**COURSE IN GEOMETRY OF THE SCIENCE FACULTY**

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# LESSONS

## ON THE GENERAL THEORY

# OF SURFACES

AND

**GEOMETRIC APPLICATIONS TO THE INFINITESIMAL CALCULUS**

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# PART ONE

## GENERALITIES. CURVILINEAR COORDINATES. MINIMAL SURFACES.


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# TABLE OF CONTENTS

# BOOK II

## VARIOUS CURVILINEAR COORDINATE SYSTEMS.

## CHAPTER VI



## CHAPTER VII



## CHAPTER VIII



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# BOOK II

## VARIOUS SYSTEMS OF CURVILINEAR COORDINATES

## CHAPTER I

### CONJUGATE SYSTEMS

Proposition of Koenigs relating to the determination of an infinitude of conjugate systems on any surface with no integration. – Application to the determination of surfaces that admit a system of planar lines of curvature whose planes pass through a line. – Orthogonal trajectories of a family of circles. – Projective and dualistic character of the definition of conjugate systems. – Link between any conjugate system and a linear partial differential equation. – Surfaces on which there exists a conjugate system that is defined by two families of planar curves.

**91.** In the various surfaces that we studied previously, we encountered and employed a wide variety of systems of curvilinear coordinates: Some were simply orthogonal, others were both orthogonal and isometric, and finally, others were defined by conjugate lines. It is obvious that there exists an infinitude of orthogonal systems of conjugate systems on any surface, because if one traces an arbitrary family of curves on a surface then their orthogonal trajectories or conjugate trajectories will be defined by a differential equation that has order and degree one, and whose integral will exist, although it is not always possible to determine it. On the contrary, the following beautiful proposition, which is due to Koenigs, establishes that no matter what surface is being considered, it will be possible to trace an unlimited number of conjugate systems on it without having to perform an integration.

Let $(\Sigma)$ be the given surface. Take an arbitrary line $D$ in space.

*The sections of the surface that are determined by all planes that contain the line $D$ will admit conjugate lines that are the curves of contact of the cones that are circumscribed on the surface and have their summits on that line.*

Indeed, if $M$ is a point of the surface then the tangent plane at $M$ will cut the line $D$ at a point $A$. The circumscribed cone with summit $A$ will admit $MA$ for a generator, and that line, which is obviously the conjugate of the tangent to $M$ at the curve of contact of the cone, is also the tangent at $M$ to the plane section of the surface that is determined by the line $D$ and the point $M$. The proposition is thus proved.

**92**. In order to give an application, we now propose to determine the surface for which the lines of curvature of one of the systems are in the planes that pass through a line *D*.

It results from the preceding proposition that the curves of contact of circumscribed cones that have their summits on the line *D* will necessarily be the lines of second curvature, and since those lines must be orthogonal to the first ones, each of them must cut the generators of the cone for which it is the curve of contact at a right angle. Consequently, the lines of second curvature are spherical, the spheres that contain them will have their centers on the line *D*, and they will be cut at a right angle by lines of the first curvature.

Conversely, take an arbitrary family of spheres (*S*) that have their centers on the line *D*. Their orthogonal trajectories will obviously be plane curves, since the tangents to those trajectories must pass through the center of one of the spheres and necessarily meet the line *D*. If one takes an arbitrary surface that is generated by those orthogonal trajectories then they will be one of the desired surfaces. Indeed, let ($t$), ($t'$), ($t''$), … be a sequence of trajectories, and let $M$, $M'$, $M''$, …, resp., be the points at which they cut the sphere (*S*) at a right angle. The tangents to the trajectories at $M$, $M'$, $M''$, … will obviously agree at the center of the sphere (*S*), and they will generate a cone that circumscribes the surface that is defined by the trajectories ($t$), ($t'$), ($t''$), … along the spherical curve $MM'M''$… The two systems of conjugate lines that are defined by Koenig's theorem cut at a right angle here, and will consequently be the two systems of lines of curvature.

That results in a very simple method for generating desired surfaces. We consider an arbitrary family of spheres that have their centers on the line *D* and propose to determine their orthogonal trajectories. Let ($t$) be one of those trajectories. If one makes it turn around *D* in such a manner as to bring its plane into a fixed plane then it will not cease to be an orthogonal trajectory:

*Find the orthogonal trajectory of a family of circles that have their centers along a straight line.*

Here is how one can solve that problem:

**93.** In a general manner, consider a family of circles that are defined by the equation:

$$(x - a)^2 + (y - b)^2 = r^2, \tag{1}$$

in which $a$, $b$, $r$ are functions of a parameter $u$. The orthogonal trajectories satisfy the equation:

$$\frac{dx}{x-a} = \frac{dy}{y-b}, \tag{2}$$

and in order to form their differential equation, one must eliminate $u$ from equations (1) and (2). That elimination is impossible, in general; it is better to use the following method:

Express $x$, $y$ as functions of $u$ and a new variable $\theta$ by the formulas:

$$x = a + r \cos \theta, \qquad y = b + r \sin \theta. \tag{3}$$

The significance of $\theta$ is obvious: $\theta$ is the angle that the $x$-axis makes with the radius of the circle that passes through the point where the circle is cut by the orthogonal trajectory. One infers the values of $dx$ and $dy$ from formulas (3), and upon substituting them in equation (2), one will obtain the following equation for $\theta$:

$$\frac{d\theta}{du} = \frac{1}{r}\frac{da}{du}\sin\theta - \frac{1}{r}\frac{db}{du}\cos\theta, \tag{4}$$

which one must integrate. Now, if one takes the unknown to be:

$$\tan\frac{\theta}{2} = t \tag{5}$$

then one will arrive at a Ricatti equation:

$$2\frac{dt}{du} = \frac{2}{r}\frac{da}{du}t - \frac{1}{r}\frac{db}{du}(1-t^2). \tag{6}$$

Several consequences result from this: If one knows just one trajectory of a system of circles then one can determine all of the other ones by two quadratures. If one knows two of them then just one quadrature will suffice. Finally, the knowledge of three orthogonal trajectories will permit one to determine all of the other ones with no integration.

From that, suppose that one would like to determine the most general orthogonal system such that one family is composed of circles. One can give two of the orthogonal trajectories $(C)$, $(C_1)$ arbitrarily. Indeed, there exists a family of circles that cut two arbitrary curves at a right angle, and their centers will be found on the curve $(L)$ that is the locus of points where one can draw equal tangents to $(C)$, $(C_1)$. Since one knows two orthogonal trajectories, one quadrature will suffice to obtain all of the other ones. One can then obtain the equation of the most general planar orthogonal system that includes a family of circles by just one quadrature ([1]).

([1]) In his remarkable thesis, "Étude géométrique des surfaces dont les lignes de courbure d'un système sont planes," Toulouse, 1882, V. Rouquet has even shown that one can obtain the equation of that orthogonal system with no integration. Indeed, consider two families of circles that correspond to the system of values $a_1$, $b_1$, $r_1$, and $a_2$, $b_2$, $r_2$ of the variables $a$, $b$, $r$. If one has:

$$\frac{da_1}{r_1} = \frac{da_2}{r_2}, \qquad \frac{db_1}{r_1} = \frac{db_2}{r_2} \tag{1}$$

for each value of $u$ then the Ricatti equations that determine the orthogonal trajectories of those two families of circles will be the same, and consequently a knowledge of the orthogonal trajectories of one of the families will imply that of the orthogonal trajectories of the other family. With Rouquet, we say that two families of circles are *similar* when they satisfy the relations (1). It is easy to interpret those relations

There is, nonetheless, an extensive particular case in which the determination of the orthogonal system will require no quadrature. It is the one in which one knows, *a priori*, that one must find a straight line or circle ($\gamma$) among the orthogonal trajectories of a family of circles, because all of the desired circles will then be doubly normal to the straight line or circle ($\gamma$), so that particular orthogonal trajectory must be counted twice and will give two solutions to the Ricatti equation. It will then suffice to give ($\gamma$) and another orthogonal trajectory ($C$) in order to have three solutions of the differential equation.

We point out a further consequence of the preceding argument: When one has a well-defined system of circles that cuts a given circle at a right angle, or has their centers along a straight line, the determination of their orthogonal trajectories will demand only one quadrature.

From another viewpoint, the following consequence results from equation (6): Let tan $\theta / 2$, tan $\theta_1 / 2$, tan $\theta_2 / 2$, tan $\theta_3 / 2$ be four arbitrary solutions of that equation. We know that their anharmonic ratio is constant. Now, that anharmonic ratio of four tangents is, by definition, the anharmonic ratio of the points where the corresponding trajectories cut any of the circles. One then has the following theorem:

*The anharmonic ratio of the four points where an arbitrary circle of the family considered is cut by four fixed orthogonal trajectories is constant.*

All of these proposition obviously apply to the systems of circles that are traced on the sphere, which can always be transformed into a system that is situated in a plane by an inversion.

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geometrically. Indeed, they express the idea that the centers of the circles that correspond in the two similar families describe curves whose tangents are parallel at each instant, and furthermore, that the radii of the two circles have the same radii as the infinitely-small arcs that are traversed by their centers during the same time interval; i.e., the same ratio as the radii of curvature of the two curves that are described by their centers at the corresponding points. That proposition obviously permits one to construct all of the families that are similar to a given family of circles *with no integration.*

From that, let an arbitrary family of circles be given that corresponds to the values $a_1$, $b_1$, $r_1$ of $a$, $b$, $r$, resp. One can always imagine that there exist three functions $a_2$, $b_2$, $r_2$ such that one has:

$$\frac{da_1}{r_1} = \frac{da_2}{r_2}, \qquad \frac{db_1}{r_1} = \frac{db_2}{r_2} \qquad a_2^2 + b_2^2 = r_2^2 .$$

As a result, any family of circles can be considered to be similar to a family that is represented by the equation:

$$(x - a_2)^2 + (y - b_2)^2 = a_2^2 + b_2^2 ,$$

for all circles pass that through a fixed point, namely, the origin. Since one can exhibit the orthogonal trajectories of that particular family without integration, the same thing will be true for the most general family, from the preceding propositions.

Moreover, in many questions, it is of little importance whether one does or does not have the quadrature sign. What is essential is that one can obtain the equation of the orthogonal system in explicit form, and the developments in the text establish that this will always be possible.

**94.** We return to the proposed question. It amounts to finding the most general orthogonal system such that one of its families is defined by circles that have their centers on a line *D*.

For that, one considers an arbitrary curve (*C*) and traces out all of the circles that are normal to (*C*) and have their centers on *D*. The orthogonal trajectories of those circles are determined without integration. Indeed, consider any of those normal circles at a point *M* of the curve (*C*). If $\theta_0$ denotes the angle between the tangent to the curve at *M* and the line *D* then the Ricatti equation will admit three particular solutions:

$$\theta_0, 0, \pi,$$

and consequently, its general integration will be given by the formula:

$$\frac{\tan\frac{\theta}{2}}{\tan\frac{\theta}{2} - \tan\frac{\theta_0}{2}} = C',$$

or, more simply:

$$\tan\frac{\theta}{2} = C\tan\frac{\theta_0}{2}, \tag{7}$$

The desired surfaces, which were first studied by Joachimsthal ([2]), admit the following mode of generation:

*In a plane that passes through the line D, one defines (by the means that we just described) the most general family of plane curves* (*t*) *that admit a family of circles that have their centers on the line D for their orthogonal trajectories. One turns those curves* (*t*) *around the line D through an angle that varies according to a given, but arbitrary, law when one passes from one curve to the other. The locus of all their possible new positions is the desired surface.*

Take the *x*-axis to be the line *D*. The formulas that are the analytical translation of the preceding mode of generation are the following ones. Let:

$$(x-a)^2 + y^2 = r^2$$

be the equation of the system of circles. Take:

$$a = F(\theta_0), \qquad r = F'(\theta_0)\sin\theta_0, \qquad \tan\frac{\theta}{2} = F_1(\psi)\tan\frac{\theta_0}{2}.$$

The coordinates of an arbitrary point of the desired surface will be:

([2]) JOACHIMSTHAL: "Demonstrationes theorematum ad superficie curvas spectantium," Crelle's Journal **30**, pp. 347 and "Sur les surfaces dont les lignes de l'une des courbures sont planes," *Ibidem*, **54**, pp. 181. Above all, see the last article.

(8)
$$\begin{cases} x = a + r\cos\theta, \\ y = r\sin\theta\cos\psi, \\ z = r\sin\theta\sin\psi. \end{cases}$$

These formulas are identical to the ones that one attributes to Joachimsthal.

**95.** After having developed an application of Koenig's proposition, we return to arbitrary conjugate systems. Those two systems possess two essential properties, upon which we shall insist, and which one can state as follows:

*Any conjugate system will not cease to be conjugate when one subjects the surface on which it is traced to either a homographic transformation or a transformation by polar reciprocals.*

First, suppose that one subjects a surface ($S$) to a homographic transformation. Consider a curve ($C$) that is traced on ($S$). The tangent planes to the surface at all points of that curve will generate a developable surface ($\Delta$) whose rectilinear generators will be conjugate to the tangents to the curve ($C$). Now, it is obvious that the homographic transformation changes nothing in all of those relations. The surface ($S$) will correspond to a surface ($S'$), the curve ($C$), to a curve ($C'$), the developable ($\Delta$), to a developable ($\Delta'$) that circumscribes ($S'$) along the curve ($C'$), the tangents to the curve ($C$), to those of the curve ($C'$), and the generators of ($\Delta$), to those of ($\Delta'$). Consequently, the homographic transformation will even make two conjugate tangents to ($S$) correspond to two conjugate tangents to ($S'$).

On the contrary, if one performs a transformation by polar reciprocals then the surface ($S$) will correspond to a surface ($S''$), the curve ($C$), to a developable ($\Delta''$) that circumscribes ($S''$), and the developable ($\Delta$), to the curve of contact ($C''$) of the developable ($\Delta''$). As a result, the tangents to the curve ($C$) will correspond to the rectilinear generators of ($\Delta''$), and the generators of ($\Delta$), to the tangents to the curve ($C''$). Here again, one sees that two conjugate lines correspond to two conjugate lines.

**96.** The preceding properties, which one further expresses by saying that the definition of conjugate systems is projective and dualistic, can also be established by the following analytical method, which will permit us to generalize a proposition that was proved already (no. **84**).

Let $\alpha$ and $\beta$ be the parameters of two families of curves that are traced on a surface ($S$). Adopt an absolutely arbitrary system of homogeneous or tetrahedral coordinates, and let:

(10)
$$u\,X + v\,Y + w\,Z + p\,T = 0$$

be the equation of the tangent plane to the surface; $u$, $v$, $w$, $p$ will be functions of the parameters $\alpha$ and $\beta$, and one will obtain the equation in point-like coordinates of the surface ($S$) by eliminating $\alpha$ and $\beta$ from equation (10) and its two derivatives with respect to $\alpha$ and $\beta$:

$$\text{(11)} \qquad \begin{cases} X\dfrac{\partial u}{\partial\alpha}+Y\dfrac{\partial v}{\partial\alpha}+Z\dfrac{\partial w}{\partial\alpha}+T\dfrac{\partial p}{\partial\alpha}=0, \\ X\dfrac{\partial u}{\partial\beta}+Y\dfrac{\partial v}{\partial\beta}+Z\dfrac{\partial w}{\partial\beta}+T\dfrac{\partial p}{\partial\beta}=0. \end{cases}$$

Consequently, if one lets $x$, $y$, $z$, $t$ denote the coordinates of the point of contact of the tangent plane then they must satisfy three equations:

$$\text{(12)} \qquad \begin{cases} ux+vy+wz+pt=0, \\ x\dfrac{\partial u}{\partial\alpha}+y\dfrac{\partial v}{\partial\alpha}+z\dfrac{\partial w}{\partial\alpha}+t\dfrac{\partial p}{\partial\alpha}=0, \\ x\dfrac{\partial u}{\partial\beta}+y\dfrac{\partial v}{\partial\beta}+z\dfrac{\partial w}{\partial\beta}+t\dfrac{\partial p}{\partial\beta}=0. \end{cases}$$

Differentiate the first of these equations with respect to $\alpha$ and $\beta$ in succession. Upon taking the other two equations into account, one will get the new relations:

$$\text{(13)} \qquad \begin{cases} u\dfrac{\partial x}{\partial\alpha}+v\dfrac{\partial y}{\partial\alpha}+w\dfrac{\partial z}{\partial\alpha}+p\dfrac{\partial t}{\partial\alpha}=0, \\ u\dfrac{\partial x}{\partial\beta}+v\dfrac{\partial y}{\partial\beta}+w\dfrac{\partial z}{\partial\beta}+p\dfrac{\partial t}{\partial\beta}=0. \end{cases}$$

Moreover, one could have written down these equations immediately. They express the idea that the tangents to the two curves $\alpha$ = const., $\beta$ = const. are in the plane that is tangent to the surface.

Equations (12) and (13) apply to any system of curvilinear coordinates. We now seek the condition for the two families ($\alpha$) and ($\beta$) to form a conjugate system. If one displaces the tangent plane along the surface $\alpha$ = const. then it will envelop a developable surface. Its intersection with the infinitely-close tangent plane will be defined by equation (10), when combined with the second of equations (11). One must express the idea that the line that is represented by those equations is the tangent to the curve $\beta$ = const. For that, one must write down the idea that these equations are verified when one replaces $X$, $Y$, $Z$, $T$ with:

$$x+\frac{\partial x}{\partial\alpha}d\alpha, \qquad y+\frac{\partial y}{\partial\alpha}d\alpha, \qquad z+\frac{\partial z}{\partial\alpha}d\alpha, \qquad t+\frac{\partial t}{\partial\alpha}d\alpha.$$

Upon taking formulas (12) and (13) that were established already into account, that will give only one new equation:

$$\frac{\partial u}{\partial \beta}\frac{\partial x}{\partial \alpha}+\frac{\partial v}{\partial \beta}\frac{\partial y}{\partial \alpha}+\frac{\partial w}{\partial \beta}\frac{\partial z}{\partial \alpha}+\frac{\partial p}{\partial \beta}\frac{\partial t}{\partial \alpha} = 0, \tag{14}$$

and that single equation will consequently express the necessary and sufficient condition for the two families ($\alpha$) and ($\beta$) to form a conjugate system.

One deduces the following identities, which are applicable to any system of curvilinear coordinates, by differentiating formulas (12) and (13):

$$\tag{15} \left\{ \begin{aligned} & x\frac{\partial^2 u}{\partial\alpha\partial\beta}+y\frac{\partial^2 v}{\partial\alpha\partial\beta}+z\frac{\partial^2 w}{\partial\alpha\partial\beta}+t\frac{\partial^2 p}{\partial\alpha\partial\beta} \\ & =-\frac{\partial u}{\partial \alpha}\frac{\partial x}{\partial \beta}-\frac{\partial v}{\partial \alpha}\frac{\partial y}{\partial \beta}-\frac{\partial w}{\partial \alpha}\frac{\partial z}{\partial \beta}-\frac{\partial p}{\partial \alpha}\frac{\partial t}{\partial \beta}, \\ & =-\frac{\partial u}{\partial \beta}\frac{\partial x}{\partial \alpha}-\frac{\partial v}{\partial \beta}\frac{\partial y}{\partial \alpha}-\frac{\partial w}{\partial \beta}\frac{\partial z}{\partial \alpha}-\frac{\partial p}{\partial \beta}\frac{\partial t}{\partial \alpha}, \\ & = \ u\frac{\partial^2 x}{\partial\alpha\partial\beta}+v\frac{\partial^2 y}{\partial\alpha\partial\beta}+w\frac{\partial^2 z}{\partial\alpha\partial\beta}+p\frac{\partial^2 t}{\partial\alpha\partial\beta}. \end{aligned} \right.$$

It follows from this that equation (14) can also be written in one of the following three forms:

$$\tag{16} \left\{ \begin{aligned} & \frac{\partial u}{\partial \alpha}\frac{\partial x}{\partial \beta}+\frac{\partial v}{\partial \alpha}\frac{\partial y}{\partial \beta}+\frac{\partial w}{\partial \alpha}\frac{\partial z}{\partial \beta}+\frac{\partial p}{\partial \alpha}\frac{\partial t}{\partial \beta}=0, \\ & u\frac{\partial^2 x}{\partial\alpha\partial\beta}+v\frac{\partial^2 y}{\partial\alpha\partial\beta}+w\frac{\partial^2 z}{\partial\alpha\partial\beta}+p\frac{\partial^2 t}{\partial\alpha\partial\beta}=0, \\ & x\frac{\partial^2 u}{\partial\alpha\partial\beta}+y\frac{\partial^2 v}{\partial\alpha\partial\beta}+z\frac{\partial^2 w}{\partial\alpha\partial\beta}+t\frac{\partial^2 p}{\partial\alpha\partial\beta}=0. \end{aligned} \right.$$

The condition for the families ($\alpha$), ($\beta$) to define a conjugate system is expressed by any of the four equations (14) or (16) indifferently.

**97.** In particular, consider equations (12) and the third of equations (16). They do not contain the derivatives of $x$, $y$, $z$, $t$, and the elimination of those coordinates will lead to the equation:

$$(17) \qquad \begin{vmatrix} u & \dfrac{\partial u}{\partial\alpha} & \dfrac{\partial u}{\partial\beta} & \dfrac{\partial^2 u}{\partial\alpha\partial\beta} \\ v & \dfrac{\partial v}{\partial\alpha} & \dfrac{\partial v}{\partial\beta} & \dfrac{\partial^2 v}{\partial\alpha\partial\beta} \\ w & \dfrac{\partial w}{\partial\alpha} & \dfrac{\partial w}{\partial\beta} & \dfrac{\partial^2 w}{\partial\alpha\partial\beta} \\ p & \dfrac{\partial p}{\partial\alpha} & \dfrac{\partial p}{\partial\beta} & \dfrac{\partial^2 p}{\partial\alpha\partial\beta} \end{vmatrix} = 0,$$

which contains only the tangential coordinates ([3]).

Conversely, whenever equation (17) is satisfied, there will exist values of $x$, $y$, $z$, $t$ that verify equations (12) and the third of equations (16). Those equations express the idea that $x$, $y$, $z$, $t$ are the coordinates of the point of contact of the plane that is defined by equation (10) with the surface that it envelops, and in addition, that the two families ($\alpha$), ($\beta$) that are traced on that envelope are conjugate. Equation (17) is then characteristic of conjugate systems, in any case.

Similarly, upon eliminating $u$, $v$, $w$, $p$ from the first of equations (12), the two equations (13), and the second equation in (16), one will find the condition that the point-like coordinates must satisfy:

$$(18) \qquad \begin{vmatrix} x & \dfrac{\partial x}{\partial\alpha} & \dfrac{\partial x}{\partial\beta} & \dfrac{\partial^2 x}{\partial\alpha\partial\beta} \\ y & \dfrac{\partial y}{\partial\alpha} & \dfrac{\partial y}{\partial\beta} & \dfrac{\partial^2 y}{\partial\alpha\partial\beta} \\ z & \dfrac{\partial z}{\partial\alpha} & \dfrac{\partial z}{\partial\beta} & \dfrac{\partial^2 z}{\partial\alpha\partial\beta} \\ t & \dfrac{\partial t}{\partial\alpha} & \dfrac{\partial t}{\partial\beta} & \dfrac{\partial^2 t}{\partial\alpha\partial\beta} \end{vmatrix} = 0,$$

and one proves, as before, that this condition, which is necessary, is also sufficient.

**98.** Upon repeating the argument that was made already in no. **84**, one will immediately obtain the following proposition:

*The necessary and sufficient condition for the two families of curves whose parameters are $\alpha$ and $\beta$ to be conjugate is that either all four homogeneous point-like coordinates or all four tangential coordinates must satisfy a partial differential equation of the form:*

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([3]) BRIOSCHI, "Sulle linee di curvatura della superficie delle onde," Annali di Tortolini **2** (1859), pp. 135.

$$\frac{\partial^2\theta}{\partial\alpha\,\partial\beta} + A\frac{\partial\theta}{\partial\alpha} + B\frac{\partial\theta}{\partial\beta} + C\theta = 0, \tag{19}$$

*in which A, B, C denote arbitrary functions of $\alpha$ and $\beta$* (4).

A homographic transformation will not change the homogeneous coordinates, provided that one varies the tetrahedron and the reference parameters, and the transformation by polar reciprocals will be equivalent to a change of point-like and tangential coordinates, so one sees that the preceding method indeed exhibits the projective and dualistic character of the definition of conjugate systems.

If one employs ordinary Cartesian coordinates then the coordinate $t$ must be equal to unity; as a result, equation (19) must be verified by the value $\theta = 1$. One must then have:

$$C = 0,$$

and one will recover the result of no. **84**.

**99.** Upon concluding these developments, I remark that it is impossible to find two equations of the form (19) that the four point-like coordinates must all satisfy when the surface reduces to a curve, or the four tangential coordinates when the surface is developable.

Indeed, suppose that the four coordinates verify two linear equations of the form (19). The elimination of $\dfrac{\partial^2\theta}{\partial\alpha\,\partial\beta}$ from those two equations will lead us to a first-order equation:

$$A'\frac{\partial\theta}{\partial\alpha} + B'\frac{\partial\theta}{\partial\beta} + C'\theta = 0,$$

which those coordinates must further satisfy. Now, the general solution of that equation has the form:

$$\theta = \theta_0\, F(\sigma_0),$$

in which $F$ denotes an arbitrary function, and $\theta_0$, $\sigma_0$ are well-defined functions. Consequently, if one divides all of the homogeneous coordinates by $\theta_0$ then one will reduce them to functions of only the variable $\sigma_0$. If the coordinates are point-like then the point will describe a curve, and if they are tangential then the plane will envelop a developable.

Thus, two and only two equations of the form (19) will correspond to any conjugate system that is traced on that surface. One of them is verified by the coordinates of an arbitrary point of the surface, and the other one, by the coordinates of an arbitrary tangent plane on the surface.

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(4) It is useful to remark that the linear equation that the four point-like coordinates must satisfy is not, in general, the same as the one that the four tangential coordinates must satisfy.

**100.** The preceding theorem obviously permits one to construct an infinitude of surfaces on which one knows conjugate systems ([5]). We shall now present an application of it by looking for the surfaces for which there exist two conjugate families that are composed exclusively of plane curves.

If the tangential coordinates satisfy the equation:

$$\frac{\partial^2 \theta}{\partial \alpha \partial \beta} = 0$$

then the corresponding most general surface will be the envelope of the planes that are defined by the equation:

$$(20) \qquad [f_1(\alpha) + \varphi_1(\beta)]\, x + [f_2(\alpha) + \varphi_2(\beta)]\, y + [f_3(\alpha) + \varphi_3(\beta)]\, z + [f_4(\alpha) + \varphi_4(\beta)]\, t = 0.$$

Now, one recognizes, with no difficulty, that this surface enjoys the indicated properties, because in order to get the envelope of the tangent planes, one must combine equation (20) with the following two:

$$(21) \qquad \begin{cases} f_1'(\alpha)x + f_2'(\alpha)y + f_3'(\alpha)z + f_4'(\alpha)t = 0, \\ \varphi_1'(\beta)x + \varphi_2'(\beta)y + \varphi_3'(\beta)z + \varphi_4'(\beta)t = 0, \end{cases}$$

which each contain only one of the variables $\alpha$, $\beta$, and also show that the two conjugate families $\alpha$ = const., $\beta$ = const. are composed of plane curves.

The solution that we just obtained is very general. One can prove that there is no other one. In order to do that, we define the desired surface to be the envelope of the planes:

$$(22) \qquad uX + vY + wZ + pT = 0.$$

If we take the derivative of that equation with respect to $\alpha$ then we will have:

$$(23) \qquad X\frac{\partial u}{\partial \alpha} + Y\frac{\partial v}{\partial \alpha} + Z\frac{\partial w}{\partial \alpha} + T\frac{\partial p}{\partial \alpha} = 0.$$

As we have seen, those two equations represent the tangent to the curve $\alpha$ = const. whenever the two families ($\alpha$), ($\beta$) are conjugate.

Now, if the parameter curves $\alpha$ are planar then they will be determined by an equation of the form:

$$(24) \qquad X f_1(\alpha) + Y f_2(\alpha) + Z f_3(\alpha) + T f_4(\alpha) = 0,$$

and consequently, the three planes that are defined by equations (22), (23), (24) will contain the same line, which is the tangent to the curve $\alpha$ = const.; one of those three equations must then become a linear combination of the other two. We write down the

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([5]) DARBOUX, "Mémoire sur la théorie des coordonées curvilignes et des systèmes orthogonaux," Annales de l'École Normale (2) **7** (1878), pp. 293.

idea that the third one can be obtained by adding the other two when they are multiplied by $\mu$ and $\lambda$, respectively. We will have the system:

$$f_1(\alpha) = \mu\, u + \lambda\, \frac{\partial u}{\partial \alpha},$$
$$f_2(\alpha) = \mu\, v + \lambda\, \frac{\partial v}{\partial \alpha},$$
$$f_3(\alpha) = \mu\, w + \lambda\, \frac{\partial w}{\partial \alpha},$$
$$f_4(\alpha) = \mu\, p + \lambda\, \frac{\partial p}{\partial \alpha}.$$

Upon eliminating the functions $f_1(\alpha)$, … by a differentiation, one will see that $u$, $v$, $w$, $p$ must satisfy the same second-order equation:

$$\frac{\partial}{\partial \beta}\left(\mu\,\theta + \lambda\frac{\partial \theta}{\partial \beta}\right) = 0;$$

i.e.:

$$\lambda\frac{\partial^2\theta}{\partial\alpha\,\partial\beta} + \frac{\partial\theta}{\partial\alpha}\frac{\partial\lambda}{\partial\beta} + \mu\frac{\partial\theta}{\partial\beta} + \theta\frac{\partial\nu}{\partial\beta} = 0. \tag{25}$$

Upon likewise considering the curves $\beta$ = const., one will find that the tangential coordinates must also satisfy the similar equation:

$$\lambda'\frac{\partial^2\theta}{\partial\alpha\,\partial\beta} + \mu'\frac{\partial\theta}{\partial\beta} + \frac{\partial\lambda'}{\partial\alpha}\frac{\partial\theta}{\partial\beta} + \theta\frac{\partial\mu'}{\partial\alpha} = 0. \tag{26}$$

Now, we have seen that $u$, $v$, $w$, $p$ cannot simultaneously satisfy two equations of the preceding form, at least, when the surface is not developable. It is then necessary that equations (25), (26) should be identical, which gives the conditions:

$$\frac{\mu'}{\lambda'} = \frac{\partial \log \lambda}{\partial \beta}, \qquad \frac{\mu}{\lambda} = \frac{\partial \log \lambda'}{\partial \alpha}, \qquad \frac{1}{\lambda}\frac{\partial \mu}{\partial \beta} = \frac{1}{\lambda'}\frac{\partial \mu'}{\partial \alpha}.$$

Upon substituting the values of $\mu$ and $\mu'$ in the last equation, we will find that:

$$\frac{\partial^2 \log \lambda}{\partial\alpha\,\partial\beta} = \frac{\partial^2 \log \lambda'}{\partial\alpha\,\partial\beta},$$

from which, we will deduce, upon integrating, that:

$$\lambda' = \lambda\, f(\alpha)\, \varphi(\beta),$$

which will give:

$$\mu = \frac{\partial \lambda}{\partial \alpha} + \lambda \frac{f'(\alpha)}{f(\alpha)}.$$

If one substitutes that value for $\mu$ in equation (25) then it will take the form:

$$\frac{\partial^2}{\partial \alpha \partial \beta}[\lambda f(\alpha)\,\theta] = 0.$$

One sees that if we multiply the four coordinates $u$, $v$, $w$, $p$ by the same function $\lambda f(\alpha)$, which is obviously permitted, then they will satisfy the equation:

$$\frac{\partial^2 \theta}{\partial \alpha \partial \beta} = 0,$$

and we will recover the solution that we gave *a priori*.

It is true that we have discarded the hypothesis that the surface might be developable from our argument. However, in order to obtain such a surface, it will suffice to suppose that all of the functions $\beta$ in formula (20) are zero. Our first solution will then give, without exception, all of the surfaces for which there can exist a conjugate system that is composed of two families of planar curves.

**101.** It remains for us to describe a simple mode of generating the surfaces that we just obtained. In order to do that, set $t = 1$, and imagine the two families of spheres that are defined by the equations:

(27) $$x^2 + y^2 + z^2 - 2\, f_1(\alpha)\, x - 2\, f_2(\alpha)\, y - 2\, f_3(\alpha)\, z - 2\, f_4(\alpha) = 0,$$

(28) $$x^2 + y^2 + z^2 + 2\, \varphi_1(\beta)\, x - 2\varphi_2(\beta)\, y - 2\varphi_3(\beta)\, z - 2\varphi_4(\beta) = 0.$$

Those two families of spheres are absolutely arbitrary, and their radical plane is precisely the tangent plane to the desired surface, moreover, which is represented by equation (20). We are then led to the following theorem:

*If one considers two families of spheres in space that are defined in the most general manner then the radical plane of one of the spheres of the first family and one of the spheres of the second one will envelop the most general surface that admits two conjugate families that are composed exclusively of planar curves.*

In order to determine the surface by points, one remarks that the two equations (21) are the derivatives with respect to $\alpha$ and $\beta$ of equations (27), (28). Therefore:

*If one associates two different spheres of the family with two infinitely-close spheres then the radical center of those four spheres will describe the desired surface. The* *radical plane of two infinitely-close spheres of the same family will contain one of the curves of one of the conjugate systems.*

# CHAPTER II.

## CONJUGATE SYSTEMS. – ASYMPTOTIC LINES.

Application of the preceding proposition to the determination of surfaces with planar lines of curvature in both systems. – Characteristics of a linear partial differential equation. – New theorem that relates to conjugate systems. – Asymptotic lines. – Simplest form of their differential equation. – Their determination in particular cases. – Lamé's tetrahedral surfaces.

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**102.** The proposition that was obtained at the end of the preceding chapter leads to a very simple method for determining surfaces whose lines of curvature are planar in both systems. Indeed, it will suffice to look among the enveloping surfaces of the planes that are represented by equation (20) for the ones that have the property that the two families of conjugate curves intersect at a right angle.

Consider the enveloping surface of the planes:

$$u\,X + v\,Y + w\,Z + p\,T = 0,$$

in which $u$, $v$, $w$, $p$ are function of $\alpha$ and $\beta$. The orthogonality condition for the curves with parameters $\alpha$ and $\beta$ that are traced on the envelope is generally complicated and contains second derivatives of the tangential coordinates. Indeed, if $x$, $y$, $z$, $t$ are the coordinates of the point of contact, and if one makes $t = 1$ then one will have the two equations:

$$u\frac{\partial x}{\partial \alpha} + v\frac{\partial y}{\partial \alpha} + w\frac{\partial z}{\partial \alpha} = 0,$$

$$\frac{\partial u}{\partial \beta}\frac{\partial x}{\partial \alpha} + \frac{\partial v}{\partial \beta}\frac{\partial y}{\partial \alpha} + \frac{\partial w}{\partial \beta}\frac{\partial z}{\partial \alpha} = 0,$$

so one can deduce the proportions:

$$(1) \qquad \frac{\partial x}{\partial \alpha} : \frac{\partial y}{\partial \alpha} : \frac{\partial z}{\partial \alpha} :: v\frac{\partial w}{\partial \beta} - w\frac{\partial y}{\partial \beta} : w\frac{\partial u}{\partial \beta} - u\frac{\partial w}{\partial \beta} : u\frac{\partial v}{\partial \beta} - v\frac{\partial u}{\partial \beta}.$$

One has analogous formulas for $\frac{\partial x}{\partial \beta} : \frac{\partial y}{\partial \beta} : \frac{\partial z}{\partial \beta}$, and upon writing down the orthogonality condition, one will find the equation ([6]):

([6]) BRIOSCHI, Annali di Tortolini, **2** (1859), pp. 135.

$$(2)\quad \begin{cases} \left(u\frac{\partial u}{\partial\alpha}+v\frac{\partial v}{\partial\alpha}+w\frac{\partial w}{\partial\alpha}\right)\left(u\frac{\partial u}{\partial\beta}+v\frac{\partial v}{\partial\beta}+w\frac{\partial w}{\partial\beta}\right) \\ -(u^2+v^2+w^2)\left(\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\beta}+\frac{\partial v}{\partial\alpha}\frac{\partial v}{\partial\beta}+\frac{\partial w}{\partial\alpha}\frac{\partial w}{\partial\beta}\right)=0. \end{cases}$$

We nonetheless remark that this equation will become illusory when $u$, $v$, $w$, $p$ do not contain $\beta$, because formulas (1) will have no meaning then; i.e., the surface will become developable.

