were invoked have an intrinsic significance in world geometry. $\delta u\ /\ r$ represents the ratio, not of the lengths of two collinear quadri-vectors (since they are isotropic, so the lengths will be zero), but of their homologous components. $\delta u\ /\ r$ is then indeed a relativistic invariant.

Under those conditions, in relativity, we agree to write the formula for the retarded (or advanced) potentials in the form:

$$\boxed{A = \frac{1}{4\pi}\iiint a\left\{\frac{\delta u}{r}\right\},} \tag{II.61}$$

in which the symbol { } is supposed to simultaneously suggest that it is a relativistic invariant and direct attention to the significance of that invariant in four-dimensional geometry; i.e., to the effective fashion by which one performs the calculations.

II.12. – **The world quadri-velocity. Examples of the effects of motion.** – Consider a material point of the universe (viz., an isolated material point or fluid "molecule") that describes a time-like trajectory $\mathcal{T}$. If v^u denote the three components of its ordinary velocity then introduce the quadri-vector V^i that is defined by:

$$\boxed{V^u = \alpha v^u, \quad V^4 = ic\alpha,} \qquad \alpha = \frac{1}{+\sqrt{1-\beta^2}}, \qquad \boxed{V^i\,V_i = -c^2.} \tag{II.62}$$

In the universe, α is nothing but the cosine of the angle between the tangent to $\mathcal{T}$ and the axis Ox^4 (eqs. 13 and 16), in such a way that if dt denotes the projection of the proper time interval $d\tau$ onto that axis then one will have:

$$\alpha = \frac{dt}{d\tau}. \tag{II.63}$$

The preceding definitions of the quadri-vector V^i then condense into the form:

$$\boxed{V^i = \frac{1}{d\tau}dx^i} \qquad (i = 1, 2, 3, 4). \tag{II.64}$$

Now suppose that one attaches a tensor to the preceding material point that enjoys certain properties with respect to V^i. We then direct our attention to three simple cases that will be found to be applied several times in the course of this book.

Quadri-vector X^i that is tangent to the world-trajectory. This case is characterized by one or the other of the relations:

(II.65) $$V^j X^i - V^i X^j = 0,$$

(II.66) $$\frac{X^1}{dx^1} = \frac{X^2}{dx^2} = \frac{X^3}{dx^3} = \frac{X^4}{dx^4}.$$

One then concludes from this that:

(II.67) $$X^u = X^4 \frac{dx^u}{dx^4} \qquad \text{or} \qquad \boxed{\mathbf{X} = -i\boldsymbol{\beta} X_4.}$$

That relation is notably encountered in the context of the quadri-vectors of *electric charge-current density* $j^u = q\, v^u$, $j^4 = ic\, q$ (no. III.5) and *finite mass-impulse* $p^u = mv^u$, $p^4 = ic\, m$ (no. IV.7). One can say that the ordinary spatial vector $\mathbf{X}$ is "generated" by the velocity of the "scalar" X^4.

Quadri-vector X^i normal to the world-trajectory. – This case is characterized by the relations:

(II.68) $$X_i V^i = 0 \qquad \text{or} \qquad X_i\, dx^i = 0,$$

from which, one concludes that:

$$X_u\, dx^u + X_4\, dx^4 \equiv (X_u \beta^u + i X_4)\, c\, dt = 0,$$

or

(II.69) $$\boxed{X^4 = i(\mathbf{X}\cdot\boldsymbol{\beta}).}$$

That relation is notably encountered in the context of the *power-force density* f^i (nos. III.7 and IV.3) and in the theory of the Dirac electron, in the context of the *spin density* σ^i.

Frenkel relation. – Consider a second-rank antisymmetric tensor $X^{ij} \equiv -X^{ji}$ such that one has:

(II.70) $$X^{ij} V_j = 0 \qquad \text{or} \qquad X^{ij} dx_j = 0.$$

It is equivalently characterized by the fact that its three components $X^{u4} \equiv -X^{4u}$ are annulled in the comoving Galilean frame and the fact that one can exhibit the first three of (70) explicitly by:

$$X^{ui}\, dx_i \equiv X^{uv}\, dv_u + X^{u4}\, dx_4 \equiv 0.$$

The last of (70) is specified by:

$$X^{4i}\, dx_i \equiv X^{4u}\, dv_u \equiv 0,$$

and that will show that *any of the four* (70) *(in which X^{ij} denotes an antisymmetric tensor) is a consequence of the other three.*

Now, introduce the two spatial tri-vectors:

$$Y^w = X^{uv}, \qquad i\,Z^u = X^{u4}.$$

The preceding relations then give:

$$\text{(II.71)} \qquad \begin{aligned} &Y^w dx^v - Y^v dx^w + cZ^u dt = 0 \quad &\text{or} \quad &\boxed{\mathbf{Z} = \mathbf{Y} \wedge \boldsymbol{\beta},} \\ &Z_u dx^u = 0 \quad &\text{or} \quad &\boxed{\mathbf{Z} \cdot \boldsymbol{\beta} = 0.} \end{aligned}$$

Here, one can say that the tri-vector **Z** is generated by the velocity of the tri-vector **Y** according to (71_1), with (71_2) as a consequence. In the case of a medium that is endowed with magnetic polarization, but not electric polarization, when it is at rest, that relation is known by the name of the *Frenkel relation* [**121**].

C. – GENERALITIES ON RELATIVISTIC PHYSICS

II.13. – **Integration elements of various orders and their duals.** – Let $\mathcal{P}$ be an arc of an arbitrary line that is drawn in the universe. If A denotes an arbitrary tensor then consider the curvilinear integral:

$$\int_{\mathcal{L}} A\,dx_i \,.$$

(Of course, one can assume that one of the indices of A is *contracted* with the index i of the integration element.) A frequent special case in that regard is concerned with the *world-circulation of a quadri-vector* A^i*:*

$$\int_{\mathcal{L}} A^i\,dx_i \,.$$

If the arc-length element dx_i is timelike then there will exist a *tangent Galilean frame* in which the differential reduces to its component:

$$A\,dx_4 \equiv ic\,A\,dt.$$

If the arc-length element dx_i is spacelike then there will exist a double infinitude of Galilean frames (which are characterized by the world-directions of their Ox_4 axes) in which the temporal component of that differential is annulled. In particular, the element of circulation of a quadri-vector will then reduce to its spatial part:

$$A^u\,dx_u \equiv \mathbf{A} \cdot d\mathbf{x}.$$

Now let $\mathcal{S}$ be a general world-area – i.e., a two-dimensional curvilinear manifold. One can refer them to a system of two curvilinear coordinates and then define two tangent quadri-vectors $\delta_1\, x^i$ and $\delta_2\, x^i$ at each of their points. Now consider the six determinants that are extracted from the matrix:

$$\left\| \begin{matrix} \delta_1 x^1 & \delta_1 x^2 & \delta_1 x^3 & \delta_1 x^4 \\ \delta_2 x^1 & \delta_2 x^2 & \delta_2 x^3 & \delta_2 x^4 \end{matrix} \right\|,$$

each affected with a sign, and denote them by the symbols $[dx^i\, dx^j]$, as was explained in no. I.8. It will then be clear that the three:

$$[dx^u\, dx^v] = \delta\mathbf{s}^w,$$

which are the projections of the $[dx^i\, dx^{\,j}]$ onto the coordinate planes $Ox^u\, x^v$, will be the components of an elementary area in the usual sense ([1]). World geometry adds the three:

$$[dx^u\, dx^4] = ic \cdot dx^u\, dt$$

to them in order to constitute a second-rank antisymmetric tensor. The $[dx^u\, dx^4]$ are the *projections* of the tensor $[dx^i\, dx^{\,j}]$ onto the coordinate planes $Ox^u\, x^4$.

If, by definition, we introduce the dual of the preceding tensor:

(II.72) $$\boxed{ic\,\delta s^{ij} = [dx_k dx_l]}$$

then we will have:

(II.73) $$\boxed{\delta s^{u4} = \frac{1}{ic}\delta s^u, \quad \delta s^{vw} = dx^u dt.}$$

$\delta\mathbf{s}$ or δs^u always denotes the ordinary elementary area.

Now, let $\mathcal{V}$ be a general world-volume – i.e., a curvilinear, three-dimensional manifold. The preceding considerations can be generalized, and one defines the antisymmetric third-rank *world-volume* tensor $[dx^i\, dx^{\,j}\, dx^k]$ at every instant-point $\mathcal{V}$, and its component is:

$$\delta u = [dx^u\, dx^v\, dx^w] \equiv dx^1\, dx^2\, dx^3,$$

which is the projection of $[dx^i\, dx^{\,j}\, dx^k]$ onto the hyperplane $x_4 = 0$, denotes a volume in the ordinary sense ([2]). As before, the other three components can be written:

$$[dx^u\, dx^v\, dx^4] = ic \cdot dx^u\, dx^v\, dt,$$

and they represent the *projections* of the tensor $[dx^i\, dx^{\,j}\, dx^k]$ onto the hyperplanes x^u = const.

([1]) On that subject, see the remarks that will be made in a note in the following no.
([2]) On that subject, see the remarks that will be made in a note in the following no.

If, by definition, we introduce the quadri-vector that is dual to the preceding tensor by way of:

(II.74) $$\boxed{ic\,\delta u = [dx_j dx_k dx_l]}$$

then we will get the following interpretation of its four components:

(II.75) $$\boxed{\delta u^4 = \frac{1}{ic}\delta u, \quad \delta u^w = \delta s^w dt.}$$

δu always denotes an elementary volume in the ordinary sense.

We shall now show that the quadri-vector δu^i is orthogonal to the three $\delta_u x^i$ that define the $[dx_j\, dx_k\, dx_l]$. One then deduces from (74) that:

$$ic\ \delta u^i\ \delta_u x_i = \sum \delta_u x_i\,[dx_j dx_k dx_l],$$

in which the sum is extended over all permutations i, j, k, l, each affected with a sign. The three right-hand sides are identically zero, since they are the developments of rank-four determinants with two identical rows.

Finally, let $\mathcal{D}$ be a four-dimensional world-domain, and consider the corresponding integration element:

(II.76) $$\boxed{\delta u_i\, dx^i = \delta\omega \equiv \frac{1}{ic}[dx^1 dx^2 dx^3 dx^4].}$$

Recall that in no. II.7, we explained the physical reasons that allowed us to consider the expression $\delta u\, dt$ as representing that same quantity. We can then write:

(II.77) $$\boxed{\delta u_i\, dx^i \equiv \delta u\, dt.}$$

Pre-relativistic physics is in the habit of taking its curvilinear integrals – whether surface or volume – "at constant time," which is a hypothesis that can only be *relative* in the new kinematics. If one wishes to preserve that hypothesis then *one must change the line, surface, or volume integral whenever one changes the Galilean frame,* and by that itself, if one starts with a well-defined tensor density then *one must change the integral tensor.* That habit of pre-relativistic physics is therefore is disaccord with the spirit of world-geometry. That is what one encounters at the origin of numerous well-known difficulties that relate to the reputed non-tensorial character of various quantities that are very important physically, such as, for example, the finite force, the finite kinetic moment, and temperature (see no. IV.19). *The difficulties in question are raised when one renounces the demand of simultaneity at a distance and gives the line, surface, and volume of integration* ***a priori*** *and independently of the Galilean frame.* Moreover, inspired by the theory of relativity, E. Cartan has already shown that it can be interesting in classical mechanics to argue on the basis of *non-simultaneous* states of a system of points or a fluid. The profound significance and symmetry of the formulas will emerge more clearly, as well ([1]).

Before passing on to another subject, we insist upon the following fact: We just saw that in world-geometry, the three-dimensional volume is defined to be a quadri-vector – exactly as in ordinary geometry – and the elementary area is defined by a tri-vector. It then follows that *in relativity, the finite and density tensors that relate to the same physical quantity will not have the same variance. The variance of the finite quantity will be one unit less or more (according to whether there is a contraction or not, resp.) from that of the density quantity.* Among the innumerable examples of that very important fact, we give the following ones in advance:

Physical quantity	*Density tensor*	*Finite tensor*
Electric charge	Quadri-vector	Invariant
Force	Quadri-vector	Second-rank, antisymmetric tensor ([2])
Mass-impulse	Second-rank tensor ([3])	Quadri-vector
Kinetic moment	Third-rank tensor, antisym. in 2 or 3 indices	Second-rank, antisymmetric tensor

II.14. – **Generalities on fluid kinematics. The hyper-endcap integral and the hyper-wall integral. Definition of the proper – or "scalar" – volume. Definition of an incompressible fluid.** – The various *molecules* of a continuous medium (or extended material points, in the purely geometric sense) generate a congruence $\mathcal{T}$ of trajectories – or *world-streamlines* – that are all time-like. Such a hyper-tube of trajectories, which is bounded by a hyper-wall $\mathcal{P}$ that is three-dimensional and said to be *timelike*, represents the evolution of the *same* portion of matter in the course of time. Now consider a three-dimensional hyper-endcap $\mathcal{E}$ of that timelike hypertube; i.e., by definition, it is such that the dx^i that are tangent to it are spacelike. If that hyper-endcap $\mathcal{E}$ is planar and orthogonal to Ox^4 in the Galilean frame Ox^i that is used then from classical kinematics it will represent the state of a fluid droplet that is followed in the course of its motion at the instant t. However, it goes without saying that the notion of the *state of a material fluid drop* is defined only in a *relative* manner. *By definition,* we say that a continuous family

([1]) In a series of very recent papers, J. Schwinger systematically introduced the consideration of *non-simultaneous states* of a system into quantum electrodynamics. Even if the physical results that are obtained do not pertain to the new covariant formulation of this author, it will still remain that the latter has the advantage that it guarantees the demands of relativity at each step in argument (*note added in proof*).

([2]) And possibly an asymmetric tensor (see below, nos. III.7 and III.8).

([3]) Either symmetric or asymmetric (see below, no. IV.12).

of arbitrary curvilinear hyper-endcaps $\mathcal{E}$ – all of which are spacelike – represents a family of successive *states* of the same fluid drop whose motion is being followed ([1]).

The laws of fluid kinematics, whether relativistic or classical, are expressed essentially by density equations at each instant-point that is occupied by the fluid. Naturally, one can infer some integral laws from this that are stated for the entire fluid volume "at the instant *t*" in classical physics. In relativistic physics, the integral laws must be stated for an arbitrary current hyper-endcap $\mathcal{E}$ that is restricted simply by the demand that it must be everywhere-spacelike. Given the considerable degree of arbitrariness in the definition of $\mathcal{E}$, it is obvious *a priori* that *one must demand that the formulation of the integral laws should be invariant when changes the family of hyper-endcaps* $\mathcal{E}$. Physically, that amounts to saying that *relativity attaches no importance and no objective significance to the instants relative to which the various molecules of the extended medium are considered when the corresponding instant-points are in the "unphysical" region; i.e., it is impossible to link to them with a signal.* That is the manner by which relativity considers the various parts of an extended system *at once* (but not simultaneously!).

We have explained the sense in which the current hyper-endcap $\mathcal{E}$ is said to be spacelike and the hyper-wall $\mathcal{P}$, timelike. Now, if one introduces the quadri-vector that is dual to the elementary volume at any instant-point on those hypersurfaces then we showed in the preceding no. that this quadri-vector will be orthogonal to that hypersurface. Hence, *the quadri-vectorial volume element* δu^i *of a hyper-endcap* $\mathcal{E}$ *is timelike* (and similarly, the quadri-velocity V^i at that instant-point). *The quadri-vectorial volume element* δu^{*i} *of the hyper-wall* $\mathcal{P}$ *is spacelike and orthogonal to the quadri-velocity at that same instant-point, moreover:*

(II.78) $$V_i\,\delta u^{*i} \equiv 0.$$

([1]) Consider the case of an infinitely-thin hypertube, and let $\delta u'$ be the evaluation of the elementary volume that is made *simultaneously* (in the classical manner) by a Galilean observer for a well-defined instant-point x^i of the mean trajectory. Let δu_0 be the δu that is found by the Galilean observer $\mathcal{G}_0$ that is *tangent* – or *proper,* and let β be the relative velocity of $\mathcal{G}$ and $\mathcal{G}_0$. One sees effortlessly that those *constant-time volumes* transform according to the law:

$$\delta u' = \delta u_0\sqrt{1-\beta^2}\,,$$

whereas, on the contrary, the fourth component of the quadri-vector that is dual to the "proper volume" δu_0^i will transform according to the law:

$$\delta u = \delta u_0\sqrt{1-\beta^2}\,.$$

The preceding $\delta u'$ is then the fourth component of a different quadri-vector. When we said in the preceding no. that $ic\ \delta u^4$ represents an elementary volume in the ordinary sense, the remark that we just made was implicit in it. When the three-dimensional manifold is timelike, the difference between $\delta u = ic\ \delta u^4$ and a volume that is considered simultaneously in the usual manner will become even more significant. Some similar remarks are true for the case of the double and simple integrals that were considered in the preceding no.

Let us say, in passing, that it might seem natural to consider the case in which the quadri-vectors V^i and δu^i are collinear at an instant-point of a hyper-endcap $\mathcal{E}$. That will always be true under the hypothesis of an infinitely-thin hypertube, but in the case of a finite hypertube, for that to be true, it would be necessary for the congruence $\mathcal{T}$ to be a normal congruence, which will be true only in the exceptional cases. There would be no reason to attach a general theory to an exceptional situation, and that is why one must leave both of the two timelike quadri-vectors V^i and δu^i completely independent of each other. Despite that, in certain inductive studies, it can be interesting to consider the exceptional case in which the congruence $\mathcal{E}$ is a normal congruence first, and consequently to choose δu^i to be everywhere-collinear to V^i (see below, nos. IV.5 and IV.6).

During the evolution of the material drop $\mathcal{E}$, the hyper-wall $\mathcal{P}$ is generated by its two-dimensional contour $\mathcal{S}$. Like $\mathcal{E}$, that contour $\mathcal{S}$ is generally curvilinear, in such a way that the six components of its $[dx^i\ dx^j]$ will occur effectively. We shall then prove the following relation, which will be quite useful in what follows:

(II.79) $$\boxed{\delta u^{*i} = -\delta s^{*ij} dx_j \equiv -\delta s^{*ij} V_j\, d\tau.}$$

$ic\ \delta u^{j*}$ always denotes the dual of the elementary volume of the hyper-wall $\mathcal{P}$, $ic\ \delta u^{ij*}$, that of the elementary of the contour $\mathcal{S}$ of the drop $\mathcal{E}$, and $dx_j \equiv V_j\, d\tau$ is the element of the world-trajectory $\mathcal{T}$. Passing to dual quantities, one writes:

$$[dx_j\ dx_k\ dx_l] = \sum dx_j [dx_k dx_l],$$

in which the summation extends over all circular permutations. In the right-hand side, one has the developments of the components of the three-dimensional volume that is constructed from two spacelike quadri-vectors $\delta_1 x^i$ and $\delta_2 x^i$ that are tangent to $\mathcal{S}$ and the timelike quadri-vector dx^i that is tangent to $\mathcal{T}$. That is, in fact, a strongly-suggested way of defining the volume element of the hyper-wall $\mathcal{P}$.

Now consider an infinitely-thin current hypertube, and let δu^i be the quadri-vector that represents the magnitude and direction of its spacelike hypersection. Exactly as in three-dimensional geometry, one will effortlessly see that the *scalar product:*

(II.80) $$\boxed{\delta u_0 \equiv V_i \delta u^i}$$

has an invariant value when one considers the various hypersections that pass through the same mean instant-point that one chooses in the hypertube. If one prefers, it is *the elementary hyper-flux of the quadri-velocity V^i at the proper instant τ.* In the Galilean frame $\mathcal{G}_0$ that is tangent to the mean streamline at the proper instant τ, one will have:

$$\delta u_0 = V_{04}\,\delta u_0^4,$$

which is precisely the definition of the value of the ordinary volume on $\mathcal{G}_0$ when one takes (62_1) and (75_1) into account. With that, one sees that the invariant δu_0 also deserves the name of the *elementary proper volume* inside of the fluid. If one multiplies the two sides of (80) by $d\tau$ then one will get:

(II.80′) $$\boxed{\delta u_0\, d\tau \equiv \delta u^i dx_0,}$$

which is a formula that confirms what we said when we established (27) and (76).

Inside of a material medium (and also in the case of a *material point* when one constricts it and passes to the limit), the definition of the *proper* or *scalar* volume will permit one give a *second definition of the finite quantities, thanks to which, those quantities will have the same variances as the homologous density quantities (to which they are even homothetic).* In several situations, that definition of the finite quantities *in the second way* (which is not general, by virtue of what was pointed out in the preceding no.) can nonetheless be of service.

By its very definition, the *elementary proper volume* is identically zero on a hyper-wall:

(II.81) $$\delta u_0^* \equiv 0.$$

Again by definition, the integral of the expression (80) over a hyper-endcap $\mathcal{E}$ will be called the *finite proper volume* of the material drop in the state $\mathcal{E}$:

(II.82) $$u_0 = \iiint_{\mathcal{E}} V^i \delta u_i \equiv \iiint_{\mathcal{E}} \delta u_0\,.$$

Obviously, the expression u_0 also deserves the name of *hyper-flux of the quadri-velocity V^i across the hyper-endcap $\mathcal{E}$.*

If one takes (81) into account then upon integrating the expression (80) over the closed, three-dimensional domain that is composed of two different *states* $\mathcal{E}_1$ and $\mathcal{E}_2$ of then same fluid drop and the corresponding hyper-wall element, transforming that into a quadruple integral that is extended over the domain $\mathcal{D}$ that is bounded by the preceding one, by virtue of the general formula [I, eq. (40)] and taking (76) into account, and finally orienting the two hyper-endcaps in the same sense relative to the hyper-streamlines $\mathcal{T}$, one will get the formula:

(II.83) $$u_0(2) - u_0(1) = \iiiint_{\mathcal{D}} \partial_i V^i \cdot \delta u^k dx_k\,.$$

One will then see that *the necessary and sufficient condition for the hyper-flux or "scalar volume" u_0 to be preserved along the entire hypertube is that one must have:*

(II.84) $$\partial_i V^i \equiv 0.$$

That will be the definition of an *incompressible fluid* ([1]) that is suggested the most strongly by relativity.

In infinitesimal form, formula (83) can also be written:

(II.85) $$d\,\delta u_0 = \partial_i V^i \cdot \delta u_0\, d\tau,$$

and one recognizes the relativistic extension of the well-known classical formula:

$$d\,\delta u = \operatorname{div} \mathbf{v} \cdot \delta u\, dt.$$

For some of the inductive arguments in Chapter IV, it will be useful to consider the case in which one orients the hyper-section of an infinitely-thin hypertube normally to the mean streamline. In that case, the quadri-vector δu^i will be collinear to V^i, and we shall call it δu_0^i. If one takes the definition (80) into account, as well as (62), then one will find with no difficulty that:

(II.86) $$\boxed{\delta u_0^i = -\frac{1}{c^2}\delta u_0 V^i \quad \text{or} \quad \delta u_0^i = -\frac{1}{c^2} V^i V^j \delta u_j.}$$

Both of those formulas will be useful to us in what follows.

We hope that in the course of the present chapter, the reader that was not further familiar with the theory of relativity would be progressively elevated, without too much effort, to an understanding of some particular cases of the general methods that permit one to enter into the various domains of relativistic physics.

Everything that we said in regard to the arbitrary character of the hyper-endcap $\mathcal{E}$ must be extended from the case of a flowing fluid to the general case of a field, which can, moreover, include singular lines inside of it that represent the material trajectories of material points.

II.15. – **Fluid kinematics (cont.): Three general formulas that will be useful in what follows.** – Let Φ be an arbitrary function of the fluid molecules. Exactly as in the pre-relativistic kinematics of fluids, one sees that if τ denotes the proper time then the variation Φ that is concomitant to the evolution of the fluid molecule can be written:

(II.87) $$d\Phi \equiv V_i\, \partial^i \Phi \cdot d\tau.$$

One can equivalently write:

(II.87′) $$\Phi' \equiv V_i\, \partial^i \Phi, \quad \text{with} \quad \Phi' \equiv \frac{d}{d\tau}\Phi.$$

([1]) For A. Lichnerowicz, the present definition of incompressible fluid is "definition A" [**151**, **152**, **153**]. For von Laue, that definition is the definition of a "fluid of least compressibility" [**4**, § 36]. – It is clear that in relativity the notions of *rigid body* and *incompressible fluid* must accommodate the *Lorentz contraction* of moving volume elements.

Φ always answers to the preceding definition, and represents the fact that for any of the physical densities that are attached to the fluid, one can write:

$$\iiint_{\mathcal{E}_2-\mathcal{E}_1} \Phi V^i \delta u_i \equiv \iiiint \partial_i (\Phi V^i) \cdot \delta u_0 \, d\tau , \tag{II.88}$$

or further, in infinitesimal form:

$$d\,(\Phi\, V^i\, \delta u_i) \equiv \partial_i\,(\Phi\, V^i) \cdot \delta u_0\, d\tau \equiv \partial_i\,(\Phi\, V^i) \cdot \delta u\, dt. \tag{II.88′}$$

That formula constitutes the relativistic symmetrization of a formula that we will appeal to in no. IV.2 in order to establish the relativistic dynamics of fluids deductively.

Finally, since, by definition, the fluid quadri-velocity V^i has constant length [eq. (62_3)], one can remark that one has:

$$V_i \; \partial_k V^i \equiv 0. \tag{II.89}$$

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CHAPTER III

RELATIVISTIC ELECTROMAGNETISM

III.1 – Electromagnetism has known the constant c since Maxwell, and experiment soon verified the relation:

$$\frac{K_1}{K_2} = c,$$

in which K_1 and K_2 denote the fundamental constants of electrostatics and magnetism, respectively. The classics treated K_1 and K_2 as universal constants, in theory and in practice. By convention, they attributed the dimension of *zero* and the numerical value of 1 to both of them, which resulted in the alternative of two systems of units E. S. U and E. M. U, as one knows. Now, the constant c plays a role that is perfectly homogeneous to that of the constants K_1 and K_2. In order to avoid the dilemma of E. S. U and E. M. U, it will suffice to attribute the dimension *zero* and the numerical value of 1 to c; i.e., to establish the physical equivalence between length and time that relativity discovered much later.

It is now commonplace to say that electromagnetism was relativistic to begin with ([1]). Electromagnetism is developed directly and effortlessly by a simple transcription of the classical formulas into four-dimensional language. The only novelty that presents itself consists of the kinematical variances that are attributed to the usual quantities. Once the "absolutes" have been supposed, most of those quantities will become "relative," and fusions of them will then define world-tensors.

If optics is the root of special relativity then electromagnetism is its trunk. It was by means of electromagnetism that relativity specified the variances that one must attribute to the force density and the force, the energy density and energy, and that brief deduction will provide the point of origin for all of relativistic dynamics. On the contrary, in the classical epoch, the notions of force and energy, as well as their essential properties that were of interest to all of physics, were obtained by means of dynamics.

Contrary to a prevailing assertion, we will show that it is possible (and even quite indicated) to define the *finite* force to be a tensor that is called a second-rank tensor [**108**]. That will allow us to give a very elegant form to relativistic point dynamics as a result. We shall also deduce a formula that will ultimately serve as the basis for our symmetric presentation of analytical dynamics (Chapter V, § B). Passing from the case of convection to that of conduction, we shall then specify what one generally says about Joule heat from the standpoint of variances. The formulas that will be obtained will comprise a general elaboration that touches upon the problems of the creation and annihilation of energy or mass, and one will recall them later on in the treatment of sources and sinks in hydrodynamics (Chapter V, § A). Finally, it goes without saying that those formulas can illustrate the definitions that relativistic thermodynamics will be

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([1]) Langevin has often insisted upon the fact that electromagnetism collectively affords an implicit verification of relativity. Indeed, the general equations that summarize it, and whose consequences are regularly verified, are not invariant under Galilei transformations, while they are under those of Lorentz-Poincaré.

called upon to pose (Chapter IV, § D). At any rate, *the brief paragraph in which relativity treats force and energy in electromagnetism will appear to be an essential piece in the entire theory*.

The last paragraph of the present chapter addresses the Maxwell elastic tensor and the couple density that are present in polarized media. We propose to introduce another elastic field tensor [**108**] and relate that notion to a couple density that E. Henriot took into consideration [**107**]. To conclude, we recall the two spin densities for the field that were discovered by Henriot. That entire paragraph can be skipped by the novice reader in relativity, and it will be returned to only after the lecture in the following chapter.

A. – THE GENERAL FIELD EQUATIONS

III.2 – **Preamble: Review of the general equations of the Maxwell-Lorentz theory. –** As usual, **E** denotes the electric field and **B** denotes the magnetic induction, so the Maxwell equations with no right-hand side can be written:

$$\text{(III.4)} \qquad \operatorname{rot} \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0, \qquad - \operatorname{div} \mathbf{B} = 0,$$

and one concludes from this that the vectors **E** and **B** can be derived from a vector potential **A** and a scalar potential V according to:

$$\text{(III.2)} \qquad \mathbf{B} = - \operatorname{rot} \mathbf{A}, \qquad \mathbf{E} = \operatorname{grad} V + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} .$$

H. A. Lorentz showed that one can profit from the arbitrariness that exists in the definition of the potentials in order to arrange that they should satisfy the condition:

$$\text{(III.3)} \qquad \operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}V = 0.$$

Now, if **P** and **M** denote the electric and magnetic polarization densities, respectively, of a material medium then the electric induction **D** and the magnetic field **H** will be related to **E** and **B** by:

$$\text{(III.4)} \qquad \mathbf{D} = \mathbf{E} + \mathbf{P}, \qquad \mathbf{H} = \mathbf{B} - \mathbf{M},$$

respectively. That being the case, if **j** denotes the current density, and q that of the charge then the Maxwell equations that have a right-hand side will be written ([1]):

$$\text{(III.5)} \qquad \operatorname{rot} \mathbf{H} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D} = \mathbf{j}, \qquad \operatorname{div} \mathbf{D} = c\, q.$$

([1]) We shall utilize what one calls Heaviside electromagnetic units.

One immediately concludes the continuity equation from that:

(III.6) $$\operatorname{div} \mathbf{j} + \frac{\partial}{\partial t} q = 0.$$

One also defines the "fine" Lorentz current and charge densities and the polarization current and charge densities:

(III.7) $$\operatorname{rot} \mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} = \mathbf{j}_1 , \quad \operatorname{div} \mathbf{E} = c\, q_1 ,$$

(III.8) $$- \operatorname{rot} \mathbf{M} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{P} = \mathbf{j}_2 , \quad \operatorname{div} \mathbf{P} = c\, q_2 ,$$

in such a way that:

(III.9) $$\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2 , \qquad q = q_1 + q_2 .$$

If one takes the Lorentz condition (3) into account then one will deduce the following equations of propagation of the potentials from formulas (2) and (5):

(III.10) $$\Box \mathbf{B} = \mathbf{j}_1 , \qquad \Box V = c\, q_1,$$

and then upon introducing (2) once more, the equations of propagation of the field:

(III.11) $$\Box \mathbf{B} = - \operatorname{rot} \mathbf{j}_1 , \qquad \Box \mathbf{E} = \operatorname{grad} c\, q_1 + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{j}_1 .$$

Recall that Kirchhoff gave to the equation of the type:

$$\Box \psi = \varphi$$

the "retarded" solution:

$$\psi = \frac{1}{4\pi}\iiint \{\varphi\}\frac{\delta u}{r} .$$

ψ is the value of the "potential" that is created at the point (0, 0, 0) at the instant 0 by the "distribution" φ, where $\{\varphi\}$ denotes the value at the point $x^u = r\, \alpha^u$ at the instant $- r / c$ ($\alpha_u\, \alpha^u \equiv 1$, $x_u\, x^u \equiv r^2$). In II.11, we analyzed the mechanism of relativistic invariance in that formula.

III.3 – **Relativistic transcription of the general field equations.** – We now propose to put equations (1) to (11), inclusive, into a relativistic form. If u, v, w always denote a circular permutation of the spatial indices 1, 2, 3 then we introduce the system of six essentially-antisymmetric components E^{ij}, and the system of six dual components B^{kl} (i, j, k, l = 1, 2, 3, 4) by the formulas:

(III.12) $$\boxed{E^{vw} \equiv iB^{u4} = E^{u}, \quad E^{u4} \equiv iB^{vw} = iB^{u}.}$$

One effortlessly verifies that the four formulas (1) lead to a single tensorial formula, which we write in the two equivalent forms:

(III.13) $$\boxed{\partial_k E^{ik} = 0, \quad \sum \partial^i B^{ik} = 0.}$$

From the definition of the quadri-operator ∂^i or ∂_i , and as a consequence of formula (II.1):

(III.14) $$\partial_u \equiv \frac{\partial}{\partial x^u}, \qquad \partial_4 \equiv \frac{1}{ic}\frac{\partial}{\partial t};$$

the summation $\sum$ extends over all circular permutations of the three indices. One or the other of formulas (13) show that the two dual systems of six components E^{ij} and B^{ij} are tensors that we shall call the *magneto-electric induction field tensor and electromagnetic induction field tensor*, respectively, or M. E .I. F and E. M. I. F, to abbreviate.

