RECIPROCAL FIGURES

IN

GRAPHICAL STATICS

BY

Prof. L. CREMONA

Translated by D. H. Delphenich

MILAN

PRINTING WORKS OF GIUSEPPE BERNADONI

—

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PREFACE

This brief work was brought to light for the first time, accompanied by a memoir of my friend FELICA CASORATI, on the solemn occasion that was a family celebration for the friends of the director of the Milanese Polytechnic Institute (1 June, p. p.). It has now been reprinted without additions, except for some minor corrections, in a more modest edition for the convenience of younger students at the institute that must pursue a course of graphical statics.

The purpose of these few pages – namely, the use of geometric methods to determine the tensions and pressures in trusses that are subjected to external systems of forces – will have merit in a more substantial study. In fact, it was my intention to follow it with a second part that would be printed on the occasion of some later date. But even now the implementation of that proposal is impeded by other obstacles, so I am forced to postpone that to another occasion.

Milan, 12 August 1872.

L. C.

1.

The theory of reciprocal figures in space that are deduced from the consideration of a system of forces that are applied to a free, rigid body is well-known. An infinite axis passes through an arbitrary point $\mathfrak{M}$ in space, with respect to which the moment of the system is zero, and the geometric locus of those axes will be a plane μ that MÖBIUS (*) called the *null plane* of the point $\mathfrak{M}$. Conversely, an arbitrary plane μ will contain an infinitude of axes of zero moment, all of which will intersect at the same point $\mathfrak{M}$, which MÖBIUS called the *null point* of the plane μ. The *null point* and *null plane* can also be called the *pole and polar plane.*

In that way, any point $\mathfrak{M}$ in space corresponds to a plane μ that passes through $\mathfrak{M}$, and conversely, any plane μ will correspond to a point $\mathfrak{M}$ that is situated in μ. If the pole $\mathfrak{M}$ moves in a given plane α then the polar plane μ will constantly pass through a fixed point $\mathfrak{A}$ that is the pole of α; conversely, the locus of the poles of all planes that pass through a point $\mathfrak{A}$ will be a plane α that is the polar to $\mathfrak{A}$.

If the pole traverses a line then the polar plane will rotate around another line. Two lines can then be called *conjugate* or *reciprocal* when they are, at the same time, the loci of the poles of the planes that pass through the other one and the envelope of the polar planes of the points of the other one. One knows that the given system can be reduced to two forces in an infinitude of ways. If one chooses the line of action of one of them arbitrarily then the line of action of the other one will determined uniquely, since the two lines of action must be conjugate lines.

2.

The poles that correspond to a pencil of parallel planes are the points of a line with a well-defined direction. That is, a line at infinity is conjugate to a line whose point at infinity $\mathfrak{J}$ is the pole of the plane at infinity. In that way, all pencils of parallel planes will correspond to all (mutually-parallel) lines that pass through $\mathfrak{J}$. Among these lines, there is one that is perpendicular to the corresponding pencil of parallel planes, and which is called the *central axis* of the system.

In other words: Any line that is parallel to the central axis will have its conjugates at infinity. Any plane that is parallel to the central axis will have its pole at infinity.

(*) "Ueber eine besondere Art dualer Verhältnisse zwischen Figure im Raume," in Crelle's Journal **10** (1833), 371, Berlin; also *Lehrbuch der Statik* (Leipzig, 1837), v. I, pp. 144. Cf., STAUDT, *Geometrie der Lage* (Nürnberg, 1847), pp. 191. – BRIOSCHI, *Statica dei sistemi di forma invariabile* (Milan, 1859), pp. 38. – See the note at the end of this work.

3.

Two arbitrary conjugate lines enjoy the following property: Their minimal distance is a line that cuts the central axis orthogonally. As a result:

If one projects the two reciprocal lines parallel to the central axis and over a plane that is perpendicular to that axis then the two projections will be parallel lines.

4.

One easily recognizes that:

1. Several lines $\mathfrak{r}$ in space whose projections coincide in just one line correspond to the line $\mathfrak{r}'$ whose projections are coincident or parallel, according to whether the $\mathfrak{r}$ (which are necessarily contained in a plane that is parallel to the central axis) are or are not parallel, respectively.

2. Several lines $\mathfrak{r}$ in space whose projections are parallel correspond to lines $\mathfrak{r}'$ whose projections are coincident or parallel, according to whether the $\mathfrak{r}$ (which are necessarily parallel to the plane of the central axis) are or are not parallel, respectively.

5.

Suppose that the central axis is horizontal, and call a plane of projection *orthographic* when it encounters the central axis at its pole. If one puts the origin of the axes of the rectangular coordinate system x, y, z at that point, the last of which will coincide with the central axis then the foregoing law of reciprocal correspondence will be expressed in the following formulas: The point x_1, y_1, z_1 is the pole of the plane:

$$xy_1 - yx_1 + k\,(z - z_1) = 0,$$

where k is a constant. Conversely, the plane:

$$ax + by + cz + d = 0$$

will correspond to the pole:

$$x = -\frac{kb}{c}, \quad y = \frac{ka}{c}, \quad z = -\frac{d}{c},$$

and the line:

$$\begin{aligned} ax + by + c &= 0, \\ px + qy + rz &= 0 \end{aligned}$$

will be conjugate to the other line:

$$\begin{aligned} ax + by + c' &= 0, \\ px + qy + r'z &= 0, \end{aligned}$$

where

$$rc' = r'c = k\,(aq - bp).$$

6.

Call two polyhedra *reciprocal* when the vertices of the first one are the poles of the faces of the second one, just as the faces of the first one will be portions of the polar planes of the vertices of the second one, and the edges of one will be the conjugates of the edges of the other.

Since any pole must lie in its respective polar plane, each of the two polyhedra will be simultaneously inscribed and circumscribed by the other one. For example, let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ be the vertices of a tetrahedron and let α, β, γ, δ be the faces of the reciprocal tetrahedron. The planes α, β, γ, δ pass through the poles $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, respectively, and the vertices $\beta\gamma\delta$, $\gamma\alpha\delta$, $\alpha\beta\delta$, $\alpha\beta\gamma$ lie in the polar planes $\mathfrak{BCD}$, $\mathfrak{CAD}$, $\mathfrak{ABD}$, $\mathfrak{ABC}$, respectively.

Now, project the two polyhedra onto the orthographic plane; the projections will be two figures that are endowed with the reciprocal property. Any side of the first figure will correspond to a parallel side of the second one, since two corresponding sides will be the projections of the conjugate edges of the two polyhedra. If a polyhedron has a solid angle in which m edges are concurrent at its vertex then the other one will have a polygonal face with m sides. Meanwhile, consider the two orthographic figures such that there are m edges on one of them that diverge from a point or *node*, so the m corresponding edges of the other one will be the edges of a closed polygon.

Any edge of a polyhedron is common to two faces and links two vertices. Any face has at least three edges. Therefore, in each of the two orthographic figures, any edge will be common to two polygons and conjugate to two nodes, while at least three edges will be concurrent at any node, just as any polygon will have three or more edges.

Suppose that one of the polyhedra, and therefore, the other one, as well, is *simply-connected*, so the sum of the numbers of vertices and faces is greater by 2 than the number of edges, according to the EULER's celebrated theorem. Therefore, if the first orthogonal figure has p nodes, p' polygons, and s edges then one will have:

$$p + p' = s + 2.$$

The second one will have p' nodes, p polygons, and s lines.

7.

If the one polyhedron has a vertex at infinity then the other one will have a face that is perpendicular to the orthographic plane, and conversely. Namely, if one of the two orthographic figures has a vertex at infinity then the other one will contain a polygon whose edges all fall along the same line, and conversely.

If the point $\mathfrak{I}$ at infinity on the central axes is the common vertex of n faces of the first polyhedron then the other polyhedron will have a polygonal face with n edges in the plane at infinity. In that case, the first orthographic figure will have $p - 1$ nodes, $p' - n$ polygons, and $s - n$ lines, and the second one (if one ignores the line at infinity) will have $p - 1$ polygons, $p' - n$ nodes, and $s - n$ lines, where the numbers p, p', s are again intended to obey the relation $p + p' = s + 2$.

8.

These *reciprocal diagrams* that are obtained as the orthographic projections of two reciprocal polyhedra are encountered by a direct path in *graphical statics.* The mechanical property of reciprocal diagrams is expressed in the following theorem, which is due to the illustrious prof. CLERK MAXWELL (*):

> "If forces represented in magnitude by the lines of a figure be made to act between the extremities of the corresponding lines of the reciprocal figure, then the points of the reciprocal figure will all be in equilibrium under the action of these forces."

The truth of the theorem is made immediately obvious when one observes that the applied force at an arbitrary node of the second diagram is parallel and proportional to the edges of a closed polygon in the first diagram. The theorem is primarily useful when one applies it to the graphic determination of the *internal forces in reticulated trusses.*

9.

The first germs of this theory can be found in the property of the *force polygon,* in which edges represent a system of forces in magnitudes and directions that are applied to a point and equilibrated, and that well-known geometric construction that given the tensions in the edges of a planar funicular polygon. However, the one who fist applied it to the reticulated truss was prof. MACQUORN RANKINE, who, in article 150 of his excellent *Manual of applied Mechanics* (1857), proved the theorem that:

> "If lines radiating from a point be drawn parallel to the lines of resistance of the bars of a polygonal frame then the sides of any polygon whose angles lie in those radiating lines will represent a system of forces, which, being applied to the joints of the frame, will balance each other; each such force being applied to the joint between the bars whose lines of resistance are parallel to the pair of radiating lines that enclose the side of the polygon of forces, representing the force in question.

(*) Philosophical Magazine, April 1864, pp. 258.

> Also, the lengths of the radiating lines will represent the stresses along the bars to whose lines of resistance they are respectively parallel (*)."

Much later (**), the very same RANKINE published an analogous theorem for *polyhedral frames.*

10.

However, the geometric theory of reciprocal diagrams is properly due to prof. CLERK MAXWELL, who first gave their general definition and deduced the projection of two reciprocal polyhedra from them in 1864 (***) and then once more on 1870 (****). However, the polyhedra of that author are reciprocal in the sense of PONCELET's *theory of reciprocal polar figures, relative to a certain paraboloid of rotation,* in such a way that the projections (orthogonal and parallel to the axis) of the corresponding edges are not parallel, but perpendicular to each other. Therefore, one of the diagrams must be rotated through 90° in its plane in order to assume the position it is required in the static problem. By contrast, with the method that I propose, the orthographic projections of two reciprocal polyhedra are given without any other diagrams that are obtained from graphical statics.

11.

The practical application of the method of reciprocal figures is the subject of a memoir that prof. FLEEMING JENKIN communicated in March 1869 to the Royal Society of Edinburgh (†), in which that author, after having cited the definition of reciprocal figures and the static property that MAXWELL announced in his paper of 1864, added that:

> "Few engineers would, however, suspect that the two paragraphs quoted put at their disposal a remarkably simple and accurate method of calculating the stresses in frameworks, and the author's intention was drawn to the method chiefly by the circumstance that it was independently discovered by a practical draughtsman, Mr. TAYLOR, working in the office of the well-known contractor, Mr. J. B. COCHRANE."

