Excerpted from G. Darboux: *Leçons sur la theorie des surfaces*, etc., Part IV, Notes by the author, Gauthier-Villars, Paris, 1896, pp. 466-488.

NOTE VIII.

ON THE ASYMPTOTIC LINES AND LINES OF CURVATURE OF THE FRESNEL WAVE SURFACE.

Translated by D. H. Delphenich

1. Ever since Fresnel and Ampère, a great number of geometers have published important works on the wave surface. A complete monograph on that surface deserves to be undertaken. However, one must not try to disguise the fact that it would have to be quite lengthy, because it would have to draw upon many different theories, and in particular, analysis and geometry. In this brief note, we only propose to define and study the differential equations of the asymptotic lines and lines of curvature upon considering the wave surface to be the *apsidal* of an ellipsoid with three unequal axes.

In Book VIII, Chapter VIII, we saw how one could attach certain contact transformations to the consideration of two equations [viz., equations (17) on page 172] in the coordinates x, y, z, X, Y, Z of two corresponding points m, M. If one takes those two equations to be the following two:

$$(1) \qquad \begin{cases} x^2 + y^2 + z^2 - X^2 - Y^2 - Z^2 = 0, \\ Xx + Yy + Zz = 0, \end{cases}$$

in which the coordinates of the two points enter symmetrically, then from the theory that we just developed, one must add the four relations:

$$(2) \qquad \begin{cases} x + pz - \lambda(X + pZ) = 0, \\ y + qz - \lambda(Y + qZ) = 0, \end{cases} \qquad (3) \qquad \begin{cases} X + PZ + \lambda(x + Pz) = 0, \\ Y + QZ + \lambda(y + Qz) = 0, \end{cases}$$

which are derived from equations (21) and (22) that were given on page 174, and in which P, Q, p, q have the significance that was pointed out already. These new equations, when combined with the preceding ones (1), define the transformation to which one gives the name of *apsidal transformation* completely. In order to know the properties of that transformation, it will suffice to interpret it geometrically.

Always let (s) be the surface that is described by the point m, and let (S) be the corresponding surface that is described by the point M. Equations (2) express the idea that the normal to (s) admits:

$$x - \lambda X, \qquad y - \lambda Y, \qquad z - \lambda Z$$

for its direction parameters, and equations (3) likewise express the idea that the direction parameters of the normal to (S) are:

$$X + \lambda x, \qquad Y + \lambda y, \qquad Z + \lambda z\,.$$

One deduces the following well-known properties of the apsidal transformation from these remarks:

Let the two equal and perpendicular radius vectors OM, Om that join the origin to the corresponding points, along with the normals at M and m, be four lines in the same plane.

The normals at M and m will be perpendicular.

2. If we assume all of those properties, which obviously define the transformation, then we can point out some simple formulas that allow one to pass from a surface to its transform.

If x, y, z denote the coordinates of an arbitrary point m of a surface (s) then one writes the equation of the tangent plane in the form:

$$p\,X + q\,Y + r\,Z = 1; \tag{4}$$

p, q, r will be tangential coordinates that verify the following two relations:

$$\begin{cases} px + qy + rz = 1, \\ p\,dx + y\,dy + r\,dz = 0. \end{cases} \tag{5}$$

If one now introduces three new quantities p', q', r' by the relations:

$$\begin{cases} qz - ry + p' = 0, \\ rx - pz + q' = 0, \\ py - qx + r' = 0 \end{cases} \tag{6}$$

then p, q, r, p', q', r' will be the six coordinates of the normal (no. **139**).

Therefore, the nine quantities x, y, z; p, q, r; p', q', r' determine the point, the tangent plane, and the normal. *The surface that is described by the point* (p, q, r) *is the polar reciprocal to the proposed one with respect to the sphere that is concentric to the origin and has a radius equal to unity.* The differential equation of the asymptotic lines is:

$$dp\,dx + dq\,dy + dr\,dz = 0, \tag{7}$$

and that of the lines of curvature (no. **139**) is:

$$dp\,dp' + dq\,dq' + dr\,dr' = 0. \tag{8}$$

Let the quantities that are analogous to the preceding ones be denoted by capital letters and let the transformed surfaces be denoted by (*S*). One must adjoin the following equations to equations (1):

$$\left\{\begin{array}{ll} px+qy+rz=1, & PX+QY+RZ=1, \\ Xp'+Yq'+Zr'=0, & Pp'+Qq'+Rr'=0, \\ Pp+Qq+Rr=0, & P'p+Q'q+R'r=0, \end{array}\right. \tag{9}$$

from which one will deduce the following values:

$$\left\{\begin{array}{lll} P=\dfrac{qr'-rq'}{\sqrt{G}}, & X=\dfrac{r'y-q'z}{\sqrt{G}}, & P'=p', \\ Q=\dfrac{rp'-pr'}{\sqrt{G}}, & Y=\dfrac{p'z-r'x}{\sqrt{G}}, & Q'=q', \\ R=\dfrac{pq'-qp'}{\sqrt{G}}, & Z=\dfrac{q'x-p'y}{\sqrt{G}}, & R'=r', \end{array}\right. \tag{10}$$

in which *G* is defined by the equation:

$$G = p'^2 + q'^2 + r'^2 = P'^2 + Q'^2 + R'^2. \tag{11}$$

The preceding formulas lead to the two relations:

$$\left\{\begin{array}{l} Pp+Qq+Rr=0, \\ P^2+Q^2+R^2=p^2+q^2+r^2, \end{array}\right. \tag{12}$$

which, when compared with formulas (1), exhibit a well-known property of the apsidal transformation: *When two surfaces correspond under that transformation, the same thing will be true for their polar reciprocals with respect to any sphere that has its center at the pole of the transformation.*

3. Apply these general properties to the case in which the surface (*s*) is an ellipsoid (*E*) that is defined by the equation:

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = 1. \tag{13}$$

Here, one will have:

$$\left\{\begin{array}{lll} p=\dfrac{x}{a}, & q=\dfrac{y}{b}, & r=\dfrac{z}{c}; \\ p'=(b-c)qr, & q'=(c-a)pr, & r'=(a-b)pq. \end{array}\right. \tag{14}$$

Take β and α to be curvilinear coordinates, which are the square of the radius *Om* and the square of the distance from the center to the tangent plane at *m*; i.e., set:

$$\left\{\begin{array}{l} x^2+y^2+z^2=\beta, \\ p^2+q^2+r^2=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=\dfrac{1}{\alpha}. \end{array}\right. \tag{15}$$