In order to apply equation (2) to the problem that we are treating, we must set:

$$u = A_1 + B_1,\quad v = A_2 + B_2,\quad w = A_3 + B_3,$$

in which $A_1$, $A_2$, $A_3$ denote functions of $\alpha$ and $B_1$, $B_2$, $B_3$ denote functions of $\beta$. One recognizes that equation (2) can be converted into one that contains only the derivatives of the function that is defined by the equation:

$$h^2 = u^2 + v^2 + w^2 = (A_1 + B_1)^2 + (A_2 + B_2)^2 + (A_3 + B_3)^2.$$

Indeed, when that equation is differentiated with respect to $\alpha$ and $\beta$, that will give successively:

$$h\frac{\partial h}{\partial\alpha} = u\frac{\partial u}{\partial\alpha}+v\frac{\partial v}{\partial\alpha}+w\frac{\partial w}{\partial\alpha},$$

$$h\frac{\partial h}{\partial\beta} = u\frac{\partial u}{\partial\beta}+v\frac{\partial v}{\partial\beta}+w\frac{\partial w}{\partial\beta},$$

$$h\frac{\partial^2 h}{\partial\alpha\,\partial\beta}+\frac{\partial h}{\partial\alpha}\frac{\partial h}{\partial\beta} = \frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\beta}+\frac{\partial v}{\partial\alpha}\frac{\partial v}{\partial\beta}+\frac{\partial w}{\partial\alpha}\frac{\partial w}{\partial\beta}.$$

Upon taking these relations in account, equation (2) will take the form:

$$\frac{\partial^2 h}{\partial\alpha\,\partial\beta} = 0,$$

and in order for this to be satisfied, one must have:

$$h = A_4 + B_4\,.$$

One can then state the following proposition:

*In order to obtain the surfaces with planar lines of curvature in the two systems, one begins by determining the functions* $\alpha$ *and* $\beta$ *that satisfy the equation:*

(3) $$(A_1 + B_1)^2 + (A_2 + B_2)^2 + (A_3 + B_3)^2 = (A_4 + B_4)^2.$$

*When these functions are known, the surface will be the envelope of the planes that are represented by the equation:*

(4) $$(A_1 + B_1)^2 + (A_2 + B_2)^2 + (A_3 + B_3)^2 = A + B,$$

*in which A, B denote two new functions that depend upon $\alpha$ and $\beta$, respectively. The lines of curvature of the two systems will be defined by the equations:*

(5) $$\begin{cases} A_1'x + A_2'y + A_3'z = A', \\ B_1'x + B_2'y + B_3'z = B', \end{cases}$$

*which each contain only one of the variables $\alpha$ or $\beta$ and represent the plane of those lines.*

**103.** The preceding method reduces all of the complexity in the problem to the determination of the most general functions that satisfy the identity (3) identically. Now, if one differentiates that equation by $\alpha$ and $\beta$, in turn, then one will come to the simpler equation:

(6) $$A_1' B_1' + A_2' B_2' + A_3' B_3' = A_4' B_4'$$

that J.-A. Serret gave all the possible solutions of in his important paper "Sur les surfaces dont les lignes de courbure sont planes ou sphériques," Journal de Liouville (1) **18**, pp. 116. Instead of following the path that Serret adopted, we shall adhere to equation (3), which we write in the form:

(7) $$(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2 = (A_4 - B_4)^2,$$

upon changing the signs of the $B$ functions.

That equation can be interpreted geometrically in the following manner: Consider the sphere ($S$), which is variable and depends upon the parameter $\alpha$, whose center has the coordinates $A_1$, $A_2$, $A_3$, and whose radius is equal in magnitude and sign to $A_4$. Consider the same sphere ($S'$), which depends upon the parameter $\beta$, and whose center has the coordinates $B_1$, $B_2$, $B_3$, and whose radius is equal to $B_4$. Equation (7) obviously expresses the idea that the two spheres ($S$) and ($S'$) are constantly tangent. It is then necessary that these two spheres, when imagined in turn, must envelop the same surface ($\Sigma$), and since that surface ($\Sigma$) is touched along a circle by each of the spheres ($S$), as well as by each of the spheres ($S'$), it is necessary that *all of its lines of curvature must be circular.*

We are then reduced to a well-known problem, which was proposed and solved for the first time by Dupin ([7]): *Determine all surfaces whose lines of curvature are circular.* The solution yields a fourth-order surface, namely, the *Dupin cyclide,* whose normals

([7]) DUPIN, *Applications de Géométrie et de Méchanique,* pp. 200, *et seq.*; 1822.

meet an ellipse and a hyperbola, which are the focal curves of each other, and which contain the centers of all spheres that are tangent to the surface along one of its lines of curvature. That surface can degenerate into a torus, and then the focal ellipse will reduce to a circle – i.e., one will have a third-order surface – and in that case, the two focal curves will become parabolas.

**104.** If one supposes that the ellipse reduces to a circle then the hyperbola will reduce to a line that passes through the center of the circle and perpendicular to the plane. Upon choosing the center of the circle to be the origin of the coordinates, and the line to be the $z$-axis, we will obtain a first solution to equation (7) that is given by the following formulas:

$$\begin{array}{lll} A_1 = 0, & A_2 = 0, & A_3 = \alpha, \\ B_1 = \cos\beta, & B_2 = \sin\beta, & B_3 = 0. \end{array}$$

The corresponding surface with planar lines of curvature is the envelope of the planes:

$$\alpha z - x\cos\beta - y\sin\beta = f(\alpha) + \varphi(\beta). \tag{8}$$

The lines of curvature $\alpha$ = const. are in the parallel planes:

$$z = f'(\alpha),$$

and consequently, that first class will include only the rolling surfaces that were studied already (no. **86**).

We now pass on to the general case, in which the focal curve is an ellipse. Since one can multiply all of the functions $A_i$, $B_i$ by the same number, one can take the equations of that focal curve to be:

$$x^2 + \frac{z^2}{\lambda^2} = 0, \quad y = 0,$$

and the corresponding hyperbola will then be represented by the system:

$$x = 0, \quad y^2 - \frac{z^2}{\lambda^2 - 1} = -1.$$

A point of the first curve will be defined by the formulas:

$$x = A_1 = \alpha, \qquad y = A_2 = 0, \qquad z = A_3 = \lambda\sqrt{1-\alpha^2}, \tag{9}$$

and likewise, a point of the second one will be defined by the analogous formulas:

$$x = B_1 = 0, \qquad y = B_2 = 0, \qquad z = B_3 = \sqrt{\lambda^2-1}\sqrt{1+\beta^2}. \tag{10}$$

The surface with planar lines of curvature that corresponds to these values of the functions $A_i$, $B_i$ will be the envelope of the planes:

$$\alpha x - \beta y + (\lambda\sqrt{1-\alpha^2} - \sqrt{\lambda^2 - 1}\sqrt{1+\beta^2})\, z = f(\alpha) - \varphi(\beta) \tag{11}$$

when $\alpha$ and $\beta$ vary.

In the case where the ellipse reduces to a parabola, one will likewise find that the corresponding surface is the envelope of the planes:

$$2\alpha x + 2\beta y + (1 - \alpha^2 - \beta^2)\, z = f(\alpha) + \varphi(\beta). \tag{12}$$

**103.** In summary, we obtain three classes of surfaces with planar lines of curvature in the two systems. However, it clearly results from the preceding argument that the first and third of them can be considered to be limiting cases of the second one, which is defined by formula (11). We shall exhibit a new mode of generating those surfaces.

If one successively differentiates equation (11) with respect to $\alpha$ and $\beta$ then one will obtain two equations:

$$x - \frac{\lambda\alpha z}{\sqrt{1-\alpha^2}} = f'(a), \tag{13}$$

$$y + \frac{\sqrt{\lambda^2-1}\,\beta z}{\sqrt{1+\beta^2}} = \varphi'(a), \tag{14}$$

which represent the planes of the lines of the first and second curvature, as we have seen. Thus, the lines of curvature of each system are in the tangent planes to a cylinder, and the cylinders that correspond to the two systems have perpendicular generators.

On the other hand, the two families of spheres that were considered in no. **101** have the equations:

$$\begin{cases} x^2 + y^2 + z^2 - 2\alpha x - 2\lambda\sqrt{1-\alpha^2}\, z + 2 f(\alpha) = 0, \\ x^2 + y^2 + z^2 - 2\beta x - 2\sqrt{\lambda^2 - 1}\sqrt{1+\beta^2}\, z + 2\varphi(\beta) = 0 \end{cases} \tag{15}$$

here. The centers of these spheres are situated on the two focal curves, respectively. Furthermore, their radii depend upon the functions $f(\alpha)$ and $\varphi(\beta)$ and consequently vary according to an arbitrary law. Upon applying the theorem of no. **101**, one will then be led to the following proposition:

*In order to obtain all of the surfaces with planar lines of curvature in the two systems, one constructs two different families of spheres whose centers are required to describe two second-degree curves that are focal curves to each other, and whose radii vary according to an arbitrary law for each of the two families. The radical planes of the two spheres* ($S$), ($\Sigma$), *which belong to the two different families, will envelop the desired surface. If one associates* ($\Sigma$) *and* ($S$) *with the infinitely-close spheres* ($S'$) *and* ($\Sigma'$) *then* *the radical centers of those four spheres will describe the surface. The radical planes of* ($S$) *and* ($S'$), ($\Sigma$) *and* ($\Sigma'$) *will be the planes of the lines of curvature of the two systems* ([8]).

Although our argument leaves aside the case of developable surfaces, the results that are obtained include those surfaces, which are, as one easily recognizes, the rolling surfaces that are defined by the tangents to a helix that is traced on an arbitrary cylinder.

**106.** We conclude this preliminary study of conjugate systems by generalizing the proposition that was given in no. **98**, and in order to do that in a precise manner, we begin by recalling the definition of the characteristics of a linear partial differential equation.

Let:

$$A\frac{\partial^2\theta}{\partial\alpha^2}+B\frac{\partial^2\theta}{\partial\alpha\,\partial\beta}+C\frac{\partial^2\theta}{\partial\beta^2}+A'\frac{\partial\theta}{\partial\alpha}+B'\frac{\partial\theta}{\partial\beta}+C'\theta=0 \tag{16}$$

be such an equation, in which the coefficients are arbitrary, given function of $\alpha$ and $\beta$. If one replaces those independent variables with the following ones:

$$\rho=\varphi(\alpha,\beta),\quad \rho_1=\psi(\alpha,\beta)$$

then the equation will keep its form and become:

$$a\frac{\partial^2\theta}{\partial\rho^2}+b\frac{\partial^2\theta}{\partial\rho\,\partial\rho_1}+c\frac{\partial^2\theta}{\partial\rho_1^2}+a'\frac{\partial\theta}{\partial\rho}+b'\frac{\partial\theta}{\partial\rho_1}+c'\theta=0,$$

in which $a$, $b$, $c$ have the following values:

$$\left\{\begin{aligned} a&=A\left(\frac{\partial\theta}{\partial\alpha}\right)^2+B\frac{\partial\rho}{\partial\alpha}\frac{\partial\rho_1}{\partial\beta}+C\left(\frac{\partial\rho}{\partial\beta}\right)^2,\\ b&=2A\frac{\partial\rho}{\partial\alpha}\frac{\partial\rho_1}{\partial\alpha}+B\left(\frac{\partial\rho}{\partial\alpha}\frac{\partial\rho_1}{\partial\beta}+\frac{\partial\rho}{\partial\beta}\frac{\partial\rho_1}{\partial\alpha}\right)+2C\frac{\partial\rho}{\partial\beta}\frac{\partial\rho_1}{\partial\beta},\\ c&=A\left(\frac{\partial\rho_1}{\partial\alpha}\right)^2+B\frac{\partial\rho_1}{\partial\alpha}\frac{\partial\rho_1}{\partial\beta}+C\left(\frac{\partial\rho_1}{\partial\beta}\right)^2.\end{aligned}\right. \tag{17}$$

If one would then like to make the two terms in $\frac{\partial^2\theta}{\partial\rho^2}$, $\frac{\partial^2\theta}{\partial\rho_1^2}$ disappear then $\rho$ and $\rho_1$ would have to be two different functions that satisfy the equation:

([8]) One can define the variation of the radii of the spheres of each of the families by requiring that those spheres must be tangent to an arbitrarily-chosen curve that is situated in the plane of the line that contains their centers. One can then geometrically construct the radical plane of each sphere and the infinitely-close sphere.

(18)
$$A\left(\frac{\partial u}{\partial\alpha}\right)^2 + B\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\beta} + C\left(\frac{\partial u}{\partial\beta}\right)^2 = 0.$$

One can state that result in the following manner:

*Consider the first-order differential equation of degree two:*

(19)
$$A\,d\beta^2 - B\,d\alpha\,d\beta + C\,d\alpha^2 = 0,$$

*to which we give the name of the* ***differential equation of the characteristics****, and which decomposes into two first-degree equations that each admit an integral. Let:*

$$\rho = \varphi(\alpha, \beta), \quad \rho_1 = \psi(\alpha, \beta)$$

*be the two integrals thus obtained. One must take $\rho$, $\rho_1$ to be the new variables if one would like to reduce the equation in $\theta$ to the form:*

(20)
$$\frac{\partial^2\theta}{\partial\rho\,\partial\rho_1} + a'\frac{\partial\theta}{\partial\rho} + b'\frac{\partial\theta}{\partial\rho_1} + c'\theta = 0.$$

One sees that the transformation will be impossible if the left-hand side of equation (19) is a perfect square. However, it results from formulas (17) that if one then takes $\rho$ to be the unique integral of equation (19) then the equation in $\theta$ will reduce to the simple form ([9]):

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([9]) One can likewise further simplify that equation as long as one knows some particular solutions, because if one lets $\theta_1$, $\theta_2$ be two of those solutions then if one takes the new independent variables to be $\rho$ and the ratio:

$$\frac{\theta_2}{\theta_1} = \rho_1',$$

and if one sets:

$$\theta = \theta_1\,\sigma$$

then the function $\sigma$ will satisfy an equation of the same form as equation (21):

$$\frac{\partial^2\sigma}{\partial\rho_1'^2} + a_1\frac{\partial\sigma}{\partial\rho} + b_1\frac{\partial\sigma}{\partial\rho_1'} + c_1\sigma = 0.$$

However, since that equation must admit the particular solutions:

$$\sigma = 1, \quad \sigma = \rho_1'$$

in which $b_1$ and $c_1$ are zero, it will reduce to the binomial form:

(21) $$\frac{\partial^2\theta}{\partial\rho_1^2} + a'\frac{\partial\theta}{\partial\rho} + b'\frac{\partial\theta}{\partial\rho_1} + c'\theta = 0.$$

**107.** After having recalled those definitions and properties, we now return to the question that we have in mind, and suppose that the four homogeneous coordinates $x$, $y$, $z$, $t$, or $u$, $v$, $w$, $p$ are expressed as functions of the two variables $\alpha$ and $\beta$. If one has obtained (by whatever process) an equation of the form (16) that these four coordinates must satisfy then one can convert into the form (20) by the process that we just pointed out, and one will then immediately recognize that the curves ($\rho$), ($\rho_1$) will trace out a conjugate system on the surface. We can then state the following proposition:

*When one has defined (in whatever manner) an equation of the form* (16) *that must be satisfied by either the four point-like coordinates or the four tangential coordinates, the characteristics of that equation will trace out a conjugate system on the surface.*

An equation of the form (16) that contains five coefficients is not determined by the condition that it must admit, for example, the four point-like coordinates for particular solutions. However, if one appends to that condition the condition that it must admit a fifth solution $\varphi$ then all of its coefficients will be determined perfectly, as well as the conjugate system that it formed by its characteristics. In that sense, one can say that each function $\varphi$ corresponds to a particular conjugate system:

Thus, suppose that one takes the Cartesian coordinates $x$, $y$, $z$. Before equation (16) can admit the solution $\theta = t = 1$, it cannot contain the term in $\theta$, and all of its coefficients will be determined completely by the condition that it must admit a new particular solution $\varphi$, along with $x$, $y$, $z$, and we suppose that the new solution is expressed as a function of $x$, $y$, $z$, for example. Convert the equation to the form (20) by taking the parameters $\rho$, $\rho_1$ of the conjugate system that is defined by its characteristics to be the new variables, and then let:

$$\frac{\partial^2\theta}{\partial\rho\,\partial\rho_1} = A\frac{\partial\theta}{\partial\rho} + B\frac{\partial\theta}{\partial\rho_1}$$

be its new form. One will have:

$$\frac{\partial^2 x}{\partial\rho\,\partial\rho_1} = A\frac{\partial x}{\partial\rho} + B\frac{\partial x}{\partial\rho_1},$$

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$$\frac{\partial^2\sigma}{\partial\rho_1'^2} + a_1\frac{\partial\sigma}{\partial\rho} = 0.$$

The preceding argument shows, moreover, that this form is not typical, and can be obtained in an infinitude of ways.

and an analogous equation in $y$ and $z$. If one now expresses the idea that $\varphi$ is a solution, and if one eliminates $\frac{\partial^2 x}{\partial\rho\,\partial\rho_1}$, $\frac{\partial^2 y}{\partial\rho\,\partial\rho_1}$, $\frac{\partial^2 z}{\partial\rho\,\partial\rho_1}$ by means of the preceding equations then one will find that:

$$\frac{\partial^2\varphi}{\partial x^2}\frac{\partial x}{\partial\rho}\frac{\partial x}{\partial\rho_1}+\frac{\partial^2\varphi}{\partial x\,\partial y}\left(\frac{\partial x}{\partial\rho}\frac{\partial y}{\partial\rho_1}+\frac{\partial y}{\partial\rho}\frac{\partial x}{\partial\rho_1}\right)+\cdots+\frac{\partial^2\varphi}{\partial z^2}\frac{\partial z}{\partial\rho}\frac{\partial z}{\partial\rho_1}=0.$$

This is a relation that the tangents to the two conjugate families must satisfy at each point of the surface.

**108.** If one takes, for example, the following value of $\varphi$:

$$\varphi = x^2 + y^2 + z^2$$

then one will have:

$$\frac{\partial x}{\partial\rho}\frac{\partial x}{\partial\rho_1}+\frac{\partial y}{\partial\rho}\frac{\partial y}{\partial\rho_1}+\frac{\partial z}{\partial\rho}\frac{\partial z}{\partial\rho_1}=0,$$

so the conjugate system will be orthogonal; i.e., it will be defined by two systems of lines of curvature, which leads us to the following theorem:

*The equation of the form:*

$$A\frac{\partial^2\theta}{\partial\alpha^2}+B\frac{\partial^2\theta}{\partial\alpha\,\partial\beta}+C\frac{\partial^2\theta}{\partial\beta^2}+D\frac{\partial\theta}{\partial\alpha}+E\frac{\partial\theta}{\partial\beta}+c'\theta=0,$$

*whose coefficients are determined by the condition that it must admit:*

$$x,\quad y,\quad z,\qquad x^2+y^2+z^2$$

*as particular solutions, in which x, y, z denote the orthogonal Cartesian coordinates of a point on the surface, which are expressed as functions of two arbitrary variables $\alpha$ and $\beta$, admits the two families of lines of curvature of the surface for its characteristics.*

The following application, which is very simple, provides a verification of that proposition. Take two of the coordinates $x$ and $y$ to be the independent variables. Before the equation:

$$A\frac{\partial^2\theta}{\partial x^2}+B\frac{\partial^2\theta}{\partial x\,\partial y}+C\frac{\partial^2\theta}{\partial y^2}+D\frac{\partial\theta}{\partial x}+E\frac{\partial\theta}{\partial y}+c'\theta=0$$

can admit the particular solutions $x$ and $y$, one must first have:

$$D = E = 0.$$

If one then expresses the idea that it must likewise admit the two solutions:

$$\theta = z, \quad \theta = x^2 + y^2 + z^2$$

then one will obtain the two relations:

$$Ar + Bs + Ct = 0, \qquad A\,(1 + p^2) + B\,pq + C\,(1 + q^2) = 0,$$

in which $p$, $q$, $r$, $s$, $t$ denote the derivatives of $z$, and they determine the ratios of $A$, $B$, $C$. The desired equation will then be:

$$[s\,(1 + q^2) - tpq]\frac{\partial^2\theta}{\partial x^2} - [r\,(1 + q^2) - t\,(1 + p^2)]\frac{\partial^2\theta}{\partial x\,\partial y} + [rpq - s\,(1 + q^2)]\frac{\partial^2\theta}{\partial y^2} = 0,$$

and the differential equation of its characteristics will provide the well-known equation of the lines of curvature.

**109.** The theory of asymptotic lines of a surface is linked very closely to that of conjugate systems. If one groups those lines into two distinct families, as one does for lines of curvature, then one can say that each of the two families that are obtained in that way will be conjugate to itself. Consequently, the asymptotic lines will be preserved when one subjects the surface to either a homographic transformation or a transformation by polar reciprocals. The following calculations will exhibit those results, moreover.

Keep all the notations of no. **96**. Always let $u$, $v$, $w$, $p$ be the coordinates of the tangent plane, and let $x$, $y$, $z$, $t$ be those of the point of contact. As we have seen, one will have the equalities:

$$(22) \qquad \begin{cases} ux + vy + wz + pt = 0, \\ u\,dx + v\,dy + w\,dz + p\,dt = 0, \\ x\,du + y\,dv + z\,dw + t\,dp = 0, \end{cases}$$

which refer to an arbitrary displacement that is performed on the surface.

We seek the differential equation of the asymptotic lines. We must write down the idea that the osculating plane of those lines coincides with the tangent plane; i.e., that the point whose coordinates are:

$$x + dx + \tfrac{1}{2}d^2x, \qquad y + dy + \tfrac{1}{2}d^2y, \;\ldots$$

must be found in the tangent plane. Upon taking the preceding equalities into account, one will then be led to the equation:

$$(23) \qquad u\,d^2x + v\,d^2y + w\,d^2z + p\,d^2t = 0.$$

The identities that one obtains by differentiating the last two equations (22) permit us to replace the preceding equation with one of the following two:

$$(24) \qquad \begin{cases} du\,dx + dv\,dy + dw\,dz + dp\,dt = 0, \\ x\,d^2u + y\,d^2v + z\,d^2w + t\,d^2p = 0, \end{cases}$$

which are equivalent to it.

The first of these two formulas immediately gives the differential equation of the asymptotic lines when the surface is defined by its equation in either point-like or tangential coordinates. However, the preceding formulas (23) and (24) also permit one to write down that differential equation if one supposes that the coordinates are expressed as functions of the two variables $\alpha$ and $\beta$. For example, if one eliminates $u$, $v$, $w$, $p$ from the first equation (22), the two equations (13) of no. **96**, and equation (23) then one will be led to the relation:

$$\begin{vmatrix} x & \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial x}{\partial \beta} & d^2x \\ y & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial y}{\partial \beta} & d^2y \\ z & \dfrac{\partial z}{\partial \alpha} & \dfrac{\partial z}{\partial \beta} & d^2z \\ t & \dfrac{\partial t}{\partial \alpha} & \dfrac{\partial t}{\partial \beta} & d^2t \end{vmatrix} = 0,$$

which constitutes the desired differential equation. If one develops it and arranges it with respect to $d\alpha$, $d\beta$ then one will find that:

$$(25) \quad \begin{vmatrix} x & \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial x}{\partial \beta} & \dfrac{\partial^2 x}{\partial \alpha^2} \\ y & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial^2 y}{\partial \alpha^2} \\ z & \dfrac{\partial z}{\partial \alpha} & \dfrac{\partial z}{\partial \beta} & \dfrac{\partial^2 z}{\partial \alpha^2} \\ t & \dfrac{\partial t}{\partial \alpha} & \dfrac{\partial t}{\partial \beta} & \dfrac{\partial^2 t}{\partial \alpha^2} \end{vmatrix} d\alpha^2 + \begin{vmatrix} x & \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial x}{\partial \beta} & \dfrac{\partial^2 x}{\partial \alpha \partial \beta} \\ y & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial^2 y}{\partial \alpha \partial \beta} \\ z & \dfrac{\partial z}{\partial \alpha} & \dfrac{\partial z}{\partial \beta} & \dfrac{\partial^2 z}{\partial \alpha \partial \beta} \\ t & \dfrac{\partial t}{\partial \alpha} & \dfrac{\partial t}{\partial \beta} & \dfrac{\partial^2 t}{\partial \alpha \partial \beta} \end{vmatrix} d\alpha\,d\beta + \begin{vmatrix} x & \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial x}{\partial \beta} & \dfrac{\partial^2 x}{\partial \beta^2} \\ y & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial^2 y}{\partial \beta^2} \\ z & \dfrac{\partial z}{\partial \alpha} & \dfrac{\partial z}{\partial \beta} & \dfrac{\partial^2 z}{\partial \beta^2} \\ t & \dfrac{\partial t}{\partial \alpha} & \dfrac{\partial t}{\partial \beta} & \dfrac{\partial^2 t}{\partial \beta^2} \end{vmatrix} d\beta^2 = 0.$$

The equation in tangential coordinates is entirely similar, and can be obtained by replacing $x$, $y$, $z$, $t$ with $u$, $v$, $w$, $p$ in the preceding.

We can deduce the proposition that relates to conjugate systems and was proved already (no. **98**) from the preceding equation, because the necessary and sufficient condition for the two families of curves ($\alpha$), ($\beta$) to be conjugate – i.e., for the tangents to the coordinates curves that pass through each point of the surface to harmonically divide the angle that is defined by the asymptotic tangents at that point – is obviously that the coefficient of $d\alpha\, d\beta$ must be zero in the differential equation (25). We then recover the condition that was given already (no. **97**).

**110.** Conversely, whenever one knows a conjugate system on a surface, one can write down the equation of the asymptotic lines in a form that no longer contains the rectangle $d\alpha\, d\beta$.

Here, we are presented with a new occasion to apply the proposition of Koenigs, because if we suppose that the line $D$ that appears in the statement of that proposition is pushed out to infinity parallel to a fixed plane then we will recognize that the plane sections that are parallel to that fixed plane have conjugates that are the curves of contact of the circumscribed cylinder and the surface whose rectilinear generators are parallel to the various lines in that plane. If we refers the points of the surface to that conjugate system then the equation of the asymptotic lines must contain only squares of the differentials.

Indeed, take a system of Cartesian coordinates $x$, $y$, $z$, and let $p$, $q$ be the derivatives of $z$, considered as functions of $x$ and $y$. Upon supposing that the fixed plane has been chosen to be the $yz$-plane, the variables that one must adopt will be the following ones:

$$x = \alpha, \quad q = \beta. \tag{26}$$

From the equation:

$$dz = p\, dx + q\, dy,$$

one deduces that:

$$d\,(z - qz) = p\, dx - y\, dq = p\, d\alpha - y\, d\beta.$$

Consequently, if one sets:

$$z - qy = z', \tag{27}$$

and if one expresses $z$ as a function of $\alpha$ and $\beta$ then the preceding equation will obviously give:

$$p = \frac{\partial z'}{\partial \alpha}, \qquad y = -\frac{\partial z'}{\partial \beta},$$

or furthermore:

$$p = p', \qquad y = -q', \tag{28}$$

in which $p'$ and $q'$ denote the derivatives of $z'$ with respect to $\alpha$ and $\beta$.

The differential equation of the asymptotic lines:

$$dp\, dx + dq\, dy = 0$$

will then become:

$$dp'\, d\alpha - dq'\, d\beta = 0 \tag{29}$$

with the variables $\alpha$ and $\beta$, and if one replaces $p'$, $q'$ with their expressions $r'\, d\alpha + s'\, d\beta$, $s'\, d\alpha + t'\, d\beta$ as functions of the second derivatives $r'$, $s'$, $t'$ of $z'$ then it will take on the form:

(30) $$r'\,d\alpha^2 - t'\,d\beta^2 = 0,$$

which no longer contain the term in $d\alpha\,d\beta$, as we predicted.

**111.** The simple form of that equation will permit us to obtain a large number of surfaces whose asymptotic lines one can determine in finite terms.

Indeed, consider both a differential equation:

(31) $$\frac{\partial^2\beta}{\partial\alpha^2} = \varphi(\alpha, \beta)$$

and the partial differential equation:

(32) $$r' - t'\varphi(\alpha, \beta) = 0.$$

If one knows how to find a function $z'$ that satisfies the latter equation then one can deduce a surface by replacing $x, y, z$ by means of the formulas:

$$x = \alpha, \qquad y = -q', \qquad z = qy + z' = z' - \beta q'.$$

Now, the asymptotic lines of that surface will be determined by the equation:

$$r'\,d\alpha^2 - t'\,d\beta^2 = 0,$$

and upon replacing $r' / t'$ with its value that is deduced from equation (32), one will recover equation (31). Whenever one has to integrate that differential equation, along with the partial differential equation (32), one will then have a surface for which one knows the asymptotic lines.

For example, suppose that one takes the function $\varphi$ to be a constant $k^2$. The finite equations of the two systems of asymptotic lines will be:

$$\beta + k\,\alpha = \text{const.}, \qquad \beta - k\,\alpha = \text{const.}$$

Equation (32) will have the general integral:

$$z' = F(\beta + k\,\alpha) + F_1(\beta - k\,\alpha),$$

and the coordinates $x, y, z$ of a point on the surface will be given as functions of $\alpha$ and $\beta$ by the formulas:

$$\begin{aligned} x &= \alpha, \\ y &= -F'(\beta + k\,\alpha) - F_1'(\beta - k\,\alpha), \\ z &= F - \beta F' + F_1 - \beta F_1'. \end{aligned}$$

**112.** Here is yet another application of formula (25).

Consider the surfaces for which the Cartesian coordinates $x$, $y$, $z$ are defined as functions of two variables $\rho$, $\rho_1$ by the following expressions:

$$(33) \qquad \begin{cases} x = A(\rho - a)^m (\rho_1 - a)^n, \\ y = B(\rho - b)^m (\rho_1 - b)^n, \\ z = C(\rho - c)^m (\rho_1 - c)^n. \end{cases}$$

I first say that the curves ($\rho$), ($\rho_1$) trace out a conjugate on these surfaces. Indeed, one verifies that the three coordinates satisfy the equation:

$$(34) \qquad (\rho - \rho_1)\,\frac{\partial^2\theta}{\partial\rho\,\partial\rho_1} + n\frac{\partial\theta}{\partial\rho} - m\frac{\partial\theta}{\partial\rho_1} = 0.$$

It follows from this that if one seeks the equation of the asymptotic lines by applying formula (25) then that equation will not contain the term in $d\rho\, d\rho_1$. Upon performing that calculation, one will indeed obtain the differential equation:

$$(35) \qquad \frac{m(m-1)\,d\rho^2}{(a-\rho)(b-\rho)(c-\rho)} = \frac{n(n-1)\,d\rho_1^2}{(a-\rho_1)(b-\rho_1)(c-\rho_1)},$$

which can be integrated by quadratures in any case, but whose integral is algebraic whenever the quotient $\frac{m(m-1)}{n(n-1)}$ is the square of a commensurable number.

In the particular case where $m$ is equal to $n$, one can eliminate $\rho$, $\rho_1$ and find the equation of the surface, which is:

$$\left(\frac{x}{A}\right)^{1/m}(b-c) + \left(\frac{y}{B}\right)^{1/m}(c-a) + \left(\frac{z}{C}\right)^{1/m}(a-b) = (a-b)\,(b-c)\,(a-c).$$

One recognizes the tetrahedral surfaces that were first studied by Lamé in his "Examen des différentes methods employées pour résoudre les problèmes de Géométrie," which was published in 1818, and which have since then been the subject of the work of numerous geometers, among which, one must cite, most especially, de la Gournerie. Equation (35) then reduces to the one that was integrated by Euler and which gives the addition of elliptic functions. Among the numerous forms that one can give to its integral, we choose the following one:

$$\sqrt{\alpha}\sqrt{\frac{(\rho-a)(\rho_1-a)}{(a-b)(a-c)}} + \sqrt{\beta}\sqrt{\frac{(\rho-b)(\rho_1-b)}{(b-a)(b-c)}} + \sqrt{\gamma}\sqrt{\frac{(\rho-c)(\rho_1-c)}{(c-a)(c-b)}} = 0,$$

in which $\alpha$, $\beta$, $\gamma$ denote three arbitrary constants whose sum is zero.

Changing the notations slightly will give the following result.

*The asymptotic lines of the tetrahedral surface:*

$$\left(\frac{x}{A}\right)^m + \left(\frac{y}{B}\right)^m + \left(\frac{z}{C}\right)^m = 1$$

*are determined by means of the equation:*

$$\sqrt{\alpha}\left(\frac{x}{A}\right)^{m/2} + \sqrt{\beta}\left(\frac{y}{B}\right)^{m/2} + \sqrt{\gamma}\left(\frac{z}{C}\right)^{m/2} = 0,$$

*in which* $\alpha$, $\beta$, $\gamma$ *are three arbitrary constants whose sum is zero. Their projections onto one of the coordinate planes – for example, the yz-plane – have the equation* ([10]):

$$\sqrt{-\alpha} = \sqrt{\beta}\left(\frac{z}{C}\right)^{m/2} - \sqrt{\gamma}\left(\frac{y}{B}\right)^{m/2}.$$

**113.** Formulas (33) determine a large number of different surfaces. In particular, they are suited to the Steiner surface for $m = n = 2$, to the surface of centers of curvature of the ellipsoid for $m = 3/2$, $n = 1/2$, … We remark that they keep that same form when one substitutes the tangential coordinates for the point-like coordinates. Indeed, let:

$$u\,X + v\,Y + w\,Z - 1 = 0$$

be the equation of the tangent plane. $u$, $v$, $w$ are determined by the three equations:

$$u\,X + v\,Y + w\,Z - 1 = 0,$$

$$u\frac{\partial x}{\partial \rho} + v\frac{\partial y}{\partial \rho} + w\frac{\partial z}{\partial \rho} = 0,$$

$$u\frac{\partial x}{\partial \rho_1} + v\frac{\partial y}{\partial \rho_1} + w\frac{\partial z}{\partial \rho_1} = 0,$$

which give:

$$u = \frac{(\rho - a)^{1-m}(\rho_1 - a)^{1-n}}{A(a-b)(a-c)},$$

([10]) The asymptotic lines of tetrahedral surfaces were determined for the first time by LIE [*see* the article "Ueber die Reciprocitäts-Verhältnisse des Reye'schen Complexes," Göttingen Nachrichten (1870), 53-66]. The method that is described here was developed by the author in the Bulletin des Sciences mathématiques (1) **1** (1870), pp. 355.

$$v = \frac{(\rho - b)^{1-m}(\rho_1 - b)^{1-n}}{B(b-a)(b-c)},$$

$$w = \frac{(\rho - c)^{1-m}(\rho_1 - c)^{1-n}}{A(c-a)(c-b)}.$$

In particular, if one has:

$$m + n = 1$$

then one will obtain the surfaces that coincide with their polar reciprocal with respect to the second-degree surface:

$$\frac{x^2}{A^2(a-b)(a-c)} + \frac{y^2}{B^2(b-a)(b-c)} + \frac{z^2}{C^2(c-a)(c-b)} = 1.$$

In this case, as well, the differential equation of the asymptotic lines will reduce to that of Euler ([11]).

**114.** Upon concluding this subject, we point out some things that the theory of asymptotic lines and that of linear partial differential equations have in common that are analogous to the ones that were the subject of nos. **84** and **107**. A family of asymptotic lines can be considered as a system that is itself its own conjugate, so the theorem of no. **107** immediately gives us the following:

*If the coordinates x, y, z, t, or u, v, w, p, when considered as functions of* $\alpha$ *and* $\beta$*, satisfy a linear equation of the form* (16) *for which the characteristics coincide then the characteristics of that equation will trace out one of the two families of asymptotic lines on the surface. In particular, if they satisfy an equation of the form:*

$$\frac{\partial^2 \theta}{\partial \beta^2} + D\frac{\partial \theta}{\partial \alpha} + E\frac{\partial \theta}{\partial \beta} + F\theta = 0 \tag{36}$$

*then the lines* $\alpha$ = const. *will be asymptotic.*

The last part of the proposition is verified immediately by inspection of equation (25), in which the coefficient of $d\beta^2$ will become zero, by virtue of the hypothesis.

Whenever one has a linear equation of the form (36) and one knows four linearly-independent solutions, the preceding theorem will permit one to obtain a surface on which one knows one of the two families of asymptotic lines.

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([11]) One can consult my "Note sur les lignes asymptotiques de la surface des ondes," Comptes rendus **97** (1883), pp. 1039, in which one will find a generalization of the method that is employed in the last number of this chapter.

# CHAPTER III

## ISOTHERMAL ORTHOGONAL SYSTEMS

Division of a surface into infinitely-small squares. – Isothermal systems and symmetric coordinates. – Geographic charts. – Resolution of the problem for surfaces of revolution and second-degree surfaces. – Isothermal systems in the plane.

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**115.** After having given the simplest properties of conjugate systems, we shall now consider orthogonal systems, and in particular, the systems that are both orthogonal and isothermal. We first look for all orthogonal coordinate systems that permit one to divide a surface into infinitely-small squares.

Let:

$$ds^2 = A^2\, du^2 + C^2\, dv^2$$

be the expression for the line element. $A$ and $C$ will be given functions of $u$ and $v$:

$$A = f(u, v), \qquad C = \varphi(u, v).$$

Consider (Fig. 6) four lines that correspond to each family, namely, $(A)$, $(A_1)$, $(B)$, $(B_1)$, which correspond to the values of $u$:

$$u_0\,, \qquad u_0 + du_0\,, \qquad u, \qquad u + du,$$

resp., and $(C)$, $(C_1)$, $(D)$, $(D_1)$, which correspond to the values of $v$:

$$v_0\,, \qquad v_0 + dv_0\,, \qquad v, \qquad v + dv,$$

resp. They obviously determine four infinitely-small rectangles (1), (2), (3), (4). We seek to find out whether it is possible to arrange $du$, $dv$, $du_0$, $dv_0$ in such a manner that those rectangles are all squares. The consideration of each of them will give us the relations:

$$\begin{aligned} f(u_0, v_0)\, du_0 &= \varphi(u_0, v_0)\, dv_0\,, \\ \varphi(u_0, v)\, dv &= f(u_0, v_0)\, du_0\,, \\ \varphi(u, v_0)\, dv_0 &= f(u, v_0)\, du\,, \\ f(u, v)\, du &= \varphi(u, v)\, dv. \end{aligned}$$

These equations, which are four in number, contain only three unknowns, namely, the ratios of $du$, $dv$, $du_0$, $dv_0$ . The elimination of those ratios will lead to one condition, which one obtains immediately, moreover, upon multiplying corresponding sides of the equations. One will then find that:

$$\frac{f(u_0, v_0)\, f(u, v)}{\varphi(u_0, v_0)\, \varphi(u, v)} = \frac{f(u, v_0)\, f(u_0, v)}{\varphi(u, v_0)\, \varphi(u_0, v)}.$$

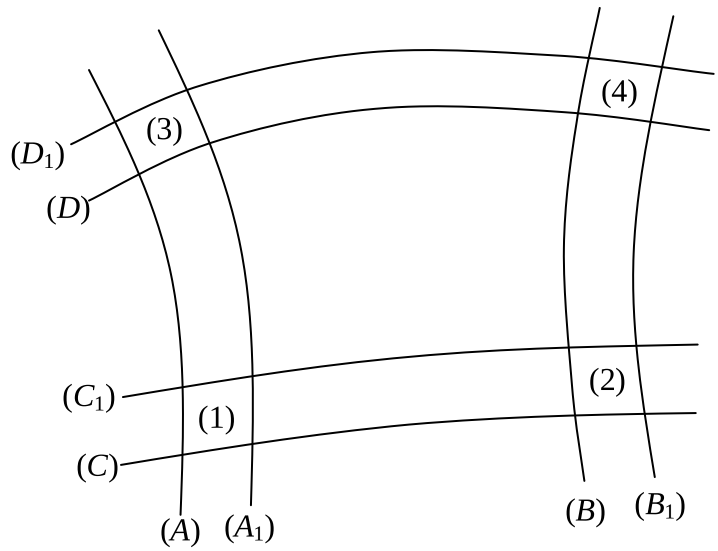


Figure 6.

