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CHAPTER I

GENERAL NOTIONS ON CONGRUENCES.

Definition of a congruence. – Limited number of curves that pass through a point. – Exceptional case. – Surfaces of the congruence. – Definition of the focal points. – Determination of the number and degree of multiplicity of the focal points. Application to congruences that are generated by algebraic plane curves. – Focal surface, its contact relations with the curves and surfaces of the congruence. – Determination of the surfaces of the congruence whose generators admit an envelope. – Particular case of rectilinear congruences. – The two series of developables that one can form with the lines of the congruence. – Particular case in which one of the sheets of the focal surface reduces to a curve. – Fundamental proposition that relates to two conjugate systems that are traced on the two sheets of the focal surface. – Relation between that proposition and the method of transforming linear partial differential equations that is due to Laplace.

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311. Consider a system of curves that is defined by the equations:

$$(1) \qquad \begin{cases} f(x, y, z, a, b) = 0, \\ \varphi(x, y, z, a, b) = 0, \end{cases}$$

where a and b denote two arbitrary constants. We refer to the set of curves that correspond to all systems of values of a and b by the name *congruence*, which is due to Plücker. It is clear that a limited number of curves of the congruence will pass through an arbitrary point in space, because if one replaces x, y, z with the coordinates x_0, y_0, z_0 of that point in the preceding equations then one will get two relations that determine a limited number of systems of values of a and b, in general. Meanwhile, it can happen that these equations admit an unlimited number of solutions for certain exceptional points. One will then have certain points through which an infinitude of curves of the congruence will pass.

For example, imagine that one considers the congruence that is formed by the lines that meet a curve (K) and are tangent to a surface (S). The lines of that congruence that pass through a point M in space are, in general, limited in number. They are the edges of intersection of two cones whose summit is M, one of which is circumscribed by the surface (S) and the other of which contains the curve (K). However, if the point M belongs to the curve (K) then an infinitude of lines of the congruence will pass through it, and they will form the cone that is circumscribed by the surface (S) that has its summit at that point.

If one likewise consider the congruence that is formed by the circles in space that pass through two fixed points A and B then one will recognize immediately that just one circle of the congruence will pass through a point *M*, in general. That conclusion will cease to be exact if the point *M* coincides with one of the points *A*, *B*; one would then obtain all circles of the congruence.

The two different examples that we just pointed out correspond to the two cases that can present themselves when there is any indeterminacy. In the first one, the two equations that *a* and *b* must satisfy will reduce to just one; in the second, they are both verified identically.

312. We now return to more general congruences. If one establishes an arbitrary relation between *a* and *b*:

$$b = F(a) \tag{2}$$

then the curves of the congruence that correspond to the values of *a* and *b* for which that relation is verified will generate a surface that one will obtain by eliminating *a* and *b* from equations (1) and (2). The tangent plane to that surface at a point (*x*, *y*, *z*) will be defined by the equation:

$$\frac{\dfrac{\partial f}{\partial x}dx+\dfrac{\partial f}{\partial y}dy+\dfrac{\partial f}{\partial z}dz}{\dfrac{\partial \varphi}{\partial x}dx+\dfrac{\partial \varphi}{\partial y}dy+\dfrac{\partial \varphi}{\partial z}dz} = \frac{\dfrac{\partial f}{\partial a}+\dfrac{\partial f}{\partial b}F'(a)}{\dfrac{\partial \varphi}{\partial a}+\dfrac{\partial \varphi}{\partial b}F'(a)}, \tag{3}$$

whose form leads us to a very interesting proposition:

To abbreviate, we give the name of *surfaces of the congruence* to the surfaces that we just defined that are generated by the curves of the congruence, which are chosen in the most arbitrary manner, moreover. Consider four of these surfaces that pass through the same curve (*C*) of the congruence and take their tangent planes at a well-defined point *M* of that curve. *x*, *y*, *z*, *a*, *b* will have the same value at that point as they do for all of the surfaces, while $F'(a)$ will vary only when one passes from one surface to the other.

From that remark and the form of equation (3), we can then conclude that the anharmonic ratio of the tangent planes to the four surfaces [which are tangent planes that all contain the tangent to the curve (*C*) at *M*] is equal to that of the values of $F'(a)$ that relate to the four surfaces, and will consequently be the same for all points of the curve (*C*). Therefore:

If one considers four arbitrary surfaces of the congruence that contain the same curve of the congruence then the anharmonic ratio of the tangent planes to those surfaces at an arbitrary point of the curve that is common to them will remain constant when the point is displaced along that curve.