The author presented a good number of examples that were illustrated with figures, and concluded with the observation that:

(*) Page 160 of the sixth edition (1872). Of course, one must understand that the *truss* or *frame* that the author speaks of is a simple polygon.

(**) Philosophical Magazine, February 1864, pp. 92.

(***) "On reciprocal figures and diagrams of forces" (Phil. Magazine, April 1864, pp. 250).

(****) "On reciprocal figures, frames, and diagrams of forces" (Transaction of the Royal Society of Edinburgh, vol. XXVI). See also a letter of M. RANKINE in the journal The Engineer, 16 February 1872.

(†) "On the practical application of reciprocal figures to the calculation of strains on frameworks" (Transaction of the Royal Society of Edinburgh, v. XXV). For the same author, see also "On braced arches and suspension bridges," read before the Royal Scottish Society of Arts (Edinburgh, 1870).

> "When compared with the algebraic methods, the simplicity and rapidity of execution of the graphic method is very striking, and algebraic methods applied to frames, such the Warren girders, in which there are numerous similar pieces, are found to result in frequent clerical errors, owing to the cumbrous notation which is necessary, and especially owing to the necessary distinction between odd and even diagonals."

12.

However, when one speaks of geometric solutions to the problems of the science of construction, one can certainly not fail to mention prof. CULMANN, the ingenious and distinguished creator of graphical statics (*), to whom is due the expeditious and elegant methods of that science, which come out of the Zurich school, and are now taught in several German polytechnic schools [and for the last five years at the Superior Technical Institute in Milan (**)]. All of the questions of theoretical statics that would pertain to the individual branches of practical applications are answered by prof. CULMANN with uniform and simple procedures that, in substance, reduce to the construction of two figures that he called the *Kräftepolygon* and the *Seilpolygon*. Although he did not consider reciprocal figures, in the sense of MAXWELL's theory, those polygons were essentially such things. In particular, almost all of the geometric constructions that CULMANN gave in the fifth section of his book, which was dedicated to reticulated systems (*das Fachwerk*), coincide with what one would deduce by MAXWELL's methods. As a matter of fact, CULMANN's constructions also subsume the cases (which have not escaped the English geometers) in which reciprocal diagrams are not possible.

13.

First of all, we would like to show that CULMANN's *Kräftepolygon* and the *Seilpolygon* (i.e., the force polygon and the funicular polygon) can be reduced to reciprocal diagrams.

If one is given n forces $P_1, P_2, \ldots, P_n$ in a plane (which is always assumed to be the orthographic plane) that are in equilibrium then the term *force polygon* is intended to

(*) *Die graphische Statik*, Zürich, 1866. Since this distinguished work is not exempt from grave difficulties, especially for those who are not familiar with projective geometry, it would be wise for one to assist the propagation of this precious doctrine by means of more elementary treatises. See: K. VON OTT: *Die Grundzüge des graphischen Rechnens und der graphischen Statik*, Prague, 1871; – J. BAUSCHINGER, *Elemente der graphischen Statik*, Munich, 1871; – and the second section of the book by F. REULAUX, *Der Constructur* (3rd ed.), Braunschweig, 1869; there are several other less extensive memoirs that consider special topics by MOHR, HARLACHER, W. RITTER, E. WINKLER, etc. In England, in addition to the authors that were cited above, some other mathematicians have directed their attention to graphical statics, which emerged in the Proceedings of the London Mathematical Society, vol. III, pp. 233, 320-322. See also: C. UNWIN: *Wrought iron bridges and roofs*, London, 1869.

(**) As well as in the engineering schools at Palermo and Padua.

mean a polygon whose edges 1, 2, …, n are *equipollent* (*) to the lines that represent the forces, taken in a given or fixed, arbitrary order of succession (**). Take a point O (in the same plane), which one calls the *pole* of the polygon that is being drawn, project from it to the vertices of that polygon, and let $(r\ s)$ be the ray that projects to the common vertex to the edges r, s. As for the *funicular polygon*, which corresponds to the pole O, that term is intended to mean another polygon whose vertices fall along the lines of action (which are enumerated by **1**, **2**, …, ***n***) of the forces P_1, P_2, …, P_n , and whose edges are parallel to the rays through O (***), respectively, such that the edge that is included between the lines of action of P_r, P_s will be parallel to the ray $O(r\ s)$, and that will be indicated by the symbol (***r s***).

The resulting funicular polygon is closed, as is the force polygon.

14.

If the lines of action of the given forces are concurrent at the same point (Fig. 1) then one would already have two reciprocal diagrams that will obviously be the orthographic projections of two pyramidal n-hedra.

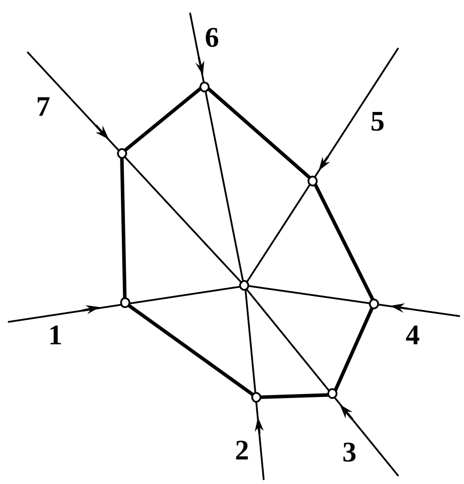

Figure 1.a.

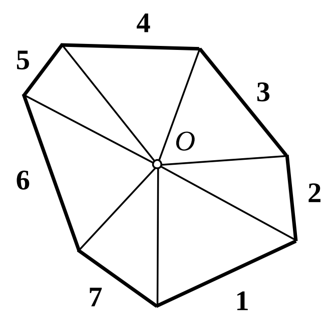

Figure 1.b.

If the forces are parallel then the force polygon will reduce to a line; that is, the base of the first pyramid will be perpendicular to the orthographic plane, and the vertex of the second one will fall at infinity. That is to say, the second polyhedron will be a prism that has just one base at a finite distance. This case is illustrated in Fig. 2, where the edges of the force polygon are indicated, not by just numbers 1, 2, 3, …, but by two numbers placed at the ends of each segment; therefore, the segments 01, 12, 23, 34, … will correspond to the lines **1**, **2**, **3**, **4**, …, resp. in the second diagram.

(*) Equal in magnitude, direction, and sense; this terminology was used by prof. BELLAVITUS.
(**) The position of the first vertex is arbitrary in the construction of this polygon.
(***) Only the direction of the first edge is given in the construction of this polygon.

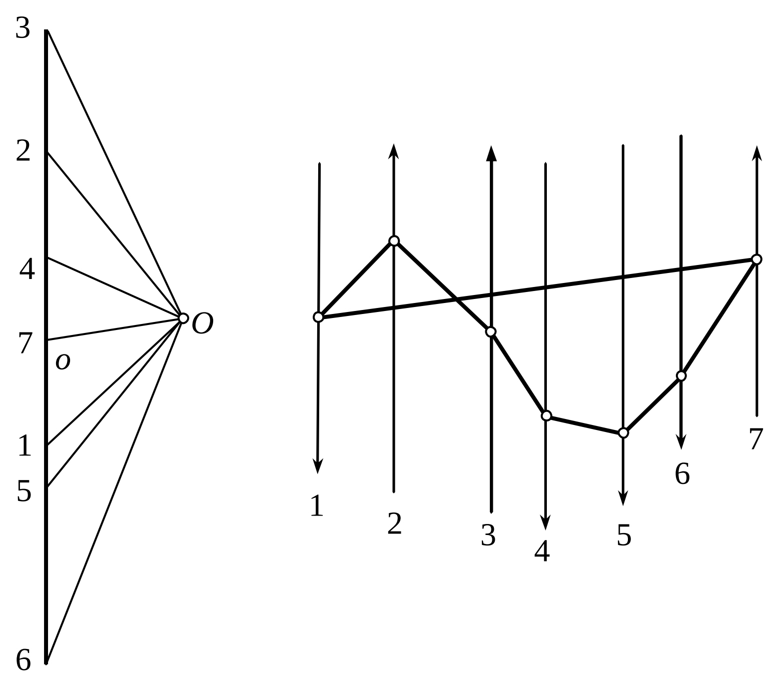

Figure 2.a. Figure 2.b

Here, and in what follows, we shall adopt two series of numbers 1, 2, 3, …, *r*, …, *s*, …, **1**, **2**, **3**, …, ***r***, …, ***s***, … *in the text* in order to distinguish the lines of one diagram from the corresponding lines of the other one.

15.

Now, consider the general case, in which the forces are not concurrent at the same point. Assume that *O′* is a second pole that is joined to the vertices of the force polygon by means of lines, and construct a second funicular polygon that corresponds to the pole *O′*, namely, a polygon whose vertices fall along the lines of action of the forces, and whose edges are parallel to the rays that diverge from *O′*, resp. See Figs. 3 and 5, in which the rays that emanate from the second pole are drawn with dashed lines, as well as in the corresponding funicular polygon.