Give arbitrary numerical values to $u_0$ and $v_0$ in that relation. It will take the form:

$$\frac{f(u, v)}{\varphi(u, v)} = \frac{\theta(u)}{\theta_1(v)},$$

and consequently one must have:

$$A = f(u, v) = \lambda\, \theta(u),$$
$$C = \varphi(u, v) = \lambda\, \theta_1(u),$$

in which $\lambda$ denotes an arbitrary function of $u$ and $v$.

If one substitutes these values for $A$ and $C$ into the line element then one will obtain the new expression:

$$ds^2 = \lambda^2\, [\theta^2(u)\, du^2 + \theta_1^2(v)\; dv^2],$$

or, more simply:

(1) $$ds^2 = \lambda^2\, (du_1^2 + dv_1^2),$$

upon setting:

$$u_1 = \int \theta(u)\, du, \qquad v_1 = \int \theta_1(v)\; dv.$$

Conversely, whenever the line element can be converted into the preceding form, the surface, as we know (no. **65**), will be divisible into infinitely-small squares by the coordinate lines.

**116.** We have already encountered some surfaces on which there exist isothermal orthogonal systems. We shall now prove that the line element of an arbitrary surface can be converted into the form (1) in an infinitude of ways. In order to understand the following method, it will suffice to remark that if one replaces $u_1$, $v_1$ with the complex variables:

$$\alpha = u_1 + iv_1, \quad \beta = u_1 - iv_1$$

then formula (1) will be presented in the form:

$$ds^2 = \lambda\, d\alpha\, d\beta. \tag{2}$$

Having said that, consider an arbitrary surface whose line element is given in the most general form:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

Set:

$$H^2 = EG - F^2,$$

to abbreviate, and exclude the case (which cannot present itself for real surfaces, moreover) in which $EG - F^2$ is zero, and in which the line element is a perfect square ([12]). One can decompose the line element into two factors and write:

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([12]) The line is a perfect square only in the case where the surface is a developable that circumscribes the imaginary circle at infinity. Indeed, suppose that one has:

$$ds^2 = (m\,du^2 + n\,dv^2).$$

Let $1 / h$ be the factor that makes:

$$\frac{m\,du + n\,dv}{h}$$

an exact differential.

One can set:

$$ds^2 = h^2\,d\beta^2. \tag{1}$$

$h$ will be a function of $\beta$ and a second curvilinear coordinate $\alpha$ that permits one to define the various points of the surface, along with $\beta$. The rectangular coordinates $x$, $y$, $z$ of an arbitrary point of the surface must satisfy the two equations:

$$\left\{\begin{aligned} &\left(\frac{\partial x}{\partial\alpha}\right)^2 + \left(\frac{\partial y}{\partial\alpha}\right)^2 + \left(\frac{\partial z}{\partial\alpha}\right)^2 = 0, \\ &\frac{\partial x}{\partial\alpha}\frac{\partial x}{\partial\beta} + \frac{\partial y}{\partial\alpha}\frac{\partial y}{\partial\beta} + \frac{\partial z}{\partial\alpha}\frac{\partial z}{\partial\beta} = 0. \end{aligned}\right. \tag{2}$$

Upon differentiating the first one with respect to $\alpha$ and $\beta$, one will obtain:

$$\left\{\begin{aligned} &\frac{\partial x}{\partial\alpha}\frac{\partial^2 x}{\partial\beta^2} + \frac{\partial y}{\partial\alpha}\frac{\partial^2 y}{\partial\beta^2} + \frac{\partial z}{\partial\alpha}\frac{\partial^2 z}{\partial\beta^2} = 0, \\ &\frac{\partial x}{\partial\alpha}\frac{\partial^2 x}{\partial\alpha\partial\beta} + \frac{\partial y}{\partial\alpha}\frac{\partial^2 y}{\partial\alpha\partial\beta} + \frac{\partial z}{\partial\alpha}\frac{\partial^2 z}{\partial\alpha\partial\beta} = 0. \end{aligned}\right. \tag{3}$$

If one differentiates the second of formulas (2) with respect to $\alpha$ then one will have, upon taking the second equation of (3) into account:

$$\frac{\partial x}{\partial \beta}\frac{\partial^2 x}{\partial \alpha^2}+\frac{\partial y}{\partial \beta}\frac{\partial^2 y}{\partial \alpha^2}+\frac{\partial z}{\partial \beta}\frac{\partial^2 z}{\partial \alpha^2}=0. \tag{4}$$

A comparison of the preceding equations shows us that one will have two different solutions for the homogeneous equations in $u$, $v$, $w$:

$$u\frac{\partial x}{\partial \alpha}+v\frac{\partial y}{\partial \alpha}+w\frac{\partial z}{\partial \alpha}=0,$$
$$u\frac{\partial x}{\partial \beta}+v\frac{\partial y}{\partial \beta}+w\frac{\partial z}{\partial \beta}=0,$$

if one takes either:

$$u=\frac{\partial x}{\partial \alpha},\qquad v=\frac{\partial y}{\partial \alpha},\qquad w=\frac{\partial z}{\partial \alpha}$$

or

$$u=\frac{\partial^2 x}{\partial \alpha^2},\qquad v=\frac{\partial^2 y}{\partial \alpha^2},\qquad w=\frac{\partial^2 z}{\partial \alpha^2}.$$

One must then have:

$$\frac{\dfrac{\partial^2 x}{\partial \alpha^2}}{\dfrac{\partial x}{\partial \alpha}}=\frac{\dfrac{\partial^2 y}{\partial \alpha^2}}{\dfrac{\partial y}{\partial \alpha}}=\frac{\dfrac{\partial^2 z}{\partial \alpha^2}}{\dfrac{\partial z}{\partial \alpha}};$$

one then deduces, by integration, that:

$$\frac{\dfrac{\partial x}{\partial \alpha}}{f(\beta)}=\frac{\dfrac{\partial y}{\partial \alpha}}{f_1(\beta)}=\frac{\dfrac{\partial z}{\partial \alpha}}{f_2(\beta)}. \tag{5}$$

If one denotes the common value of these ratios by $\partial\alpha'/\partial\alpha$ then one will have:

$$\begin{cases} x = f(\beta)\alpha'+\varphi(\beta), \\ y = f_1(\beta)\alpha'+\varphi_1(\beta), \\ z = f_2(\beta)\alpha'+\varphi_2(\beta). \end{cases}$$

If we write down the idea that $x$, $y$, $z$ satisfy equation (2) then we will find that:

(7) [*sic*]
$$\begin{cases} f^2(\beta)+f_1^2(\beta)+f_2^2(\beta)=0, \\ f(\beta)\varphi'(\beta)+f_1(\beta)\varphi_1'(\beta)+f_2(\beta)\varphi_2'(\beta)=0. \end{cases}$$

If one adds equations (6), after multiplying them by $f(\beta)$, $f_1(\beta)$, $f_2(\beta)$, respectively, then one will have:

$$f(\beta)\,x+f_1(\beta)\,y+f_2(\beta)\,z=f\,\varphi+f_1\,\varphi_1+f_2\,\varphi_2. \tag{8}$$

If one multiplies them by $f'$, $f_1'$, $f_2'$, resp., then one will similarly find upon adding them that:

$$x\,f'(\beta)+y\,f_1'(\beta)+z\,f_2'(\beta)=\varphi f'+\varphi_1 f_1'+\varphi_2 f_2',$$

$$ds^2 = \left(\sqrt{E}du + \frac{F+iH}{\sqrt{E}}dv\right)\left(\sqrt{E}du + \frac{F-iH}{\sqrt{E}}dv\right).$$

If we equate the two factors to zero in succession then we will get two differential equations. Let:

$$\varphi(u, v) = \alpha, \quad \psi(u, v) = \beta$$

be the integrals of those equations. They define two families of imaginary lines that are traced on the surface, and whose arc length is equal to zero. As one knows, one will have:

$$(3) \qquad \begin{cases} d\alpha = \mu\left(\sqrt{E}du + \dfrac{F+iH}{\sqrt{E}}dv\right), \\ d\beta = \nu\left(\sqrt{E}du + \dfrac{F-iH}{\sqrt{E}}dv\right), \end{cases}$$

in which $\mu$ and $\nu$ are suitably-chosen factors. $\alpha$, $\beta$ will obviously be mutually-independent functions of $u$ and $v$, because their functional determinant:

$$-2\mu\nu\, i\, H$$

is non-zero. If one takes them to be the new variables then multiplying the preceding two formulas will give us the expression:

$$ds^2 = \frac{1}{\mu\nu}\, d\alpha\, d\beta$$

for the line element of the surface, or, upon changing the notations:

$$(4) \qquad ds^2 = \lambda^2\, d\alpha\, d\beta.$$

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or, upon taking the second of equations (7) into account:

$$(9) \qquad x\, f'(\beta) + y\, f_1'(\beta) + z\, f_2'(\beta) = \frac{\partial}{\partial\beta}(\varphi f + \varphi_1 f_1 + \varphi_2 f_2).$$

The equation (9) is obtained by taking the derivative of equation (8) with respect to $\beta$. The surface is then the envelope of the plane that is defined by equation (8). Now, from the first of formulas (7), that plane will be tangent to the circle at infinity.

The only surfaces for which the line element is a perfect square are then the developables that circumscribe the circle at infinity. The edges of regression of those developables are curves whose tangents all meet the circle at infinity, and which satisfy the equation:

$$dx^2 + dy^2 + dz^2 = 0.$$

We will often have to employ the system of variables $\alpha$, $\beta$, which has been given the name of *symmetric coordinates*. In the case of a real line element, one can obviously suppose that the variables $\alpha$ and $\beta$ are conjugate imaginaries, as well as the factors $\mu$ and $\nu$. For example, suppose that one has obtained some functions $\alpha$ and $\mu$ that verify the first of equations (3). If one changes $i$ into $-i$ then one will see that the conjugate imaginaries of $\alpha$ and $\mu$ give a solution to the second equation.

**117.** Suppose that the line element has been put into the form (4) in two different ways, and that one has both:

(5) $$ds^2 = \lambda^2\, d\alpha\, d\beta = \lambda'^2\, d\alpha'\, d\beta'.$$

We shall show that this equation can be true only if $\alpha'$, $\beta'$ depend upon just one of the variables $\alpha$, $\beta$, respectively.

Indeed, if one supposes that $\alpha'$, $\beta'$ are expressed as functions of the independent variables $\alpha$, $\beta$, and if one replaces the differentials $d\alpha'$, $d\beta'$ with their values:

$$\frac{\partial\alpha'}{\partial\alpha}d\alpha+\frac{\partial\alpha'}{\partial\beta}d\beta, \qquad \frac{\partial\beta'}{\partial\alpha}d\alpha+\frac{\partial\beta'}{\partial\beta}d\beta$$

then the identity (5) will give three equations:

$$\frac{\partial\alpha'}{\partial\alpha}\frac{\partial\beta'}{\partial\alpha}=0, \quad \frac{\partial\alpha'}{\partial\beta}\frac{\partial\beta'}{\partial\beta}=0, \quad \frac{\partial\alpha'}{\partial\alpha}\frac{\partial\beta'}{\partial\beta}+\frac{\partial\alpha'}{\partial\beta}\frac{\partial\beta'}{\partial\alpha}=\frac{\lambda^2}{\lambda'^2}.$$

The first one can be true only if $\alpha'$ or $\beta'$ depends upon only the variable $\alpha$, and upon taking the second one into account, one will have two solutions:

$$\alpha' = \mathcal{F}(\alpha), \qquad \beta' = \mathcal{F}_1(\beta)$$

or

$$\alpha' = \mathcal{F}(\beta), \qquad \beta' = \mathcal{F}_1(\alpha).$$

We then obtain the following theorem:

*When one has exhibited the line element in the form:*

$$ds^2 = \lambda^2\, d\alpha\, d\beta,$$

*one can preserve that form for the line element only if one replaces the variables $\alpha$, $\beta$ with the variables $\alpha'$, $\beta'$, which are determined by one or the other of the systems:*

1. $\alpha' = \mathcal{F}(\alpha)$, $\beta' = \mathcal{F}_1(\beta)$,
2. $\alpha' = \mathcal{F}(\beta)$, $\beta' = \mathcal{F}_1(\alpha)$.

**118.** From the symmetric coordinates, one passes on immediately to the isothermal systems. Indeed, recall formula (4), and replace $\alpha$, $\beta$ with the following expressions:

$$\alpha = u + iv, \qquad \beta = u - iv\ ;$$

it will give:

(6) $$ds^2 = \lambda^2\,(du^2 + dv^2).$$

That is the form of the line element that characterizes isothermal systems. Upon applying the propositions that were proved already for symmetric coordinates to those systems, one can state the following theorems:

1. *There exists an infinitude of isothermal orthogonal systems on any surface. One will obtain them by the complete integration of the equation:*

$$ds^2 = 0.$$

2. *When one has obtained an isothermal system* ($u$, $v$), *one can pass to any other isothermal system* ($u'$, $v'$) *by the use of one or the other system of formulas:*

*a*.

(7) $$\begin{cases} u' + iv' = f(u + iv), \\ u' - iv' = f_1(u - iv); \end{cases}$$

*b*.

(8) $$\begin{cases} u' + iv' = f(u - iv), \\ u' - iv' = f_1(u + iv), \end{cases}$$

*and consequently the knowledge of just one isothermal orthogonal system that is traced on the surface will imply that of all the other such systems that are traced on that surface.*

**119.** The theory of symmetric coordinates and isothermal systems, which one can trace back to the first paper of Gauss ([13]), which was published in 1825, had its origins in the study of a beautiful question of practical geometry, namely, that of the geographic tracing of one surface on another one, and more particularly the plane. The theory of geographic charts was the subject of some important work of Lambert, Euler, and Lagrange. Since it is impossible (no. **72**) to represent a portion of the sphere or any other non-developable surface on the plane in such a manner that it preserves arc lengths, one will always appeal to the modes of representation that preserve angles, such as stereographic projection and Mercator projection. Those modes of representation have the fundamental property that they establish similitude between the corresponding infinitely-small elements on the two surfaces. Indeed, if one considers two corresponding infinitely-small triangles, one can take them to be two rectilinear triangles, and since they

([13]) GAUSS, "Allgemeine Auflösung der Aufgabe die Theile einer gegeben Fläche auf einer andern gegeben so abzubilden dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird," *Gesammelte Werke*, v. IV, pp. 193.

are equi-angular, their homologous sides will be proportional. Conversely, if two surfaces correspond point-by-point in such a manner that their line elements are coupled by the relation:

$$ds^2 = \lambda^2 \, ds'^2,$$

and if one considers a region on one of them that is small enough that one can ignore the variation of $\lambda$ then the corresponding lines that are traced on the two surfaces will have a constant ratio. Consequently, two corresponding infinitely-small triangles will be similar, and the angles will be preserved when one passes from one surface to the other one.

One can establish that proposition in a more rigorous manner by looking for the angle between the two curves that are traced on an arbitrary surface. Suppose that one is displaced in two different directions upon starting from a point of the surface, and let $d$, $\delta$ denote the characteristics of the differentials that relate to those two displacements. Let:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

be the expression for the line element. The formulas:

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv, \qquad \delta x = \frac{\partial x}{\partial u} \delta u + \frac{\partial x}{\partial v} \delta v,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots.., \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

give us:

$$dx\,\delta x + dy\,\delta y + dz\,\delta z = E\,du\,\delta u + F\,(du\,\delta v + du\,\delta v) + G\,dv\,\delta v, \tag{9}$$

and consequently the angle $V$ between the two directions will be determined by the formula:

$$\cos V = \frac{E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v}{\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}\sqrt{E\,\delta u^2 + 2F\,\delta u\,\delta v + G\,\delta v^2}}. \tag{10}$$

One sees that it depends upon only the ratios of $E$, $F$, $G$, and remains the same when the entire line element is multiplied by an arbitrary function of $u$ and $v$.

**120.** Since the line element in the plane is reducible to the form:

$$ds^2 = d\alpha^2 + d\beta^2,$$

the problem of the geographic tracing of an arbitrary surface onto the plane can be formulated as follows:

*Put the line element of the surface into the form:*

$$ds^2 = \lambda^2\,(d\alpha^2 + d\beta^2);$$

*i.e., determine an isothermal orthogonal system on the surface.*

It results from the preceding developments that when one knows one solution to that problem, one can obtain all of the other ones with no integration.

We have seen that the meridian and parallels on any surface of revolution define an isothermal system. One will then know how to solve the problem for all surfaces of revolution.

For example, consider the sphere for which one has:

$$ds^2 = du^2 + \sin^2 u\, dv^2 = \sin^2 u \left(\frac{du^2}{\sin^2 u} + dv^2\right). \tag{11}$$

Set:

$$\int \frac{du}{\sin u} = kx = \log \tan \frac{u}{2}, \qquad v = ky,$$

so the line element will become:

$$ds^2 = \frac{4k^2\, e^{2kx}}{(1+e^{2kx})^2}(dx^2 + dy^2). \tag{12}$$

If one makes the point $(u, v)$ on the sphere correspond to the point on the plane whose rectangular coordinates are $x$, $y$ then one will obtain a method of tracing for which the meridians that make equal angles will be represented by equidistant parallel lines, and the parallels, by lines that are perpendicular to the latter; that is the Mercator projection. It offers the advantage, which was formerly very appreciated in marine charts, of making loxodromes (viz., curves that cut all meridians at a constant angle) correspond to lines on the chart.

On the contrary, if one sets:

$$\frac{du}{\sin u} = \frac{d\rho}{\rho}, \quad v = \omega, \quad \rho = k \tan \frac{u}{2}$$

then the line element will become:

$$ds^2 = \frac{4k^2}{(\rho^2 + k^2)^2}(d\rho^2 + \rho^2\, d\omega^2).$$

Upon making the point $(u, v)$ correspond to the point in the plane whose polar coordinates are $\rho$ and $\omega$, one will have a geographic trace in which the meridians corresponds to concurrent lines, and the parallels, to concentric circles in the plane that cuts all of those lines at a right angle. It is the trace that one will obtain by making a stereographic projection of the sphere from a viewpoint that is placed at the pole.

**121.** Now consider a second-degree surface that is represented by the equation:

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1. \tag{13}$$

One can regard it as a tetrahedral surface (no. **112**), and take the coordinates $x$, $y$, $z$ of any of its points to have the following expressions:

$$\left\{ \begin{aligned} x &= \sqrt{\frac{a(a-\rho)(a-\rho_1)}{(a-b)(a-c)}}, \\ y &= \sqrt{\frac{b(a-\rho)(b-\rho_1)}{(b-a)(b-c)}}, \\ z &= \sqrt{\frac{c(c-\rho)(c-\rho_1)}{(c-a)(c-b)}}. \end{aligned} \right. \tag{14}$$

We already know that the curves ($\rho$), ($\rho_1$) define a conjugate system. That system is also orthogonal, because the preceding formulas permit one to verify the equation:

$$\frac{\partial x}{\partial \rho}\frac{\partial x}{\partial \rho_1} + \frac{\partial y}{\partial \rho}\frac{\partial y}{\partial \rho_1} + \frac{\partial z}{\partial \rho}\frac{\partial z}{\partial \rho_1} = 0.$$

Since the system ($\rho$, $\rho_1$) is both orthogonal and conjugate, it will then be composed of lines of curvature of the surface. Formulas (14) also permit us to calculate the line element for which one obtains the following expression:

$$ds^2 = \frac{\rho - \rho_1}{4}\left[\frac{\rho\, d\rho^2}{(a-\rho)(b-\rho)(c-\rho)} - \frac{\rho_1\, d\rho_1^2}{(a-\rho_1)(b-\rho_1)(c-\rho_1)}\right]. \tag{15}$$

Set:

$$\frac{\sqrt{\rho}\, d\rho}{\sqrt{(a-\rho)(b-\rho)(c-\rho)}} = d\alpha, \qquad \frac{\sqrt{\rho_1}\, d\rho_1}{\sqrt{(a-\rho_1)(b-\rho_1)(c-\rho_1)}} = d\beta,$$

and formula (15) will become:

$$ds^2 = \frac{\rho - \rho_1}{4}(d\alpha^2 + d\beta^2). \tag{16}$$

One will then have an isothermal orthogonal system; its use will permit one to make a chart of any region that is traced on the second-degree surface ([14]).

([14]) On the subject of that representation, one can read the paper of Jacobi, "Ueber die Abbildung eines ungleichaxigen Ellipsoids auf einer Ebene, bei welcher die kleinsten Theile ähnlich bleiben," Crelle's Journal **59** (1861), pp. 74.

It results from the preceding calculation that the second-degree surfaces are divisible into infinitely-small squares by their lines of curvature. As we have seen already, that property also belongs to the surfaces of revolution.

We further point out an essential property of the line element of second-degree surfaces. In formula (16), $\rho$ is a function of $\alpha$, and $\rho_1$ is a function of $\beta$. The line element then belongs to the following type:

$$ds^2 = [f(\alpha) - F(\beta)]\,(d\alpha^2 + d\beta^2), \tag{17}$$

which will be presented in its most general form in the theory of geodesic lines.

**122.** The theory of homofocal surfaces of degree two leads to a method of performing the geographic trace of a second-degree surface that is more elegant than the preceding one.

Let:

$$\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} - 1 = 0 \qquad (a > b > c > 0)$$

be the equation of a system of homofocal surfaces. Three surfaces of the system will pass through an arbitrary point of space. One of them will be an ellipsoid that corresponds to a value $\rho_2$ of $\lambda$ that is less than $c$, another will be a hyperboloid with one sheet that corresponds to a value $\rho_1$ of $\lambda$ that is found between $b$ and $c$, and the third one will be a hyperboloid with two sheets that corresponds to a value $\rho$ of $\lambda$ that is found between $a$ and $b$. Consequently, $\rho$, $\rho_1$, $\rho_2$ constitute a curvilinear coordinate system – namely, Lamé's *elliptical coordinates* – that are properly defined at any point of space. One knows that this system is orthogonal; the expression for the line element is the following one:

$$ds^2 = \frac{1}{4}\left[\frac{(\rho-\rho_1)(\rho-\rho_2)}{f(\rho)}d\rho^2 + \frac{(\rho_1-\rho)(\rho_1-\rho_2)}{f(\rho_1)}d\rho_1^2 + \frac{(\rho_2-\rho)(\rho_2-\rho_1)}{f(\rho_2)}d\rho_2^2\right], \tag{18}$$

in which one has set:

$$f(\rho) = (a-\rho)(b-\rho)(c-\rho),$$

to abbreviate. Upon setting $\rho_2 = 0$, one will recover formula (15).

The plane $z = 0$ will correspond to the hypothesis that $\rho_2 = c$.

Having said that, consider any of the ellipsoids of the orthogonal system and make an arbitrary point $(\rho, \rho_1)$ of that ellipsoid correspond to the point of the $xy$-plane that has the elliptic coordinates $\rho'$, $\rho_1'$ which are quantities that are defined by the following two equations:

$$\int_b^{\rho'} \frac{d\rho'}{\sqrt{(a-\rho')(\rho'-b)}} = \int_b^{\rho} \frac{\sqrt{\rho-\rho_2}\,d\rho}{\sqrt{(a-\rho)(b-\rho)(c-\rho)}},$$

$$\int_c^{\rho_1'} \frac{d\rho_1'}{\sqrt{(a-\rho_1')(b-\rho_1')}} = \int_c^{\rho_1} \frac{\sqrt{\rho_1 - \rho_2}\, d\rho}{\sqrt{(a-\rho_1)(b-\rho_1)(\rho_1 - c)}}.$$

It results from formula (18) that the line elements of the two surfaces will be proportional. If $ds$, $ds'$ denote the corresponding arc lengths on the ellipsoid and the plane, resp., then one will have:

$$\frac{ds^2}{ds'^2} = \frac{\rho - \rho_1}{\rho' - \rho_1'}.$$

The correspondence that is established then gives a new geographic trace of the ellipsoid on the plane, which is a trace that one can characterize by remarking that if the ellipsoid is flattened while remaining constantly homofocal to itself then the region that is represented will coincide with the chart itself in the limit.

**123.** Let us return to the general properties of isothermal systems. If one knows just one of those systems on an arbitrary surface then one can trace not only other isothermal systems on that surface, but also an infinitude of orthogonal systems. Indeed, knowing just one isothermal system will permit one to make a chart of the surface on the plane with preservation of the angles and similitude of infinitely-small elements. Consequently, any orthogonal system that is traced in the plane will correspond to an orthogonal system on the surface. Moreover, if that system is isothermal and divides the plane into infinitely-small squares then the property that belongs to the corresponding system on the surface will be that is it isothermal, as a result.

It results from this that we can confine ourselves to a planar surface for the study of questions that relate to the substitution of one isothermal system for another. Let $X$, $Y$ denote the rectangular coordinates of a point in the plane, and set:

$$Z = X + iY, \qquad Z' = X - iY.$$

The line element in the plane will have the expression:

$$dS^2 = dZ\, dZ', \tag{19}$$

and if one similarly lets $z$, $z'$ denote the complex variables:

$$z = x + iy, \qquad z' = x - iy$$

then the formulas that permit one to pass to the most general isothermal system will be:

$$Z = f(z), \qquad Z' = f_1(z') \tag{20}$$

or

$$Z = f(z'), \qquad Z' = f_1(z). \tag{21}$$

If one prefers that the new variables $x$, $y$ should be real, along with the old ones, then it is obviously necessary that the functions $f$, $f_1$ must be conjugate imaginary in both cases. In order to geometrically study the preceding formulas, we consider them as defining a method of transformation that will make a point $m$ ($x$, $y$) in a plane correspond to another point $M$ ($X$, $Y$) in that same plane.

We already know that the two methods of transformation that are defined by formulas (20) or (21) preserve the angles and assure the similitude of infinitely-small corresponding elements. However, there is an essential distinction to be made between the two systems of formulas here:

Suppose that the point $m$ describes a certain curve; let $ds$ be the differential of the arc length of that curve, and let $\omega$ be the angle that its tangent makes with the $x$-axis. The formulas:

$$dx = ds \cos \omega, \qquad dy = ds \sin \omega$$

give us:

(22) $$dz = ds\, e^{i\omega}, \qquad dz' = ds\, e^{-i\omega}.$$

Similarly, let $dS$ and $\Omega$ denote the analogous quantities that relate to the curve that is described by the point $M$ ($X$, $Y$). One will likewise have:

(23) $$dZ = dS\, e^{i\Omega}, \qquad dZ' = dS\, e^{-i\Omega}.$$

We now refer to the formulas of the first system (20). They give:

$$dZ = f'(z)\, dz, \qquad dZ' = f_1'(z')\, dz' ;$$

one then deduces, upon multiplying, that:

(24) $$dS^2 = f'(z)\, f_1'(z')\, ds^2,$$

and upon dividing:

(25) $$e^{2i\Omega} = \frac{f'(z)}{f_1'(z')}\, e^{2i\omega}.$$

Hence, if one considers two corresponding points $m$, $M$ of the two figures then the tangents at those points to two arbitrary corresponding curves must define a constant angle between them. As a result, two curves of the first figure that cross at $m$ will correspond to two curves in the second one that cross at $M$, and will define an angle that that is not only equal to that of the first two curves, but will also have the same sense of rotation. If the point of the first figure describes a small closed curve around the point $m$ then the corresponding point of the second figure will also describe a closed curve around $M$, and furthermore, the two corresponding curves will be traversed in the same sense.

The same thing will no longer be true when one employs formulas (21). Indeed, the transformation that it defines reduces to the one that corresponds to formulas (20), when preceded or followed by a rotation of 180° around the $x$-axis. Consequently, under the second transformation, the sense of rotation of all angles and all traversals will be changed.

**124.** We content ourselves by studying formulas (20) and considering only the real isothermal systems; i.e., the ones for which the functions $f$, $f_1$ are conjugate imaginary. We can then say that any function of the complex argument $z$ will correspond to an isothermal system, and we will have the following proposition, which we shall use frequently:

*The planar curves that one obtains by equating to zero the real and imaginary parts of an arbitrary function $f(z)$ of the complex variable $z = x + yi$ define an isothermal orthogonal system. Moreover, the formula:*

$$Z = f(z),$$

*which gives two real equations, will define a method of transformation with preservation of the magnitude and sense of rotation of the angles.*

For example, consider the function:

$$f(z) = \frac{k^2}{z}; \tag{26}$$

one will have:

$$X + Yi = k^2 \frac{x - yi}{x^2 + y^2}$$

or

$$X = \frac{k^2 x}{x^2 + y^2}, \qquad Y = \frac{-k^2 y}{x^2 + y^2}.$$

These are the formulas of the transformation by reciprocal radius vectors in which one changes $y$ into $-y$. The latter transformation will be defined by the relation:

$$Z = \frac{k^2}{z'},$$

which is attached to formulas (21). Consequently, as the name *inversion* implies (which is what it is often called), it will belong to the general group of conformal representations that change the sense of rotation of the angles and the traversals.

The transformation that is defined by the formula:

$$Z = \frac{az + b}{cz + d},$$

in which $a$, $b$, $c$, $d$ are constants, approximates an inversion, in that it makes a circle correspond to a circle. However, it is distinguished by the fact that it preserves the senses of rotation of the angles and of traversal; i.e., it assures the direct similitude of the infinitely-small elements. That transformation plays an important role in the research that relates to the modern theory of functions. In order to distinguish it from an inversion, in the next chapter, we shall give it the name of *circular transformation*. Moreover, it reduces to the one that is defined by formulas (26), when it is preceded and followed by a translation; it can then also be replaced by an even number of inversions.

The general theorem that we have stated can take a very important new form. Suppose that one no longer applies it to $f(z)$, but to $\log f(z)$. The real part of that logarithm is the logarithm of the modulus of $f(z)$ and the imaginary part of the argument of $f(z)$. One will then obtain this new proposition:

*If one is given an arbitrary function of the complex variable z then the curves of equal modulus and equal argument of that function will define an isothermal orthogonal system.*

**125.** The isothermal orthogonal systems that are defined by the preceding two theorems play a very important role in mathematical physics. I will point out the following applications:

Let $A_k$ and $M$ be two points that correspond to values $a_k$ and $z$ of the complex variable. One will have:

$$z - a_k = \rho_k\, e^{i\theta_k},$$

in which $\rho_k$ denotes the length of the segment $A_kM$, and $\theta_k$ denotes the angle that the segment makes with the $x$-axis. Similarly, one will have:

$$z - b_k = \rho'_k\, e^{i\theta'_k}$$

for a point $B_k$ that corresponds to the value $b_k$ of the complex variable, in which $\rho'_k$ denotes the length of $B_kM$, and $\theta'_k$ denotes the angle that the radius vector makes with the $x$-axis.

If one considers the function:

$$f(z) = \prod_{k=1}^{n} \frac{z - a_k}{z - b_k}$$

then the curves of equal modulus will have the equation:

$$\frac{\rho_1\,\rho_2\cdots\rho_n}{\rho'_1\,\rho'_2\cdots\rho'_n} = \text{const.},$$

and the curves of equal argument will have:

$$\theta_1 - \theta'_1 + \theta_2 - \theta'_2 + \ldots + \theta_n - \theta'_n = \text{const.}$$

That gives the following theorem, which is easy to generalize:

*If one considers two groups of n poles* $A_1, A_2, \ldots, A_n$ *and* $B_1, B_2, \ldots, B_n$ *then the curves that are the loci of points for which the product of the distances to the first poles* $A_i$ *is proportional to the product of the distances to the n poles* $B_i$ *will have orthogonal trajectories that consist of curves that the loci of points from which one will see the n segments* $A_i B_i$ *form angles whose sum is constant.*

The theorems will be further applicable in the case where the two groups do not contain the same number of poles, provided that one introduces a multiple pole that is situated at infinity in an arbitrary, but well-defined, direction into the group that contains the least points.

For example, the Cassinian, which is the locus of points such that the product of their distances to two fixed foci $F$, $F'$ is constant, admits orthogonal trajectories that take the form of the curves that are the loci of the points $M$ such that the sum of the angles that are defined by the radius vectors $MF$, $MF'$ and the $x$-axis are constant. The latter curves are equilateral hyperbolas that pass through the foci $F$, $F'$.

In order to prove that, it will suffice to consider the curves of equal modulus and equal argument, that relate to the function:

$$z^2 - c^2.$$

**126.** Now, consider the integral:

$$\int_0^z \frac{dz}{\sqrt{c^2 - z^2}},$$

in which $c$ denotes a real, positive constant, and look for the isothermal orthogonal system that is defined by the curves on which the real part or the imaginary part of that function remains constant.

Set:

$$(27) \qquad z = c \cos(\alpha + \beta i), \qquad \sqrt{c^2 - z^2} = -c \sin(\alpha + \beta i).$$

The integral will have the value $\alpha + \beta i$, and the two families of the isothermal system will be defined by the equations:

$$\alpha = \text{const.}, \qquad \beta = \text{const.}$$

Let $F$, $F'$ be the points of the plane that are affixed to $c$ and $-c$. If $M$ denotes the point that is affixed to $z$ then one will have:

$$(28) \qquad z - c = FM\, e^{i\widehat{MFx}}, \qquad z + c = F'M\, e^{i\widehat{MF'x}},$$

and consequently:

$$\begin{aligned} FM &= r = \operatorname{mod}(z - c) = c \operatorname{mod}[\cos(\alpha + \beta i) - 1] = c\,[\cos \beta i - \cos \alpha], \\ F'M &= r' = \operatorname{mod}(z + c) = c \operatorname{mod}[\cos(\alpha + \beta i) + 1] = c\,[\cos \beta i + \cos \alpha]; \end{aligned}$$

one deduces from these equations that:

$$r + r' = 2c \cos \beta i,$$
$$r - r' = 2c \cos \alpha .$$

One sees that the curves of the family ($\beta$) are ellipses that admit $F$, $F'$ for foci; the curves of the family ($\alpha$) are the homofocal hyperbolas.

On the other hand, if we, with Weierstrass, employ the symbol $\mathcal{R}$ to indicate the real part of a function then the equation of the ellipses will obviously be:

$$\mathcal{R}\, i \int \frac{dz}{\sqrt{c^2 - z^2}} = -\beta,$$

and their differential equation will be:

$$\mathcal{R}\, \frac{i\, dz}{\sqrt{z^2 - c^2}} = 0.$$

The use of formulas (22) and (28) will permit us to transform that equation and give it the form:

$$\mathcal{R}\, \frac{ds}{\sqrt{rr'}} e^{i\left(\omega - \frac{\widehat{MF\,x} + \widehat{MF'\,x}}{2}\right)} = 0,$$

in which $\omega$ denotes the angle between the tangent to the curve and the $x$-axis. One will then has:

$$\omega = \frac{\pi}{2} + \frac{\widehat{MF\,x} + \widehat{MF'\,x}}{2},$$

which will give the well-known construction of the tangent to the ellipse.

We will conclude these applications (which go on *ad infinitum*) by taking the function:

$$f(z) = \int \frac{dz}{\sqrt{z(z-c)\left(z - \frac{c}{k^2}\right)}},$$

in which $c$ and $k^2$ denote two real, positive constants.

The curves that are obtained by equating the real and imaginary parts of that function to zero define an algebraic isothermal system that one can define as follows:

Set:

$$f(z) = \frac{2k}{\sqrt{c}}(\alpha + \beta i),$$

so one will have:

$$(29)\qquad \begin{cases} z = \ \ c \ \operatorname{sn}^2(\alpha+\beta i), & z' = \ \ c \ \operatorname{sn}^2(\alpha-\beta i), \\ z - c = - \ c \ \operatorname{cn}^2(\alpha+\beta i), & z' - c = - \ c \ \operatorname{cn}^2(\alpha-\beta i), \\ z - \dfrac{c}{k^2} = -\dfrac{c}{k^2}\operatorname{dn}^2(\alpha+\beta i), & z' - \dfrac{c}{k^2} = -\dfrac{c}{k^2}\operatorname{dn}^2(\alpha-\beta i), \end{cases}$$

and consequently if one lets $r$, $r'$, $r''$ denote the distances from the point ($x$¸ $y$) to the three point on a straight line:

$$z = 0, \quad z = c, \quad z = \frac{c}{k^2},$$

respectively, then one will have:

$$(30)\qquad \begin{cases} r = \ c \ \operatorname{sn}(\alpha+\beta i)\operatorname{sn}(\alpha-\beta i), \\ r' = \ c \ \operatorname{cn}(\alpha+\beta i)\operatorname{cn}(\alpha-\beta i), \\ r'' = \dfrac{c}{k^2}\operatorname{dn}(\alpha+\beta i)\operatorname{dn}(\alpha-\beta i). \end{cases}$$

Write down the well-known formulas:

$$\operatorname{cn} x \operatorname{cn}(x + a) + \quad \operatorname{dn} a \operatorname{sn} x \operatorname{sn}(x + a) = \operatorname{cn} a,$$
$$\operatorname{cn} x \operatorname{dn}(x + a) + k^2 \operatorname{cn} a \operatorname{sn} x \operatorname{sn}(x + a) = \operatorname{dn} a,$$

which relate to the addition of elliptic functions. If we replace $x$ and $x + a$ with the following values:

$$x = \alpha + \beta i, \qquad x + a = \alpha - \beta i$$

then we will find, upon taking formulas (30) into account, that:

$$(31)\qquad \begin{cases} r' + r\operatorname{dn}(2\beta i) = \ c \ \operatorname{cn}(2\beta i), \\ r'' + r\operatorname{cn}(2\beta i) = \dfrac{c}{k^2}\operatorname{dn}(2\beta i). \end{cases}$$

If one then replaces $x$ and $x + a$ in the same formulas with the following values:

$$-x = \alpha + \beta i, \qquad x + a = \alpha - \beta i$$

then that will give the two new relations:

(32)
$$\begin{cases} r' - r\,\mathrm{dn}(2\alpha) = \;c\;\mathrm{cn}(2\alpha), \\ r'' - r\,\mathrm{cn}(2\alpha) = \dfrac{c}{k^2}\mathrm{dn}(2\alpha). \end{cases}$$

Equations (31) and (32) define two isothermal families.