From the properties of the apsidal transformation, these variables α and β keep the same significance when one passes from *m* to the corresponding point *M*; however, they have the inconvenience that they do not lead to simple expressions in *x*, *y*, *z*. Upon solving equations (13) and (15), for example, one is led to expressions such as the following one for the coordinates *x*, *y*, *z*:

$$x^2 = \frac{a^2}{(a-b)(a-c)}\left(\frac{bc}{\alpha}-b-c+\beta\right).$$

In order to avoid that difficulty, we introduce two new variables α' and β', whose geometric significance we shall see later on, and which are related to the preceding ones by some relations that one can combine into the following identity:

The equation:

$$\xi\,(\xi-\beta)\,(\xi-\beta')-f\,(\xi) = M\,(\xi-\alpha)\,(\xi-\alpha'), \tag{16}$$

in which M is independent of ξ, and $f(\xi)$ is the polynomial of degree three:

$$f(\xi) = (\xi-a)\,(\xi-b)\,(\xi-c), \tag{17}$$

must be true for all values of ξ.

Indeed, if one sets:

$$a+b+c=h, \qquad ab+ac+bc=k, \qquad abc=l, \tag{18}$$

to abbreviate, then the identity (16) is equivalent to the relations:

$$(19)\qquad \begin{cases} M\alpha\alpha' = l, \\ M(\alpha+\alpha') = k - \beta\beta', \\ M = h - \beta - \beta', \end{cases}$$

which determine M, α', β' as functions of α and β. One can, moreover, do the calculations by appealing to that identity. If one replaces ξ with α in it then one will have:

$$(20)\qquad \alpha - \beta' = \frac{f(\alpha)}{\alpha(\alpha-\beta)}$$

to determine β.

Since equations (19) give:

$$(21)\qquad M = h - \alpha - \beta', \qquad \alpha' = \frac{l}{M\alpha}.$$

In what follows, we shall simultaneously employ the four variables α, β, α', β', and return, whenever it becomes necessary, to the preceding relations or to the identity (16), which contains all three. In particular, we shall often appeal to the following relations:

$$(22)\quad \begin{cases} \alpha(\alpha-\beta)(\alpha-\beta') = f(\alpha), \\ \alpha'(\alpha'-\beta)(\alpha'-\beta') = f(\alpha'), \end{cases} \qquad (23)\quad \begin{cases} f(\beta) = -M(\beta-\alpha)(\beta-\alpha'), \\ f(\beta') = -M'(\beta'-\alpha)(\beta'-\alpha'), \end{cases}$$

which are obtained by replacing ξ by α, β, α', β', in turn, and the relations:

$$(24)\qquad \begin{cases} a(a-\beta)(a-\beta') = M(a-\alpha)(a-\alpha'), \\ b(b-\beta)(b-\beta') = M(b-\alpha)(b-\alpha'), \\ c(c-\beta)(c-\beta') = M(c-\alpha)(c-\alpha'), \end{cases}$$

which are likewise are obtained by replacing ξ with a, b, c, resp.

4. Upon taking all of these relations into account, one will easily find the following expressions for p, q, r, x, y, z, p', q', r' *in the form of products:*

$$(25)\qquad \begin{cases} p = \dfrac{x}{a} = \sqrt{\dfrac{(\beta-\alpha)(\beta'-a)}{(\alpha-a)\,f'(a)}} = \sqrt{\dfrac{l\,(a-\alpha')(\beta-\alpha)}{a\alpha\alpha'(a-\beta)\,f'(a)}}, \\[2ex] q = \dfrac{y}{a} = \sqrt{\dfrac{(\beta-\alpha)(\beta'-b)}{(\alpha-b)\,f'(b)}} = \sqrt{\dfrac{l\,(b-\alpha')(\beta-\alpha)}{b\alpha\alpha'(b-\beta)\,f'(b)}}, \\[2ex] r = \dfrac{z}{a} = \sqrt{\dfrac{(\beta-\alpha)(\beta'-c)}{(\alpha-c)\,f'(c)}} = \sqrt{\dfrac{l\,(c-\alpha')(\beta-\alpha)}{c\alpha\alpha'(c-\beta)\,f'(c)}}, \end{cases}$$

(26)
$$\left\{\begin{aligned} p' &= K\sqrt{\frac{a-\alpha}{(a-\beta')\,f'(a)}},\\ q' &= K\sqrt{\frac{b-\alpha}{(b-\beta')\,f'(b)}},\\ r' &= K\sqrt{\frac{c-\alpha}{(c-\beta')\,f'(c)}},\end{aligned}\right.$$

in which K has the value:

(27)
$$K = (\beta - \alpha)\sqrt{\frac{-f(\beta')}{f(\alpha)}} = \frac{1}{\alpha}\sqrt{\frac{l}{\alpha'}(\beta-\alpha)(\beta'-\alpha')}\,.$$

If one substitutes these values for p, q, r, p', q', r' in the fundamental formulas (10) and remarks that here one has:

(28)
$$G = p'^2 + q'^2 + r'^2 = \frac{\beta-\alpha}{\alpha}$$

then one will find the following values for the elements X, Y, Z, P, Q, R that relate to the wave surface:

(29)
$$\left\{\begin{aligned} X &= \frac{p(a-\beta)}{\sqrt{G}} = \sqrt{\frac{l(a-\beta)(a-\alpha')}{a\alpha'\,f'(a)}},\\ Y &= \frac{q(b-\beta)}{\sqrt{G}} = \sqrt{\frac{l(b-\beta)(b-\alpha')}{b\alpha'\,f'(b)}},\\ Z &= \frac{r(c-\beta)}{\sqrt{G}} = \sqrt{\frac{l(c-\beta)(c-\alpha')}{c\alpha'\,f'(c)}},\end{aligned}\right.$$

(30)
$$\left\{\begin{aligned} P &= \frac{p(a-\alpha)}{\sqrt{G}} = \sqrt{\frac{(\beta'-\alpha)(\alpha-a)}{\alpha\,f'(a)}},\\ Q &= \frac{q(b-\alpha)}{\sqrt{G}} = \sqrt{\frac{(\beta'-b)(\alpha-b)}{\alpha'\,f'(b)}},\\ R &= \frac{r(c-\alpha)}{\sqrt{G}} = \sqrt{\frac{(\beta'-c)(\alpha-c)}{\alpha\,f'(c)}},\end{aligned}\right.$$

in which, as we have seen, P, Q, R are equal to p', q', r' in any case.