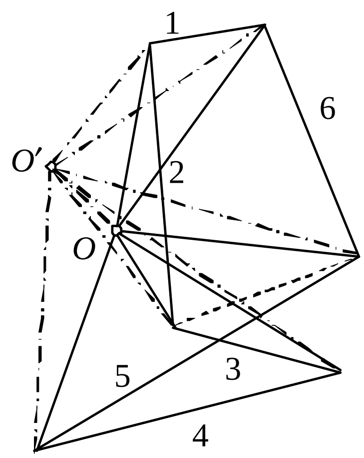

Figure 3.a

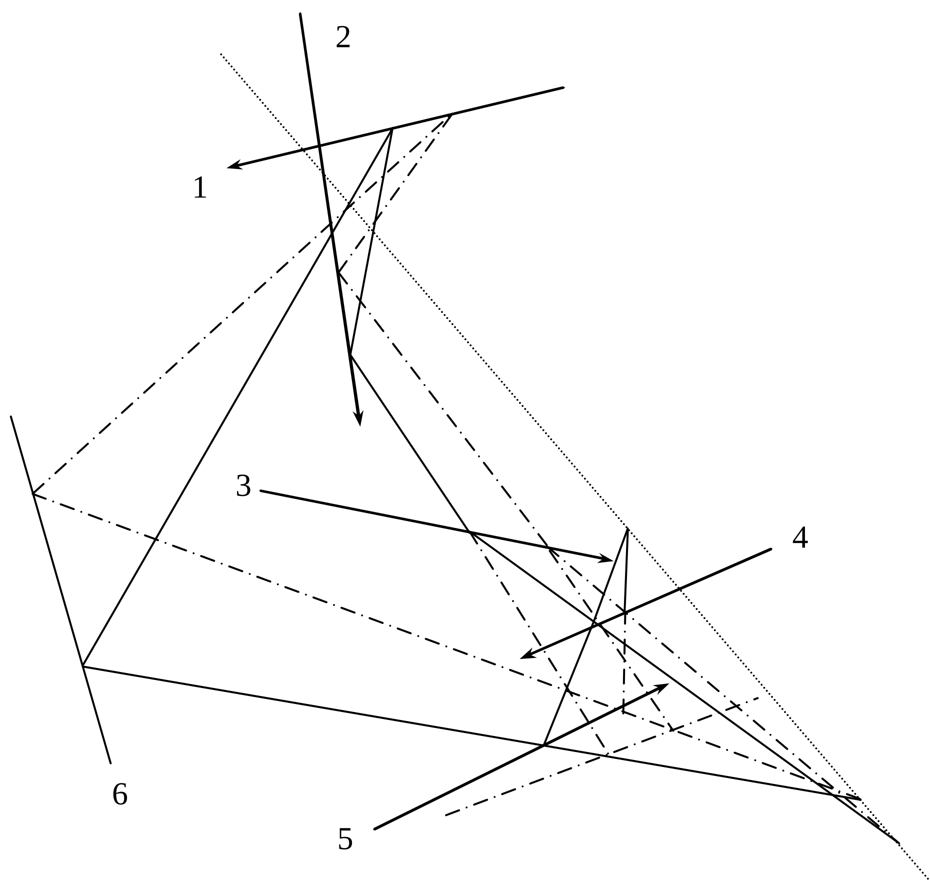

Figure 3.b

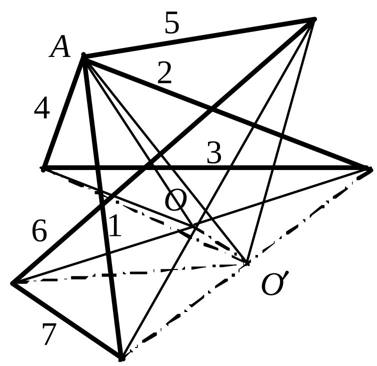

Figure 4.a

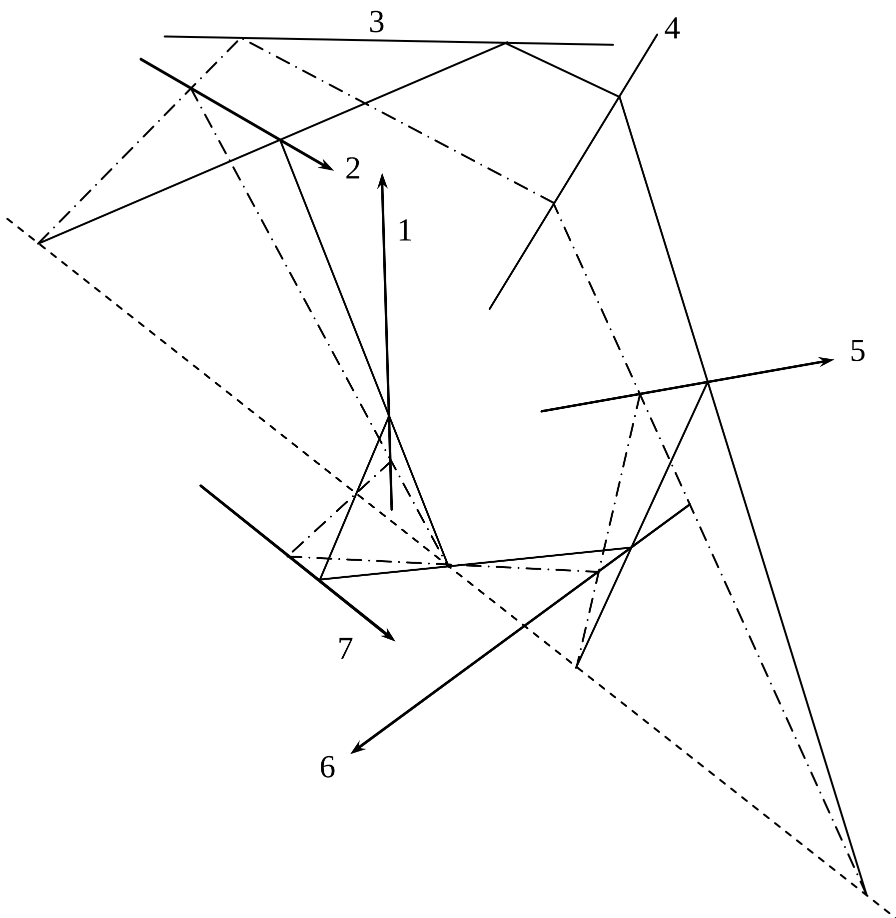

Figure 4.b

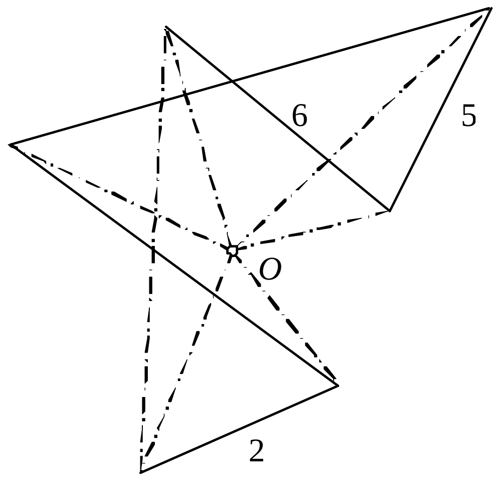

Figure 5.a

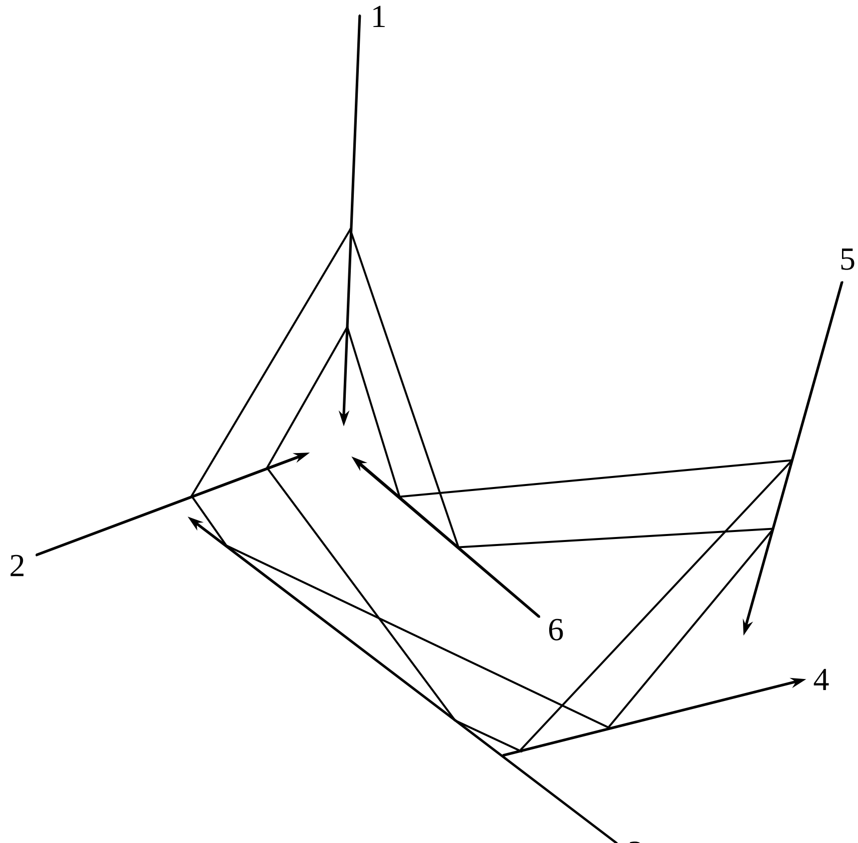

Figure 5.b

In that way, the diagram that is comprised of the force polygon and the rays that project from O and O', and the diagram that is comprised of the two funicular polygons and the lines of action of the forces are obviously reciprocal. The first one is the projection of a polyhedron (*) that is formed from two solid n-hedral angles whose corresponding faces intersect along the contour of a non-convex (*gobbo*) polygon with n sides; the second one is the projection of a polyhedron that is comprised of two planar polygons with n sides, such that the sides of one meet the corresponding sides of the other one in sequence.

The line in space that joins the vertices of the two n-hedral angles of the first polyhedron is the conjugate of the one that is common to the planes of the two bases of the second polyhedron. Anticipating the property that two conjugate lines have to project orthographically into two parallel lines, it then follows that two arbitrary corresponding edges $(\boldsymbol{r}\ \boldsymbol{s})$, $(\boldsymbol{r}\ \boldsymbol{s})'$ of two funicular polygons will intersect above a fixed line that is parallel to the line that joins to the two poles O, O'. This theorem is fundamental to CULMANN's methods.

16.

If the poles O, O' coincide then the corresponding edges of the two funicular polygons will be parallel (Fig. 4). In that case, the line that joins the vertices of the n-hedral angles in the first polyhedron is perpendicular to the orthographic plane; that shows that the bases of the second polyhedron are parallel.

17.

The diagonal between the vertices of the two tetrahedral angles of the first polyhedron (no. **15**) – namely, the diagonal between two vertices of the convex polygon – is conjugate to the intersection of the corresponding quadrilateral faces of the second polyhedron, which joins the points that are common to two edges of a base with common point to the corresponding edges of the other base. In the orthographic projection of the first line, one is given a diagonal between two vertices $(r \cdot r + 1)$, $(s \cdot s + 1)$ of the force polygon, namely, a line that is equipollent to the resultant of the forces P_{r+1}, P_{r+2}, …, P_s ; the second line will give the line of action of that resultant. Therefore, the line of action of the resultant of an arbitrary number of consecutive forces P_{r+1}, P_{r+2}, …, P_s will pass through the point that is common to the edges $(\boldsymbol{r} \cdot \boldsymbol{r} + \mathbf{1})$, $(\boldsymbol{s} \cdot \boldsymbol{s} + \mathbf{1})$ of the funicular polygon; this is another fundamental theorem of graphical statics. See, for example, the resultant of the forces 6, 1, 2 in Fig. 3.

(*) This polyhedron has $3n$ edges, $2n$ triangular faces, two n-hedral angles, and n tetrahedral angles. The other polyhedron has $3n$ edges, $2n$ trihedral angles, 2 bases that are polygons with n sides, and n quadrilateral faces.

18.

If the aforementioned diagonal of the first polyhedron is perpendicular to the orthographic plane then its conjugate line will be at infinity. Therefore, the two vertices $(r \cdot r + 1)$, $(s \cdot s + 1)$ of the force polygon will coincide in just one point A (see Fig. 5, in which $r = 1$, $s = 4$), and the edges $(\boldsymbol{r} \cdot \boldsymbol{r} + \mathbf{1})$, $(\boldsymbol{s} \cdot \boldsymbol{s} + \mathbf{1})$ in the funicular polygon will be parallel. The resultant of the forces P_{r+1}, P_{r+2}, …, P_s will then have a vanishing magnitude, and its line of action will fall along the line at infinity of the orthographic plane. This resultant will be a force that is infinitely small and far away that is equivalent to a couple of forces that act along the predicted parallel edges of the funicular polygon amd is represented in magnitude by the line that joins the corresponding pole O to the point A. Since these two forces are equivalent to the system of P_{r+1}, P_{r+2}, …, P_s , the sense of the one that acts along the edge $(\boldsymbol{r} \cdot \boldsymbol{r} + \mathbf{1})$ is from A to O, and the sense of the other one that acts along the edge $(\boldsymbol{s} \cdot \boldsymbol{s} + \mathbf{1})$ is from O to A.