As one would expect, these two families are represented by the same equation, but with different values for the arbitrary constant. They consist of Descartes ovals that have the three points:

$$z = 0, \quad z = c, \quad z = \frac{c}{k^2}$$

for their common foci.

The double equation that is obtained for each family exhibits a beautiful property of ovals that was given by Chasles in his *Aperçu historique*, pp. 352:

The differential equation that is common to the two families of ovals is obviously:

(33)
$$\frac{dz}{\sqrt{z(z-c)\left(z-\dfrac{c}{k}\right)}} \pm \frac{dz'}{\sqrt{z'(z'-c)\left(z'-\dfrac{c}{k}\right)}} = 0.$$

It then leads to a very simple geometric construction of the tangents to the two ovals that pass through a point $M$ of the plane. The angle between one of those tangents and the focal axis will be one-half the sum of the angles that the three radius vectors that are drawn from the point $M$ to the three foci make with that axis. That sum will be defined only up to a multiple of $\pi$, so the construction will in fact give two rectangular tangents.

**127.** A family of isothermal curves can be defined by an equation of the form:

$$\lambda = f(z) + f_1(z'),$$

in which the parameter $\lambda$ of that family satisfies the partial differential equation:

(33)
$$\frac{\partial^2 \lambda}{\partial z\, \partial z'} = 0,$$

and conversely, any function $\lambda$ that verifies that equation will give an isothermal family. That remark will permit us to treat the following problem:

Determine all isothermal families that are composed of circles. If one writes the equation of a circle with variable $z, z'$ in the form:

(34)
$$zz' + az + bz' + c = 0$$

then one will obtain the most general family of circles by taking $a$, $b$, $c$ to be arbitrary functions of one parameter $\lambda$. If one demands that the family should be isothermal then it will be necessary that the function $\lambda$ should satisfy equation (33). One will then be led, by a simple calculation, to the equation of condition:

$$(35) \qquad (ab - c)(a''z + b''z + c'') - (ab' + ba' - c')(a'z + b'z' + c') = 0,$$

in which $a'$, $b'$, $c'$; $a''$, $b''$, $c''$ denote the first and second derivatives of $a$, $b$, $c$ with respect to $\lambda$, resp., and which must be a consequence of equation (34). Since equation (35) has degree one only with respect to $z$ and $z'$, it must be verified identically, and one will have:

$$\frac{a''}{a'} = \frac{b''}{b'} = \frac{c''}{c'} = \frac{ab' + ba' - c'}{ab - c}.$$

Upon neglecting the last ratio in this and integrating, one will deduce that:

$$a = l\,c + l_0\,,$$
$$b = mc + m_0\,,$$

in which $l$, $l_0$, $m$, $m_0$ denote constants. Equation (34) will then take the form:

$$z\,z' + l_0\,z + m\,z' + c\,(l\,z + m\,z' + 1) = 0$$

and will necessarily represent a family of circles that pass through two distinct or coincident points. It is now pointless to continue the calculations and to determine the expression for $c$ as a function of $l$ because one knows that all of the families of circles that pass through two distinct or coincident points can be deduced by an inversion from one family of parallel lines, or concurrent lines, or concentric circles, and will consequently be isothermal. Moreover, their orthogonal trajectories, which are circles, likewise constitute an isothermal family that is conjugate to the first one. We recover that last property as a particular case of a general theorem that relates to geodesic circles that are traced on an arbitrary surface.

In his paper "Sur la construction des cartes géographiques," which was published in 1779 ([15]), Lagrange studied a beautiful question in a detailed manner that one can now answer in a few words. If one considers the Earth to be a sphere or a spheroid of revolution then Lagrange proposed to look for all of the geographic traces in which the meridians and the parallels were represented by arcs of circles. Since the meridians and parallels define two families of isothermal conjugates, the preceding results will lead us to the following proposition, which then gives the complete solution to Lagrange's problem:

*The only geographic traces for which the meridians or the parallels are represented by arcs of circles are the ones for which those two systems of lines are drawn by arcs of circles on the chart. In the case where the Earth is assumed to be spherical, one will obtain all of those traces by combining the stereographic projection or Mercator projection with planar inversions.*

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([15]) LAGRANGE, *Oeuvres complètes*, t. IV, pp. 637.

# CHAPTER IV

## CONFORMAL REPRESENTATION OF PLANAR AREAS

Statement of the problem. – Analytical principle upon which the solution is based. – Conformal representation on the region of the plane that is situated above the real axis of a simply-connected planar area that is bounded by straight lines or by arcs of a circle. – Method of Schwarz. – Application to the planar triangle that is bounded by three arcs of a circle and a spherical triangle.

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**128.** In the preceding chapter, we saw that one can make any function $Z = f(z)$ of the complex variable $z$ correspond to a method of transformation with direct similitude of infinitely-small elements, and we discussed the most elementary properties of the unlimited number of transformations that one can thus obtain. We now propose to study (in a very extensive case) the solution to the following problem:

*If one is given two planar areas* $(A)$, $(A_1)$ *then determine the function* $Z = f(z)$ *that permits one to perform a conformal representation of one of the areas on the other one, in such a manner that a point that is taken from the interior of either of the two areas will correspond to just one point that is taken from the interior of the other one, and points that are taken from the contour of one of the areas will correspond to points that are taken from the contour of the other one.*

The examination of this beautiful question, when given its most general statement, is attached to the solution of the some of the most important problems of analysis and mathematical physics. Riemann showed that it is always possible to solve it in article **21** of his *Inaugural Dissertation* ([16]). Riemann's proof appealed to a postulate to which that illustrious geometer gave the name of *Dirichlet's principle.* Schwarz established Riemann's theorem without employing Dirichlet's principle in various papers, and in particular in an article that was included in the Monatsberichte of the Berlin Academy ([17]). However, the proof of that eminent geometer has not yet been published in full detail.

One must also credit Schwarz ([18]) with some very far-reaching research that relates to the case, which is very important for the theory of minimal surfaces, in which the planar

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([16]) RIEMANN, *Gesammelte mathematische Werke*, pp. 39.

([17]) H.-A. SCHWARZ, "Ueber die Integration der partiellen Differentialgleichung $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ unter vorgeschriebenen Grenz- und Unstetigkeits-Bedingungen," Monatsberichte der Berliner Akademie, October 1870, pp. 767.

([18]) H.-A. SCHWARZ, "Ueber einige Abbildungsaufgaben," Crelle's Journal **70** (1869), 105-120.

"Ueber diejenigen Fälle in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt," Crelle's Journal **75** (1872), 292.

areas that one needs to find a conformal representation of are bounded by circular arcs or straight lines. We propose to exhibit the principles of Schwarz's method here.

If one can represent two planar areas ($A$), ($A'$) on a third area ($A''$) then one can obviously refer the one to the other with similitude of the infinitely-small elements. The Riemann problem can then be converted into the following one:

*Represent an arbitrary area* ($A$) *on a well-defined area* ($A''$) – *for example, on the surface of a circle of given radius or on the part of the plane that is found above the x-axis.*

The problem thus-posed is clearly not well-defined, since if one considers, for example, the region ($K$) of the plane that is found above the $x$-axis then it will be easy to see that it can be mapped to itself in an infinitude of ways with similitude of the infinitely-small elements. Indeed, let $z$ denote the complex variable and consider the transformation that is defined by the formula:

$$Z = \frac{az+b}{cz+d}, \tag{1}$$

in which $a$, $b$, $c$, $d$ are real constants. If one lets $z_0$ and $Z_0$ be the conjugate imaginary variables of $z$ and $Z$ then one will have:

$$Z_0 = \frac{a\,z_0+b}{c\,z_0+d},$$

and consequently:

$$Z - Z_0 = \frac{(ab-bc)(z-z_0)}{(c\,z+d)(c\,z_0+d)}.$$

If the determinant $ad - bc$ is positive then the transformation, which will make real values of $Z$ correspond to real values of $z$, will also make values of $z$ with positive imaginary parts correspond to values of $Z$ that enjoy the same property. In other words, it will constitute a conformal representation of the region ($K$) on itself. One easily proves – either by analysis or geometry – that it is always possible to find a transformation of that type that makes three given points on the $x$-axis correspond to three other, likewise-given, points on that axis, or which makes an arbitrary given point in the interior of ($K$) correspond to another, likewise-given, point in the interior, along with making a point on the $x$-axis correspond to another point on the that axis.

If one assumes (since it is possible to prove this) that the transformation that is defined by formula (1) is the most general one that realizes the conformal representation of the region ($K$) onto itself then one will see that the Riemann problem can be converted into the following one, which will be perfectly well-defined:

*If one is given a simply-connected area* ($A$) *then represent it on the upper part* ($K$) *of the plane in a conformal manner such that three points that are taken on the contour of the area* ($A$) *will correspond to three given points on the x-axis.*

**129.** Let:

$$Z = f(z)$$

denote the function of a complex argument that gives the solution to the problem. We shall enumerate the conditions to which it is subject:

1. It must be uniform and continuous for all values of $z$ that are represented by points in the interior of the area ($K$). If $z_0$ denotes one of those values then that function must be developable into a positive, integer powers of $z - z_0$ in a neighborhood of $z_0$ .

2. The derivative $f'(z)$ cannot be zero for any point $z_0$ that is found inside of the area ($K$), because if the derivative were zero for $z = z_0$ then there would have to be at least two points in a neighborhood of $z_0$ for which the function $Z$ will have the same value, and consequently a point in the area ($A$) would correspond to several points in the area ($K$), which would contradict the hypothesis.

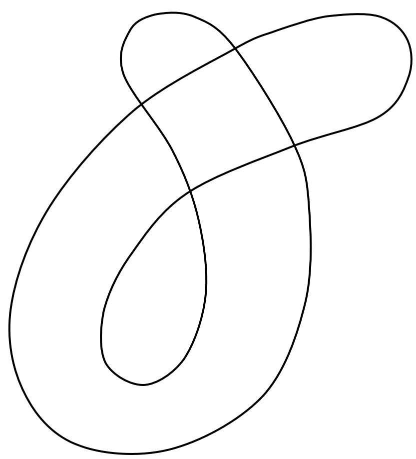

Figure 7.

3. The function $Z$ must not cease to be continuous for real values of $z$, which are represented by points on the real axis. However, one does suppose that $Z$ is necessarily developable in positive, integer powers of $z - z_0$ at those points, because it is defined only for values of $z$ whose imaginary part is positive in their neighborhood.

4. Finally, when $z$ is considered to be a function of $Z$, it must satisfy the same conditions as $Z$, when it is considered to be a function of $z$; i.e., it must be a uniform, continuous function of $Z$ in a neighborhood of the contour of the area ($A$) that takes on real values when the point $Z$ is located on the contour.

Conversely, if a function $Z$ satisfies all of those conditions then one can easily prove that it will give the solution to the problem. More generally, if one finds a function $Z$ that is defined in an arbitrary simply-connected area ($A$) that is uniform inside of that area and satisfies the conditions that we just stated then it will yield a conformal representation of ($A$) on an area ($A'$) that is simply-connected, like the first one, but which must cover the plane more than one time in certain subsets when the function $Z$ takes the same values several times inside of ($A$). (See Fig. 7.)

After having pointed out the conditions that the function $Z$ must satisfy, we shall discuss how Schwarz gave a means of determining that function in the case where the area ($A$) is bounded by lines or circular arcs.

**130.** Here is the analytical principle upon which the solution is based ([19]):

Consider a function $Z$ of $z$ that is defined only on the upper part of the plane and satisfies all of the conditions that were listed in the preceding number, moreover. If the function is real for all real values of $z$ that are close to a real value $z_0$ then it will be developable in a neighborhood of $z_0$ into a series that is ordered by positive, integer powers of $z_0$, and the coefficients of the series will all be real.

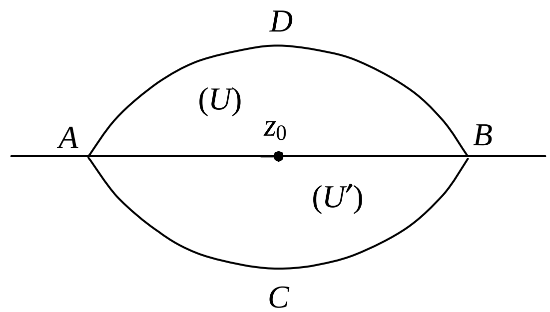


Figure 8.

Indeed, consider (Fig. 8) an area ($U$) that is bounded by the contour $Az_0BD$ and is found entirely in the upper part of the plane, and let ($U'$) be the area that is symmetric to the latter one with respect to the $x$-axis. By hypothesis, the function $Z$ is known only for the upper part of the plane. However, one can also define it for the lower part by agreeing that two conjugate imaginary values of the variable (which are represented by two points that are located symmetrically with respect to $Ox$) will correspond to two conjugate imaginary values of the function. That *analytic continuation* of the function obviously preserves continuity, since the function $Z$ is real and continuous for real values of $z$, by definition.

Since the function is then defined in the interiors of the areas ($U$) and ($U'$), consider the two integrals:

$$\frac{1}{2\pi i}\int_{(U)}\frac{Z\,dz}{z-\zeta}, \qquad \frac{1}{2\pi i}\int_{(U')}\frac{Z\,dz}{z-\zeta},$$

which are taken along the contours $ABDA$, $ACBA$ of the two areas. From Cauchy's theorem, the first of those integrals will be equal to $f(\zeta)$, while the second one will be

([19]) This principle was stated and proved by Schwarz in an article that was cited above (Crelle's Journal, v. **70**, pp. 107). It was also employed by Riemann in a paper on minimal surfaces (*Gesammelte Werke*, pp. 297).

zero when the point $\zeta$ is found in the interior of the area $(U)$. On the contrary, if the point $\zeta$ is found in the interior of $(U')$ then the result will be the opposite; i.e., the first integral will be zero, and the other one will be equal to $f(\zeta)$ ([20]). Hence, whenever the point $\zeta$ is found inside of the area $(U) + (U')$, the sum of the two integrals will be equal to $f(\zeta)$. However, if one adds the two integrals, the parts that relate to the common portion of the contour $AB$ will mutually cancel, since they are equal an opposite, and all that will remain is the integral:

$$\frac{1}{2\pi i}\int \frac{Z\,dz}{z-\zeta},$$

which is taken along the contour $ACBDA$. Now, one knows (and this is obvious) that the integral is developable into a series in a neighborhood of all points interior to the contour, and in particular, for real values of $\zeta$, which are represented by points on the line $AB$.

In particular, for $\zeta = z_0$, one will have:

$$Z - Z_0 = a\,(z - z_0) + b(z - z_0)^2 + c\,(z - z_0)^3 + \ldots,$$

and since real values of $z$ must correspond to real values of $Z$, the coefficients $a$, $b$, $c$ will all be real. If it happens, moreover, as in the examples that we shall treat, that $z$, when considered to be a function of $Z$, must satisfy the same conditions as $Z$, when it is considered to be a function of $z$, then the preceding equation must give a series of increasing integer powers of $Z - Z_0$ when it is solved for $z - z_0$, and consequently the coefficient $a$ will always be non-zero.

**131.** Now that this preliminary lemma has been established, first consider an area $(A)$ that is bounded by straight lines $(L_1), \ldots, (L_n)$. Let $Z_0$ be affixed to a point that is located on one of the lines $(L)$, and let $h\pi$ be the angle that this line makes with the real axis. Consider the function:

$$(Z - Z_0)\,e^{-ih\pi}.$$

It will have the same properties as the function $Z$ for a point $Z$ that is located inside of the contour. If the point $Z$ is found on the line $(L)$ in the neighborhood of $Z_0$ then it will be real and will change sign when $Z$ passes through the value $Z_0$. It follows from this that one can apply the lemma that was proved in the preceding number and set:

$$e^{-ih\pi}(Z - Z_0) = (z - z_0)\,p\,(z - z_0), \tag{3}$$

in which the symbol $p\,(z - z_0)$ denotes a series that is ordered in positive, integer powers of $z - z_0$ whose coefficients are all real, and the first of them is non-zero.

([20]) There is a slight difficulty here, which we shall be content to point out, and which is, moreover, easy to make disappear, if one assumes that one applies Cauchy's theorem to a function that is not assumed to be developable in a series for points on the contour. One easily recognizes that the theorem will still be applicable whenever the function $f(z)$ is assumed to be continuous in the neighborhood of the contour of the area.

Now, study the function $Z$ in the neighborhood of the value $Z_0$ that corresponds to the point of intersection of two consecutive lines $(L_k)$, $(L_{k+1})$ (Fig. 9) that make an angle of $\alpha\,\pi$ between them.

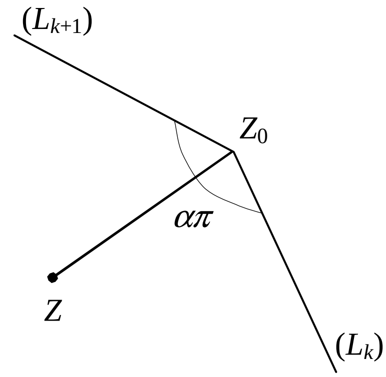


Figure 9.

Consider the function $Z - Z_0$ . Its argument, which is the angle between the line $ZZ_0$ (Fig. 9) and the real axis, varies between the two limits:

$$h_k\,\pi, \qquad h_k\,\pi - \alpha\,\pi$$

when the point $Z$ is displaced inside of the area and is directed from $(L_k)$ to $(L_{k+1})$. It follows from this that the function:

$$[(Z_0 - Z)\,e^{-i\pi h_k}]^{1/\alpha}$$

will be real and positive on the side $(L_k)$, real and negative on the side $(L_{k+1})$, and it will have the same properties as the function $Z$ inside of the area $(A)$, moreover. An application of the preceding lemma will then give us:

$$[(Z_0 - Z)\,e^{-i\pi h_k}]^{1/\alpha} = (z - z_0)\ p\ (z - z_0),$$

in which $p\ (z - z_0)$ has the same significance as the preceding. Upon raising both sides of the equality to the power $\alpha$, one can further write:

$$Z - Z_0 = e^{i\pi h_k}\,(z - z_0)\ p\ (z - z_0). \tag{4}$$

Finally, for all points inside the contour, since the derivative of $Z$ is never zero, one will have:

$$Z - Z_0 = (z - z_0)\ P\ (z - z_0), \tag{5}$$

in which $P\ (z - z_0)$ denotes a series that is analogous to the series $p\ (z - z_0)$, but whose coefficients are not necessarily real.

It remains for us to consider the point of the contour that corresponds to the infinite value of $z$. Since one can always perform the substitution:

$$z = \frac{-1}{z_1},$$

which gives a conformal representation of the upper portion of the plane onto itself, that case will be converted into the preceding one, and one will have:

$$Z - Z_0 = \frac{e^{ih\pi}}{z}\, p\left(\frac{1}{z}\right) \tag{6}$$

when the point is not a summit of the contour, and:

$$Z - Z_0 = \frac{e^{ih\pi}}{z^{\alpha}}\, p\left(\frac{1}{z}\right) \tag{7}$$

when the point is a summit where the two consecutive edges meet at an angle of $\alpha\pi$, when it is measured in the interior of $(A)$.

**132.** The preceding developments embrace all possible hypotheses. In order to eliminate the constants $Z_0$ and $h$, which change value when one passes from one to the other, consider (with Schwarz) the function:

$$\frac{d}{dz}\log\frac{dZ}{dz} = E(z). \tag{8}$$

One finds, by an easy calculation, that:

1. For a point inside of the area:

$$E(z) = P_1(z - z_0). \tag{9}$$

2. For a point that taken from one of the edges of the contour:

$$E(z) = p_1(z - z_0). \tag{10}$$

3. For a summit of the contour that corresponds to the angle $\alpha\pi$:

$$E(z) = \frac{\alpha - 1}{z - z_0} + p_1(z - z_0), \tag{11}$$

in which $P_1(z - z_0)$, $p_1(z - z_0)$ denote powers series, and the series $p_1(z - z_0)$ has real coefficients, moreover.

4. Finally, for the point that corresponds to the infinite value of $z$:

(12) $$E(z) = -\frac{\alpha}{2} + \frac{1}{z^2}\, p_1\left(\frac{1}{z}\right)$$

if that point is not a summit of the contour.

The last three formulas show us that the function $E(z)$ is real for all real values of $z$, and that it can, consequently, be continued analytically by the method that was described in no. **130**. Moreover, from the preceding developments, it has only a limited number of poles, which correspond to the summits of the contour, and it will become infinitely small for infinitely large $z$. From the known theorems of the theory of functions, it will then be a rational fraction.

Let $a, b, c, \ldots, l$ be values of $z$ that correspond to summits of the contour, and let $\alpha\pi$, $\beta\pi$, $\gamma\pi$, …, $\lambda\pi$ be the angles that are defined at those summits, when measured in the interior of the polygon. One will have:

(13) $$E(z) = \sum \frac{\alpha \cdot 1}{z - a} = \frac{d}{dz} \log \frac{dZ}{dz},$$

with the condition that:

$$\sum (\alpha - 1) = -2,$$

which is only the analytical expression of the theorem that relates to the sum of the angles of a polygon.

The integration of equation (13) then gives us:

$$Z = C \int (z-a)^{\alpha-1} (z-b)^{\beta-1} \ldots (z-l)^{\lambda-1}\, dz + C',$$

in which $C$ and $C'$ denote two arbitrary real or imaginary constants. Upon displacing the area ($A$) without changing either its form or its magnitude, one can convert the expression for $Z$ into the form:

(14) $$Z = H \int (z-a)^{\alpha-1} (z-b)^{\beta-1} \ldots (z-l)^{\lambda-1}\, dz,$$

in which $H$ denotes a real constant.

That is the formula that was given by Schwarz ([21]) and Christoffel ([22]).

Since one can take the values of $z$ that correspond to three summits of the polygon arbitrarily (no. **128**), in reality, it will contain $2n - 3$ constants. One can arrange those constants in such a manner as to obtain the conformal representation of an arbitrary polygon, but that essential result is deduced only from the general theorem that proved by Riemann and Schwarz on the conformal representation of arbitrary planar areas, and we know of no well-developed work in which the determination of the constants $a, b, c, \ldots, l, H$ was studied in the case where the polygon is given.

([21]) SCHWARZ, "Ueber einige Abbildungsaufgaben," pp. 114; 1864, 1866.

([22]) CHRISTOFFEL, "Sul problema delle temperature stazionarie e la rappresentazione di una data superficie," Annali di Matematica **1** (1867), pp. 97.

In the case of the triangle, whose form is determined by the values of the angles, the solution is obvious. One will obtain all triangles that are similar to a given triangle upon varying the constant $H$, and one can determine that constant in such a manner as to obtain any of those triangles.

If the polygon is a rectangle then one will have:

$$\alpha = \beta = \gamma = \delta = \tfrac{1}{2}.$$

$Z$ will become an elliptic integral, which one assumes has been converted into the normal form:

$$Z = H \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{15}$$

The sides of the rectangle will be:

$$a = 2HK, \qquad b = HK',$$

in which $K$ and $K'$ denote the complete integrals that enter into the definition of the periods. Consequently, if one sets:

$$q = e^{-\frac{\pi K'}{K}} = e^{-\frac{2\pi b}{a}}$$

then the modulus will be defined by the equation:

$$k = 4\sqrt{q}\left[\frac{(1+q^2)(1+q^4)(1+q^6)\cdots}{(1+q)(1+q^3)(1+q^5)\cdots}\right]^k,$$

which will then give the complete solution to the problem in this particular case.

**133.** We now pass to the examination of the case in which the contour is composed of arcs of circles. For more clarity, we suppose that two consecutive circles are never tangent.

Since one can always transform any of the circles that constitute the contour into a straight line by means of the circular transformation (no. **124**) that is defined by the formula:

$$Z = \frac{aZ_1 + b}{cZ_1 + d}, \tag{16}$$

and similarly, for two consecutive circles of that contour, we can immediately apply the results that were obtained in no. **131**, and we will see that one can always choose the real or imaginary constants $a$, $b$, $c$, $d$ in such a manner that $Z_1$ takes the form (3) on one of the sides and the form (4) at one of the summits of the contour. One will then have the following expressions for $Z$:

1. At an arbitrary point of the contour:

$$Z = \frac{a(z-z_0)\,p(z-z_0)+b}{c(z-z_0)\,p(z-z_0)+d}. \tag{17}$$

2. At a summit where two consecutive circles make an angle of $\alpha\,\pi$, when measured inside the area:

$$Z = \frac{a(z-z_0)^{\alpha}\,p(z-z_0)+b}{c(z-z_0)^{\alpha}\,p(z-z_0)+d}. \tag{18}$$

3. At the point of the contour that corresponds to the value $\infty$ of $z$:

$$Z = \frac{\frac{a}{z}\,p\left(\frac{1}{z}\right)+b}{\frac{c}{z}\,p\left(\frac{1}{z}\right)+d} \tag{19}$$

if the point is not one of the summits of the contour, and:

$$Z = \frac{\frac{a}{z^k}\,p\left(\frac{1}{z}\right)+b}{\frac{c}{z^k}\,p\left(\frac{1}{z}\right)+d} \tag{20}$$

if the point is a summit where two consecutive sides make an angle of $\alpha\pi$.

4. Finally, for a point inside of the area, on will have:

$$Z - Z_0 = (z - z_0)\,P\,(z - z_0), \tag{21}$$

as before.

**143.** In these various formulas, $a$, $b$, $c$, $d$ denote real or imaginary constants that have different values according to the development that one considers. Here is the ingenious artifice by which Schwarz eliminated everything that was concerned with those constants:

In a general manner, let:

$$Z = \frac{aT+b}{cT+d}, \quad cZT + dZ - aT - b = 0$$

be a relation between two functions $Z$ and $T$ of one variable $z$. If one eliminates the constants by differentiation then one will be led to the relation:

$$\begin{vmatrix} (ZT)' & Z' & T' \\ (ZT)'' & Z'' & T'' \\ (ZT)''' & Z''' & T''' \end{vmatrix} = 0,$$

which will take the elegant form:

$$\text{(22)} \qquad \frac{d^2}{dz^2}\left(\log\frac{dZ}{dz}\right) - \frac{1}{2}\left(\frac{d}{dz}\log\frac{dZ}{dz}\right)^2 = \frac{d^2}{dz^2}\left(\log\frac{dT}{dz}\right) - \frac{1}{2}\left(\frac{d}{dz}\log\frac{dT}{dz}\right)^2,$$

in which the variables are separated. If one adopts a notation of Cayley ([23]) and sets:

$$\text{(23)} \qquad \{Z, z\} = \frac{d^2}{dz^2}\left(\log\frac{dZ}{dz}\right) - \frac{1}{2}\left(\frac{d}{dz}\log\frac{dZ}{dz}\right)^2$$

then one will have:

$$\{Z, z\} = \{T, z\}.$$

Upon appealing to the preceding results, we shall study the development of the function $\{Z, z\}$ for all points that are located on the interior or the contour of the area ($A$):

1. For an arbitrary point of the contour, one must replace $T$ with the development:

$$(z - z_0)\, p(z - z_0),$$

which enters into formula (17). One will then find a result of the form:

$$\{Z, z\} = h\,(z - z_0) + k\,(z - z_0)^2 + \ldots$$

2. For a summit, the value of $T$ is the one that figures in formula (18):

$$T = (z - z_0)^{\alpha}\, p(z - z_0).$$

An easy calculation will then give us:

$$\{Z, z\} = \frac{1}{2}\frac{1-\alpha^2}{(z-z_0)^2} + \frac{h}{z - z_0} + k + l\,(z - z_0) + \ldots$$

3. For the point of the contour that corresponds to the value $\infty$ of $z$, one will likewise find that:

$$\{Z, z\} = \frac{h}{z^4} + \frac{k}{z^5} + \ldots$$

([23]) CAYLEY, "On the schwarzian derivative and the polyhedral functions," Cambridge philosophical Transactions, March 1880.

when the point is not a summit, and if one employs the corresponding value of $T$ in the development (19). On the contrary, if the point is a summit then one must give $T$ the value $\frac{1}{z^\alpha} p\left(\frac{1}{z}\right)$, which will give:

$$\{Z, z\} = \frac{1}{2}\frac{1-\alpha^2}{z^2} + \frac{h}{z^3} + \frac{k}{z^4} + \ldots$$

We remark, in a general manner, that the coefficients will be real in all of these developments. The function $\{Z, z\}$ will then be real for all real values of $z$. It can be analytically continued by the method in no. **130**, and as a result, it will be defined over the entire extent of the plane.

4. Finally, the derivative $dZ / dz$ will never be zero for a point inside of $(A)$, and $\{Z, z\}$, like $z$, will be a function that is developable for all values of $z$.

The function $\{Z, z\}$ has all of the properties of a rational fraction for finite values of $z$, and will become a rational fraction for infinite $z$ since it will become infinitely small. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the values of $z$ that correspond to the summits of the contour, let $\alpha_1\pi$, $\alpha_2\pi, \ldots, \alpha_n\pi$, be the corresponding angles that are defined by the two consecutive sides. If one then has:

$$\{Z, z\} = \frac{1}{2}\frac{1-\alpha_i^2}{(z-a_i)^2} + \frac{h_i}{z-a_i} + k_i + \ldots$$

then the function:

$$\{Z, z\} - \sum \frac{1}{2}\frac{1-\alpha_i^2}{(z-a_i)^2} - \sum \frac{h_i}{z-a_i},$$

which will remain finite for all finite values of $z$ and will become infinitely small for infinite $z$, will necessarily be equal to zero. Consequently, one will have:

$$\{Z, z\} = \sum \frac{1}{2}\frac{1-\alpha_i^2}{(z-a_i)^2} - \sum \frac{h_i}{z-a_i} = F(z). \tag{24}$$

If a point on the contour that corresponds to an infinite value of $z$ is not a summit then, as we have seen, it will be necessary that the development of the right-hand side in powers of $1 / z$ must begin with the term in $1 / z^4$, which will give the equalities:

$$\left\{\begin{aligned} &\sum h_i = 0, \\ &\sum \left(a_i h_i + \frac{1-\alpha_i^2}{2}\right) = 0, \\ &\sum [\alpha_i^2 h_i + a_i(1-\alpha_i^2)] = 0 \end{aligned}\right. \tag{25}$$

that the constants $h_i$, $\alpha_i$ must satisfy.

On the contrary, the point of the contour that correspond to the value $z = \infty$ is a summit where the angle between the two consecutive sides is $\beta\pi$, so the development will commence with the term $\frac{1-\beta^2}{2z^2}$, which will give only the two relations:

$$(26) \qquad \begin{cases} \sum h_i = 0, \\ \sum \left( a_i h_i + \frac{1-\alpha_i^2}{2} \right) = \frac{1-\beta}{2}. \end{cases}$$

**135.** Once the value of $\{Z, z\}$ has been obtained, the train of reasoning will lead us to consider the third-order equation:

$$(27) \qquad \{Z, z\} = F(z),$$

and we will then attempt to integrate it.

Knowing the origin and properties of that equation will simplify the solution of that problem greatly.

Indeed, from the way that the expression $\{Z, z\}$ was defined, the differential relation:

$$\{Z, z\} = \{Z_1, z\}$$

is *equivalent* to the finite relation:

$$Z = \frac{aZ_1 + b}{cZ_1 + d},$$

and in turn, the knowledge of *just one* particular solution $Z_1$ to equation (27) will entail that of the general integral, which will be given by the preceding formula, in which the constants $a$, $b$, $c$, $d$ can take on arbitrary values. That very remarkable property of equation (27) approximates that of linear equations, and it is easy to show that the integration of that integration can be effectively converted into that of a second-order linear equation.

Indeed, consider a second-order linear equation:

$$(28) \qquad \frac{d^2\theta}{dz^2} + p\frac{d\theta}{dz} + q\,\theta = 0,$$

in which $p$ and $q$ are given functions of $z$, and look for the differential equation that the ratio:

$$(29) \qquad Z = \frac{\theta_2}{\theta_1}$$

of two particular integrals must satisfy. One will find, by an easy calculation, that:

(30) $$\{Z, z\} = 2q - \tfrac{1}{2}p^2 - \frac{dp}{dz}.$$

The equation thus-obtained has the same form as the proposed one (27). In order to determine $q$ by the relation:

(31) $$2q - \tfrac{1}{2}p^2 - \frac{dp}{dz} = F(z),$$

it will suffice to choose $p$ arbitrarily, and the integration of equation (27) will be converted into that of the linear equation (28). If one takes, for example, $p = 0$, then the linear equation will be converted into the following one:

(32) $$\frac{d^2\theta}{dz^2} + \tfrac{1}{2}F(z)\,\theta = 0.$$

Furthermore, one can explain how a certain indeterminacy can remain in regard to the linear equation, since the ratio of the particular integrals will not change when one multiplies one and the other by the same given, but arbitrary, function of $z$.

All of its coefficients and singular point for equation (32) will be real, as well as all of the equations that one obtains by taking the following value for $p$:

$$p = \sum \frac{\beta_i}{z - a_i},$$

in which the constants $\beta_i$ are arbitrary real numbers; all of its integrals will be *regular*, moreover.

Conversely, if one considers *a priori* an arbitrary second-order linear equation that possesses all of those properties then one can establish that the ratios of its integrals will give the conformal representation of an area that is more or less complex and limited by arcs of circles on the upper part of the plane.

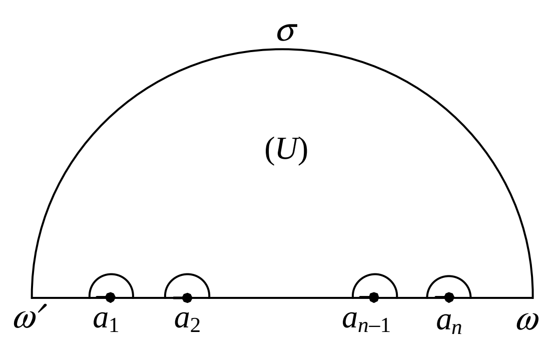


Figure 10.

Indeed, mark out the singular points $a_1, a_2, \ldots, a_{n-1}, a_n$ of the equation on the real axis (Fig. 10), and let:

$$Z = \frac{\theta_2}{\theta_1}$$

be the ratio of two arbitrary particular integrals, which, from the hypotheses that were made in regard to the equation, are uniform functions of $z$ in the area ($U$) that is limited by the real axis and the semi-circle $\omega\sigma\omega'$ of radius infinity. One can obtain two particular integrals in each of the intervals $a_1\,a_2$, $a_2\,a_3$, …, $a_{i-1}\,a_i$, …, $a_n\,\infty\,a_1$ that will both be real for real values of $z$. For example, if one lets $t_{i-1}$ denote the ratio of those two integrals in the interval $a_{i-1}\,a_i$ then one will obviously have:

$$Z = \frac{\alpha_{i-1} + \beta_{i-1}\,t_{i-1}}{\gamma_{i-1} + \delta_{i-1}\,t_{i-1}},$$

in which $\alpha_{i-1}$, $\beta_{i-1}$, $\gamma_{i-1}$, $\delta_{i-1}$ denote real or imaginary constants, and since the variable $t_{i-1}$ will remain real when the point $z$ describes the segment $a_{i-1}\,a_i$, one will see that the point $Z$ describes an arc of the circle. The arcs of the circles that are described by the point $Z$, which then correspond to $n$ intervals, will define a closed polygon in which the consecutive sides will cut at angles whose magnitudes are arbitrary. Two consecutive sides can even be tangent when the developments of the integrals in a neighborhood of a singular point contain logarithms. However, we shall defer the examination of all cases, and the precise definition of the area in which one then obtains a conformal representation to much later.

We content ourselves by remarking that equation (27) indeed contains the number of constants that is necessary if one would like to perform the conformal representation of an arbitrary polygon that is composed of arcs of circles on the upper part of the plane. Indeed, the function $F(z)$ depends upon $3n$ real constants that are linked by three equations (25), and since one can take three of the quantities $a_i$ arbitrarily, moreover, (no. **128**), only $3n - 6$ real parameters will remain. However, one must append six other parameters to them that serve to define the three imaginary constants that figure in the general integral of equation (27). The number $3n$ of real constants thus-obtained is equal to precisely the number of arbitrary parameters upon which a polygon that is defined by $n$ circular arcs will depend.

It results from a general proposition to which Schwarz arrived by a most elegant method in the article that was cited above ([24]) that one can always determine those constants in such a manner as to effectively obtain the solution to the problem that was posed.