By reason of their homogeneity, formulas (14) then permit us to define the *electromagnetic polarization tensor* P^{ij} and its dual M^{kl} by ([1]):

(III.15) $$\boxed{-P^{vw} \equiv iM^{u4} = -P^{u}, \quad -P^{u4} \equiv iM^{vw} = iM^{u},}$$

as well as their *magneto-electric electric field tensor* (M. E. I. F.) H^{kl} and its dual D^{ij} (E. M. I. F), by:

(III.16) $$\boxed{D^{vw} \equiv iH^{u4} = D^{u}, \quad D^{u4} \equiv iH^{vw} = iH^{u}.}$$

Formulas (4) can then be condensed into one or the other of the equivalent form:

(III.17) $$\boxed{H^{ij} \equiv B^{u4} - M^{u}, \quad D^{kl} \equiv E^{ki} + P^{kl}.}$$

As always, by means of the definitions (14), one then easily verifies that the four equations (5) can be condensed into the formula:

([1]) In regard to that point, we remark that the terminology (which has been, unfortunately, enshrined by its usage) of *fields* and *inductions* proves to be troublesome. It is contrary to the desire for both simple names for relativistic tensors and elegance in the defining formulas. It would seem desirable, if possible, to invert the two terms in either the electric or the magnetic domain and to change the sign of the corresponding polarization.

We have sought to give our defining formulas (12) and (16) maximum elegance, so we have written H^{ij} and B^{ij}, where R. Becker, for example, wrote F^{ij} and H^{ij}, respectively. The letter F hardly seems appropriate in that position because it leaves behind the way that E, B, D, H are used pre-relativistically. On the other hand, we shall need to reserve it for our tensorial definition of the finite force (cf., *infra*, C).

We have likewise changed the sign in the definition of the tensor M^{ij} with respect to the symbols of R. Becker [**6**], in order to preserve the similitude of formulas (17) and (4).

(III.18) $$\boxed{\partial_l H^{kl} = j^k,}$$

under the simple condition that one must set:

(III.19) $$\boxed{j^4 = icq.}$$

One then sees that the charge density q fuses with the three components of the current density j^u to constitute a world-quadri-vector that is called the *quadri-current* j^k, to abbreviate. As a pre-relativistic theory, one likewise defines the other two quadri-vectors:

(III.20) $$\boxed{\partial_l B^{kl} = j_1^k, \quad \partial_l M^{kl} = -j_2^k,}$$

in such a way that one has:

(III.21) $$\boxed{j^k = j_1^k + j_2^k.}$$

j_1^k is called the *fine (or hidden) quadri-current*, and j_2^k is the *polarization quadri-current.*

One deduces the E. M. I. F. tensor B^{ij} from one or the other of the equivalent formulas (13). It is derived from a quadri-potential A^i, and defined up to a quadri-gradient $\partial^i U$, by ([1]):

(III.20) $$\boxed{B^{ij} = \partial^i A^j - \partial^j A^i.}$$

With the simple condition that one must set:

(III.23) $$\boxed{A^4 = iV,}$$

that formula will condense (2), and show that the quadri-potential A^i results from the fusion of the vector potential A^u and the scalar potential V into a single geometric entity.

If one substitutes (22) into (20) then one will get:

(III.24) $$\partial_l^l A^k - \partial_l^k A^l = j_1^k.$$

A^k is defined only up to a quadri-gradient, so one (with Lorentz) can impose the condition (3), or:

(III.25) $$\boxed{\partial_l A^l = 0.}$$

Indeed, that amounts to the equation:

$$\partial_l (A_0^l - \partial^i U) = 0 \qquad \text{or} \qquad \partial_l^l U = \partial_l A_0^l,$$

([1]) That assertion does not constitute anything besides the relativistic formulation of what one usually proves in Lorentz's theory. Here, we have given it a formulation that is symmetric in space-time from the proof itself.

which is classical, and which one knows how to solve. Thanks to the Lorentz condition (25), equation (24) takes the form:

(III.26) $$\boxed{\partial_l^l A^k = j_1^k,}$$

which condenses (10), and which shows that the potential propagates from the source j_1^k with the velocity *c*. Finally, if one again takes (22) into account then that will likewise give the equation of field propagation that condenses (11):

(III.27) $$\boxed{\partial_i^i B^{kl} = \partial^i j_1^k - \partial^k j_1^i.}$$

As for equation (18), which condenses the Maxwell equations with right-hand side, from the fact that the tensor H^{kl} is antisymmetric, it will imply the equation of continuity:

(III.28) $$\boxed{\partial_k j^k = 0,}$$

which condenses the pre-relativistic form (6). One similarly deduces from (20) that:

(III.29) $$\boxed{\partial_k j_1^k = 0, \quad \partial_k j_2^k = 0.}$$

Finally, before leaving this subject, we point out that the quadri-vector l^i (or its dual l^{jkl}):

(III.30) $$\boxed{l^i = \partial_k P^{ik} \equiv \partial_k D^{ik}, \quad l^{jkl} = -\sum \partial^l M^{jk} \equiv \sum \partial^l H^{jk},}$$

which is non-zero, in general, is referred to as the *magnetic polarization current.*

III.4 – **Examples of new variances in relativistic electromagnetism. Some important invariants.** For example, since it is classical, we give the law of partially-reciprocal transformation of the electric field into the magnetic induction by a change of Galilean frame. In the case of a special Lorentz transformation, one has:

$$E_1' = E_1, \qquad B_1' = B_1,$$

$$E_2' = \frac{E_2 - \beta B_3}{\sqrt{1-\beta^2}}, \qquad B_2' = \frac{B_2 + \beta E_3}{\sqrt{1-\beta^2}},$$

$$E_3' = \frac{E_3 + \beta B_2}{\sqrt{1-\beta^2}}, \qquad B_3' = \frac{B_3 - \beta E_2}{\sqrt{1-\beta^2}}.$$

We also point out the existence of the relativistic invariants:

$$- E_{ij}\, E^{ij} \equiv B_{kl}\, B^{kl} = 2\,(\mathbf{B}^2 - \mathbf{E}^2), \qquad E_{ij}\, B^{ij} = 4\,\mathbf{E}\cdot\mathbf{B}, \qquad A_i\, A^i = \mathbf{A}^2 - V^2.$$

One knows that the first two of these are zero in the case of a monochromatic plane wave. As for the third one, if ox_1 is the direction of the rays of the plane wave then the known relation:

$$A_1 + V = 0$$

will show that it is positive, and as a consequence, the invariant considered will be made zero, and the quadri-vector A^i will be isotropic ([1]).

B. – INVARIANCE AND CONSERVATION OF ELECTRIC CHARGE

III.5 – **Convection current and conduction current.** – First consider an unpolarized, charged fluid that flows through the universe. In that case, one will have:

$$H^{ij} = B^{ij}, \qquad j^k = j_1^k$$

in all space-time. If $\mathbf{v}$ or v^u denotes the usual velocity field of the fluid then, from Lorentz, one will have the formula (31_1), beside which, we rewrite formula (19):

$$j^u = qv^u, \qquad j^4 = ic\, q. \tag{III.31}$$

If one introduces the proper charge density q_0, or the value that q takes in the locally-Galilean frame that follows instantaneously, which is such that:

$$q = \frac{q_0}{\sqrt{1-\beta^2}}\,, \tag{III.32}$$

as well as the quadri-vector V^l that is defined by (II.62), then the preceding formulas will condense into the form:

$$\boxed{j^l = q_0 V^l.} \tag{III.33}$$

In the case that is presently under consideration, the two quadri-vectors j^l and V^l will then be collinear, and one will say that the electrified matter is in a "regime of pure convection."

On the contrary, now take a portion of a conducting fluid body, where V^l denotes its quadri-velocity field. We say that the body is in a "regime of pure conduction" if the charge density q_0 is zero, but the current $\mathbf{j}_0$ is not at the origin of any locally-Galilean frame that instantaneously follows the fluid. Physically, that amounts to saying that a flux of electrons crosses the lattice of positive ions that composes the conducting body with a mean quadri-velocity that is not coincident with V^l. The mass density of that

(1) In L. de Broglie's *Théorie du photon*, the small, positive value for the invariant $A^i A_i$ is related to that of the proper mass of the photon. (Paris, Hermann, 1940, v. I, pp. 169).

electronic flux is extremely weak, and one can neglect it in a schematic treatment of the problem, which amounts to considering the quadri-current of the electrons to be completely immaterial. As for the nullity of q_0 that is assumed, by hypothesis, it results from the exact compensation of the electronic charge density, which is moving and negative, by the charge density of the matter, which is fixed and positive. Moreover, nothing will prevent us from relocating it into the context of Lorentz's theory of electrons and assuming that the conduction quadri-current results from the addition of a fine (or hidden) quadri-current to a polarization quadri-current that is due to the vorticity of the hidden charges.

The general case of a medium that is both convective and conductive results from the superposition of the two preceding notions. If there is no compensation between the charge density of matter and the electronic charge density in the co-moving frame then the excess of the first will general a convection quadri-current that is collinear to V^k and has the type that was studied in the first paragraph. Finally, in the general case, the total quadri-current j^k can be considered to be the sum of two orthogonal quadri-currents j'^k and j''^k, the former of which is collinear to V^k, while the latter is orthogonal to it.

III.6. – **Invariance and conservation of charge.** – As in Chapter II, § C, Let $\mathcal{T}$ be the congruence of world streamlines of a flow V^k, let $\mathcal{E}$ be a current hyper-endcap (which is generally curvilinear and space-like) that represents a fluid drop in space-time and follows its motion, and let $\mathcal{P}$ be the lateral hyper-wall of the tube that is generated by the two-dimensional fluid surface $\mathcal{S}$ of the drop $\mathcal{E}$. Always let $ic\ \delta u^1$ and $ic\ \delta s^{kl}$ be the dual tensors with elements $[dx_i\ dx_j\ dx_k]$ and $[dx_i\ dx_j]$, such that $ic\ \delta u^4 \equiv \delta u$ and $ic\ \delta s^{u4} \equiv \delta s^u$ represent the elementary volume and area, respectively, in the usual sense. Finally, recall the definition [II, (80)] of the *elementary scalar volume* of the hyper-endcap and the expression [II, (79)] for the elementary volume of the hyper-wall.

Now consider the tensorial invariant:

$$\delta Q = j_k\ \delta u^k. \tag{III.34}$$

On a hyper-endcap $\mathcal{E}$, one can give it the form:

$$\delta Q_{\mathcal{E}} = q_0\ \delta u_0\,. \tag{III.35}$$

Moreover, under constant-time integration, its expression will reduce to the one that is well-known in classical physics for the "finite electric charge at the instant t":

$$\delta Q_{\mathcal{E}} = q\ \delta u\,. \tag{III.36}$$

The same result will be obtained in the local frame of the field V^k if the quadri-vectors j^k and V^k are collinear, but will no longer be true in the general case in which we have placed ourselves, and that remark is closely attached to an apparent paradox that is well-known in Minkowski's electrodynamics [**4**, § 21.c]. In the general case of an arbitrary hyper-endcap $\mathcal{E}$, if one takes [II, (75)] into account then the expression (36) can be specified by:

(III.36′) $$\delta Q = q\ \delta u + j_u\ \delta s^u\ \delta t,$$

and the significance of the last group of terms will be clear: They amount to three terms for the flux of electricity when one takes into account the fact that the infinitesimal drop δu is not considered in a simultaneous manner. If one prefers, that flux constitutes a *correction for non-simultaneity.*

On the hyper-wall P, and taking [II, (79)] into account, the invariant δQ can be written:

(III.37) $$-\ \delta Q_{\mathcal{P}} = j_k\ V_l\ \delta s^{kl}\ d\tau.$$

From the fact of the antisymmetry of the tensor δs^{kl}, the necessary and sufficient condition for $\delta Q_{\mathcal{P}}$ to be zero is that the quadri-vectors j_k and V_l must be collinear. In the local proper frame, the expression for $\delta Q_{\mathcal{P}}$ will reduce to:

(III.38) $$\delta Q_{\mathcal{P}} = \mathbf{j}\cdot\delta\mathbf{s}\ dt.$$

Everything that we just said clearly shows that the tensorial invariant (34), when calculated on a hyper-endcap element, merits the name of an *electric charge* whose *state* corresponds to material drop, and when it is calculated over a hyper-wall element, it will merit the name of *electric flux that leaves* across the contour of the same material drop between two of its consecutive *states.* In the two cases, the expression considered is that of a *world hyper-flux* with a quadri-vector j^k that crosses a three-dimensional hypersurface.

Now consider two successive *states* $\mathcal{E}_1$ and $\mathcal{E}_2$ of the same finite drop $\mathcal{E}$; i.e., two hyper-endcaps that are not cut by a hyper-tube that is bounded laterally by a hyper-wall $\mathcal{P}$, and integrate the expression (34) over the closed contour $\mathcal{E}_2 - \mathcal{E}_1 + \mathcal{P}$. If one transforms the quadruple integral:

$$\iiint\!\!\int \partial_k j^k \delta\omega ,$$

when it is extended over the domain that is enclosed in the preceding one, and finally takes the equation of continuity into account (28), then one will get the important result that:

(III.39) $$\boxed{Q_2 - Q_1 = -Q_{\mathcal{P}}.}$$

The variation of the charge of the drop $\mathcal{E}$ from the state 1 *to the state* 2 *is equal to the flux of electricity that enters through the contour $\mathcal{S}$ of $\mathcal{E}$.*

The necessary and sufficient condition for one to have $Q_2 \equiv Q_1$ (i.e., $Q_{\mathcal{P}} \equiv 0$) is that the quadri-vectors j^k and V^k must be collinear; i.e., that one must be in a regime of pure convection. In that case, the charge on a given material drop will be independent of the manner by which one calculates it. Notably, it will be the same for a "classical" integration at constant time and for a "relativistic" integration over an arbitrary $\mathcal{E}$.

C. – FORCE DENSITY AND FINITE FORCE

III.7 – **Theory of the Lorentz force in the convection regime.** – We shall study, in parallel, on the one hand, an electrified world-fluid whose charge density is q, and on the other hand, a rigorously point-like charge Q. We shall show that an argument that is entirely similar to the one by which one usually defines the world force density to be a quadri-vector will permit us to define the finite world-force to be a second-rank antisymmetric tensor. Obviously, if one is given the third-rank variance of the three-dimensional volume element then those two notions will be formally compatible with each other. Meanwhile, we see that in order to establish the correspondence in question, we essentially suppose that the quadri-velocity V^i of the fluid q tends to a limit along an infinitely-thin world-tube, and that it will coincide with that of the point-like charge Q, which will then describe the mean streamline of the tube, We pass to the limit by constricting the tube, and assume essentially that the charge δQ that is attached to the tube (with hyper-flux being conserved) does not vary under that operation. Finally, we suppose that $Q = \delta Q$.

Consider the well-known formulas that give the Lorentz force density and the finite Lorentz force when they are applied to the fluid q and the point-like charge Q, respectively:

(III.40) $$\mathbf{f} = q\,(c\,\mathbf{E} + \mathbf{v} \wedge \mathbf{B}), \qquad \mathcal{F} = Q\,(c\,\mathbf{E} + \mathbf{v} \wedge \mathbf{B}).$$

They imply the consequences:

(III.41) $$\mathbf{f} \cdot \mathbf{v} = cq\,\mathbf{v} \cdot \mathbf{E}, \qquad \mathcal{F} \cdot \mathbf{v} = cQ\,\mathbf{v} \cdot \mathbf{E},$$

respectively. With the single condition that one must set:

(III.42) $$\boxed{f^4 = \frac{i}{c}\mathbf{f} \cdot \mathbf{v},}$$

which is a formula that imposes the condition upon the quadri-vector f^i that:

(III.43) $$\boxed{V_i\, f^i = 0,}$$

and if one takes the definition (12) for the tensor B^{ij} ([1]) into account then formulas (40_1) and (41_1) can be condensed into the form:

([1]) Recall that, for homogeneity of notation, and in order to preserve the letter F, we write B^{kl} and H^{kl} where some authors – notably, R. Becker [**6**] – write H^{kl} and F^{kl}, respectively.

(III.44) $$\boxed{f^k = B^{kl} j_l .}$$

We remark that formula (43), which we have posed by hypothesis, is a consequence of (44) and (33). That then seems to be a link between the two hypotheses (33) and (43) whose sense will be made precise in the following no. Hence, the *world force density* quadri-vector results from the association of the components of the usual force density with the power density that it provides. For the classical theory, only the latter quantity will be *relative.*

We shall now make an argument in regard to formulas (40_2) and (41_2) that is completely parallel to the preceding one. Since Q is a scalar quantity (tensorial invariant), in order to give those formulas a covariant form, we must first multiply them by dt. By definition, we set the elementary *impulse-energy quadri-vector* that is provided by the point-like charge Q equal to:

(III.45) $$\boxed{dp^u = \mathcal{F}^u dt, \quad dp^4 = \frac{i}{c}\boldsymbol{\mathcal{F}} \cdot d\mathbf{M} \equiv \frac{i}{c}\boldsymbol{\mathcal{F}} \cdot \mathbf{v}\, dt.}$$

Formulas (40_2) and (41_2) are condensed into the form:

(III.46) $$dp^k = Q\, B^{kl}\, dx_l \, .$$

By definition, we then set the *world force* that is applied to the point-like charge Q by the field B^{kl} :

(III.47) $$\boxed{F^{kl} = Q\, B^{kl},}$$

and then:

(III.48) $$\boxed{F^u = ic\, F^{u4}, \quad K^u = F^{vw}.}$$

One sees that $\mathbf{F}$ and $\mathbf{v} \wedge \mathbf{K}$ are the Coulomb force and the Laplace force, respectively, that are applied to Q:

$$\mathbf{F} = e\, Q\, \mathbf{E}, \qquad \mathbf{v} \wedge \mathbf{K} = Q\, \mathbf{v} \wedge \mathbf{B}.$$

We say – as always, by definition – that $\mathbf{F}$ is the *force* and $\mathbf{K}$ is the *coforce* that are applied to the point Q by the field, and we see that the *world force* will result from the fusion of the *force* and the *coforce* into just one tensor. It is only the fault of not having introduced the motion of *coforce* that the relativists have not further explicitly defined the finite force to be a tensor. The relation between the *apparent force* $\boldsymbol{\mathcal{F}}$ (with no definite variance), the force $\mathbf{F}$, and the *coforce* $\mathbf{K}$ is (49_1), which results from (49_2):

(III.49) $$\boxed{\boldsymbol{\mathcal{F}} = \mathbf{F} + \mathbf{v} \wedge \mathbf{K}, \quad \boldsymbol{\mathcal{F}} \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v}.}$$

If one substitutes (47) into (46) then one will obtain the purely dynamical relation:

(III.50) $$\boxed{dp^k = F^{kl} dx_l.}$$

By definition, and to abbreviate the discussion, we often call the elementary impulse-energy dp^k the *elementary quadri-work* that is done by the world-force F^{kl} under the quadri-displacement dx^k (in which dp^4 is the elementary work, properly speaking, up to a factor of i / c). Always by definition, we say that *two world-forces* F^{kl} *and* F'^{kl} *are* equivalent *for a quadri-displacement* dx_l *if they provide the same elementary energy-impulse;* i.e., if one has:

(III.51) $$\boxed{(F'^{kl} - F^{kl})\, dx_l = 0.}$$

The transcription of that formula into ordinary language is:

$$\overline{F' - F} + \mathbf{v} \wedge (\overline{K' - K}) = 0, \qquad \mathbf{v} \cdot \overline{F' - F} = 0.$$

Finally, electromagnetism leads us quite naturally to define the force density to be a space-like quadri-vector f^i, and the finite force to be a second-rank antisymmetric tensor F^{ij}. Formally, those two definitions are perfectly coherent, and it seems that if $ic\ \delta u^l$ always denotes the elementary quadri-vectorial volume then one must expect to verify the formula:

$$\delta F^{ij} = f^i\ \delta u^j - f^j\ \delta u^i.$$

Now, if one takes formulas (44), (47), and (34) into account then one can easily be assured that the tensor:

(III.52) $$\boxed{\delta F'^{ij} = f^i\, dx^j - f^j dx^i}$$

will differ irreducibly from the tensor δF^{ij} that was defined by (47), when one performs the passage to the limit from a thin hyper-tube of charge δQ to a filamentary trajectory by keeping δQ constant, by hypothesis. Since the hyper-tube and the dual $ic\ \delta u^l$ of its hyper-section are time-like, to say that the hyper-tube is infinitely thin amounts to saying that the instant-points of the hyper-section are contained in a quadri-parallelepiped whose dimensions will all vanish when one passes to the limit.

The inequality of the two tensors δF^{ij} and $\delta F'^{ij}$ seems difficult to justify on first glance. However, *if we appeal essentially to the relation* (43), which one necessarily introduces into the present theory, as well as the hypothesis (which is necessary for the passage to the limit) that the quadri-vector j^l in formula (44) and the dx^l in formula (50) are collinear:

(III.53) $$j^k\, dx^l - j^l\, dx^k = 0,$$

then we shall show that the two tensors δF^{ij} *and* $\delta F'^{ij}$ *are* ***equivalent*** *in the sense of formula* (51).

We first calculate the elementary quadri-work $d\delta p'^i$ that is provided by the force $\delta F'^{ij}$ during the quadri-displacement dx^i. *If one takes* (43) and (52) *into account* then formula (50) can be written:

(III.54) $$d\delta p'^i = f^i \; \delta u^j \; dx_j \, ;$$

$\delta u^j \, dx_j \equiv \delta\omega$ is nothing but the dual of the four-dimensional hyper-volume element. In passing, we make an interesting remark that will find its application in what follows: *If we do not consider the antisymmetry to be an essential characteristic of the second-rank world-force tensor then we can say that, in the sense of formula* (51), *there is an equivalence between the tensors* $\delta F'^{ij}$ *and* δF_1^{ij} *that are defined by* (52) *and:*

(III.55) $$\boxed{\delta F_1^{ij} = f^i \delta u^j ,}$$

respectively.

Now, take the tensor δF^{ij} that is defined by (47). We will then be led to represent the force density by a ternary tensor that is antisymmetric in i, k:

(III.56) $$\boxed{f^{ikl} = B^{ik} j^l ,}$$

and to relate the finite force to that density by:

(III.57) $$\boxed{\delta F^{ik} = f^{ikl} \delta u_l \, .}$$

Upon comparing them with (44) and (52), formulas (56) and (57) will show the irreducible difference between the tensors δF^{ij} and $\delta F'^{ij}$. *If we take* (53) *into account* then the quadri-work that that is done by the force δF^{ij} can be written, in succession:

$$d\delta p^i = B^{ik} \, dx_k \cdot j^l \; \delta u_l = B^{ik} \, j_k \cdot dx^l \; \delta u_l \, ,$$

and consequently, if one takes (44) into account:

(III.58) $$\boxed{d\delta p^i = f^i \delta u^k dx_k \equiv f^i \delta\omega.}$$

As we have said, that expression coincides with the one (54) for the elementary quadri-work that is done by δF^{ij}.

If we replace the field B^{ij} as a function of its quadri-potential A^k in (22) in the expression for the quadri-work done by F^{ik} in (47) and (50) then that will give the very important relation:

(III.59) $$\boxed{d(p^k - QA^k) = -Q \partial^k A^l dx_l \, ,}$$

which can be used as a basis for all of the relativistic analytical dynamics of the point (no. V.6). If one introduces the useful notion of *total energy-impulse* (inertial + electromagnetic):

(III.60) $$\boxed{P^k \equiv p^k - QA^k}$$

then one will introduce, *ipso facto*, an asymmetric definition:

(III.61) $$F_1'^{kl} = -Q\,\delta^k A^l$$

for the finite force.

III.8 – **Case of conduction. Laplace force and Joule heat.** – If one is given a conducting body that is traversed by a current density **j** then the classical expression for the force density that is applied by the field will be:

(III.62) $$\mathbf{f} = \mathbf{j} \wedge \mathbf{B}\,.$$

The experiment with the Barlow wheel, for example, shows that it is indeed the matter of the conducting body to which the force density is applied. If j''^k denotes the conduction quadri-current, as always, then the expression that is covariant from the four-dimensional viewpoint and becomes coincident with (62) in the local proper Galilean frame will be:

(III.63) $$j''^k = B^{kl}\, j''_l\,.$$

Of course, if there also exists a proper charge density inside the body, and consequently, a convection quadri-current, then one must add the expression f'^k, whose theory was given in the preceding no., to f''^k, and one will then set:

(III.64) $$f^k = f'^k + f''^k\,,$$

in the general case. *Since* $j^k = j'^k + j''^k$ *is not collinear with* V^k *in the present case, the force* f^k *will no longer satisfy the relation* (43). Now, one recalls that it is that relation that is the basis for the equivalence of the two expressions for the finite force [eq. (54)]:

$$\delta F'^{ij} = f^i\,\delta u^j - f^j\,\delta u^i \quad\text{and}\quad \delta F_1^{ij} = f^i\,\delta u^j.$$

In what follows, it will be shown that, contrary to what one might think *a priori, it is the second, asymmetric, one of the two preceding expressions that must be retained.*

Now, if r denotes the resistivity of a material medium, and consequently $1 / r$ denotes its conductivity, consider the density expression of Ohm's law:

(III.65) $$\frac{1}{cr}\mathbf{E} = \mathbf{j}.$$

If one assumes that r is a relativistic invariant then one will see, with Minkowski, that the only covariant relation that one can establish between the tensors B^{ij} and j_k that will agree with (65) in the local proper frame is [cf., eq. (12_1)]:

(III.66) $$\boxed{V_k B^{ik} = r\, j''^i.}$$

V^k always denotes the quadri-velocity field of the conducting body. By definition, V_k / r is the *quadri-vectorial conductivity*, and $1 / r$ is the *scalar conductivity* of the body under consideration.

Finally, always in density form, consider the classical expression for the Joule heat that is released in the elementary volume δu during the elementary time dt ([1]):

(III.67) $$J\, d\delta Q = \mathbf{j}^2\, \delta u\, dt.$$

Naturally, in relativity, we will make sure to preserve the scalar character of the universal constant J. If we first adopt the quadri-vectorial concept of conductivity then we will see that one covariant way of writing it that will agree with (67) locally will be:

(III.68) $$\boxed{-\frac{1}{c^2 r} J\, V_k\, d\delta Q^k = j''_l\, j''^l\, (\delta u_k dx^k).}$$

The finite heat is then defined to be the temporal component of a certain quadri-vector, as was done already for energy [eq. (45_2)]. The three spatial components of that quadri-vector (whose world-direction we leave indeterminate, for the moment, and whose study we shall pursue in no. IV.20) will obviously deserve the name of *caloric impulse.*

In no. IV.1, we shall see that it is quite useful to define the notion of the *proper* – or *scalar* – *energy* of a material drop $\mathcal{E}$, along with that of quadri-vectorial impulse-energy. In the case of an infinitesimal drop, the proper energy will be nothing but the length of the energy-impulse quadri-vector. Here, the notion of *proper heat* that is homogeneous to the proper energy is introduced quite naturally by starting with that of the scalar conductivity; indeed, the notation:

(III.69) $$\boxed{\frac{1}{r} J\, d\delta Q_0 = j''_l\, j''^l\, (\delta u_k dx^k)}$$

will obviously agree with (67) locally. For a conducting droplet δu^k or $\delta u_0 \equiv V_k\, \delta u^k$, the relation between the *quadri-vectorial heat* and the *proper heat* will be:

(III.70) $$\boxed{V_k \delta Q^k = -c^2 \delta Q_0.}$$

Upon taking into account formulas (63) and (66), in succession, one can write:

([1]) Integrate formula (67) over du for a filamentary element of the conductor of length l and section s, and take into account the known formulas:

$$R = rl/\,s, \qquad I = sj,$$

in which R denotes the resistance, and I, the intensity. One will then get the well-known integral formula:

$$J\, dQ = RI^2\, dt.$$

(III.71) $$V^i\, f_i'' = -\, r\, j''^k\, j_k'',$$

in such a way that the expression (69) for the proper Joule heat can be further written:

(III.74)[*sic*] $$\boxed{J\, d\delta Q_0 = -V_i\, f''^i (\delta u_k dx^k).}$$

In no. IV.5, we shall see how that result is interpreted in the dynamical theory of volume forces, and, conforming to what we have said, it will appear that the definition of the finite force that we must retain is the asymmetric definition (55).

D. THE ELASTIC TENSORS OF THE FIELD. SPIN DENSITY AND ELECTROMAGNETIC COUPLE DENSITY ([1]).

III.9 – **The asymmetric Maxwell-Minkowski tensor and the density of electromagnetic ponderomotor couples.** – If B^{kl} and D^{kl} denote the duals of the two arbitrary antisymmetric tensors E_{ij} and H_{ij} , respectively, then one will effortlessly verify the identity ([2]):

(III.75) $$\boxed{E^{ik} D^j_{\ ,k} + B^{jk} H^i_{\ ,k} \equiv \tfrac{1}{2} E^{kl} D_{kl} \delta^{ij} \equiv \tfrac{1}{2} B^{kl} H_{kl} \delta^{ij}.}$$

That being the case, start with Maxwell's equations – with and without right-hand sides – namely, (18) and (13_1), resp., as well as Lorentz's formula (44). One can then write:

$$f^{\,i} = -\, B^{ik}\, \partial^l H_{lk} + D^{ik}\, \partial^l E_{lk}\,,$$

and if one takes the identity (75) into account then one can transform it into:

$$f^{\,i} = -\tfrac{1}{2} B^{ik}\, \partial^l H_{lk} + \tfrac{1}{2} E_{lk}\, \partial^l D^{ik} - \tfrac{1}{4} B^{kl}\, \partial_l H_{lk}\, \delta^{ij} + \tfrac{1}{2} D^{ik}\, \partial^l E_{lk} - \tfrac{1}{2} H_{lk}\, \partial^l B^{ik} + \tfrac{1}{4} H^{kl}\, \partial_l B_{lk}\, \delta^{ij}.$$

Then set, by definition, the relativistic *elastic tensor of the field*, or *Maxwell-Minkowski tensor:*

(III.76) $$\boxed{M^{ij} = \tfrac{1}{2}[D^{ik} E^j_{\ ,k} - B^{ik} H^j_{\ ,k}] \equiv -B^{ik} H^j_{\ ,k} + \tfrac{1}{4} B^{kl} H_{kl} \delta^{ij}.}$$

The result that was obtained can then be written:

([1]) The novice reader of relativity should defer reading this § D until he is familiar with the theory of inertial tensors of elastic type in Chapter IV, § A, as well as that of ponderomotor and proper kinetic moments in Chapter IV, § B. All of the elements that are necessary for one to establish the relativistic dynamics in Chapter IV, § A are contained in the preceding § C.

([2]) For $i \neq j$, take $i, j = u, v = 1, 2, 3$, for example. One will have:

$$(H^{uw}\, D^v_{\ ,w} - H^{u4} D^v_{\ ,4}) - (H^{u4}\, D^v_{\ ,4} - {}^{uw}D^v_{\ ,w}) \equiv 0.$$

For $i = j$, take $i = j = 4$, for example. One will have:

$$H^{u4}\, D_u^{\ 4} + \tfrac{1}{2}\, H^{vw}\, D_{vw} \equiv \tfrac{1}{2} H^{kl}\, D_{kl}\,.$$

(III.76) $$\boxed{f^i = \partial_k M^{ik} + \tfrac{1}{4}[H^{kl}\partial^i B_{kl} - B_{kl}\,\partial^i H^{kl}].}$$

To our knowledge, Minkowski was the first to exhibit the presence of the $\frac{1}{4}[^i]$ term in the right-hand side of the expression for f^i in the general case ([1]).

The Maxwell-Minkowski tensor is defined for any instant-point, inside the polarized matter, as well as outside it; however, the force density f^i is, of course, zero *in vacuo*. $B^{kl} = H^{kl}$ *in vacuo* or in non-polarized matter, and the $\frac{1}{4}[^i]$ will term disappear. The same thing will be true under the hypothesis that is usually adopted that in a certain Galilean frame $\mathcal{G}_0$:

$$\mathbf{D}_0 = \varepsilon\,\mathbf{E}_0\,, \qquad \mathbf{B}_0 = \mu\,\mathbf{H}_0\,,$$

with ε = const. and μ = const.; indeed, since the quadri-vector $\frac{1}{4}[^i]$ is then zero in $\mathcal{G}_0$, it will be intrinsically zero ([2]).

It is important to remark that *the tensor M^{ik} is defined only by starting with its divergence in k; i.e., only up to an additive tensor whose divergence in k is zero.* The interpretations of the components of M^{ik} that we shall give, and which are generalizations of concepts that are due to especially Maxwell and Poynting (not to mention Abraham and Poincaré), are physically ambiguous then, in the sense that an arbitrary additive constant will be implied in them. That remark, we repeat, arrives at the same calculation principle that is found in Maxwell [**99**]. Hence, in particular, *one can consider the energy density w of the Maxwell field to be defined only up to an additive constant.* That remark, when combined with the presence of the Minkowski term $\frac{1}{4}[^i]$ in formula (77), confers a somewhat fictitious character upon the set of classical interpretations that we shall present as a coherent whole, thanks to relativity.

Recall the expression (76) for the elastic tensor of the universe M^{ij}. For $i = j = u = v$ = 1, 2, 3, one will get the nine well-known expressions for the pre-relativistic Maxwell-Heaviside-Hertz tensor:

(III.78) $$\boxed{M^{uv} = H^u B^v + E^u D^v + \tfrac{1}{2}(\mathbf{D}\cdot\mathbf{E} - \mathbf{B}\cdot\mathbf{H})\,\delta^{uv};}$$

for $i, j = u$, 4, and $i, j = 4, u$, one will get the two Poynting vectors:

(III.79) $$\boxed{R^u_{(1)} \equiv -i\,M^{u4} = [\mathbf{D}\wedge\mathbf{B}]^{vw}, \quad R^u_{(2)} \equiv -i\,M^{4u} = [\mathbf{E}\wedge\mathbf{H}]^{vw}.}$$

The first of them is interpreted as the *impulse density* of the field, and the second one is its *energy current density*. Finally, for i, j = 4, 4, one will get the well-known expression for the *energy density* of the field:

([1]) Gott. Nachr., (1908), pp. 53 or Math. Ann. **68** (1910), pp. 472; see also § 13 of the study in question.