19.

Given the forces P_1, P_2, P_3, …, P_{n-1} (no. **13**), the two polygons (viz., force and funicular) serve to determine P_n, which is equal and opposite to the resultant of the given ones (see Fig. 3, in which $n = 5$). In fact, if one constructs the broken line $1 \cdot 2 \cdot 3 \ldots n - 1$, whose edges are equipollent to the given forces then the line n that joins its extremities (directed from the end to the origin of the break) will be equipollent to P_n . Then, assuming a pole O, construct a (funicular) polygon whose first $n - 1$ vertices $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$, …, $\boldsymbol{n} - \mathbf{1}$ fall along the lines of action of the give forces P_1, P_2, P_3, …, P_{n-1}, resp., and whose edges $(\boldsymbol{n} \cdot \mathbf{1})$, $(\mathbf{1} \cdot \mathbf{2})$, $(\mathbf{2} \cdot \mathbf{3})$, …, $(\boldsymbol{n} - \mathbf{1} \cdot \boldsymbol{n})$ are parallel to the rays that project from O to the vertices of the first polygon that have similar names. The line that passes through the last vertex $\boldsymbol{n}$ of the funicular polygon – that is to say, through the point of concurrence of the first edge $(\boldsymbol{n} \cdot \mathbf{1})$ with the last one $(\boldsymbol{n} - \mathbf{1} \cdot \boldsymbol{n})$ and is parallel to the last edge $\boldsymbol{n}$ of the force polygon – will the be the line of action of P_n .

Suppose that the first edge of the funicular polygon must pass through a fixed point, and that one moves the pole along a straight line, and that the other edges rotate around just as many fixed points that are aligned with the first one in a line that is parallel to the locus of the pole (no. **15**). This recalls the celebrated aphorism (*porisma*) of PAPPO (*):

> "Si quoteumque rectæ lineæ sese mutuo secent, non plures quàm duæ per idem punctum, omnia qutem in una ipsarum data sint, et reliquorum mulitudenem habentium triangulum numerum, hujus latus singular habet puncta tangential rectam lineam positione datam, quorum trium non ad angulum existens trianguli spacii unumquodque reliquum punctum rectam lineam positione datum tanget."

(*) *Mathematicæ Collectiones*, preface to book VII, pp. 162 in the COMMANDINO edition (Venice, 1589). See the translation or paraphrase of the aphorism that PONCELET made in no. 498 of his *Traité des propriétés projectives* (Paris, 1822).

20.

If one considers the point O as being capable of taking any position in the plane then the property of the polygons (viz., force and funicular) can be summarized in the following geometric statement:

> Let a planar polygon be given that has n edges 1, 2, 3, …, n – 1, n, and in addition, let n – 1 lines **1**, **2**, **3**, …, $\boldsymbol{n}$ **– 1** be given in that plane that are parallel to the first n – 1 edges of the polygon, resp. We wish to project the vertices of the given polygon from a point or pole that moves in the plane (with no restrictions). Now, imagine a variable polygon with n edges, the first n – 1 of which, namely, **1**, **2**, **3**, …, $\boldsymbol{n}$ **– 1**, must lie in succession along the given lines with the similar names, while the n edges ($\boldsymbol{n} \cdot \mathbf{1}$), ($\mathbf{1} \cdot \mathbf{2}$), ($\mathbf{2} \cdot \mathbf{3}$), …, ($\boldsymbol{n} - \mathbf{1} \cdot \boldsymbol{n}$) must be parallel to the rays that project from the vertices of the given polygon that have similar names. The point of concurrence of two arbitrary edges ($\boldsymbol{r} \cdot \boldsymbol{r} + \mathbf{1}$), ($\boldsymbol{s} \cdot \boldsymbol{s} + \mathbf{1}$) of the variable polygon will fall along a well-defined line that is parallel to the diagonal between the vertices ($r \cdot r + 1$), ($s \cdot s + 1$) of the given polygon.

This theorem, which is proved by means of just *plane geometry* and does not seem to be obvious, will nonetheless result from a straightforward line of reasoning if one considers the plane figures to be the orthographic projections of reciprocal polyhedra.

21.

The force polygon is the projection of a planar polygon or a skew polygon according to whether the force P is or is not concurrent at a point, resp. As we saw in no. **14**, in the former case the two reciprocal diagrams are composed simply, in the one case, of the force polygon and the pole O, and in the other case, of the lines of action of the forces and the funicular polygon that corresponds to O. By contrast, in the latter case (no. **15**) it is necessary to add another pole O' to the first diagram and another funicular polygon that corresponds to O' to the second one. As we have seen (no. **16**), the two poles can be assumed to coincide, and then the first diagram will become as simple as possible. However, if one desires to simplify the second one then one must put the pole O' at infinity in an arbitrary direction. The polyhedron whose orthographic projection is the first diagram will have the vertex of one of its n-hedral angles at infinity, and since the polar plane to a point at infinity is parallel to the central axis, the new funicular polygon that corresponds to O' will have all of its extended edges along the same straight line (whose point at infinity will be O'). The absolute position of this line in the orthographic plane will be, once more, arbitrary, although one must place it at infinity.

One gets an even simpler result in the following way: Suppose that the first polyhedron has the vertex of the aforementioned n-hedral angle at the point at infinity along the central axis. In the first diagram, the pole O' will then disappear absolutely, since the edges of the n-hedral angle will project orthographically onto the vertices of the forces polygon. The polar plane of the vertex will then be the plane at infinity; therefore the second funicular polygon will lie at infinity entirely (no. **7**).

22.

We conclude the even simpler fact from that, by which, one can regard the force polygon and the funicular polygon of a system of n equilibrating forces as reciprocal diagrams when the forces are situated in a plane (viz., the orthographic plane), but not concurrent at one of its points. The one diagram is comprised of the force polygon and the rays the project from a pole O to the vertices. The other is comprised of the lines of action of the forces, the funicular polygon, and the line at infinity. The first diagram is the projection of a polyhedron whose faces are obtained by projecting the n sides of a non-convex polygon that is perpendicular to the orthographic plane from a finite point in space (?). The reciprocal polyhedron whose projection is the second diagram is the (infinite) portion of the space that is contained in a planar polygon and n planes that successively pass through the sides of it and are extended to infinity.

23.

We now go on to more complicated diagrams, as they might present themselves in the theory of reticulated trusses. Let $\mathfrak{S}$, $\mathfrak{S}'$ be two reciprocal polyhedral surfaces that are simply-connected and endowed with a boundary (*). Let $\mathfrak{P}$ be the polyhedron that is composed of $\mathfrak{S}$ and the pyramidal surface whose vertex is a point $\mathfrak{O}$ that is fixed, but arbitrary, in space, and whose directrix is the polygonal contour of $\mathfrak{S}$. Let $\mathfrak{P}'$ be the reciprocal polyhedron to $\mathfrak{P}$, namely, the polyhedron that contains the faces of $\mathfrak{S}'$, the planes of the angles of the boundary of $\mathfrak{S}'$, and the plane ω that is polar to $\mathfrak{O}$. If one projects the two polyhedra orthographically then one will get two reciprocal diagrams that one would like to take into consideration.

Let the boundary of $\mathfrak{S}$ have n sides, and let that surface have m other edges (**) and p faces. The polyhedron $\mathfrak{P}$ will have $n + p$ faces and $2n + m$ edges, and therefore $m + n - p + 2$ vertices. Therefore $\mathfrak{S}$ will have $m - p + 1$ vertices outside of the boundary (***).

Reciprocally, $\mathfrak{P}'$ will have $m + n - p + 2$ faces, $n + p$ vertices, and $2n + m$ edges.

24.

Now, suppose that the projection of $\mathfrak{S}'$ is the drawing of a given reticulated truss with *p nodes* and *m members* – or rectilinear pieces – for which the external forces have lines

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(*) If the boundary of $\mathfrak{S}$ is a planar polygon with n sides then that of $\mathfrak{S}'$ will be point that is the vertex of an n-hedral angle.

(**) Obviously, $m \geq n$.

(***) So m cannot be less than $p - 1$.

of action that are the projections of the edges of $\mathfrak{S}'$, and are represented in magnitude by the n sides of the polygon that is the projection of the boundary of $\mathfrak{S}$ (*). The projection of the face of $\mathfrak{S}'$ that is in the plane ω will then be the funicular polygon of external forces that corresponds to the pole O that is the projection of $\mathfrak{O}$, and the projections of the m edges of $\mathfrak{S}$ that do not belong to the boundary will give the measure of the internal stresses that the corresponding members of the truss are subjected to as a consequence of the given system of external forces.

25.

If the point $\mathfrak{O}$ is moved to infinity in the direction that is perpendicular to the orthographic plane then the plane ω will coincide with the plane at infinity. The first diagram will then reduce to the projection of $\mathfrak{S}$, namely, to the totality of lines that measure the external forces and the internal stresses. When the funicular polygon disappears completely, the second diagram will contain only the drawing of the truss (namely, the lines of action of the internal forces) and the lines of action of the external forces. In the figures that are at the basis of this paper, the former diagram will indicated by the letter b and the latter one, by the letter a.

26.

If the external forces are all mutually parallel, as often happens in practical applications, then the boundary of $\mathfrak{S}$ will be a polygon that is contained completely in a plane that is perpendicular to the orthographic plane. Thus, the sides of the external force polygon will all fall along one and the same line.

27.

Other degenerate polygonal forms can offer diagrams that produce analogous degenerations in the spatial figure.

Suppose, e.g., that one has a solid tetrahedral angle in space that corresponds to the reciprocal figure to a quadrilateral face, and that two (non-opposite) edges of the solid angle approach each other indefinitely closely in their plane until the one overlaps the other. The final solid angle will then be reduced to a system that is composed of a trihedron and a plane that passes through one of its edges. Therefore, any of the quadrilateral faces of the reciprocal figure will have two edges that, will be found to have the same or opposite directions without ceasing to have a common vertex.

(*) Therefore, $\mathfrak{S}$ must not have any vertex at infinity; i.e., $\mathfrak{S}'$ must not have any face that is perpendicular to the orthographic plane.

If we now pass on to the orthographic projection then we will have a point in one diagram from which four lines emanate, two of which have overlapping directions, and in the other diagram we will have a quadrilateral with three vertices that lie along a straight line (*).

28.

Suppose that the drawing of a reticulated truss is given, and suppose that the system of external forces (**) is known, so one must, above all, construct the external force polygon, i.e., a polygon whose sides are equipollent to the external forces. In the accompanying figures, the external forces and the sides of the polygon are indicated by 1, 2, 3, …, in such a way that as one traverses the contour of the polygon in order of increasing numbers, the sides will be traversed in the sense of the forces that they represent. This way of traversing the contour of the polygon is called the *cyclic ordering of that contour.*

When one treats the construction of the reciprocal diagram and the one that is composed of the drawing of the truss and its system of external forces, the order in which the forces follow each other in the construction of the relevant polygon is not arbitrary. The order in which they are treated is determined from the following argument: In the external force polygon, which is part of diagram b, the successive sides must be equipollent to two forces, since their lines of action belong to the contour of the same polygon in diagram a. That polygon will then correspond to the vertex that is common to those two sides.