**136.** As an application, we propose to determine the conformal representation of a triangle that is defined by three circular arcs. We can always suppose that the three summits of that triangle correspond to the values 0, 1, $\infty$ of $z$. Let $\lambda\pi$, $\mu\pi$, $\nu\pi$ be the angles of the triangle at those three summits. Here, we will have:

([24]) Monatsberichte, (1870) 768-784.

$$\{Z, z\} = \frac{1}{2}\frac{1-\lambda^2}{z^2} + \frac{a_1}{z} + \frac{1}{2}\frac{1-\mu^2}{(1-z)^2} + \frac{a_2}{z-1},$$

and the development in positive powers of 1 / $z$ must begin with the term $\frac{1}{2}\frac{1-\nu^2}{z^2}$, moreover. That condition will determine $a_1$, $a_2$, and one will find that:

$$\{Z, z\} = \frac{1}{2}\frac{1-\lambda^2}{z^2} + \frac{1}{2}\frac{1-\mu^2}{(1-z)^2} + \frac{1}{2}\frac{1-\lambda^2-\mu^2+\nu^2}{z(1-z)}. \tag{33}$$

Now, if one considers the equation:

$$z\,(1-z)\frac{d^2\theta}{dz^2} + [\gamma - (\alpha + \beta + 1)]\,\frac{d\theta}{dz} - \alpha\beta\theta = 0, \tag{34}$$

which defines Gauss's hypergeometric series, then one will easily see, upon applying formula (30), that the ratio of its two integrals will satisfy equation (33) if one takes:

$$\lambda^2 = (1-\gamma)^2, \qquad \mu^2 = (\gamma - \alpha - \beta)^2, \qquad \nu^2 = (\alpha - \beta)^2. \tag{35}$$

One can then express $Z$ as the quotient of two particular integrals of equation (34); those integrals are well-known. One can determine the variation that they experience when one follows an arbitrary path in the plane ([25]), and one will verify that they effectively provide the desired representation.

Among the four systems of values $\alpha$, $\beta$, $\gamma$ that are determined by equations (35), we choose the following one, for example:

$$\begin{cases} \alpha = \frac{1}{2}(1-\lambda-\mu+\nu), \\ \beta = \frac{1}{2}(1-\lambda-\mu-\nu), \\ \gamma = 1-\lambda. \end{cases} \tag{36}$$

The differential equation (34) will admit several particular solutions, among which, we distinguish the following ones:

$$\begin{cases} \theta_1 = F(\alpha, \beta, \gamma, \delta), \\ \theta_2 = z^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, z), \\ \theta_3 = F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-z), \\ \theta_4 = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma+1-\alpha-\beta, 1-z), \end{cases} \tag{37}$$

([25]) KUMMER, "Ueber die hypergeometrische Reihe," Crelle's Journal **15** (1836).
GOURSAT (E.), "Sur l'équation différentielle lineaire qui admet pour intégrale la série hypergéométrique," Annales de l'École Normale (2) **10** (1881), supplement.

in which the symbol $F$ denotes Gauss's hypergeometric series. If one agrees that the arguments of $z$ and $1 - z$ will be taken to be zero when the variable $z$ is real and between 0 and 1 then those integrals will be unambiguously determined for the entire upper region of the plane, and the formulas that one finds on pages 20 and 21 of the beautiful paper by Goursat will permit one to calculate the value for each point of that region. Furthermore, they will satisfy the two equations (which one will find on page 28 of that paper):

$$\left\{ \begin{aligned} \theta_1 &= a\,\theta_3 + b\,\theta_4, \\ \theta_2 &= a'\,\theta_3 + b'\,\theta_4 \end{aligned} \right. \tag{38}$$

in all of the region considered, in which one has:

$$\left\{ \begin{aligned} a &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, & b &= \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \\ a' &= \frac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)}, & b' &= \frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha+1-\gamma)\Gamma(\beta+1-\gamma)}. \end{aligned} \right. \tag{39}$$

Having assumed those things, let $C$ denote a real or imaginary constant and set:

$$CZ = \frac{\theta_2}{\theta_1}. \tag{40}$$

When $z$ varies between 0 and 1, the ratio $\theta_2 / \theta_1$ will be real, and the argument of $Z$ will be constant and equal to that of $1 / C$. The point $Z$ will then describe a segment of the line $OA$ (Fig. 11). On the contrary, if the point $z$ passes through the upper part of the plane to values between 0 and $-\infty$ then $\theta_1$ will remain real, and the argument of $\theta_2$ will become equal to $\pi(1-\gamma)$ or $\pi\lambda$. The argument of $Z$ will then increase to $\pi\lambda$, and since it will still remain constant, the point $Z$ will describe a segment $OB$ that has its origin at $O$ and makes the angle $\lambda\pi$ with $OA$.

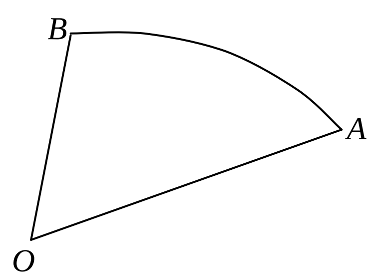

Figure 11.

Now, suppose that $z$ passes through the upper region in the plane to values between 1 and $+\infty$. The integral $\theta_3$ will be real. As for the integral $\theta_1$, it will be imaginary, and since the argument of $1 - z$ becomes equal to $-\pi$, that of the integral will be:

$$-\pi(\gamma-\alpha-\beta) \qquad \text{or} \qquad -\mu\pi.$$

If one sets:

$$\frac{\theta_4}{\theta_3} = e^{-i\mu\pi}\,T$$

then the variable $T$ will be real. Upon dividing corresponding sides of equations (38), one will have:

$$CZ = \frac{a' + b' e^{-i\mu\pi} T}{a + b e^{-i\mu\pi} T}.$$

If one changes $i$ into $-i$, and if one lets $C_0$, $Z_0$ denote the conjugate imaginaries of $C$ and $Z$, resp., then one will find that:

$$C_0\,Z_0 = \frac{a' + b' e^{i\mu\pi} T}{a + b e^{i\mu\pi} T}.$$

All that remains is for us to eliminate $T$ from the two preceding two equations, and we will then obtain the equation:

$$(a'b'\,CC_0\,ZZ_0 + ab)(1 - e^{2i\mu\pi}) + CZ\,(ba' e^{2i\mu\pi} - ab') + C_0Z_0\,(ab'\,e^{2i\mu\pi} - ba') = 0,$$

which represents the circular arc that passes through the points $A$, $B$ and is described by the point $Z$ when $z$ varies between 1 and $+\infty$.

The power $t^2$ of the origin with respect to the preceding circle has the expression:

$$t^2 = \frac{1}{CC_0}\frac{ab}{a'b'},$$

or, upon replacing $a$, $b$, $a'$, $b'$ with their values:

$$(41) \qquad t^2 = \frac{1}{CC_0}\frac{\Gamma^2(1+\lambda)}{\Gamma^2(1-\lambda)}\frac{\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right)}{\Gamma\left(\frac{1+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1+\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)}.$$

The calculation of that power $t^2$ is interesting, because if it is positive then one will describe a circle that has the origin for its center and cuts the side $AB$ at a right angle; i.e., a circle that is orthogonal to the three sides of the triangle $OAB$. On the contrary, if it is negative then no real circle can satisfy those conditions.

Since the arguments of the $\Gamma$ functions that figure in the preceding formula are all greater than $-1$, those functions will have the same sign as the variables that they depend upon. One then sees that the power $t^2$ will be negative if one has:

$$(42)\qquad \begin{cases} \lambda+\mu+\nu>1, \\ \nu+1>\lambda+\mu, \\ \mu+1>\lambda+\nu, \\ \lambda+1>\mu+\nu; \end{cases}$$

i.e., if the angles of the triangle *OAB* satisfy all of the inequality relations that exist between the angles of a spherical triangle. On the contrary, the power $t^2$ will be positive if the preceding inequalities are not all verified. That result indeed conforms to the one that geometry gives: In order for a triangle that is composed of three circular arcs to be the stereographic projection of a spherical triangle, as one knows, it will be necessary and sufficient that the circle that is orthogonal to the three sides of the triangle should be imaginary.

If we choose the following variable:

$$Z_1 = \frac{aZ+b}{cZ+d},$$

in which $a$, $b$, $c$, $d$ denote arbitrary constants, instead of the variable $Z$ that is defined by formula (40), then we will obtain a triangle that has the same angles as the triangle *OAB*, but whose sides will be circular arcs, in general, because it will be derived from the triangle *OAB* by an arbitrary circular transformation.

Suppose that the angles $\lambda$, $\mu$, $\nu$ satisfy the inequality relations (42) and take the value:

$$(43)\begin{cases} \sqrt{CC_0} = \\ \dfrac{\Gamma(1+\lambda)}{\Gamma(1-\lambda)}\sqrt{\dfrac{\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right)}{\Gamma\left(\frac{1+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1+\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)}} \end{cases}$$

for the modulus of $C$.

$t^2$ will become equal to – 1, and the three sides of the triangle *OAB* will be orthogonal to the circle of radius $i$ that has its center at the origin. One knows that, in this case, the three sides can be considered to be the stereographic projections of the three arcs of the great circles that are traced on the sphere of radius 1 that has its center at the origin. Hence, if one represents the variable $Z$ by a point on that sphere using Riemann's method that was discussed in no. **30** then the preceding results will give the conformal representation of the area of a spherical triangle on the upper part of the plane. The summit of that triangle that corresponds to the angle $\lambda\pi$ will be located on the lowest point of the sphere and diametrically opposite to the pole of the stereographic projection.

# CHAPTER V.

## THE ORTHOGONAL SYSTEM THAT IS FORMED BY THE LINES OF CURVATURE.

Differential equations of the lines of curvature. – Application to the surface $x^m\, y^n\, z^p = C$. – Formula of Olinde Rodrigues. – Gauss's spherical representation. – Linear equation whose characteristics are the lines of curvature. – Lines of curvature of cyclides. – Inversion preserves the lines of curvature. – Dupin's theorem that relates to triply-orthogonal systems.

———————

**137.** The properties of isothermal orthogonal systems depend upon only the form of the linear element and are preserved when one deforms the surface without altering the lengths of its arcs. The same thing is no longer true for the orthogonal system that is defined by the two families of lines of curvature. However, that system is distinguished from all others by an essential property: It is both orthogonal and conjugate, so it plays an extremely important role in the examination of a great number of problems that relate to the theory of surfaces, and from all those viewpoints, it merits the fact that we have made a very detailed study of it up to now.

As one knows, a line of curvature can be defined by the property that the normals to the surface at its various points form a developable surface. The edge of regression of that developable is obviously one of the developments of the line of curvature, so the point of contact of each normal with the edge of regression will be the center of principal curvature that corresponds to the line of curvature being considered. We shall first show how one obtains the differential equations of the lines of curvature.

**138.** Let $x$, $y$, $z$ be the rectangular coordinates of an arbitrary point of the surface in question, while $u$, $v$, $w$ are quantities that are proportional to the direction cosines of the normal at that point. The coordinates $X$, $Y$, $Z$ of an arbitrary point of the normal will have the expressions:

$$X = x + u\lambda, \qquad Y = y + v\lambda, \qquad Z = z + w\lambda, \tag{1}$$

in which is $\lambda$ an arbitrary number whose variation will give all points of the normal. We express the idea that there exists a displacement for which that point describes a curve that is tangent to the normal; we will have the equations:

$$\frac{d(x+u\lambda)}{u} = \frac{d(y+v\lambda)}{v} = \frac{d(z+w\lambda)}{w},$$

or, more simply, upon substracting $d\lambda$, three equal ratios:

$$\frac{dx+\lambda\, dv}{u} = \frac{dy+\lambda\, dv}{v} = \frac{dz+\lambda\, dw}{w}. \tag{2}$$

The elimination of $\lambda$ will give us the differential equation:

$$\begin{vmatrix} dx & du & u \\ dy & dv & v \\ dz & dw & w \end{vmatrix} = 0, \tag{3}$$

which is that of the lines of curvature. Upon developing it, one will find that:

$$du\,(v\,dz - w\,dy) + dv\,(w\,dx - u\,dz) + dw\,(u\,dy - v\,dx) = 0. \tag{4}$$

That equation determines the directions of the two lines of curvature that pass through each point of the surface. Formulas (2) exhibit the value of $\lambda$ that relates to each line of curvature, and the corresponding center of curvature will then be defined by formulas (1).

**139.** One will also be led to equation (4) if one employs another method that is based exclusively upon the use of the *coordinates* of the normal. One knows that Plücker considered the straight line to be a spatial element that he defined by its coordinates, as one does for a point or a plane. We shall point out the system of determination that leads to the most symmetric calculations.

Let:

$$\begin{cases} bz - cy + a' = 0, \\ cx - az + b' = 0 \end{cases} \tag{5}$$

be the equation of a straight line. One can append the following equation:

$$ay - bx + c' = 0 \tag{5′}$$

to these equations, provided that $c'$ is determined by the equation:

$$aa' + bb' + cc' = 0. \tag{6}$$

Equations (5), (5′) represent the projections of the line onto the three coordinate planes. That line is determined perfectly when one knows that six quantities $a$, $a'$, $b$, $b'$, $c$, $c'$. We say that these six quantities, which must always satisfy the condition (6), are the *homogeneous coordinates* of the straight line.

Suppose that these six coordinates are given functions of one parameter. The line will generate a ruled surface. In order for that surface to be developable, it is necessary that there should exist a curve that is tangent to all positions of the line; in other words, it is necessary that one must be able to determine the coordinates $x$, $y$, $z$ of a variable point that verifies the equations of the line and satisfies the conditions:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}.$$

If one differentiates equations (5) and (5′), while taking into account the preceding relations, then one will find that the coordinates $x'$, $y'$, $z'$ must verify the three equations:

$$(7) \qquad \begin{cases} z\,db - y\,dc + da' = 0, \\ x\,dc - z\,da + db' = 0, \\ y\,da - x\,db + dc' = 0, \end{cases}$$

which do not contain $dx$, $dy$, $dz$. If one adds them, after multiplying them by $da$, $db$, $dc$, respectively, then one will get the condition:

$$(8) \qquad da\,da' + db\,db' + dc\,dc' = 0$$

that the coordinate differentials must satisfy. One easily proves that this condition, which is necessary, is also sufficient, and when it is fulfilled, formulas (5) and (7) will exhibit the point of contact of the generator with the edge of regression for each value of the independent variable.

In the case that we are treating, the equations of the normal are:

$$\frac{X-x}{u} = \frac{Y-y}{v} = \frac{Z-z}{w}.$$

Consequently, the six coordinates of the normal are $u$, $v$, $w$, and the quantities $u'$, $v'$, $w'$, which are defined by the equalities:

$$(9) \qquad \begin{cases} vz - wy + u' = 0, \\ wx - uz + v' = 0, \\ uy - vx + w' = 0. \end{cases}$$

The condition for the normal to generate a developable surface will then be expressed by the equation:

$$(10) \qquad du\,du' + dv\,dv' + dw\,dw' = 0,$$

which one easily recognizes to be equivalent to equation (4).

**140.** In order to give an example of the preceding method, we propose to determine the lines of curvature of the surface:

$$(11) \qquad x^m\,y^n\,z^p = C,$$

in which $m$, $n$, $p$, $C$ are arbitrary constants. The differential equation of the surface will be:

$$m\,\frac{dx}{x} + n\,\frac{dy}{y} + p\,\frac{dz}{z} = 0.$$

Formulas (1) will become:

$$X = x - \frac{m\lambda}{x}, \quad Y = y - \frac{n\lambda}{y}, \quad Z = z - \frac{p\lambda}{z}$$

here, and equations (2) will give:

(12)
$$\frac{dx\left(\lambda + \dfrac{x^2}{m}\right)}{x} = \frac{dy\left(\lambda + \dfrac{y^2}{n}\right)}{y} = \frac{dz\left(\lambda + \dfrac{z^2}{p}\right)}{z}.$$

If we substitute the values of $dx$, $dy$, $dz$ in the differential equation for the surface then we will have the second-degree equation:

(13)
$$\frac{m}{\lambda + \dfrac{x^2}{m}} + \frac{n}{\lambda + \dfrac{y^2}{n}} + \frac{p}{\lambda + \dfrac{z^2}{p}} = 0,$$

which will exhibit the values of $\lambda$ that correspond to the two lines of curvature.

Instead of making each root of that equation correspond to the line of curvature whose direction is defined by formulas (12), one can consider the perpendicular line of curvature. Upon letting $dx$, $dy$, $dz$ denote the differentials that relate to that second line, one will have the equation:

(14)
$$\frac{x\,dx}{\lambda + \dfrac{x^2}{m}} + \frac{y\,dy}{\lambda + \dfrac{y^2}{n}} + \frac{z\,dz}{\lambda + \dfrac{z^2}{p}} = 0,$$

which will determine the ratios of $dx$, $dy$, $dz$ when it is combined with the differential equation of the surface. Thus, in order to obtain the differential equations of the two families of lines of curvature, it will suffice to replace $\lambda$ in equation (13) with the two roots of equation (13) in succession.

Having done that, the integration will be easy. Indeed, multiply equation (14) by 2 and add it to equation (13), when it is multiplied by $d\lambda$. We will then obtain an exact differential:

$$d\left[m\,L\left(\lambda + \frac{x^2}{m}\right) + n\,L\left(\lambda + \frac{y^2}{n}\right) + p\,L\left(\lambda + \frac{z^2}{p}\right)\right] = 0,$$

and upon integrating this, we will have:

(15)
$$u = \left(\lambda + \frac{x^2}{m}\right)^m \left(\lambda + \frac{y^2}{n}\right)^n \left(\lambda + \frac{z^2}{p}\right)^p,$$

in which $u$ denotes the parameter of the line of curvature. In order to obtain both families, one must successively replace $\lambda$ with the two roots of equation (13). If one now remarks that equation (13) is obtained by setting the derivative of equation (15) with respect to $\lambda$ equal to zero then one will be led to the following theorem:

*If one considers u to be a constant and $\lambda$ to be a variable parameter in equation* (15) *then the envelopes of the surfaces that are represented by that equation will correspond to the various values of u that cut the proposed surface along its lines of curvature.*

One sees that these lines of curvature will be algebraic whenever the proposed surface is – i.e., whenever $m$, $n$, $p$ are commensurable. Furthermore, one recognizes by an easy calculation that the family of surfaces is represented by equation (11), in which one gives $C$ all possible values, and that the two families of envelopes that are in question in the preceding statement will form a triply-orthogonal system ([26]).

Suppose, for example, that one takes:

$$m = n = - p = 1.$$

Equation (11) will take the form:

$$\frac{xy}{z} = C. \tag{16}$$

Equation (13) will admit the roots:

$$z^2 \pm \sqrt{(x^2 + y^2)(y^2 + z^2)}\,.$$

It will then result that $u$ has the following two values:

$$\sqrt{u'} = \sqrt{z^2 + x^2} + \sqrt{z^2 + y^2}\,, \tag{17}$$

$$\sqrt{u''} = \sqrt{z^2 + x^2} - \sqrt{z^2 + y^2}\,. \tag{17$'$}$$

The surfaces that are represented by the last two equations are the loci of points such that the sum or the difference of their distances to the $x$ and $y$ axes – i.e., the two rectangular lines that cut them – is constant.

If one takes:

$$m = n = p = 1$$

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([26]) In his "Mémoire sur les surfaces orthogonales," which was included in the Journal de Liouville (1) **12** (1847), pp. 246, J.-A. Serret developed a remark by Bouquet and showed for the first time that the surfaces that are represented by equation (11) constitute one of the families of a triply-orthogonal system, and he showed the means by which one could determine the other two families that complete the system. However, he developed the calculations only in the two cases:

$$m = 1, \quad n = 1, \quad p = 1, \quad \text{and} \quad m = 1, \quad n = 1, \quad p = -1.$$

The method that is followed in this book was presented, along with the generalizations that it entails, in a "Mémoire sur la Théorie des coordinées curvilignes et des systèmes orthogonaux," that was published by the author in Annales de l'École Normale supérieure (2) **7** (1878), pp. 227.

then the equations of the three families will take the form:

$$(19)\qquad \begin{cases} xyz = C, \\ 3\sqrt{3}\sqrt{u'} = (x^2 + \omega y^2 + \omega^2 z^2)^{3/2} + (x^2 + \omega^2 y^2 + \omega z^2)^{3/2}, \\ 3\sqrt{3}\sqrt{u''} = (x^2 + \omega y^2 + \omega^2 z^2)^{3/2} - (x^2 + \omega^2 y^2 + \omega z^2)^{3/2}, \end{cases}$$

in which $\omega$ denotes an imaginary cube root of unity. This result is due to Cayley.

**141.** We now return to the general theory. Formulas (2) take a particularly remarkable form when one supposes that $u$, $v$, $w$ are not simply proportional to the direction cosines of the normal, but equal to those cosines, which we shall call $c$, $c'$, $c''$. Formulas (1) will then take the form:

$$(20)\qquad X = x + cR, \qquad Y = y + c'R, \qquad Z = z + c''R$$

and define a point on the normal that is situated at a distance of $R$ from the foot of that normal. That distance $R$ will have a sign, moreover, so it must be measured in the sense that is defined by the cosines $c$, $c'$, $c''$ if it is positive and in the contrary sense if it is negative. Formulas (2) will give us the following ones:

$$\frac{dx + R\,dc}{c} = \frac{dy + R\,dc'}{c'} = \frac{dz + R\,dc''}{c''}.$$

If we add the numerators and denominators, after multiplying them by $c$, $c'$, $c''$, respectively, then we will find, upon taking into account the obvious equation:

$$c\,dx + c'\,dy + c''\,dz = 0,$$

that the common value of these ratios is equal to zero.

One can then write the following formulas:

$$(21)\qquad dx + R\,dc = 0, \qquad dy + R\,dc' = 0, \qquad dz + R\,dc'' = 0,$$

which are due to Olinde Rodrigues, and which play an essential role in the theory of lines of curvature. $R$ denotes the radius of principal curvature that corresponds to the line considered, and the coordinates of the corresponding center of curvature are defined by formulas (20).

**142.** One also obtains the equations of Olinde Rodrigues by making use of a very important notation that is due to Gauss, namely, that of *spherical representation.* Imagine an arbitrary portion of a given surface and attribute a sense to the normal at every point in that region by requiring that it must satisfy the condition that the direction cosines $c$, $c'$, $c''$ of the normal should be continuous functions of the two parameters that define the foot of that normal. Now, construct the point whose rectangular coordinates are $c$, $c'$, $c''$. It obviously belongs to the sphere of radius 1 that has its center at the coordinate origin, and one will see that we establish a point-by-point correspondence between that sphere and the given surface. A point $M$ of the surface will correspond to a point $m$ of the sphere, a curve on the surface to a curve on the sphere, and a continuous region of the surface, to a likewise continuous region of the sphere. That kind of correspondence has received the name of *spherical representation* or *spherical image* of the surface, and, as we shall see later, Gauss used it to establish and formulate one of the most important propositions of the theory that we address.

If we consider an arbitrary point $M$ of the surface and its spherical image $m$ then it will result from our definition that the tangent plane to the surface at the point $M$ is parallel to the tangent plane to the sphere at $m$. The normals to the two surfaces are parallel, and as far as their senses are concerned, one sees that the positive sense of the normal to the surface corresponds to that of the external normal to the sphere.

Imagine that the point $M$ is displaced by starting with its initial position on the surface and describing an element of the curve $MM'$, so the point $m$ that serves as its image will be displaced by starting with $m$ and describing an element of the curve $mm'$. We now seek the relationship between those two corresponding elements.

It is obvious at first that the arc $mm'$ measures the magnitude of the angle between the normals at $M$ and $M'$, or – what amounts to the same thing – the infinitely-small angle between the planes that are tangent to the surface at those two points. That is a very important primary property of the spherical representation; viz., it permits one to study the variation of the tangent plane geometrically.

On the other hand, the tangent $mm'$ to the sphere and the tangent $MM'$ to the surface obviously have parallel lines for their conjugates, since the tangent planes to the corresponding points of the two surfaces will always be that way. Now, for the sphere, the conjugate to a tangent is perpendicular to that tangent. One can then state the following proposition:

*The angle between the tangents at $m$ and $M$ to the corresponding curves $mm'$ and $MM'$ is complementary to the angle that is formed by the tangent to the surface with its conjugate.*

In other words:

*A tangent $MT$ to the surface that has $MT'$ for its conjugate will correspond to a tangent $mt$ to the sphere that is perpendicular to $MT'$.*

We apply that general remark to the particular case that is most interesting to us.

First, suppose that $MT$ is an asymptotic tangent: It will coincide with its conjugate $MT'$ and will be consequently perpendicular to its spherical image. One will then get the differential equation for the asymptotic lines by differentiating the fact that the corresponding displacements on the surface and the sphere are perpendicular. One will then be led to the equation:

$$dx\,dc + dy\,dc' + dz\,dc'' = 0,$$

which one deduces, in fact, from the first of formulas (24) [pp. 123] by setting $t = 1$ in it.

Now, take $MT$ to be a principal tangent. $MT$ will be perpendicular to $MT'$, and consequently parallel to $mt$, and conversely, if $MT$ is parallel to $mt$ then it will be perpendicular to its conjugate. Thus, the principal tangents are characterized by the property of being parallel to their spherical representations. Upon writing the conditions for parallelism, we recover the equations:

$$\frac{dx}{dc} = \frac{dy}{dc'} = \frac{dz}{dc''},$$

which are those of Olinde Rodrigues, when one has eliminated that radius of curvature $R$ ([27]).

We will frequently have occasion to employ the spherical representation, and for the moment, we shall content ourselves with the preceding elementary remarks. Before continuing on to the general study of the properties of lines of curvature, we shall show how one can define their differential equations in the very diverse cases that can present themselves.

**143.** First, suppose that the surface is considered to be a locus of points, and that the rectangular coordinates $x$, $y$, $z$ are given functions of two parameters $\alpha$, $\beta$. One can employ equation (4); the following method leads to the result in a more symmetric manner.

The equations of the normal to the point $(x, y, z)$ will become:

$$(X - x)\frac{\partial x}{\partial \alpha} + (Y - y)\frac{\partial y}{\partial \alpha} + (Z - z)\frac{\partial z}{\partial \alpha} = 0,$$

$$(X - x)\frac{\partial x}{\partial \beta} + (Y - y)\frac{\partial y}{\partial \beta} + (Z - z)\frac{\partial z}{\partial \beta} = 0,$$

or

$$(22) \qquad \begin{cases} X\dfrac{\partial x}{\partial \alpha} + Y\dfrac{\partial y}{\partial \alpha} + Z\dfrac{\partial z}{\partial \alpha} - \dfrac{\partial r}{\partial \alpha} = 0, \\ X\dfrac{\partial x}{\partial \beta} + Y\dfrac{\partial y}{\partial \beta} + Z\dfrac{\partial z}{\partial \beta} - \dfrac{\partial r}{\partial \beta} = 0, \end{cases}$$

upon setting:

$$(23) \qquad r = \frac{x^2 + y^2 + z^2}{2},$$

to abbreviate.

We express the idea that the normal generates a developable surface – i.e., that there exists a displacement for which a conveniently-chosen point $(X, Y, Z)$ on that line will satisfy the equations:

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([27]) *See* J. BERTRAND, *Traite de Calcul differential*, pp. 665 and 697.

$$\frac{\partial x}{\partial\alpha}dX+\frac{\partial y}{\partial\alpha}dY+\frac{\partial z}{\partial\alpha}dZ=0,$$

$$\frac{\partial x}{\partial\beta}dX+\frac{\partial y}{\partial\beta}dY+\frac{\partial z}{\partial\beta}dZ=0.$$

Differentiate equations (22), while taking the preceding into account; we will get:

$$(24)\qquad \begin{cases} X\,d\dfrac{\partial x}{\partial\alpha}+Y\,d\dfrac{\partial y}{\partial\alpha}+Z\,d\dfrac{\partial z}{\partial\alpha}-d\dfrac{\partial r}{\partial\alpha}=0, \\ X\,d\dfrac{\partial x}{\partial\beta}+Y\,d\dfrac{\partial y}{\partial\beta}+Z\,d\dfrac{\partial z}{\partial\beta}-d\dfrac{\partial r}{\partial\beta}=0. \end{cases}$$

The elimination of $X$, $Y$, $Z$ from formulas (22) and (24) will give us the differential equation of the lines of curvature in the form of a determinant:

$$(25)\qquad \begin{vmatrix} \dfrac{\partial x}{\partial\alpha} & \dfrac{\partial x}{\partial\beta} & d\dfrac{\partial x}{\partial\alpha} & d\dfrac{\partial x}{\partial\beta} \\ \dfrac{\partial y}{\partial\alpha} & \dfrac{\partial y}{\partial\beta} & d\dfrac{\partial y}{\partial\alpha} & d\dfrac{\partial y}{\partial\beta} \\ \dfrac{\partial z}{\partial\alpha} & \dfrac{\partial z}{\partial\beta} & d\dfrac{\partial z}{\partial\alpha} & d\dfrac{\partial z}{\partial\beta} \\ \dfrac{\partial r}{\partial\alpha} & \dfrac{\partial r}{\partial\beta} & d\dfrac{\partial r}{\partial\alpha} & d\dfrac{\partial r}{\partial\beta} \end{vmatrix}=0,$$

and formulas (22) and (24) will exhibit the center of principal curvature that corresponds to each line of curvature.

The preceding result can also be presented in the following form: Consider a linear partial differential equation in the form:

$$(26)\qquad A\frac{\partial^2\theta}{\partial\alpha^2}+B\frac{\partial^2\theta}{\partial\alpha\,\partial\beta}+C\frac{\partial^2\theta}{\partial\beta^2}+A'\frac{\partial\theta}{\partial\alpha}+B'\frac{\partial\theta}{\partial\beta}=0,$$

and say that it admits the four particular solutions $x$, $y$, $z$, $r$. We will then have four equations that determine the mutual ratios of $A$, $B$, $C$, $A'$, $B'$. The linear equation can also be written in the form of a determinant:

$$\begin{vmatrix} \frac{\partial^2\theta}{\partial\alpha^2} & \frac{\partial^2\theta}{\partial\alpha\,\partial\beta} & \frac{\partial^2\theta}{\partial\beta^2} & \frac{\partial\theta}{\partial\alpha} & \frac{\partial\theta}{\partial\beta} \\ \frac{\partial^2 x}{\partial\alpha^2} & \frac{\partial^2 x}{\partial\alpha\,\partial\beta} & \frac{\partial^2 x}{\partial\beta^2} & \frac{\partial x}{\partial\alpha} & \frac{\partial x}{\partial\beta} \\ \frac{\partial^2 y}{\partial\alpha^2} & \frac{\partial^2 y}{\partial\alpha\,\partial\beta} & \cdots & \cdots & \cdots \\ \frac{\partial^2 z}{\partial\alpha^2} & \frac{\partial^2 z}{\partial\alpha\,\partial\beta} & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

It suffices to compare this equation with the determinant (25) for one to recognize that the differential equation of the lines of curvature, when ordered with respect to $d\alpha$, $d\beta$ can be written:

(27) $$A\,d\beta^2 - B\,d\alpha\,d\beta + C\,d\beta^2 = 0,$$

in which $A$, $B$, $C$ are the coefficients that appear in equation (26). We can then state the first proposition:

*When one has obtained (by whatever process) a partial differential equation of the form* (26) *that must be satisfied by x, y, z, and* $x^2 + y^2 + z^2$*, the differential equation of the lines of curvature will be given by formula* (27). *In other words, the lines of curvature will be the characteristics of that partial differential equation.*

That is the proposition that we obtained already along a different path in no. **108**.

**144.** We attach the following theorem to the preceding result:

*If one is given:*

$$\frac{\partial^2\theta}{\partial\alpha\,\partial\beta} = A\frac{\partial\theta}{\partial\alpha} + B\frac{\partial\theta}{\partial\beta} + C\,\theta,$$

*in which A, B, C are arbitrary functions of* $\alpha$, $\beta$*, then if one knows five particular solutions that are coupled by a homogeneous equation of degree two with constant coefficients, one can obtain a surface for which one has determined the lines of curvature.*

Indeed, let $\theta_1$, $\theta_2$, …, $\theta_5$ be five solutions that satisfy the homogeneous relation of degree two:

$$\varphi(\theta_1, \theta_2, \ldots, \theta_5) = 0.$$

Upon performing a linear substitution with constant coefficients on these solutions, one can reduce the preceding relation to the form:

(28) $$\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\theta_4\theta_5 = 0.$$

Make the substitution:

$$\theta = \sigma\,\theta_5$$

in the equation for $\theta$. $\sigma$ will satisfy a linear equation like $\theta$, but that equation, while admitting the solution $\sigma = 1$, will have the form:

$$\frac{\partial^2\sigma}{\partial\alpha\,\partial\beta} = A_1\frac{\partial\sigma}{\partial\alpha} + B_1\frac{\partial\sigma}{\partial\beta}$$

and no longer contain the term in $\sigma$. It will admit the particular solutions:

(29) $$x = \frac{\theta_1}{\theta_5}, \qquad y = \frac{\theta_2}{\theta_5}, \qquad z = \frac{\theta_3}{\theta_5}, \qquad r = \frac{\theta_4}{\theta_5},$$

which are coupled by the relation:

$$\frac{x^2+y^2+z^2}{2} = r.$$

Thus, by virtue of the proposition that was proved in the preceding number, the surface that is the locus of points $(x, y, z)$ will admit $\alpha$ and $\beta$ for its parameters of its lines of curvatures.

**145.** In order to give an application of the theorem that we just established, we choose the equation:

(30) $$2\,(\rho - \rho_1)\,\frac{\partial^2\theta}{\partial\rho\,\partial\rho_1} + \frac{\partial\theta}{\partial\rho} - \frac{\partial\theta}{\partial\rho_1} = 0,$$

which admits the particular solution:

$$\theta = A\sqrt{(\rho - a)(\rho_1 - a)}\,,$$

no matter what the constants $A$, $a$ are. We shall take five systems of values for those constants in the following manner:

Set:

$$f(u) = (u - a_1)\,(u - a_2)\,\dots\,(u - a_5).$$

The five solutions $\theta_i$ that are defined by the general formula:

(31) $$\theta_i = \sqrt{\frac{(a_i-\rho)(a_i-\rho_1)(a_i-h)}{f(a_i)}} \qquad (i = 1, 2, .., 5)$$

satisfy the identity:

$$\theta_1^2 + \ldots + \theta_5^2 = 0.$$

Take:

$$x_1 = \theta_1, \qquad x_2 = \theta_2, \qquad x_3 = \theta_3, \qquad x_5 = -\frac{R}{2}(\theta_4 - i\,\theta_5), \quad x_5 = (\theta_4 + i\,\theta_5),$$

in which $R$ denotes an arbitrary constant; the five solutions $x_i$ satisfy the identity (28). Here, formulas (29) will give us:

$$(32) \qquad x = \frac{R\,\theta_1}{\theta_4 + i\,\theta_5}, \quad y = \frac{R\,\theta_2}{\theta_4 + i\,\theta_5}, \quad z = \frac{R\,\theta_3}{\theta_4 + i\,\theta_5}, \quad x^2 + y^2 + z^2 = -R^2\,\frac{\theta_4 - i\theta_5}{\theta_4 + i\,\theta_5}$$

and define a surface that is referred to the system of curvilinear coordinates that is defined by the lines of curvature. One can, moreover, find the equation of that surface in the following manner:

Equations (32) give us:

$$\frac{\theta_1}{x} = \frac{\theta_2}{y} = \frac{\theta_3}{z} = \frac{\theta_4}{\dfrac{x^2 + y^2 + z^2 - R^2}{2R}} = \frac{\theta_5}{\dfrac{x^2 + y^2 + z^2 + R^2}{2Ri}}.$$

Furthermore, the functions $\theta_i$ also satisfy the identity:

$$\sum_{i=1}^{5} \frac{\theta_i^2}{a_i - h} = 0,$$

and upon replacing them with quantities that are proportional to them, one will obtain the desired equation in the form:

$$(33) \qquad \frac{x^2}{a_1 - h} + \frac{y^2}{a_2 - h} + \frac{z^2}{a_3 - h} + \frac{\left(\dfrac{x^2 + y^2 + z^2 - R^2}{2R}\right)^2}{a_4 - h} + \frac{\left(\dfrac{x^2 + y^2 + z^2 + R^2}{2Ri}\right)^2}{a_5 - h} = 0.$$

One will likewise find the equations that determine each line of curvature. Indeed, the roots $\theta_i$ contain the three quantities $h$, $\rho$, $\rho_1$ in a symmetric manner. The preceding equation must then be once more verified when one replaces $h$ with $\rho$ and $\rho_1$ in it. That remark leads immediately to the following proposition:

*If one considers h to be a variable parameter in equation* (33) *then the surfaces that correspond to two distinct values of h will mutually intersect along a line of curvature that is common to those surfaces.*

If one clears the denominators in equation (33) then one will recognize that since the coefficient of $h^4$ is zero, it will be only of degree three with respect to $h$; consequently, if one gives $h$ all possible values in such a manner as to obtain a family of surfaces then one will have three surfaces of that family that pass through each point of space. While those three surfaces mutually intersect along common lines of curvature, they will be necessarily orthogonal. Equation (33) then defines a triply-orthogonal system that is analogous to the one that is defined by second-degree surfaces and composed of three families that are represented by the same equation. If one supposes, for example, that $a_1$, $a_2$, $a_3$, $a_4$ are real and are arranged by order of magnitude then the three families that correspond to the values of $h$ will be included between $a_1$ and $a_2$, between $a_2$ and $a_3$ and between $a_3$ and $a_4$ .

The surfaces that are represented by equation (33) have order four and admit the circle at infinity for a double line; they have received the name of *cyclides.*

**146.** We shall now describe an application of a different nature. One knows that *inversion*, or the transformation by reciprocal radius vectors, which is defined in the simplest case by formulas such as the following ones:

$$(34) \qquad X = \frac{K^2 x}{x^2+y^2+z^2}, \qquad Y = \frac{K^2 y}{x^2+y^2+z^2}, \qquad Z = \frac{K^2 z}{x^2+y^2+z^2},$$

will make a sphere or a plane correspond to a sphere or a plane, resp., and that it preserves angles and the ratios of similitude of infinitely-small elements. We shall show that it also preserves lines of curvature.