([2]) In the case where the system contains permanent magnetism, the preceding relations between the fields and the inductions will not be satisfied inside of the magnets, and the term $\frac{1}{4}[^i]$ must be taken into consideration. If one omits it then one will, in particular, arrive at an erroneous expression for the energy of the system and certain incorrect physical predictions. (Course taught by L. de Broglie in 1948-1949 at l'Institut Henri Poincaré. *Note added during correction of the proofs.*)

(III.80) $$\boxed{w \equiv -M^{44} = \tfrac{1}{2}(\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}).}$$

In regard to the definition (76_2) of the tensor M^{ij}, one sees immediately that its trace M_i^i, which is physically homogeneous to the *proper* (or *scalar*) *energy density* of a continuous material medium, is zero:

(III.81) $$\boxed{M_i^i = 0.}$$

Moreover, if one constructs the *proper ponderomotor couple density of the universe* tensor from the theory of elasticity:

(III.82) $$\boxed{\mu^{ij} = M^{ji} - M^{ij}}$$

or

(III.83) $$\boxed{\mu^{ij} = B^{jk}H^{i}_{\cdot k} - B^{ik}H^{j}_{\cdot k} \equiv D^{jk}E^{i}_{\cdot k} - D^{ik}E^{j}_{\cdot k}}$$

then one will effectively recognize, with Maxwell and Heaviside, that its components:

(III.84) $$\boxed{\mu^{w} \equiv \mu^{uv} = [\mathbf{H}\wedge\mathbf{B}+\mathbf{E}\wedge\mathbf{D}]^{uv}}$$

are the well-known components of the density of couples that are applied to the polarized body. In passing, we point out the expressions:

(III.85) $$i\,\mu^{\mu 4} = [\mathbf{D}\wedge\mathbf{B} - \mathbf{E}\wedge\mathbf{H}]^{vw}.$$

III.10. – **Another elastic tensor for the field. Its relationship to a couple density that was considered by E. Henriot.** – We always start from Lorentz's formula (44) and substitute in it, not just the Maxwell-Minkowski equations with a right-hand side, but also their consequences (28) and (22), respectively. We can write:

$$f^{i} = (\partial^{k}A^{i} - \partial^{i}A^{k})\,j_k = \partial_k(A^{i}j^{k}) - j_k\,\partial^{i}A^{k},$$

and the last group of equations will be transformed into:

$$-j_k\,\partial^{i}A^{k} = -\tfrac{1}{2}\,j_k\,\partial^{i}A^{k} - \tfrac{1}{2}\partial^{i}(A^{k}j_k) - \tfrac{1}{2}\,j_k\,\partial^{i}A^{k}.$$

Finally, if we set:

(III.86) $$\boxed{N^{kl} = A^{k}j^{l} - \tfrac{1}{2}A^{i}j_i\,\delta^{ij}}$$

then we will get the following expression for f^{i}:

(III.87) $$\boxed{f^{i} = \partial_k N^{ik} + \tfrac{1}{2}[A^{k}\partial^{i}j_k - j_k\partial^{i}A^{k}].}$$

The forms of the relations (86) and (87) are curiously parallel to those of (76) and (77), resp.

Contrary to the Maxwell-Minkowski tensor, the new elastic tensor N^{ik} is not defined *in vacuo*, but only in matter where a quadri-current of convection, conduction, or polarization is present; the same thing will be true for the quadri-vector $\frac{1}{2}[^i]$, which is not zero, in general ([1]).

We study the various components of the tensor N^{kl}. For k, $l = u$, $v = 1, 2, 3$, one will get the nine components of an elastic tensor in ordinary space:

(III.88) $$\boxed{N^{uv} = A^u j^v - \tfrac{1}{2}(\mathbf{A}\cdot\mathbf{j} - cVq)\,\delta^{uv}.}$$

For k, $l = u$, 4 and k, $l = 4$, u, one will get an *impulse density* vector ([2]), and an *electromagnetic energy-current density* vector:

(III.89) $$\boxed{R^{*u}_{(1)} \equiv -iN^{u4} = cqA^u, \quad R^{*u}_{(2)} \equiv -iN^{4u} = Vj^u.}$$

Finally, for k, $l = 4, 4$, one will recognize the classical expression for the electromagnetic energy density, which is assumed to be concentrated in the distribution of currents and charges:

(III.90) $$\boxed{w^* \equiv -N^{44} = \tfrac{1}{2}(\mathbf{A}\cdot\mathbf{j} + cVq).}$$

Of course, just like the components of the tensor M^{kl}, one must not consider the preceding expressions to be those of the "true" physical values, because the tensor N^{kl} is defined only up to an additive tensor whose divergence in l is zero. Moreover, a $\frac{1}{2}[i]$ term is present in (87) that is not identically zero.

Contrary to the situation for the tensor M^{kl}, the trace of the tensor N^{kl} is not zero, and one will have:

(III.91) $$\boxed{-N_i^i = A^k j_k = \mathbf{A}\cdot\mathbf{j} - cVq.}$$

That expression is homogeneous to what one calls a *proper* (or *scalar*) *energy density* of the distribution.

Finally, consider the tensor v^{kl}, which is physically homogeneous to a proper ponderomotor density of the universe:

(III.92) $$\boxed{v^{kl} = N^{lk} - N^{kl}}$$

or

([1]) In L. de Broglie's *Théorie du photon*, a quadri-current $j^k = \mu_0^2 A^k$ prevailed, for which the coefficient μ_0^2 was very small.

([2]) $R^{*u}_{(1)}$ is indeed the density quantity that is homologous to the electromagnetic impulse $c\,Q\,A^u$ of a point-like charge.

(III.93) $$\boxed{\nu^{kl} = A^l j^k - A^k j^l.}$$

That formula can be written explicitly as:

(III.94) $$\boxed{\nu^w \equiv \nu^{uv} = [\mathbf{A} \wedge \mathbf{j}]^{uv},} \qquad \nu^{u4} = i\,(V\,\mathbf{j} - cq\,\mathbf{A})^u.$$

The three ν^w correspond to a couple density, properly speaking.

Physically, it is certain that the couple density ν^w is not manifested, which should not seem absurd in light of the remarks that were made. Nevertheless, the consideration of the tensor ν^{kl} is not entirely devoid of interest. E. Henriot encountered that same tensor, and with the same interpretation, along another path [**107**, pp. 47].

III.11. – **A formula of E. Henriot. The two spin densities of the electromagnetic field.** – To conclude these considerations, we shall prove a formula of E. Henriot in our manner and generalize it slightly. We start with the expression $\mu^{kl} - \nu^{kl}$ and infer from (83) and (93) that:

$$\mu^{kl} - \nu^{kl} = -\left\{H^{k}_{\cdot i} B^{li} + A^{i \cdot k}_{j}\right\} + \{\text{symmetric in } k, l\}.$$

If one replaces B^{kl} with (22) and j^k with (18) then one will get:

$$\begin{aligned} &-\left\{H^{k}_{\cdot i}(\partial^i A^l - \partial^l A^i) + A^l \partial_i H^{ki}\right\} + \{\text{symmetric in } k, l\} \\ &= -\left\{(\partial^i A^l H^{ki}) - \partial^l (H^{ki} A_i) + A_i \partial^l H^{ki}\right\} + \{\text{symmetric in } k, l\} \\ &= \partial_i (A^k H^{li} - A^l H^{ki}) \partial^l (H^{ki} A_i) - \partial^k (H^{li} A_i) - A_i [\partial^l H^{ki} - \partial^k H^{li}]. \end{aligned}$$

However, if one sums over all cyclic permutations and takes into account the definition (20_2) of the magnetic polarization current, as well as Maxwell's equation (13) with a right-hand side, then:

$$\partial^l H^{ki} - \partial^k H^{li} = \sum \partial^i H^{kl} + \partial^i H^{kl} = l^{ikl} + \partial^i H^{kl}.$$

Finally, if one takes the Lorentz condition (25) into account then one can write:

$$A_i\,[\partial^l H^{ki} - \partial^k H^{li}] = -A_i\, l^{ikl} + \partial_i\,(A^i H^{kl}).$$

Then set:

(III.95) $$\left\{\begin{array}{l} \boxed{\sigma^{ijk}_{(2)} = A^i H^{jk} - A^j H^{ik} \equiv \sum A^k H^{ij} - 2A^k H^{ij},} \\ \boxed{\sigma^{i}_{(2)} = H^{ik} A_k,} \end{array}\right.$$

in which the summation is again over all cyclic permutations, and one will arrive at *E. Henriot's formula* [**107**, pp. 13-14, formula B], which has been generalized by taking the magnetic polarization current into consideration:

(III.96)
$$\boxed{\mu^{ij} - \nu^{ij} + A_k l^{ijk} = \partial_k \sigma^{ijk}_{(1)} + [\partial^j \sigma^i_{(2)} - \partial^i \sigma^j_{(2)}].}$$

The antisymmetric tensors μ^{ij} and v^{ij} are physically homogeneous in the proper ponderomotor couple densities, while the ternary tensor $\sigma^{ijk}{}_{(1)}$, which is antisymmetric in *i*, *j*, and the quadri-vector $\sigma^i{}_{(2)}$, is homogeneous in the spin densities of electromagnetic origin. E. Henriot called these two groups of tensors *torques* and *momentors*, respectively.

CHAPTER IV

RELATIVISTIC DYNAMICS

IV.1. – If the deduction of relativistic dynamics imposes that one must appeal to electromagnetism then that is because, before Einstein, dynamics ignored the role that was played by c in its proper domain ([1]).

The most fundamental intervention of the constant c in dynamics is produced by the law of universal proportionality between mass and energy:

$$W = c^2 m$$

that was deduced by Einstein and Langevin at almost the same time. That law constitutes the great physical discovery of the theory.

Since Einstein, the traditional manner of basing relativistic dynamics consisted of imposing the laws of variance of force upon electromagnetism. In regard to that, one remarks that the argument is much more incongruous than the much easier one that one finds upon starting with relativistic electromagnetism. Now, to us, that situation is provided *uniquely* by the fact that one generally bases relativistic dynamics by starting with the dynamics of points. On the contrary, we show that the deduction in question can be performed without the least incongruity, and *by simple transcription*, if one works in terms of continuous media. The equations that one must use for that – whose generality is perfect – are not usually given in the treatises on Newtonian mechanics. We must then start by establishing them at the onset of the present chapter by a brief argument from Newtonian mechanics. To our knowledge, von Laue was the first to write down those equations ([**4**], 28 *b* and Appendix *c*). The ease by which one thus founds relativistic dynamics must be compared with the ease by which one relativistically transcribes the Maxwell-Lorentz field theory. Indeed, all of the equations of the latter theory are density equations in which the fields are homogeneous to polarization densities. Conforming to a

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([1]) In his course at the Collège de France (which is, unfortunately, unpublished), P. Langevin showed by an ingenious argument that that certain very general energetic postulates permit one to establish deductively the expression for the *vis viva* of a material point, and starting from that, all of dynamics. If the kinematics used is that of Galileo then one will get back to the classical expression $mv^2 / 2$; on the contrary, with that of Lorentz, one will get the Einsteinian expression for mass $m = m_0 / \sqrt{1-\beta^2}$.

In his likewise-unedited papers, Allard has generalized Langevin's arguments considerably. He showed that the entire theory of the dynamics of a point can be constructed abstractly without appealing to kinematics. After postulating the existence of the *vis viva* theorem, he then showed that the theories kinematics that were still possible were three in number: That of Galileo-Newton, that of Lorentz-Poincaré, with a universal constant that he identified with c, and a symmetric kinematics in which the ds^2 is positive-definite.

Along the same order of ideas, an argument that is even more general, and is based upon the theory of groups, was presented by V. Lalan [**89**]. Lalan specified some very general postulates that implied either only classical kinematics or the alternative of classical kinematics and that of relativity as a consequence.

The interest that these abstract considerations present is uncontestable. From the viewpoint of the theoretical physicist, they amount to constituting an *a posteriori* classification of results that are found by much simpler inductions.

remark that is becoming classical, the extreme affinity between relativity and the theories of continuous media and fields is verified once more.

Once we have written the well-known fundamental equations of the relativistic dynamics of fluids, which are found by starting with the universal law $w = c^2\rho$ in its density form, we will direct our attention to *proper mass density* ρ_0 of the fluid. That will give rise to a double series of inductive generalizations, one of which concerns the volumetric force density, and the other of which concerns the force density of surface origin. In the former case, we shall give an absolute generalization of a result that was found in the context of Joule's theory of heat (no. III.8) that relates to the creation and annihilation of energy or mass. In the latter case, we begin by giving a theory of isotropic pressure in fluids that will find some applications later on (nos. IV.21 and V.2). To conclude sub-chapter A, we will deduce the relativistic dynamics of points from that of continuous media. We put the classical results into covariant form that is made possible by our definition of the rank-two *finite force* tensor (no. III.7). Recall that in the relativistic dynamics of points, the *vis viva* theorem, as well as the variation of mass with velocity, are essentially kinematical effects, or with the established terminology, *relative* ones.

Our sub-chapter B is dedicated to the problem of the relativistic theory of the proper kinetic moment, or *spin*. Like the other authors that are interested in that subject, and whose work will be mentioned in the bibliography, it is naturally the interpretation of certain equations in quantum theory – and mainly Dirac's theory of the electron – that we have in mind. On that subject, the tendency of some authors has been to argue in terms of integrals, rather than densities. True to our chosen path, we have, on the contrary, made all of our arguments in terms of continuous media, in order to deduce the case of the material point as only a result of them. In that regard, the work of Weyssenhoff and Raabe is very close to ours. It is satisfying that two inductive arguments that are compelled by the nature of things and completely independent of each other are essentially compatible in all of their common results. In that same sub-chapter B, we have introduced the consideration of general surface forces of elastic type. The reader will perhaps be surprised to see a theory of essentially microscopic origin – viz., that of *spin* – together with a theory of an essentially macroscopic and statistical nature – viz., that of elastic forces; the single justification for such an agreement resides in the great kinship (which is wrong, except as a formal identity) between the mathematical concepts that are implied by one theory and the other. The formulas that are obtained are of great generality, but it results from the entire system of current knowledge that they have two radically distinct domains of application: On the one hand, the microphysical one, in which the theory of *fictitious probability fluids* essentially ignores the elastic forces, and on the other hand, the macrophysical one, in which no possible manifestation of *proper kinetic moments* has ever been confirmed. In that way, the two series of applications of our general formulas will be based upon the identical annihilation of entire groups of terms.

In sub-chapter *C*, we shall begin the unresolved problem of the relativistic dynamics of systems of points in interaction. To commence, once we have generalized a fundamental theorem of the theory of torsors into four-dimensional terms, we shall give the mean technique for consistently defining the notions of *barycenter* and *moment about* *the barycenter* [1]. Guided, in turn, by electromagnetism, as is expressed by the notions of the *density of energy that is distributed throughout the Maxwell field* and Abraham-Poincaré's *potential impulse density*, we shall show that in order to arrive at the relativistic statement of some classical *general theorems*, one must take into consideration the distribution and step-by-step transmission of energy and impulse (as well as spin) in "potential" form throughout the field. It will then appear that *by its very nature, the relativistic dynamics of systems prohibits the approximations that permit one to use* ***general theorems*** *as intermediaries in the reasoning of Newtonian dynamics.* It turns out that the problem considered must be essentially a field problem, which is qualitatively apparent in the *N*-body problem of general relativity.

Our last sub-chapter is dedicated to some general considerations of relativistic thermodynamics. That subject seems to have been somewhat forsaken between the epoch of the fundamental papers of Einstein and Planck, which both appeared in 1907, and the epoch of the modern work of Tolman, Van Dantzig, Eckhart, Bergmann, and other authors. Now, there is a curious divergence regarding a main question that appears between these two series of papers, namely, the question of the *variance of temperature.* For Einstein and Planck, the variance of temperature, which has no tensorial character, seems to be that of the volume of a portion of matter that is considered at a single moment. On the contrary, the modern authors, while directing all of their attention to the problem of continuous media, have defined the inverse of temperature. For one group (Tolman, Eckhart), it was an invariant θ_0, while for the other (Van Dantzig, Bergmann), it was the θ^4 component of a quadri-vector θ^i. That dilemma, as well as the one that we spoke of in the Foreword, presents itself in close connection with the one that is posed by the practice of *integrating at constant time*, or with the use of *complete differential forms*, in the sense of E. Cartan; we shall discuss that question in no. IV.19 in a particular case. Then, in IV.20, we shall systematically present the viewpoint of integral invariants, which is that of all our work. For the sake of illustrating some new tensorial definitions of temperature, IV.21 and IV.22 will give covariant expressions for the fundamental equations of the theory of thermal conduction and that of perfect gases, respectively. For a more adequate development of relativistic thermodynamics, we refer the reader to the work that is especially devoted to that subject.

A. – GENERAL EQUATIONS OF THE DYNAMICS OF INVISCID FLUIDS WITHOUT SPIN. DEDUCTION OF THE DYNAMICS OF POINTS WITHOUT SPIN.

IV.2 – **Preamble: General form that one can give to the equations of continuous media in Newtonian mechanics. On the classical principle of the conservation of mass. –** We begin by establishing a general formula from Newtonian fluid mechanics that we shall give several applications of. Let **v** or v^u (u = 1, 2, 3) be the velocity field of a fluid at the instant t, let δu be an infinitesimal volume of matter that follows its motion, and let Φ be an arbitrary function of the fluid molecule; i.e., it follows the motion from

[1] We have just been made aware of an interesting, and quite recent, work by H. L. Pryce that is dedicated to that double question, and thanks to it, the existence of some prior studies of Fokker and Papepetrou (*note added in correction*).

the geometric standpoint. Indeed, Φ will be the volumetric density of one of the quantities that are taken under consideration. For our purposes, we must essentially suppose that the material volume δu remains infinitesimal in all of its dimensions in the course of time, which is a hypothesis that is frequently made in fluid mechanics, and that one can call "anti-ergodic."

The increases in the function Φ and the material volume δu during the infinitesimal time dt have the expressions:

$$d\Phi = (v_u\, \partial^u \Phi + \partial^t \Phi)\, dt,$$
$$d\,(\delta u) = \operatorname{div} \mathbf{v}\ \delta u\ dt \equiv \partial^u v_u\ \delta u\ dt,$$

respectively. When combined, these two relations will give the general formula that we have in mind:

$$d\,(\Phi\ \delta u) = \{\partial^u (\Phi\, v_u) + \partial^t \Phi\}\, \delta u\ dt. \tag{IV.1}$$

We then take the well-known fundamental equations of point dynamics:

$$\mathfrak{F}\ dt = d\,(m\,\mathbf{v}), \qquad \mathfrak{F}\ d\mathbf{M} = dW. \tag{IV.2}$$

We have taken the expression $\mathfrak{F}\ dt$ in a form that is appropriate for our purposes, and not in the form $m\ d\mathbf{v}$, which is equivalent in Newtonian mechanics. We shall deduce the general equations of continuous media from the preceding equations by an application of formula (1). Ultimately, in relativistic dynamics, we shall proceed in the opposite order and deduce the equations of a point from those of continuum mechanics.

Let δm be the mass of the droplet δu, let $\delta\mathfrak{F}$ be the total ponderomotor force that is applied to it, let δW be the work that it has done since an arbitrary initial time, and define the corresponding densities ρ, f, w by:

$$\delta m = \rho\ \delta u, \qquad \delta\mathfrak{F} = \mathbf{f}\ \delta u, \qquad \delta W = w\ \delta u. \tag{IV.3}$$

The fundamental equations (2) for the droplet δu are written:

$$\mathbf{f}\ \delta u\ dt = d\,(\rho\,\mathbf{v}\ \delta u), \qquad (\mathbf{f}\cdot\mathbf{v})\ \delta u\ dt = d\,(w\ \delta u). \tag{IV.4}$$

There is no reason to add the corresponding equations "in moments" to them, because *in the case considered of an inviscid fluid without spin*, the kinetic and ponderomotor moments that relate to the droplet δu are infinitely small of fifth order, while all of the $\Phi\ \delta u$ in the problem are of third order.

Upon applying formula (1) to the right-hand side of equations (4) and dividing both sides by $\delta u\ dt$ ([1]), one will get the four equations:

([1]) Recall that $\delta u\ dt$ is a relativistic invariant [no. II.6, eqs. (11), (27)].

(IV.5) $$\boxed{\begin{aligned} f^u &= \partial_v(\rho v^u v^v) + \partial_t(\rho v^v), \\ \mathbf{f}\cdot\mathbf{v} &= \partial_u(w v^u) \quad + \partial_t(w). \end{aligned}}$$

One sees an *inertia tensor* $v^u\, v^v$ enter into them that is symmetric and of rank two, and which provides a contribution of "elastic" type to the force density. That tensor has a degenerate form, moreover, since it is the general product of a vector by itself.

Newtonian mechanics assumes the conservation of the mass δm of each droplet δu *as a principle*. An application of formula (1) will then immediately provide the classical *continuity equation:*

(IV.6) $$\boxed{\partial_u(\rho v^u) + \partial_t(\rho) = 0.}$$

From an interesting remark that is due to E. Durand, we then take out the quantity $\rho\ \delta u$ in the ( ) on the right-hand side of (4_1) that is reputed to be conservative and make it appear in the ( ) in the right-hand side of (4_2) so we can treat it likewise. Moreover, we scalar multiply the two sides of (4_1) by **v**, so we will get the relation:

$$\mathbf{v}\, d\mathbf{v} = d\left(\frac{w}{\rho}\right)$$

in a general manner, which will give:

(IV.7) $$\boxed{\frac{w}{\rho} = \frac{1}{2}v^2 + \text{const.}}$$

by integration. We point out that, up to the present, relativity rejects both formulas (6) and (7).

IV.3. – **Deduction of the fundamental laws of the relativistic dynamics of continuous media.** – Our first group of postulates, which is completely similar to the one that has classically been the basis for relativistic electromagnetism since Minkowski, consists of:

α) *Taking and preserving density equations such as* (5).

β) *Employing the notion of the quadri-vector differential* ∂^i, with:

$$\partial^4 \equiv \frac{1}{ic}\partial^{\,t}.$$

However, as has been known since the beginning, a second group of postulates that are imposed by the lessons of electromagnetism are necessary if one is to base relativistic dynamics. Here, we take them in the form:

γ) The ponderomotor force density f^u is of purely volumetric origin.

δ) The power density:

$$f^4 = \frac{i}{c}\mathbf{f}\cdot\mathbf{v}$$

must be associated with the three f^u in order to form a quadri-vector f^i that is found, ipso facto, to be orthogonal to the quadri-vector V_i :

(IV.8) $$\boxed{V_i\, f^i = 0.}$$

We point out that in the course of some ultimate inductions, we will liberate ourselves of the two postulates δ and γ.

By means of this set of postulates, and under the simple condition that we set:

(IV.9) $$\boxed{\begin{gathered} T^{uv} = T^{vu} = \rho v^u v^w, \\ T^{u4} = ic\rho v^u, \quad T^{u4} = \frac{i}{c} w v^u, \\ T^{44} = -w, \end{gathered}}$$

the four equations (5) can be condensed into the form:

(IV.10) $$\boxed{f^i = \partial_j T^{ij},}$$

and one will see that the 16 functions T^{ij} are the components of a second-rank tensor. Since the nine T^{uv} are essentially symmetric, the tensor T^{ij} is necessarily symmetric ([1]). From the fact that:

$$T^{u4} = T^{4u},$$

one concludes the extremely important relation:

(IV.11) $$\boxed{w = c^2\rho,}$$

([1]) Indeed, consider the transformation formula:

$$\bar{T}^{u'v'} = \bar{o}_i^{u'}\,\bar{o}_j^{v'}\,T^{ij}.$$

The entire contribution from the right-hand side is essentially symmetric in u', v', except perhaps the ones from the terms:

$$\bar{o}_u^{u'}\,\bar{o}_4^{v'}\,T^{u4} + \bar{o}_4^{u'}\,\bar{o}_u^{v'}\,T^{4u}.$$

Upon expressing the idea that the latter contribution is invariant under a permutation of u' and v', one will get:

$$T^{u4} = T^{4u}.$$

Q. E. D.

which constitutes the great physical discovery of relativity, and which establishes a universal proportionality in the ratio c^2 between the mass and energy (or, what amounts to the same thing, between their densities). That universal relation substitutes for the universal relation (7) that was established by classical theory. If one likewise takes the indeterminate classical constant equal to c^2 then the two relations will coincide when $v \to 0$. Upon substituting the fourth of (5) in (11), one will get the relation:

(IV.12) $$\boxed{\partial_u(\rho v^u) + \partial_t(\rho) = \frac{1}{c^2}\mathbf{f}\cdot\mathbf{v},}$$

which will replace the classical equation of continuity (6). The new right-hand side, which is extremely small, is defined *relative* to the Galilean frame that is used.

Now, introduce the *proper* values ρ_0 and w_0 of the mass and energy densities, resp.; i.e., the values that ρ and w take in the co-moving Galilean frame locally and instantaneously. Naturally, the law of equivalence:

(IV.13) $$\boxed{w_0 = c^2\rho_0}$$

must be true for these *proper* densities, in particular. It is clear from (9) that one will have:

(IV.14) $$\rho = \frac{\rho_0}{1-\beta^2}\,, \quad w = \frac{w_0}{1-\beta^2} \qquad \left(\beta = \frac{v}{c}\right).$$

If the quadri-vector V^i is still defined by (II.62) then the expressions (9) for the inertia tensor condense into the form:

(IV.15) $$\boxed{T^{ij} = \rho_0 V^i V^j.}$$

One has the following two equivalent expressions for the *trace of the inertia tensor:*

(IV.16) $$\boxed{T_i^i = \rho(v^2 - c^2) = -c^2\rho_0.}$$

Up to the factor – c^2, the trace can then be interpreted as the proper mass density of the fluid.

If one takes the expression (15) for T^{ij} into account then the fundamental equation (10) can be written:

(IV.17) $$f^i = \partial_j(\rho_0 V^i V^j) = V^i \partial_j(\rho_0 V^j) + \rho_0 V'^i .$$

If one multiplies all terms by V_i, while taking into account the hypothesis (8) that is presently adopted, as well as (II.62), one will get the relation:

(IV.18) $$\boxed{\partial_j(\rho_0 V^j) = 0,}$$

which will imply that:

$$f^i = \rho_0 V'^i, \tag{IV.19}$$

when it is substituted in the preceding equation.

One sees that equation (18), which is already in its density form, expresses the idea that there is conservation of proper mass within the fluid that answers to the present theory, and that this result will already seem paradoxical, since from the relativistic proportionality between mass and energy, it seems clear that the work that is done by the forces of pressure and viscosity must contribute to the proper mass of the fluid drop. It will soon appear that the very simple explanation for that paradox is that all of the present theory ignores the forces of surface origin, in principle. Upon introducing the theory of those forces, we will see that formula (18) can be completed by a right-hand side of appropriate form. Before we do that, however, we must generalize the theory of volumetric forces in such a manner that it makes the case in which there is (at least approximately) creation or annihilation of energy or mass understandable, since that takes place in Joule's electro-thermal effect, and even in the hydrodynamical theory of sources and sinks.

As for formula (19), which is a consequence of the theory in its present state, it might seem seductive at the moment to regard it as something that yields a covariant generalization of the general formula from Newtonian mechanics:

$$\mathbf{f} = \rho\, \boldsymbol{\gamma}.$$

In reality, in what follows, it will appear clearly that this formula hass value here only by a fortuitous set of compensations that result from the too-restrictive postulates upon which the theory presently rests.

IV.4 – **Continuation of the argument: integral forms.** – Multiply both sides of equation (10) by the four-dimensional element $\delta\omega$ and integrate over the domain that is enclosed by the lateral hyper-wall $\mathcal{P}$ of a current world-tube and two spacelike hyper-endcaps $\mathcal{E}_1$ and $\mathcal{E}_2$. From what was said at the end of no. III.7, the expression:

$$\iiint\int f^i\,\delta\omega$$

that appears in the left-hand side is interpreted as the quadri-work that is done by the finite world-force; i.e., as the ponderomotor energy-impulse:

$$\Delta p^i = \iiint_{\mathcal{E}}\int_{\mathcal{T}} \delta F^{ij} dx_j\,. \tag{IV.20}$$

As always, $\mathcal{E}$ denotes a spacelike current hyper-endcap that must initially coincide with $\mathcal{E}_1$ and finally with $\mathcal{E}_2$, and $\mathcal{T}$ is a segment of the general world-streamline. Once more, by virtue of what was said in no. III.7, and essentially taking into account the hypothesis (8) that was placed at the head of the entire present theory, one can take the tensor δF^{ij} to be the antisymmetric expression (21_1) or the asymmetric expression (21_2) indifferently, since they are *equivalent* in the sense of equation (III.51):

(IV.21) $$\boxed{\delta F^{ij} = f^i \delta u^j - f^j \delta u^i, \quad \delta F^{ij}_{(1)} = f^i \delta u^j.}$$

The quadruple integral that is provided by the right-hand side of equation (10) is transformed into a triple integral that is taken over the contour $\mathcal{E}_2 - \mathcal{E}_1 + \mathcal{P}$ of the preceding domain, which is written (for the third form, see eq. II.80):

$$\iiint T^{ij}\,\delta u_j = \iiint \rho_0 V^i V^j\,\delta u_j = \iiint \rho_0 V^i\,\delta u_0\,.$$

The scalar product $V^j\,\delta u_i$ is identically zero on the hyper-wall $\mathcal{P}$, so the contribution of $\mathcal{P}$ to the triple integral will be identically zero, and what will finally remain in the right-hand side is:

$$\iiint_{\mathcal{E}_2} T^{ij}\,\delta u_j - \iiint_{\mathcal{E}_1} T^{ij}\,\delta u_j\,.$$

Suppose, first of all, that in the Galilean frame that is being used, the current hyper-endcap that represents the material drop $\mathcal{E}$ as it follows it in its motion is planar and horizontal; i.e., that the *state of the current* is defined *at constant time*. By definition, we therefore provisionally place ourselves under the "simultaneity hypothesis" that we shall have to consider in what follows with several repetitions. Of the four δu_j , only:

$$\delta u^4 = \frac{1}{ic}\,\delta u = \frac{1}{ic}\,dx^1\,dx^2\,dx^3$$

will be zero then, in such a way that if one takes (9) into account, as well as (11) then the four components of the triple integral considered can be written simply:

(IV.22) $$p^u = \iiint \rho\, v^u\,\delta u\,, \qquad p^4 = ic\iiint \rho\,\delta u \equiv ic\iiint \delta m\,,$$

One recognizes the classical definitions of the inertial impulse and mass m of the drop $\mathcal{E}$, resp., in this. In the general case where the current hyper-endcap $\mathcal{E}$ is arbitrary (but spacelike), one can continue to say that, by definition, the *integral:*

(IV.23) $$\boxed{p^i = \iiint_{\mathcal{E}} T^{ij}\delta u_j \equiv \iiint_{\mathcal{E}} \rho_0 V^i V^j \delta u_j \equiv \iiint_{\mathcal{E}} \rho_0 V^i \delta u_0}$$

represents the inertial mass-impulse of the material drop $\mathcal{E}$. Taking into account (II.75), and in infinitesimal form, that relation can be made explicit as follows:

(IV.24) $$p^u = \iiint (\rho v^u\, \delta u + \rho v^u v^v\, \delta s_v\, dt)\,, \qquad m = \iiint (\rho\, \delta u + \rho v^u \delta s_u\, dt)\,.$$

Just as was true for equation (III.36′), the last groups of terms can be interpreted as a flux of impulsion and energy, resp., and correspond to a *relativistic correction from non-simultaneity.*

Finally, if one integrates in the way that was just described and takes into account the definitions (20) and (23), equation (10) can be written:

(IV.25) $$\Delta p^i = p^i_{(2)} - p^i_{(1)}\,,$$

or if one prefers, more explicitly:

(IV.26) $$\boxed{\iiint\!\int f^i \delta\omega = \iiint_{\mathcal{E}_2-\mathcal{E}_1} T^{ij} \delta u_j \equiv \iiint_{\mathcal{E}_2-\mathcal{E}_1} \rho_0 V^i V^j \delta u_j = \iiint_{\mathcal{E}_2-\mathcal{E}_1} \rho_0 V^i \delta u_0.}$$

On the left-hand side, one has the ponderomotive expression for the energy-impulse that is provided to the drop $\mathcal{E}$ between the state 1 *and the state* 2, *and on the right-hand side, one has the variation of its inertial mass-impulse between those same two states. That equation then summarizes the fundamental theorems of impulse and energy in a relativistic form.*

Now, take equation (18), and integrate it over the preceding four-dimensional domain. The left-hand side will then transform into the triple integral:

$$\iiint_{\mathcal{E}_2-\mathcal{E}_1+\mathcal{P}} \rho_0\, V^i \delta u_i \equiv \iiint_{\mathcal{E}_2-\mathcal{E}_1+\mathcal{P}} \rho_0\, \delta u_0\,,$$

and for the same reason as before, the contribution of $\mathcal{P}$ to that integral will be identically zero. On a current hyper-endcap $\mathcal{E}$, the expression $V^i \delta u_i \equiv \delta u_0$ is nothing but the *elementary scalar* (or *proper*) *material volume* (eq. II.80). It is then clear that the expression:

(IV.27) $$\boxed{m_0 = \iiint_{\mathcal{E}} \rho_0\, V^i\, \delta u_i \equiv \iiint_{\mathcal{E}} \rho_0\, \delta u_0}$$

deserves the name of *proper* (or *scalar*) *mass* of the drop $\mathcal{E}$, because each of the elements:

(IV.28) $$\delta m_0 = \rho_0\, \delta u_0$$

can be interpreted as the proper mass of the droplet δu^i ; i.e., as the value that is taken by that mass in the co-moving Galilean frame.