One is therefore given the index 1 and any of the external forces; the line of action of the chosen force is the common side to two polygons of the diagram a. Each of them has the line of action of another external force in its contour. It will therefore have two external forces that can be regarded as *contiguous* to force 1, and will be indifferent to the attributes of one or the other index 2. Naturally, the other will then have the index *n,* if *n* is the number of external forces. After doing that, nothing will be arbitrary or uncertain in the order by which one has to arrange the sides of the external force polygon.

Suppose that the nodes to which the external forces are applied are all found on the contour of the drawing of the truss (***), so those forces can be taken in the order in which they are encountered as one traverses that contour. When one does not follow these rules, but the other ones that were exhibited before, one can still solve the problem of the graphic determination of the internal stresses, but one will not have *reciprocal*

(*) Examples of these degenerate forms can be found on page 444 and in the first of two tables in the cited memoir of prof. FLEEMING JENKIN (1869), and also in our own Figure 9.

(**) The external forces are:

1. The proper weight of the truss, the accidental, or transitory, weight, the action of wind, etc., and everything that one can regard as effectively given and distributed over the nodes of the truss.

2. The supports reactions.

For the determination of these, forces one can confer the method that was taught by CLERK MAXWELL in his memoir "On the calculation of the equilibrium and stiffness of frames" (Phil. Magazine, April 1864).

(***) The contour of the truss is composed of the two things in that *table* (*Gurten, Streckbäume*), the upper table and the lower table. The pieces that join the nodes of one to those of these other one are called *shafts* and *arrows.*

diagrams, but rather, figures that are more complicated or disconnected, in which the same segment that is not found at it *proper* place must repeated or brought back in order to give rise to the final constructions (*), as was the case in the old method of separately constructing a force polygon for each node of the truss.

29.

If one formats the external force polygon in this way then one will complete diagram b by constructing successively the polygons that correspond to the nodes of the truss. The problem of constructing a polygon whose sides must have given directions is determined whenever just two sides are unknown. Therefore, one must start from a node at which just three lines are concurrent: viz., the line of resistance of two members of the truss and the line of action of one external force. The equipollent segment to that force will be a side of the triangle that corresponds to that node; the triangle can then be constructed. No ambiguity will persist in this construction if one understands that a member of the truss at which intersect the lines of action of two external forces that belong to the contour of a polygon in diagram a will correspond to a lines in b that passes through the common vertex to the sides that are equipollent to two forces.

One then passes on to the other nodes successively in such a way that just two unknown sides will remain to be constructed for any new polygon.

In the accompanying figures, all of the lines in each diagram are denoted by numbers in order to indicate the order of operations.

> "The figure can be drawn in five minutes, whereas the algebraic computation of the stresses, though offering not mathematical difficulty, is singularly apt, from mere complexity of notation, to result in error (**)."

30.

A superficial consideration might suggest that the solution of the problem is possible and determinate, even in the case in which none of the nodes is a point of concurrence of just three lines. Suppose that, e.g., the drawing of the truss is comprised of the sides **5**, **6**, **7**, **8** of a quadrilateral and the lines **9**, **10**, **11**, **12** that join the vertices to a fifth point, and the external forces are **1**, **2**, **3**, **4**, which are applied to the vertices (**8** · **5** · **9**), (**5** · **6** · **10**), (**6** · **7** · **11**), (**7** · **8** · **12**) of the quadrilateral (***). Formatting the external force polygon 1 · 2 · 3 · 4, then the points (1 · 2), (2 · 3), (3 · 4), (4 · 1), will leave the order of 5, 6, 7, 8 indefinite. One then poses the problem of constructing a quadrilateral whose sides 9, 10,

(*) For that reason, Fig. 1 and 3 in Table 16 of the atlas of CULMANN's *Graphische Statik* are not reciprocal diagrams, nor are Fig. 7 and 7_1 of Table 19, etc. By contrast, diagrams 168 and 169 on page 422 of the text, etc., are perfectly reciprocal.

(**) Prof. FLEEMING JENKIN on page 443 of the cited volume of the Transactions of the Royal Society of Edinburgh.

(***) These considerations persist without alteration when the truss is, by contrast, comprised of an arbitrary polygon and the lines that join its vertices to a fixed point. The figures that relate to this paragraph are not given, but the reader can supply them with no difficulty.

11, 12 are parallel to the respective lines in the given diagram that have similar names, and whose vertices (9 · 10), (10 · 11), (11 · 12), (12 · 11) fall upon 5, 6, 7, 8, resp. The problem of constructing a quadrilateral whose sides have given directions (or pass through given points along the same line) and whose vertices fall upon just as many fixed lines, will then admit one and only one solution, *in general*, so one can believe the first notion that the force diagram becomes perfectly determinate.

However, the illusion will evaporate when one considers that the geometric problem has some impossible and indeterminate cases. In fact, when one omits one of the proposed conditions – i.e., one requires only that the quadrilateral have all of its sides in given directions and its first three vertices on the given lines 5, 6, 7 – the fourth vertex will describe a line *r* (*), and the common point to it and the given line 8 must give the desired solution when it is taken to be the fourth vertex. Now, if the givens in question are such that *r* becomes parallel to 8 then one will be dealing with one of the impossible cases. However, if the line *r* coincides with 8 precisely under an even more specialized hypothesis then the problem will become indeterminate – namely, an infinitude of quadrilaterals will satisfy the proposed conditions.

In order to persuade oneself that in the construction of the reciprocal diagram to a given one, one will come to precisely its impossibility or its indeterminacy, it is enough to reflect upon the face that if you consider the given diagram to be a force polygon whose magnitudes are expressed by the segments **5**, **6**, **7**, **8**, and whose pole is the point (**9** · **10** · **11** · **12**), then the figure that one seeks, 9 · 10 · 11 · 12, will be the corresponding funicular polygon. Now, in order for the construction of the funicular polygon to be possible, it is necessary that the forces be in equilibrium. For that purpose, if one supposes that the their magnitudes **5**, **6**, **7**, **8** are given, along with the lines of action of three of them 5, 6, 7, then the line of action of the fourth one will be determined uniquely, and it will be precisely the line *r* that was mentioned above. Therefore, if *r* and 8 do not coincide then the fictitious forces **5**, **6**, **7**, **8** will not be in equilibrium, but will be equivalent to an infinitely small force at infinity, and the solution of the problem will be impossible. If *r* then coincides with 8 – namely, the fictitious forces are in equilibrium – then the problem will be indeterminate, since one can construct an infinitude of funicular polygons for a given system of forces and for a given pole (no. **16**).

In the first of these two cases, if equilibrium is attained then one must add an equal and opposite force to the (infinitely small and distant) resultant of the forces **5**, **6**, **7**, **8**; that is, one considers the polygon **5** · **6** · **7** · **8** to be the projection, not of a quadrangle, but a pentagon, two consecutive vertices of which will project onto the same point (**7** · **8** · **12**). The line **12** is then the projection of two distinct lines in space, so in the diagram that is reciprocal to the point (**9** · **10** · **11** · **12**), will correspond an (open) pentagon 9 · 10 11 · 12 · 12′ that has the vertices (9 · 10), (10 · 11), (11 · 12), (12′ · 9), which are situated along 5, 6, 7, 8, resp., and the vertex (12 · 12′) at infinity.

(*) PONCELET, *Traité des propriétés projectives* (Paris, 1822), no. 500.

31.

Any member or rectilinear piece of the truss is the line of action of two equal and opposite forces that are applied to the two nodes that are adjoined to that piece. The common magnitude of these two forces – namely, the measure of the stress that the piece considered is subject to – is given by the corresponding line of the diagram b. Those two forces can be considered to be *actions* or *reactions*. In order to pass from one case to the other, it is enough to invert the senses. When considered to be *actions,* if they act on the respective points of application towards the inside of the piece then one will call them *pressures or compressions;* by contrast, if they act towards the outside then they will be called *tensions or stretchings*. Often the compressed pieces are given the name of *struts*, while the pieces in a state of tensions are called *ties* (*).

32.

Each node of a truss is the point of application of a system of equilibrated forces that are at least three in number. One of them can be an external force, while all of the other ones will be *reactions* in the pieces that are concurrent at that node. It will be enough to know the sense of one of the forces of the system in order to deduce the sense of all the other ones. It will not be necessary for one to traverse the contour of the polygon that corresponds to that node. If an external force is applied to the node, and one traverses the equipollent side of the polygon in the same sense, then each of the other sides of the contour will be traversed in the sense that belongs to the corresponding internal force, when regarded as a reaction that is applied to the node that is considered. By contrast, if the internal forces tend to act in the sense that competes with them as *actions* then when one traverses the contour of the polygon, one must reverse the sense of the external force.

If no external force is applied to a node, but only internal forces, then it will also suffice to know the sense of one of them in order to get the sense each of the remaining ones by the process that we now discuss.

One refers to the *cyclic ordering of the contour* of a polygon in diagram b when it corresponds to the internal forces considered as *actions*.

In that way, if one starts from a node at which an external force is applied then one will successively determine the magnitudes and the sense of all internal forces. If one considers an internal force to be an *action* that is applied to one of the two nodes between which it acts then one will soon recognize whether the piece that is limited by those nodes is compressed or tensed.

Any line in diagram b is the common side of two polygons. If one traverse their contours in their respective cyclic orders then that side will be described, once in one sense, and the other time in the opposite sense (**). That line will then correspond to the

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(*) In the figures that are the basis for this article, the ties are indicated by lines that are fainter than the ones that represent the struts. In the figures of CULMANN and REULAUX, the struts are denoted by double lines, and the ties, by simple lines.

(**) This property is in accord with that of the *law of edges* (*Kantengesetz*) for polyhedra that are endowed with an internal surface and an external surface. See MÖBIUS, “Ueber die Bestimmung des Inhalts eines Polyeders” [in the Berichte über die Verhandlungen der Königl, Sächs. Gesellschaft d.

existence of a measuring line for two equal and opposite forces that act along the corresponding member of the truss.

33.

Note that the algebraic sum of the projection of the faces of a polyhedron is equal to zero. If one applies that theorem to the polyhedron 𝔓 (no. **23**) and observes that the projection of the surface 𝔖 is composed of polygons of the diagram b that correspond to the nodes of the truss, while the projection of the remaining surfaces of 𝔓 is nothing but the external force polygon, then one will get the following result:

If one regards the area of a polygon as positive or negative according to whether it lies to the left or the right when one traverses the contour in its cyclic order then the sum of the areas of the polygons of the diagram b *that correspond to the nodes of the truss will be equal and opposite to the area of the external force polygon.*

MAXWELL arrived at this theorem by a different path by proving it for an arbitrary planar *frame*, regardless whether it is or is not possible to construct a force diagram (*).