5. One deduces a great number of relations from the preceding formulas, among which we point out the following ones:

$$(31)\qquad \begin{cases} X^2+Y^2+Z^2=\beta, & \dfrac{X^2}{\beta-a}+\dfrac{Y^2}{\beta-b}+\dfrac{Z^2}{\beta-c}=1, \\ \dfrac{a\,X^2}{\beta-a}+\dfrac{bY^2}{\beta-b}+\dfrac{c\,Z^2}{\beta-c}=0, & \dfrac{a\,X^2}{\beta-a}+\dfrac{bY^2}{\beta-b}+\dfrac{c\,Z^2}{\beta-c}=0, \end{cases}$$

and also:

$$(32)\qquad \begin{cases} P^2+Q^2+R^2=\dfrac{1}{\alpha}, & \dfrac{X^2}{\beta-a}+\dfrac{Y^2}{\beta-b}+\dfrac{Z^2}{\beta-c}=1, \\ \dfrac{P^2}{a-\beta'}+\dfrac{Q^2}{b-\beta'}+\dfrac{R^2}{c-\beta'}=0, & \dfrac{P^2}{a-\alpha}+\dfrac{Q^2}{b-\alpha}+\dfrac{R^2}{c-\alpha}=0. \end{cases}$$

The elimination of β from the two equations in the first row of (31), for example, will give the equation of the surface. The analogy between the other two equations (31) will show the geometric significance of α' immediately. The extended ray OM will cut the surface at a second point M', and one will have:

$$(33)\qquad OM = \sqrt{\alpha'}\,, \qquad\qquad \text{since} \quad OM = \sqrt{\beta}\,.$$

Thus, β and α' are the squares of the two radius vectors that are directed along the diameter OM.

Equations (32) can be deduced from the preceding ones if one replaces:

$$X, \quad Y, \quad Z, \quad a, \quad b, \quad c, \quad \beta, \quad \alpha',$$

with

$$P, \quad Q, \quad R, \quad \frac{1}{a}, \quad \frac{1}{b}, \quad \frac{1}{c}, \quad \frac{1}{\alpha}, \quad \frac{1}{\beta'}$$

respectively. The surface that is described by the point (P, Q, R) is then the wave surface that relates to the ellipsoid (E'):

$$(34)\qquad a\,x^2 + b\,y^2 + c\,z^2 = 1.$$

Since that ellipsoid (E') is the polar reciprocal of the ellipsoid (E) relative to the sphere of radius 1 that has its center at the origin, the preceding result is simply a consequence of the general proposition that was pointed out above that related to the apsidal of the polar reciprocal.

One then sees that if one draws a tangent plane to the first wave surface that is parallel to the tangent plane at M then the distance from the origin to the tangent plane will be $\sqrt{\beta'}$.

Therefore $\sqrt{\alpha}$, $\sqrt{\beta'}$ *are the distances between two parallel tangent planes.* The two remarks that we just made thus attach the variables α', β' that were introduced to the original variables α and β.

6. Having established those points, we shall first define the differential equation of the asymptotic lines of the wave surface:

$$dP\,dX + dQ\,dY + dR\,dZ = 0. \tag{35}$$

In order to perform the calculations most simply, we make use of formulas such as the following one:

$$PX = \frac{(a-\beta)(a-\beta')}{f'(a)} = \frac{M\,(a-\alpha)(a-\alpha')}{a\,f'(a)}, \tag{36}$$

which the reader can establish with no difficulty. Since one can write the equation (35) in the form:

$$\mathrm{S}\; PX\,\frac{dP}{P}\frac{dX}{X} = 0,$$

upon employing formulas (29) and (30), one will get the differential equation:

$$\mathrm{S}\; PX\left[\frac{d\beta}{a-\beta}+\frac{a\,d\alpha'}{\alpha'(a-\alpha')}\right]\left[\frac{d\beta'}{a-\beta'}+\frac{a\,d\alpha'}{\alpha(a-\alpha)}\right] = 0.$$

Upon utilizing the two different expressions (36) for *PX*, one will obtain the following result:

$$\frac{\alpha-\beta'}{f(\alpha)}\,d\alpha\,d\beta + \frac{\alpha'-\beta}{f(\alpha')}\,d\alpha'\,d\beta' = 0. \tag{37}$$

The two relations (22) further permit one to eliminate $f(\alpha)$, $f(\alpha')$, and to convert the preceding equation into the form:

$$\frac{d\alpha\,d\beta}{\alpha(\alpha-\beta)} = -\frac{d\alpha'\,d\beta'}{\alpha'(\alpha'-\beta')}. \tag{38}$$

However, that differential equation will always contain four variables α, β, α', β'. We shall see how one can eliminate two of them.

From the identity (16), α and α' will be roots of the equation:

$$t\,(t-\beta)\,(t-\beta') - f(t) = 0,$$

which has degree two in t. Totally differentiate this equation. Upon denoting its left-hand side by $\varphi(t)$, we will have:

$$\varphi'(t)\,dt = t\,(t-\beta')\,d\beta + t\,(t-\beta)\,d\beta'$$

for each of its two roots, which will give us:

$$(39)\qquad \begin{cases} \varphi'(\alpha)\,d\alpha = \alpha(\alpha-\beta')\,d\beta + \alpha(\alpha-\beta)\,d\beta', \\ \varphi'(\alpha')\,d\alpha' = \alpha'(\alpha'-\beta')\,d\beta + \alpha'(\alpha'-\beta)\,d\beta', \end{cases}$$

upon replacing t with α and α', in turn. One will have, moreover:

$$(40)\qquad \varphi(t) = \frac{l}{\alpha\alpha'}(t-\alpha)\,(t-\alpha'),$$

and as a consequence:

$$(41)\qquad \varphi'(\alpha) = -\,\varphi'(\alpha') = l\left(\frac{1}{\alpha'} - \frac{1}{\alpha}\right).$$

Upon then dividing the two sides of the two equations (39), one will have:

$$\frac{d\alpha}{d\alpha'} = -\,\frac{\alpha(\alpha-\beta')\,d\beta + \alpha(\alpha-\beta)\,d\beta'}{\alpha'(\alpha'-\beta')\,d\beta + \alpha'(\alpha'-\beta)\,d\beta'},$$

which establishes a homographic relationship between the two differential coefficients $d\alpha / d\alpha'$, $d\beta / d\beta'$. Upon canceling the denominators, one will find that:

$$\alpha'(\alpha'-\beta')\,d\alpha\,d\beta + \alpha(\alpha-\beta)\,d\alpha'\,d\beta' + \alpha'(\alpha'-\beta)\,d\alpha\,d\beta' + \alpha(\alpha-\beta')\,d\alpha'\,d\beta = 0.$$