The equation of the tangent plane also leads us to another essential consequence. Determine the points along the common curve (*C*) for which one has:

(4) $$\frac{\partial f/\partial a}{\partial \varphi/\partial a} = \frac{\partial f/\partial b}{\partial \varphi/\partial b}.$$

The tangent plane will be independent of $F'(a)$ for those points, and consequently, it will be the same for all surfaces of the congruence that contain the curve (C). Therefore:

If one considers all of the surfaces of the congruence that contain the same curve (C) *of the congruence then there will exist a certain number of points on* (C) *for which all of those surfaces admit the same tangent plane, no matter which rule one has used to assemble the curves that generate them.*

We refer to those points by the name of *focal points.*

313. The number of focal points that are situated along each curve depends upon the form of equations (1) with regard to x, y, z, rather than the manner by which a and b figure in them. In other words, it depends, above all, upon the form and the definition of the curves of the congruence, rather than the manner by which they are assembled. Suppose, for example, that the equations (1) are of degree one with respect to x, y, z, and that they represent a line. Equation (4) will be of degree two, in general, and will define two focal points on each line of the congruence. If f has degree m, and φ has degree one then the congruence will be composed of planar curves of order m. Equation (4) will then have degree $m + 1$, and there will generally be $m(m + 1)$ focal points along each curve of the congruence.

When the curves of the congruence meet a fixed curve (K), their points of intersection with the fixed curve will obviously be focal points. The same thing will be true when the curves pass through fixed points, but we shall show that in the latter case, each of those fixed points is equivalent to at least two focal points.

Indeed, let M be a point through which all curves of the congruence pass. If one takes it to be the origin of the coordinates then equations (1) will not contain constant terms, and the lower-degree terms will be of degree at least one, and at least two of the points of intersection of the surface that is represented by that equation with the curve (C) of the congruence will coincide with the origin of the coordinates; i.e., with the point M.

Let us point out some applications.

For a congruence of conics, there will be six focal points on each conic.

If the conics become circles – i.e., if they meet the circle at infinity twice – then two of the focal points will be pushed out to the intersection of each circle with the circle at infinity; only four of them will remain at a finite distance.

If the circles pass through a fixed point A then, apart from A, only two focal points will remain on each circle that are at a finite distance. One will recognize that immediately, moreover, by performing an inversion whose pole is at A.

314. With those perfunctory remarks about the number of focal points, we now go on to study their distribution in space. If one eliminates a and b from equation (1) and (4) then one will obtain just one equation, in general, which represents the locus of focal points of all the curves of the congruence. We give that locus the name of *focal surface of the congruence.* In particular, it is composed of two sheets that have focal points on each curve of the congruence. The focal surface can be decomposed; i.e., it can be reduced, in total or in part, into curves or points. We put aside the examination of all those cases, which we shall address much later, and now establish the most general properties.

Write equations (1) and (4) in the form:

$$(5) \qquad f = 0, \quad \varphi = 0, \qquad \frac{\partial f}{\partial a} - k\frac{\partial \varphi}{\partial a} = 0, \qquad \frac{\partial f}{\partial b} - k\frac{\partial \varphi}{\partial b} = 0,$$

in which k denotes an auxiliary unknown. If one regards a and b as given then these equations will determine the focal points of a certain curve (C) of the congruence. The tangent plane to any of the surfaces of the congruence that contain the curve (C) at one of those points will be defined by equation (3), which takes the form:

$$(6) \qquad \left(\frac{\partial f}{\partial x} - k\frac{\partial \varphi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} - k\frac{\partial \varphi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} - k\frac{\partial \varphi}{\partial z}\right)dz = 0$$

here.

We shall show that this plane is also tangent to the focal surface at the point considered.

Indeed, one obtains the equation of the focal surface by eliminating a, b, k from the four equations (5). However, instead of performing that elimination, one can preserve all of these equations by agreeing to regard a, b, k as functions to be determined in them. As a result, in order to get the tangent plane to the focal surface at the point considered, one must differentiate equations (5), while regarding a, b, k as variables in them. Now, if one differentiates only the first two of them and then forms the combination:

$$df - k\, d\varphi = 0$$

then the coefficients of da, db will disappear by virtue of the last two equations, and one will recover equation (6). As we have asserted, that equation will then represent the tangent plane to the focal surface, and we can state the following proposition:

The curves of the congruence are tangent to the focal surface at all of their focal points. The various surfaces of the congruence that contain a well-defined curve of that congruence will all be tangent to the focal surface at all of the focal points that are situated along that curve, and consequently, as we have established already, they must all be tangents to those points.

315. One again recovers the focal surface by studying the following problem:

Assemble the curves of the congruence in such a manner that they have an envelope – i.e., that they are all tangent to a certain curve.