Finally, the integration of equation (18) for a material drop along its motion yields the result:

(IV.29) $$\boxed{m_0 = \text{const.}}$$

This is, in fact, the *conservation of proper mass* that we have stated.

We return to the expressions (23) for the mass-impulse of the drop $\mathcal{E}$. If one takes the expression (28) for the proper mass into account then one can write:

$$\boxed{p^i = \iiint_{\mathcal{E}} V^i \delta m_0.} \tag{IV.30}$$

It is important to remark that the *quadri-vector* p^i *is a function of the integration hyper-endcap* $\mathcal{E}$. Two different Galilean observers that would like to define the quadri-vector p^i *in the same current world-hypertube* by an integration at constant time will be in disaccord, not only in regard to the components of p^i, which is obvious, *but also in regard to the quadri-vector itself.* The necessary and sufficient condition for such observers to be identically in accord is that the quadri-vector p^i must remain invariant in the course of the evolution of the system; i.e., that one must have the four relations:

$$\partial_j T^{ij} = 0.$$

In other words, the *form* and *position* of the hyper-endcap $\mathcal{E}$ are not disjoint concepts, since *the non-invariance of the definition of* p^i *and the non-conservation of* p^i *are two related facts.* If such circumstances are not realized in the problem of electric charge (no. III.6) then that is solely by virtue of the equation of continuity (III.28). Here, we have a very clear illustration of what we said in no. II.14, namely, that the integrals of relativity must be invariant with respect to the form of the hyper-endcaps $\mathcal{E}$, and that they must then be capable of being taken arbitrarily.

Before we leave this subject genre, we make one last remark. It is clear that the tensorial invariant:

$$\mathcal{A} = \int_{\mathcal{T}} \iiint_{\mathcal{E}} \delta p^i dx_i \tag{IV.31}$$

constitutes the relativistic definition of action. By virtue of (23), and taking (II.80′), (II.76), and (II.624) into account, we can write:

$$\boxed{\mathcal{A} = -c^2 \iiint \int \rho_0 \delta\omega.} \tag{IV.32}$$

L. de Broglie made use of that remark in his theory of the electron ([1]).

IV.5. – **Two inductions that generalize the preceding theory of volume force.** – Considering the state that was attained by the deductive theory of the two preceding nos., we propose to free ourselves of the restrictive hypothesis (8) that was essential to its basis. This time, we take our point of departure to be the integral formulas of the preceding no., and we will be led quite naturally to consider two successive hypotheses,

([1]) *L'électron magnétique*, Paris, 1934, pp. 223-224.

according to which the finite force is defined by (21_1) or (21_2), resp. In fact, those two definitions cease to be *equivalent* under the present hypothesis:

(IV.33) $$\boxed{V_i f^i \neq 0.}$$

We begin with the first one; i.e., *we adopt the antisymmetric definition* (21_1) *for the finite force.* The fundamental dynamical law:

(IV.34) $$\iiiint \left\{\delta F^{ij} dx_j - \partial_i (\rho_0 V^i V^j)\,\delta\omega\right\} = 0$$

will then be written:

(IV.35) $$\iiiint \left\{ f^i - \partial_i (\rho_0 V^i V^j) \right\} \delta\omega = \int_{\mathcal{T}} \iiint_{\mathcal{E}} f_j \, dx^j \delta u^i .$$

From the form of its right-hand side, it will have the inconvenience of needing to keep the integral form, and of not being able to take a density form that is analogous to (10). To avoid that difficulty, we shall reason by induction.

Under the name of "hypothesis H," *consider the case in which the streamlines $\mathcal{T}$ admit orthogonal trajectories $\mathcal{E}_0$, and suppose that one has taken two of those $\mathcal{E}_0$ to be $\mathcal{E}_1$ and $\mathcal{E}_2$* . The quadri-volume δu_0^i will then be collinear with V^i on the initial and final hypercaps and will be provided by formula ($\text{II}.86_1$) that we established for that situation, moreover. The preceding right-hand side is then written:

$$-\frac{1}{c^2}\iiiint f_j V^j V^i \delta\omega ,$$

in such a way that the equation considered can be put into the density form:

(IV.36) $$\boxed{f^i + \frac{1}{c^2} f_j V^j V^i = \partial_j (\rho_0 V^i V^j).}$$

The stated induction consists of *postulating that the form* (36) *is true in the general case, and consequently, that the integral form* (35) *that we appealed to as an intermediary in our argument is valid only approximately in general.* (Its validity will become rigorous under the previously-formulated "hypothesis H.") We call the foregoing postulate "postulate P." We shall employ the same mode of inductive reasoning several times in what follows and appeal to "postulate P" and "hypothesis H" that we just discussed.

Hence, *if, under the new hypothesis* (33), *we adopt (as an approximation) the antisymmetric definition* (21_1) of the *finite force then the fundamental dynamical equation will have the form* (36), and its right-hand side will always decompose into the same form as in (17). If we then multiply all terms by V_i , and take ($\text{II}.62_4$) into account then *we will find once more that the law* (18) *of the conservation of proper mass is a consequence of that equation* (36); by contrast, formula (19) will become:

(IV.37) $$f^i + \frac{1}{c^2} f_k V^k V^i = \rho_0 V'^i.$$

We shall now – always under the new hypothesis (33) – *likewise attempt to use the asymmetric definition* (21_2) *of the finite force.* This time, the integral form of the dynamical law will differ from (35) by a zero right-hand side, in such a way that *its density form will remain identical to the old form* (10). Upon multiplying all of its terms by V_i , *we will get the law:*

(IV.38) $$\boxed{\partial_i(\rho_0 V^i) = -\frac{1}{c^2} V_i\, f^i,}$$

instead of (18), and furthermore, (37), instead of (19).

Therefore, along with formula (38), recall the integral calculation in no. IV.4 that gave the variation of the proper mass of a material drop $\mathcal{E}$ between an *initial state* $\mathcal{E}_1$ and a *final state* $\mathcal{E}_2$, which are represented by two hyper-endcaps of the same current world-tube. One will get:

(IV.39) $$\boxed{m_{0(2)} - m_{0(1)} = -\frac{1}{c^2} \iiint\!\!\int V_i\, f^i \delta\omega,}$$

or, in infinitesimal form:

(IV.40) $$\boxed{d\delta m_0 = -\frac{1}{c^2} V_i\, f^i \delta\omega.}$$

This result is quite interesting. Indeed, *if one takes into account the relativistic equivalence of mass and energy and the thermodynamic equivalence of energy and heat then it will coincide with formula* (III.74) *for the Joule heat.* We then assume, as is quite natural, that *formulas* (21_2), (33), *and* (38) *agree in all cases where the there is creation or annihilation of heat, energy, or mass. The scalar product $V_i f^i$ will not be zero in the corresponding regions of space-time. It will be negative or positive according to whether one is dealing with creation or annihilation, respectively.* An example of the creation of heat was given in no. III.8 by Joule's electro-thermal effect. An example of the creation or annihilation of mass in hydrodynamics is given by the consideration of volumetric distributions of sources or sinks (see below, no. V.2). Therefore, *in order to be able to take into account of the phenomena of creation or annihilation of mass or energy, we must no longer adopt the antisymmetric definition* (21_1) *or* (III.52) *for the finite force that appeared to begin with, but precisely the asymmetric definition* (21_2) *or* (III.55). An entirely similar situation will appear later on in the theory of viscous surface forces. We remark, in passing, that in this second and last induction, we must bring into play our "postulate P" and "hypothesis H," which amounts to saying that *the starting integral formula, and consequently, formula* (21_2), *are rigorously valid here.*

Finally, *the most general fluid without spin that is subject to the action of forces of purely volumetric origin will satisfy equation* (38), *whose right-hand side will be non-zero only in the regions of space-time that are ruled by distributions of sources and sinks.*

Outside of those regions, equation (38) will reduce to the form (18), which can be written equivalently:

$$\rho_0\, \partial_j\, V^j + \rho_0' = 0.$$

One will then see upon referring to the purely kinematic definition (II.84) of the incompressible fluid that an equivalent definition is that:

(IV.41) $$\rho_0 = \text{const.}$$

along the trajectories $\mathcal{T}$. More generally, if ρ_0 = const. in all of space-time then one will have the relation (II.84), and one will have the right to say that the fluid is *incompressible.*

We then make one last remark, which is due, in principle, to E. Durand. With the dynamical definition (41) of an incompressible fluid, the relation (10), which the case that we are considering must satisfy, by hypothesis, can be written:

$$f^i = \rho_0\, (\sigma V^i + \theta^{ij}\, V_j).$$

We have set, by definition:

$$\sigma = \partial_j\, V^j, \qquad \theta^{ij} = \partial^j\, V^i - \partial^i\, V^j.$$

Indeed, the expression $\partial_j\, \partial^i\, V^j = 0$ follows from the fact that $V_j\, V^j = -c^2$.

IV.6. – **Introduction and theory of surface force in the simple case of one pressure.** – As always, let $ic\ \delta s^{kl}$ be the dual of the world-area element $[dx_i\ dx_j]$, so the three $ic\ \delta s^{u4} = [dx_v\ dx_w]$ will represent area in the usual sense. It is clear that the classical definition:

$$\delta \mathbf{F} = -\,\varpi\,\delta\mathbf{s}$$

of the normal pressure ϖ will admit the four-dimensional generalization:

(IV.42) $$\boxed{\delta F^{ij} = -\varpi\,\delta s^{ij}.}$$

Here again, the elementary force δF^{ij} is essentially antisymmetric. As in classical theory, *we say, by definition, that the fluid is* ***inviscid*** *if the elementary surface force on it has essentially the form* (42).

Therefore, let $\mathcal{C}$ be the two-dimensional contour of a spacelike three-dimensional domain $\mathcal{E}$ that represents a fluid drop. If one integrates the expression (42) over the contour $\mathcal{C}$ and transforms it into a triple integral then one will get the formula:

(IV.43) $$\iint_{\mathcal{C}} \delta F^{ij} = -\iiint_{\mathcal{E}} [\partial^i \varpi\, \delta u^j - \partial^j \varpi\, \delta u^i]\,,$$

which, when compared to (21_1), will permit one, in a sense, to consider the quadri-vector:

(IV.44) $$f^i = -\partial^i \varpi$$

to be the volumetric force density of surface origin.

Although the two *force systems* δF^{ij} and f^i have the same sum, as we just saw, they are not integrally *equivalent*, in the sense that they do not yield the same elementary quadri-work for the deformation of a droplet δu^i. In pre-relativistic elasticity, we recall that this situation is true only for work, properly speaking ([1]). Indeed, to simplify, take the proper time $d\tau$ to be equal along all trajectories $\mathcal{T}$ that issue from $\mathcal{E}$. The quadri-force of the surface forces is written:

(IV.45) $$d\tau \iint_{\mathcal{C}} V_j\, \delta F^{ij} = -\,d\tau \iiint_{\mathcal{E}} V_j (f^i \delta u^j - f^j \delta u^i) - d\tau \iiint_{\mathcal{E}} \varpi(\partial^i V_j \delta u^j - \partial^j V_j \delta u^i)\,;$$

the last integral represents the stated difference.

As always, let $\mathcal{P}$ be the lateral hyper-wall that is generated by the contour $\mathcal{C}$ of $\mathcal{E}$, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be an *initial state* and a *final state* of the drop $\mathcal{E}$, respectively. The left-hand side of (46) is nothing but another form of the left-hand side of (45) when it is integrated over $d\tau$, and if one takes (II.79) into account then an integral transformation will permit one to write:

(IV.46) $$-\iiint_{\mathcal{P}} \varpi\, \delta u^j = -\iiiint \partial^i \varpi\, \delta\omega + \iiint_{\mathcal{E}_2 - \mathcal{E}_1} \varpi\, \delta u^j\,.$$

Upon assuming, for the moment, that the forces of surface origin are the only ponderomotive forces that act upon the fluid, the fundamental dynamical equation will permit one to replace the left-hand side of this with the quadruple integral:

$$\iiiint \partial_j (\rho_0 V^i V^j)\, \delta\omega\,.$$

We then find ourselves confronting a problem that is similar to the one that we posed with equation (35), namely, that of an integral equation that does not admit an equivalent density equation. We shall eliminate that difficulty by the same inductive process that we used before, by appealing to the *hypothesis H* and *postulate P* that were formulated on that occasion.

In the particular case where the $\mathcal{T}$ admit orthogonal trajectories $\mathcal{E}_0$, and one takes two $\mathcal{E}_0$ to be $\mathcal{E}_1$ and $\mathcal{E}_2$, the volume element δu_0^i on $\mathcal{E}_0$ will admit the expression (II.86$_2$). An

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([1]) For a finite, material drop, the classical expression for the total surface force, its impulse, and its work are:

$$-\iint \varpi\, \delta\mathbf{s} = -\iiint \operatorname{grad}\cdot\varpi\, \delta u\,, \quad -\,dt \iint \varpi\, \delta\mathbf{s} = -\,dt \iiint \operatorname{grad}\cdot\varpi\, \delta u\,,$$

$$-\,dt \iint \varpi\, \mathbf{v}\, \delta\mathbf{s} = -\,dt \iiint \operatorname{div}\cdot(\varpi \mathbf{v})\, \delta u = -\,dt \iint \varpi\, \delta\mathbf{s} = -\,dt \iiint \operatorname{grad}\cdot\varpi\cdot\mathbf{v}\, \delta u - dt \iiint \varpi \operatorname{div}\cdot\mathbf{v}\, \delta u\,,$$

respectively. The situation that was stated in the text seems to be, in fact, a direct consequence of the classical notion of *universal time.*

integral transformation will then permit one to replace the ∭ in the right-hand side of (46) with:

$$-\frac{1}{c^2}\iiiint \partial_i(\varpi V^i V^j)\,\delta\omega\,.$$

In this particular case, formula (46) will then admit:

(IV.47) $$\boxed{\partial^i\varpi + \partial_i\left\{\left(\rho_0 + \frac{\varpi}{c^2}\right)V^i V^j\right\} = 0}$$

as an equivalent density form. By virtue of "postulate P," we shall assume the general validity of (47), while that of (46) is rigorous only under "hypothesis H." Formula (46) will then serve as only an intermediary in the inductive argument. *In formula* (47), *one will recover the homogeneity between pressure and energy density that is well-known in classical physics.* In regard to equation (10) (in which the force density is of proper volume origin), equation (47) presents a new term in ϖ / c^2. It is that term, which has the very small coefficient $1 / c^2$ as a factor, whose presence must imply a *variation of the proper mass that is due to the work that is done by surface forces.*

The relation (47) can be transformed into:

$$\partial^{\,i}\varpi + V^i\,\partial_j\,\{\,(\ )\,V^j\} + (\ )\,V'^i = 0.$$

If one multiplies this by V^i and takes (II.62) and (II.87′) into account then one will get:

$$\varpi' - c^2\,\partial_j\,\{\,(\ )\,V^j\} = 0,$$

and since:

$$\partial_i\,(\varpi V^{\,i}) = \varpi\,\partial_i\,V^{\,i} + \varpi',$$

the new relation can be finally written:

(IV.48) $$\boxed{\partial_i(\rho_0 V^i) = -\frac{\varpi}{c^2}\partial_i V^i.}$$

If one takes the existence of pressure forces into account then that is the new law of adiabatic compression that replaces (18). If one continues to call a fluid such that ρ_0 = const. in all of space-time *incompressible*, by definition, then if one discards the physically-uninteresting hypothesis $\varpi = -c^2\rho_0$ = const. in all of space-time, one will see that a fluid will continue to be characterized by formula (II.84).

If one integrates (47) over the usual four-dimensional domain that is enclosed by the hyper-contour $\mathcal{E}_2 - \mathcal{E}_1 + \mathcal{P}$ then one will get:

$$-\iiint_{\mathcal{P}}\varpi\,\delta u^i = \iiint_{\mathcal{E}_2-\mathcal{E}_1}\left\{\varpi\,\delta u^i + \left(\rho_0 + \frac{\varpi}{c^2}\right)V^i\,\delta u_0\right\}.$$

Similarly, if one integrates (48) then one will get the *equation of the variation of proper mass:*

(IV.49) $$m_{0(2)} - m_{0(1)} = -\frac{1}{c^2}\iiint\!\!\int \varpi\,\partial_i(\delta u^k dx_k)\,,$$

which can be further written:

(IV.50) $$\boxed{m_{0(2)} - m_{0(1)} = -\frac{1}{c^2}\iiint\!\!\int \varpi\, d\,\delta u_0}$$

when one takes (II.85) into account. The expression $\iiiint \varpi\, d\ \delta u_0$, which is analogous to the classical $\int p\, dv$, deserves the name of *scalar work that is done by pressure*. As one would naturally expect, *the variation of the proper mass of the drop $\mathcal{E}$ is related directly to the scalar work that is done by pressure.*

IV.7. – **Deduction of some general equations of the dynamics of a point without spin.** – We return to the integral calculation of no. IV.4, and take an infinitely-thin current hypertube with a scalar hypersection δu_0, as well as two infinitely-close initial and final hypersections $\delta u^i_{(1)}$ and $\delta u^i_{(2)}$. Either of the hypotheses that were made in no. IV.5 for the general case $V_i f^i \neq 0$ (eq. 33) will yield the same integral formula:

$$\delta F^{ij}\, dx_j = d\ \delta p^i.$$

The only difference is that with the first hypothesis, the tensor δF^{ij} will be antisymmetric, and with the second one, it will be asymmetric; dx_j denotes the mean trajectory element of the elementary hypertube. In the right-hand side, δp^i represents the inertial expression for the mass-impulse that is attached to the current hyper-wall δu^i. If one takes (22) into account and lets v^u denote the mean value of the ordinary velocity on the hyper-endcap δu^i then, under the hypothesis of simultaneity, one can write:

$$\delta p^u = \delta\, m v^u, \quad \delta p^4 = ic\ \delta m.$$

Moreover, if V^i likewise denotes the mean value of the quadri-velocity on the hyper-endcap δu^i, and δm_0 denotes the proper mass that is attached to that hyper-endcap then, by virtue of (23) and (28), one will generally have:

$$\delta p^i = V^i\ \delta m_0\,.$$

If one then passes to the limit by constricting the hypertube δu_0 , and supposes that the various quantities δ keep their finite values then the preceding equations will become:

(IV.51) $$\boxed{F^{ij} dx_j = dp^i,}$$

(IV.52) $$\boxed{p^u = mv^i, \quad p^4 = icm,}$$

(IV.53) $$\boxed{p^i = m_0 V^i,}$$

respectively, in which m denotes the mass of the material point considered, which is no longer an invariant in relativity, and which is related to the *proper* (or *scalar*) *mass* m_0 by:

(IV.54) $$\boxed{m = \frac{m_0}{\sqrt{1-\beta^2}}.}$$

The quadri-vector p^i, when defined by (52) or (53), obviously deserves the name of *mass-impulse quadri-vector of the material point.* One sees that this mass-impulse quadri-vector p^i is essentially defined to be tangent to the mean hyper-current line V^i of the hypertube that we started with; i.e., as the *tangent to the world-trajectory at the point.* Furthermore, at a given instant-point on the hypertube, *the length of that quadri-vector is independent of the orientation of the hyper-section* δu^i, because as we observed in no. II.14, *the proper volume* δu_0 is invariant under these conditions. In the left-hand side of equation (51), one has the expression for the *elementary ponderomotive quadri-work* that is done by the *world-force* F^{ij}.

The necessary and sufficient condition for the proper mass m_0 (i.e., the length of the quadri-vector p^i) *to be conserved is that the tensor* F^{ij} *must be anti-symmetric.* One will see that immediately in equation (51) when one recalls that the quadri-vectors p_j and dx_j are collinear, and infers the consequence that:

$$p_i\, dp^i = F^{ij}\, p_i\, dx_j .$$

One can also see that from equation (53), when one returns to the considerations of no. IV.5 and remembers that the condition for the conservation of proper mass within the fluid is the antisymmetry of the tensor F^{ij}.

Consider equation (51) more closely in the case where the tensor F^{ij} is antisymmetric, and refer to the definitions (III.49) of the *force* **F** and *co-force* **K**. The tensorial equation (51) can be made explicit in the form:

(IV.55) $$\boxed{(\mathbf{F} + \mathbf{v} \wedge \mathbf{K})\, dt = d(m\mathbf{v}), \quad dW \equiv \mathbf{F} \cdot d\mathbf{M} = c^2 dm.}$$

(55_1) are nothing but the well-known fundamental equations of dynamics, when put into the generalized form that relativistic covariance demands [**120**]. As for equation (55_2), it expresses the relativistic equivalence of mass and energy in point-like language.

One can then pose an entirely natural question: *Would one be unable to deduce the relativistic equations of point dynamics* (49) *or* (55) *directly from the theory of finite forces that is given by equations* (III.47) *and the following ones?* The necessary and sufficient condition for that to be true is that one must be able to infer equation (52_2) from (52_1) *deductively.* Now, one can pass from (52_1) to (52_2) by an induction that is not equivalent to a deduction if it is to be extremely natural. When we reasoned with continuous media in nos. IV.2 and IV.3, that induction was indeed replaced by a deduction, thanks to the symmetry that was imposed on the tensor T^{ij}. That is why, conforming to what we said, part of the induction will be reduced when one establishes relativistic dynamics by starting with continuous media.

By virtue of (51), if the derivatives x'^i, x''^i, … are taken with respect to τ, one will have:

(IV.56) $$p^i = m_0\, x'^i, \qquad p'^i = m_0\, x''^i.$$

Formula (49) can then be written:

(IV.57) $$F^{ij}\, p_j = m_0^2\, x''^i .$$

That is the covariant expression for a formula that is used quite currently in the study of the curved trajectories of electrified particles in a magnetic field. Upon assuming that the force **F** is zero in the Galilean frame that is used, and the co-force **K**(x) is constant in time, the world-trajectory will be a helix, and one can write the formula:

$$\mathbf{K} \wedge \mathbf{p} = m_0^2\, \mathbf{x}''_\tau = m^2\, \mathbf{x}''_t .$$

It is used constantly in order to determine the proper mass of particles in the Wilson chamber when one knows their impulse, and *vice versa*, in the form:

(IV.58) $$K = p_1\, \rho,$$

in which ρ denotes the curvature of the cylinder around which the spatial helix is wrapped, and p_1 is the constant modulus of the projection of the impulse normal to the generators.

To conclude, here is one last form for the equations of point dynamics that is used occasionally. Return to formula (9), which *is written essentially under the hypothesis* (8). Conforming to a remark in no. II.14, we introduce the *finite force of the second type*, whose elementary definition is:

(IV.59) $$\boxed{\delta F^{ij} = f^i \delta u_0,}$$

and which is collinear with the quadri-vector x''^i_τ or V'^i_τ. It is with that definition, which is generally posed explicitly, that some classical treatises on relativity have introduced the finite force ([**2**], pp. 115-166). If one takes (28) into account and passes to the limit as before then the fundamental equation of point dynamics will take the form:

(IV.60) $$\boxed{F^{ij} = m_0 V'^i = p'^i,}$$

in which the derivatives are taken with respect to proper time τ. That formula generalizes a classical three-dimensional formula into four-dimensional form in a very simple manner.

IV.8. – **Applications of relativistic point dynamics. Purely kinematical interpretation of the *vis viva* theorem. Deviations from Newtonian dynamics at large velocities.** – Formula (54) is developed into:

(IV.61) $$m = m_0 \left(1+\tfrac{1}{2}\beta^2+\cdots\right) \qquad \text{where} \qquad \boxed{W = m_0c^2+\tfrac{1}{2}m_0v^2+\cdots}$$

The classical *vis viva theorem* is then recovered in the first approximation, with the remarkable situation that the integration constant $m_0 c^2$ is kept fixed. One should note the very new interpretation that the *vis viva* theorem takes on in relativity: *It amounts to a purely kinematical theorem,* since the kinetic energy appears when one is not in the proper system of the point. In the final analysis, that new interpretation is much more satisfying than the old one. In classical theory, one hardly sees how the *relative* character of the velocity – and consequently, the kinetic energy – is consistent with the alleged *absolute* character of the force, mass, and energy, in general.

In relativity, a material point that is animated with the limiting velocity c will have an infinitely large energy. In order to accelerate a given proper mass point up to the velocity c, one must provide an infinite amount of energy to it. One sees that in the dynamical confirmations of the limiting character of the constant c in all physical velocities.

Recall the formula:

$$\mathfrak{F} = \frac{d}{dt}(m\mathbf{v}),$$

which involves ordinary force and velocity, and develop it, while taking into account the fact that the mass is now variable:

$$\mathfrak{F} = m\frac{d\mathbf{v}}{dt}+\frac{dm}{dt}\mathbf{v}\,.$$

That form suggests the decomposition of force, like acceleration, tangentially and normally to the trajectory. If one takes the well-known formula:

$$\gamma_l = \frac{dv}{dt} = v'$$

into account then one will the following expressions for the "longitudinal force" and the "transversal force":

$$\mathfrak{F}_l = mv' + m'\,v, \qquad \mathfrak{F}_t = m\,\gamma_t\,.$$

The expression for m' is provided by (54) as follows:

$$m' = \frac{m_0\beta\beta'}{(1-\beta^2)^{3/2}} = \frac{m\beta\beta'}{1-\beta^2}, \qquad m'v = mv'\,\frac{\beta^2}{1-\beta^2}\,.$$

Finally, the desired laws for longitudinal acceleration and transversal acceleration are:

$$\mathfrak{F}_l = \frac{m\gamma l}{1-\beta^2} = \frac{m_0\gamma_l}{(1-\beta^2)^{3/2}} = m_0\,\gamma_l\left(1+\frac{3}{2}\beta^2+\cdots\right),$$

$$\mathfrak{F}_t = m\gamma_t = \frac{m_0\gamma_t}{(1-\beta^2)^{1/2}} = m_0\,\gamma_t\left(1+\frac{1}{2}\beta^2+\cdots\right).$$

In the early days of relativity, one called the two acceleration coefficients:

$$m_l = \frac{m_0}{(1-\beta^2)^{3/2}}, \qquad m_t = \frac{m_0}{(1-\beta^2)^{1/2}}$$

the *longitudinal and transversal mass*, respectively. The only interesting notion in the present theory is the relativistic mass *m* that we have used up to now, and which one can call the *Maupertuisian mass*.

Be that as it may, the preceding formulas provide practical expressions for the laws of acceleration of a material point of given proper mass at large velocities. One knows that the study of large velocities in the dynamics of the electron has plainly confirmed the relativistic dynamics of a material point [**112**, **113**, **117**].

IV.9 – **Another application of the relativistic dynamics of points: the collision of two particles with large velocities. –** Consider two particles, which are first assumed to be point-like, and are found to coincide in space-time at an instant-point x^i. Before and after that encounter, they are assumed to be rigorously non-interacting, in such a way that they each describe a rectilinear world-trajectory. The two trajectories will break at the instant-point x^i, and there will be an exchange of mass-impulse between the two particles. If those particles have no spin then their mass-impulse will certainly be collinear to their world-trajectory. Later on, we shall see that the same thing is true for particles with spin in the particular case of uniform, rectilinear motion.

We have four mass-impulses $p^i_{\mu\nu}$ to consider. The index ν takes the values 1 or 2 according to the particle considered is before or after the collision, resp. There is obviously conservation of total mass-impulse under the collision, which is written:

(IV.62) $$p^i_{11} + p^i_{21} = p^i_{12} + p^i_{22}$$

when the four quadri-vectors are concurrent. If the internal characteristics of the particles are not altered by the collision then one will have:

(IV.63) $$|\,p^i_{11}\,| = |\,p^i_{12}\,|, \qquad |\,p^i_{21}\,| = |\,p^i_{22}\,|,$$

moreover.

If one sets:

(IV.64) $$\Delta p^i_1 = p^i_{12} - p^i_{11}, \qquad \Delta p^i_2 = p^i_{22} - p^i_{21},$$

by definition, then (62) can be written:

(IV.65) $$\Delta p_1^i + \Delta p_2^i = 0.$$

As always, by definition, we say that the spacelike axis that is defined by the quadri-vector Δp_1^i or $-\Delta p_2^i$ is the *instantaneous axis of the collision.*

Take that axis to be the Ox^1 axis, and complete the tetrahedron $Ox^1x^2x^3x^4$ to be timelike. One will have:

(IV.66) $$\Delta p_1^1 + \Delta p_2^1 = 0 \quad \text{and} \quad \Delta p_1^k = \Delta p_2^k = 0 \qquad \text{for} \qquad k = 2, 3, 4.$$

One concludes from (66_2) that *for each particle, there is conservation of the component of mass-impulse that is normal to the instantaneous axis.* If one takes that into account, along with (63), then it will result from (66_1) that one must have:

$$p_{11}^1 = \varepsilon\, p_{12}^1, \qquad p_{21}^1 = \varepsilon\, p_{22}^1 \qquad \text{with} \qquad \varepsilon = \pm 1.$$

Only the case of $\varepsilon = -1$ will correspond to an actual collision. We then set, by definition:

(IV.67) $$p_{11}^1 = -p_{12}^1 = p_1^1, \qquad p_{21}^1 = -p_{22}^1 = p_2^1,$$

so (66_1) will show that one has:

(IV.68) $$p_1^1 + p_2^1 = 0.$$

Therefore, *the collision exchanges components that are, moreover, equal in modulus between the components of the mass-impulse that are parallel to the instantaneous axis* ([1]).

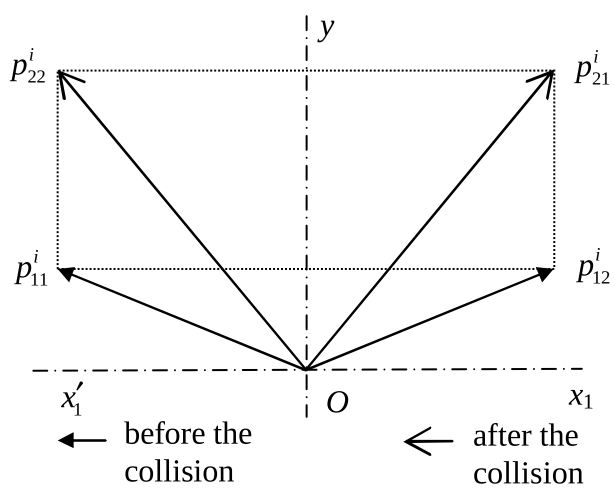

Figure 3.

([1]) One effortlessly recognizes the analogy between the two preceding relativistic statements and the Newtonian statements that were used by the kinetic theory of gases. (See, for example, E. BOREL, *Traité de Calcul des Probabilités*, t. II, fasc. III, Paris, 1925, pp. 36-40.)

We recapitulate all of these results by giving the four quadri-vectors $p^i_{\mu\nu}$ the same origin x^i. The triangles p^i_{11}, p^i_{12} and p^i_{21}, p^i_{22} are isosceles, with bases that are parallel in space-time. It then results that the four extremities are contained in the same world-plane π, which we take to be the plane of the figure, and onto which we project the instant-point x^i of the collision from O. The instantaneous axis of the collision projects onto $x'_1 x_1$ parallel to the two bases. In the general case where the two corpuscles have unequal proper masses, the axis Oy, which is perpendicular to the *instantaneous axis* and directed along the quadri-vector:

$$p^i_{21} - p^i_{12} = p^i_{22} - p^i_{11},$$

can be either timelike or spacelike. If the two corpuscles have equal proper masses then the axis Oy will necessarily be spacelike, and the point x^i will project from O to the center of the rectangle $p^i_{\mu\nu}$. If the two corpuscles are *indistinguishable*, in the quantum sense, then the question of knowing what the true association of emergent trajectories with incident ones will be absurd, and the two possible modes of association will be equivalent. Each set of four trajectories will then correspond to two possible *instantaneous axes* that are mutually orthogonal in space-time and project onto π along the symmetry axes of the rectangle $p^i_{\mu\nu}$.

If one is given two concurrent mass-impulses before the collision *a priori* then how are the possible *instantaneous axes* distributed? First of all, it is clear that the various possible planes π are spacelike hyperplanes that contains the two instant-points p^i_{11} and p^i_{21}. One of them, which we call the *principal* one, contains the two quadri-vectors p^i_{11} and p^i_{21}. If that is the case then every plane π will contain one and only one possible instantaneous axis. It is the axis that passes through the projection O of x^i and is such that the components of p^i_{11} and p^i_{21} along that axis will be opposite.