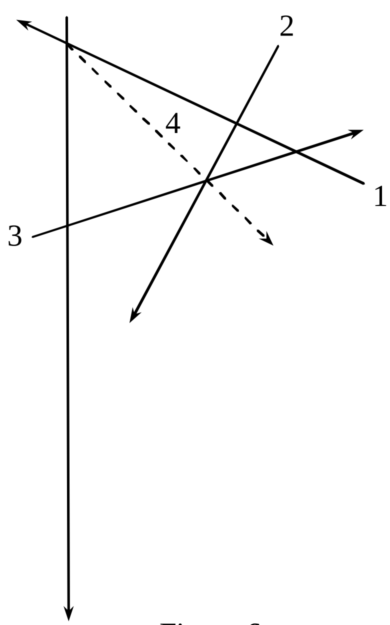

Figure 6.a

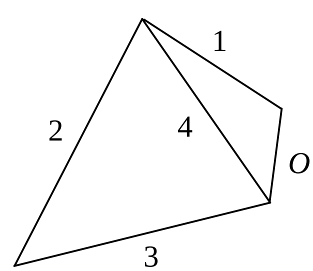

Figure 6.b

Wissenschaft, math.-phys. Klasse **17** (1865), pp. 33, *et seq.*] and BALTZER, *Stereometria* (pp. 143 of the Italian translation, Genoa, 1867).

(*) Cited paper of 1870, pp. 30.

34.

The method of sections that is generally adopted in the treatment of variable systems offers the designer a valuable means of verification.

If one considers one of the two parts into which the truss is divided to be an *ideal section* then the external force that is applied to that part will equilibrate the reactions of the pieces that meet at the section.

If just three of these reactions are unknown then it can be deduced from the equilibrium condition that the problem is determinate and admits just one solution, like the problem of decomposing a force *P* into three components whose lines of action **1**, **2**, **3** are given and form a complete quadrilateral with **0**, the line of action of *P*. In fact (Fig. 6), it suffices to choose one of the diagonals of the quadrilateral – e.g., the line **4**, which joins the points **01**, **23** – decompose the given force **0** into two components along the lines **1**, **4** (and then construct the force triangle 0 4 1, the side 0 of which is given in magnitude and direction), and to finally decompose the force **4** along the lines **2**, **3** (analogously construct the force triangle 4 3 2).

This method, which can be called *static*, suffices by itself for the graphic determination of the internal forces, just like the *geometric* method that was discussed previously, which derived from the theory of reciprocal figures and consisted in the successive construction of the polygons that correspond to the various nodes of the truss. The static method then seems to be less simple to me, and one might benefit from combining it with the other one, above all, for the verification of the validity of the graphic operations that were performed already. The external force that is applied to a portion of the truss that is separated by an arbitrary section and the reactions of the pieces that were cut must have the property that the corresponding lines in diagram b are the sides of a closed polygon. That polygon must be the projection of a non-convex polygon, which is likewise closed, and which is not yet broken, and whose extremities lie along a line that is perpendicular to the orthographic plane: That condition is equivalent to the reciprocal non-convex polygon also being closed – namely, to being able to connect the corresponding lines in diagram a with a closed funicular polygon.

The method of sections can also be presented in another form. Once more, let **0** denote the resultant of all forces (that are applied to the portion of the truss) that are known, and let **1**, **2**, **3** denote the three unknown reactions, so the sum of the moments of the four forces will be zero. It then follows that if one places the center of the moments at the point of intersection of two lines of resistance – e.g., at the point **23** – then the moment of the third reaction **1** will be equal and opposite to the moment of the force **0**. One will then have a proportion of four magnitudes (the two forces and their lever arms), between which, the only unknown is the magnitude of the force **1**. In this, one finds the *method of static moments*, which is adopted when one desires to *numerically calculate* (not construct graphically) the internal stresses in the reticulated truss (*).

(*) See A. RITTER, *Elementare Theorie und Berechnung eisener Dach- and Brücken-Constructionen*, 2nd ed. (Hannover, 1870).

35.

We now go on to give some examples that are designed to exhibit the simplicity and elegance of the graphical method. In these examples, I do not always adhere to regularity or symmetry in form, although in practice one does not give them up. *However, the symmetric form in practice is not that of a special case of the irregular forms of abstract geometry*. Therefore, the treatment of it will include all of the possible practical cases in it. Here, the term *reticulated truss* is taken in the general, theoretical sense that MAXWELL gave to the term *frame* (*).

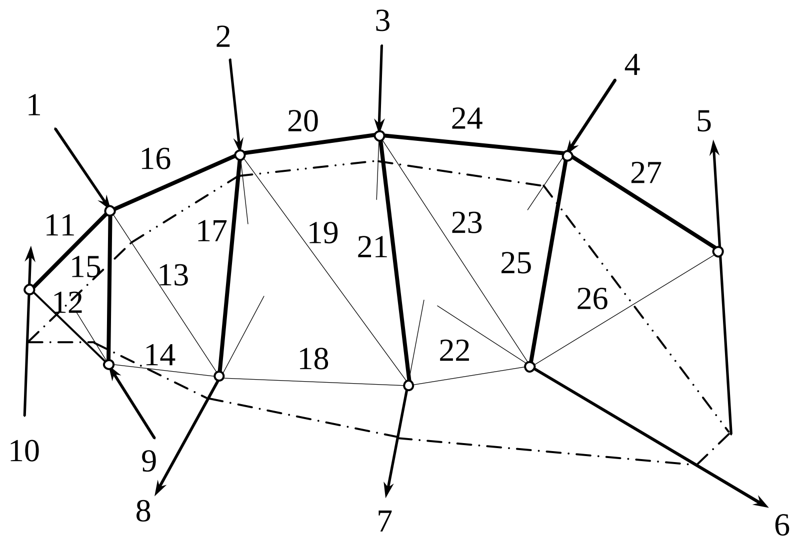

Figure 7.a

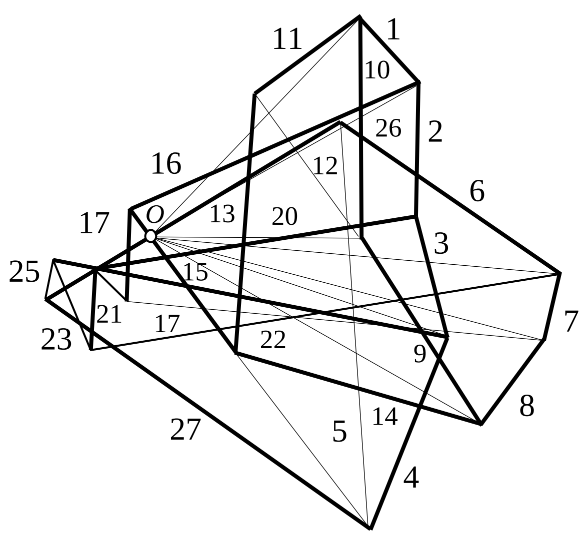

Figure 7.b

(*) Page 294 of the Philosophical Magazine, April 1864.

> "A frame is a system of lines connecting a number of points. A stiff frame is one in which the distance between any two points cannot be altered without altering the length of one or more of the connecting lines of the frame…A frame of *s* points in a plane requires in general 2*s* – 3 connecting lines to render it stiff."

I intend to limit myself to the consideration of only *planar* figures (*).

36.

As a first (theoretical, general) example, let **1**, **2**, …, **10** (Fig. 7.a) be a system of ten (external) forces in equilibrium, so that when one constructs the polygon of these forces and projects the vertices from a pole *O* (Fig. 7.b), in which the force polygon is denoted by double lines), one can trace a funicular polygon whose vertices are along the lines of action **1**, **2**, …, and whose sides (which are denoted by broken lines in Fig. a) are, in turn, parallel to the rays (which are denoted by broken lines in Fig. b) that emanate from *O*. The given forces are applied to the nodes of a reticulated truss whose rectilinear members are indicated by **11**, **12**, **13**, …, **27** (Fig. a).

Start by constructing the triangle that corresponds to the node (**10** · **11** · **12**), draw two lines 11, 12 that are parallel to **11**, **12**, resp., from it to the ends of 10. If necessary, observe that 11 must pass through the point (1 · 10), which is why in diagram a the lines **1**, **10**, **11** belong to the contour of the same polygon (**); for the same reason, 12 must pass through the point (9 · 10). If one traverses the contour of the triangle that is thus obtained in the sense that is contrary to that of the force **10** (no. **32**) then one will get the senses of the actions that are exerted on the nodes in question along the lines **11**, **12**, and one will see that the piece **11** is compressed, while **12** is tensed.

If one constructs the quadrilateral that corresponds to the node to which the force **9** is applied, drawing 13 through the point (11 · 12) and 14 through the point (8 · 9), then the piece **13** will be compressed and **14** will be tensed.

If one constructs the pentagon that corresponds to the node to which the force **1** is applied, drawing 15 through the point (13 · 14) and 16 through the point 1 · 2, then the pentagon thus-obtained will have an intertwined contour. The piece **15** will be tensed and **16** will be compressed.

One constructs the pentagon that corresponds to the node to which the external force **8** is applied; if necessary, draw 17 through the point (15 · 16) and 18 through the point (7 · 8). The piece **17** will be compressed, while **18** will be tensed.

(*) On the practical importance of the reticulated truss, one can read: No. 106 (first of Abschnitt 5, *das Fachwerk*) in CULMANN's *Graphische Statik*. The first pages of the brochure by O, HENRICI, *Skeleton structures* (London, 1866). The *Vorbermerkungen* with which prof. A. RITTER began his previously-cited book. The *Teoria elemetare delle travature reticolari* (Milan, in the journal il Politechnico of 1866) of my esteemed colleague prof. CLERICETTI. The introduction to Abschnitt 3 (*Gitterträger*) of the *Vorträge über Brückenbau* (Vienna, 1870) of prof. E. WINKLER. etc., etc.

(**) Which is a quadrangle whose fourth side is the side of the funicular polygon that is found between the forces **1**, **10**. As was already said elsewhere (nos. **21**, **25**), the funicular polygon can also lie completely at infinity.

Continuing in that way, one will find all of the other internal stresses. The final construction will give the triangle that corresponds to the point of application of the force **5**.

20, **21**, **24**, **25**, **27** will be compressed, while **19**, **22**, **23**, **26** will be tensed.

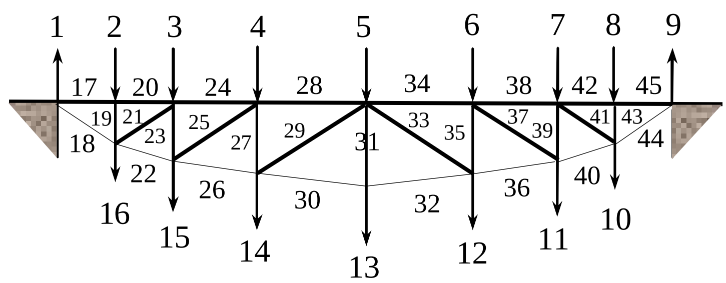

Figure 8.a

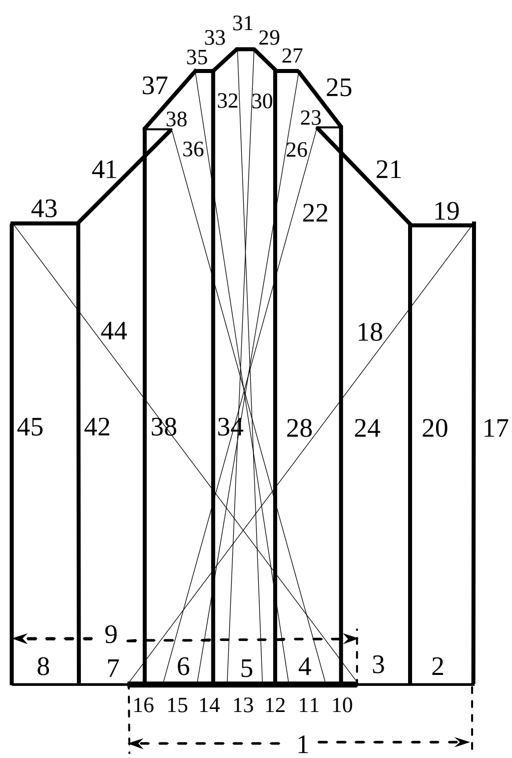

Figure 8.b

37.