Take b to be a function of a: We obtain a family of curves that are represented by the equations:

$$f = 0, \quad \varphi = 0, \tag{7}$$

which contains only one parameter a. In order for these curves to have an envelope, it is necessary that for any value of a, one can determine the values of x, y, z that satisfy the two equations (7), along with their derivatives with respect to a:

$$\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\frac{db}{da} = 0, \qquad \frac{\partial \varphi}{\partial a} + \frac{\partial \varphi}{\partial b}\frac{db}{da} = 0. \tag{8}$$

If one eliminates db / da from these equations then one will recover equation (4), which characterizes focal points. Therefore:

If one wishes to assemble the curves of the congruence in such a manner that they have an envelope then that envelope will be necessarily formed from the focal points of those curves, and consequently, it will be situated on one of the sheets of the focal surface.

On the other hand, if one eliminates x, y, z from equations (7) and (8) then one will be led to a relation of the form:

$$\Phi\left(a, b, \frac{db}{da}\right) = 0, \tag{9}$$

which shows that the complete solution to the problem generally demands the integration of a first-order differential equation.

The propositions that were established previously once more lead to the latter result. We have seen that if the curves of the congruence admit an envelope (E) then that envelope (E) must be found on one of the sheets of the focal surface. However, it is clear that it must also satisfy one further condition: It is necessary that at each of its points it is tangent to the curve of the congruence that touches that point of the focal surface. The latter condition, which is obviously necessary and sufficient, is equivalent to a differential equation whose integration will permit one to solve the problem completely. That differential equation will be of order and degree one for each of the two sheets of the focal surface. However, if those sheets cannot be separated analytically then one will have to consider just one first-order equation whose degree will be equal to the number of sheets of the surface.

In the case where one of the sheets of the focal surface reduces to a curve (K), in order to arrive at a partial solution to the problem, it will suffice to assemble all of the curves of the congruence that pass through the same point of (K). For example, if one considers the congruence that is formed by the circles that meet four given curves then one will get all the solution by assembling the circles (which are infinite in number) that pass through an arbitrary point of one of those curves.

316. When the curves of the congruence are simply tangent to one of the sheets of the focal surface, the contact points of these curves must be considered as simple focal points. However, if the curves have a contact of order p with the sheet considered then each contact point will be a multiple focal point that consists of the locus of p ordinary focal points. Here is how one can establish that proposition:

Let:

$$z = f(x, y)$$

be the equation of the sheet considered. If one replaces z with $z + f(x, y)$ then that substitution will not change the order of contact of the curve and the surface. However, the preceding sheet is replaced with the xy-plane. With no loss of generality in the argument, we can then suppose that all of the curves of the congruences are tangent to the xy-plane, and we shall study them in the neighborhood of the plane.

Replace the parameters a and b that enter into the equations of each curve of the congruence with the coordinates x_0, y_0 of the contact point of the curve with the xy-plane. The equations that define that curve will then give values for y and z that can be developed in series and will have the form:

$$y = y_0 + A\,(x - x_0) + B\,(x - x_0)^2 + \ldots,$$
$$z = A_1(x - x_0)^{p+1} + B_1(x - x_0)^{p+2} + \ldots,$$

in which A, A_1, B, B_1, … are arbitrary functions of x_0, y_0 , and p denote the order of contact of the curve with the xy-plane. Write the equation:

$$\frac{\partial f}{\partial x_0}\frac{\partial \varphi}{\partial y_0} - \frac{\partial f}{\partial y_0}\frac{\partial \varphi}{\partial x_0} = 0,$$

which determines the focal points here. It will have the form:

$$-(p + 1)\,A_1\,(x - x_0)^p + \ldots = 0,$$

in which the unwritten terms all contain a power of $x - x_0$ with an exponent greater than p as a factor. One see that the value x_0 of x will have an order of multiplicity that is equal to precisely the order of contact of each curve of the transformed congruence with the xy-plane; i.e., of each curve of the original congruence with the sheet considered of the focal surface. That is the proposition that we wanted to establish.