Indeed, consider an arbitrary surface ($\Sigma$) and let $\rho$, $\rho_1$ be the parameters of its lines of curvature. We know that $x$, $y$, $z$ satisfy an equation of the form:

$$(35) \qquad \frac{\partial^2\theta}{\partial\rho\,\partial\rho_1} = A\frac{\partial\theta}{\partial\rho} + B\frac{\partial\theta}{\partial\rho_1},$$

and that equation is distinguished from all of the ones that relate to other conjugate systems by the property that was pointed out already that it also admits $x^2 + y^2 + z^2$ for a solution. These four solutions to equation (35) are expressed in terms of $X$, $Y$, $Z$ in the following manner:

$$\frac{K^2 X}{X^2+Y^2+Z^2}, \qquad \frac{K^2 Y}{X^2+Y^2+Z^2}, \qquad \frac{K^2 Z}{X^2+Y^2+Z^2}, \qquad \frac{K^4}{X^2+Y^2+Z^2}.$$

If one then performs the substitution:

$$\theta = \frac{\sigma}{X^2+Y^2+Z^2}$$

in equation (35) then the equation in $\sigma$ will admit the particular solutions:

$$X, Y, Z, 1,$$

and will consequently have the form:

$$(36) \qquad \frac{\partial^2 \sigma}{\partial \rho \, \partial \rho_1} = A_1 \frac{\partial \sigma}{\partial \rho} + B_1 \frac{\partial \sigma}{\partial \rho_1}.$$

Furthermore, equation (35) admits the obvious solution $\theta = 1$, and consequently, equation (36) will admit the solution:

$$\sigma = X^2 + Y^2 + Z^2,$$

along with $X$, $Y$, $Z$. It results from this that the surface that is the locus of points ($X$, $Y$, $Z$) will have $\rho$, $\rho_1$ for parameters of its lines of curvature. That result is precisely the one that had to be established.

**147.** One proves the preceding proposition in the usual way by appealing to Dupin's theorem that relates to lines of curvature of a surface that define part of a triply-orthogonal system. To conclude this chapter, we shall give a new proof of that theorem.

Let:

$$(40) \qquad \rho = f(x, y, z), \qquad \rho_1 = f_1(x, y, z), \qquad \rho_2 = f_2(x, y, z)$$

be the equations of three families of surfaces that mutually intersect at a right angle. If one solves those equations with respect to $x$, $y$, $z$ then the relations:

$$(41) \qquad \begin{cases} \mathrm{S} \dfrac{\partial x}{\partial \rho} \dfrac{\partial x}{\partial \rho_1} = \dfrac{\partial x}{\partial \rho} \dfrac{\partial x}{\partial \rho_1} + \dfrac{\partial y}{\partial \rho} \dfrac{\partial y}{\partial \rho_1} + \dfrac{\partial z}{\partial \rho} \dfrac{\partial z}{\partial \rho_1} = 0, \\ \mathrm{S} \dfrac{\partial x}{\partial \rho} \dfrac{\partial x}{\partial \rho_2} = \dfrac{\partial x}{\partial \rho} \dfrac{\partial x}{\partial \rho_2} + \dfrac{\partial y}{\partial \rho} \dfrac{\partial y}{\partial \rho_2} + \dfrac{\partial z}{\partial \rho} \dfrac{\partial z}{\partial \rho_2} = 0, \\ \mathrm{S} \dfrac{\partial x}{\partial \rho_1} \dfrac{\partial x}{\partial \rho_2} = \dfrac{\partial x}{\partial \rho_1} \dfrac{\partial x}{\partial \rho_2} + \dfrac{\partial y}{\partial \rho_1} \dfrac{\partial y}{\partial \rho_2} + \dfrac{\partial z}{\partial \rho_1} \dfrac{\partial z}{\partial \rho_2} = 0, \end{cases}$$

in which we employ Lame's $\mathrm{S}$ sign in order to indicate a sum that extends over the three coordinates at the same point, must be true identically. If we differentiate the first one with respect to $\rho_2$, the second one with respect to $\rho_1$, and the third one with respect to $\rho$ then we will have:

$$\mathrm{S} \frac{\partial x}{\partial \rho} \frac{\partial^2 x}{\partial \rho_1 \, \partial \rho_2} + \mathrm{S} \frac{\partial x}{\partial \rho_1} \frac{\partial^2 x}{\partial \rho \, \partial \rho_2} = 0,$$

$$\mathrm{S} \frac{\partial x}{\partial \rho} \frac{\partial^2 x}{\partial \rho_1 \, \partial \rho_2} + \mathrm{S} \frac{\partial x}{\partial \rho_2} \frac{\partial^2 x}{\partial \rho \, \partial \rho_1} = 0,$$

$$\mathrm{S} \frac{\partial x}{\partial \rho_1} \frac{\partial^2 x}{\partial \rho \, \partial \rho_2} + \mathrm{S} \frac{\partial x}{\partial \rho_2} \frac{\partial^2 x}{\partial \rho \, \partial \rho_1} = 0,$$

and, in turn:

(42) $$\mathrm{S}\frac{\partial x}{\partial \rho}\frac{\partial^2 x}{\partial \rho_1\,\partial \rho_2} = \mathrm{S}\frac{\partial x}{\partial \rho_1}\frac{\partial^2 x}{\partial \rho\,\partial \rho_2} = \mathrm{S}\frac{\partial x}{\partial \rho_2}\frac{\partial^2 x}{\partial \rho\,\partial \rho_1} = 0.$$

One sees from this that one will have three different systems of solutions for the equation in $u$, $v$, $w$:

$$\frac{\partial x}{\partial \rho}u + \frac{\partial y}{\partial \rho}v + \frac{\partial z}{\partial \rho}w = 0$$

if one takes either:

$$u = \frac{\partial x}{\partial \rho_1}, \qquad v = \frac{\partial y}{\partial \rho_1}, \qquad w = \frac{\partial z}{\partial \rho_1}$$

or

$$u = \frac{\partial x}{\partial \rho_2}, \qquad v = \frac{\partial y}{\partial \rho_2}, \qquad w = \frac{\partial z}{\partial \rho_2}$$

or finally:

$$u = \frac{\partial^2 x}{\partial \rho_1\,\partial \rho_2}, \qquad v = \frac{\partial^2 y}{\partial \rho_1\,\partial \rho_2}, \quad w = \frac{\partial^2 z}{\partial \rho_1\,\partial \rho_2}.$$

These systems cannot be linearly independent, so it is necessary that the latter must be a linear combination of the first two – i.e., that $x$, $y$, $z$ must be particular solutions of an equation of the form:

(43) $$\frac{\partial^2 \theta}{\partial \rho_1\,\partial \rho_2} = m\frac{\partial \theta}{\partial \rho_1} + n\frac{\partial \theta}{\partial \rho_2}.$$

The variables $\rho_1$, $\rho_2$ then define a conjugate system on the surface ($\rho$), and since, by the nature of the question, that system is orthogonal, it will necessarily be composed of lines of curvature. Moreover, one easily verifies that the linear equation (43) also admits the particular solution:

$$\theta = x^2 + y^2 + z^2.$$

**148.** The preceding proof leads to a new method of analysis for orthogonal systems. We just saw that the coordinates $x$, $y$, $z$, and the sum $x^2 + y^2 + z^2$ satisfy equation (43), and it is clear that these solutions will satisfy two other similar equations in $\rho$, $\rho_2$, and $\rho_1$, $\rho$. Conversely, we shall show that:

*If three linear equations of the form:*

$$(44)\qquad \begin{cases} \dfrac{\partial^2\theta}{\partial\rho_1\,\partial\rho_2} = m\dfrac{\partial\theta}{\partial\rho_1} + n\dfrac{\partial\theta}{\partial\rho_2}, \\[2ex] \dfrac{\partial^2\theta}{\partial\rho\,\partial\rho_2} = m_1\dfrac{\partial\theta}{\partial\rho_2} + n_1\dfrac{\partial\theta}{\partial\rho}, \\[2ex] \dfrac{\partial^2\theta}{\partial\rho_1\,\partial\rho} = m_2\dfrac{\partial\theta}{\partial\rho} + n_2\dfrac{\partial\theta}{\partial\rho_1} \end{cases}$$

*admit three particular solutions $x, y, z$, along with the sum of their squares $x^2 + y^2 + z^2$, then the system of curvilinear coordinates that is defined by the expressions for $x, y, z$ as functions of $\rho, \rho_1, \rho_2$ will necessarily be orthogonal.*

Indeed, consider one of the coordinate surfaces ($\rho$). The system of curvilinear coordinates ($\rho_1$, $\rho_2$) that is determined on that surface by the other two families is composed of lines of curvature of that surface, because $x$, $y$, $z$, when considered to be functions of $\rho_1$, $\rho_2$, will satisfy the first of the preceding equation, along with $x^2 + y^2 + z^2$. The surfaces of three families that mutually intersect along their lines of curvature will necessarily be orthogonal.

**149.** In order to not treat this subject in an incomplete manner, we remark that one can easily obtain the coefficients $m$, $n$, … when one knows the expression for the linear element:

$$ds^2 = H^2\, d\rho^2 + H_1^2\, d\rho_1^2 + H_2^2\, d\rho_2^2$$

in the orthogonal system. Indeed, if one differentiates the equation:

$$\left(\frac{\partial x}{\partial\rho}\right)^2 + \left(\frac{\partial y}{\partial\rho}\right)^2 + \left(\frac{\partial z}{\partial\rho}\right)^2 = H^2$$

with respect to $\rho_1$ then one will get:

$$H\frac{\partial H}{\partial\rho_1} = \mathbf{S}\frac{\partial x}{\partial\rho}\frac{\partial^2 x}{\partial\rho\,\partial\rho_1},$$

and upon replacing $\dfrac{\partial^2 x}{\partial\rho\,\partial\rho_1}$ with the value that one deduces from the last of equations (44):

$$H\frac{\partial H}{\partial\rho_1} = m_2\,\mathbf{S}\left(\frac{\partial x}{\partial\rho}\right)^2 + n_2\,\mathbf{S}\frac{\partial x}{\partial\rho}\frac{\partial x}{\partial\rho_1} = m_2\, H^2,$$

0ne will then have:

$$m_2 = \frac{1}{H}\frac{\partial H}{\partial \rho_1}.$$

If one substitutes that expression and the analogous values for $n_2$, $m$, $n$, … in equations (44) then one will obtain them in the form:

$$(45)\qquad \left\{\begin{aligned} \frac{\partial^2\theta}{\partial\rho_1\,\partial\rho_2} &= \frac{1}{H_1}\frac{\partial H_1}{\partial\rho_2}\frac{\partial\theta}{\partial\rho_1} + \frac{1}{H_2}\frac{\partial H_1}{\partial\rho_1}\frac{\partial\theta}{\partial\rho_2}, \\ \frac{\partial^2\theta}{\partial\rho_1\,\partial\rho} &= \frac{1}{H_2}\frac{\partial H_1}{\partial\rho}\frac{\partial\theta}{\partial\rho_2} + \frac{1}{H}\frac{\partial H_1}{\partial\rho_2}\frac{\partial\theta}{\partial\rho}, \\ \frac{\partial^2\theta}{\partial\rho\,\partial\rho_1} &= \frac{1}{H}\frac{\partial H_1}{\partial\rho_1}\frac{\partial\theta}{\partial\rho} + \frac{1}{H_1}\frac{\partial H_1}{\partial\rho}\frac{\partial\theta}{\partial\rho_1}, \end{aligned}\right.$$

which is due to Lamé.

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# CHAPTER VI.

## PENTA-SPHERICAL COORDINATES.

The system of five orthogonal spheres. – Relationship to an orthogonal linear substitution in five variables. – Main formulas that relate to distances and angles. – Use of penta-spherical coordinates in the theory of lines of curvature and in that of orthogonal systems. – Inversion. – Study of the system of two spheres. – The six coordinates of the sphere, compared to those of the straight line. – The transformation of Sophus Lie.

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**150.** In the study of conjugate systems, we saw that the use of homogeneous and tangential coordinates exhibited projective and dualistic properties that belonged to those systems. If one desires to give a satisfactory exposition of the analytic theory of lines of curvature, moreover, then one will be led to introduce a particular system of coordinates to which we have given the name of *penta-spherical* coordinates. In this chapter, we propose to define that system of coordinates and to point out its role in the theory of lines of curvature.

Consider an arbitrary sphere that is referred to rectangular axes:

$$K(x^2 + y^2 + z^2) + 2Ax + 2By + 2Cz + D = 0.$$

Its radius $\rho$ will be given by the formula:

$$\rho^2 = \frac{A^2 + B^2 + C^2 - DK}{K^2}.$$

The quadratic form that appears in the numerator planes a fundamental role in the theory of the sphere. It is natural to convert it into a sum of squares, and to that end, we write the equation of the sphere in the form:

$$(1) \qquad 2\alpha x + 2\beta y + 2\gamma z + \delta\,\frac{x^2 + y^2 + z^2 - R^R}{R} + i\varepsilon\,\frac{x^2 + y^2 + z^2 + R^R}{R} = 0.$$

If one lets $\rho$ denote the radius and $x_0$, $y_0$, $z_0$, the coordinates of the center of that sphere then one will obtain the following expressions for those quantities:

$$(2) \qquad \left\{\begin{array}{l} x_0 = \dfrac{-\alpha R}{\delta + i\varepsilon}, \quad y_0 = \dfrac{-\beta R}{\delta + i\varepsilon}, \quad z_0 = \dfrac{-\gamma R}{\delta + i\varepsilon}, \\[2ex] \rho = \dfrac{R\sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2}}{\delta + i\varepsilon}, \quad x_0^2 + y_0^2 + z_0^2 - \rho^2 = -R^2\,\dfrac{\delta - i\varepsilon}{\delta + i\varepsilon}. \end{array}\right.$$

If the sphere does not reduce to a point then one can always suppose that one has:

(3) $$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 = 1,$$

and the expression for the radius will take the simple form:

(4) $$\rho = \frac{R}{\delta + i\varepsilon}.$$

That formula gives a well-defined sign to the radius; we shall return to that point later on.

If one substitutes the coordinates of an arbitrary point in equation (1) then the left-hand side will have the value:

(5) $$\frac{S}{\rho},$$

in which $S$ denotes the power of the point with respect to the sphere considered. We remark, once and for all, that if the sphere reduces to a plane then one will have:

$$\delta + i\varepsilon = 0,$$

and the left-hand side of equation (1) will become equal to twice the distance from the point to that plane.

Now suppose that one considers, along with the sphere that is represented by equation (1), another sphere ($S'$) that is represented by the similar equation:

$$2\alpha' x + 2\beta' y + \ldots = 0.$$

Let $\rho'$ be the radius, and let $x_0'$, $y_0'$, $z_0'$ be the coordinates of the center of the second sphere. Formulas (2) give us:

(6) $$\begin{cases} (x_0 - x_0')^2 + (y_0 - y_0')^2 + (z_0 - z_0')^2 - \rho^2 - \rho'^2 \\ \quad = -\dfrac{2R^2(\alpha\alpha' + \beta\beta' + \gamma\gamma' + \delta\delta' + \varepsilon\varepsilon')}{(\delta + i\varepsilon)(\delta' + i\varepsilon')}, \end{cases}$$

and, in turn, the equation:

(7) $$\alpha\alpha' + \beta\beta' + \gamma\gamma' + \delta\delta' + \varepsilon\varepsilon' = 0$$

will express the necessary and sufficient condition for the two spheres to cut at a right angle. That condition will persist when one or the other of the spheres reduces to a plane. Its form will permit us to give a very simple theory of a system of five spheres that are pair-wise orthogonal.

**151.** Indeed, consider five spheres $(S_1)$, $(S_2)$, …, $(S_5)$ of radii $R_1$, $R_2$, …, $R_5$, resp., and write their equations in the form:

$$2\alpha_k\,x + 2\beta_k\,y + 2\gamma_k\,z + \delta_k\,\frac{x^2+y^2+z^2-R^2}{R} + i\varepsilon_k\,\frac{x^2+y^2+z^2+R^2}{R} = 0, \tag{8}$$
$$k = 1, 2, \ldots, 5.$$

We first have, by hypothesis:

$$\alpha_k^2+\beta_k^2+\gamma_k^2+\delta_k^2+\varepsilon_k^2 = 1, \tag{9}$$

and furthermore, since the spheres are orthogonal:

$$\alpha_k\,\alpha_{k'} + \beta_k\,\beta_{k'} + \gamma_k\,\gamma_{k'} + \delta_k\,\delta_{k'} + \varepsilon_k\,\varepsilon_{k'} = 0. \tag{10}$$

*These two groups of formulas associate the theory of systems of spheres with that of an orthogonal linear substitution in five variables.* Any substitution of this kind will provide a group of five orthogonal spheres, and *vice versa*.

One knows that relations (9) and (10) imply the following consequences:

$$\alpha_1^2+\alpha_2^2+\cdots+\alpha_5^2 = 1, \tag{11}$$
$$\alpha_1\,\beta_1 + \ldots + \alpha_5\,\beta_5 = 0, \tag{12}$$

and all of the ones that one would obtain by replacing $\alpha$ and $\beta$ with $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ in an arbitrary manner.

One can deduce a first property from that remark that is fundamental in the theory that we are treating. Let $S_k$ denote the power of an arbitrary point with respect to the sphere $(S_k)$; the left-hand side of equation (8) will be $S_k / R_k$. If one takes the square of that equation, and if one adds all of the equations thus-obtained then one will find, upon applying formulas (11) and (12), that:

$$\sum_{k=1}^{5}\left(\frac{S_k}{R_k}\right)^2 = \left(\frac{x^2+y^2+z^2-R^2}{R}\right)^2 + \left(\frac{x^2+y^2+z^2-R^2}{Ri}\right)^2 + 4x^2 + 4y^2 + 4z^2 = 0.$$

The homogeneous relation:

$$\sum\left(\frac{S_k}{R_k}\right)^2 = 0 \tag{13}$$

then exists between the powers of an arbitrary point with respect to five spheres.

We now recall that if one of the spheres $(S_k)$ reduces to a plane $(P_k)$ then $S_k / R_k$ must be replaced by $2P_k$, in which $P_k$ denotes the distance from the point $(x, y, z)$ to that plane.

If one likewise multiplies the left-hand side of equation (8) by $\delta_k + i\varepsilon_k$ then one will find that:

$$\sum (\delta_k + i\varepsilon_k)\frac{S_k}{R_k} = -\,2R$$

or also:

$$\sum \frac{S_k}{R_k^2} = -\,2, \tag{14}$$

upon remarking that, from formula (4), one will have:

$$\delta_k + i\varepsilon_k = \frac{R}{R_k}.$$

**152.** We can now define the system of coordinates that we propose to study. We call the five quantities $x_k$ that are proportional to $S_k / R_k$ *penta-spherical coordinates* and set:

$$x_k = \lambda\,\frac{S_k}{R_k}. \tag{15}$$

Since we employ only homogeneous equations, the factor $\lambda$ will have no influence on the results. One will find, moreover, by virtue of formula (13) that:

$$x_1^2 + x_2^2 + \ldots + x_5^2 = 0. \tag{16}$$

Our five coordinates will always be coupled by one homogeneous relation then, as one could expect. It is easy to show that there is no other relation, and that five quantities that satisfy equation (15) will determine one and only one point.

Indeed, we remark that equations (13) and (14) contain all of the possible relations between the quantities $S_k$, because in order to determine a point, three of those quantities must be chosen arbitrarily. Moreover, if one substitutes the expression for $S_k$ in terms of $x_k$ in these relations then the first one will reduce to equation (16), which is verified by hypothesis, while the second one will become:

$$-\,2\lambda = \sum \frac{x_k}{R_k}, \tag{17}$$

which will tell one what the proportionality factor $\lambda$ is.

Moreover, one can obtain expressions for $x$, $y$, $z$ as functions of the variables $x_k$ ; it suffices to solve the system:

$$2\alpha_k\,x + 2\beta_k\,y + 2\gamma_k\,z + \delta_k\,\frac{x^2+y^2+z^2-R^2}{R} + i\varepsilon_k\,\frac{x^2+y^2+z^2+R^2}{R} = \frac{S_k}{R_k} = \frac{x_k}{\lambda}, \tag{18}$$

in which one gives $k$ the values 1, 2, …, 5. Upon adding these equations, after multiplying them by $\alpha_k$, and then by $\beta_k$, $\gamma_k$, $\delta_k$, $\varepsilon_k$, one will find that:

$$(19)\qquad \begin{cases} 2\lambda x = \sum_{k=1}^{5} \alpha_k x_k, & \lambda(x^2+y^2+z^2-R^2) = \quad R\sum_{k=1}^{5}\delta_k x_k, \\ 2\lambda y = \sum_{k=1}^{5} \beta_k x_k, & \lambda(x^2+y^2+z^2+R^2) = -iR\sum_{k=1}^{5}\varepsilon_k x_k, \\ 2\lambda z = \sum_{k=1}^{5} \gamma_k x_k. \end{cases}$$

The last two equations, when subtracted, will exhibit the factor $\lambda$ in the form:

$$(20)\qquad 2\lambda R = -\sum (\delta_k + i\,\varepsilon_k)\,x_k\,,$$

which is equivalent to equation (17); the other ones will give $x$, $y$, $z$, and likewise $x^2 + y^2 + z^2$ ([28]).

Along with a point $M$, whose coordinates $x$, $y$, z and $x_k$ are coupled by formulas (18), consider another point $M$ whose coordinates we denote by the same letters with primes $x'$, $y'$, $z'$, and $x'_k$. One will find, with no difficulty, that:

([28]) If one would like to make a more detailed study of this then one would have to point an exceptional case. Formula (13) shows that the five radii satisfy the relation:

$$(20)_a\qquad \sum_{k=1}^{5} \frac{1}{R_k^2} = 0\,.$$

If one looks for the point at which one has:

$$x_k = \frac{1}{R_k}$$

then one will find that it is indeterminate and is subject to only the condition that it must be in the plane at infinity.

On the other hand, a point on the circle at infinity has an infinitude of coordinates, which are determined from the formula:

$$x_k + \frac{h}{R_k}\,,$$

in which $h$ is arbitrary, and the $x_k$ satisfy the relation:

$$\sum \frac{x_k}{R_k} = 0,$$

along with equation (16).

(21) $$\overline{MM'}^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 = -\frac{\sum x_k x_k'}{2\lambda\lambda'} = \frac{-2\sum x_k x_k'}{\sum \frac{x_k}{R_k} \sum \frac{x_k'}{R_k}};$$

this is the formula that gives the distance between the two points. If one takes into account the identity relations between the coordinate then one can give it the form:

(22) $$\overline{MM'}^2 = \frac{\sum (x_k - x_k')^2}{\sum \frac{x_k}{R_k} \sum \frac{x_k'}{R_k}}.$$

When the two points are infinitely close, one will see the following expressions for the linear element:

(23) $$ds^2 = \frac{\sum dx_k^2}{\left(\sum \frac{x_k}{R_k}\right)^2}.$$

It is pointless to insist upon the analogy that these two formulas present with the ones that refer to the geometry of Descartes.

**153.** We now propose to establish the orthogonality relations in the system of coordinates $x_i$ .

When the coordinates are functions of two variables $\rho$, $\rho_1$, the curves ($\rho$), ($\rho_1$) will be perpendicular if the coefficient of $d\rho\, d\rho_1$ in the linear element is zero; i.e., from formula (23), in the sum:

$$\sum dx_k^2 .$$

That condition translates into the relation:

(24) $$\sum \frac{\partial x_k}{\partial \rho} \frac{\partial x_k}{\partial \rho_1} = 0,$$

which is entirely similar to the one that one obtains in Cartesian coordinates.

If one is now given two surfaces by the homogeneous equations:

$$\varphi(x_1, \ldots, x_5) = 0, \qquad \psi(x_1, \ldots, x_5) = 0$$

then one will seek the condition for them to cut at a right angle. We first point out the following relations between the powers of a point with respect to the five orthogonal spheres, which are easy to verify:

(25)
$$\begin{cases} \left(\dfrac{\partial S_k}{\partial x}\right)^2 + \left(\dfrac{\partial S_k}{\partial y}\right)^2 + \left(\dfrac{\partial S_k}{\partial z}\right)^2 = 4(S_k + R_k^2), \\ \dfrac{\partial S_k}{\partial x}\dfrac{\partial S_{k'}}{\partial x} + \dfrac{\partial S_k}{\partial y}\dfrac{\partial S_{k'}}{\partial y} + \dfrac{\partial S_k}{\partial z}\dfrac{\partial S_{k'}}{\partial z} = 2(S_k + S_{k'}). \end{cases}$$

With that, replace the $x_i$ with the proportional quantities $S_i$ / $R_i$ in the homogeneous equations for the two surface and set:

$$(\varphi\,,\ \psi) = \frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z}\,,$$

to abbreviate.

One will obviously have:

$$(\varphi\,,\ \psi) = \sum\sum \frac{\partial\varphi}{\partial S_k}\frac{\partial\psi}{\partial S_{k'}}(S_k, S_{k'})\ ;$$

i.e., upon taking into account formulas (25):

$$(\varphi\,,\ \psi) = 2\left(\sum S_k \frac{\partial\varphi}{\partial S_k}\right)\left(\sum \frac{\partial\psi}{\partial S_{k'}}\right) + 2\left(\sum S_k \frac{\partial\psi}{\partial S_k}\right)\left(\sum \frac{\partial\varphi}{\partial S_{k'}}\right) + 4\sum R_k^2 \frac{\partial\varphi}{\partial S_k}\frac{\partial\psi}{\partial S_k}\,.$$

Since the functions are homogeneous, one will have:

$$\sum S_k \frac{\partial\varphi}{\partial S_k} = m\varphi = 0, \qquad \sum S_k \frac{\partial\psi}{\partial S_k} = m\psi = 0,$$

and the orthogonality condition will take the form:

$$(\varphi\,,\ \psi) = 4\sum R_k^2 \frac{\partial\varphi}{\partial S_k}\frac{\partial\psi}{\partial S_k} = 0,$$

or, upon introducing the quantities $x_i$ :

(26)
$$(\varphi\,,\ \psi) = 4\sum_{k=1}^{5} \frac{\partial\varphi}{\partial x_k}\frac{\partial\psi}{\partial x_k} = 0.$$

More generally, the cosine of the angle $V$ at which two surfaces intersect is given by the formula:

$$\cos V = \frac{(\varphi, \psi)}{\sqrt{(\varphi, \varphi)(\psi, \psi)}}\,,$$

which will have the expression:

(27) $$\cos V = \frac{\sum \frac{\partial \varphi}{\partial x_k} \frac{\partial \psi}{\partial x_k}}{\sqrt{\sum \left(\frac{\partial \varphi}{\partial x_k}\right)^2} \sqrt{\sum \left(\frac{\partial \psi}{\partial x_k}\right)^2}} .$$

**154.** Before continuing with the study of the coordinates $x_i$, we point out their role in the theory of lines of curvature.

Consider an arbitrary surface and suppose that the coordinates $x_i$ of an arbitrary point on that surface are expressed as functions of two independent variables $\alpha$, $\beta$. Form the linear equation:

(28) $$A \frac{\partial^2 \theta}{\partial \alpha^2} + B \frac{\partial^2 \theta}{\partial \alpha \partial \beta} + C \frac{\partial^2 \theta}{\partial \beta^2} + D \frac{\partial \theta}{\partial \alpha} + E \frac{\partial \theta}{\partial \beta} + F\theta = 0$$

that the five coordinates satisfy. We shall show that its characteristics are the lines of curvature of the surface.

Indeed, if one sets:

$$\theta = \lambda\, \sigma,$$

in which $\lambda$ is the proportionality factor that appears in formulas (15), and which is defined by equation (17), then the linear equation will take the form:

$$A \frac{\partial^2 \sigma}{\partial \alpha^2} + B \frac{\partial^2 \sigma}{\partial \alpha \partial \beta} + C \frac{\partial^2 \sigma}{\partial \beta^2} + D' \frac{\partial \sigma}{\partial \alpha} + E' \frac{\partial \sigma}{\partial \beta} = 0,$$

in which the term in $\sigma$ has disappeared, since $\lambda$, being a linear function of the $x_i$, is a particular solution of equation (28). The equation in $\sigma$, which admits the five solutions $x_i$ / $\lambda$ or $S_i$ / $R_i$, which are linearly-independent functions of $x^2 + y^2 + z^2$, $x$, $y$, $z$, 1, must admit the same functions:

$$1, x, y, z, x^2 + y^2 + z^2$$

as particular solutions.

It will then become the equation that was considered in no. **143**, whose characteristics are the lines of curvature. Since the characteristics of the equation in $\sigma$ are the same as those of the equation in $\theta$, our proposition is proved. One immediately deduces the following consequence, which basically amounts to the theorem in no. **144**:

*If one knows five particular solutions $x_1, \ldots, x_5$ of a linear equation:*

(29) $$\frac{\partial^2 \theta}{\partial \alpha \partial \beta} = A \frac{\partial \theta}{\partial \alpha} + B \frac{\partial \theta}{\partial \beta} + C\, \theta$$

*that are coupled by the relation:*

$$\sum x_i^2 = 0$$

*then the quantities* $x_i$ *will be the penta-spherical coordinates of a point on a surface for which* $\alpha$ *and* $\beta$ *are the parameters of the lines of curvature.*

Likewise, the theorem that relates to orthogonal systems that was given in no. **148** will receive the new expression:

*If one is given three linear equations:*

$$(30)\qquad \begin{cases} \dfrac{\partial^2\theta}{\partial\rho_2\,\partial\rho_1} = m\dfrac{\partial\theta}{\partial\rho_1} + n\dfrac{\partial\theta}{\partial\rho_2} + p\theta, \\[2ex] \dfrac{\partial^2\theta}{\partial\rho\,\partial\rho_2} = m_2\dfrac{\partial\theta}{\partial\rho_2} + n_1\dfrac{\partial\theta}{\partial\rho} + p_1\theta, \\[2ex] \dfrac{\partial^2\theta}{\partial\rho\,\partial\rho_1} = m_2\dfrac{\partial\theta}{\partial\rho} + n_2\dfrac{\partial\theta}{\partial\rho_1} + p_2\theta, \end{cases}$$

*and one knows five particular solutions* $x_i$ *that satisfy the condition:*

$$\sum x_i^2 = 0$$

*then they can be regarded as the penta-spherical coordinates of a point in space, and* $\rho$, $\rho_1$, $\rho_2$ *will be the parameters of three families of surfaces that mutually intersect at a right angle* ([29]).

In particular, consider the three equations:

$$(31)\qquad \begin{cases} 2(\rho_1 - \rho_2)\dfrac{\partial^2\theta}{\partial\rho_1\,\partial\rho_2} + \dfrac{\partial\theta}{\partial\rho_1} - \dfrac{\partial\theta}{\partial\rho_2} = 0, \\[2ex] 2(\rho_2 - \rho)\dfrac{\partial^2\theta}{\partial\rho\,\partial\rho_2} + \dfrac{\partial\theta}{\partial\rho_2} - \dfrac{\partial\theta}{\partial\rho} = 0, \\[2ex] 2(\rho - \rho_1)\dfrac{\partial^2\theta}{\partial\rho\,\partial\rho_1} + \dfrac{\partial\theta}{\partial\rho} - \dfrac{\partial\theta}{\partial\rho_1} = 0, \end{cases}$$

which admits the common solution:

$$\theta = A\sqrt{(a-\rho)(a-\rho_1)(a-\rho_2)}\;;$$

if one sets:

([29]) G. DARBOUX, "Mémoire sur la Théorie des coordinees curvilignes et des systèmes orthogonaux," Annales de l'École Normale (2) **7** (1878), pp. 297.

(32) $$f(u) = (u - a_1)(u - a_2) \dots (u - a_5)$$

then the five solutions that are defined by the general formula:

(33) $$x_i = \sqrt{\frac{(a-\rho)(a-\rho_1)(a-\rho_2)}{f(a_i)}} \qquad (i = 1, 2, \dots, 5)$$

will satisfy the identity:

$$\sum x_i^2 = 0.$$

Consequently, formulas (33) determine a triply-orthogonal system. The surfaces that comprise it – which have received the name of *cyclides* – are nothing but the transforms of the ones that were defined in no. **145** by inversion; they are represented by the unique equation:

(34) $$\sum \frac{x_i^2}{a_i - \lambda} = 0,$$

in which one replaces $\lambda$ successively by $\rho$, $\rho_1$, $\rho_2$ . Moreover, one verifies immediately, upon applying the orthogonality condition (26), that two surfaces that correspond to two different values of $\lambda$ will mutually intersect at a right angle. Upon appealing to formulas (33), one will find the formula:

(35) $$4\sum dx_k^2 = \frac{(\rho-\rho_1)(\rho-\rho_2)}{f(\rho)}d\rho^2 + \frac{(\rho_1-\rho)(\rho_1-\rho_2)}{f(\rho_1)}d\rho_1^2 + \frac{(\rho_2-\rho)(\rho_2-\rho_1)}{f(\rho_2)}d\rho_2^2,$$

which permits one to obtain the expression for the linear element in the orthogonal system considered. Formula (23) will give:

(36) $$M^2\, ds^2 = \frac{(\rho-\rho_1)(\rho-\rho_2)}{f(\rho)}d\rho^2 + \frac{(\rho_1-\rho)(\rho_1-\rho_2)}{f(\rho_1)}d\rho_1^2 + \frac{(\rho_2-\rho)(\rho_2-\rho_1)}{f(\rho_2)}d\rho_2^2,$$

in which $M$ has the value:

(37) $$M = 2\sum_{k=1}^{5}\frac{1}{R_k}\sqrt{\frac{(a_k-\rho)(a_k-\rho_1)(a_k-\rho_2)}{f(a_k)}}.$$

If one makes $\rho_2$ = constant, in particular, then one will obtain the linear element of one of the cyclides that the system is composed of in the form:

$$M^2\, ds^2 = (\rho - \rho_1)\left[\frac{\rho-\rho_2}{f(\rho)}d\rho^2 - \frac{\rho_1-\rho_2}{f(\rho_1)}d\rho_1^2\right].$$

One sees that cyclides possess the property that we have already recognized in second-degree surfaces (no. **121**), namely, that they are divided into infinitesimal squares by their lines of curvature. One can then make the chart an arbitrary region that is traced on one of those surfaces.

**155.** In the theory of conjugate systems and asymptotic lines, the use of homogeneous coordinates allows us to recognize immediately, and with no calculation, that the properties of those systems and those lines will persist when one subjects the surface to a homographic transformation or a transformation by polar reciprocals. Indeed, the homogeneous coordinates of an arbitrary point do not change when one performs an arbitrary homographic transformation, provided that one supposes that the transformation is performed on the reference tetrahedron at the same time as on the points of space. The system of penta-spherical coordinates enjoys an analogous property under inversion. Here is how one can prove that:

First, recall that the theory of contact of spheres, their centers, and axes of similitude has led geometers to consider the radius of a sphere to be a quantity that can take on a sign, in such a way that in many studies, there is a great advantage to regarding two spheres as distinct when they have the same center, but their radii are equal, but with opposite signs. In particular, we shall see that if one subjects an arbitrary sphere to an inversion then one can rationally determine the radius of the transformed sphere as a function of the proposed sphere.

Indeed, let:

$$x = \frac{k^2 X}{X^2+Y^2+Z^2}, \qquad y = \frac{k^2 Y}{X^2+Y^2+Z^2}, \qquad z = \frac{k^2 Z}{X^2+Y^2+Z^2} \tag{38}$$

be the formulas that define the inversion.

Upon applying these formulas to the transformation of a sphere ($S$) with a center $x_0$, $y_0$, $z_0$ and radius $R$ that is defined by the equation:

$$S = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - r^2 = 0, \tag{39}$$

one will obtain the identity:

$$S = \frac{x_0^2 + y_0^2 + z_0^2 - r^2}{X^2+Y^2+Z^2} [(X - X_0)^2 + (Y - Y_0)^2 + (Z - Z_0)^2 - R^2], \tag{40}$$

in which $X_0$, $Y_0$, $Z_0$ are give by the formulas:

$$\begin{cases} X_0 = \dfrac{k^2 x_0}{x_0^2 + y_0^2 + z_0^2 - r^2}, \\[2ex] Y_0 = \dfrac{k^2 y_0}{x_0^2 + y_0^2 + z_0^2 - r^2}, \\[2ex] Z_0 = \dfrac{k^2 z_0}{x_0^2 + y_0^2 + z_0^2 - r^2}, \end{cases} \tag{41}$$

$$R = \frac{\pm k^2 r}{x_0^2 + y_0^2 + z_0^2 - r^2}.$$

One can take an arbitrary sign in the latter formula. However, if one agrees to constantly take the same sign (for example, the + sign) then formulas (41) and the following one:

(42) $$R = \frac{k^2 r}{x_0^2 + y_0^2 + z_0^2 - r^2}$$

will determine not only the position of the transformed sphere, but the sign of its radius, as well.

With that hypothesis, formula (40) will become:

$$\frac{S}{r} = \frac{k^2}{X^2 + Y^2 + Z^2} \frac{S'}{R},$$

in which $S'$ denotes the power of the point ($X$, $Y$, $Z$) with respect to the transformed sphere, and one can state the following result:

*If one divides the power S of a point with respect to a sphere by the radius r of that sphere, and one then subjects the entire figure to an inversion then the quotient S / r will be reproduced, but multiplied by a quantity that does not depend upon the sphere and contains only coordinates of the point.*

In particular, the five quantities $S_i / R_i$ will be reproduced, and all of them will be multiplied by the same number, and since the coordinates $x_i$ are proportional to them, one can say that those coordinates will remain invariable, while only the proportionality factor changes, provided that one refers to the new figure to the orthogonal spheres that are derived from the original spheres by the inversion considered. One then sees that all of the properties that are established for a figure independently of the choice of coordinate spheres will necessarily persist for all of the inverse figures of the figure considered.