That is then the analysis of the collision of two corpuscles whose proper masses remain unaltered in four-dimensional geometry. To our knowledge, that process has yielded two good verifications of the relativistic dynamics of the electron. One of them, which was only qualitative, was by means of a plate from a Wilson chamber that was due to Joliot. The other one, which was quantitative, was by means of a Wilson plate of Leprince-Ringuet [**116**, **117**]. In the two cases, one is dealing with the collision of an electron with an electron at rest. The three trajectories – viz., that of the incident particle and those of the emergent particles – are curved by a known magnetic field, which gives one a means of calculating the three impulses, as well as the three velocities, or even better, the three corresponding β's; one deduces the three corresponding values of the relativistic mass from them. One can, moreover, write down the conservation laws for the two components in question of the impulses with no difficulty, as well as that of the masses, and calculate the angle that must exist between the two emergent Wilson trajectories. If the two proper masses are equal then one will find that this angle is such that ([**117**], eq. 12):

(IV.69) $$\cos\theta = \sqrt{\frac{(m_{12}-m_0)(m_{22}-m_0)}{(m_{12}+m_0)(m_{22}+m_0)}}, \qquad |\,\theta\,| < \frac{\pi}{2},$$

and consequently, it will be acute and will get more acute as the β's get larger. In the Newtonian approximation, one will have:

(IV.70) $$\cos \approx \theta, \qquad |\theta| \approx \frac{\pi}{2},$$

which is a result that is easy to establish by direct calculation. Formula (69) has been verified in a very satisfactory way in a plate of Leprince-Ringuet ([1]).

IV.10 – **Finite kinetic and orbital ponderomotor moments and their densities.** – Recall the fundamental equation of point dynamics (IV.51), adopt an arbitrary origin in space-time, and form the expression:

$$(x^i F^{jk} - x^j F^{ik})\, dx_k = x^i\, dp^j - x^j\, dp^i.$$

For the point without spin that we are presently studying, the quadri-vectors dx^i and p^i will be collinear, and the preceding equation can also be written:

(IV.71) $$\boxed{(x^i F^{jk} - x^j F^{ik})\, dx_k = d(x^i p^j - x^j p^i).}$$

We shall study the two sides of this equation in succession.

The second-rank antisymmetric tensor $x^i p^j - x^j p^i$ deserves the name of *orbital barycentric and kinetic moment*, or more simply, *orbital kinetic world-moment* of the material point. Indeed, its three components in u, v:

$$C_{(0)}^{uv} = x^u p^v - x^v p^u$$

are nothing but those of the ordinary orbital kinetic moment, while its three components in u, 4 can be written:

$$C_{(0)}^{u4} = x^u p^4 - x^4 p^u = ic\, m\, (x^u - v^u t).$$

If $t = 0$ – i.e., if the material point is considered simultaneously with the coordinate origin of space-time – then one will recognize the three components of the barycentric moment in the usual sense in the latter expression. That notion will persist when one takes an infinitesimal value dt for t, because the three $- dx^u = - v^u\, dt$ will then constitute an obvious "correction for non-simultaneity." In the general case where t is arbitrary, the

([1]) One should note that the two emergent trajectories are *indistinguishable* in the quantum sense; i.e., nothing will permit one to attribute them to the incident electron and the electron that was initially at rest in the Wilson chamber. Formula (69) accounts for that fact by its symmetry in m_{12} and m_{22}.

It is easy to make a four-dimensional diagram of the phenomenon by taking the plane of the Wilson figure to be Galilean space, and the time axis to the corresponding time. One will then see that the spatial trajectory of the electron that is put into motion will coincide with the spatial projection of the *instantaneous axis* that is defined in the text. Furthermore, the ambiguity in assigning the emerging trajectories is recovered in the definition of the projection of the axis.

three C^{u4} will constitute a generalized definition of the classical barycentric moment. Finally, the tensor:

(IV.72) $$\boxed{C^{ij}_{(0)} = x^i p^j - x^j p^i}$$

indeed deserves the name that we have given it.

As for the third-rank tensor, which is essentially antisymmetric in i, j (no matter what definition is adopted for F^{ij}):

(IV.73) $$\boxed{M^{ij}_{(0)} = x^i F^{jk} - x^j F^{ik},}$$

it is easy to convince oneself that it deserves the name of *orbital ponderomotor world-moment.* Indeed, if one takes the antisymmetric definition of F^{ij}, to simplify, which is suitable for a point with a conservative proper mass, then for $ij = uv$, the left-hand side of formula (71) can be written:

$$\{\mathbf{x} \wedge [\mathbf{F} + \mathbf{v} \wedge \mathbf{K}]\}\, dt.$$

For $ij = u4$, one will get another vectorial expression that Newtonian mechanics does not take into consideration:

$$\{t\,(\mathbf{F} + \mathbf{v} \wedge \mathbf{K}) - \frac{1}{c^2}\,\mathbf{x}\,(\mathbf{F} \cdot \mathbf{v})\}\, dt.$$

For $t = 0$, the latter expression is interpreted as an infinitesimal barycentric moment of the ponderomotor work

Finally, formulas (72), (73), and (71) clearly constitute the relativistic generalization of the definitions of *orbital kinetic moment*, *orbital ponderomotor moments*, and *the theorem of kinetic moment*, as it is written for a point without spin, resp. (71) condenses into the form:

(IV.74) $$M^{ijk}_{(0)}\, dx_k = dC^{ij}_{(0)}\,.$$

We now pass on to the study of the same question in terms of continuous media, and in order to do that, we recall equation (10), which is valid for a continuous medium that is subject to forces of volumetric origin. One assumes that the relation (8) is verified in all of the fluid, except perhaps in certain regions of space-time that contain sources and sinks.

We then form the expression:

$$x^i f^j - x^j f^i = x^i\, \partial_k\, T^{jk} - x^j\, \partial_k\, T^{ik}.$$

If one takes into account the facts that $\partial_k x^i \equiv \delta^i_k$ *and that the tensor* T^{ik} *is symmetric in the present case* then that equation will admit the equivalent form:

(IV.75) $$\boxed{x^i f^j - x^j f^i = \partial_k (x^i T^{jk} - x^j T^{ik}).}$$

If one takes the four-dimensional integral of both sides of this inside of a domain $\mathcal{E}_2 - \mathcal{E}_1 + \mathcal{P}$ of the usual type, transforms the right-hand side into a triple integral and takes into account the fact that, due to the form (15) of T^{ij}, the contribution from the hyper-wall $\mathcal{P}$ to that triple integral will be zero, then one can write:

$$\text{(IV.76)} \qquad \iiiint (x^i f^j - x^j f^i)\,\delta\omega = \iiint_{\mathcal{E}_2-\mathcal{E}_1} (x^i T^{jk} - x^j T^{ik})\,\delta u_k\,.$$

That is obviously the relativistic expression for the *theorem of orbital kinetic moment*, when it is written for the entire material drop $\mathcal{E}$. One then sees that the second-rank antisymmetric tensor:

$$\text{(IV.77)} \qquad \boxed{\mu_{(0)}^{ij} = x^i f^j - x^j f^i}$$

and the third-rank antisymmetric tensor in i, j:

$$\text{(IV.78)} \qquad \boxed{\sigma_{(0)}^{ijk} = x^i T^{jk} - x^j T^{ik}}$$

deserve the name of *orbital ponderomotor moment density* and *orbital kinetic world-moment density*, resp. With those notations, equation (75) can be written:

$$\text{(IV.79)} \qquad \mu_{(0)}^{ij} = \partial_k \sigma_{(0)}^{ijk}\,.$$

In a sense, it might seem more logical to have studied the force – density and finite – in no. III.7 of the chapter that was devoted to electromagnetism and the ponderomotor moment – density and finite – in no. IV.10 of the chapter that was devoted to dynamics. In fact, that is a better way of presenting things, since the study of the ponderomotor moment seems clearer when one couples it with the study of the kinetic moment. The remarks that we just made constitute an indispensible preparation for the problems that we will treat in the next paragraph.

B. – GENERAL EQUATIONS OF THE DYNAMICS OF VISCOUS FLUIDS ENDOWED WITH SPIN. DEDUCTION OF THE DYNAMICS OF POINTS ENDOWED WITH SPIN.

IV.11. – **The introduction of proper kinetic moments into fluid dynamics.** – It is well-known in classical elasticity that the density of ponderomotor force does not suffice to account for the existence of proper ponderomotor moment density, and that the theory of the latter must be introduced by itself. Similarly, it is easy to convince oneself that the distribution of translational force densities of inertia within the fluid does not permit one to account for the existence of a distribution of proper kinetic moment density that one can postulate. Since the moment of inertia of a homogeneous spherical droplet relative to an axis that issues from its center is infinitely small of fifth order, the corresponding kinetic moment density will be infinitely small of second order and tend to zero at the same time that the radius *r* of the droplet does.

We then postulate (as we certainly have the right to do) that classical fluids do not provide us with an illustration of the fact that a proper kinetic moment density exists within certain material fluids. Due to the homogeneity in the definition (78) of the orbital kinetic moment density $\sigma^{ijk}_{(0)}$, we assume that the proper kinetic moment density is a third-rank tensor $\sigma^{ijk}_{(p)}$ that is essentially antisymmetric in *i*, *j*, and whose integral must be taken in the form of:

(IV.80) $$C^{ij}_{(p)} = \iiint_{\mathcal{E}} \sigma^{ijk}_{(p)}\, \delta u_k\, .$$

The integral tensor $C^{ij}_{(p)}$, which we call the *proper barycentric and kinetic world-moment* of the material droplet, will then be defined to be of rank two and antisymmetric, in accord with the definition (72) of the orbital kinetic moment.

We study the proper kinetic moment $C^{ij}_{(p)}$ when the drop $\mathcal{E}$ passes from state 1 to state 2. An integral transformation will permit one to write [upon taking formula (II.79) into account]:

(IV.81) $$\iiint_{\mathcal{E}_2-\mathcal{E}_1} \sigma^{ijk}_{(p)}\, \delta u_k = \iiiint \partial_k \sigma^{ijk}_{(p)}\, \delta\omega - \iiint_{\mathcal{P}} \sigma^{ijk}_{(p)}\, \delta u_{kl}\, ds^l\, .$$

A reference to formulas (76) and (71) then shows that in the quadruple integral, $\partial_k \sigma^{ijk}_{(p)}$ can be interpreted as a volume density of proper ponderomotor moment, and that in the triple integral in the right-hand side, $-\sigma^{ijk}_{(p)}$ is interpreted as a surface density of proper ponderomotor moment. Taking both of these ponderomotor moments into consideration is necessary if one is to account for the existence and variation of the proper kinetic moment.

As one knows, the physical reasons that impose the consideration of proper kinetic moments are of quantum origin. Dirac's theory of the electron contains, among other things, a theory of a fictitious statistical fluid that is endowed with a proper kinetic moment – or *dynamical spin* – and a proper electromagnetic moment – or *electromagnetic spin*. It was the need to understand the sense of the equations of that theory that induced various authors to study the relativistic dynamics of media that are endowed with spin for its own sake.

From the outset, Dirac defined the proper kinetic moment density σ^{ijk} in his theory as a completely-antisymmetric third-rank tensor that is, consequently, the dual of a quadri-vector σ^i. Tetrode defined the inertia tensor T^{ij} in Dirac's theory ([1]) to be asymmetric and proved the two relations on that occasion:

(IV.82) $$T^{ij} - T^{ji} = \partial_k \sigma^{ijk}, \qquad \partial_j T^{ji} = \partial_j T^{ij} = f^i.$$

f^i denotes the symbolic Lorentz force density that is applied to the fictitious electronic fluid by the prevailing field. If one takes into account the well-known homogeneity between an inertia tensor and an elastic tensor, as well as the homogeneity between $\partial_k \sigma^{ijk}$ and a proper ponderomotor moment density then Tetrode's formula (82_1) will appear to be quite natural from the standpoint of the classical theory of elasticity. The asymmetry of the inertia tensor and the existence of a spin density seem to be linked in a fashion that is convenient *a priori*.

It is the second equality in formula (82_2) that proves to be essential. (We shall return to this point in the following nos.) The first one, and consequently, the equality of the two divergences of the asymmetric inertia tensor, is not imposed by abstract dynamics, and can then be considered from that viewpoint as being a particular trait of Dirac's theory. From what was said above, the same thing will be true for the complete antisymmetry of the tensor σ^{ijk}. Only the antisymmetry in i, j is imposed by abstract dynamics. Now, it is easy to see that these two particular traits of Dirac's theory are linked with each other in a necessary and sufficient manner. If one compares one to the other then Tetrode's two tensors will, in fact, permit one to write:

$$\partial_{jk} \sigma^{ijk} = 0,$$

and since the symbolic tensor ∂_{jk} is essentially symmetric, that relation will be equivalent to the essential antisymmetry of the tensor σ^{ijk} in j, k.

Along with the well-known spin density σ^{ijk}, another spin density τ^{ijk} is introduced into Dirac's theory. E. Durand was first to point out its existence and proved the relation ([2]):

(IV.83) $$\partial_{jk} \tau^{ijk} = \mu^{ij},$$

in which μ^{ij} denotes the proper ponderomotor moment density that is symbolically applied to the fictitious polarized medium by the electromagnetic field [cf., (III.83)]. Now, without there being any need for us to insist upon it here, moreover, we agree that ferromagnetism, which is due to the spin of the electron, offers a macroscopic manifestation of the Durand's couple, and therefore all of its physical importance, as well. Conforming to the general model that is provided by abstract dynamics, we finally point out that Durand's density τ^{ijk} is anti-symmetric in only the indices i, j.

([1]) Zeit. Phys. **49** (1928), pp. 858.
([2]) C. R. Acad. Sci. **218** (1944), pp. 36, eq. (8).

Formulas (82) and (83) of Dirac's theory essentially constitute the formulas that physically justify the study of the dynamics of continuous media that are endowed with spin. Notably, their precise interpretation is the goal of all of the theory that we shall now present.

IV.12. – **Hypothesis of an asymmetric inertia tensor and a mass-impulse that is oblique to the trajectory.** – We return to the equations of Newtonian mechanics (4), but postulate the existence of not only the kinematic velocity **v**, or v^u, but also a *pseudo-velocity* **u** or u^u that is not collinear with the preceding one, and is such that equations (5) must be generalized in the form:

$$\text{(IV.84)} \qquad \begin{cases} f^u = \partial_v(\rho u^u u^v) + \partial_t(\rho u^u), \\ \mathbf{f}\cdot\mathbf{v} = \partial_u(w v^u) + \partial_t(w). \end{cases}$$

We will see later on that the vector $\rho\,\mathbf{u}$ intervenes globally in the expression for the impulse $\delta\mathbf{p}$ of a droplet δu, and that this vector is oblique to the trajectory. Physically, each of the elements ρ and **u** are then defined only up to a factor, and that factor will be chosen in such a manner that it will simplify the formulas.

With (84), the generalized expressions for the components of the inertia tensor, which is now asymmetric, will become:

$$\text{(IV.85)} \qquad \begin{cases} T^{uv} = \rho u^u v^v, \\ T^{u4} = ic\,\rho u^u, \quad T^{4u} = \dfrac{i}{c} w v^u, \\ T^{44} = -w. \end{cases}$$

This tensor then seems to be the general product of the quadri-vector world-velocity:

$$V^u = \alpha v^u, \qquad V^4 = ic\,\alpha$$

with another quadri-vector:

$$\rho_0\, U^u = \rho\, u^u, \quad \rho_0\, U^4 = \frac{i}{\alpha c} w.$$

With these generalized definitions, the law of universal proportionality:

$$w = c^2\,\rho$$

will no longer be posed deductively. In order to recover it, one must postulate that the three u^u are the direction cotangents of the quadri-vector $\rho_0\, U^i$, which will then imply that:

$$\rho_0\, U^4 = \frac{ic}{\alpha}\rho.$$

That postulate amounts to making an appropriate definition of the mass density. Similarly, in order to define the proper mass density ρ_0 , one must postulate that the length of the quadri-vector U^i is such that:

(IV.86) $$\boxed{U_i V^i = -c^2.}$$

Finally, the covariant expression for the tensor T^{ij} will be:

(IV.87) $$\boxed{T^{ij} = \rho_0 U^i V^j ,}$$

and one will have two expressions for its trace:

(IV.88) $$\boxed{T_j^i = \rho(\mathbf{u}\cdot\mathbf{v} - c^2) = -c^2\rho_0 .}$$

At some points, we shall insist upon the new and curious circumstances that result from the preceding definitions. One will naturally continue to assume that the quantities ρ_0 and ρ are essentially positive; since one has:

(IV.89) $$c^2 \rho = - \rho_0 U^4 V^4,$$

and since the quadri-vector V^i is essentially timelike, $-i V^4$ will be essential positive, the second postulate will imply that $-i U^4$ is essentially positive, and therefore that U^i is also timelike ([1]). Under these conditions, there effectively exists a second Galilean frame such that $\rho = \rho_0$; the three U^u will be zero in that frame. With those postulates, one easily convinces oneself that one essentially has $|u| < c$, and consequently (since the scalar product $\mathbf{u}\cdot\mathbf{v}$ is capable of taking on negative values):

(IV.90) $$-c^2 < \mathbf{u}\cdot\mathbf{v} < + c^2.$$

Therefore, the lower bound on the possible values of:

(IV.91) $$\rho = \frac{\rho_0}{1 - \dfrac{\mathbf{u}\cdot\mathbf{v}}{c^2}}$$

will no longer be ρ_0, but, in fact, $\rho_0 / 2$.

The minimum value of $\mathbf{u}\cdot\mathbf{v}$ in the set of Galilean frames that have a given instant-point for their origin is obviously attained for a certain direction of Ox_4 that is coplanar to U^i and V^i, and interior to the angle between those quadri-vectors. For the Galilean frames that satisfy that condition, with δ denoting a constant, and φ, a real variable, one can set:

([1]) Recall that in Dirac's theory the Gordon quadri-current, which must be contrasted with the present quadri-vector U^i in several regards, is not essentially timelike. [W. Gordon, Zeit. Phys. **50** (1928), pp. 630. O. Costa de Beauregard, Jour. de Math. (2) **22** (1943), pp. 113 and 174.]

$$v = c \tanh(\varphi - \delta), \qquad u = c \tanh(\varphi + \delta).$$

One will then effortlessly see that the desired minimum value is:

$$\mathbf{u} \cdot \mathbf{v} = c^2 \frac{1 - \cosh \delta}{1 + \cosh \delta} \leq 0,$$

and that it is attained when Ox_4 is the lower bisector of U^i and V^i. The *minimum minimorum* is then attained only in the limiting cases $\delta = \pm \infty$. By contrast, the maximum value of ($\mathbf{u} \cdot \mathbf{v}$) in the Galilean frame that is being considered is never bounded.

The preceding equations imply that the fundamental formula for the dynamics of the medium under consideration, when it is subject to the action of only volumetric forces, will be:

(IV.92) $$\boxed{f^i = \partial_j T^{ij} \equiv \partial_j (\rho_0 U^i V^j).}$$

If one takes the quadruple integral of the two sides of this inside the usual domain $\mathcal{E}_2 - \mathcal{E}_1 + \mathcal{P}$ and transforms it into a triple integral then the contribution of the hyper-wall will still be zero, as in the classical case, and one will see that the expression:

(IV.93) $$\boxed{p^i = \iiint_{\mathcal{E}} \rho_0 U^i V^j \equiv \iiint_{\mathcal{E}} \rho_0 U^i \delta u_0 \equiv \iiint_{\mathcal{E}} U^i \delta m_0}$$

represents the mass-impulse of the material drop $\mathcal{E}$. The single, but still essential, difference between this and the classical case is that for a droplet δu_0, the *mass-impulse:*

(IV.94) $$\delta p^i = \delta m_0\, U^i$$

is no longer collinear with the quadri-vector V^i. If one so desires, one can decompose this *oblique mass-impulse* into a *longitudinal component* that is collinear with V^i and a *transversal component* that is orthogonal to V^i. We also note that at a given instant-point in an infinitesimal hypertube, *the length of the quadri-vector* δp^i *will remain defined independently of the orientation of the hypersection.*

Equation (92) can be developed in the form:

(IV.95) $$f^i = U^i\, \partial_j (\rho_0 V^i) + \rho_0 U'^i,$$

so one can conclude, upon taking (86) into account, that:

$$V_i f^i = -c^2 \partial_j (\rho_0 V^j) + \rho_0 V_i U'^i \equiv -c^2 \partial_j (\rho_0 V^j) - \rho_0 U^i V_i'.$$

The necessary and sufficient condition for formula (38) (which we have considered to be characteristic of continuous media that are subject to volumetric forces and admit the possible presence of sources and sinks) *to once more result is that one must have essentially:*

(IV.96) $$\boxed{U_i V'^i = 0;}$$

i.e., the quadri-vector $U^i - V^i$, *which is already orthogonal to* V^i (or x'^i), *must also be orthogonal to* V^i (or x''^i). By means of that hypothesis, the entire theory of proper mass in fluids that was developed in no. IV.5 will persist in the present case.

We shall now give a new form to the definition (87) of the tensor T^{ij} that will be useful in what follows. Speaking kinematically, the quadri-vector U^i has no real existence. It then seems natural to attach its definition to that of the quadri-vector V^i by setting:

(IV.97) $$\boxed{\rho_0 U_i = \rho^{ij} V_j}$$

as the definition of a new second-rank tensor ρ^{ij} that is called the *tensorial mass density* and supposing that is it asymmetric, in general. The definition (87) then becomes:

(IV.98) $$\boxed{T^{ij} = \rho^{ik} V_k V^j,}$$

and it will result from (86) that one has:

(IV.99) $$\boxed{\rho^{ij} V_i V_j = -c^2.}$$

Just like the tensor T^{ij}, the tensor $c^2 \rho^{ij}$ is homogeneous to an elastic tensor, and what follows will show that this homogeneity is not purely formal.

IV.13. – **The bases for the theory of surface forces of elastic type.** – When one develops the relativistic theory of surface forces in the case of an arbitrary elastic tensor E^{ij}, one will encounter serious difficulties when one seeks to preserve the hypothesis that the second-rank *world-force* tensor F^{ij} must be antisymmetric. The four-dimensional extension of the well-known formula:

$$\delta F^u = E^{uv} \delta s_v ,$$

will then be, in fact:

$$\delta F^{ij} = E^{ik} \delta s^j_{\cdot k} - E^{jk} \delta s^i_{\cdot k} ,$$

and if, as always, δu^*_k denotes the hyper-wall volume element $\delta s^{kl} dx_l$ then one will have:

$$d\,\delta p^i \equiv \delta F^{ij} dx_j = E^{ik} \delta u^*_k - E^{jk} \delta s^i_{\cdot k} dx_j$$

as an expression for the elementary quadri-work that is done by the surface force. When one starts with that formula and seeks to extend the arguments and formulas of no. IV.6, the presence of the second group of terms in the right-hand side will imply numerous difficulties in calculation and interpretation. Furthermore, that same presence will imply that the value of the expression $d\,\delta p^i$ will not be independent of the world-orientation of the initial and final δs^{ij}. Now, all of the experience that we have acquired up to now in relativistic theories makes us consider that situation to be quite unsatisfactory in itself.

The simple and radical means of avoiding all of those difficulties is *to assume that the tensor δF^{ij} is no longer antisymmetric in the general elastic case.* That idea seems much more acceptable to us than the volumetric theory that we were already led to in the case where we desire to be able to account for the existence of *sources* or *sinks* (no. IV.5). We then assume that the expression:

$$\boxed{\delta F^{ij} = E^{ik}\delta s^{j}_{\cdot k}} \tag{IV.100}$$

will give the general definition of surface force of elastic origin. For:

$$E^{ik} = \varpi\,\delta^{ik},$$

that formula will coincide with formula (42), which was given for the case of normal pressure. In the general case, it will imply that:

$$\boxed{\delta F^{ij} dx_j = E^{ik}\delta u^*_k} \tag{IV.101}$$

as the expression for the elementary quadri-work, in which, as always, δu^*_k denotes the hyper-wall volume element of $\mathcal{P}$ that is expressed by (II.79).

Without wishing to explicitly extend some results from the simple case of pressure, and for which the concomitant considerations are presently valid *mutatis-mutandis,* in the following no., we shall extend formula (46), as well as the analogous one that one can write down for the moments. Indeed, those formulas are absolutely fundamental to the theory that we have in mind.

IV.14. – **General fundamental equations for the dynamics of continuous media endowed with spin and viscosity.** – We shall give a generality to the formulas that we shall now establish that is complete in regard to all of the preoccupations that we have had since the beginning of Chapter IV. The present theory will then have the maximum level of comprehensiveness that we have claimed that we can give it.

From the inertial viewpoint, we shall assume that the fluid considered enjoys:

– A *volumetric mass-impulse density* that is represented by an *asymmetric tensor* that responds to the equivalent definitions (87) or (99).

– A *volumetric proper kinetic moment density* whose theory was given in no. IV.11, and is represented by a third-rank tensor σ^{ijk} that is antisymmetric in *i, j*.

From the ponderomotor viewpoint, we shall assume that this fluid is subject to:

– Forces of *volume* origin that are derived from a *density* f^i that is not orthogonal to the quadri-velocity V^i in the regions of space-time where there exist sources or sinks, and whose theory was given in no. IV.5 (second hypothesis).

– Forces of *surface* origin that are derived from an *elastic tensor* E^{ij} and conform to the theory that was just given in the preceding no.

– Moments of volume origin that are derived from a *proper ponderomotor moment density* that are represented by an antisymmetric tensor μ^{ij}, for which one physical example is provided by the classical polarized electromagnetic media.

– Moments of surface origin that are derived from a *surface density of proper ponderomotor moment* σ^{*ijk} that is, as we have seen, homogeneous to a volume density of spin, and whose presence proves to be indispensible if one is to equilibrate the presence of the spin density precisely.

We begin by writing the dynamical equation that relates to the sum of forces – both inertial and ponderomotive – that are applied to the material drop $\mathcal{E}$. In integral form, one will obviously have:

$$\iiiint \partial_k T^{ik}\,\delta\omega = \iiiint f^i\,\delta\omega + \iiint_{\mathcal{P}} E^{ik}\,\delta u_k^* .$$

The triple integral, which relates to the quadri-work done by surface forces, transforms into:

$$\iiiint \partial_k E^{ik}\,\delta\omega - \iiint_{\mathcal{E}_2-\mathcal{E}_1} E^{ik}\,\delta u_k .$$

Just as we did in no. IV.6 for the case of pressure, we shall transform the latter $\iiint$ into $\iiiint$ by appealing to "hypothesis H" and "postulate P," which were stated in IV.5. Under "hypothesis H," one will have:

$$\delta u_{0k} = -\frac{1}{c^2} V_k\,V_l\,\delta u^l,$$

in such a way that if one takes into account the obvious vanishing of a hyper-wall integral then the triple integral in question can be written:

$$-\frac{1}{c^2}\iiiint \partial^l E^{ik}\,V_k\,V_l\,\delta\omega .$$

Finally, if one takes "postulate P" into account, as well as the expression (99) for T^{ij}, then one will have:

(IV.102) $$\boxed{\partial_k T^{ik} \equiv \partial^l(\rho^{ik}\,V_k\,V_i) = f^i + \partial_k E^{ik} - \frac{1}{c^2}\partial^l(E^{ik}\,V_k\,V_l)}$$

for the desired equation. Conforming to what we have said, there is physical homogeneity between the three tensors T^{ij}, $c^2\,\rho^{ij}$, and E^{ij}.

We now write down the dynamical equations for moments, and first in integral form. If we take into account the balance of everything that was stated before then we will have:

$$\iiint_{\mathcal{E}_2-\mathcal{E}_1}(x^iT^{jk}-x^jT^{ik}+\sigma^{ijk})\,\delta u_k$$
$$=\iiiint (x^i f^j - x^j f^i + \mu^{ij})\,\delta\omega + \iiint_{\mathcal{P}}(x^iE^{jk}-x^jE^{ik}+\sigma^{*ijk})\,\delta u_k^* .$$

Taking into account the obvious vanishing of a hyper-wall integral, the $\iiint$ on the left-hand side will be transformed into:

$$\iiiint \partial_k\{x^iT^{jk}-x^jT^{ik}+\sigma^{ijk}\}\,\delta\omega - \iiint_{\mathcal{P}}\sigma^{ijk}\,\delta u_k^* .$$

Similarly, under "hypothesis H," and taking into account the obvious vanishing of a hyper-wall integral, the first two terms in the $\iiint_{\mathcal{P}}$ will transform into:

$$\iiiint\left[\partial_k\{x^iE^{jk}-x^jE^{ik}\}-\frac{1}{c^2}\partial^l\{x^iE^{jk}-x^jE^{ik}\}V_k\,V_l\right]\delta\omega .$$

Finally a $\iiint_{\mathcal{P}}$ will remain on both sides of the equation, and one must necessarily have:

(IV.103) $$\boxed{\sigma^{*ijk}-\sigma^{ijk}=0}$$

identically at every instant-point of the contour $\mathcal{C}$ of the drop, which amounts to saying that *in a medium that is endowed with spin σ^{ijk}, the simple fact of considering an isolated portion $\mathcal{E}$ of the medium demands that for dynamical equilibrium to exist, one must fictitiously apply a surface density of ponderomotor moment that is precisely opposite to the volume density of spin to all of the contour $\mathcal{C}$* ([**127**], pp. 127). From everything that was said before, it is clear that the two meanings for the density σ^{ijk} are compatible from the standpoint of dimensional analysis.

Once the two $\iiint_{\mathcal{P}}$ are assumed, they will cancel identically, from what was just said, and what will remain for the equation that we are studying in its density form is:

$$\partial_k\,\{x^i\,T^{jk}-x^j\,T^{ik}+\sigma^{ijk}\}$$
$$=x^i f^j - x^j f^i + \mu^{ij} + \partial_k\{x^i\,E^{jk}-x^j\,E^{ik}\}-\frac{1}{c^2}\partial^l\{(x^i\,E^{jk}-x^j\,E^{ik})\,V_k\,V_l\}.$$

The various expressions of the form $\partial_k\,x^i\,A^{jk}\ldots$ or $\partial_k\,x^j\,A^{ik}\ldots$ transform according to the schema:

$$\begin{aligned}\partial_k x^i A^{jk}\ldots &= x^i \partial_k A^{jk}\ldots + \delta^i_k A^{jk}\ldots \\ &= x^i \partial_k A^{jk}\ldots + A^{ji}\ldots\end{aligned}$$

If one takes equation (98) into account then what will remain of the equation in question will be simply:

$$\text{(IV.104)}\qquad \boxed{\begin{aligned}&T^{ji} - T^{ij} + \partial_k \sigma^{ijk} \equiv [\rho^{jk}V^i - \rho^{ik}V^j]V_k + \partial_k \sigma^{ijk} \\ &= \mu^{ij} + E^{ji} - E^{ij} - \frac{1}{c^2}[E^{jk}V^i - E^{ik}V^j]V_k.\end{aligned}}$$

In the absence of proper kinetic moments, the two tensors:

$$T^{ji} - T^{ij} \qquad \text{and} \qquad \sigma^{ijk},$$

which are antisymmetric in i, j, will be zero identically. Similarly, in pre-relativistic elasticity, the two tensors:

$$\mu^{ij} \qquad \text{and} \qquad E^{ji} - E^{ij},$$

which are antisymmetric in i, j, will be zero identically in the absence of proper ponderomotor moments. In relativity, we must correct that statement slightly by referring to the two tensors:

$$\mu^{ij} \qquad \text{and} \qquad E^{ji} - E^{ij} - [E^{ji}V^i - E^{ij}V^j]\,V_k\,.$$

Under those conditions, *in the absence of proper moments (whether kinetic or ponderomotor), equation* (104) *will be satisfied identically,* which amounts to saying that *the equation of dynamical equilibrium for the moments is a consequence of the equation for the resultants.* That is a well-known result of pre-relativistic hydrodynamics, and that justifies *a posteriori* the fact that we have given all of our theory in § A without being preoccupied with moments.

It is quite easy for us to interpret the two equations (82_1) and (83) of Dirac's theory – the first of which is due to Tetrode, and second of which, to Durand – with the aid of the general equation (104). One obtains Tetrode's equation by annulling the entire right-hand side of (104), and that amounts to saying that Dirac's spin density is defined independently of any consideration of ponderomotor moments. More precisely: *The Dirac spin density is the spin density that is induced by the asymmetry of the inertia tensor.* The Durand equation is obtained similarly by annulling the terms that contain the inertia tensor, and of course, the ones that contain the elastic tensor. That amounts to saying that: *The Durand spin density globally contains all of the dynamical effects that result from the (fictitious) application of the electromagnetic ponderomotor couple density to the electronic fluid.*

IV.15. – **Deduction of the dynamics of a point endowed with spin. Summary of a theory of Weyssenhoff and Raabe.** – As we did in no. IV.7 in order to deduce the dynamics of a point without spin, we shall now essentially suppose in what follows that there are no forces of surface origin. Exactly as before, we will deduce the relation (51), which we now write:

(IV.105) $$dp^i = F^{ij}\, dx_j .$$

This time, by virtue of (93), the quadri-vector:

(IV.106) $$\boxed{p^i = m_0 U^i}$$

will not be collinear with the quadri-vector V^i. One can decompose it parallel and orthogonal to V^i (i.e., by virtue of (86), along V^i and $U^i - V^i$) into a *longitudinal mass-impulse* and a *transverse mass-impulse.* Here, we rewrite the relation (86) in the form:

(IV.107) $$\boxed{V_i p^i \equiv -c^2 m_0 .}$$

If one essentially adopts the hypothesis that the world-force F^{ij} *is antisymmetric* (which is a hypothesis that corresponds to the one where there is conservation of proper mass, in the language of continuous media) then one can deduce from formula (105) that:

(IV.108) $$V_i\, dp^i = 0.$$

Upon comparing formulas (107) and (108), one will then see that *the necessary and sufficient condition for there to be conservation of proper mass* m_0 *is that the quadri-vector* $U^i - V^i$*, which is already orthogonal to* V^i*, must also be orthogonal to* V'^i :

(IV.109) $$p^i\, dV_i = 0.$$

That corroborates what we said in no. IV.12 precisely.