317. One can argue in an analogous manner when the curves of the congruence all meet a fixed curve (K). If:

$$z = f(x), \qquad y = \varphi(x)$$

are the equations of that curve then one will begin by replacing z with $z + f(x)$ and y with $y + \varphi(x)$, and the curve will be transformed into the x-axis without changing the order of contact. If one replaces that one of the parameters a and b that enter into the equation for each curve of the congruence with the abscissa x_0 of the point at which that curve meets the x-axis then the equations of the curve will take the form:

$$y = A\,(x - x_0) + B\,(x - x_0)^2 + \ldots,$$
$$z = A_1\,(x - x_0) + B_1(x - x_0)^2 + \ldots,$$

in which $A, B, A_1, B_1, \ldots$ denote functions of x_0 and the other parameter a upon which the curves of the congruence. The equation that determines the focal points will be:

$$(x - x_0)\left(A\frac{\partial A_1}{\partial \alpha} - A_1\frac{\partial A}{\partial \alpha}\right) + \ldots = 0$$

here, while the unwritten terms will contain $(x - x_0)^2$ as a factor. One sees that the point of intersection will be a simple focal point, unless one has:

$$A\frac{\partial A_1}{\partial \alpha} - A_1\frac{\partial A}{\partial \alpha} = 0,$$

or, upon integrating:

$$A_1 = A\,\varphi(x_0).$$

That condition is easily interpreted: It expresses the idea that the tangents to all of the curves of the transformed congruence that intersect at the same point of the x-axis will be in the same plane that passes through Ox. Upon extending that conclusion to the original congruence, we will get the following result:

When the curves of the congruence meet a fixed curve (K), *the points of intersection will be simple focal points, unless the tangents to all of the curves of the congruence that pass through an arbitrary point of* (K) *are not all situated in a plane that passes through the tangent to the curve* (K) *at that point.*

If one extends this line of inquiry to the case in which all of the curves of the congruence pass through a fixed point then one will likewise prove that this fixed point (which consists of at least two focal points, as we have seen) can have a higher order of multiplicity only if the tangents to all curves of the congruence that are drawn at that point form a cone (which is arbitrary, moreover), instead of filling up all of space. We likewise leave it to the reader to establish an elegant proposition that was communicated to us by G. Koenigs by a simple application of the preceding methods:

If a congruence is such that each of the curves that it is comprised of is met by all of the infinitely-close curves of the congruence then the various curves of the congruence will be tangents to one or more fixed curves, or else they might pass through one or more fixed points ([1]).

([1]) One passes over the exceptional case in which all curves of the congruence are traced on the same surface, and in which, as a result, all of their points will satisfy the definition of a focal point.

318. Apply the preceding propositions to the congruences of lines or *systems of rectilinear rays.* The number of lines of the congruence that passes through an arbitrary point of space has been given the name of the *order* of the congruence in this case. One calls the number of lines that are in an arbitrary plane the *class* of the congruence. The system of normals to an ellipsoid forms a congruence of order 6 and class 2.

There will be two real or imaginary focal points on each line, in general. When the line is displaced, those two focal points will describe the two sheets of the focal surface. The lines of the congruence are double tangents to that focal surface, but the converse is not true, in general. Not every double tangent of the focal surface is a line of the congruence. For example, the surface of centers of curvature of an ellipsoid admits the normals to that ellipsoid as double tangents; however, it also admits some other double tangents that do not belong to the congruence of normals.

Let (*d*) be an arbitrary line of the congruence that touches the first sheet of the focal surface (*F*) at *M* and the second sheet at *M′*. Let (*P*), (*P′*) be the tangent plane to (*F*) at *M* and *M′*. If one wants to assemble the lines of the congruence in such a manner that they have an envelope – i.e., they generate a developable surface – then, from the general propositions that were previously established, the edges of regression of all the developables thus obtained must be found on one or the other of the sheets of (*F*). It will then follow from this that the line (*d*) will belong to only the two developables, one of which will have its edge of regression on the first sheet and tangent to the line (*d*) at *M*, while the other one will have its edge of regression on the second sheet and tangent to (*d*) at *M′*. Moreover, the determination of these two series of developables will demand the integration of two differential equations of order and degree one, or – what amounts to the same thing – the integration of one equation of order and degree one.

All of the ruled surfaces that are generated by the lines of the congruence and contain the line (*d*) will be tangent to one or the other of the two focal points *M* and *M′*. One must nevertheless make a special remark in relation to the two developables that admit (*d*) for their generators. For example, the one whose edge of regression passes through *M* and is situated on the first sheet of the focal surface will then admit tangent plane to the second sheet of the focal surface at *M′* at *M′*, and consequently, at all other points of (*d*). True, that plane is not tangent to the first sheet at *M*. However, that exception to the general theorem, which also presents itself for the most general congruences, will disappear when one remarks that the edge of regression of a developable surface is a multiple line at all points at which the tangent plane must be regarded as indeterminate.