Having accepted that proposition, the theorem of no. **154** will show immediately that the inversion will preserve the lines of curvature of the surface.

**156**. We conclude by pointing out the main formulas that relate to the sphere. Equation (21), which gives the distance between two points, permits us to write down immediately the equation of a sphere whose radius is $\rho$ and whose center has coordinates $\alpha_1, \ldots, \alpha_5$ in the form:

(43) $$2\sum \alpha_k x_k + \rho^2 \sum \frac{\alpha_k}{R_k} \sum \frac{x_k}{R_k} = 0.$$

That equation is linear with respect to the coordinates $x_i$ . Conversely, any equation of the form:

$$\sum_{k=1}^{5} m_k x_k = 0 \tag{44}$$

represents a sphere or a plane. In order to see that, it will suffice to replace the $x_k$ with their expressions (18) in terms of $x, y, z$. In order to obtain the center and radius of that sphere, one must identify equations (43) and (44). One will then have the equations:

$$\mu\, m_k = 2\alpha_k + \frac{\rho^2}{R_k}\sum\frac{\alpha_k}{R_k} \qquad (k = 1, \ldots, 5), \tag{45}$$

in which $\mu$ denotes the proportionality factor.

Since one can multiply the $\alpha_k$ by an arbitrary number, one can replace the factor $\mu$ with an arbitrarily-chosen number; for example, we set:

$$\mu = 2.$$

Multiply equation (45) by $1 / R_k$ and add the equations that correspond to the various values of the index $k$. Upon taking into account the relation:

$$\sum\frac{1}{R_k^2} = 0,$$

which was pointed out already on page 189, we will have:

$$\sum\frac{\alpha_k}{R_k} = \sum\frac{m_k}{R_k}.$$

On the other hand, if we once more add equations (45), after having squared both sides, then we will find that:

$$\sum_{k=1}^{5} m_k^2 = \rho^2\left(\sum\frac{\alpha_k}{R_k}\right)^2.$$

The agreement of the preceding two formulas will give us:

$$\rho = \frac{\sqrt{\sum m_k^2}}{\sum\frac{m_k}{R_k}}, \tag{46}$$

and, upon substituting that value of $\rho$ in equation (45), we will have:

(47) $$\alpha_k = m_k - \frac{1}{2R_k}\frac{\sum m_k^2}{\sum \frac{m_k}{R_k}}.$$

This is the formula that will give us the penta-spherical coordinates of the center.

A sphere is determined completely if one knows the ratios of the five quantities $m_k$, and for that reason, one can call then the *homogeneous coordinates of the sphere.* However, in all questions where the sign of the radius enters into consideration, it is necessary to introduce a new coordinate that will serve to show the value of the radical that appears in the expression for the radius. If we let $m_6$ denote that sixth coordinate then we set:

(48) $$i\, m_6 = \sqrt{\sum_{k=1}^{5} m_k^2}\,,$$

in order that the relation between the six coordinates will take the very symmetric form:

(49) $$\sum_{k=1}^{6} m_k^2 = 0.$$

Two spheres with the same center and radii that are equal, but opposite in sign, will have the same coordinates $m_1, \ldots, m_5$ ; however, the coordinates $m_6$ will be equal and opposite in sign. From formula (46), one will have:

(50) $$\rho = \frac{i\, m_6}{\sum_{k=1}^{6} \frac{m_k}{R_k}}.$$

Having said that, consider two spheres $(S)$, $(S')$ whose coordinates are $m_k$, $m'_k$, respectively, and let $d$ be the distance between their centers, while $\rho$, $\rho'$ are their radii. Formulas (21) and (47) allow us to calculate $d$, and they give us:

(51) $$d^2 - \rho^2 - \rho'^2 = \frac{-2\sum_{k=1}^{5} m_k m'_k}{\sum_{k=1}^{5} \frac{m_k}{R_k} \sum_{k=1}^{5} \frac{m'_k}{R_k}}.$$

If we would like to calculate the angle $V$ between the two spheres, which we define precisely by the relation:

(52) $$d^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos V,$$

then, upon applying formulas (50), (51), we will have:

(53) $$\cos V = -\frac{\sum_{k=1}^{5} m_k m'_k}{m_6 m'_6},$$

and consequently:

(54) $$2\sin^2\frac{V}{2} = \frac{\sum_{k=1}^{6} m_k m'_k}{m_6 m'_6}.$$

These formulas lead us to several consequences, and in particular, to the geometric definition of the coordinates of the sphere.

Suppose that the sphere ($S'$) reduces to the coordinate sphere ($S_k$), so one will have:

$$m'_1 = 0, \qquad \ldots, \qquad m'_k = 1, \qquad \ldots, \qquad m'_5 = 0,$$

and formula (50), in which one sets $\rho = R_k$, which will give:

$$m'_6 = -i\,;$$

equation (53) will then become:

$$m_k = i\, m_6 \cos V_k\,,$$

in which $V_k$ denotes the angle between the sphere ($S$) and the sphere ($S_k$).

Thus, the five coordinates $m_1, \ldots, m_5$ of an arbitrary sphere are proportional to the cosines of the angles between that sphere and the coordinate spheres. Moreover, the relations between the six coordinates will take the form:

$$\sum_{k=1}^{5} \cos^2 V_k = 1,$$

which is completely analogous to the one that relates the angles between a plane and the three coordinate planes in the geometry of Descartes. Formula (53) can also be written in the form:

(55) $$\cos V = \sum_{k=1}^{5} \cos V_k \cos V'_k,$$

and its analogy with the one that gives the cosine of the angle between two planes is likewise obvious.

If one takes the identity relations between the coordinates into account then formula (54) can be written:

(56) $$4\sin^2\frac{V}{2} = \frac{-\sum_{k=1}^{6}(m_k - m'_k)^2}{m_6 m'_6}.$$

If one supposes that the two spheres are infinitely close then the angle between them $dv$ will be provided by the relation:

$$dv^2 = \frac{-\sum_{k=1}^{6} dm_k^2}{m_6^2}. \tag{57}$$

Whenever two spheres are tangent, they will intersect at an angle 0 or $\pi$. However, in the theories where one takes the sign of the radius into account, there is some advantage to considering them to be tangent only when they intersect at an angle of zero. When thus extended, the condition of contact will be expressed by the relation:

$$\sum_{k=1}^{6} (m_k - m'_k)^2 = -2\sum_{k=1}^{6} m_k m'_k = 0. \tag{58}$$

When the two spheres are infinitely close, the distinction in regard to the sign of the radius will disappear, and the condition of contact will become:

$$\sum_{k=1}^{6} dm_k^2 = 0. \tag{59}$$

**157.** The form of the condition of contact ultimately leads to a crucial point of agreement between the geometry of spheres and that of straight lines. Recall the equation of the straight line, when written in the form that was already employed in no. **139**:

$$\begin{cases} qz - ry + p_1 = 0, \\ rx - pz + q_1 = 0, \\ py - qx + r_1 = 0. \end{cases} \tag{60}$$

As we have seen, the six homogeneous coordinates of the straight line satisfy the identity equation:

$$pp_1 + qq_1 + rr_1 = 0, \tag{61}$$

and conversely (no. **139**), six quantities that satisfy that relation will always define a straight line. Equation (61) is quadratic, like the relation (49) between the six coordinates of the sphere, and one can convert these relations into each other by setting:

$$\begin{cases} p = m_1 + im_2, \quad q = m_3 + im_4, \quad r = m_3 + im_6, \\ p_1 = m_1 - im_2, \quad q_1 = m_3 - im_4, \quad r_1 = m_3 - im_6, \end{cases} \tag{62}$$

or, what amounts to the same thing:

(63)
$$\begin{cases} m_1 = \dfrac{p+p_1}{2}, & m_3 = \dfrac{q+q_1}{2}, & m_5 = \dfrac{r+r_1}{2}, \\ m_2 = i\dfrac{p-p_1}{2}, & m_4 = i\dfrac{q-q_1}{2}, & m_6 = i\dfrac{r-r_1}{2}. \end{cases}$$

Formulas (62) or (63), which lead to the identity:

$$pp_1 + qq_1 + rr_1 = \sum_{k=1}^{6} m_k^2 ,$$

make any line correspond to a sphere, and *vice versa*. Moreover, if one considers two spheres with coordinates $m_k$, $m'_k$, respectively, and for the two corresponding lines with coordinates $p, q, r, \ldots; p', q', r', \ldots$, it results from the preceding identity and the linear form of the equations of correspondence that there is a more general identity:

(64)
$$(p - p_1)(p_1 - p'_1) + (q - q_1)(q_1 - q'_1) + (r - r_1)(r_1 - r'_1) = \sum_{k=1}^{6} (m_k - m'_k)^2 .$$

The right-hand side of this formula will be annulled when the two spheres are tangent, and only in that case; the left-hand side will be annulled when the two lines considered intersect. One then sees that the transformation that is defined by the formulas (62) or (63) makes two tangent spheres correspond to two lines that intersect, and *vice versa*.

Furthermore, one can define that transformation without passing to homogeneous coordinates, which have been the object of our studies in this chapter, because if one takes the equations of a line in the form:

(65)
$$\begin{cases} x = az + p, \\ y = bz + q \end{cases}$$

then the condition for the two different lines to intersect will be expressed by the well-known condition:

$$(a - a')(q - q') - (b - b')(p - p') = 0,$$

and if one sets:

(66)
$$\begin{cases} a = x + yi, & b = z + R, \\ q = x - yi, & p = R - z, \end{cases}$$

and similarly:

$$\begin{aligned} a' &= x' + y' i, & b' &= z' + R', \\ q' &= x' - y' i, & p' &= R' - z', \end{aligned}$$

then it will become:

$$(x - x')^2 + (y - y')^2 + (z - z')^2 - (R - R')^2 = 0.$$

Hence, if one considers formulas (66) as establishing a correspondence between the arbitrary line that is represented by equations (65) and the sphere whose radius is $R$ and whose center has the Cartesian coordinates $x$, $y$, $z$ then one will see that these formulas make two lines that intersect correspond to two spheres that touch, and conversely.

That transformation, which establishes a link between straight lines and spheres – i.e., between the most essential elements in space – is one of the most beautiful discoveries of modern geometry. It is due to Sophus Lie, who presented it with some important consequences in a paper that was included in volume V of the *Mathematischen Annalen* ([30]). Among these consequences, one must, above all, cite the following one:

*Lie's transformation makes the set of lines that are tangent to a surface* ($S$) *correspond to the set of spheres that are tangent to another surface* ($S'$). *All lines that are tangent to a point M of* ($S$) *will correspond to all spheres that are tangent to a point M′ of* ($S'$). *When the point M describes an asymptotic line of* ($S$), *the point M′ will describe a line of curvature of* ($S'$). *As a result, Lie's transformation can make any surface for which one knows how to determine the asymptotic lines correspond to another surface for which one knows the lines of curvature, and* vice versa.

We studied that theorem with all of the necessary details in our course during 1881-82, which had the geometric theory of partial differential equations for its objective. We will be led to it later on by an indirect path. However, in order to establish it with all desirable breadth, we would be obliged to develop a theory of contact here that would take us far from our objective, and which we reserve for another occasion.

———

([30]) SOPHUS LIE, "Ueber Complexe, insbesondere Linien- und Kugel-Complexe, mit Anwendung auf die Theorie partieller Differentialgleichungen," Mathematischen Annalen **5** (1871), 145-256.

# CHAPTER VII

## LINES OF CURVATURE IN TANGENTIAL COORDINATES.

Case in which the surface is defined by its tangential equation. – Application to the surface of class four that is normal to all positions of a invariable line, three points of which describe three rectangular planes. – Case in which the tangential coordinates are expressed as functions of two parameters. – First solution to the problem that has the objective of determining the surfaces that admit a spherical representation that is given by their lines of curvature. – Developments on a particular system of tangential coordinates that was employed by O. Bonnet in the study of surfaces.

________

**158.** We shall now pass on to the examination of the case in which the surface is defined by a certain property of its tangent planes, and we first suppose that one knows the homogeneous equation:

(1) $$f(u, v, w, p) = 0$$

that couples the coordinates of the tangent plane, in which the axes are rectangular. The point of contact of the tangent plane will have the coordinates:

(2) $$x = \frac{\partial f / \partial u}{\partial f / \partial p}, \quad y = \frac{\partial f / \partial v}{\partial f / \partial p}, \quad z = \frac{\partial f / \partial w}{\partial f / \partial p},$$

and the direction cosines of the normal will be proportional to $u$, $v$, $w$. In order to obtain the differential equation of the line of curvature, it will then suffice to apply equation (4) of no. **138**, which will give:

$$\begin{vmatrix} \frac{\partial f}{\partial p} d\frac{\partial f}{\partial u} - \frac{\partial f}{\partial u} d\frac{\partial f}{\partial p} & u & du \\ \frac{\partial f}{\partial p} d\frac{\partial f}{\partial v} - \frac{\partial f}{\partial v} d\frac{\partial f}{\partial p} & v & dv \\ \frac{\partial f}{\partial p} d\frac{\partial f}{\partial w} - \frac{\partial f}{\partial w} d\frac{\partial f}{\partial p} & w & dw \end{vmatrix} = 0,$$

or, in a more symmetric form:

(3) $$\begin{vmatrix} \frac{\partial f}{\partial u} & d\frac{\partial f}{\partial u} & u & du \\ \frac{\partial f}{\partial v} & d\frac{\partial f}{\partial v} & v & dv \\ \frac{\partial f}{\partial w} & d\frac{\partial f}{\partial w} & w & dw \\ \frac{\partial f}{\partial p} & d\frac{\partial f}{\partial p} & 0 & 0 \end{vmatrix} = 0.$$

It is important to remark that from the standpoint of applications the preceding equation still keep its form even when equation (1) is not homogeneous, provided that one supposes that $u$, $v$, $w$ are equal to the direction cosines of the normal and that they are coupled by the equation:

$$u^2 + v^2 + w^2 = 1.$$

In order to prove this, suppose that equation (1) is not homogeneous. One can always make it homogeneous and of degree zero – for example, by dividing $u$¸ $v$, $w$, $p$ by the quantity:

$$h = \sqrt{u^2 + v^2 + w^2}\,,$$

which is equal to unity. One must then replace $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}$ with:

$$\frac{\partial f}{\partial u} + \frac{\partial f}{\partial h}\frac{u}{h}, \qquad \frac{\partial f}{\partial v} + \frac{\partial f}{\partial h}\frac{v}{h}, \qquad \frac{\partial f}{\partial w} + \frac{\partial f}{\partial h}\frac{w}{h}$$

in equation (3), which is equivalent to adding the last two columns of the determinant (3) to the first two, after multiplying them by suitably-chosen coefficients and not changing the value of the determinant.

**159.** For example, consider the surface of class four that is defined by the equation:

(4) $$p = \frac{au^2 + bv^2 + cw^2}{2},$$

in which $u$, $v$¸ $w$ are the direction cosines of the normal, so $p$ denotes the distance from the origin to the tangent plane. Upon making the equation homogeneous, for the moment, and applying formulas (2), one will find that the coordinates of the point of contact of the tangent plane have the values:

(5) $$x = (p - a)\,u, \quad y = (p - b)\,v, \quad z = (p - c)\,w,$$

and those of a point on the normal that is situated at the distance $\lambda$ from the foot of that normal will be:

(6) $$X = (p + \lambda - a)\,u, \qquad Y = (p + \lambda - b)\,v, \qquad Z = (p + \lambda - c)\,w.$$

The points where the normal cuts the three coordinate planes correspond to the values:

(7) $$\lambda = a - p, \qquad \lambda = b - p, \qquad \lambda = c - p,$$

whose differences are constants. One will then already have this elegant proposition:

*When three points of an invariable line describe three rectangular planes, and consequently all of the other points describe ellipsoids, the line will constantly remain normal to a family of parallel surfaces that are represented by an equation of the form:*

$$(8) \qquad p = \frac{(a+k)u^2 + (b+k)v^2 + (c+k)w^2}{2},$$

*in which k denotes the constant that varies when one passes from a surface to a parallel one* ([31]).

One can, moreover, obtain one construction from points of these surfaces very easily. Look for the foot of the perpendicular to the normal that is based at the coordinate origin. The value $\lambda$ that corresponds to that point will be determined by the equation:

$$uX + vY + wZ = 0,$$

which will give:

$$\lambda = p$$

upon applying formulas (6).

Let $P$ be that point, and let $M$ be the point where the normal cuts the $yz$-plane, and which will have the value $\lambda = a - p$, as we have seen. The midpoint of the segment $PM$ will obviously correspond to a value of $\lambda$ that is one-half the sum of the preceding values, and as a result, equal to $a / 2$. Since that value is constant, the midpoint of the segment will describe a surface that is parallel to the proposed one. Therefore:

*If one considers all positions of the moving invariable line then the midpoint of the segment that is defined by the point where that line cuts one of the coordinate planes and the foot of the perpendicular that is based at the origin on the line will describe one of the surfaces that are normal to the line in its various positions* ([32]).

We now propose to determine the lines of curvature. Equation (3) takes the form:

$$\begin{vmatrix} u & du & a\,du \\ v & dv & b\,dv \\ w & dw & c\,dw \end{vmatrix} = 0$$

here, and its integral, which one finds easily, will be defined by the equation:

$$\frac{u^2}{a-\rho} + \frac{v^2}{b-\rho} + \frac{w^2}{c-\rho} = 0,$$

([31]) G. DARBOUX, "Sur une nouvelle définition de la surface des ondes," Comptes rendus **92** (1881), pp. 446.

([32]) Mannheim recovered all of these results by the considerations of pure geometry in an article that was included in Bulletin des Sciences mathematiques (2) **9** (1885), pp. 137.

in which $r$ denotes the arbitrary constant. That result is interpreted as follows:

*The spherical representation of two families of lines of curvature of the surface is given by a homofocal system of spherical ellipses* ([33]).

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([33]) We would not like to insist upon that particular study, and we shall content ourselves here by stating the following two propositions:

*Consider the ellipsoid* ($E$) *that is defined by the equation:*

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1.$$

*The homographic transformation that is defined by the formulas:*

$$(2) \qquad \begin{cases} \dfrac{x}{\sqrt{ma+n}} = \dfrac{X}{m\sqrt{a}}, \\[2ex] \dfrac{x}{\sqrt{mb+n}} = \dfrac{Y}{m\sqrt{b}}, \\[2ex] \dfrac{z}{\sqrt{mc+n}} = \dfrac{X}{m\sqrt{c}}, \end{cases}$$

*in which m and n denote two arbitrary constants, makes the normals to* ($E$) *correspond to those of the ellipsoid* ($E_1$) *that has the equation:*

$$\frac{X^2}{ma+n} + \frac{Y^2}{mb+n} + \frac{Z^2}{mc+n} = 1.$$

*If one sets:*

$$n = m^2 k^2,$$

*and if one keeps* $k^2$ *fixed, but increases m indefinitely, then the ellipsoid* ($E_1$) *will be transformed into a sphere of very large radius, and its normals will become invariable lines, three points of which will describe the symmetry planes of* ($E$). *Those lines are derived from the normals to* ($E$) *by the homographic transformation:*

$$x = \frac{kX}{\sqrt{a}}, \qquad y = \frac{kY}{\sqrt{b}}, \qquad z = \frac{kZ}{\sqrt{c}},$$

*which is included in the first one* ($\alpha$) *as a limiting case, and will then be deduced when one introduces the hypotheses that were made on m and n.*

The surface that is studied in the text can then be considered to be a surface that is parallel to an ellipsoid whose axes are infinitely long.

*If an arbitrary surface enjoys the property that there exists a homographic transformation that transforms its normals into the normals to another surface and preserves the plane at infinity then the lines of curvature of that surface can always be determined and will admit a system of homofocal ellipsoids for a spherical representation.*

**160.** Now, suppose that one is given an arbitrary surface and knows the expressions for the tangential coordinates as functions of two parameters $\alpha$, $\beta$. The coordinates of the point of contact of the tangent plane will satisfy (no. **96**) the three equations:

$$
(9) \qquad \begin{cases} ux + vy + wz + p = 0, \\ x\dfrac{\partial u}{\partial \alpha} + y\dfrac{\partial v}{\partial \alpha} + z\dfrac{\partial w}{\partial \alpha} + \dfrac{\partial p}{\partial \alpha} = 0, \\ x\dfrac{\partial u}{\partial \beta} + y\dfrac{\partial v}{\partial \beta} + z\dfrac{\partial w}{\partial \beta} + \dfrac{\partial p}{\partial \beta} = 0. \end{cases}
$$

A point that is situated on the normal at a distance $\lambda$ from its foot will have the coordinates:

$$
(10) \qquad X = x + \frac{u\lambda}{h}, \quad Y = y + \frac{v\lambda}{h}, \quad Z = z + \frac{w\lambda}{h},
$$

in which $h$ denotes the radical:

$$
(11) \qquad h = \sqrt{u^2 + v^2 + w^2}.
$$

Replace $x$, $y$, $z$ in equations (9) with their expressions that one infers from formulas (10); the equations thus-obtained, viz.:

$$
(12) \qquad \begin{cases} uX + vY + wZ + p = h\lambda, \\ X\dfrac{\partial u}{\partial \alpha} + Y\dfrac{\partial v}{\partial \alpha} + Z\dfrac{\partial w}{\partial \alpha} + \dfrac{\partial p}{\partial \alpha} = \lambda\dfrac{\partial h}{\partial \alpha}, \\ X\dfrac{\partial u}{\partial \beta} + Y\dfrac{\partial v}{\partial \beta} + Z\dfrac{\partial w}{\partial \beta} + \dfrac{\partial p}{\partial \beta} = \lambda\dfrac{\partial h}{\partial \beta}, \end{cases}
$$

will define the point considered on the normal. In order to find the differential equation for the lines of curvature, we will further write down the idea that there exists a displacement for which the point that corresponds to a suitably-chosen value of $\lambda$ will describe a curve that is tangent to the normal; i.e., for which we have:

$$
(13) \qquad \frac{dX}{u} = \frac{dY}{v} = \frac{dZ}{w} = d\theta,
$$

into which $d\theta$ is introduced for homogeneity.

If one differentiates formulas (12) under those hypotheses then the first one will give:

$$
u\,dX + v\,dY + w\,dZ + X\,du + Y\,dv + Z\,dw + dp = \lambda\,dh + h\,d\lambda,
$$

or, upon taking into account the following two equations, along with formulas (13):

(14) $$h^2\, d\theta = h\, d\lambda, \qquad d\lambda = h\, d\theta.$$

The differentiation of the last two formulas in (12) will then lead us to the two equations:

(15) $$\begin{cases} X\, d\dfrac{\partial u}{\partial \alpha} + Y\, d\dfrac{\partial v}{\partial \alpha} + Z\, d\dfrac{\partial w}{\partial \alpha} + d\dfrac{\partial p}{\partial \alpha} = \lambda\, d\dfrac{\partial h}{\partial \alpha}, \\ X\, d\dfrac{\partial u}{\partial \beta} + Y\, d\dfrac{\partial v}{\partial \beta} + Z\, d\dfrac{\partial w}{\partial \beta} + d\dfrac{\partial p}{\partial \beta} = \lambda\, d\dfrac{\partial h}{\partial \beta}. \end{cases}$$

Finally, the elimination of $X$, $Y$¸ $Z$, $\lambda$ from equations (12) and (15) will give us the desired equation in the form of the determinant:

(16) $$\begin{vmatrix} u & \dfrac{\partial u}{\partial \alpha} & \dfrac{\partial u}{\partial \beta} & d\dfrac{\partial u}{\partial \alpha} & d\dfrac{\partial u}{\partial \beta} \\ v & \dfrac{\partial v}{\partial \alpha} & \cdots & \cdots & \cdots \\ w & \dfrac{\partial w}{\partial \alpha} & \cdots & \cdots & \cdots \\ p & \dfrac{\partial p}{\partial \alpha} & \cdots & \cdots & \cdots \\ h & \dfrac{\partial h}{\partial \alpha} & \dfrac{\partial h}{\partial \beta} & d\dfrac{\partial h}{\partial \alpha} & d\dfrac{\partial h}{\partial \beta} \end{vmatrix} = 0.$$

**161.** That differential equation provides the greatest degree of analogy with the one that we defined (no. **143**) for the point-like coordinates, and the repetition of the arguments that were employed in nos. **143**, **144** will lead us to the following propositions:

*Form the linear partial differential equation:*

(17) $$A\frac{\partial^2 \theta}{\partial \alpha^2} + B\frac{\partial^2 \theta}{\partial \alpha\, \partial \beta} + C\frac{\partial^2 \theta}{\partial \beta^2} + D\frac{\partial \theta}{\partial \alpha} + E\frac{\partial \theta}{\partial \beta} + F\,\theta = 0,$$

*which admits the five functions $u$, $v$, $w$, $p$, $h$ of $\alpha$ and $\beta$ as particular solutions. When one has obtained them in an arbitrary manner, the characteristics of that equation, which are defined by the differential equation:*

(18) $$A\, d\beta^2 - B\, d\alpha\, d\beta + C\, d\alpha^2 = 0,$$

*will be the lines of curvature of the surface. As a result, if the coefficients $A$ and $C$ are zero then $\alpha$ and $\beta$ will be the parameters of the lines of curvature.*

That general proposition implies the following theorem as a consequence:

*If one is given the equation:*

$$\frac{\partial^2\theta}{\partial\alpha\,\partial\beta} = A'\frac{\partial\theta}{\partial\alpha} + B'\frac{\partial\theta}{\partial\beta} + C'\theta, \tag{19}$$

*in which $A'$, $B'$, $C'$ are arbitrary functions of $\alpha$ and $\beta$, and one knows four particular solutions $u$, $v$, $w$, $h$ that are coupled by the relation:*

$$u^2 + v^2 + w^2 = h^2, \tag{20}$$

*then the enveloping surface of the planes:*

$$uX + vY + wZ + \theta = 0,$$

*in which $\theta$ denotes an arbitrary solution of equation* (19), *will be referred to the system of curvilinear coordinates* ($\alpha$, $\beta$) *that is defined by its lines of curvature.*

Indeed, the linear equation of the form (17), which must then be satisfied by the five quantities $u$, $v$, $w$, $p$, $h$ that relate to that surface, will be equation (19), and consequently, $\alpha$ and $\beta$ will be the parameters of the lines of curvature.

We have shown (no. **98**) that if one has obtained an arbitrary conjugate system on a surface then the tangential coordinates $u$, $v$, $w$, $p$, when considered to be functions of the parameters $\alpha$ and $\beta$ of the two conjugate families, must satisfy a linear equation of the form (19). As one sees, the linear equation that relates to the conjugate system that is defined by the lines of curvature is distinguished from all of the other ones by the property of admitting the solution:

$$h = \sqrt{u^2 + v^2 + w^2}\,,$$

in addition.

**462.** The preceding theorems permit one to define very easily the partial differential equation upon which one bases the search for surfaces that admit two families of orthogonal curves that are chosen arbitrarily on the sphere of radius 1 for their spherical representation.

Indeed, let $u$, $v$, $w$, $h$ be the homogeneous coordinates of a point on the sphere, which are coupled by the equation:

$$u^2 + v^2 + w^2 = h^2,$$

and which we suppose are expressed as functions of the parameters $\alpha$, $\beta$ of two orthogonal families. An orthogonal system that is traced on the sphere is, by that fact itself, a conjugate system, so $u$, $v$, $w$, $h$ will satisfy an equation of the form:

$$\frac{\partial^2\theta}{\partial\alpha\,\partial\beta} + A\frac{\partial\theta}{\partial\alpha} + B\frac{\partial\theta}{\partial\beta} + C\theta = 0, \tag{21}$$

which is an equation that is easy to obtain in an explicit manner, since one knows four particular solutions of it, namely, $u$, $v$, $w$, $h$. Having said that, since the tangent plane to the surface is parallel to the tangent plane that corresponds to the sphere, it will be represented by an equation of the form:

$$uX + vY + wZ + p = 0,$$

in which $p$ must satisfy the same linear equation as $u$, $v$, $w$ – i.e., equation (21). In order to solve the problem, it will then suffice to integrate equation (21), and each particular solution of that equation will give a particular solution of the problem that was posed.

For example, suppose that one proposes to determine the surfaces that admit a system of homofocal spherical ellipses for the spherical representation of their lines of curvature. If $\rho$, $\rho_1$ denote the parameters of those ellipses then one will have:

$$u^2 = \frac{(a-\rho)(a-\rho_1)}{(a-b)(a-c)}, \quad v^2 = \frac{(b-\rho)(b-\rho_1)}{(b-a)(b-c)}, \quad w^2 = \frac{(c-\rho)(c-\rho_1)}{(c-a)(c-b)},$$

here.

The equation that $u$, $v$, $w$ must satisfy will be the following one:

$$2\,(\rho-\rho_1)\,\frac{\partial^2\theta}{\partial\rho\,\partial\rho_1} + \frac{\partial\theta}{\partial\rho} + -\frac{\partial\theta}{\partial\rho_1} = 0, \tag{22}$$

and it will suffice to integrate that equation in order to obtain the complete solution of the proposed problem.

The surface that was studied in no. **159** corresponds to the solution:

$$\theta = \rho + \rho_1 .$$

We will have occasion to return to the general problem for which we just pointed out one solution.

**163.** Instead of pursuing those particular applications, we shall now show how one determines the principal radii and the lines of curvature when one adopts a special system of tangential coordinates that was employed by O. Bonnet, along with some other systems that are worthy of interest, in the beautiful "Mémoire sur l'emploi d'un nouveau système de variables dans l'étude des propriétés des surface courbes," Journal de Liouville (2) **5**(1860), 153-266. Here is how one is led to choose the variables that were considered by that eminent geometer.

When one studies the spherical representation, it is natural to seek the geometric definition of the curves on the surface that admit the various rectilinear generators of the sphere for their spherical representation. Let $d$ be the one of those generators that cuts the circle at infinity at a point $\mu$. The curve that corresponds to it on the surface will obviously be the locus of points of contact of the tangent planes that are parallel to *d*. In other words, that will be the curve of contact of the cone whose summit is $\mu$ that circumscribes the surface.

Those curves of contact of the circumscribed cones whose summits are found on the circle at infinity enjoy an important property in regard to the lines of curvature that we shall point out. First, two of them pass through each point *M* of the surface, because if *A* and *B* are the points where the tangent plane at *M* cuts the circle at infinity then the circumscribed cones with summits *A* and *B* will touch the surface along two curves that pass through *M*. The two tangents to those curves of contact will have the generators of those two cones for their conjugates; i.e., the two lines *MA*, *MB* of length zero in the tangent plane. Now, those two lines *MA*, *MB* are placed symmetrically with respect to the two arbitrary perpendicular tangents, and in particular, with respect to the directions of the lines of curvature. The same thing will be true for their conjugates, which are tangents to the two curves of contact. That implies the following theorem:

*The curves of contact of the circumscribed cones that have their summits on the circle of infinity determine a system of curvilinear coordinates on the surface that admit the system of rectilinear generators of the sphere for their spherical image. The tangents to the two coordinate curves that pass through an arbitrary point of the surface will admit the directions of the lines of curvature for their bisectors.*

Consequently, if one employs the coordinate system that we just defined then the equation for the lines of curvature can be converted into the simple form:

$$A\,d\alpha^2 + C\,d\beta^2 = 0,$$

and will no longer contain the term in $d\alpha\,d\beta$.

**164.** Here is how one verifies that important result: Always denote the direction cosines of the normal at a point *M* on the surface by $c$, $c'$, $c''$; $c$, $c'$, $c''$ will be the coordinates of the point *m* that serves as the spherical representation of *M*.

The expressions for those coordinates as functions of the parameters $\alpha$¸ $\beta$ of the rectilinear generators of the sphere have been given already (no. **15**). They are:

$$(23) \qquad c = \frac{1-\alpha\beta}{\alpha-\beta}, \quad c' = i\,\frac{1+\alpha\beta}{\alpha-\beta}, \quad c'' = \frac{\alpha+\beta}{\alpha-\beta}.$$

We write the equation of the tangent plane in the form:

$$cx + c'z + c''z + \frac{\xi}{\alpha-\beta} = 0,$$

or, more simply:

$$(24) \qquad (1 - \alpha\beta)\,x + i\,(1 + \alpha\beta)\,y + (\alpha + \beta)\,z + x = 0.$$

One will then have:

$$u = 1 - \alpha\beta, \qquad v = i\,(1 + \alpha\beta), \qquad w = \alpha + \beta, \qquad p = \xi.$$

The application of formulas (9) first gives us the coordinates of the point of contact. One will then find:

$$(25) \qquad \begin{cases} x - iy = \dfrac{q - p}{\alpha - \beta}, \\[2ex] x + iy = \dfrac{\alpha^2 q - \beta^2 p}{\alpha - \beta} - \xi, \\[2ex] z = \dfrac{\beta q - \alpha p}{\alpha - \beta}, \end{cases}$$

in which $p$ and $q$ denote the first derivatives of $\xi$. Upon similarly calling the second derivatives $r, s, t$, equation (16) of the lines of curvature will become:

$$(26) \qquad dp\, d\alpha - dq\, d\beta = r\, d\alpha^2 - t\, d\beta^2 = 0$$

here, and the general formulas (12) and (15), which tell us the center of principal curvature and its corresponding radius, will give us:

$$(27) \qquad 2R = (s + \sqrt{rt}\,)\,(\alpha - \beta) + p - q,$$

$$(28) \qquad \begin{cases} X - iY = s + \sqrt{rt}, \\ \quad 2Z = (\alpha + \beta)(s + \sqrt{rt}) - p - q, \\ X + iY = -\alpha\beta(s + \sqrt{rt}) + \alpha p + \beta q - \xi, \end{cases}$$

in which $X, Y, Z, R$ denote the coordinates of the center and the radius of curvature, resp. In all of these formulas, we have taken the value $\sqrt{\dfrac{r}{t}}$ for $\dfrac{d\beta}{d\alpha}$, in such a way that $\sqrt{rt}$ replaces:

$$t\frac{d\beta}{d\alpha} = r\frac{d\alpha}{d\beta}.$$

**165**. The preceding formulas lend themselves to a host of interesting applications, due to their simplicity, as well as the choice of variables to which they refer. One can give them another form that offers some advantages in certain research.

We have taken the expressions (23) for the direction cosines of the normal. The variables $\alpha$, $\beta$ possess the great advantage of being transformed by the same linear substitution when one performs either a change of axes or a displacement of the surface.

However, in certain applications where one is dealing with the determination of real surfaces, the complex variables $\alpha$ and $\beta$ will yield an inconvenient result, due to the fact that they are conjugate imaginaries. We have seen that for any real point, $\alpha$ will have $-1/\beta$ for its conjugate. We then change $\beta$ into $-1/\beta$ in formulas (23), which will give the following expressions for the direction cosines, which have been employed already in no. **31**:

(29) $$c = \frac{\beta+\alpha}{1+\alpha\beta}, \qquad c' = i\frac{\beta-\alpha}{1+\alpha\beta}, \quad c'' = \frac{\alpha\beta-1}{1+\alpha\beta},$$

and we take the equation of the tangent plane to be:

(30) $$(\alpha+\beta)\,x + i\,(\beta-\alpha)\,y + (\alpha\beta-1)\,z + \xi = 0;$$

$\xi$ will now be a real variable, and $\alpha$, $\beta$ will be conjugate imaginary variables whenever the surface is real, and one will be dealing with real tangent planes.

The coordinates of the point of contact of the tangent plane will now have the expressions:

(31) $$\begin{cases} z = \dfrac{\xi - p\alpha - q\beta}{1+\alpha\beta}, \\ x - iy = -\dfrac{\beta(\xi - p\alpha - q\beta)}{1+\alpha\beta} - p, \\ x + iy = -\dfrac{\alpha(\xi - p\alpha - q\beta)}{1+\alpha\beta} - q. \end{cases}$$

The differential equation of the lines of curvature will be:

(32) $$dp\,d\alpha - dq\,d\beta = r\,d\alpha^2 - t\,d\beta^2 = 0,$$

as in the preceding case.

Finally, the formulas that exhibit the center and radius of curvature will become:

(33) $$\begin{cases} 2R = \xi - p\alpha - q\beta + (1+\alpha\beta)(s+\sqrt{rt}), \\ 2Z = \xi - p\alpha - q\beta + (1-\alpha\beta)(s+\sqrt{rt}), \\ X - iY = -p + \beta(s+\sqrt{rt}), \\ X + iY = -q + \alpha(s+\sqrt{rt}). \end{cases}$$

In that form, one recognizes immediately that one has essentially real expressions for $X$, $Y$, $Z$, $R$.

Moreover, one passes from the first system of formulas to the second one by making the very simple substitution:

(34) $$\beta = -\frac{1}{\beta'}, \qquad\qquad \xi = \frac{\xi'}{\beta'}.$$

Finally, one will likewise obtain the differential equation for the asymptotic lines with no difficulty, which will be:

$$(35) \qquad (1 + \alpha\beta)\,(r\, d\alpha^2 + 2s\, d\alpha\, d\beta + t\, d\beta^2) + 2\, d\alpha\, d\beta\,(\xi - p\alpha - q\beta) = 0,$$

and the expression for the linear element:

$$(36) \qquad ds^2 = (z\, d\alpha + dq)\,(z\, d\beta + dp),$$

which is thus presented as being decomposed into its two factors.