The first two terms have a zero sum for the asymptotic lines, so by virtue of equation (38), one will see that this equation is further equivalent to the following one:

$$(42)\qquad \frac{d\alpha\,d\beta'}{\alpha(\alpha-\beta')} = -\frac{d\alpha'\,d\beta}{\alpha'(\alpha'-\beta)}.$$

All that remains is to multiply or divide the corresponding sides of the two equations (38) and (42) in order to obtain the following two:

$$\frac{d\alpha^2}{\alpha^2(\alpha-\beta)(\alpha-\beta')} = \frac{d\alpha'^2}{\alpha'^2(\alpha'-\beta)(\alpha'-\beta')},$$

$$\frac{d\beta^2}{(\alpha-\beta)(\alpha'-\beta)} = \frac{d\beta'^2}{(\alpha-\beta')(\alpha'-\beta')},$$

and upon taking the identities (22) and (23) into account, this will come down to the following ones:

$$(43)\qquad \frac{d\alpha^2}{\alpha\,f(\alpha)} = \frac{d\alpha'^2}{\alpha'\,f(\alpha')},$$

$$\frac{d\beta^2}{f(\beta)} = \frac{d\beta'^2}{f(\beta')}, \tag{44}$$

in which the variables have been separated. One recognizes that either of these equations are Euler's equation, whose integrals can take various forms.

Therefore, the asymptotic lines of the wave surface are algebraic curves. That important result is due to Lie, who pointed it out in his note: "Sur une transformation géométrique," which was presented to the Académie des Sciences in 1870 (Comptes rendus, t. LXXI, pp. 579). Lie established it for Kummer surfaces, which include the wave surface as a special case. The asymptotic lines of that surface were studied by Klein and Lie in a note: "Ueber die Haupttangentencurven der Kummer'schen Fläche vierten Grades mit 16 Knotenpunkten," which was inserted into the Berlin *Monatsberichte* in 1870 on pp. 891-899.

7. Before studying the integral of the preceding equation, we shall extend the results that were obtained somewhat, while always keeping the four variables α, β, α', β', which are coupled by the identities (22) and (24), and search for surfaces for which the six variables X, Y, Z, P, Q, R are expressed by the following formulas:

$$\begin{cases} X = CA(a-\alpha)^m (a-\alpha')^{m'} (a-\beta)^n (a-\beta')^{n'}, \\ Y = C_1 A(b-\alpha)^m (b-\alpha')^{m'} (b-\beta)^n (b-\beta')^{n'}, \\ Z = C_2 A(c-\alpha)^m (c-\alpha')^{m'} (c-\beta)^n (c-\beta')^{n'}, \end{cases} \tag{45}$$

$$\begin{cases} P = \dfrac{1}{CA\,f'(a)} (a-\alpha)^{-m} (a-\alpha')^{-m'} (a-\beta)^{1-n} (a-\beta')^{1-n'}, \\ Q = \dfrac{1}{C_1 A\,f'(a)} (b-\alpha)^{-m} (b-\alpha')^{-m'} (b-\beta)^{1-n} (b-\beta')^{1-n'}, \\ R = \dfrac{1}{C_2 A\,f'(a)} (c-\alpha)^{-m} (c-\alpha')^{-m'} (c-\beta)^{1-n} (c-\beta')^{1-n'}, \end{cases} \tag{46}$$

in which A denotes an arbitrary function of two of the variables α, β, α', β'; C, C_1, C_2, m, n, m', n' are arbitrary constants. These formulas include the ones that relate to the wave surface as a special case. It is necessary and sufficient that the preceding values verify the two relations:

$$PX + QY + RZ = 1,$$
$$P\,dX + Q\,dY + R\,dZ = 0,$$

identically.

The first one results immediately from the formulas:

$$(47)\qquad \begin{cases} PX = \dfrac{(a-\beta)(a-\beta')}{f'(a)}, \\ QY = \dfrac{(b-\beta)(b-\beta')}{f'(b)}, \\ RZ = \dfrac{(c-\beta)(c-\beta')}{f'(c)}, \end{cases}$$

which are consequences of equations (45) and (46).

As for the second one, after a simple calculation, it will give us the following condition:

$$\frac{dA}{A} + m\frac{(\alpha-\beta)(\alpha-\beta')}{f(\alpha)}d\alpha + m'\frac{(\alpha'-\beta)(\alpha'-\beta')}{f(\alpha')}d\alpha' = 0,$$

or, upon taking the identities (22) into account:

$$\frac{dA}{A} + m\frac{d\alpha}{\alpha} + m'\frac{d\alpha'}{\alpha'} = 0.$$

One can then take:

$$A = \alpha^{-m}\,\alpha'^{-m'},$$

and one will then have the definitive formulas for the desired surface:

$$(48)\qquad \begin{cases} X = C\left(\dfrac{a-\alpha}{\alpha}\right)^m \left(\dfrac{a-\alpha'}{\alpha'}\right)^{m'} (a-\beta)^n (a-\beta')^{n'}, \\ Y = C_1\left(\dfrac{b-\alpha}{\alpha}\right)^m \left(\dfrac{b-\alpha'}{\alpha'}\right)^{m'} (b-\beta)^n (b-\beta')^{n'}, \\ Z = C_2\left(\dfrac{c-\alpha}{\alpha}\right)^m \left(\dfrac{c-\alpha'}{\alpha'}\right)^{m'} (c-\beta)^n (c-\beta')^{n'}, \end{cases}$$

$$(49)\qquad \begin{cases} P = \dfrac{1}{C\,f'(a)}\left(\dfrac{a-\alpha}{\alpha}\right)^{-m} \left(\dfrac{a-\alpha'}{\alpha'}\right)^{-m'} (a-\beta)^{1-n} (a-\beta')^{1-n'}, \\ Q = \dfrac{1}{C_1\,f'(a)}\left(\dfrac{b-\alpha}{\alpha}\right)^{-m} \left(\dfrac{b-\alpha'}{\alpha'}\right)^{-m'} (b-\beta)^{1-n} (b-\beta')^{1-n'}, \\ R = \dfrac{1}{C_2\,f'(a)}\left(\dfrac{c-\alpha}{\alpha}\right)^{-m} \left(\dfrac{c-\alpha'}{\alpha'}\right)^{-m'} (c-\beta)^{1-n} (c-\beta')^{1-n'}. \end{cases}$$