319. Let the letters (*C*), (*C′*) denote the edges of regression that are situated on the first and second sheet, respectively, of the two series of developables that we just defined. The first developables, which have the curve (*C*) for their edge of regression, touch the second sheet along the curve (*D′*). The second ones, which have the curves (*C′*) for their edges of regression, likewise touch the first sheet along the curve (*D*). There is an almost obvious (but still quite essential) geometric relation between the two families of curves that are traced on the same sheet (*C*) and (*D*) or (*C′*) and (*D′*). By hypothesis, the tangents to the various curves (*C*) at the points where these curves meet the same curve (*D*) will generate a developable surface that has its edge of regression on the second sheet. Therefore, the curves (C) and (D) will form a conjugate system on the first sheets, and likewise, the curves (C') and (D') form a conjugate system on the second sheet. Thus:

The curves on each sheet of the focal surface will correspond to two series of developables that form a conjugate system. One of them will be the edges of regression of one of the two families of developables, while the other one will be the contact curves of the developables of the other series.

320. The detailed study of the preceding relations is very rich in consequences and it permits one to answer various questions. For example, we propose the following problem:

Consider a family of curves (C) *on a surface* (Σ). *The tangents to those curves form a congruence in which one already knows one of the two families of developables, which are the ones that are generated by the tangents to the various curves* (C). *We propose to define the developables of the other family.*

From the preceding, it is clear that one must associate the curves (C) with their conjugates (D) on (Σ). The tangents to the curves (C) at all points of a curve (D) generate the developables of the second family. The determination of those developables will demand the integration of an equation of order and degree one, namely, that of the curves (D).

The preceding construction shows immediately that if the curves (C) are asymptotic lines – and only in that case – then they will coincide with the curves (D). Thus, the congruences in which the two focal points constantly coincide and the two sheets of the focal surface reduce to just one (Σ) are formed from one of the two systems of asymptotic tangents to (Σ). That result is in perfect accord with the one that was proved in no. **316**.

321. The preceding propositions submit to some modifications that are easy to predict in the case where one or the other of the two sheets of the focal surface reduces to a curve. The developables that have their edge of regression on that sheet will then reduce to cones that are generated by the lines of the congruence that cut the curve to which the sheet considered reduces at the same point.

We propose to determine the second family of developables of the congruence in that special case. We give the equations of the *focal* curve in the form:

$$x = f(z), \qquad y = \varphi(z),$$

while the equations that determine a line of the congruence will have the form:

$$X - x = \lambda (Z - z), \qquad Y - y = \mu (Z - z),$$

in which X, Y, Z are variable coordinates, and μ is a function of λ and z that is determined from the definition of the congruence. In order to obtain the developables, we express the idea that the line that is represented by the preceding equation is met by an infinitely-close line of the congruence; i.e., that we associate the preceding equations with the following ones:

$$dx + d\lambda\,(Z - z) - \lambda\,dz = 0,$$
$$dy + d\mu\,(Z - z) - \mu\,dz = 0,$$

which one obtains by varying λ and μ infinitely little. If we eliminate $Z - z$ then we will obtain the differential equation:

$$(dx - \lambda\,dz)\,d\mu - (dy - \mu\,dz)\,d\lambda = 0,$$

whose integration will make known the developables of the congruence. One immediately perceives a first solution:

$$dx = dy = dz = 0.$$

It corresponds to the cone that is composed of all lines of the congruence that pass through the same point of the focal curve. If we discard that obvious solution, which we have already pointed out, and if we replace $d\mu$ with its value $\frac{\partial\mu}{\partial z}dz + \frac{\partial\mu}{\partial\lambda}d\lambda$ then we will be led to the first-degree differential equation:

$$\frac{d\lambda}{dz} = \frac{(\lambda - x')\dfrac{\partial\mu}{\partial z}}{\mu - y' - (\lambda - x')\dfrac{\partial\mu}{\partial\lambda}}, \tag{10}$$

whose integration seems impossible in the general case.

If one supposes that all of the lines of the congruence that pass through the same point of the focal point form a plane then the second sheet of the focal surface will obviously be the developable (Δ), which is the envelope of that plane. In that case, the relation between μ, λ, z will take the form:

$$\mu = A\,\lambda + B,$$

in which A and B are functions of z, and the equation (10) will reduce to the following one:

$$\frac{d\lambda}{dz} = \frac{(\lambda - x')(A'\lambda + B')}{Ax' + B - y'},$$

which is a Ricatti equation. Thus:

The determination of all curves that are traced on a developable (Δ) *and whose tangents meet an arbitrary given curve* (K) *depends upon the integration of a Ricatti equation.*

Consequently, that determination can be performed (nos. **16** and **17**) by $3 - n$ quadratures as soon as one knows n particular curves that give a solution to the problem. The reader will apply this to the case in which the developable (Δ) is the envelope of the normal planes to the curve (R) by himself. The desired curves then become the developables of (R). One can, moreover, obtain two of those curves with no integration: They are the edges of regression of the two sheets of the developable that circumscribed by the curve (R) and the circle at infinity. As a result, the most general developables can be determined by a simple quadrature, which conforms to the results of no. **12** [I, pp. 18].