**166.** From the standpoint of ultimate applications, it would not be pointless to examine what the coordinates $\alpha$, $\beta$, $\xi$ will become when one changes the coordinate axes, or – what amounts to the same thing – when one displaces the surface. First consider the original system, in which the equation of the tangent plane is:

$$(1 - \alpha\beta)\, X + i\,(1 + \alpha\beta)\, Y + (\alpha + \beta)\, Z + \xi = 0.$$

When one imparts a translation with components $\lambda$, $\mu$, $\nu$ on the surface, one must replace $X$, $Y$, $Z$ with $X - \lambda$, $Y - \mu$, $Z - \nu$ in the preceding equation; the new values $\xi'$ of $\xi$ will then be:

$$(37) \qquad \xi' = \xi - \lambda\,(1 - \alpha\beta) - i\mu\,(1 + \alpha\beta) - (\alpha + \beta)\,\nu.$$

Now, imagine that one turns the surface around the origin of the coordinates. If we remark that, from their definition, $\alpha$ and $\beta$ are the symmetric coordinates of the point $m$, which serves as the spherical representation of the tangent plane on the sphere of radius 1 then it will result from some propositions that were developed above (Book I, Chap. III) that the new values $\alpha_1$, $\beta_1$ of the coordinates $\alpha$, $\beta$ will be obtained by the same linear substitution that was performed on $\alpha$ and $\beta$. One will have:

$$(38) \qquad \alpha = \frac{m\alpha_1 + n}{p\alpha_1 + q}, \qquad \beta = \frac{m\beta_1 + n}{p\beta_1 + q}.$$

Denote the new value of $\xi$ by $\xi'$. Since the distance from the origin to the tangent plane will not change, one will have:

$$(39) \qquad \frac{\xi_1}{\alpha_1 - \beta_1} = \frac{\xi}{\alpha - \beta},$$

or, upon replacing $\alpha$, $\beta$ with their values:

$$(40) \qquad \xi = \frac{\xi_1 (mq - np)}{(p\alpha_1 + q)(p\beta_1 + q)}.$$

The proposed question is then resolved completely, as far as the first system of coordinates is concerned.

**167.** Upon reusing the same method for the second one, in which the tangent plane has the equation:

$$X(\alpha+\beta) + i\,Y(\beta-\alpha) + Z(\alpha\beta-1) + \xi = 0,$$

or, upon passing from the first system to the second one by the substitution (34), one will see that a translation $(\lambda, \mu, \nu)$ of the surface will give the new value for $\xi$:

$$\xi_1 = \xi - \lambda(\alpha+\beta) - i\,\mu(\beta-\alpha) - \nu(\alpha\beta-1). \tag{41}$$

Similarly, a rotation around the coordinate origin will be defined by the formulas:

$$\alpha = \frac{m\alpha_1+n}{p\alpha_1+q}, \qquad \beta = \frac{p-q\beta_1}{n\beta_1-m}, \tag{42}$$

$$\xi = \frac{\xi_1(mq-np)}{(m-n\beta_1)(p\alpha_1+q)}, \tag{43}$$

in which $\alpha_1$, $\beta_1$, $\xi_1$ denotes the new coordinates. If the rotation is real then one can take (no. **29**):

$$q = m_0, \qquad p = -n_0,$$

in which $m_0$, $n_0$ denote the conjugate imaginaries to $m$ and $n$. The preceding equations will then give:

$$\alpha = \frac{m\alpha_1+n}{-n_0\alpha_1+m_0}, \qquad \beta = \frac{m_0\beta_1+n_0}{-n\beta_1+m}, \qquad \xi = \xi_1\frac{mm_0+nn_0}{(m-n\beta_1)(m_0-n_0\alpha_1)}. \tag{44}$$

These formulas will be useful to us in the theory of minimal surfaces.

# CHAPTER VIII

## VARIOUS APPLICATIONS

Applications of the formulas that relate to the line of curvature that were given in the preceding chapter. – Lie's transformation when the lines of curvature of a surface correspond to the asymptotic lines of the transform. – Transformation by reciprocal directions. – Relations between the elements that correspond under that transformation. – Inversion in the system of coordinates ($\alpha$, $\beta$, $\xi$).

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**168.** We need to exhibit some applications of the systems of formulas that were developed in the preceding chapter. We shall now study the following ones, and since we will be doing research of a general nature, we shall prefer to employ the first system of formulas that were studied in nos. **164** and **166**.

We first remark that the differential equation (26) [pp. 212] of the lines of curvature is identical to the differential equation for asymptotic lines that was given in no. **110**. That implies a first result:

*One can make any surface whose asymptotic lines one knows correspond to a surface for which one knows how to determine the lines of curvature, and vice versa* ([34]).

We have already pointed out (no. **157**) that beautiful proposition, which is due to Lie. Here, we shall be content to remark that in equation (30) [pp. 125] for the asymptotic lines, $\alpha$ and $\beta$ denote real quantities for any real point of the surface, while $\alpha$ and $\beta$ are complex variables for the lines of curvature, and even for a real point. Consequently, the correspondence that is defined by the preceding proposition cannot exist between the real elements of two real surfaces.

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([34]) A comparison of the formulas that were given in nos. **110** and **161** will lead us to the following result: Let $x$, $y$, $z$ denote the coordinates of a point on the surface for which one knows the asymptotic lines, and let $p$ and $q$ be the derivatives of $z$, when they are considered to be functions of $x$ and $y$. Let $X$, $Y$, $Z$, $P$, $Q$ be the analogous quantities that relate to the transformed surface whose lines of curvature correspond to the asymptotic lines of the first one. One will have:

$$X + iY = -z - x\,\frac{px+qy}{q-x}, \qquad P = \frac{qx-1}{x+q},$$

$$X - iY = \frac{p+y}{q-x}, \qquad Q = -i\,\frac{1+qx}{x+q},$$

$$Z = \frac{px+qy}{q-x}.$$

Moreover, the theory of contact transformations permits one to deduce all of these formulas from the relations:

$$X + i\,Y = -z - x\,Z, \qquad x\,(X - iY) = Z - y,$$

which contain only the coordinates of the corresponding points.

**169.** We now imagine the differential equation of the lines of curvature:

$$dp\, d\alpha - dq\, d\beta = r\, d\alpha^2 - t\, d\beta^2 = 0. \tag{1}$$

It possesses numerous properties that all give rise to theorems in geometry.

First of all, it does not change form when one replaces $\xi$ with the new variable:

$$\xi' = \xi + A\, \alpha\beta + B\, \alpha + C\, \beta + D, \tag{2}$$

in which $A, B, C, D$ denote arbitrary constants.

We propose to define that transformation geometrically.

One can obviously obtain it by composing the following two:

$$\xi' = \xi + A\, (\alpha + \beta) + C\, \alpha\beta + D,$$
$$\xi' = \xi + h\, (\alpha - \beta)\,.$$

As one easily recognizes, the first of them is equivalent to a transformation of the origin of the coordinates; the second one replaces the surface with a parallel surface that is drawn at a distance $h$ from the first one. Indeed, one knows that the lines of curvature correspond to each other on two parallel surfaces. Thus, the first property of the differential equation for the lines of curvature that presents itself to us is only the analytical translation of an important, but well-known, geometric proposition.

**170.** The most general displacement of the proposed surface translates into an arbitrary linear substitution that is performed on $\alpha$ and $\beta$. That will lead us to the following general proposition, which one can verify with no difficulty:

*The differential equation* (1) *again preserves its form when one replaces $\alpha$, $\beta$, and $\xi$ with the variables $\alpha'$, $\beta'$, and $\xi'$, which are defined by one or the other of the substitutions:*

$$\alpha = \frac{A\alpha' + B}{C\alpha' + D}, \qquad \beta = \frac{A_1\beta' + B_1}{C_1\beta' + D_1}, \qquad \xi' = H\xi\, (C\alpha' + D)\, (C_1\, \beta' + D_1), \tag{3}$$

$$\alpha = \frac{A\beta' + B}{C\beta' + D}, \qquad \beta = \frac{A_1\alpha' + B_1}{C_1\alpha' + D_1}, \qquad \xi' = H\xi\, (C\beta' + D)\, (C_1\, \alpha' + D_1), \tag{4}$$

*where $A, A_1, \ldots, H$ denote arbitrary constants.*

Consequently, formulas (3) or (4) exhibit a transformation of surfaces that preserves the lines of curvature. We shall leave to the reader the task of proving that the transformation (when it is real) can always be obtained by the combined use of a displacement, a homothetic transformation, and the following one, which can seem very specialized at first:

If $k$ denotes an arbitrary constant then take:

$$\alpha' = \frac{1+k}{1-k}\beta, \qquad \beta' = \frac{1-k}{1+k}\alpha, \qquad \xi' = \xi. \tag{5}$$

Those formulas make the plane ($P$) that is defined by the equation:

$$(1 - \alpha\beta)\, X + i\,(1 + \alpha\beta)\, Y + (\alpha + \beta)\, Z + \xi = 0$$

correspond to another plane ($P'$) that has the equation:

$$(1 - \alpha\beta)\, X + i\,(1 + \alpha\beta)\, Y + \left[\frac{1+k}{1-k}\beta + \frac{1-k}{1+k}\alpha\right] Z + \xi = 0.$$

We have already seen that two corresponding planes will intersect in a fixed plane, namely, the $xy$-plane. Here is how one can succeed in defining the transformation:

Associate the plane ($P$) of the first figure with the point $m$ whose coordinates are the direction cosines of the normal to the plane:

$$c = \frac{1-\alpha\beta}{\alpha-\beta}, \qquad c' = i\,\frac{1+\alpha\beta}{\alpha-\beta}, \quad c'' = \frac{\alpha+\beta}{\alpha-\beta}.$$

That point is found on the sphere of radius 1, and if the planes ($P$) envelop a surface ($\Sigma$) then that point will be the spherical representation of the point of contact of ($P$) and ($\Sigma$).

Similarly, associate the plane ($P'$) with the point $m'$ whose coordinates are:

$$\frac{1-\alpha'\beta'}{\alpha'-\beta'}, \qquad i\,\frac{1+\alpha'\beta'}{\alpha'-\beta'}, \qquad \frac{\alpha'+\beta'}{\alpha'-\beta'},$$

and whose relationship to ($P'$) is the same as that of $m$ to ($P$). As one verifies with no difficulty, formulas (5) express the idea that the points $m$, $m'$ are in a straight line with a fixed point that is situated at the distance $k$ along the $z$-axis. The complete geometric definition of the transformation will result from that.

*Let* ($\Sigma$), ($\Sigma'$) *be two corresponding surfaces. The tangent planes to the corresponding points intersect in a fixed plane* ($\Pi$). *The spherical images of the corresponding points on the sphere of radius* 1 *that is located in space in an arbitrary manner are inverse to each other with respect to a fixed point A that is situated on the diameter of the sphere that is perpendicular to the plane* ($\Pi$).

That proposition obviously permits one to construct tangent planes to ($\Sigma'$) when one knows those of ($\Sigma$). As far as the points of contact are concerned, we add the following remark, which one can verify, but which will also result from some propositions that will be established later on:

*The line that connects the two corresponding points of contact is parallel to the one that links the spherical images of those two points.*

The relationships between corresponding elements that we just pointed out are obviously reciprocal, and consequently the transformation is *involutive*. That property is almost like inversion. For that reason, Laguerre, who has studied it in detail, gave it the name of *the transformation by reciprocal directions* ([35]), which we shall adopt in what follows. However, the preceding construction gives rise to another essential remark: The transformation is not *single-valued*, and in general it will make one surface (Σ) correspond to two surfaces (Σ′), or rather, two different sheets of the same surface. Indeed, if one considers a region of the surface (Σ) for which the sense of the normal is perfectly defined then each point of that region will have a spherical representation that is determined completely by the sense of the normal, and the construction that was shown above will exhibit the tangent plane to the corresponding surface with no ambiguity. However, that construction will obviously give different elements if one changes the sense of the normal at all points of (Σ). Moreover, the following formulas, which are equivalent to the relations (5), will show that:

Let:

$$u\,x + v\,y + w\,z + p = 0,$$
$$u'\,x + v'\,y + w'\,z + p' = 0$$

be the equation of two corresponding planes. Denote the radicals:

$$\pm\sqrt{u^2+v^2+w^2}\,, \qquad \pm\sqrt{u'^2+v'^2+w'^2}$$

by $h$ and $h'$, to abbreviate.

If we set:

$$u = 1 - \alpha\beta, \quad v = i\,(1 + \alpha\beta), \quad w = \alpha + \beta, \quad p = \xi, \quad h = \alpha - \beta,$$
$$u' = 1 - \alpha'\beta', \quad v' = i\,(1 + \alpha'\beta'), \quad w' = \alpha' + \beta', \quad p' = \xi', \quad h' = \alpha' - \beta'$$

then formulas (5) will give us:

$$\text{(6)} \qquad \begin{cases} u' = u, & w' + h' = \dfrac{1+k}{1-k}(w-h), \\ v' = v, & w' - h' = \dfrac{1-k}{1+k}(w+h). \\ p' = p, & \end{cases}$$

([35]) LAGUERRE, "Sur la transformation par directions réciproques," Comptes rendus **92** (1881), pp. 71.

The presence of the radical $h$ indeed shows that a given plane will correspond to two different planes, which will be impossible to separate analytically as long as the variable $h$ is not a perfect square.

**171.** Since the preceding transformation preserves lines of curvature, it will necessarily make a sphere correspond to a sphere. One verifies that proposition in the following manner.

The tangential equation for a sphere whose radius of $R$, and whose center has the coordinates $x$, $y$, $z$ is:

$$ux + vy + wz + p = R\,h, \tag{7}$$

in which $h$ has the significance that it was given before, because that equation expresses the idea that the distance from the center to the tangent plane is constant. In order to obtain the corresponding surface, perform the substitution that is defined by formulas (6). We will obtain the equation:

$$u'\,x + v'\,y + \left[\frac{1+k}{1-k}(w'-h') + \frac{1-k}{1+k}(w'+h')\right]\frac{z}{2} + p$$
$$= \frac{R}{2}\left[\frac{1+k}{1-k}(w'-h') - \frac{1-k}{1+k}(w'+h')\right],$$

or, upon rearranging terms and dropping the primes:

$$u\,x + v\,y + \left(\frac{1+k^2}{1-k^2}z - \frac{2k\,R}{1-k^2}\right)w = h\left(\frac{2k\,R}{1-k^2} - \frac{1+k^2}{1-k^2}R\right).$$

That equation, which has the same form as equation (7), represents a sphere whose center $(x', y', z')$ and radius $R$ are defined by the formulas:

$$\begin{cases} x' = x, & z' = \dfrac{1+k^2}{1-k^2}z - \dfrac{2k\,R}{1-k^2}, \\[2ex] y' = y, & R' = \dfrac{2k\,R}{1-k^2} - \dfrac{1+k^2}{1-k^2}R, \end{cases} \tag{8}$$

which also give:

$$\begin{cases} z' + R' = \dfrac{1+k}{1-k}(z-R), \\[2ex] z' - R' = \dfrac{1-k}{1+k}(z+R), \end{cases} \tag{9}$$

and consequently:

$$x'^2 + y'^2 + z'^2 - R'^2 = x^2 + y^2 + z^2 - R^2.$$

It follows from this that a sphere ($S$) of the first figure, which is represented by the point-like equation:

$$X^2 + Y^2 + Z^2 - 2xY - 2y\,Y - 2z\,Z + x^2 + y^2 + z^2 - R^2 = 0,$$

will correspond to a sphere ($S'$) of the second one that has the equation:

$$X^2 + Y^2 + Z^2 - 2xY - 2y\,Y - 2\left[\frac{z(1+k^2)-2kR}{1-k^2}\right] Z + x^2 + y^2 + z^2 - R^2 = 0.$$

One sees that the two spheres ($S$) and ($S'$) intersect the $xy$-plane – i.e., the plane ($\Pi$) of the transformation – along the same circle.

Let $V$, $V'$ denote the angles that the two spheres make with the plane ($\Pi$), and which are defined by the formulas:

$$\cos V = \frac{z}{R}, \qquad \cos V' = \frac{z'}{R'}.$$

Formulas (9) give us the relation:

$$\frac{1+\cos V'}{1-\cos V'} = \left(\frac{1+k}{1-k}\right)^2 \frac{1-\cos V'}{1+\cos V'}$$

between those angles, which one can convert into the simple form:

$$\tan\frac{V}{2}\tan\frac{V'}{2} = \frac{1-k}{1+k}. \tag{10}$$

Suppose that one of the two angles is constant, so the same thing will be true for the other. That will yield a new method of determining the surface ($\Sigma$) that corresponds to a surface ($\Sigma'$):

*Construct the spheres* ($S$) *that are tangent to* ($\Sigma$) *and cut the plane* ($\Pi$) *at a constant angle* $\alpha$. *Pass a sphere* ($S'$) *through the intersection of each sphere* ($S$) *and the plane* ($\Pi$) *that cuts* ($\Pi$) *at a given angle* $\beta$. *The envelope of the spheres* ($S'$) *will give the surface* ($\Sigma'$) *that corresponds to* ($\Sigma$).

A sphere of radius zero must be considered to be one that cuts an arbitrary plane or sphere at an *infinite* angle. The preceding construction will then contain the following one as a special case:

*Construct the spheres* ($S$) *that are tangent to* ($\Sigma$) *and cut the plane* ($\Pi$) *at a constant angle* $\alpha_1$ ([36]). *The spheres of radius zero that pass through the intersection of each sphere* ($S$) *and the plane* ($\Pi$) *will describe a surface* ($\Sigma'$) *that corresponds to* ($\Sigma$) *with preservation of the lines of curvature.*

Even before the recent studies on transformations that preserve the lines of curvature, and in an era when one did not know that those transformations included the inversion and *dilatation* by which one passes from one surface to a parallel surface, O. Bonnet exhibited a transformation that is included in the preceding ones, and which corresponds to the particular case in which the angle $\alpha_1$ is a right angle ([37]).

However, if one employs spheres then one can obtain a geometric construction of the transformation that is even simpler. Indeed, formulas (9) show us that a sphere will coincide with its transform whenever one has:

$$z + R = \frac{1+k}{1-k}(z - R)$$

or

$$\frac{z}{R} = \frac{1}{k}. \tag{11}$$

From that:

*If one considers all spheres* ($S$) *that are tangent to a surface* ($\Sigma$) *and cut the plane* ($\Pi$) *at a constant angle whose cosine is equal to* $1 / k$ *then they, along with* ($\Sigma$), *will envelop the surface* ($\Sigma'$) *that is homologous to* ($\Sigma$) *under the transformation that is considered* ([38]).

**172.** The last proposition that we just stated permits us to give a simple geometric definition of the most general transformation that one obtains if one subjects a figure and its transform to the same inversion. For all spheres that cut the plane ($\Pi$) at a constant angle, the inversion will, in fact, make spheres or planes correspond when the cut a fixed sphere ($S$) at a constant and equal angle. Hence:

*If one constructs all spheres that are tangent to a surface* ($A$) *and cut a fixed sphere* ($S$) *at a constant angle then they, along with* ($A$), *will envelop a surface* ($A'$) *that corresponds to* ($A$) *with preservation of the lines of curvature.*

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([36]) In order to obtain the angle $\alpha_1$, it will suffice to set $\tan V' / 2 = \pm i$ in formula (10). One will then have:

$$\tan\frac{\alpha_1}{2} = \pm i\,\frac{k-1}{k+1}.$$

([37]) O. BONNET, "Note sur un genre particulier de surfaces réciproques," Comptes rendus **42** (1856), pp. 485.

([38]) If the constant $k$ is smaller than unity then the spheres ($S$) will not cut the plane ($\Pi$), but the ratio $z / R$ will always be constant. Fig. 12 is drawn on the basis of that hypothesis.

If the sphere ($S$) reduces to a plane ($\Pi$) then one will recover the transformation by reciprocal directions.

It would not be pointless to prove that preceding proposition in a manner that is entirely elementary. In the first case, we shall examine the case in which the constant angle is a right angle.

Consider all of the spheres ($U$) that have their center on a surface ($\Sigma$) and cut a sphere ($S$) of radius $R$ at a right angle. Since the center $O$ of ($S$) has the constant power $R^2$ with respect to all spheres ($U$), the radical axes of three arbitrary spheres ($U$), and in turn, the chord of contact of each sphere with its envelope, must pass through the point $O$. The following construction of the envelope will result from this:

*The two points of contact of the sphere* ($U$) *whose center is M with its envelope are found at the intersection of that sphere and the perpendicular that is based at the center O of* ($S$) *on the tangent plane at M to the surface that is the locus of centers* ($\Sigma$).

Let $\mu$, $\mu'$ be those two points, which are located symmetrically with respect to the tangent plane to ($\Sigma$). One will obviously have:

$$\overline{O\mu} \cdot \overline{O\mu'} = R^2,$$

in which $R$ denotes the radius of ($S$), and consequently the two sheets of the envelope will be inverse to each other with respect to the point $O$.

Moutard gave the name of *anallagmatic* ([39]) to the surfaces that do not change when one subjects them to a well-defined inversion. The two sheets of the preceding envelope will then constitute a surface that is anallagmatic with respect to the pole $O$, and conversely, it is easy to show that any anallagmatic surface can be obtained by the preceding manner of generation. Furthermore, since the two sheets of the envelope are inverses to each other, the lines of curvature that are traced on those two sheets will correspond to each other.

We now study the general case in which the variable spheres ($U$) cut the fixed sphere ($S$) at a constant angle that is not a right angle. We begin by establishing the following lemma:

*If two variable spheres* ($U$) *cut the same fixed sphere* ($S$) *at a constant angle that is not equal to zero or* $\pi$ *then one can always transform them into spheres* ($U'$) *that cut a fixed sphere* ($S'$) *that is concentric to* ($S$) *at a right angle by adding a constant to their radii.*

Indeed, let $\rho$ and $R$ be the radii of the spheres ($U$) and ($S$), let $d$ be the distance between their centers, and let $\alpha$ be the constant angle at which they intersect. One will have:

([39]) MOUTARD, "Note sur la transformation par rayons vectors réciproques," Nouvelles Annales de Mathématiques (2) **3** (1864), pp. 306.

"Sur les surfaces anallagmatiques du quatrième ordre," *ibid*, **3** (1864), pp. 536.

"Lignes de courbure d'une classe de surfaces du quatriéme ordre," Comptes rendus **59** (1864), pp. 243.

$$d^2 = \rho^2 + R^2 - 2rR \cos \alpha$$

or also:

$$d^2 = (\rho - R \cos \alpha)^2 + R^2 \sin^2 \alpha .$$

As a result, if one adds the constant quantity – $R$ cos $\alpha$ to the radius $\rho$ of ($U$), which will give a concentric sphere ($U'$), then that sphere ($U'$) will cut the fixed sphere ($S'$) that is concentric to ($S$) and has a radius of $R$ sin $\alpha$ at a right angle.

Having established that lemma, if we imagine all of the spheres ($U$) that depend upon two parameters and cut ($S$) at a given angle then they will envelop a surface with two sheets ($A$), ($A'$). The concentric spheres ($U'$), which cut the sphere ($S'$) at a right angle, envelop a surface with two sheets ($B$), ($B'$) that are parallel to ($A$), ($A'$), respectively, and since the lines of curvature correspond on the two sheets ($B$), ($B'$), which are inverse to each other with respect to the common center of the fixed spheres ($S$), ($S'$), the same thing will be true as far as the two sheets ($A$) and ($A'$) are concerned. The proposition that we have in mind is then found to be established in full generality.

Subject the preceding figure to an inversion whose pole is on the sphere ($S$). That sphere will be transformed into a plane ($\Pi$), while the two sheets ($A$), ($A'$) will be transformed into two sheets ($C$), ($C'$) that are the common envelope of a family of spheres ($U''$) that are the transforms of the spheres ($U$) and consequently cut the plane ($\Pi$) at a constant angle. In other words, the sheets ($C$) and ($C'$) are deduced from each other by means of the most general transformation by reciprocal directions. Now, one can pass from ($C$) to ($C'$) by performing the following transformations:

1. An inversion that transforms ($C$) into ($A$).
2. A dilatation that transforms ($A$) into ($B$).
3. An inversion that transforms ($B$) into ($B'$).
4. A dilatation that transforms ($B'$) into ($A'$).
5. A final inversion that transforms ($A'$) into ($C'$).

That result conforms to a general proposition of Lie ([40]), from which all of the contact transformations that preserve the lines of curvature will be converted into inversions and dilatations.

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([40]) Lie exhibited all of the contact transformation that preserve the lines of curvature in the paper that was cited above, which was included in Bd. V of Mathematische Annalen. He even pointed out (pp. 186) the particular case of the transformation by reciprocal directions, but that transformation had been given before in various papers by Ribaucour. In particular, *see* RIBAUCOUR, "Note sur la déformation des surfaces," Comptes rendus **70** (1870), pp. 332.

In a different form, it was the subject of some studies by the author that were published in Notes V and IX of the "Mémoire sur une classe remarquable de courbes et de surfaces algébriques," (1873).

The general transformations that were considered by Lie in his paper can be defined in an elegant manner if one employs the six coordinates of the sphere that are related by the equation:

$$\sum_{i=1}^{6} m_i^2 = 0,$$

**173.** We return to the results that we obtained in regard to two spheres that correspond under a transformation by reciprocal directions. That will permit us to complete the preceding constructions and to exhibit some new relations between the corresponding elements.

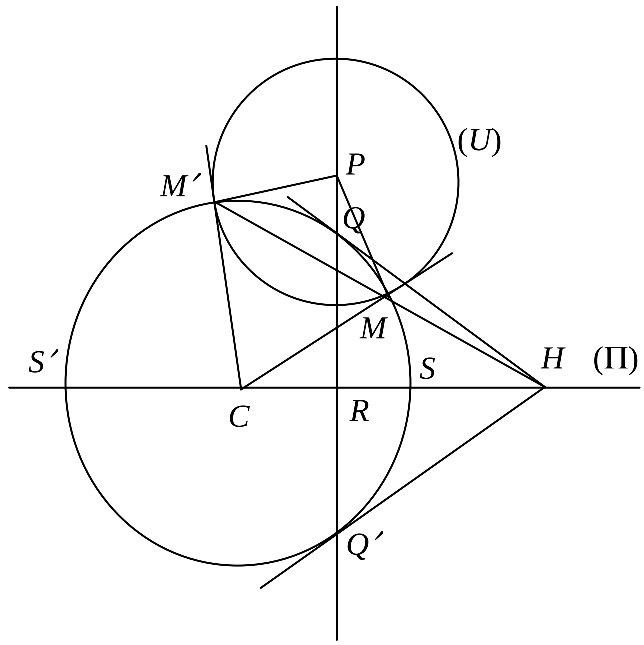


Figure 12.

Consider a surface (Σ) in the first figure and take the plane of picture (Fig. 12) to be the plane that is drawn through an arbitrary point *M* of that surface that is perpendicular to both the plane (Π) of the transformation and the tangent plane to (Σ) at *M*. Let *MC* be the trace of the latter plane. If we construct the sphere (*U*) that is tangent to (Σ) at *M* and cuts the plane (Π) at a constant angle whose cosine is 1 / *k* then we will know that it, along with (Σ), will envelop the surface (Σ′) that corresponds to (Σ). Let *M′* be the point of contact with (Σ′). The tangent plane to (Σ′) at *M′* and the tangent plane to (Σ) at *M* must cut the plane (Π) along the same line, so the point *M′* will be in the plane of the picture, and one will obtain it by drawing the second tangent to the circles along which the sphere (*U*) cuts the plane of figure from the point *C* where the line *MC* meets the plane (Π). It follows from this that the circle that is described with the point *C* as its center and a radius of *CM* will pass through the point *M′* and cut the surfaces (Σ), (Σ′) normally at *M* and *M′*, resp. Thus:

*If one constructs all of the circles that are normal to both* (Σ) *and the plane* (Π) *then the surface* (Σ′) *will cut each of those circles at a right angle.*

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and were defined in no. **156**. In order to obtain them, it will suffice to make a sphere with coordinates $m_i$ correspond to a new sphere whose coordinates $m_i'$ are deduced from the preceding ones by an orthogonal linear substitution with constant coefficients.

Furthermore, since the two tangent planes *CM*, *CM′* correspond under the transformation, one can apply formula (10) to them, which was established for two arbitrary corresponding spheres, and if one lets $V$, $V'$ denote the angles between those two planes and the plane (Π) then one will have:

$$\tan\frac{V}{2}\tan\frac{V'}{2} = \frac{1-k}{1+k}.$$

If one takes the senses of the normals to the two planes into account then one will have:

$$V = \widehat{MCS}, \qquad V' = \widehat{M'CS} \pm \pi,$$

and consequently:

$$\tan\frac{\widehat{M'CS}}{2} = \frac{k+1}{k-1}\tan\frac{\widehat{MCS}}{2}.$$

That equation, which defines the point $M'$ when one knows the point $M$, expresses the idea that the anharmonic ratio that is defined on the circle by the four points $M'$, $M$, $S$, $S'$ is constant and equal to $\frac{k+1}{k-1}$. One then obtains the following theorem, which is due to Ribaucour, and which is included as a special case in a general proposition to which we shall return:

*If one is given a surface* (Σ) *then one can construct all circles that are normal to both the surface and a fixed plane* (Π)*. Those circles are normal to a family of surfaces* (Σ′) *that are defined in the following manner: For each of them, the anharmonic ratio of the point where it cuts each circle normal to* (Σ) *and three other points where that same circle is cut by* (Σ) *and* (Π) *is a constant number. The surfaces* (Σ′) *are the ones that are derived from* (Σ) *under the various transformations by reciprocal directions that admit the same plane* (Π).

One can further point out some other geometric relations. If one drops a perpendicular *PR* from the center *P* of the sphere that envelops both (Σ) and (Σ′) onto the plane (Π) then it will cut the circle at a point *Q* that describes a surface that is normal to the circle, which is easy to prove. Indeed, the tangent at *Q* and the line *MM′* cut the axis *SS′* at the same point *H*, so by virtue of a proposition of elementary geometry, one will have:

$$\tan^2\frac{\widehat{QCS}}{2} = \tan\frac{\widehat{MCS}}{2}\;\tan\frac{\widehat{M'CS}}{2},$$

or, upon taking into account the formula that was given above:

$$\tan\frac{\widehat{QCS}}{2} = \sqrt{\frac{k+1}{k-1}}\tan\frac{\widehat{MCS}}{2}.$$

Since the anharmonic ratio of the points $Q$, $M$, $S$, $S'$ is constant, by virtue of that equation, the surface that is described by the point $Q$ will, in fact, be normal to the circle, and it will correspond to ($\Sigma$) with preservation of the lines of curvature. Moreover, one will have:

$$\overline{MP}^2 = MQ \times MQ' = \overline{PR}^2 - \overline{RQ}^2,$$

or, upon remarking that $MP$, $PR$ are coupled by equation (11):

$$RQ = PR\,\sqrt{1-k^2}\ .$$

One will then have the surface that is the locus of the points $Q$ by reducing the ordinates that are perpendicular to the plane ($\Pi$) of the surface that is the locus of the points $P$ by the ratio of $\sqrt{1-k^2}$ to 1. That implies the following theorem:

*If one considers all spheres* ($U$) *whose centers describe a surface* ($S$) *and cut a plane* ($\Pi$) *at a constant angle whose cosine is equal to* $1 / k$ *then they will envelop a surface with two sheets* ($\Sigma$), ($\Sigma'$) *whose lines of curvature will correspond point-by-point with those of the surface* ($S'$) *that is obtained by reducing the ordinates of* ($S$) *that are perpendicular to the plane* ($\Pi$) *by the ratio of* $\sqrt{1-k^2}$ *to* 1.

For example, if the surface ($S$) has degree two then that theorem will permit one to immediately determine the lines of curvature of the two sheets ($\Sigma$), ($\Sigma'$). Indeed, they correspond to the lines of curvature of the surface ($S'$), which will have degree two here.

**174.** We shall now try to find what the formulas that relate to the transformation by reciprocal radius vectors will become when we employ the coordinate system ($\alpha$, $\beta$, $\xi$).

Let:

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \frac{k^2}{x^2+y^2+z^2}$$

be the formulas for the transformation. The plane:

(12) $$uX + vY + wZ + p = 0$$

will correspond to the sphere:

$$ux + vy + wz + \frac{p}{k^2}(x^2 + y^2 + z^2) = 0.$$

We seek the tangential equation of that sphere – i.e., the condition for the plane:

$$u'x + v'y + w'z + p' = 0$$

to be tangent to it. An application of some elementary methods will lead us to the desired equation:

$$-\frac{2pp'}{k^2} + uu' + vv' + ww' \pm \sqrt{u^2+v^2+w^2}\sqrt{u'^2+v'^2+w'^2} = 0.$$

Introduce the coordinates $\xi$, $\alpha$, $\beta$, in place of $u$, $v$, $w$, $p$, and $\xi'$, $\alpha'$, $\beta'$, in place of $u'$, $v'$, $w'$, $p'$. The preceding equation will take the form:

$$\frac{2\xi\xi'}{k^2} - 2\alpha\beta + 2\alpha'\beta' - (\alpha+\beta)(\alpha'+\beta') \pm (\alpha-\beta)(\alpha'-\beta') = 0.$$

Upon successively taking the + and – sign, we will get the two equations:

(13) $$\xi\xi' = -k^2(\alpha-\alpha')(\beta-\beta'),$$
(14) $$\xi\xi' = -k^2(\beta-\alpha')(\alpha-\beta'),$$

resp., which will realize the doubling of the inversion, in a way. These formulas will be converted into each other when one exchanges $\alpha$ and $\beta$, and that exchange will modify nothing in the equation of a plane in the system ($\alpha$, $\beta$, $\xi$). We can then confine ourselves to just one of the two equations; we choose formula (13).

When the plane that is defined by equation (12) envelops a surface ($\Sigma$), $\xi$ will be a given function of $\alpha$ and $\beta$, and the sphere that is defined by equation (13) in $\alpha'$, $\beta'$, $\xi'$ will envelop the surface ($\Sigma'$) that corresponds to ($\Sigma$). In order to have the point (or rather, the plane) of contact of that sphere with its envelope, one must apply the general principles of the theory of envelopes and combine equation (13) with its two derivatives with respect to $\alpha$ and $\beta$, in which $\xi$ is considered to be a function of $\alpha$ and $\beta$. Upon letting $p$ and $q$ denote the first derivatives of $\xi$, one will then find the system:

(15) $$\begin{cases} \xi\xi' = -k^2(\alpha-\alpha')(\beta-\beta'), \\ p\xi' = -k^2(\beta-\beta'), \\ q\xi' = -k^2(\alpha-\alpha'), \end{cases}$$

which determines $\alpha'$, $\beta'$, $\xi'$ as functions of $\alpha$ and $\beta$.

If one now differentiates the first equation in (15), while taking the other two into account, then one will find that:

$$\xi\,d\xi' = k^2(\beta-\beta')\,d\alpha' + k^2(\alpha-\alpha')\,d\beta'.$$

Upon letting $p'$, $q'$ denote the derivatives of $\xi'$ with respect to $\alpha'$, $\beta'$, one will then have:

(16) $$\begin{cases} \xi p' = k^2(\beta-\beta'), \\ \xi q' = k^2(\alpha-\alpha'). \end{cases}$$

Formulas (15), (16) define all of the relations between corresponding elements of the two surfaces. They provide the following table:

$$(17)\qquad \begin{cases} \xi' = -\dfrac{k^2\xi}{pq}, \quad p' = \dfrac{k^2}{q}, \quad \dfrac{\xi'}{p'} = -\dfrac{\xi}{p}, \\ \alpha' = \alpha - \dfrac{\xi}{p}, \quad q' = \dfrac{k^2}{p}, \quad \dfrac{\xi'}{q'} = -\dfrac{\xi}{q}, \\ \beta' = \beta - \dfrac{\xi}{p}, \end{cases}$$

so one will deduce, by an easy calculation, that:

$$dp'\,d\alpha - dq'\,d\beta' = -\frac{k^2}{pq}(dp\,d\alpha - dq\,d\beta).$$

That theorem once more establishes that inversion preserves lines of curvature.

It is important to establish carefully what distinguishes formulas (13) and (14) from the geometric standpoint. The equation of the tangent plane will not change when one exchanges $\alpha$ and $\beta$, and consequently one will always have the same surface; however, the positive sense of each normal will obviously change. That results from the expressions:

$$c = \frac{1-\alpha\beta}{\alpha-\beta}, \quad c' = i\frac{1+\alpha\beta}{\alpha-\beta}, \quad c'' = \frac{\alpha+\beta}{\alpha-\beta}$$

for the direction cosines of the normal. Consequently, formulas (13) or (14), which are deduced from each other by the exchange of $\alpha$ and $\beta$, will make a surface ($\Sigma$) correspond to the same surface ($\Sigma'$), but with a definite sense of a normal to the surface, while ($\Sigma$) will correspond to the opposite sense of the normal to the surface ($\Sigma'$), depending upon whether one adopted formula (13) or (14), respectively. In order to characterize each of those formulas from the geometric viewpoint, it will then suffice to give the relationship between the positive senses of the two normals, which correspond by formula (13), for example. In order to do that, consider two homologous surfaces ($\Sigma$), ($\Sigma'$) and two corresponding points $M$, $M'$ on those two surfaces. A point of ($\Sigma$) that describes an infinitely-small curve around $M$ in a well-defined sense will correspond to the point of ($\Sigma'$) that likewise turns around $M'$ in a well-defined sense. Attribute a sense to the two normals such that the two curves seem to be traversed in opposite senses around their respective normals. One will then have the correspondence between the senses of the normals that is defined by formula (13) and the ones that we have deduced from it.

In order to prove that, it will suffice to consider the sphere ($S$) of radius $R$ whose center is at the coordinate origin, and whose equation is:

$$\xi = R\,(\alpha - \beta).$$

Formulas (17) give us:

$$\beta' = \alpha, \qquad \alpha' = \beta, \qquad \xi' = -\frac{k^2}{R}(\alpha' - \beta')$$

for the corresponding sphere.

The first two formulas show that the positive sense of the normal is found to change, and the preceding proposition will be verified in that special case. That will suffice, because if one progressively deforms the sphere (*S*) in such a manner as to make it coincide with an arbitrary surface (Σ) then the proposition that was established for the sphere (*S*) must necessarily be preserved for the surface (Σ) by virtue of continuity.