Now, consider the density formula (104), which we rewrite upon neglecting the terms of surface origin and making the expression for the tensor T^{ij} explicit from (87):

(IV.110) $$\rho_0\,(U^j V^i - U^i V^j) + \sigma^{ijk} = \mu^{ij},$$

and then multiply all of the terms by the elementary scalar volume δu_0 . In the left-hand side, one will get the proper mass m_0 as a factor, and if one takes (106) into account then that term can be written:

$$\delta p^j V^i - \delta p^i V^j .$$

The second term:

$$\frac{1}{d\tau}\,\partial_k \sigma^{ijk} \delta u_0\, d\tau$$

can be replaced with:

$$\frac{d}{d\tau}(\sigma^{ijk}\, \delta u_0) \equiv \delta C'^{ij}_{(p)} ,$$

in which $\delta C'^{ij}_{(p)}$ denotes the elementary proper kinetic world-moment and the derivative is taken with respect to the proper time. In order to do that, one performs a transformation of the quadruple integral into a triple integral at the infinitesimal scale, and one neglects the hyper-wall integral by virtue of what was said in the context of formula (103). Ultimately, the *finite proper ponderomotor moment*, *as defined in the second way* (no. II.14):

(IV.111) $$\boxed{\delta M^{ij} = \mu^{ij}\delta u_0}$$

will appear in the right-hand side of equation (110). Finally, if one passes to the usual limit then one will see that the density formula (110) admits the equivalent integral:

(IV.112) $$\boxed{p^j V^i - p^i V^i + C'^{ij} = M^{ij}.}$$

That is the one formula that several authors have posed, by hypothesis, as the basis for the dynamics of a point that is endowed with spin {[**132**], eq. (2), [**134**], pp. 28, eq. (11)}. Here, we obtain that formula deductively. One therefore verifies once more the primacy that relativity accords the density theorems with respect to the finite theories.

If one sets:

(IV.113) $$\boxed{\delta M^{ijk} = \mu^{ij}\delta u^k}$$

for the *general* definition of the finite ponderomotor moment then if one takes the definition (II.80) of the elementary scalar volume into account, one will see that the relation (112) can be further written:

(IV.114) $$p^j\, dx^i - p^i\, dx^j + C'^{ij} = M^{ijk}\, V_k\,,$$

or, if one prefers:

(IV.115) $$\boxed{p^j dx^i - p^i dx^i + dC'^{ij} = M^{ij} dx_k.}$$

Under the hypothesis that the world-force F^{ij} is antisymmetric [cf., (108)], it is clear that formula (60), which uses the *finite force of the second kind*, will remain valid. We rewrite it as:

$$p'^i = F^i,$$

and we can then effortlessly conclude that:

(IV.116) $$x^i\, p'^j - x^j\, p'^i = x^i\, F^j - x^j\, F^i\,.$$

Upon adding corresponding sides of (112) and (116), we will obtain the formula:

(IV.117) $$\boxed{(x^j p^i - x^i p^i + C^{ij})' = x^j F^i - x^i F^i + M^{ij},}$$

which conforms to the reasoning of the cited authors and is in accord with those of the previous no., or equivalently, upon returning to the ordinary finite force and ponderomotor moment:

(IV.118)
$$\boxed{d(x^j p^i - x^i p^i + C^{ij}) = (x^i F^{jk} - x^j F^{ik} + M^{ijk})\, dx_k.}$$

That is the complete set of fundamental formulas for the dynamics of a point that is endowed with spin. We point out that in the following paragraph some reasons will appear for assuming that the general relation:

(IV.119)
$$C^{ij} p_j = 0$$

exists between the spin C^{ij} and the mass-impulse p_j of a material point, by virtue of which the three components C^{u4} of the barycentric moment will be annulled in the Galilean frame in which the three p_u are zero. As far as that is concerned, Weyssenhoff and Raabe have made another hypothesis, and we shall summarize their theory [**133**, **134**] to conclude.

Instead of the preceding relation, one postulates that one has essentially (the *Weyssenhoff-Raabe postulate*):

(IV.120)
$$\boxed{C^{ij} V_j = 0}$$

and consequently:

(IV.121)
$$C'^{ij} V_j + C^{ij} V'_j = 0.$$

If one multiplies all of the terms in (112) by V_j, under the hypothesis that $M^{ij} \equiv 0$, and takes into account the definition (107), as well as (120), then one will get:

(IV.122)
$$p^i = m_0 V^i + \frac{1}{c^2} C^{ij} V'_j ,$$

and then, with no difficulty, the relation (108), which shows that *the proper mass m_0 will then be constant.* One deduces from the same formula (112), as always, by virtue of (120) and under the hypothesis that $M^{ij} \equiv 0$, that:

(IV.123)
$$C_{ij} C'^{ij} = 0, \qquad C_{ij} C^{ij} = \text{const.},$$

which shows that *under the hypotheses that were made, the proper kinetic moment is preserved in modulus.*

Under the hypothesis that $F^i \equiv 0$, the quadri-vector p^i will remain constant. With Weyssenhoff and Raabe, we then agree to call the Galilean frame $\mathcal{G}_c$ in which the three p^u are annulled the *proper system of the circle.* In that frame, and by virtue of (107), one will have:

$$p_c^4 = -\, ic\, m_0 \sqrt{1-\beta_c^2}\ ,$$

from which, one will conclude that:

$$\beta_c = \text{const.}, \qquad V_c^4 = \text{const.}$$

Weyssenhoff and Raabe called the constant quantity:

$$M_c = m_0 \sqrt{1-\beta_c^2}$$

the *proper mass of the circle*. Since V_4' is zero in the frame $\mathcal{G}_c$, the expressions for the three p^u, which are zero, will reduce to:

$$m_0\, V^u + \frac{1}{c^2} C^{uv} V_v' = 0,$$

or furthermore, if one transforms the derivative with respect to proper time for the particle into a derivative with respect to "proper time of the circle" then:

$$M_0\, V^u + \frac{1}{c^2} C^{uv} V_v' = 0.$$

If we then return to the ordinary velocity $v^u = V^u \sqrt{1-\beta^2}$ and the notation of the common vector calculus then the preceding relation will be written:

$$M_0\, \mathbf{v} + \frac{1}{c^2} \mathbf{C} \wedge \mathbf{v}' = 0,$$

in which $\mathbf{C}$ denotes a constant vector, and M_0 is a constant. The integration of that equation is classical. One finds very easily that in the frame $\mathcal{G}_c$, and with a well-defined angular velocity $\omega = c^2 M_0' / C$, one has a circular helix with an axis that is parallel to $\mathbf{C}$ and has a radius $r = Cv/(c^2 M_0)$ that is proportional to v. On that subject, Weyssenhoff and Raabe observed that no macroscopic material point will enjoy the preceding properties. They then assumed that certain reasons would oblige the radius r_0 to remain extremely small.

Those authors completed their theory by treating the case in which the particle was a pre-quantum electron that was endowed with a proper magnetic moment (Uhlenbeck, Goudsmit, Frenkel), and then the case in which that particle has a vanishing proper mass, and consequently, a velocity that would be indistinguishable from c.

C. – ON THE PROBLEM OF THE DYNAMICS OF SYSTEMS OF INTERACTING POINTS.

IV.16. – **Four-dimensional extension of a theorem from the theory of torsors. Combined definitions of the barycenter and moment around the barycenter. –** Consider a system of N torsors, each of which is, by definition, composed of a *resultant shift* p^i and a *couple* $s^{ij} \equiv - s^{ji}$. Suppose, to fix ideas, that all of the quadri-vectors p^i are timelike, with positive temporal components. By the definition of the *sum* P^i of the *system* of N torsors, set:

(IV.124) $$\boxed{P^i = \sum p^i.}$$

As soon as we have determined its line of action or *axis*, P^i will be the *resultant shift* of the *resultant torsor*. By virtue of the indicated restriction, the quadri-vector P^i will certainly be timelike with a positive temporal component.

Now let X^i be the four coordinates of the point of application of a quadri-vector that is equipollent to P^i, and let $S^{ij} = - S^{ji}$ be the six components of an antisymmetric tensor, to which we impose the *a priori* condition:

(IV.125) $$\boxed{S^{ij} P_j = 0,}$$

which is expressed by four equations, only three of which are independent. Then define the instant-point X^i and the tensor S^{ij} together by means of the tensorial equation:

(IV.126) $$\boxed{X^i P^j - X^j P^i + S^{ij} = \sum (x^i p^j - x^j p^i + s^{ij}),}$$

which is equivalent to six algebraic equations. If the P^i are determined by (124) then one will see that (125) and (126) define a linear system of rank nine in the ten unknowns X^i and S^{ij}.

We provisionally place ourselves in the Galilean frame $\mathcal{G}_0$ in which, from (125), the three P^u, as well as the three S^{u4}, are annulled:

(IV.127) $$P_0^u = 0, \qquad S_0^{u4} = 0.$$

In $\mathcal{G}_0$, equations (126) will reduce to:

(IV.128) $$\begin{cases} S_0^{uv} = \sum [x_0^u p_0^v - x_0^v p_0^u + s_0^{uv}], \\ X_0^u P_0^4 = \sum [x_0^u p_0^4 - x_0^4 p_0^u + s_0^{u4}]. \end{cases}$$

The tensor S^{ij} is determined completely by (127_2) and (128_1), and the line of action of the quadri-vector P^i is determined by (128_2). Of the ten unknowns X_0^i and S_0^{ij}, only the X_0^i remain arbitrary, for the moment.

Finally, equations (124), (125), and (126) completely determine the *resultant torsor* of the *system* of N torsors p^i, $s^{ij} \equiv - s^{ji}$ in terms its *resultant shift* P^i and its *couple* $S^{ij} \equiv - S^{ji}$.

We now (and optionally) suppose that the N quadri-vectors p^i no longer slide, but are linked. It will then also be possible to link the quadri-vector P^i by virtue of the new equation:

(IV.129) $$\boxed{X_i P^i = \sum x_i p^i ,}$$

which is written:

(IV.130) $$X_{0i} P_0^i = \sum x_{0i} \, p_0^i$$

in the frame $\mathcal{G}_0$, and determine the tenth unknown X_0^4. One sees that the equations (125), (126), and (129) form a linear system of rank ten in the ten unknowns X^i and S^{ij}.

The preceding calculations provide the means for defining, first, the *total mass-impulse* P^i in a covariant manner, then collectively, *the barycenter* X^i and the *moment about the barycenter* S^{ij} of a *system of material points* that are endowed with spin s^{ij} in the general case. The fact that the last two notions are combined with each other is what makes the remark that was made in no. IV.10 stand out *a priori*, namely, that the three components in u, 4 of the tensor $x^i p^j - x^j p^i$ generalize the classical notion of *barycentric moment* of a material point about the origin of the universe.

In order to define the barycenter of a swarm of N material points *in the large*, we temporarily assume that they are non-interacting and fictitiously regard each of them as sliding along a timelike axis that is collinear with its mass-impulse p^i (hence, not tangent to its world-trajectory in the general case of spin). By definition, the *sum* P^i will be the *total mass-impulse* of the swarm, while the *couple* S^{ij} will be its *kinetic, barycentric moment about the barycenter*, and finally, the *axis* will be the *world-locus of the barycenters* X^i. The justification for those definitions results from the fact that if one sets:

(IV.131) $$P^4 = ic\, M$$

then (124) can be written:

(IV.132) $$P^u = \sum p^u , \quad M = \sum m ,$$

as well as the fact that, in $\mathcal{G}_0$, the (128_1) define the *kinetic moment about the barycenter* (so the *barycentric moment* will then be zero), and that (note well!) it is *independent of the origin of space*, since the three P_0^u are zero, and finally that (always in $\mathcal{G}_0$) equations (128_2) generalize the usual definition of the barycenter by:

$$M_0 X_0^u = \sum \left\{ m_0 (x_0^u - v_0^u \, t_0) + \frac{1}{ic} s_0^{u4} \right\},$$

which is a formula that will coincide with the classical formula exactly if all of the points are taken at the same instant $t_0 = 0$ and in the absence of spin.

In order to define the barycenter in the *strict* manner, one links each of the N points of the swarm to the instant-point of its *axis* and its world-trajectory (in the general case, not all of those instant-points are taken *simultaneously*). In $\mathcal{G}_0$, equation (129) can then be written:

$$M_0\, T_0 = \sum\left(m_0\, t_0 - \frac{1}{c^2}\, x_{0u}\, p_0^u \right),$$

and the *time of the barycenter* $T_0 = X_0^4 / ic$ will then appear like the weighted mean of the times of the points of the swarm, up to a very small correction term that corresponds to the *virial* of those points. Hence, even if one takes all of the points at the same instant, their barycenter, when defined in the *strict* manner, will not be at that instant. That unsettling fact will seem quite singular to the practitioners of Newtonian mechanics. If one prefers, one can avoid it by appealing to the definition of the barycenter *in the large.*

The spirit of Newtonian mechanics, as well as that of the quantum theory of particles with spin will be respected if we fictitiously call P^i the *mass-impulse of the barycenter* and S^{ij} the *spin of the barycenter.* Note well that even if the constituent points are devoid of spin, the barycenter will have spin, which will be nothing but the classical *moment of the system about the barycenter*. To us, that remark seems to justify the necessity of a relativistic study of points with spin (no. IV.15), even in the absence of the results of quantum mechanics.

IV.17. – **The potential energy-impulse and the potential spin of the field. Statement of some general theorems of dynamics. –** As we have said in the Foreword to this Chapter, according to what we have learned about electromagnetism from Maxwell, Abraham, Poincaré (and even, we might add, from the paper by E. Henriot that relates to the couple density of the field [**107**]), we shall assume that the existence of a field of interaction between N material points translates, in the entire universe, into the existence of a continuous distribution of a *potential mass-impulse density* T^{ij} (which is asymmetric in the general case of spin) and a *potential spin density* σ^{ijk} (which is essentially antisymmetric in i, j). To commence, replace the N filamentary hyper-trajectories with N infinitely-thin *material world-hypertubes.* We assume that each of the two tensors T^{ij} and σ^{ijk} each take the form of a sum of two tensors, the first of which – which is called the *inertial* or *material* part – is not identically zero only inside of the preceding hypertubes and suffers a discontinuity upon traversing their hyper-walls, and the second of which – which is called the *field* part – is continuous in the whole universe. From the definitions that were given in the preceding paragraphs, the elementary *mass-impulse* and *spin* that are attached to a hyper-section of one of the preceding hypertubes will be attached to their material tensors by the formulas:

$$\delta p^i = \underset{1}{T}^{ik}\, \delta u_k, \qquad\qquad \delta s^{ij} = \underset{1}{\sigma}^{ijk}\, \delta u_k .$$

That being the case, and by virtue of formulas (10) and (104), which we rewrite ([1]), the *force density* f^i and the *proper ponderomotor moment density* μ^{ij} will have the following expressions at every instant-point:

([1]) The second one, by annulling all of the terms of elastic origin identically.

$$f^i \equiv \partial_k T^{ik}, \qquad \mu^{ij} \equiv T^{ji} - T^{ij} + \partial_k \sigma^{ijk}.$$

Now, from d'Alembert's principle, the *force density* and the *couple density* (orbital + proper) must be identically zero at any instant-point, which is written:

$$f^i \equiv 0, \qquad\qquad x^i f^i - x^j f^i + \mu^{ij} \equiv 0.$$

If one takes the obvious relation:

$$T^{ik} \partial_k x^j \equiv T^{ij}$$

into account then the equations above will admit the following two consequences:

(IV.133) $$\partial_k T^{ik} = 0, \qquad \partial_k \{x^i T^{jk} - x^j T^{ik} + \sigma^{ijk}\} = 0.$$

Those are the two *purely local* general equations that express dynamic equilibrium, in the d'Alembert sense, of the matter + field system as a *sum* and a *moment*. We shall infer the relativistic statements of the general theorems of dynamics from them.

We *arbitrarily* introduce a continuous family of three-dimensional, spacelike hypersurfaces $\mathcal{E}(\theta)$ of the usual kind. Suppose that when one goes to infinity along an arbitrary direction on a $\mathcal{E}(\theta)$, the mean density of matter decreases in such a way that if the intensity of the interaction field decreases sufficiently fast with the distance from the matter then the two density tensors T^{ij} and σ^{ijk} will decrease sufficiently fast at spatial infinity. Then take a closed, four-dimensional domain that lies between two $\mathcal{E}$ – e.g., $\mathcal{E}(\theta_1)$ and $\mathcal{E}(\theta_2)$ – and inside of a hyper-wall $\mathcal{P}$ that does not meet any material hypertube. Integrate (133) in that domain, transform it into a triple integral, and stretch the hyper-wall $\mathcal{P}$ out to spatial infinity in all directions. From the hypotheses that were made, the triple hyper-wall integral will tend to zero, and if one orients $\mathcal{E}(\theta_1)$ and $\mathcal{E}(\theta_2)$ in the same sense relative to the time axis then one will get the following two integral equations, which are equivalent to (133):

(IV.134) $$\left\{\begin{aligned} &\iiint_{\mathcal{E}(\theta)} T^{ik}\,\delta u_k = P_0^i, \\ &\iiint_{\mathcal{E}(\theta)} [x^i T^{jk} - x^j T^{ik} + \sigma^{ijk}]\,\delta u_k = C_0^{ij}, \end{aligned}\right.$$

which are equations in which the right-hand sides P_0^i and $C_0^{ij} \equiv -\, C_0^{ji}$ are constant tensors.

Now, make the *material* terms in δp^i and δs^{ij} of the corresponding tensors appear under the $\iiint$ sign, and then constrict the hypertubes in order to return to the case of *material points.* In that passage to the limit, one supposes essentially that:

a) The $p^i = \delta p^i$ and $s^{ij} = \delta s^{ij}$ that are attached to each hypertube remain finite.

b) The *interior* portions of the integrals of the field tend to zero and their *exterior* portion remains finite.

(134) will then take on the form:

$$
\text{(IV.135)} \qquad \boxed{\begin{cases} \sum\left\{p^i + \iiint T^{ik}\delta u_k\right\}_\theta = P_0^i, \\ \sum\left\{[x^i p^j - x^j p^i + s^{ij}] + \iiint [x^i T^{jk} - x^j T^{ik} + \sigma^{ijk}]\delta u_k\right\}_\theta = C_0^{ij}. \end{cases}}
$$

The terms in $\sum$ – or *kinetic terms* – relate to the material points of the swarm whose mass-impulses are p^i and whose spins are s^{ij}. The terms in $\iiint$ – or *potential terms* – relate to the interaction field. P_0^i and $C_0^{ij} \equiv - C_0^{ji}$ are two tensorial constants. The indices θ signify that the triple integrals are calculated over a current hypersurface of the family $\mathcal{E}(\theta)$ and that all points of the swarm are taken from that same hypersurface. The two tensorial equations (135) contain a condensed statement of the general theorems of the relativistic dynamics of those systems of points.

In the absence of spin, one will have, on the one hand:

$$\sigma^{ijk} \equiv 0, \qquad s^{ij} \equiv 0,$$

where the second equation is a finite consequence of the first one, and likewise (cf., the end of IV.15):

$$T^{ji} - T^{ij} \equiv 0, \quad x'^i p^j - x'^j p^i \equiv 0.$$

One will then effortlessly see that the second of (135) is a consequence of the first one, with:

$$C_0^{ij} = X^i P_0^j - X^j P_0^i,$$

X^i denotes the current instant-point of a certain *axis* that is collinear with the constant covector P_0^i.

We return to the general case in order to infer the detailed statements of the *general theorems* that were stated from (135). We arbitrarily decompose the current hypersurface $\mathcal{E}(\theta)$ into elements δu_k that are infinitely small in all of their dimensions and vary as a function of θ in a continuous manner. It will result from (135) and the theory that was developed in no. IV.16 that *the resultant torsor P^i, S^{ij} of the system of N torsors p^i, s^{ij} and ∞ torsors T^{ik} δu_k , σ^{ijk} δu_k is conservative.* It is conservative in the double sense that it will not be altered by either a change of $\mathcal{E}$ inside the family $\mathcal{E}(\theta)$ or by a change of the family $\mathcal{E}(\theta)$. That will amount to saying that *the resultant dynamical torsor is:*

a) *Conservative throughout the mechanical evolution of the system points + field.*

b) *Defined independently of the choice of family $\mathcal{E}(\theta)$.*

In passing, we insist upon the essential importance that the second result has *a priori*, and we come to the detailed explanation for the first one.

THEOREM I. – *In the absence of external forces, the total mass-impulse of the system points + field – which is called the* ***mass-impulse of the barycenter****, moreover – is conservative.*

For $i = 4$, one will then obtain both the statement of the *theorem* (and not the *principle*) *of the conservation of total mass* and that of the *vis viva theorem.* The main novelty of the classical theorem consists of attaching a *potential mass* to the field of interaction that correlates with the *potential energy*, whose *constant* is then found to be fixed. The *potential mass* must obviously be negative in a stable system, and as one knows, that is an established result of nuclear chemistry ([1]). The secondary novelty consists of the fact that *the law of distribution of that mass or that potential energy in the field is physically well-defined.*

For $i = u = 1, 2, 3$, Theorem I generalizes the classical *impulse theorem*. Since it is certain *a priori*, the older equality of action and reaction at a distance will have to be rejected as absurd. Correlatively, one introduces the notions of a *potential-impulse* that is distributed throughout the field as a density.

THEOREM II. – *In the absence of external forces, the kinetic and barycentric moment about the barycenter of the system points + field – which is called the* ***spin of the barycenter****, moreover – is conservative.*

For $i, j = u, v = 1, 2, 3$, one will then obtain the extension of the classical *theorem of kinetic moments.* Along with the *kinetic spin* that is due to the points, a *potential spin* ([2]) that is coupled with the field will come about. Each of those two moments will itself be the sum of an *orbital moment* and a *proper moment* in the general case of spin.

For $i, j = u, 4$, the same statements can be repeated for the barycentric moment.

THEOREM III. – *When θ varies, the instant-point X^i that is associated with the strict definition of each hypersurface $\mathcal{E}(\theta)$ in no.* IV.16 *will describe a fixed axis that is collinear with the constant quadri-vector P^i in the universe. One will then have every right to say that in the absence of external forces, the quadri-velocity V^i of the world-barycenter of the system points + field will be conservative and collinear with the total mass-impulse (which is itself conservative by virtue of Theorem I), moreover.*

One clearly recognizes that this is an extension of the classical *theorem of the barycenter*, and that it has a very interesting specialization for the theory of the point with spin. Since the barycenter, conforming to the spirit of mechanics – whether classical or quantum – is considered fictitiously to be a material point that is endowed with spin, one would not expect that its mass-impulse and its world-velocity would be collinear, in general. However, it will result from the corollary that is implied in Theorem III that *in the very important special case of the free motion of a system, the two quadri-vectors V^i and P^i are collinear (and constant, moreover).* It seems quite natural to inductively

([1]) We have already mentioned that the mass-energy balance in nuclear chemistry provides an excellent verification of the theory of relativity [**114**, **115**, **117**].

([2]) Unfortunately, the words in the two expressions *kinetic kinetic moment* and *potential kinetic moment* clash with each other.

extend that result to the case of an elementary point that is endowed spin. One knows that in the wave mechanics of particles with spin, and in the very important special case of the monochromatic plane wave, the mass-impulse (which is well-defined) is collinear with the world-rays. That fact constitutes the quantum transposition of the preceding result.

All of the preceding theory was formulated under the hypothesis of the absence of external forces. In order to pass to the general case, one must necessarily superimpose the previously-considered *interaction field* with an *external action field,* whose resultant torsor will be valid with the change of the hypersurface $\mathcal{E}(\theta)$. The resultant torsor of the torsor that is "interior to the system" and the "external action" torsor must be conservative.

IV.18. – **Limitations of the preceding theory. Some words on the general problem of the relativistic dynamics of systems.** – In the preceding two nos., we have successively been able to make relativistic extensions of the basic notions that permit one to characterize a system dynamically and the statements of some *general theorems* of dynamics. It now seems clear that we have done little more than to clear the path to the true problem, namely, the field problem that Newtonian mechanics treats by passing over it by making a whole group of physical approximations, namely:

– *Instantaneous transmission of the interaction,* which is the postulate that the entire system is taken at the same instant *t* of the so-called *universal time.*

– *Equality of action and reaction at a distance,* which is a hypothesis that is conceivable only in the case of *universal time,* which is a consequence of the preceding one, and will render the notion of a *potential impulse* useless.

– *The absence of an equivalent mass to the potential energy,* which will render the specification of its law of distribution inside the field useless for the definition of the barycenter, for example.

All of those approximations amount to letting $c \to \infty$, in various forms, and that itself will show one the corresponding ways that the constant *c* enters into dynamics. They form a perfectly-coherent body amongst themselves that keeps its value as a technical method for the treatment of a whole group of problems. However, *by its very nature, relativity forbids the use of that body of approximations.* It is essentially their use that permits Newtonian mechanics to appeal to its *general theorems* as logical intermediaries in the treatment of some problems, and it is the suppression of their use in relativity that will imply the suppression of the *general theorems.* Indeed, in order for the *general theorems* in relativity to be able to serve as logical intermediaries, one must previously know the law of evolution of the field of interaction, but that will be controlled by the law that allows one to find the system of points.

Hence, the field problem that Newtonian dynamics solves by omission comes to the fore in relativity, and is the one that essentially constitutes the problem of the *dynamics of systems*. To be sure, that class of problems has been encountered before, notably in the electromagnetism of Maxwell-Abraham-Poincaré ([1]) and also in the relativistic theory of gravitation ([2]). It appears as an obligatory consequence of taking into consideration a finite velocity of propagation for that interaction. To touch upon the advantage that the present problem can infer from the example of analogous problems, we think that by the power and generality of the methods that are brought into play, it is, above all, the study of general relativity in which that example can be profitable.

It seems to us that here we have a strong argument *a priori* in favor of the introduction of a metric in order to treat the problem that we consider: *In relativity, the motion of a force of interaction between two points as a function of their* ***finite*** *separation is* ***a priori*** *as unacceptable as that of the equality of action and reaction at a distance.* However, consider the problem in Newtonian mechanics of two material points that attract each other by the intermediary of a *massive* elastic filament. The force of interaction is transmitted between them in the step-by-step sense, by waves and at finite velocity. One can say that it exists virtually at every point of the filament. In fact, it is realized only at the extremities A and B, where there is no equality between action and reaction (except in the limiting case where the linear density of the filament tends to zero and the wave velocity tends to infinity). Moreover (and this is the point that we are trying to reach), the *local* law of tension in the filament will yield, step-by-step, a substitute for the motion of a force of attraction that is a function of the *finite* separation distance. Furthermore, nothing is easier than to put that problem into equations: One defines a *proper metric for the filament* by graduating it into equal divisions *that pertain to its molecules* when it is taken at rest. Let ξ be that gradation, while ξ_A and ξ_B are the *proper abscissas* of the extremities (which are constant, by hypothesis), ρ is the linear density of the filament relative to the proper gradation ξ, and γ is the coefficient of elasticity. To simplify, argue with just one Galilean dimension x to space, so one will effortless arrive at the equation of evolution of the vibrating string:

$$\frac{\partial^2 x}{\partial \xi^2} - \frac{\rho}{\gamma}\frac{\partial^2 x}{\partial t^2} = 0,$$

which is rigorously valid for large deformations of the filament, due to the precautions that were taken. Moreover, if m_A and m_B denote the Newtonian masses of the two material points then the two equations of condition:

$$\rho \frac{\partial x}{\partial \xi} : \frac{\partial^2 x}{\partial \xi^2} = \begin{cases} +m_A & \text{for} \quad \xi = \xi_A, \\ -m_B & \text{for} \quad \xi = \xi_B \end{cases}$$

must be satisfied at the extremities.

([1]) On that subject, one should read RICHARDSON, *The electron theory of matter*, Cambridge, 1914, chaps. XI and XII. A very important phenomenon that pertains to the case of the electromagnetic field is the irreversible creation of radiation by accelerated charges.

([2]) On that subject, one might read G. DARMOIS, Mém. Sci. Math. 25, 1927. RACINE, *La problème des n corps dans la théorie de la relativité*, Paris, 1934. LICHNEROWICZ, *Sur certains problèmes globaux relatifs au système des équations d'Einstein*, Paris, 1939, and J. de Math. (1) **23** (1944), 37.

It seems to us that the preceding problem of Newtonian mechanics provides a pretty fair simplified picture of what the relativistic problem of the dynamics of systems of interacting points must be, which is a problem that might be answerable to the methods of general relativity.

Despite that suggestion, we hope that our very modest contribution to the question – namely, the relativistic extension of some notions and classical results of the Newtonian mechanics of systems of points – might lead to more work being done on the true problem.

D. – ON THE FOUNDATIONS OF RELATIVISTIC THERMODYNAMICS

IV.19. – **On heat and temperature in relativity: classical definitions of Planck-Einstein and modern covariant definitions.** – Consider a body $\mathcal{C}$ in uniform, rectilinear translation (hence, it is indeformable) whose proper mass at absolute zero is M_{00} . If it has the *proper temperature* T_0 in the proper Galilean system $\mathcal{G}_0$ then it will acquire a certain *quantity of proper heat* $\mathcal{Q}_0$, whose mass equivalent is $J\mathcal{Q}_0 / c^2$. Its proper mass will then be increased, and it will become:

$$M_0 = M_{00} + \Delta M_0 , \qquad \Delta M_0 = \frac{1}{c^2}\, J\mathcal{Q}_0 \ . \tag{IV.136}$$

Now let $P^{\,i}$ be the mass-impulse of the body $\mathcal{C}$, and let P_0^i be the value of $P^{\,i}$ at absolute zero. If $V^{\,i}$ denotes the constant quadri-velocity of the body $\mathcal{C}$ then:

$$P^{\,i} = M_0\, V^{\,i}, \qquad P_0^i = M_{00}\, V^{\,i},$$

in such a way that the contribution to the mass-impulse that is due to the proper heat $\mathcal{Q}_0$ is:

$$\Delta P^{\,i} \equiv J\, \mathcal{Q}^{\,i} = \Delta M_0 \cdot V^{\,i}, \qquad \mathcal{Q}^{\,i} = \frac{1}{c^2}\, \mathcal{Q}_0\, V^{\,i}. \tag{IV.137}$$

We agree to call the quadri-vector $\mathcal{Q}^{\,i}$ the *caloric-heat-impulse.*

In an arbitrary Galilean frame $\mathcal{G} \neq \mathcal{G}_0$, the three ΔP^u will represent an impulse that must be provided to the body $\mathcal{C}$ in order to keep its velocity v^u constant when its proper temperature passes from absolute zero to T_0 . When that impulse has a well-defined direction, it will correspond to a certain amount of work:

$$\mathcal{T} \equiv v_u\, \Delta P^u = \frac{\Delta M_0 \cdot v^2}{\sqrt{1-\beta^2}}. \tag{IV.138}$$

Upon subtracting $\mathcal{T}$ from the energy $\Delta W = - ic\, \Delta P^4$ that is provided to the body $\mathcal{C}$ during the transformation in question, one will get a certain energy $J\mathcal{Q}_P$ with no associated impulse – namely, the "disordered energy" – that gets added to the terms in the pre-relativistic theory. It is the *Planck heat*, which satisfies the relations:

(IV.139) $$J\mathcal{Q}_P + \mathcal{T} = \Delta W, \qquad \mathcal{Q}_P = \mathcal{Q}_0 \sqrt{1-\beta^2}\ .$$

The transformation law for Planck heat under a change in Galilean frame, which is inverse to that of a temporal component of the quadri-vector, is the same as that of a material volume considered simultaneously (II.26$_2$). We agree to call the expression $\mathcal{Q} = - ic\, \mathcal{Q}^4$ that is defined by (1372) the *covariant heat*, and if one takes (II.62) into account, it will transform according to the law:

(IV.140) $$\mathcal{Q} = \frac{\mathcal{Q}_0}{\sqrt{1-\beta^2}}\ .$$

It is indeed clear that there is an equivalence between being given the three $\mathcal{Q}^u$ and $\mathcal{Q}^4 = (i / c)\ \mathcal{Q}$, on the one hand, and being given the work $\mathcal{T}$, with its direction, and $\mathcal{Q}_P$, on the other. (137) and (139) will always permit one to pass from one language to the other invertibly. Being given the work $\mathcal{T}$ and the Planck heat $\mathcal{Q}_P$ is more physical, in the sense that the notion $\mathcal{Q}_P$ conforms more rigorously to the notion of a heat that is defined to be a *disordered energy*. By contrast, being given the *caloric-heat-impulse* quadri-vector $\mathcal{Q}^i$ is more mathematical, and one will see that only that quadri-vector lends itself to writing the formulas in a tensorial way.