If the curve (R) is planar then its plane will cut (Δ) along a planar curve whose tangents meet (R). One will then know a solution to the problem, and in turn, one will obtain the general integral by means of two quadratures.

If the curve (R) is arbitrary and the developable (Δ) is a cone with its summit at A then one will also know a particular solution that is furnished by the cone whose summit is A and contains the curve (R). Therefore, one can determine by means of two quadratures only the curves that are traced on an arbitrary cone whose tangents will meet a curve that is entirely arbitrary.

Equation (10) permits us to recognize the case in which the two focal points on each line of the congruence are coincident. For that to be the case, it is obviously necessary that the equation must reduce to:

$$dz = 0;$$

i.e., that one have:

$$\mu - y' - (\lambda - x')\,\frac{\partial \mu}{\partial \lambda} = 0.$$

Upon integrating, one will have:

$$\mu - y' = h(\lambda - x'),$$

in which h is a function of z. The geometric interpretation of that result leads to the following theorem, which is, moreover, a special case of the one that was established in no. **317**.

The lines that meet a curve (C) *and belong to a congruence will have coincident focal points only in the case where all of the lines that pass through a point of* (C) *generate a tangent plane to* (C) *at that point.*

322. Those are the simplest properties of systems of rectilinear rays. One sees that if such a system is well-defined then it will always be possible to obtain the equation of the focal surface by algebraic calculation; however, the determination of the two families of developables into which one can distribute all of the lines will generally demand the integration of two differential equations. The integration of those differential equations will imply the knowledge of a conjugate system on each of the sheets of the focal surface.

Conversely, whenever one knows a system that is composed of two families of conjugate curves (*C*) and (*D*) on a surface, one will deduce two different systems of rectilinear rays for which one will know the two series of developables. Indeed, consider the tangents to all of the curves of one of the families – for example, the curves (*C*). They form a congruence for which the two series of developables will be known: One of them will be composed of the tangents at all points of the same curve (*C*), while the other one will be composed of the tangents to the various curves (*C*) at the points where they are cut by the same curve (*D*) of the second family. That will result in the following new property:

Any time one knows a conjugate system on a surface (Σ), *one will obtain a new (generally unlimited) sequence of surfaces on which one will likewise know a conjugate system.*

Indeed, keep the preceding notations and let (*C*) and (*D*) be the two series of conjugate curves that are traced on (Σ). The tangents to the curves (*C*) form a congruence whose focal surface is composed of (Σ) and another surface (Σ_1), which generally will not reduce to either a curve or a developable surface. The curves (*C*) and (*D*) of (Σ) correspond to curves (D_1) and (C_1) of (Σ_1) that form a conjugate net on that surface, and the lines of the congruence considered will be tangent to the curves (C_1). Now, construct the tangents to the curves (D_1) of (Σ_1). In addition to (Σ_1), they touch another surface (Σ_2), whose relationship to (Σ_1) will be the same as that of (Σ_1) to (Σ). One can continue these operations indefinitely until one does not arrive at a surface that degenerates into a curve or a developable, and one will then obtain a sequence of surfaces:

$$(\Sigma),\ (\Sigma_1),\ (\Sigma_2),\ (\Sigma_3),\ \ldots$$

that one can prolong indefinitely, in general.

We now return to the surface (Σ). If one draws tangents to the curves (*D*), instead of tangents to the curves (*C*), then upon operating as we just described, one will obtain a series of surfaces:

$$(\Sigma_{-1}),\ (\Sigma_{-2}),\ (\Sigma_{-3}),\ \ldots,$$

which will not terminate, in general. If one combines the two sequences in order to form a unique sequence:

$$\ldots,\ (\Sigma_{-2}),\ (\Sigma_{-1}),\ (\Sigma),\ (\Sigma_1),\ (\Sigma_2),\ \ldots$$

then each of the surfaces that one obtains will be deduced from the following one or the preceding one by a uniform construction, and one will know a conjugate system on each of them that corresponds to the original system of (Σ).

Thus, whenever one has to integrate the differential equations that determine the developables that are formed from the lines of a given congruence, the preceding method will provide a series of new congruences whose developables will be determined with no new integration. We similarly add that in order to define all of those congruences, it will never be necessary to have found the developables of the first one, because if *M* denotes an arbitrary point of the surface (Σ), and *Mt* denotes the line of the congruence that is tangent to (Σ) at M then one can always construct the conjugate tangent to the line and consequently define the new congruence that must succeed the previous one.