Fig. 8.a represents a bridge, to whose nodes are applied the forces **1**, **2**, …, **16**, which are all vertical. **1** and **9** are directed from the bottom up and express the support reactions. The forces **2**, **3**, …, **8** are weights that are applied to the nodes of the lower slab. These forces are taken in the order in which they are encountered when one

traverses the contour of the truss, and the sides of the external force polygon are arranged in that order in diagram b. That polygon has all of its sides extended in the same vertical direction. In that line, the sum of the segments 1, 9 will be equal and opposite to the sum of the segments 2, 3, …, 8, 10, 11, …, 16, since the system of external forces must be in equilibrium.

Diagram b is now completed by precisely the same rule that was already stated. Start with the node (**1** · **17** · **18**), draw 17 through the point (1 · 2), then through the point where the end of segment 1, which is directed from down to up, and where segment 2 starts, which is directed from the up to down, and 18 through the point (16 · 1), namely, through the point where the segment 16 ends, which is directed from up to down, and where the segment 1 begins.

Passing on to the node (**2** · **17** · **19** · **20**), draw 19 through the point (17 · 18) and 20 through the point (2 · 3), namely, the point where 2 ends and 3 begins, which are segments that are directed from up to down. One then gets the polygon 2 · 17 · 19 · 20, which is a rectangle.

Now, go on to the node (**16** · **18** · **19** · **21** · **22**), and draw 21 through the point (19 · 20) and 22 through the point (15 · 16). One then gets a pentagon with an interwined contour. Continuing in the same way, one operates successively with the nodes or points of application of the forces **3**, **15**, **4**, **13**, **5**, **12**, **6**, **11**, **7**, **10**, **9**. Since the drawing of the truss and the complex of external forces in diagram a have a common symmetry axis (viz., the middle vertical), diagram b will also be symmetric (around the middle horizontal). Thus, e.g., the triangle 9 · 45 · 44 will be symmetric to the triangle 1 · 17 · 18, the rectangle 8 · 45 · 43 · 42 will be symmetric to the rectangle 2 · 17 · 19 · 20, etc.

All of the truncations of the upper part of structure are compressed, while all of those in the lower part of the structure are tensed. The oblique arrows are all compressed. Of the vertical axes, two of them – namely, **23**, **39** – are tensed, while the other ones are compressed.

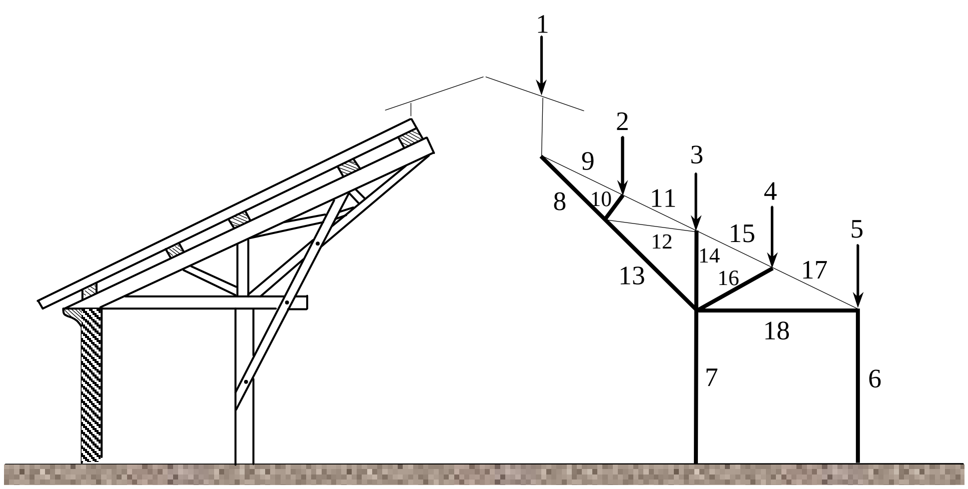

Figure 9.a

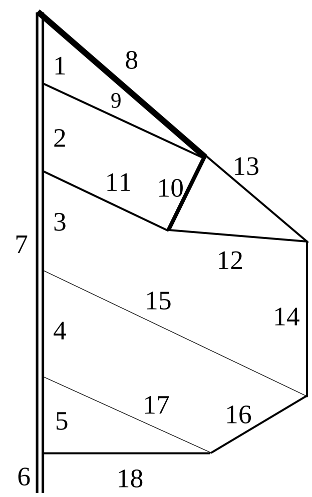

Figure 9.b

38.

In Fig. 9.a, one contemplates the one-half of a locomotive shed (*). The external forces are the weights **1**, **2**, **3**, **4**, **5**, which are applied to the upper nodes of the canopy, and the reactions **6**, **7** of the wall and column. The external forces are also all parallel here, so their polygon in diagram b will reduce to a straight line. The force 6 is (when taken in the opposite sense) equal to part of the weight 5; if one adds the difference of the other weights to that then one will get the magnitude of the force 7.

In diagram b, the lines 8, 13 will coincide in direction; the first one is part of the second one. Here, one then presents the polygon that corresponds to the node (**8** · **10** · **12** · **13**) as one of degenerate form, as was discussed in no. **27**. Namely, one has a quadrilateral 8 · 10 · 12 · 13, three vertices (13 · 8), (8 · 10), (12 · 13) of which are in a line with each other.

An analogous degenerate form is also exhibited by the quadrilateral 5 · 17 · 18 · 6 that corresponds to the point where the wall supports the roof. Indeed, the vertex (6 · 5), the point below the segment 6, and the vertices (5 · 17), (18 · 6) are along the same straight line.

The lower pieces **8**, **13**, **18**, the arrows **10**, **14**, **16**, the column **7**, and the wall **6** are compressed, while the upper parts **9**, **11**, **15**, **17**, and the arrow **12** are tensed.

(*) This example is taken from tab. 19 in the atlas of CULMANN's *Graphische Statik*. However, as was noted before, the two diagrams in it are not rigorously reciprocal.

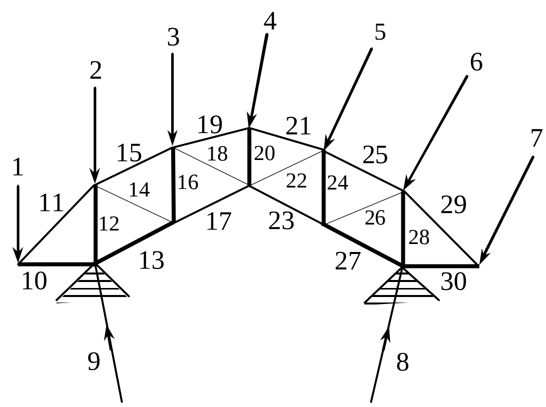

Figure 10.a

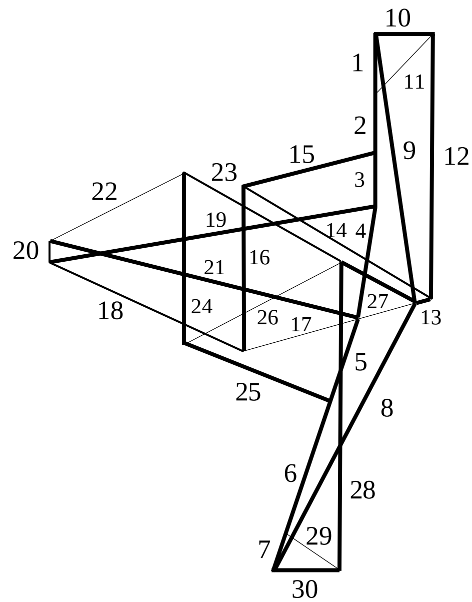

Figure 10.b

39.

Diagram a in Fig. 10 represents a truss to whose upper nodes are applied the oblique forces **1**, **2**, …, **7**, which can be considered to be the resultants of the combined actions of gravity and wind. The forces **8**, **9** represent the support reactions.

The external force polygon is drawn with double lines in diagram b. One successively constructs the triangle 1 · 10 · 11, the quadrilateral 9 · 10 · 12 · 13, the pentagon 2 · 11 · 12 · 14 · 15, the quadrilateral 13 · 14 · 16 · 17, the pentagon with an intertwined contour 3 · 15 · 16 · 18 · 19, the quadrilateral 4 · 19 · 20 · 21, which is also intertwined, and the pentagon 17 · 18 · 20 · 22 · 23, etc.

One finds that the upper pieces **15**, **19**, **21**, **25** are compressed, while the lower pieces **10**, **13**, **30** and the arrows **12**, **16**, **24**, **28** are tensed; all of the other members are tensed.

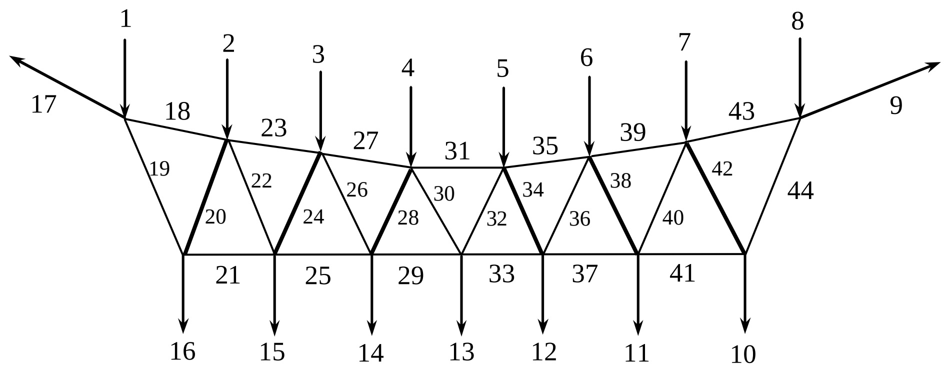

Figure 11a.

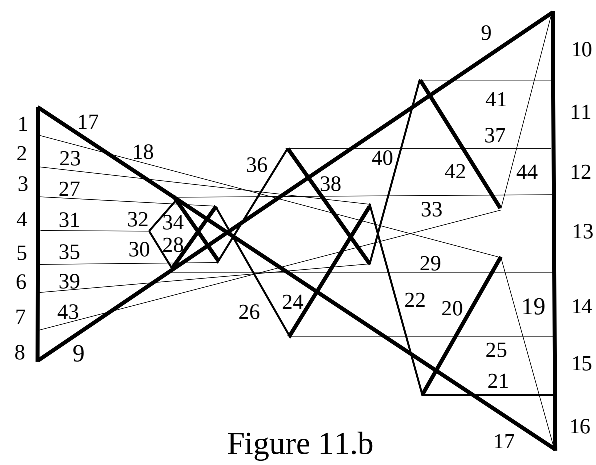

Figure 11.b

40.