Let S be the entropy of the body $\mathcal{C}$. From kinetic theory, it is a pure number, and the logarithm of a whole number; it can vary only by discrete (as well as quite small) quantities, and since the formulas for the change of Galilean frame are continuous formulas, one will see that *the entropy S is necessarily a relativistic scalar*. If θ_0 denotes the inverse of the *proper temperature* T_0 of the body $\mathcal{C}$ then, from pre-relativistic thermodynamics, one can write:

(IV.141) $$\boxed{dS = \frac{1}{T_0}\, d\mathcal{Q}_0 = \theta_0\, d\mathcal{Q}_0 .}$$

That being the case, introduce the definition of the *temperature quadri-vector* θ^i of the body $\mathcal{C}$ by way of:

(IV.142) $$\boxed{\theta^i = \theta_0 V^i ,}$$

and call the expression:

(IV.143) $$\frac{1}{T} = -\frac{i}{c}\theta^4, \qquad T = T_0\sqrt{1-\beta^2}$$

the *temperature* or *Planck temperature*. It is clear from the preceding that the elementary entropy dS can be expressed as a function of the quantities that are attached to an arbitrary Galilean frame $\mathcal{G}$ in the following two forms:

(IV.144) $$\boxed{dS = -\theta_i \, d\mathcal{Q}^i ,}$$

(IV.145) $$dS = \frac{1}{T} d\mathcal{Q}_P ,$$

the first of which has a tensorial form, while the second one, which is classical, does not. The new form (144) can be specified by:

(IV.146) $$dS = -\,\theta_u \, d\mathcal{Q}^u + \frac{1}{T} d\mathcal{Q},$$

and if one compares that with (145) then one will see that *taking the temperature quadri-vector θ^i into consideration is an obligatory corollary to the quadri-vectorial definition of heat.*

Before passing on to other subjects, we make one last remark: Let q_0 be the density of proper heat that is contained in the body $\mathcal{C}$. If one integrates the expression q_0 at constant time over the entire body $\mathcal{C}$ by way of:

$$\mathcal{Q}_P = \iiint_{i=i_0} q_0 \, \delta u$$

then the expression that will be obtained is nothing but the Planck heat, by virtue of the transformation law (II.62$_2$) for a material volume when it is considered simultaneously. On the contrary, if one introduces the notion of a *tensorial heat density* $q_0 \, V^i \, V^j$ and takes the integral:

$$c^2 \, \mathcal{Q}^{\,i} = \iiint_{\mathcal{E}} q_0 V^i V^j \delta u_j \; = \mathcal{Q}_0 \, V^i,$$

over an arbitrary spacelike hypersurface $\mathcal{E}$ then the expression that is obtained will be the *caloric-heat-impulse* quadri-vector, up to the factor c^2.

Having thus specified the distinctions that one must establish between the *Planck heat* and the *caloric-heat-impulse* quadri-vector, as well as between the *Planck temperature* and the *Van Dantzig-Bergmann temperature* quadri-vector in a particularly simple case – first from the finite viewpoint, and then from the density viewpoint – in the rest of our presentation, we shall utilize only the last two notions. Their physical interpretation is perhaps less direct that that of the notions that were defined by Planck and Einstein, but they are the only notions that permit one to write complete differential forms, which is a rule that we shall be inclined to respect in the entire course of this book.

IV.20. – **General covariant definitions of heat and temperature.** – The *first principle of thermodynamics* asserts the *equivalence* of heat and energy. It is from it that *any definition of energy – whether finite or as a density – will bijectively correspond to a homologous definition of heat.* In particular, one then introduces a *caloric inertia tensor* T^{*ij} that is homogeneous to T^{ij}, as well as a *density of proper heat* $w_0^* \equiv c\,\rho_0^{2*}$ that is homogeneous to the *density of proper energy* $W_0 \equiv c^2\,\rho_0$. It is quite natural to assume that the tensor T^{*ij} takes the form (82), which we write:

(IV.147) $$T^{*ij} = \rho_0^* U^i V^j, \qquad (U_i V^i = V_i V^i = -c^2,\; T_i^{*i} = -c^2\,\rho^*).$$

Indeed, in a heat-conducting material medium, one will have to distinguish the heat quadri-current U^i from the quadri-velocity V^i of the medium.

Upon integrating the densities T^{ij} and $\rho_0^* = w_0^* / c^2$ over a spacelike hyper-end-cap, one will cause the *caloric-energy-impulse* – or *caloric heat-impulse* – quadri-vector to appear, as well as the *proper caloric energy* – or *proper heat* – of a finite portion of the medium, which we write in infinitesimal form:

(IV.148) $$\delta p^{*i} = T^{*ij} u_j = \rho_0^* V^i\,\delta u_0 = \delta m_0^* V^i$$
$$\left(\delta p^{*i} = \frac{i}{c}\delta W^* = \frac{i}{c} J\,\delta\mathcal{Q}\right),$$

(IV.149) $$\delta m_0^* = \rho_0^* V^i\,\delta u = \rho_0^*\,\delta u_0$$
$$\left(\delta p^{*i} = \frac{i}{c}\delta W^* = \frac{i}{c} J\,\delta\mathcal{Q}\right).$$

We have let $\delta\mathcal{Q}$ denote the *covariant quantity of heat* that is attached to a material droplet δu^i, when evaluated in an arbitrary Galilean frame, and let $\delta\mathcal{Q}_0$ denote the *quantity of proper heat* in that same droplet. It is interesting to remark that *with the definition* (147) *of the caloric tensor, the caloric-heat-impulse quadri-vector δp^i of a given material droplet will be defined independently of the orientation of the hyper-section δu^i of the infinitesimal hypertube that is described by that droplet.* Moreover, *it is collinear to the total heat quadri-current (convection + conduction):*

(IV.150) $$U^i \equiv V^i + (U^i - V^i),$$

which seems completely satisfactory.

From the *second principle* of thermodynamics – and above all, from its interpretation in terms of statistical mechanics – the entropy:

$$S = \iiint \frac{\delta\mathcal{Q}}{T}$$

is certainly a scalar quantity. Indeed, if k denotes the Boltzmann constant then the "reduced entropy" S / k will be the logarithm of a probability – i.e., a pure number that is the logarithm of a whole number. Since the components of a tensor are continuous functions of σ^i_j under a change of Galilean frame, and since the variation of a whole number cannot be continuous, it is necessary that S should be a tensorial invariant.

If one recalls the classical definition of S in the definitions (148) and (149) of *caloric-heat-impulse* and *proper heat* then one will be naturally led to write the elementary entropy in one or the other of the covariant forms:

(IV.151) $$\boxed{\delta S = -\frac{1}{J}\theta_i\,\delta p^i, \quad \delta S = \frac{c^2}{J}\theta_0 \delta m_0^*.}$$

From the first one, which is specified by:

$$-J\,\delta S = \theta_u\,\delta p^u + \frac{i}{c}\,\theta_4\,\delta Q,$$

the inverse of the temperature T is defined to be the temporal component of a quadri-vector. From the second, the temperature is defined to be a scalar quantity T_0 :

(IV.152) $$\boxed{\theta_4 = \frac{ic}{T}, \quad \theta_0 = \frac{1}{T_0}.}$$

Hence, in relativity, one will be quite naturally led to distinguish the two notions of *temperature* and *proper temperature*, which are defined in connection with *heat* and *proper heat*, respectively. Moreover, if one takes the last expression (148) for δp^{*i} into account then one have:

(IV.153) $$J\,\delta S = -U^i\,\theta_i\,\delta m_0^* = c^2\,\theta_0\,\delta m_0^*,$$

which will very strongly suggest that one must assume the relation:

(IV.154) $$\boxed{\theta^i = \theta_0 V^i}$$

when one takes into account that $U^i\,V_i = -c^2$. By means of that hypothesis, the three spatial components of the quadri-vector θ^i will be annulled in the Galilean frame that follows the hot body, and the notion of *proper temperature* will be found to be justified in the usual sense.

IV.21. – **Relativistic form of the fundamental equations of the theory of thermal conduction.** – In the theory of thermal conduction, heat is associated with a conservative fluid whose density Jq we denote by w^*. If one considers the particular case in which the conducting body is a rigid body at rest in a certain Galilean frame that one can take to be a reference frame. Let **u** be the velocity of the *caloric fluid* relative to that frame, and then let $\gamma(T)$ and $C(T)$ be essentially positive quantities, namely, the coefficient of conduction and the specific heat of the medium, respectively. Naturally, T denotes the absolute temperature. The fundamental equations of the theory are ([1]):

(IV.155) $$w^* \mathbf{u} = -\gamma(T)\,\mathrm{grad}\cdot T,$$

(IV.156) $$w^* = \int^T C(T)\,dT,$$

(IV.157) $$\mathrm{div}\cdot(w^* \mathbf{u}) + \frac{\partial}{\partial t} w^* = 0.$$

In the particular case where:

(IV.158) $$\gamma(T) = \gamma = \text{const.}, \qquad C(T) = C = \text{const.},$$

one will see that the temperature satisfies the equation that is called *the heat equation:*

(IV.159) $$\frac{C}{\gamma}\frac{\partial}{\partial t} T = \sum_{u=1}^{3} \frac{\partial^2}{\partial x_u^2} T.$$

We remark that if, as one always implicitly assumes in the classical theory, T, γ, and C are considered to be invariant under a change of Galilean frame then the *heat equation* will not be covariant. Moreover, all of the treatises on analysis show that this equation does not attribute a finite speed of propagation to temperature. If all of the caloric energy at the instant O is assumed to be contained in a finite domain $\mathcal{D}$ then the density $w^*(dt)$ at the instant dt will no longer be nowhere-zero. Those two remarks show clearly that the classical formulas are certainly truncated forms of the relativistic formulas that are being sought.

Let V^i be the quadri-velocity of a conducting medium that is assumed to be a fluid, for sake of generality, let θ^i be the Van Dantzig-Bergmann temperature quadri-vector, and let θ_0 be the Tolman-Eckhart scalar. Taking our inspiration from Minkowski's theory of electric conduction, and upon remarking that the quadri-vector $(\partial^i \theta^j - \partial^j \theta^i) V_j$ is essentially orthogonal to V^i, we will first generalize (155) into the form:

(IV.160) $$\boxed{c^2 \rho_0^* (U^i - V^i) = -\kappa(\theta_0)(\partial^i \theta^j - \partial^j \theta^i) V_j \equiv x(\theta_0)(\partial^i \theta_0 + \frac{1}{c^2}\theta'^i),}$$

with

(IV.161) $$\kappa(\theta_0) \equiv T_0^2\,\gamma(T_0).$$

U^i denotes the quadri-velocity of the "caloric fluid," which is, by hypothesis, such that:

$$U_i V^i = V_i V^i = -c^2,$$

([1]) See, for example, J. BOUSSINESQ, *La théorie analytique de la chaleur*, Paris, Gauthier-Villars, 1901, eqs. (40), pp. 120, (109), pp. 168, (133), pp. 194.

so if one recalls the covariant four-dimensional formula (160) for the classical three-dimensional formula (155) then one will see that relativity introduces a very small correction term θ'^i / c^2 (1). Moreover, we generalize the classical formula (140) to:

(IV.162)
$$\boxed{c^2 \rho_0^* = \int^{T_0} C(T_0)\, dT_0 .}$$

That being the case, from an integral transformation, one will have:

$$\iiint_{\mathcal{E}_2 - \mathcal{E}_1} \rho_0^* (U^i - V^i)\, \delta u_i = \iiint\int \partial_i (\rho_0^* U^i)\, \delta u\, dt - \iiint_{\mathcal{P}} \rho_0^* U^i\, \delta u_i \, .$$

The last integral, which can also be written:

$$- \iiint_{\mathcal{P}} \rho_0^* (U^i - V^i)\, \delta u_i \, ,$$

if one prefers, is immediately interpreted as the heat flux that enters into the portion of matter $\mathcal{E}$ that is between its state $\mathcal{E}_1$ and its state $\mathcal{E}_2$ (see no. III.6, eq. III.39). As for the last integral on the left-hand side, under the *hypothesis H* that was utilized many times in the course of this chapter (see no. IV.5), it can be written:

$$\iiint_{\mathcal{E}_2 - \mathcal{E}_1} \rho_0^* V^i\, \delta u_i \equiv \iiint_{\mathcal{E}_2 - \mathcal{E}_1} \rho_0^*\, \delta u_0 \, ,$$

and it can then be interpreted as the augmentation of the *proper heat* of the material drop $\mathcal{E}$ between its state $\mathcal{E}_1$ and its state $\mathcal{E}_2$. By *hypothesis P*, we assume that this interpretation will remain valid in the general case.

We then make the hypothesis that the *heat current quadri-density* $\rho_0^* U^i$ *is conservative*, which is the generalization of the classical hypothesis (157). One will have:

(IV.163)
$$\boxed{\partial_i (\rho_0^* U^i) = 0,}$$

or

(IV.164)
$$\partial_i \{\rho_0^* (U^i - V^i)\} + \partial_i (\rho_0 V^i) = 0;$$

i.e., if one takes (160) and (162) into account:

(IV.165)
$$\boxed{\partial_i \left\{ \kappa(\theta_0) \left(\partial^i \theta_0 + \frac{1}{c^2} \theta'^i \right) \right\} + C(T_0) T_0 + \rho_0 \partial_i V^i = 0.}$$

That is the equation that generalizes Fourier's equation (159) in relativity.

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(1) Compare this with [**142**], eq. (37), while taking (12) and (15) into account.

Indeed, suppose that the conducting medium is incompressible, and neglect the relativistic term $\partial_i\,(\kappa\,\theta'^i)\,c^2$:

(IV.166) $$\partial_i\,V^i \equiv 0, \qquad \frac{1}{c^2}\partial_i\,(\kappa\,\theta'^i) \approx 0.$$

If one takes (161) into account then what will remain is:

(IV.167) $$C\,(T_0)\,T_0 \approx \partial_i\{\gamma(T_0)\,\partial^i T_0\},$$

or further, under the hypothesis that:

(IV.168) $$C\,(T_0) = C = \text{const.}, \quad \gamma(T_0) = \gamma = \text{const.}$$

one will have:

(IV.169) $$\frac{C}{\gamma}\frac{d}{d\tau}T_0 \approx \partial_i^i\,T_0\,.$$

If one remembers that the classical formula (159) is written essentially in the Galilean frame that is linked with the conducting body, which is assumed to be rigid, then the left-hand sides of (159) and (169) will be equivalent. Furthermore, the term $-\;\partial_i^i\,T_0\,/\,c^2$ in the right-hand side of (169) can be interpreted as a very small "relativistic correction." Formula (169) – or even better, formula (165), which is free of the restrictions (166), (167), and (168) – will then indeed constitute the relativistic generalization of Fourier's formula (159).

IV.22. – **Some words on the relativistic transposition of the theory of perfect gases.** – To commence, we seek the covariant form of the well-known equation of state:

$$p\,v = NkT,$$

in which N denotes the number of molecules that are contained in the volume v, and k is the Boltzmann constant. If the temperature is not uniform inside of the volume considered then one must obviously write:

$$\iiint \frac{p\,dv}{kT} = N.$$

That expression is invariant under all transformations that affect a gaseous volume that is composed of the same N molecules.

In no. IV.6, we defined the relativistic notion of the *scalar work done by pressure* by:

$$\iiiint \varpi\,d\delta u_0$$

and indicates that it translates the classical $\int p\,dv$ exactly, in which dv denotes the variation of a finite material volume v during the time dt. Thanks to the preceding notion, as well as that of the *proper* – or *scalar* – *temperature* that was defined in no. IV.19, the *covariant form of the equation of state of the perfect gas:*

$$\boxed{\iiint_{\mathcal{E}} \frac{\varpi\,\delta u_0}{kT_0} = N} \tag{IV.170}$$

will present itself. N denotes the number (which is conservative) of molecules that form the same gaseous volume and follows its motion – i.e., the conservative number of trajectories that encircle the hyper-tube that is generated by $\mathcal{E}$.

From the viewpoint of the kinetic theory of gases, the quadri-velocity V^i that is implicitly contained in δu_0 has only a statistical significance. It is the velocity of the origin of the Galilean frame with respect to which the local mean impulse of the droplet δu_0 is zero. To simplify, suppose that all of the molecules are identical. Let m_0 be their proper mass, and let m be the local mean value of their relativistic mass at the proper temperature T_0 in the preceding Galilean frame. We suppose that this temperature is high enough that the quantum degeneracies will be masked, and we assume that the molecules are rigorously rigid. The expression $c^2(m - m_0)$ represents the mean kinetic energy of a molecule – i.e., the thermal energy per molecule. From kinetic theory, it is a simple multiple of the proper temperature that has the form ([1]):

$$\boxed{kT_0 = \frac{2}{3}\nu c^2 (m - m_0).} \tag{IV.171}$$

One will have $\nu = 1, \frac{3}{5}, \frac{1}{2}$, according to whether the gas is mono-, di-, or tri-atomic, resp.

Then let δu_0 be the mean scalar molecular volume. By definition of the proper mass density ρ_0 of that volume, one will have:

$$m = \rho_0\,\delta u_0, \qquad m_0 = \rho_{00}\,\delta u_0. \tag{IV.172}$$

We have let ρ_{00} denote the value that ρ_0 takes at absolute zero; i.e., in the absence of any thermal agitation to the molecules. Moreover, the equation of state provides the relation:

([1]) A precise analysis of the question shows that $\frac{3kT_0}{2\nu}$ is equal, not to the true mean kinetic energy $c^2(m - m_0)$, but to a *mean pseudo-kinetic energy* $\frac{mv^2}{2}$ [**146**, pp. 31-32]. Those two expressions are equivalent to each other, up to 4th order in β, as the following calculation shows:

$$\begin{aligned}\frac{1}{2}mv^2 &\equiv c^2 m - c^2 m\left(1 - \tfrac{1}{2}\beta^2\right) = c^2 m - c^2 m_0\left(1 - \tfrac{1}{2}\beta^2\right)\left(1 - \beta^2\right)^{-1/2} \\ &= c^2 m - c^2 m_0\left(1 - \tfrac{1}{4}\beta^4 + \cdots\right).\end{aligned}$$

(IV.173) $$k\,T_0 = \varpi\,\delta u_0\,.$$

When substituted into (171), the latter relations will permit one to write:

(IV.174) $$\boxed{\frac{\varpi}{c^2} = \frac{2}{3}\nu(\rho_0 - \rho_{00}).}$$

That equation expresses the proper mass of the molecules universally – viz., independently of the temperature – and from the molecular concentration, one can express the pressure ϖ *of the gas as a function of its proper thermal energy density* c^2 $(\rho_0 - \rho_{00})$.

This is the place for a very satisfying illustration and verification of both the thermodynamic formula that one arrives at with formula (174) and the theory of forces of surface tension in no. IV.6. *Upon substituting* (174) *into the general formula* (48) *of adiabatic compression, one will recover the well-known law of adiabatic compression of perfect gases.*

By virtue of the equation of state (170), the pressure ϖ is zero at absolute zero. It will then result from the considerations that were developed in no. IV.3 that one must have the relation:

(IV.175) $$\partial_i\,(\rho_{00}\,V^i) = 0.$$

Under those conditions, the formula that is obtained by substituting (174) into (48) is written:

$$\frac{3}{2\nu}V^i\partial_i\varpi + \left(\frac{3}{2\nu}+1\right)\varpi\,\partial_i V^i = 0.$$

From kinetic theory, as always, one then sets:

(IV.176) $$\boxed{\gamma = 1 + \frac{2}{3}\nu,}$$

and upon taking the kinematical formulas (II.74′) and (II.73) into account, one will indeed see the expected relation:

$$\frac{d\varpi}{\varpi} + \gamma\frac{d\delta u_0}{\delta u_0} = 0,$$

which will be written:

(IV.177) $$\boxed{\varpi(\delta u_0) = \text{const.}}$$

when it is integrated. One will have γ = 5/3, 7/5, 4/3, according to whether the gas is mono-, di-, or tri-atomic, resp.

REMARK. – If V^i denotes the *statistical – or macroscopic – velocity* of the gas in the sense that was just indicated then one can introduce the temperature quadri-vector $\theta^i = \theta_0 V^i$ and put the equation of state (170) into the form:

$$\iiint_{\mathcal{E}} \varpi\, \theta^i \delta u_i = k\, N. \qquad \text{(IV.178)}$$

CHAPTER V

COMPLEMENTS TO RELATIVISTIC DYNAMICS

V.I. – The sub-chapters of the last chapter were concerned with some physical questions that were very distinct, but mathematically quite similar. Sub-chapter A dealt with the relativistic hydrodynamics of perfect fluids, whose principles are due to Eisenhart [**148**] and Synge [**149**, **150**], and to which A. Lichnerowicz gave a very elegant form [**151**, **152**]. Sub-chapter B dealt with the analytical mechanics of an electrically-charged point that is subject to the action of a quadri-potential, which we presented some years ago in a symmetric relativistic form [**8**, no. IV].

The hydrodynamics of Eisenhart-Synge-Lichnerowicz is presented by those authors as being essentially a general-relativistic theory. In fact, it is upon adopting a particular metric that one will give its equations the maximum elegance. Of course, that beautiful theory remains valid in the particular case of special relativity. It would seem to be unduly frustrating to the reader to not present it in elementary terms at the end of a book that does not appeal to the methods of the general tensor calculus. As we have already pointed out, we shall deduce its starting equation from our general theory of pressure (no. IV.6). The relativistic extension of the hydrodynamics of Lagrange-Helmholtz will then be found to be attached to an authentic theory of surface pressure. We shall conclude our presentation by pointing out how a hypothesis that is even more restrictive than that of Synge (and is called *definition B* for the incompressible fluid by Lichnerowicz) will permit us to extend to relativity the entire classical theory of velocity fields that are induced by point-sources or vortex filaments, as well as Poincaré's notion of the vortex potential.

The symmetric presentation of the analytical mechanics of charged points that are subject to the action of a quadri-potential that we gave in § B constitutes the natural form for relativistic analytical point mechanics. In the preceding chapter, we have shown that it is possible in relativity to define the world-force that is applied to a material point as an antisymmetric second-rank tensor F^{ij}, and then that that relativistic force, which is defined in a general manner, will follow the laws of the Lorentz force (notably, see nos. III.7 and IV.7). It should not be surprising then that the case for which that tensor F^{ij} is a world-rotation constitutes the natural extension of the classical case in which the force F^{ij} is a gradient. Speaking physically, the symmetric presentation of analytical point mechanics has the advantage that it lets us see that it is not by virtue of some happy accident that a charged point that is subject to the action of a quadri-potential will obey the equations of analytical mechanics.

L. de Broglie (who, as one knows, established his theory of *wave mechanics* in 1924) has been to a large extent guided by the formal analogy that exists between analytical mechanics and geometrical optics. We have oriented the presentation in our § B in such a way as to prepare the reader for the guiding ideas of wave mechanics, and will insist especially upon the compatibility of the most important theorems of analytical mechanics with the *quantum conditions* that are demanded by the pre-wave form of the Planck-Bohr-Sommerfeld theory.

Our § C is dedicated to a brief survey of the principles of L. de Broglie's [**163**] first *wave mechanics*, which, as one knows, has given the go-ahead to a complete renovation of the *theory of quanta*. Despite first appearances, we have followed the presentation that the author himself gave very closely. It seems to us that in an era when one of the major problems that been posed by theoretical physics is that of describing the relationships between the theories of relativity and quanta, any achievement will no longer point to a presentation of special relativity, as much as it will relate to the elegant theory of L. de Broglie, which is both essentially relativistic and essentially quantum. With it, one can even say that relativity has returned to its roots, and that just as Einstein came to know of the optical constant c, dynamics, with L. de Broglie, had effectively discovered the intervention of waves in their proper domain.

A. – INVISCID FLUIDS AND THE EISENHART-SYNGE-LICHNEROWICZ THEORY OF VORTICES.

V.2. – **Basic hypotheses. Synge's hypothesis.** – Consider an inviscid fluid, inside of which a normal pressure ϖ of surface origin is exerted, and which conforms to the theory of no. IV.6, and is such that a force density f^i rules in certain space-time domains, which is collinear with the quadri-velocity V^i and corresponds to the presence of volume distributions of sources or sinks (no. IV.5). From what was said in no. IV.1, the fundamental equation of dynamics will be written:

(V.1) $$-f^i + \partial^i \varpi + \partial_i (\mu V^i V^j) = 0$$

under those conditions, with:

(V.2) $$\boxed{\mu = \rho_0 + \frac{\varpi}{c^2}.}$$

Then let F and G be two functions of instant-points that are undetermined, for the moment, and are such that one has:

(V.3) $$FG \equiv \mu,$$

and define the two *pseudo-quadri-velocities* U^i and W^i, which are collinear with V^i, by:

(V.4) $$U^i = F V^i, \qquad W^i = G V^i,$$

so:

(V.5) $$\boxed{\mu V^i V^j \equiv U^i W^j.}$$

Then set:

(V.6) $$\boxed{\sigma \equiv \partial_i W^i, \quad \tau^{ij} \equiv \partial^j U^i - \partial^i U^j.}$$

With those definitions, the fundamental equation (1) can be written and developed into:

$$f^i - \partial^i \varpi = \partial_j (U^i W^j) = s\, U^i + \tau^{ij} W_j + W_j \partial^j U^i.$$

Taking account of the fact that $V_i V^i = -c^2$, as well as (4) and (3), the last term in the right-hand side can be transformed into:

$$G\,V_j\,(V^j\,\partial^i F + F\,\partial^i V^j) = -\,c^2\,G\,\partial^i F = -\,c^2\,\mu\,\frac{\partial^i F}{F}\,.$$

It is then quite natural to arrange things in such a way that this term cancels the last term in the left-hand side identically, which first demands that the fluid must satisfy an equation of state of isothermal type:

(V.7) $$\mu = \mu\,(\varpi).$$

That being the case, the necessary and sufficient condition for the result to be true is that, with Synge, one must set:

(V.8) $$F = \exp\cdot\int_{\varpi_0}^{\varpi}\frac{d\varpi}{c^2\mu}, \qquad \text{hence} \qquad G = \exp\cdot\int_{\varpi_0}^{\varpi}\frac{d\rho_0}{\mu}\,.$$

With that choice of F and G, the fundamental dynamical equation will reduce to:

(V.9) $$\boxed{(\sigma U^i - f^i) + \tau^{ij} W_j = 0.}$$

Since one of the two quadri-vectors that are present in the left-hand side is collinear with V^i and the other one is orthogonal to it, each of them must be separately zero. If one considers the first one then one will see that the divergence of the pseudo-velocity W^i is zero, in general, except in the space-time regions in which one finds sources or sinks:

(V.10) $$\partial_i W^i = \begin{cases} 0 & \text{in general,} \\ \sigma & \text{in the regions of the unverse in which sources or sinks prevail.} \end{cases}$$

Obviously, that is the generalization of a fundamental result from classical hydrodynamics.

If one then considers the second quadri-vector, one will have the relation:

(V.11) $$\boxed{(\partial^j U^i - \partial^i U^j)\,dx_j = 0,}$$

from which, we, with Lichnerowicz, will infer the generalization of the entire classical Lagrange-Helmholtz theory of vortices.

V.3. – **Some general theorems. Existence of a potential for a pseudo-velocity U^i in the irrotational case.** – Let δx_i be a spacelike quadri-displacement. Equation (11) will then imply the consequence:

$$\tau^{ij}\,dx_j\,\delta x_i \equiv \tfrac{1}{2}\,\tau^{ij}\,[dx_j\,\delta x_i - dx_i\,\delta x_j] = \tfrac{1}{2}\,\tau^{ij}\,[dx_i\,dx_j]^* = 0,$$

in which $[dx_i\, dx_j]^*$ denotes a two-dimensional area element $\mathcal{S}$ that is formed by a sheet of streamlines $\mathcal{T}$. Conversely, since the quadri-displacement δx_i is arbitrary, the relation:

(V.12) $$\tau^{ij}\,[dx_i\, dx_j]^* \equiv (\partial^j U^i - \partial^i U^j)\,[dx_i\, dx_j]^* = 0$$

will imply the relation (11). If one sets, quite naturally:

(V.13) $$dU^i = \partial^j U^i\, dx_j \qquad \text{and} \qquad \delta U^i = \partial^j U^i\, \delta x_j$$

then the preceding relation will assume the equivalent form:

(V.14) $$\boxed{dU^i \delta x_i - \delta U^i dx_i = 0,}$$

in which the d relate to a displacement along a world-streamline, and the δ, to a change of streamline. The left-hand side of (14) is an *absolute integral invariant*, in the Poincaré-Cartan sense ([1]).

The fundamental equation (11) is written:

$$dU^i = \partial^i U^j\, dx_j\,.$$

When it is integrated along one trajectory $\mathcal{T}$ from an instant-point [1] to an instant-point [2], it will become:

(V.15) $$U^i_{(2)} - U^i_{(1)} = \int \partial^i U^j\, dx_j\,.$$

If we then associate the current instant-point of $\mathcal{T}$ with a spacelike quadri-displacement δx_i that is a continuous function of x_i and suppose that each contour $\int \delta x_i$ (or $\mathcal{L}$) is closed. One will then conclude from (167) that:

$$\int_{(2)} U^i \delta x_i - \int_{(1)} U^i \delta x_i = dx_i \int_{\mathcal{L}} \partial^i U^j\, \delta x_j \equiv 0,$$

in which the symbol δ always relates to a change of trajectory. Finally, one sees that *the circulation of the pseudo-velocity U^i along a closed, spacelike streamline that is carried by the fluid is conservative:*

(V.16) $$\boxed{\int_{\mathcal{L}} U^i \delta x_i = \text{const.}}$$

(viz., the Poincaré-Cartan *relative integral invariant*).

([1]) An integral invariant is called *absolute* or *relative* according to whether it must be calculated on an arbitrary contour or a closed contour in order for the formula to be true.

Now, let $\mathcal{C}$ be a two-dimensional endcap that is encircled by the preceding linear contour $\mathcal{L}$. The relation (16) transforms into:

(V.17) $$\iint_{\mathcal{C}} \partial_j U^i [dx_i\, dx_j] \equiv \tfrac{1}{2}\iint_{\mathcal{C}} (\partial^j U^i - \partial^i U^j)[dx_i\, dx_j] = \text{const.},$$

and one will conclude from that, as with (12):

(V.18) $$\boxed{\iint_{\mathcal{C}} \delta_1 P_i \cdot \delta_2 x^i - \delta_2 P_i \cdot \delta_1 x^i = \text{const.}}$$

(viz., an *absolute integral invariant*). Hence, *the flux of the rotation of the pseudo-velocity U^i that crosses a two-dimensional fluid surface is conservative.*

Conversely, in order to return to (17) or (18) from (11), it will suffice to take the integral (17) over a closed two-dimensional contour $\mathcal{C}_2 - \mathcal{C}_1 - \mathcal{L}$ [where $\mathcal{L}$ always has the sense that was defined by (12)], and remark that the transformed triple integral:

$$\iiint \partial^{jk} U^i [dx_i\, dx_j\, dx_k]$$

will be identically zero (since ∂^{jk} is symmetric in j, k, while $[dx_i\ dx_j\ dx_k]$ is antisymmetric).

Set:

(V.19) $$\boxed{d\Phi = U^j dx_j}$$

as the definition of a *symbolic action* Φ along the streamline $\mathcal{T}$. *The symbolic action* Φ *is extremal along the streamline* $\mathcal{T}$. A classical calculation from the calculus of variations will permit one to write:

$$\int_1^2 U^i\, \delta x_i = \int_1^2 \delta U^i\, dx_i + \int_1^2 U^i\, d\delta x_i = (U^i\, \delta x_i)_2 - (U^i\, \delta x_i)_1 + \int_1^2 (\delta U^i\, dx_i - dU^i\, \delta x_i)\,.$$

Since the initial and final instant-points are fixed, the necessary and sufficient condition for the variation δ to be zero is that one must have the relation (14).

If that relation is now assumed to be satisfied, and if, by hypothesis, the quadri-displacement $\delta_1 x^i$ is such that one will have (20_1) then it will result from the preceding calculation that one has (20_2):

(V.20) $$U^i\, \delta_1 x_i = 0, \qquad U^i\, \delta_2 x_i = 0.$$

One concludes from this that *if the congruence of $\mathcal{T}$ admits a three-dimensional orthogonal trajectory $\mathcal{E}$ then it will be a normal congruence.*

Finally, we adopt that hypothesis and take Φ = constant on the hypersurface $\mathcal{E}_1$. (That is obviously a definition that will change nothing intrinsically.) It will then result from the preceding calculation that the $\mathcal{E}$ whose orthogonal trajectories are the $\mathcal{T}$ will be the hypersurfaces:

$$\Phi = \text{const.},$$

and (19) will show that one will have:

$$U^i = \partial^i \Phi. \tag{V.21}$$

That particular case is nothing but the case of irrotational motion, so equation (24) will imply that:

$$\partial^j U^i - \partial^i U^j \equiv 0, \tag{V.22}$$

and conversely.