323. We shall now seek the analytical translation of the preceding geometric operations. For that, we employ homogeneous coordinates. We let x, y, z, t denote the coordinates of an arbitrary point of (Σ), and we take the parameters ρ, ρ_1 of the two families of conjugate curves (C) and (D) to be independent parameters. We know (no. **98**) [I, pp. 122] that x, y, z, t are four particular solutions to a linear equation of the form:

$$\frac{\partial^2\theta}{\partial\rho\,\partial\rho_1}+a\frac{\partial\theta}{\partial\rho}+b\frac{\partial\theta}{\partial\rho_1}+c\theta=0. \tag{11}$$

Consider the congruence that is formed by the tangents to the curves (C). The tangents to (Σ) will touch another surface (Σ_1), which amounts to defining it analytically. For this, we take an arbitrary point M (x, y, z, t) on (Σ). A curve (C) passes through that point. One can define an arbitrary point P of the line Mt that is tangent to that curve at M by the formulas:

$$X=\lambda\,x+\frac{\partial x}{\partial\rho_1},\qquad Y=\lambda\,y+\frac{\partial y}{\partial\rho_1},\qquad Z=\lambda\,z+\frac{\partial z}{\partial\rho_1},\qquad T=\lambda\,t+\frac{\partial t}{\partial\rho_1}, \tag{12}$$

in which λ denotes an arbitrary number whose variation will give all of the points of that tangent. In order to determine λ, one must express the idea that the point P describes a curve that is tangent to the line Mt on which it is found when the point M is displaced along the curve (D) – i.e., when only the parameter ρ varies. That condition will translate into equations such as the following ones:

$$\frac{\partial X}{\partial\rho}=px+q\frac{\partial x}{\partial\rho_1},\qquad \frac{\partial Y}{\partial\rho}=py+q\frac{\partial y}{\partial\rho_1},\ \ldots, \tag{13}$$

in which p and q are indeterminate and can take on arbitrary values. Consider just the equation that relates to X, and replace $\partial X / \partial\rho$ in it with its value that is inferred from the first equation in (12). One will have:

$$x\frac{\partial\lambda}{\partial\rho}+\lambda\frac{\partial x}{\partial\rho}+\frac{\partial^2 x}{\partial\rho\,\partial\rho_1}=px+q\frac{\partial x}{\partial\rho_1},$$

or, upon replacing $\dfrac{\partial^2 x}{\partial\rho\,\partial\rho_1}$ with its value that one infers from equation (11):

$$x\left(\frac{\partial\lambda}{\partial\rho}-c-p\right)+\frac{\partial x}{\partial\rho}(\lambda-a)-\frac{\partial x}{\partial\rho_1}(b+q)=0.$$

That equation must persist when one replaces x with y, z, t. It must then be verified identically (no. **90**), which will give:

$$\lambda = a, \qquad q = -b, \qquad p = \frac{\partial a}{\partial \rho} - c.$$

Formulas (12) and (13) thus transform into the following ones:

$$X = \frac{\partial x}{\partial \rho_1} + a\,x, \ldots, \tag{14}$$

$$\frac{\partial X}{\partial \rho} = \left(\frac{\partial a}{\partial \rho} - c\right) x - b\frac{\partial x}{\partial \rho_1}, \qquad \ldots, \tag{15}$$

in which everything is known, and which defines the new surface (Σ_1) completely. In order to obtain (Σ_{-1}), it will suffice to switch the two variables ρ and ρ_1 .

321. Suppose, for example, that the coordinates x, y, z, t are given by the formulas:

$$\left\{\begin{aligned} x &= A(\rho - a)^m (\rho_1 - a)^n, \\ y &= B(\rho - b)^m (\rho_1 - b)^n, \\ z &= C(\rho - c)^m (\rho_1 - c)^n, \\ t &= D(\rho - d)^m (\rho_1 - d)^n, \end{aligned}\right. \tag{16}$$

in which a, b, c, t now denote arbitrary constants that were employed already in no. **112** [I, pp. 142]. x, y, z, t are particular solutions of the partial differential equation:

$$(\rho - \rho_1)\frac{\partial^2 \theta}{\partial \rho\, \partial \rho_1} + n\frac{\partial \theta}{\partial \rho} - m\frac{\partial \theta}{\partial \rho_1} = 0.$$

In order to define the derived surface, one employs formulas (14), which will give us:

$$X = \frac{n(\rho - a)}{(\rho_1 - a)(\rho - \rho_1)}x, \qquad Y = \frac{n(\rho - b)}{(\rho_1 - b)(\rho - \rho_1)}y, \ldots,$$

here.