In particular, one will recover the wave surface by taking:

$$(50)\qquad \begin{cases} m = 0, \quad m' = \dfrac{1}{2}, \quad n = \dfrac{1}{2}, \quad n' = 0; \\[2ex] C = \dfrac{\sqrt{l}}{\sqrt{a\,f'(a)}}, \; C_1 = \dfrac{\sqrt{l}}{\sqrt{b\,f'(b)}}, \; C_2 = \dfrac{\sqrt{l}}{\sqrt{c\,f'(c)}}. \end{cases}$$

We apply the method that used in the case of a wave surface to these more general formulas (48) and (49) and define the differential equation of the asymptotic lines:

$$\sum \frac{(a-\beta)(a-\beta')}{f'(a)} \left[\frac{n\,d\beta}{\beta - a} + \frac{n'\,d\beta'}{\beta' - a} + \frac{ma\,d\alpha}{\alpha(\alpha - a)} + \frac{m'a\,d\alpha'}{\alpha'(\alpha' - a)} \right]$$
$$\left[\frac{(1-n)\,d\beta}{\beta - a} + \frac{(1-n')\,d\beta'}{\beta' - a} - \frac{ma\,d\alpha}{\alpha(\alpha - a)} - \frac{m'a\,d\alpha'}{\alpha'(\alpha' - a)} \right] = 0.$$

We perform the summations, while replacing the product $(a - \beta)(a - \beta')$ with the equal quantity:

$$\frac{l\,(a-\alpha)(a-\alpha')}{a\,\alpha\alpha'}$$

in all of the terms where one finds only the differentials $d\alpha$, $d\alpha'$. We will get the differential equation:

$$- n\,(1 - n)\,(\beta - \beta')\,\frac{d\beta^2}{f(\beta)} - n'(1 - n')\,(\beta' - \beta)\,\frac{d\beta'^2}{f(\beta')}$$
$$+ m\,(2n - 1)\,\frac{d\alpha\,d\beta}{\alpha(\alpha - \beta)} + m\,(2n' - 1)\,\frac{d\alpha\,d\beta'}{\alpha'(\alpha' - \beta')}$$
$$+ m'\,(2n - 1)\,\frac{d\alpha'\,d\beta}{\alpha'(\alpha' - \beta)} + m'\,(2n' - 1)\,\frac{d\alpha'\,d\beta'}{\alpha'(\alpha' - \beta')}$$
$$+ \frac{lm^2}{\alpha^2\alpha'}(\alpha - \alpha')\frac{d\alpha^2}{f(\alpha)} + \frac{lm'^2}{\alpha\alpha'^2}(\alpha' - \alpha)\frac{d\alpha'^2}{f(\alpha')} = 0.$$

Now replace $d\alpha$, $d\alpha'$ with their values that are deduced from equations (39), in which one replaces $\varphi'(\alpha)$, $\varphi'(\alpha')$ with their expressions (41). After the reductions, the term in $d\beta$, $d\beta'$ will contain a numerical coefficient that will be:

$$2\,(m - m')\,(m + m' + n + n' - 1).$$

If one desires that the differential equation should again have separated variables then the preceding coefficient must be zero. Discarding the hypothesis $m = m'$, which will lead to the surfaces that were studied already in no. **112** [I, pp. 142], we suppose that:

$$(51)\qquad m + m' + n + n' = 1.$$

After one suppresses the factor:

$$\frac{(m+n)(m'+n')(\alpha-\beta')(\alpha'-\beta)+(m+n')(m'+n)(\alpha-\beta)(\alpha'-\beta')}{\alpha-\alpha'},$$

the differential equation will reduce even further to the simple form:

(52) $$\frac{d\beta^2}{f(\beta)}-\frac{d\beta'^2}{f(\beta')} = 0,$$

which is identical to the one that we obtained in the case of the wave surface, for which, moreover, the relation (51) is found to be verified by the corresponding values of m, n, m', n'. We then obtain a class of surfaces that are attached to the wave surface whose asymptotic lines are all determined by integrating the Euler equation. Those surfaces will be algebraic, as well as their asymptotic lines, whenever the exponents m, n, m', n' are measurable.

8. Among the different processes of integration for the Euler equations, here is the one that seems to us to be the most convenient for our subject:

If θ_1, θ_2, θ_3 denote functions of two variables then consider the family of curves that are defined by the equation:

(53) $$\theta_1\sqrt{c_1}+\theta_2\sqrt{c_2}+\theta_3\sqrt{c_3} = 0,$$

in which c_1, c_2, c_3 denote three constants whose sum is zero:

(54) $$c_1 + c_2 + c_3 = 0.$$

The differential equation of that family of curves can be defined with no difficulty, since one has:

$$d\theta_1\sqrt{c_1}+d\theta_2\sqrt{c_2}+d\theta_3\sqrt{c_3} = 0,$$

which gives:

$$\frac{\sqrt{c_1}}{\theta_2 d\theta_3-\theta_3 d\theta_2} = \frac{\sqrt{c_2}}{\theta_3 d\theta_1-\theta_1 d\theta_3} = \frac{\sqrt{c_3}}{\theta_1 d\theta_2-\theta_2 d\theta_1},$$

and in turn:

(55) $$(\theta_2\, d\theta_3 - \theta_3\, d\theta_2)^2 + (\theta_3\, d\theta_1 - \theta_1\, d\theta_3)^2 + (\theta_1\, d\theta_2 - \theta_2\, d\theta_1)^2 = 0,$$

or furthermore:

(55)′ $$(\theta_1^2+\theta_2^2+\theta_3^2)(d\theta_1^2+d\theta_2^2+d\theta_3^2) - (\theta_1\, d\theta_1 + \theta_2\, d\theta_2 + \theta_3\, d\theta_3)^2 = 0.$$

If one supposes that one has:

$$\theta_1^2+\theta_2^2+\theta_3^2 = 1$$

then the differential equation will take the even simpler form:

(56) $$d\theta_1^2 + d\theta_2^2 + d\theta_3^2 = 0,$$

and if one sets:

(57) $$\begin{cases} \theta_1 = \sqrt{\dfrac{(a-\beta)(a-\beta')}{f'(a)}}, \\ \theta_2 = \sqrt{\dfrac{(b-\beta)(b-\beta')}{f'(b)}}, \\ \theta_3 = \sqrt{\dfrac{(c-\beta)(c-\beta')}{f'(c)}}, \end{cases}$$

it will become:

(58) $$\frac{d\beta^2}{f(\beta)} = \frac{d\beta'^2}{f(\beta')}.$$

This is precisely the equation that we met before, and whose integral can, in turn, be put into the form:

(59) $$\sqrt{PX}\sqrt{(a-k)(b-c)} + \sqrt{QY}\sqrt{(b-k)(c-a)} + \sqrt{RZ}\sqrt{(c-k)(a-b)} = 0,$$

in which k denotes an arbitrary constant, and in which we have taken formulas (47) above into account.