Diagram a in Fig. 11 represents a suspension bridge that is loaded by the weights **1**, **2**, …, **8** at the upper nodes and the weights **10**, **11**, …, **16** at the lower nodes. The weights are equilibrated by the oblique reactions **9**, **17** at the extreme points of the truss (*).

The external force polygon has its first eight sides pointing along one vertical line and the sides 10, 11, …, up to 16 point likewise along a vertical line. The oblique sides 9 and 17 intersect, so the resulting polygonal contour will intertwine.

One successively constructs the polygons:

$$1 \cdot 17 \cdot 19 \cdot 18, \quad 16 \cdot 19 \cdot 20 \cdot 21, \quad 2 \cdot 18 \cdot 20 \cdot 22 \cdot 23,$$
$$15 \cdot 21 \cdot 22 \cdot 24 \cdot 25, \quad 3 \cdot 23 \cdot 24 \cdot 26 \cdot 27, \ldots,$$

which have intertwined contours, for the most part.

The two diagrams are symmetric in this example.

(*) This example is analogous to the one that MAXWELL gave in his 1870 paper.

Diagram b shows that the members of the upper part are all tensed, and that the tension decreases as one goes from the extremities towards the middle. All of the members in the lower part of the structure are also tensed, but there the tension diminishes as one goes from the middle towards the extremities. The arrows are alternately tensed and compressed, except for the central one, where two consecutive arrows are both tensed. Considering only the tensed arrows, or just the compressed ones, one sees that the internal stress decreases from the extremities towards the middle of the truss.

41.

The diagram in Fig. 12 (cf. *infra*) represents a reticulated crane. The proper weight of the crane is distributed over various forces **1**, **2**, **3**, **4**, …, **9** that are applied to the nodes; **5** also includes the accidental weight that the crane might sustain. All of these weights are equilibrated by the support reactions, the magnitude of which is obtained by decomposing the resultant weight into three forces that are directed along the lines **10**, **11**, **12**. When they are taken in their opposite senses, that will already given the pressures on the point **10**, the column **11**, and the tension in the tie **12**.

The determination of the external forces is also expressed in the figure. One successively takes the segments of the same vertical that represent the weights 1, 2, …, 9 and fixes a pole, and then projects from it to the points (0 · 1), (1 · 2), (2 · 3), …, (8 · 9), (9 · 0) (*), and thus constructs the corresponding funicular polygon. The vertical that passes through the common point of the extreme sides (**0** · **1**), (**9** · **0**) will be the line of action of the resultant weight, whose magnitude is, moreover, indicated by the segment that has a common origin with the segment 1 and a common end with segment 9. If one now decomposes that resultant weight, which is a known force into all of its elements – namely, three components whose lines of action are **10**, **11**, **12** – and thus employs the construction that was already mentioned in no. **34** and Fig. 6 then one will get the three forces 10, 11, 12. When taken in the opposite sense, these forces, along with the given weights, will constitute of complete system of external forces.

In order to obtain diagram b, start with the construction of the external force polygon, which is taken in the order in which one encounters the forces as one traverses the contour of the truss. Then, successively construct the polygons that correspond to the nodes (**5** · **13** · **14**), (**4** · **13** · **15** · **16**), (**6** · **14** · **15** · **17** · **18**), …, by the method that was explained above.

The diagrams that result inform us that pieces in the upper part are tensed and the lower members are compressed; as for the arrows, they are alternately compressed and tensed (**).

(*) Here, (0 · 1) indicates the origin of the segment 1, and (9 · 0) indicates the end of the segment 9. The rays in the figure that emanate from the pole in diagram B and the funicular polygon of diagram A are indicated by dashed lines.

(**) I must thank the engineer C. SAVIOTTI, who has done me the courtesy of conceiving and drawing the figures that accompany this paper.

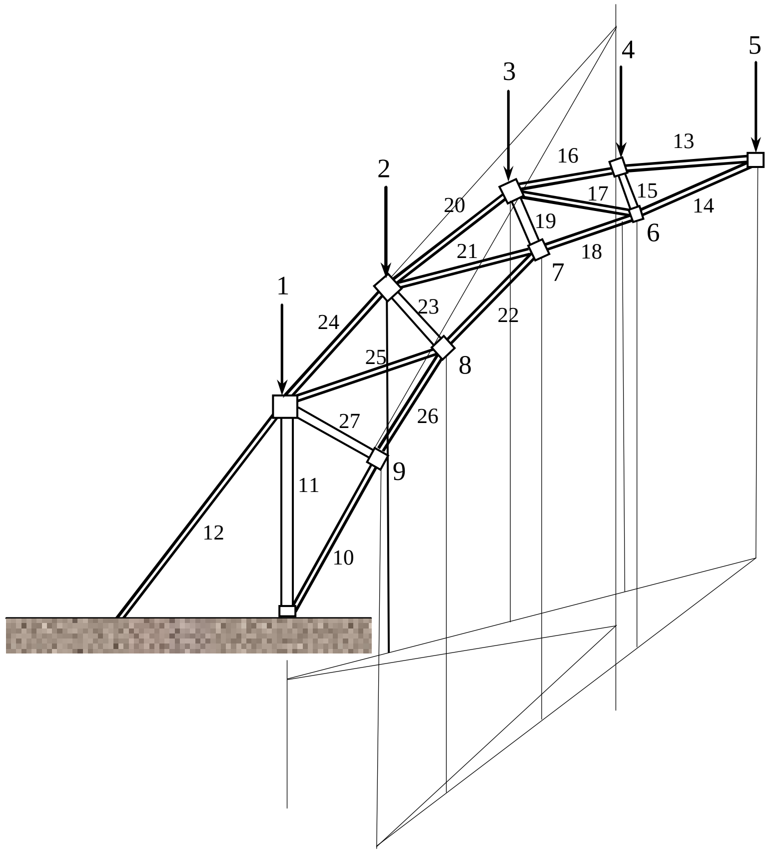

Figure 12.a

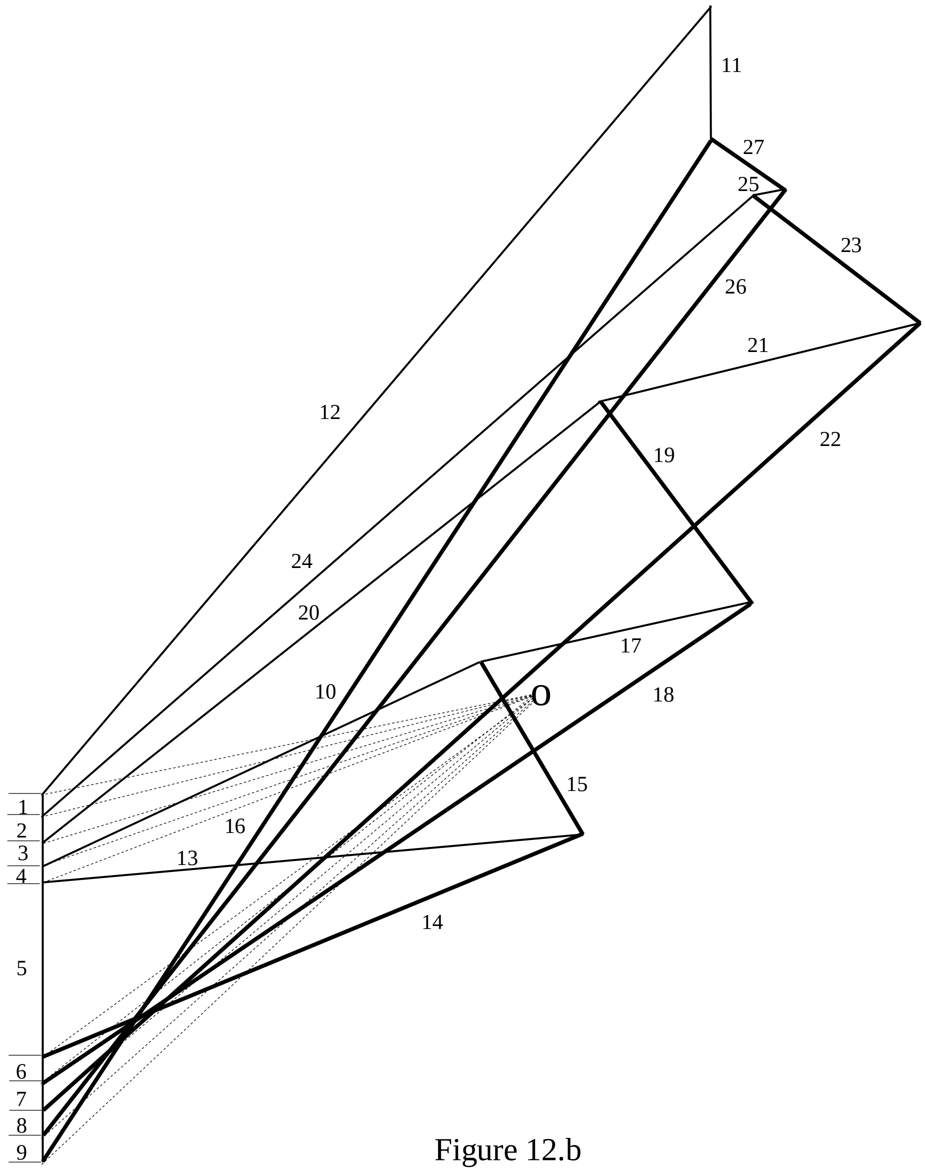

Figure 12.b

Note.

We Italians too often diminish the merits of our compatriots. After my most learned friend CHELINI had seen the first edition of this pamphlet, he reminded me that the theory of reciprocal figures, which came from the reduction of forces in statics, was first presented by MÖBIUS, and then by GAETANO GIORGINI (now a senator of the realm) in a memoir that had the title "Sopra alcune proprietà dei piani de'momenti principali e delle coppie di forze equivalenti," and which he presented to the Italian scientific society in 1828 and published in t. 20 (Modena, 1828) of the Memorie of the society. On pp. 247 of that journal one reads the theorem on the correspondence between points and planes. On pp. 249, one finds the one on reciprocal lines, and assuming that the central axis was the z-axis, the author obtained formulas that were no different, in substance, from the ones in no. **5**.

The same prof. CHELINI had the goodness to inform me that:

"GIORGINI is the author of a book *Elementi di statica* (Florence, 1835), where he proves several propositions that were contained in the 1828 paper,"

and that

"in the *Correspondence mathématique et physique* of A. QUETELET, t. 11 (Brussels, 1830), pp. 112, CHASLES, speaking of GIORGINI, who had already been his fellow student at the French polytechnic school, alluded to the aforementioned memoir, and said that the two old fellow students also both arrived at the same views as a result of that research."

The illustrious CHASLES also presented the theory of reciprocal figures that we discussed here in his paper "Sur deux principes généraux de la science: la dualité et l'homographie," which he present to the Belgian academy of science in 1830 and inserted into t. II of the *Mémoires couronnés* (Brussels, 1837); see pp. 679.

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