One can multiply the four homogeneous coordinates by the factor $(\rho - \rho_1) / n$, which will lead to the defining formulas:

(17)
$$\begin{cases} X = A(\rho - a)^{m+1}(\rho_1 - a)^{n-1}, \\ Y = B(\rho - b)^{m+1}(\rho_1 - b)^{n-1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

These are the original equations in which m, n are replaced with $m + 1$, $n - 1$, respectively. Upon pursuing the application of the method, one will obtain the system:

(18)
$$\begin{cases} X_i = A(\rho - a)^{m+i}(\rho_1 - a)^{n-i}, \\ Y_i = B(\rho - b)^{m+i}(\rho_1 - b)^{n-i}, \\ Z_i = C(\rho - c)^{m+i}(\rho_1 - c)^{n-i}, \\ T_i = D(\rho - d)^{m+i}(\rho_1 - d)^{n-i} \end{cases}$$

for the surface (Σ_i), which is true for all integer values of i, whether positive or negative, as one easily assures oneself. The sequence that is obtained will be unlimited in the two senses whenever the numbers m and n are not integers. However, if n, for example, is a positive integer then the surface (Σ_n) will reduce to a curve, and the application of the method will terminate with (Σ_n).

325. Return to the general formulas (14) and (15). The coordinates x, y, z, t of a point on the original surface (Σ) are particular solutions to the equation (11): We seek the equation with the same form that must be satisfied by the coordinates X, Y, Z, T of a point on the surface (Σ_1). Formula (14) shows us that one must make the substitution:

(19)
$$\sigma = a\theta + \frac{\partial \theta}{\partial \rho_1}.$$

Equation (11) can be put into the form:

$$\frac{\partial}{\partial \rho}\left(a\theta + \frac{\partial \theta}{\partial \rho_1}\right) + b\frac{\partial \theta}{\partial \rho_1} + \left(c - \frac{\partial a}{\partial \rho}\right)\theta = 0,$$

so one will have:

(20)
$$\frac{\partial \sigma}{\partial \rho} = \left(\frac{\partial a}{\partial \rho} - c\right)\theta - b\frac{\partial \theta}{\partial \rho_1},$$

and that formula is, moreover, identical to equation (15), which we had to pose *a priori*. In order to obtain the equation that σ must satisfy one must eliminate θ from equations (19) and (20).

First, suppose that the quantity:

(21)
$$h = \frac{\partial a}{\partial \rho} + ab - c$$

is zero. It will then be impossible to solve equation (19) and (20) with respect to θ and $\partial\theta / \partial\rho_1$. However, if one adds them after multiplying the first one by b then one will have:

$$\frac{\partial\sigma}{\partial\rho} + b\sigma = 0.$$

The general solution to that first-order equation will have the form:

$$e^{-\int b\,d\rho}\,\varphi(\rho_1),$$

in which φ denotes an arbitrary function of ρ_1 . Since X, Y, Z, T are particular values of σ, one sees that the mutual ratios of those quantities will be functions of ρ_1, and in turn, the surface (Σ_1) will reduce to a curve.

Now, suppose that the function h is not zero. One can deduce the values of θ, $\partial\theta / \partial\rho_1$ from (19) and (20), and they will be:

$$h\theta = b\sigma + \frac{\partial\sigma}{\partial\rho}, \tag{22}$$

$$h\,\frac{\partial\theta}{\partial\rho_1} = \left(\frac{\partial a}{\partial\rho} - c\right)\sigma - a\frac{\partial\sigma}{\partial\rho}. \tag{23}$$

Upon equating the value of $\partial\theta / \partial\rho_1$ that is provided by the second formula to the one that one obtains by differentiating the first one, one will find the following equation for σ:

$$\frac{\partial^2\sigma}{\partial\rho\,\partial\rho_1} + \left(a - \frac{\partial\log h}{\partial\rho_1}\right)\frac{\partial\sigma}{\partial\rho} + b\frac{\partial\sigma}{\partial\rho_1} + \left(c - \frac{\partial a}{\partial\rho} + \frac{\partial b}{\partial\rho_1} - b\frac{\partial\log h}{\partial\rho_1}\right)\sigma = 0. \tag{24}$$

This is the equation that the coordinates X, Y, Z, T of an arbitrary point of (Σ_1) must satisfy.

We are then led to a method of transforming linear partial differential equations along a purely geometric path. That very important method was developed in the *Mémoires de l'Académie des Sciences* for 1773 ([1]). Since it plays a fundamental role in the study of a great number of geometric questions, we shall present it in the following chapter, along with all of the developments that it entails.

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([1]) "Recherches sur le Calcul intégral aux differénces partielles," by DE LA PLACE. Mémoires de Mathématique et de Physique de l'Académie des Sciences for 1773, pp. 341-403. Printed in 1777.