One will then arrive at the relativistic statement of the two well-known theorems: *The irrotational motion of characterized by the existence of a potential for the pseudo-velocity U^i. If the motion of an inviscid fluid is irrotational on a certain spacelike three-dimensional hypersurface $\mathcal{E}_0$ then it will be irrotational in all of space-time.*

V.4. – **The relativistic formulation of the Lagrange-Helmholtz theory of vortices by A. Lichnerowicz.** – With A. Lichnerowicz, define the spacelike vorticity quadri-vector τ^i as the dual of the completely-antisymmetric rank-3 tensor:

$$\tfrac{1}{2}\,\tau^{jkl} \equiv \frac{1}{ic}\sum \tau^{jk} V^l\ ; \tag{V.23}$$

the summation is extended by circular permutation. In the locally-Galilean frame that instantaneously follows the fluid, one will have:

$$\tau_0^w = \tau_0^{uv}, \qquad \tau_0^4 = 0,$$

which justifies the term of the quadri-vector τ^i and helps us see that one has the relation:

$$\boxed{V_i\,\tau^i \equiv 0.} \tag{V.24}$$

The differential equation for the vortex lines is obviously:

$$\tau^j\,\delta x^i - \tau^i\,\delta x^j = 0 \qquad \text{or} \qquad \tau^{jkl}\,\delta x_k = 0. \tag{V.25}$$

If one takes the definition (23) into account then one will have:

$$\tau^{ij}\,V^k\,\delta x_k + \tau^{jk}\,\delta x_k\,V^i - \tau^{ik}\,\delta x_k\,V^j = 0,$$

which is a relation whose first term is zero, from what was just said. If one multiplies the remaining terms by the streamline element dx_j then it will follow that:

$$\tau^{jk}\, dx_j\, \delta x_k\, V^i - \tau^{ik}\, \delta x_k\, V^j\, dx_j = 0,$$

which is a relation whose first term is zero, by virtue of what was said in the preceding no. Since the expression $V^j\, dx_j$ is essentially non-zero, one will finally see that *the differential equation for the vortex lines is formally identical to the one for the streamlines* (11):

$$\boxed{(\partial^j U^i - \partial^i U^j)\, \delta x_j = 0.} \tag{V.26}$$

All of the calculations of no. V.3 can then be repeated for the spacelike vortex tubes. Notably, one will then see *the circulation of the pseudo-velocity U^i around a vortex tube, and that its flux along a vortex tube is conservative.* It will necessarily follow from this that the vortex tubes are either closed or they extend up to spatial infinity or up to timelike endcaps that form a channel for the flowing fluid.

It is clear that any quadri-vector Δx_j that is coplanar with the two orthogonal elements dx_j and δx_j, such that the former is collinear with the streamlines $\mathcal{T}$, and the latter is collinear with the vortex lines τ, will satisfy the relation:

$$(\partial^j U^i - \partial^i U^j)\, \Delta x_j = 0 \tag{V.27}$$

identically. Conversely, it results from the theory of forms with exterior multiplication that no Δx_j that is coplanar to dx_j and δx_j will satisfy that relation ([1]). Equation (27) will then define a family of two-dimensional manifolds that each support a sheet of lines $\mathcal{T}$ and a sheet of lines τ. In relativity, one will then recover the classical property from which *the vortex lines τ are fluid lines.*

If we combine the preceding results with the ones in no. V.3 then we will see that *the circulation and the flux that are attached to a vortex tube and follow its motion will be conservative.* One thus achieves the relativistic generalization of the classical Lagrange-Helmholtz theory of vortices.

Finally, *in an inviscid fluid, it is permissible to consider a flow that is collectively irrotational, except in the interior of certain "conservative" domains; i.e., ones that are bounded by sheets of streamlines.*

([1]) In the particular case under consideration, we see how that result can be proved in a more elementary manner.

Recall that any antisymmetric determinant of even order is a perfect square, and if it has odd order then it will be zero. It follows from this that in four-dimensional space, the rank of a system of linear equations:

$$A^{ij}\, x_j = y^i,$$

in which the A^{ij} are antisymmetric, can be 4, 2, or 0, *a priori*. The rank of system (27) in Δx_j is not 4, since, from (11) and (26), one knows two systems of non-zero solutions of those equations. It is not 0 either, since all of the components $\partial^j U^i - \partial^i U^j$ would be zero then. It will then be 2, and consequently, all of the solutions Δx_j to (27) will be mutually linearly dependent. Q. E. D.

V.5. – **A hypothesis that is more restrictive than that of Synge: relativistic extension of the classical theory of perfect fluids.** – *Now make the hypothesis that the two pseudo-velocities U^i and W^i that are defined by* (4) *are identically equipollent*; i.e., that one will have:

(V.28) $$U^i \equiv W^i, \qquad F = G = +\sqrt{\mu}\,.$$

(8) shows that this hypothesis is equivalent to:

(V.29) $$c^2 \rho_0 = \varpi = \text{const.}$$

along the streamlines. Lichnerowicz called a fluid that satisfies that condition an *incompressible fluid B* and showed that such a fluid is characterized by the fact that the speed of pressure waves in it is c [**153**]. If one would like to have an intrinsic definition then one might assume that the relation (29) is satisfied identically in the whole universe.

For us, an inviscid relativistic fluid that satisfies the definition of Synge and the more restrictive definition (28), as well, will be called a *perfect fluid*. Indeed, we shall show that its properties generalize the properties of the classical fluid with that name exactly.

It results from the hypothesis (28), as well as what was contained in nos. V.2, 3, and 4, that the single pseudo-velocity U^i that characterizes the *perfect fluid* will satisfy the two relations:

(V.30) $$\boxed{\partial_i U^i = 0, \quad \partial^j U^i - \partial^i U^j = \tau^{ij}.}$$

The scalar density σ is zero in space-time, in general, except for certain regions of arbitrary form that contain density distributions of sources and sinks. The antisymmetric tensor density τ^{ij}, which satisfies the relation:

(V.31) $$\sum \partial^i \tau^{jk} = 0$$

identically, in which the summation is extended by circular permutation, can also be considered to be zero, in general, except inside of certain “vorticial” fluid volumes that extend to temporal infinity in both directions. Their spatial hypersections are either closed or extend to infinity or up to a timelike wall that makes a channel for the world-flow.

We shall show that it is possible to derive the field quadri-vector U^i from two potentials, one of which is a scalar P, while the other one is an antisymmetric tensor R^{ij} that satisfies the condition:

(V.32) $$\boxed{\sum \partial^i R^{jk} = 0}$$

identically, by way of:

(V.33) $$\boxed{U^i = \partial^i P + \partial_k R^{ik}.}$$

If one substitutes (33) in (30) and takes into account the essential hypothesis (32) then one will indeed get the generating formulas for the potentials:

(V.34) $$\boxed{\partial^i P = \sigma, \quad \partial_i^i R^{jk} = \tau^{jk}.}$$

Finally, upon applying the d'Alembertian expression ∂_i^i to the terms in (33) and taking (34) into account, one will get:

(V.35) $$\boxed{\partial_i^i U^k = \partial^k \sigma + \partial_l \tau^{kl}.}$$

Formulas (34) and (35) admit *retarded* solutions of the well-known type. The various preceding equations generalize the well-known ones of the classical theory of perfect fluids. As for the electromagnetic field, the relativistic calculations are performed with the greatest of ease in the general case of variable regimes.

It is also easy to extend Poincaré's notion of a vortex potential to the present theory, and in relativity, it will likewise be a source/sink potential. The fundamental hypothesis (32) shows that R^{jk} is a rotation whose generating quadri-vector is naturally defined up to a quadri-gradient. Upon solving the equation:

$$\partial_i^i Y = P - \partial_i A_0^i,$$

one will be in a position to have both:

(V.36) $$\boxed{P = \partial_i A^i, \quad R^{ij} = \partial^j A^i - \partial^i A^j.}$$

Finally, one concludes the generating formula for the Poincaré potential ([1]) from (36) and (33):

(V.37) $$\boxed{\partial_i^i A^k = U^k.}$$

That is the relativistic extension of all the classical equations of the theory of perfect fluids that are valid in the most general case and for variable regimes. In pre-relativistic physics, one knows that that entire theory presents great formal analogies with electromagnetism, such that the theory of sources/sinks resembles electrostatics, and the theory of vortices resembles the electrodynamics of permanent regimes. That analogy will disappear in the tensorial language of the universe, since the so-called "homologous" quantities will not have the same variances, and the so-called "homologous" equations will be different *a fortiori*. Here again, the theory that one constructs in the relativistic context seems to be perfectly clear.

B. – THE ANALYTICAL MECHANICS OF A MATERIAL POINT SUBJECT TO THE ACTION OF A FORCE DERIVED FROM A QUADRI-POTENTIAL.

V.6. – **Establishing the starting formula. Some words about first integrals.** – Consider a material point – with or without spin – that is subject to the action of an

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([1]) The quadri-divergence of A^i will be zero in the absence of source/sinks.

antisymmetric force F^{ij} that is a function of a pre-determined instant-point that is independent of the world-trajectory of a point, and which is the rotation of a certain quadri-vector that is called the *world-potential*, moreover. For example, one deals with an electrically-charged point Q that is subject to the force $Q\,H^{ij}$ that is due to a pre-established field $H^{ij} = \partial^j A^i - \partial^i A^j$.

The fundamental dynamical equation of motion of such a point is equation (III.59), which we write as:

(V.38) $$dp^i = Q\,(dA^i - \partial^i A^j\,dx_j).$$

In the right-hand side, we have the ponderomotive expression for the elementary energy-impulse. On the left-hand side, we assume that we are dealing with the inertial expression for that same quantity (nos. IV.7 and IV.15).

I now say that whether the material point does or does not have spin, one will have the right to assume the presence of a term on the left-hand side of (38) that is identically zero, namely, $-\partial^i p^j\,dx_j$. It is initially necessary that a term of that form should make sense. Now, one of the essential traits of analytical point mechanics is precisely that it should associate an entire congruence of virtual trajectories to a real trajectory, which is also contained in that congruence. Under those conditions, the mass-impulse p^i of the point will be a quadri-vector field, so it will make sense to speak of its derivatives.

For the point without spin, on the one hand, p^i is collinear with dx^i, and on the other hand, its length icm_0 is constant. It will then be clear that the term $-\partial^i p^j\,dx_j$ will be zero. For the point without spin, we assume, as was explained in no. IV.15, that the projection of p^i onto the quadri-velocity V^i is constant, and that p^i is orthogonal to the normal acceleration V'^i, moreover. Under those conditions, one will have:

$$p'^j\,V_j = -p^j\,V'_j = 0. \qquad \text{Q. E. D.}$$

Finally, we indeed have the right to replace formula (38) with the formula ([1]):

$$dp^i - \partial^i p^j\,dx_j = Q\,(dA^i - \partial^i A^j\,dx_j)$$

in a general way, in which:

(V.39) $$\boxed{\partial^i p^j dx_j \equiv 0.}$$

We then set, by definition, the *total energy-impulse* of the point (viz., *inertial* + *electromagnetic*) to:

(V.40) $$\boxed{P^i = p^i - QA^i.}$$

The preceding formula will then be written:

([1]) In the case of a point with spin, the present theory neglects the proper ponderomotor couple that results from the interaction of the ambient field with the proper electromagnetic moment of the point.

(V.41) $$\boxed{dP^i - \partial^i P^j dx_j \equiv (\partial^j P^i - \partial^i P^j)\, dx_j = 0,}$$

and that will be our starting formula. It is formally identical to formula (11), and all of the calculations of no. V.3 can be repeated verbatim.

Before we do that, we stop for a moment to consider the case in which the quadri-potential A^i remains invariant in magnitude and direction along a parallel to any of the four axes Ox^i:

(V.42) $$\partial^i P^j \equiv 0 \qquad \text{for a well-defined } i.$$

It then results from (41) that one will have:

(V.43) $$P^i = p^i - Q\,A^i = \text{const.} \qquad \text{for the preceding } i.$$

A well-known example of that state of affairs is provided by the case of a permanent field:

$$\partial^i P^j = 0,$$

in which:

$$m - \frac{1}{c} QV = \text{const.} \qquad \text{with} \qquad m = m_0\,(c^2 + \tfrac{1}{2} v^2 + \ldots).$$

By definition, one says that under the hypothesis (42), the component P^i of the total *mass-impulse* is a *first integral* of the motion ([1]).

V.7. – **Some general theorems. The Hamilton-Jacobi theorem.** – As we said, consider a congruence of virtual world-trajectories $\mathcal{T}$ that one can fictitiously regard as the streamlines of a hypothetical *possibility fluid;* as one knows, it is a sort of preliminary representation of the *probability fluid* of wave mechanics.

Of course, the quadri-vector P^i is not tangent to the $\mathcal{T}$ here, so all of the calculations of no. V.3 can be repeated verbatim ([2]). One will then see that *the double integral:*

$$\iint (\partial^j P^i - \partial^i P^j)[dx_i\, dx_j]$$

is identically zero on one sheet of $\mathcal{T}$, and conservative over an endcap that follows the possibility fluid. Conversely, one knows how to pass from those statements to the starting formulas (41) and (40). The preceding integral will then be equivalent to:

$$\iint dP^i \delta x_i - \delta P^i dx_i$$

([1]) Since we have neglected the theory of kinetic and proper ponderomotor moments in this present § B, we shall not treat first integrals that are kinetic moments here.

([2]) By contrast, the calculations of no. V.4, which essentially assume that V^i is collinear with dx^i, will no longer be valid here.

on a sheet of $\mathcal{T}$, and one will have:

$$\iint \delta_1 P^i \delta_2 x_i - \delta_2 P^i \delta_1 x_i$$

on a fluid endcap.

The *action* along the trajectories will be defined by:

$$\boxed{d\Phi = P^i dx_i.} \tag{V.44}$$

As in the context of (16), one proves that *the action along a closed fluid line is conservative*, and one can conversely get back to (41).

One also proves that *the action along the trajectories of $\mathcal{T}$ is extremal* (i.e., the relativistic generalization of Hamilton's theorem), and that *if the congruence of total energy-impulse P^i lines admits a (three-dimensional) orthogonal trajectory then it will be a normal congruence* (i.e., the generalization of Jacobi's theorem). Upon adopting the latter hypothesis, one will make the hypersurfaces:

$$\Phi = \text{const.}$$

coincide with the orthogonal trajectories in question, which will imply the relation:

$$\partial^i \Phi = P^i = p^i - Q\,A^i. \tag{V.45}$$

In the particular case of a point without spin, which is such that $p_i\, p^i = -\, c^2 m_0^2$, one will conclude the following partial differential equation from (45):

$$\boxed{(\partial_i \Phi + QA_i)(\partial^i \Phi + QA^i) = -c^2 m_0^2.} \tag{V.46}$$

That is the *relativistic Jacobi equation*, which is a generalization to four indices of the well-known equation of geometrical optics. Its solutions Φ = const. will be called *world hypersurfaces* in wave mechanics.

Recall that the theory of quanta, in its pre-wave form, imposes the restriction on the integrals:

$$\int_{\mathcal{E}} P^i dx_i\,, \qquad \int_{\mathcal{L}} P^i \delta x_i\,, \qquad \iint \delta_1 P^i \delta_2 x_i - \delta_2 P^i \delta_1 x_i$$

that they must be integer multiples of the universal constant h. As one sees, *relativistic point mechanics* will then be *compatible with the quantum conditions.* It would not be surprising at all then that relativistic point mechanics provided L. de Broglie with the basis for some ideas that allowed him to inaugurate his *wave mechanics* in 1924, which was a complete renovation of the *theory of quanta.*

V.8. – **Lagrange equations. On the Hamilton equations.** – In order to integrate the equations of motions, one can suppose *a priori* that the P^i and x^i can be expressed as functions of a single parameter η, in such a way that the elementary action will be written:

(V.47) $$d\Phi = P_i\, x'^i\, d\eta = \mathcal{L}\,(\eta)\, d\eta,$$

with, by definition, the *Lagrange function* or *Lagrangian* (tensorial invariant):

(V.48) $$x'^i = \frac{dx^i}{d\eta}, \qquad \boxed{\mathcal{L} = P_i\, x'^i.}$$

It is classical to consider $\mathcal{L}$ to be a function of η by the intermediary of the x^i and x'^i. If we let $\partial^{\,i}\mathcal{L}$ and $\partial'^i\mathcal{L}$ denote the partial derivatives with respect to those eight variables then, from a well-known calculation, and by virtue of Hamilton's theorem:

$$\begin{aligned}\delta\int_1^2 \mathcal{L}\, d\eta &= \int_1^2 (\partial^i\mathcal{L}\,\delta x_i + \partial'^i\mathcal{L}\,\delta x'^i)\, d\eta \\ &= \int_1^2 (\partial^i\mathcal{L}\,\delta x_i\, d\eta + \partial'^i\mathcal{L}\, d\,\delta x_i) \\ &= \left(\partial^i\mathcal{L}\,\delta x_i\right)_1^2 + \int_2^1\left[\partial^i\mathcal{L} - \frac{d}{d\eta}(\partial'^i\mathcal{L})\right]\delta x_i\, d\eta \;= 0.\end{aligned}$$

For the same reasons as before, the Lagrange equations will result from this in the perfectly-symmetric "parametric form" that was pointed out by de Donder [**21**, pp. 176]:

(V.49) $$\boxed{\frac{d}{d\eta}\frac{\partial\mathcal{L}}{\partial x_i'} = \frac{\partial\mathcal{L}}{\partial x_i}.}$$

Conversely, in order to pass from (49) to the fundamental equation (41), if one takes the definition (48) into account then one can write:

$$\begin{aligned}\partial^{\,i}\,\mathcal{L} &= x_k'\,\partial^i P^k = x_k'\,\partial^i p^k + Q\, x_k'\,\partial^i A^k\,, \\ \partial'^i\,\mathcal{L} &= P^i + x_k'\,\partial'^i P^k = P^i + x_k'\,\partial'^i p^k\,.\end{aligned}$$

As we explained in no. V.6, the terms $x_k'\,\partial^i p^k$ and $x_k'\,\partial'^i p^k$ are zero, in such a way:

(V.50) $$\frac{\partial}{\partial x_i'}\mathcal{L} = P^i, \qquad\qquad \frac{\partial}{\partial x_i}\mathcal{L} = Q\,\partial^i A^k x_i'\,.$$

If one substitutes those expressions into (49) then one will indeed recover (41).

More generally, the authors take ordinary time t to be the parameter. If one then introduces the three components v^u of ordinary velocity, and m_0 always denotes the proper mass then one will get the very asymmetric Lagrangian:

$$\mathcal{L} = p_u\, v^u - c^2\, m - Q\,(A^u\, v_u - c\,\mathcal{V}) = -\,c^2\, m_0\,\sqrt{1-\beta^2}\, - c\,Q\,(\mathcal{V} - \boldsymbol{\beta}\cdot\boldsymbol{A}).$$

On the contrary, if one adopts proper time τ as the parameter then upon introducing the four components V^i of the world-velocity, one will have the symmetric Lagrangian:

$$\mathcal{L} = P_i\, V^i = m_0\, V_i\, V^i - Q\, A_i\, V^i.$$

However, as a result of certain particular simplifications, one will verify that this symmetric Lagrangian is not appropriate for the deduction of the equations of motion. R. Becker [**6**, pp. 380] replaced it with the arbitrarily-posed Lagrangian:

$$\mathcal{L}' = \mathcal{L} - \tfrac{1}{2} m_0\, V_i\, V^i.$$

However, one can no longer believe that the classical theory has been transposed into four-dimensional language then.

Return to the general Lagrangian $\mathcal{L}(\eta)$ and assume that it is no longer expressed as a function of the x^i and x'^i, but of x^i and P^i. We leave to the reader the task of verifying that if one defines the *Hamiltonian invariant* by:

$$\mathcal{H}\,(x^i, P^j) = \mathcal{L} - P^k x'_k$$

then one will *formally* arrive at the equations in Hamiltonian form:

$$\frac{\partial\mathcal{H}}{\partial x^i} = \frac{d}{d\eta} P_i\,, \qquad\qquad \frac{\partial\mathcal{H}}{\partial P^i} = -\frac{d}{d\eta} x_i$$

by a simple transposition of the classical calculations. *However, that result is purely fictitious, since the function $\mathcal{H}$ is identically zero.* R. Becker [**6**, pp. 380], who employed proper time τ as the parameter, showed that one can utilize eight symmetric equations of the preceding type, on the condition that one must replace $\mathcal{H}$ with:

$$\mathcal{H}' = \mathcal{L}' - P^k\, x_k = \mathcal{H} - \tfrac{1}{2} m_0\, V_i\, V^i.$$

However, one must then take into account only the fact that the function $\mathcal{H}'$ is identically constant in the calculation, and not replace $V_i\, V^i$ with its constant value $-\,c^2$.

For an asymmetric relativistic formulation of the Hamiltonian theory that corresponds to a literal transposition of the classical theory, we refer the reader to Von Laue [**4**, §§ 27 *e*, *f*, *g*]. Even then, the constant $-\,\tfrac{1}{2} c^2 m_0$ must be introduced arbitrarily when the parameter is proper time [**4**, pp. 238-239].

Finally, the literal search for a relativistic transposition of Lagrange-Hamilton theory leaves a rather artificial impression. In our sense, the essence of relativistic analytical point mechanics, and what is most interesting from the physical viewpoint, is provided by the general theorems that we have recalled in the preceding no. ([1]).

C. – LOUIS DE BROGLIE'S FIRST WAVE MECHANICS.

V.9. – The profound analogies between analytical mechanics and geometric optics have not escaped the classics [**156-160**]. Those analogies are due to the existence of a physical link whose formulation, which was discovered by L. de Broglie in the years leading up to 1924, involves the universal constants *h* of quanta and *c* of special relativity in an essential way [**163**]. As an application of his theory, L. de Broglie recovered Bohr's atomic quantization rule, and as a special case, the formula that Einstein gave for the photon. Soon after that, the experiments on electron diffraction by Davisson-Germer, G. P. Thomson, Ponte, and other authors afforded a very direct and increasingly precise verification of the concepts and formulas of theory. We also point out that Heisenberg illustrated his celebrated proof of the *uncertainty relations* by some considerations from classical wave optics that were associated with the fundamental ideas of L. de Broglie [**167**, pp. 15-19, **168**], and that L. de Broglie's thesis based one of its essential starting points upon some memorable work of Schrödinger ([2]).

In the realm of optics, properly speaking, and in the context of special relativity that he himself had to create, in 1905, Einstein stated the principle of a quantum synthesis of the traditional *wave-like* and *corpuscular* conceptions of light [**161**, **162**]. It is, moreover, quite interesting to see, by following L. de Broglie, how the possibility itself of a parallel synthesis will depend upon the formulation of the relativistic theses of relativistic dynamics ([3]), and expressly, the concept of a *quantum of light*, or *photon*. Einstein postulated that the energy of a light wave that is planar and monochromatic with frequency ν will transport the *quantum* of energy $h\nu$ at the velocity c. From relativity, the proper mass of those *light quanta* (which were called *photons* only later) must be zero or negligible, while their kinetic mass is given by the formula:

$$\boxed{c^2 m = h\nu.} \tag{V.50}$$

That formula accounts for the newly-discovered photoelectric effect, as well as the laws of thermal radiation that were discovered by Planck in 1900; it received a remarkable confirmation from the work of Bohr (1913).

The profound idea of L. de Broglie consisted of the fact that if it is necessary to introduce mechanics into optics then, conversely, it is also necessary to introduce optics into mechanics. To L. de Broglie's way of thinking, the discrete numbers that Planck and Bohr found in their formulas for the quantization of electronic oscillators – whether linear

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([1]) Nevertheless, we point out that Dirac gave a symmetric, relativistic extension of Hamilton's theory by starting with the *Poisson brackets* [Annales de l'Institut Henri Poincaré **9**, 2 (1939), 29-31].

([2]) *Mémoires sur la mécanique ondulatoire*, Fr. trans., Paris, 1933.

([3]) In addition to the celebrated *Thése* that was cited above, one should also read *La mécanique ondulatoire du photon*, Paris, 1940, pp. 36.

or turning – evoked the discrete numbers that one encounters in the theory of stationary waves. Those remarks can only give great weight to the formal analogy between analytical mechanics and geometrical optics. Effectively, it was by making each of them play precisely that role that L. de Broglie gave the formulation of his famous *wave mechanics* in 1924, and its fundamental equations contained Einstein's equation (50) as a special case.

V.10. – **Generalizations of the notion of world-hyper-wave. Phase velocity and group velocity.** – Consider a family of hypersurfaces $\mathcal{O}$ of the equation:

(V.51) $$\varphi(x^i) = \text{const.}$$

By hypothesis, the *phase* φ of the *hyperwave* $\mathcal{O}$ will determine certain physical properties at the instant-point x^i by the intermediary of the complex function:

(V.52) $$\psi = A\, e^{2\pi i\,\varphi},$$

in such a way that those properties will be periodically reproduced when one changes the hyperwave $\mathcal{O}$ in the universe by varying φ by a whole number. One can give an arbitrary direction to the quadri-displacement in the identity:

(V.53) $$d\varphi = \partial^{\,i}\varphi\, dx_i\,,$$

and one will see that the projection of the quadri-gradient $\partial^{\,i}\varphi$ onto that direction will measure the *number of waves* that are encountered per unit length, while the φ are in an arithmetic progression with difference 1. For that reason, the quadri-vector $\partial^{\,i}\varphi$ merits the name of *wave number quadri-vector* that L. de Broglie gave to it, or also that of *spatio-temporal frequency quadri-vector*. One then sets, by definition:

(V.54) $$\lambda^i = \partial^{\,i}\varphi.$$

We let $\mathcal{R}$ denote the congruence of curves that are orthogonal to the hypersurfaces $\mathcal{O}$ and are directed by the quadri-vector field λ^i and call them *world-rays;* until further notice, the rays in $\mathcal{R}$ will not necessarily be timelike.

If α^u denote the three direction cosines of the spatial projection of λ^i then the *wave length L* and the *period T* will always be defined as functions of the components of λ^i according to formulas (II.41), which we write as:

(V.55) $$\boxed{\lambda^u = \frac{1}{L}\alpha^u\,, \quad \lambda^u = \frac{i}{cT}.}$$

However, in the general case that we are treating here, the quadri-vector λ^i is not isotropic, and one will no longer have the relation (II.40). By virtue of a classical definition, the modulus of the ratio L / T will be the *phase velocity* w of the wave $\mathcal{O}$:

$$\boxed{L = wT.} \tag{V.56}$$

If the α^u denote the three components of that phase velocity then one will obviously have:

$$\alpha^u = \frac{1}{w} w^u, \tag{V.57}$$

and consequently:

$$\frac{ic}{\lambda_4} \lambda_u = \frac{c^2}{w^2} w_u . \tag{V.58}$$

Moreover, it results from relativistic kinematics that if v^u denotes [at least, symbolically ([1])] the ordinary velocity of a fluid whose world-trajectories are $\mathcal{R}$ then one will have:

$$\frac{ic}{\lambda_4} \lambda_u = v_u . \tag{V.59}$$

It will then appear that *the two spatial tri-vectors v^u and w^u are collinear, and that their moduli will satisfy the relation:*

$$\boxed{v \cdot w = c^2.} \tag{V.60}$$

That is one of the essential laws that L. de Broglie stated. One should recall that in world-geometry, the three v^u can be interpreted as the direction cotangents of the $\mathcal{R}$ at each instant-point. Analogously, the three w^u can be interpreted as the inverses of the slope coefficients of the $\mathcal{O}$ relative to the axis Ox^4.

Now imagine that the quadri-vector λ^i is affected with a certain *dispersion* $d\lambda^i$ at each instant-point that does not alter its length; i.e., such that:

$$\lambda^i \, d\lambda^i = 0. \tag{V.61}$$

It is obvious that the wave figure will not be altered by means of that condition. The phase shift $d\varphi$ that is implied by the preceding $d\lambda^i$ at each instant-point x^i will be:

$$d\varphi = x_i \, d\lambda^i, \tag{V.62}$$

([1]) For the moment, we have not imposed the constraint upon the congruence $\mathcal{R}$ that it must be timelike. We shall also insist upon the fact that the "hydrodynamical" image that we have just suggested is meaningful only in the *geometrical optics approximation* of *wave mechanics*.

in such a way that the most general quadri-displacement δx_i for which that $d\varphi$ has a well-defined value must satisfy the condition:

$$\delta(d\varphi) = \delta x_i \, d\lambda^i = 0. \tag{V.63}$$

In order for that condition to be sufficient for any sort of dispersion $d\lambda^i$, it is necessary and sufficient that the quadri-displacement δx_i must be collinear with λ^i. Now, since, from the definition itself of the *wave group* or *wave packet,* it must displace with constant phase, it will result from the preceding argument that *the quadri-velocity* V^i *that is attached to the ray congruence* $\mathcal{R}$ *is nothing but the group quadri-velocity of the wave* $\mathcal{O}$*.* That is another very remarkable law that L. de Broglie discovered in the context of his *wave mechanics.*

V.11. – **Identification of the notion of phase and action. Wave mechanics.** – Now that the theory of the world-hyperwave has been generalized in the manner that was just described, recall the results that were obtained in § C regarding analytical point mechanics. Moreover, appeal to the fact that, from Planck and Bohr's *theory of quanta*, *action* Φ *is counted in integer multiples of the universal constant h*. Then set:

$$\boxed{\Phi = h\varphi,} \tag{V.64}$$

and one will see that *there is a complete parallelism between the hyperwaves* $\mathcal{O}$ *and their world-rays* $\mathcal{R}$ *of no.* V.10, *on the one hand, and the equal-action hypersurface and their orthogonal trajectories of no.* V.7, *in the other.* In order to realize the physical identification that appeals to that analogy quite crucially, it will suffice to set, with L. de Broglie [**163**, pp. 51]:

$$\boxed{P^i \equiv p^i - QA^i = h\lambda^i;} \tag{V.65}$$

i.e., *one must institute a universal proportionality of ratio h between the total energy-impulse quadri-vector of a material point and the spatio-temporal frequency of the wave that wave mechanics associates with it, by hypothesis.* That relation, from the way that it was established, essentially amounts to the geometrical optics approximation. Naturally, one of the primary goals of the new mechanics has been to go on to the *wave-like* level upon starting from the Newtonian arena – or even better, the classical relativistic one.

In the *case of the free material point* – i.e., in the absence of a governing quadri-potential – the formula of L. de Broglie will reduce to:

$$\boxed{p^i = h\lambda^i.} \tag{V.66}$$

That is the wave-like expression for the law of inertia, which bijectively couples the uniform, rectilinear motion of a material point with the monochromatic-planar character of the associated wave. As one knows, ever since Fresnel's classical calculations and the application of the Fourier integral to the phenomena of diffraction and interference, the latter phenomena have been treated by the superposition of monochromatic plane waves. It is in that way that L. de Broglie's formula (60) implicitly predicted the phenomena of the diffraction and interference of material corpuscles that Davisson and Germer exhibited for electrons that crossed a crystal, as well as the fact that Heisenberg and Bohr could infer the classical proofs of the celebrated *uncertainty relations* [**167**, **168**].

Always in the absence of a governing quadri-potential, and in the very important special case of a monochromatic plane wave (viz., "uniform, rectilinear motion"), the quadri-velocity of a point will be tangent to the world-rays of the associated wave ([1]), which will then be timelike. It will then result from what was said in the preceding no. that the phase velocity of that *material wave* is greater than *c*, but that is no contradiction with the principles at the basis of relativity, since from what was said in the preceding no., the phase velocity does not have the significance of a kinematical velocity for the point. Moreover, the group quadri-velocity of the world *material-wave* that was defined in the preceding no. will then coincide with the quadri-velocity of the point, and will be timelike. That is one of the remarkable results that were obtained by L. de Broglie in his thesis.

Now, return to the general case and substitute the λ^i in (65) in the expression (52) for the phase, so the *wave function* can be written:

$$\psi = A e^{\frac{2\pi i}{h} P_k x^k} . \tag{V.67}$$

Upon applying the operator $\frac{h}{2\pi i}\partial_k$ to this, one will find that:

$$\frac{h}{2\pi i}\partial_k \psi = P_k\ \psi. \tag{V.68}$$

That remark, when duly generalized, is the origin of the fact that wave mechanics makes the notion of energy-impulse correspond to the quadri-operator $\frac{h}{2\pi i}\partial_k$ ([2]).

One knows that, mathematically speaking, the passage from geometrical optics to wave optics consists of replacing the first-order, second degree, partial differential equation that is called the *geometrical optics* equation with the second-order, first-degree, partial differential equation that one calls the equation of *wave optics*. If one briefly applies that process to the relativistic expression (46) for the Hamilton-Jacobi equation then one will get the Gordon equation:

$$\left\{\left(\frac{h}{2\pi i}\partial^k + Q\,A^k\right)\left(\frac{h}{2\pi i}\partial_k + Q\,A_k\right) + m_0^2\,c^2\right\}\psi = 0, \tag{V.69}$$

([1]) That will be true for a point with or without spin (no. IV.17). In the particular case of the point without spin, which is physically less interesting, one will not demand the monochromatic-planar character of the wave.

([2]) Up to certain difficulties that relate to the fourth component that we shall not dwell upon here.

which was proposed independently by several theoreticians in 1926, and which L. de Broglie himself utilized in order to describe the point charge without spin relativistically and as a wave.

For various reasons, one of which is experimental and the others of which are theoretical, it was soon recognized that this equation was not appropriate to the general case in which a quadri-potential was present. By contrast, it is valid for the free point, and in that case, it can serve to give a theory of interference and diffraction phenomena. It will then be written:

$$\left(-\frac{h^2}{4\pi^2}\partial_i + m_0^2\, c^2\right)\psi = 0, \tag{V.70}$$

and it will obviously admit monochromatic plane solutions:

$$\psi = A e^{\frac{2\pi i}{h} p_j x^j}, \tag{V.71}$$

which will permit one to reconstruct the general case by superposition.

Physically speaking, one of the main reasons that has prevented the Gordon equation from becoming accepted has been the fact that it ignores spin, by the origin itself of the train of reasoning that led up to it ([1]). In summation, spin is a foreign element that was introduced arbitrarily in the sequence of experimental facts in wave mechanics up to the year 1925, just as the notion of polarization itself had been for a long time in optics. It was the work of Dirac in 1927 that definitively formulated the *wave* mechanics of the *relativistic* electron *with spin*. In 1930 and 1936, L. de Broglie was then led to that from his *theory of the photon*, in which the notion of *polarization* played an integral role.

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([1]) In its place, we have pointed out that the relativistic Hamilton-Jacobi equation is written essentially for a point without spin.

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