9. The geometric interpretation of the latter equation is easy to give if one introduces the complex to which all of the normals to a family of homofocal ellipsoids belong, and which is composed of all of the lines that cut the three symmetry planes and the plane at infinity at four points whose anharmonic ratio is constant. We call such a complex a *Chasles complex*, and we can then translate the result that we just obtained into a geometric form:

The tangent plane to the surface at every point of every asymptotic line to the surface is tangent to the cone of a Chasles complex whose fundamental tetrahedron is composed of the symmetry planes and the plane at infinity.

If one wishes to obtain, for example, the two asymptotic lines that pass through a point of the surface then one constructs the two second-order cones that have their summits at that point and pass through the four summits of the fundamental tetrahedron and which are tangent to the tangent plane of the surface at that point. *Each of those cones corresponds to a Chasles complex and one of the two asymptotic lines that are sought.*

10. It is natural to demand that the preceding construction should apply to other surfaces. Upon returning to the Monge notations and now letting p and q denote the derivatives of z, when it is considered to be a function of x and y, the problem can obviously be formulated as follows:

Find the surfaces for which the equation:

$$\sqrt{\alpha z} + \sqrt{-\beta px} + \sqrt{-\gamma qy} = 0, \tag{60}$$

in which the constants α, β, γ *satisfy the relation:*

$$\alpha + \beta + \gamma = 0, \tag{61}$$

is the general integral of the differential equation of the asymptotic lines.

We saw above in no. **8** how one eliminates the constants α, β, γ. One is then led to the differential equation:

$$\left[\frac{dz^2}{z} - \frac{(dp\,x)^2}{px} - \frac{(dq\,y)^2}{qy}\right](z - px - qy) = [d\,(z - px - qy)]^2. \tag{62}$$

Now one has:

$$\left\{\begin{array}{l} dp\,dx + dq\,dy = 0, \\ dp^2 = (s^2 - rt)dy^2,\; dq^2 = (s^2 - rt)dx^2,\; dp\,dq = -(s^2 - rt)dx\,dy, \end{array}\right. \tag{63}$$

and as a result:

$$[d(z - px - qy)]^2 = (s^2 - rt)\,(x\,dy - y\,dx)^2. \tag{64}$$

Taking these various relations into account, one can put the differential equation (62) into the following form:

$$[(s^2 - rt)xyz + pq(z - px - qy)]\left[(z - px)\frac{y}{q}dx^2 + (z - qy)\frac{x}{p}dy^2 - 2xy\,dx\,dy\right] = 0. \tag{65}$$

Here, one must choose between two quite distinct hypotheses: If the first factor is non-zero then the second one must be identical to the polynomial:

$$r\,dx^2 + 2s\,dx\,dy + t\,dy^2,$$

which will give the two relations:

$$\frac{y(z - px)}{qr} = \frac{x(z - qy)}{pt} = \frac{-xy}{s},$$

which can be integrated by inspection, and which give us:

(66) $$z - px - qy = \frac{q}{Y'} = \frac{p}{X'},$$

in which X and Y denote functions of x and y, respectively. These two first-order equations are integrated in turn and, as the reader will easily see, lead to *Lamé's tetrahedral surfaces* (no. **112**) *and their limits.*

11. However, one can satisfy equation (65) in another manner. It suffices that the first factor should be zero. One is then led to the partial differential equation:

(67) $$(s^2 - rt)\, xyz + pq\,(z - px - qy) = 0,$$

which will make known a very extensive class of surfaces that enjoy the stated property.

That equation, whose characteristics are the asymptotes of the desired surfaces, can be integrated completely and in an elegant manner. We content ourselves with interpreting it geometrically.

Let α_0, β_0, γ_0 be the direction cosines of the normal, which are defined by the relations:

(68) $$\frac{\cos\alpha_0}{p} = \frac{\cos\beta_0}{q} = \frac{\cos\gamma_0}{-1} = \frac{1}{\sqrt{1+p^2+q^2}}.$$

The coordinates X, Y, Z of a point that is situated at a distance N from the foot of the normal will be given by the formulas:

(69) $$X - x = N\cos\alpha_0\,, \qquad Y - y = N\cos\beta_0\,, \qquad Z - z = N\cos\gamma_0,$$

from which, one deduces that if N_x, N_y, N_z, ϖ denote the segments of the normal that are limited by the three coordinate planes and the projection of the origin onto the normal then one will have:

(70) $$\begin{cases} N_x = \dfrac{-x}{\cos\alpha_0}, \; N_y = \dfrac{-y}{\cos\alpha_0}, \; N_z = \dfrac{-z}{\cos\alpha_0}, \\ \varpi = -x\cos\alpha_0 - y\cos\beta_0 - z\cos\gamma_0; \end{cases}$$

ϖ further denotes the distance from the origin to the tangent plane, *but with a well-defined sign.*

On the other hand, if ρ' and ρ'' denote the principal radii of curvature then one will have:

(71) $$\rho'\rho'' = \frac{(1+p^2+q^2)^2}{rt - s^2}.$$

Upon taking all of these relations into account, one will see that equation (67) can be replaced with the geometric relation:

(72) $$\varpi\,\rho'\rho'' = N_x\,N_y\,N_z\,,$$

which is, in turn, verified for all wave surfaces.

Upon looking for the tetrahedral surfaces that correspond to the first hypothesis and satisfy that relation, one will find that for the surface that is represented by the equation:

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m + \left(\frac{z}{c}\right)^m = 1, \tag{73}$$

one will have:

$$\varpi \rho' \rho'' = (m-1)^2 N_x N_y N_z . \tag{74}$$

Therefore, among the tetrahedral surfaces, only the second-degree surfaces satisfy equation (67).

Moreover, among the surfaces that solve the problem that was posed, the tetrahedral surfaces are the only ones for which *the generator of the contact of the cone of the complex that is defined above relative to a point of the surface and the tangent plane at that point will coincide with an asymptotic tangent to the surface.*

12. After that digression relating to an entire class of surfaces whose asymptotic lines are determined like the ones on the wave surface, we return to that particular surface in order to study and determine its lines of curvature (if that is possible). We recall the original notations and employ the formulas of Olinde Rodrigues in order to form the differential equation of the lines of curvature.

Here, the direction cosines of the normal are:

$$P\sqrt{\alpha}, \quad Q\sqrt{\alpha}, \quad r\sqrt{\alpha}.$$

The desired equations are then presented in the simple form:

$$\left\{ \begin{aligned} dX + \rho\, d(P\sqrt{\alpha}) &= 0, \\ dY + \rho\, d(Q\sqrt{\alpha}) &= 0, \\ dZ + \rho\, d(R\sqrt{\alpha}) &= 0, \end{aligned} \right. \tag{75}$$

in which ρ denotes the principal radius of curvature.

Add the preceding equations, after multiplying them by $2X$, $2Y$, $2Z$, respectively. We find the relation:

$$d\beta + 2\rho\; d\sqrt{\alpha} = 0,$$

from which, one will deduce:

$$r = -\sqrt{\alpha}\,\frac{d\beta}{d\alpha}. \tag{76}$$

That expression is in perfect accord with the one that we gave in no. **1071**, in which we have already employed the system of curvilinear coordinates α, β for an arbitrary surface.

Upon substituting the expression for ρ into the first equation in (75), for example, we will have the differential equation for the lines of curvature in the form:

$$dX - \sqrt{\alpha}\,\frac{d\beta}{d\alpha}\,d(P\sqrt{\alpha}) = 0.$$

Replacing X and P with their values will give:

$$a\,(a-\beta)\,(a-\beta')\,d\alpha\,d\alpha' - \alpha'(a-\alpha)\,(a-\alpha')\,d\beta\,d\beta' = 0,$$

or, upon taking one of equations (24) into account:

(77) $$\frac{l\,d\alpha\,d\alpha'}{\alpha\alpha'^2} = d\beta\,d\beta'.$$

This is the desired differential equation, but it contains four variables. One easily eliminates α' and β', for example, and one will then be led to the following equation:

(78) $$f(\beta)\,d\alpha^2 + f(\alpha)\,d\beta^2 - \left\{2f(\alpha) + (\beta-\alpha)\left[f'(\alpha) - \frac{f(\alpha)}{\alpha}\right]\right\} d\alpha\,d\beta = 0,$$

which has some analogy to the Euler equation. Since the differential equation (77) is symmetric in β and β', one can keep the variables α and β', and one will then be led to the equation:

(79) $$f(\beta')\,d\alpha^2 + f(\alpha)\,d\beta'^2 - \left\{2f(\alpha) + (\beta'-\alpha)\left[f'(\alpha) - \frac{f(\alpha)}{\alpha}\right]\right\} d\alpha\,d\beta' = 0,$$

which is entirely similar to the preceding one.

These equations admit particular solutions that are defined by the relations:

$$f(\alpha) = 0, \qquad f(\beta) = 0, \qquad f(\beta') = 0, \qquad (\alpha-\beta)\,(\alpha-\beta') = 0,$$

which correspond to the lines of curvature that are evident on the wave surfaces, the principal sections, and the circle of the surface, resp.

If one replaces $d\beta / d\alpha$ with its expression as a function of ρ in equation (78) then one will have the second-degree equation:

(80) $$f(\alpha)\,\rho^2 + \sqrt{\alpha}\left\{2f(\alpha) + (\beta-\alpha)\left[f'(\alpha) - \frac{f(\alpha)}{\alpha}\right]\right\}\rho + \alpha f(\beta) = 0,$$

which will make the principal radii of curvature known at each point. That equation permits one to verify the relation (72) that was established already and to find a new one.

Indeed, one now has, while preserving the notations and conventions of no. **11**:

(81)
$$\begin{cases} N_x = -\dfrac{X}{P\sqrt{\alpha}} = -\sqrt{\alpha}\,\dfrac{a-\beta}{a-\alpha}, \\ N_y = -\dfrac{Y}{Q\sqrt{\alpha}} = -\sqrt{\alpha}\,\dfrac{b-\beta}{b-\alpha}, \qquad \varpi = -\sqrt{\alpha}, \\ N_z = -\dfrac{Z}{R\sqrt{\alpha}} = -\sqrt{\alpha}\,\dfrac{c-\beta}{c-\alpha}, \end{cases}$$

Thus, if ρ' and ρ'' denote the two principal radii of curvature then one will have:

(82)
$$\rho'\rho'' = \frac{\alpha\, f(\beta)}{f(\alpha)} = \frac{N_x N_y N_z}{\varpi}.$$

This is the formula that was established above. One will similarly have:

$$\rho' + \rho'' = -2\sqrt{\alpha} - \sqrt{\alpha}\,(\beta - \alpha)\left[\frac{f'(\alpha)}{f(\alpha)} - \frac{1}{\alpha}\right],$$

which will give:

(83)
$$\rho' + \rho'' = N_x + N_y + N_z - \frac{\beta}{\varpi},$$

which is an entirely geometric relation, since β is the square of the radius vector.

13. Let us return to the differential equation (78). It is more complicated that that of Euler, but it is close to the latter equation in the sense that, like the latter, it is defined by means of a polynomial $f(\alpha)$ whose coefficients do not appear in the equation. We shall first perform some transformations that will be useful to us.

If one sets:

(84)
$$\beta = \alpha + \frac{t}{\alpha},$$

upon substituting the variable t for β, then the equation will become:

(85)
$$\varphi(\alpha)\frac{d^2t}{d\alpha^2} - \varphi'(\alpha)\,t\,\frac{dt}{d\alpha} + t\varphi(\alpha) + \frac{t^2}{2}\varphi''(\alpha) + \frac{t^3}{24}\varphi^{(?)}(\alpha) = 0,$$

in which $\varphi(\alpha)$ denotes the polynomial:

(86)
$$\varphi(\alpha) = \alpha f(\alpha).$$

Now, replace t by the variable u:

(87) $$u = \frac{\varphi(\alpha)}{t}.$$

The equation will take the form:

(88) $$\varphi(\alpha)\frac{d^2u}{d\alpha^2} - \varphi'(\alpha)\,u\frac{du}{d\alpha} + u^3 + \frac{u^2}{2}\varphi''(\alpha) + \frac{u^3}{24}\varphi(\alpha)\varphi^{(?)}(\alpha) = 0.$$

These various transformations permit us to establish the following curious proposition:

The differential equation (78) *will be integrated as soon as the polynomial* $f(\alpha)$ *– which is of degree three, in general – reduces to a second-degree polynomial.*

Indeed, in that case, the degree of the polynomial $\varphi(\alpha)$, which is defined by formula (86), will reduce to three. The last term in equation (88) will disappear, and one can give it the following form when one divides it by $d^2u / d\alpha^2$:

(89) $$\varphi\left(\alpha - u\frac{d\alpha}{du}\right) + u^2\left(\frac{d^3\alpha}{du^3} + \frac{d^2\alpha}{du^2}\right) = 0,$$

upon supposing (as is permissible) that the coefficient of x^3 in $\varphi(x)$ is equal to unity ([1]).

If one then performs the substitution that is defined by the formulas:

(90) $$\frac{d\alpha}{du} = \eta, \qquad \alpha - u\,\frac{d\alpha}{du} = \xi,$$

which gives us:

(91) $$u = -\,\frac{d\xi}{d\eta},$$

then the equation will take the form:

(92) $$\varphi(\xi) = \frac{d^3\xi}{d\eta^3}(\eta^3 + \eta^2),$$

which can be integrated immediately by separating the variables and will give us:

(93) $$\int \frac{d\xi}{\sqrt[3]{\varphi(\xi)}} = \int \eta^{-2/3}(1+\eta)^{-1/3}\,d\eta.$$

One will then have:

([1]) If $\varphi(x)$ has degree less than three then one must suppress the term in $d^2\alpha / du^2$, which will again facilitate the integration.

(94)
$$\begin{cases} \alpha = \xi - \eta \dfrac{d\xi}{d\eta}, \\[2ex] \beta = \alpha + \dfrac{f(\alpha)}{u} = \alpha - \dfrac{f(\alpha)\, d\eta}{d\xi}. \end{cases}$$

The integral is thus presented in a very complicated form, and it is not algebraic, in general.

14. We shall now see what the geometric consequences of the preceding analytical result are. One of them is almost obvious.

First, suppose that the ellipsoid (E) reduces to a cylinder whose axis $\sqrt{c}$ increases indefinitely. The wave surface will become the apsidal of an elliptic cylinder. In order to know what the differential equation of the lines of curvature then becomes, one can take the coefficient of c in equation (78), which amounts to replacing $f(\alpha)$ with the second-degree polynomial:

(95)
$$f(\alpha) = (\alpha - a)\,(\alpha - b).$$

Therefore, *one knows how to determine the lines of curvature of the apsidal surface of a surface, and these lines of curvature are not algebraic.*

When one has integrated equation (100), the coordinates of a point on the surface will be determined as functions of u_1 and α_1 by the formulas:

(101)
$$\begin{cases} X = \sqrt{\dfrac{(a_1-\alpha_1)(a_1-\alpha_1-u_1)}{(a_1-b_1)(a_1-b_1)}}, \\[2ex] Y = \sqrt{\dfrac{(b_1-\alpha_1)(b_1-\alpha_1-u_1)}{(b_1-b_1)(b_1-b_1)}}, \\[2ex] Z = \sqrt{\dfrac{(c_1-\alpha_1)(c_1-\alpha_1-u_1)}{(c_1-b_1)(c_1-b_1)}}. \end{cases}$$

These values, which satisfy the relation:

$$X^2 + Y^2 + Z^2 = 1,$$

show that the surface then reduces to a sphere.

15. One will obtain the same result by supposing that only two of the axes – a and b, for example – tend to become equal, while the third one c will remain different from the first two. The wave surface will decompose into an ellipsoid and a sphere. α and β will differ only slightly from the radius of the sphere for the component that approaches a sphere. One will thus be led to further set:

$$(102) \qquad \begin{cases} a = 1 + \varepsilon a_1, \\ b = 1 + \varepsilon b_1, \\ \alpha = 1 + \varepsilon \alpha_1, \end{cases}$$

$$(103) \qquad \varphi(\alpha) = \varepsilon^2 (\alpha_1 - a_1)(\alpha_1 - b_1)(1 - c) + \ldots = \varepsilon^2 \varphi_1(\alpha_1) + \ldots;$$

u will remain finite, and equation (88) will reduce to the following one:

$$\varphi_1(\alpha_1) \frac{d^2 u}{d\alpha_1^2} - \varphi_1'(\alpha_1) u \frac{du}{d\alpha_1} + u^2 + \frac{u^2}{2} \varphi_1''(\alpha_1) = 0,$$

which is nothing but equation (88), with a change of notations, in which the degree of $\varphi(\alpha)$ no longer reduces to three, but to two.

Therefore: *When the wave surface decomposes into an ellipsoid and a sphere, one will know how to determine the limiting position of the lines of curvature on the sphere.*

16. Although we cannot obtain a determination of the lines of curvature in finite terms in the general case, nevertheless, the preceding analysis provides several essential results. We see, not only that the lines of curvature are not algebraic, but furthermore that if those lines are of some interest in the optical study of the surface then we can make them known to a sufficient approximation, since the wave surfaces that relate to various crystals are only slightly different from the sphere, and above all they always have at least two of their axes only slightly different from the other one. The reader will easily convince himself of that if he directs his attention to the following table, which gives the indices of some crystals (relative to the ray D):

Gypsum…………….	1.529	1.522	1.520
Orthoclase………….	1.526	1.523	1.519
Aragonite…………..	1.685	1.681	1.530
Diopside……………	1.700	1.678	1.671
Stilbite……………..	1.500	1.498	1.494
Oligoclase…………	1.542	1.538	1.534
Amblygonite………	1.597	1.593	1.578
Sphène…………….	2.009	1.894	1.888
Epidote……………	1.768	1.754	1.730
Staurotide…………	1.746	1.741	1.736

The lines of curvature of the wave surface have been the object of a large number of studies, moreover. Following an inexact confirmation that these lines are the contact curves of a developable that circumscribes the surface and a concentric sphere, Combescure defined their differential equation in an elegant article that was inserted into volume II of *Annali di Matematica* in 1859 on pp. 278. In some remarks that followed that article, Brioschi showed that the differential equation that was obtained by Combescure could be reduced to the form:

$$\psi(p)\frac{d^2\omega}{dp^2} = e^{\omega} + e^{-\omega} - \psi''(p),$$

in which one sets:

$$\psi(p) = \sqrt{\varphi(p)} = \sqrt{p(p-a)(p-b)(p-c)}\,.$$

However, all of the research that has been done up to now to obtain the complete solution to the problem has stalled completely.

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