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The relativistic variance of the kinetic moment of a rotating body

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In the theory of the spinning, magnetic electron that is due to Dirac, the electron possesses a kinetic moment (moment of quantity of motion) that is provided by its proper moment of rotation. The kinetic moment – or "spin" – is represented mathematically by a vector density $\boldsymbol{\sigma}$: The spatial integral of that vector density gives the kinetic moment in question. One shows that when one performs a Lorentz transformation, the three rectangular components of the vector $\boldsymbol{\sigma}$ transform like the three spatial components of whose fourth component – viz., the temporal component – is zero in the proper system of the electron. That result can seem strange at first because one is accustomed (and, in a sense, with good reason, as we shall see) to consider the three components of a kinetic moment as defining the components *yz*, *zx*, and *xy* of an antisymmetric space-time tensor of rank two, which seems to disagree with the variance that is found for the spin density. In order to try to shed some light on that question here, we shall study the relativistic variance of the kinetic moment of a rotating body.

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Consider a body in which a point O' describes a line in uniform motion. Take that line to be the *z*-axis. If the body is rotating then it will possess a kinetic moment about O'.

First of all, we can define an antisymmetric tensor of rank two $\vec{\vec{M}}$ such that its components *yz*, *zx*, and *xy* are the three rectangular components of the total kinetic moment of the body in motion with respect to the origin *O*. The non-zero components of that tensor will be:

$$(1)\qquad \left\{\begin{array}{ll} M_{yz} = -M_{zy} = \sum (z\, p_y - y\, p_z), & M_{zx} = -M_{xz} = \sum (x\, p_z - x\, p_x), \\ M_{xy} = -M_{yx} = \sum (y\, p_x - x\, p_y), & M_{xt} = -M_{tx} = \sum (ct\, p_x - xW/c), \\ M_{yt} = -M_{ty} = \sum (ct\, p_y - yW/c), & M_{zt} = -M_{zt} = \sum (ct\, p_z - zW/c), \end{array}\right.$$

in which the $\sum$ sign indicates summation over all the molecules that comprise the body, and p_x, p_y, p_z, and W / c are the components of the world-impulse quadri-vector of a molecule. How must one take these $\sum$? Each molecule of a body has its world-line, and one can take a point on each world-line that corresponds to the eight quantities x, y, z, t, p_x, p_y, p_z, and W. However, the observer that is linked to a reference system will always naturally perform the summation over the states of molecules that are simultaneous for him; i.e., ones that correspond to the same value of his proper time.

Suppose, to simplify (and this will change nothing in the results), that the body in motion is homogeneous, that its molecules are all similar, and that they all have the same proper mass m_0. The observer that is linked with the system $Oxyz$ attributes a certain velocity $\mathbf{v}$ to each molecule that varies from one molecule to the other, and he sees a number of molecules $\rho\, d\tau$ in the volume element $d\tau$. He then sets:

$$
(2) \qquad \left\{
\begin{aligned}
M_{yz} &= \int_V m_0 \left[z \frac{\rho v_y}{\sqrt{1-\frac{v^2}{c^2}}} - y \frac{\rho v_z}{\sqrt{1-\frac{v^2}{c^2}}} \right] d\tau, \\
M_{zx} &= \int_V m_0 \left[x \frac{\rho v_z}{\sqrt{1-\frac{v^2}{c^2}}} - z \frac{\rho v_x}{\sqrt{1-\frac{v^2}{c^2}}} \right] d\tau, \\
M_{xy} &= \int_V m_0 \left[y \frac{\rho v_x}{\sqrt{1-\frac{v^2}{c^2}}} - x \frac{\rho v_y}{\sqrt{1-\frac{v^2}{c^2}}} \right] d\tau, \\
M_{xt} &= \int_V m_0 \left[ct \frac{\rho v_x}{\sqrt{1-\frac{v^2}{c^2}}} - x \frac{\rho c}{\sqrt{1-\frac{v^2}{c^2}}} \right] d\tau, \\
M_{yt} &= \int_V m_0 \left[ct \frac{\rho v_y}{\sqrt{1-\frac{v^2}{c^2}}} - y \frac{\rho c}{\sqrt{1-\frac{v^2}{c^2}}} \right] d\tau, \\
M_{zt} &= \int_V m_0 \left[ct \frac{\rho v_z}{\sqrt{1-\frac{v^2}{c^2}}} - z \frac{\rho c}{\sqrt{1-\frac{v^2}{c^2}}} \right] d\tau.
\end{aligned}
\right.
$$

The integrals are extended over the entire volume V of the body in motion.

We would now wish that the M_{ik} thus-defined should transform like the components of an antisymmetric tensor of rank two. In order to do that, we consider a "proper observer" that is linked to the point O' of the body in motion. He employs the coordinates x_0, y_0, z_0 , and a time t_0 . If $V = \beta c$ denotes the velocity of O' with respect to the system $Oxyz$, and if we suppose that the axes $O'x_0\, y_0\, z_0$ are parallel to the axes $Oxyz$ then we will have the following formulas for the simple Lorentz transformation between the coordinates of the two systems:

$$(3) \qquad x_0 = x, \qquad y_0 = y, \qquad z_0 = \frac{z - \beta ct}{\sqrt{1-\beta^2}}, \qquad t_0 = \frac{t-(\beta/c)z}{\sqrt{1-\beta^2}}.$$

It is well-known that upon passing from the system $Oxyz$ to the system $O'x_0\, y_0\, z_0$, the four quantities $\rho\, v_x$, $\rho\, v_y$, $\rho\, v_z$, and ρ transform like the components of a space-time vector, namely, the current-density quadri-vector. One will then have:

$$(4) \qquad \rho_0\, v_x^0 = \rho\, v_x, \quad \rho_0\, v_y^0 = \rho\, v_y, \quad \rho_0\, v_z^0 = \frac{\rho v_z - \beta c\rho}{\sqrt{1-\beta^2}}, \quad \rho_0 = \frac{\rho-(\beta/c)\rho v_z}{\sqrt{1-\beta^2}}.$$

Naturally, the observer properly defines the components of the tensor $\vec{\vec{M}}$ by the formulas:

$$(5) \qquad M_{yz}^0 = \int_{V_0} m_0 \left[z_0 \frac{\rho v_y^0}{\sqrt{1-\frac{v_0^2}{c^2}}} - y_0 \frac{\rho v_z^0}{\sqrt{1-\frac{v_0^2}{c^2}}} \right] d\tau_0, \ \ldots,$$

in which $\mathbf{v}_0$ is the velocity in the proper system of the molecule that occupies the position x_0, y_0, z_0 at the instant t_0, and ρ_0 is the number of molecules in the element $d\tau_0$.

If one wishes to express the quantities M_{xy}^0, … with the aid of the variables x, y, z, t then one must make use of the following relations:

$$(6) \qquad d\tau_0 = \frac{d\tau}{\sqrt{1-\beta^2}}\left(1-\frac{\beta}{c}v_z^0\right),$$

$$(7) \qquad \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1+(\beta/c)v_z^0}{\sqrt{1-\frac{v_0^2}{c^2}}\sqrt{1-\beta^2}}.$$

The relation (6) is proved by taking into account the fact that the states of the molecules that occupy the volume element $d\tau_0$ at the instant t_0 for the proper observer are not simultaneous for the observer that is linked to the axes $Oxyz$, in such a way that one cannot apply the usual formula $d\tau_0 = d\tau/\sqrt{1-\beta^2}$ here.

The relation (7) results from the relativistic formulas for the transformation of velocities, namely:

(8) $$v_x^0 = \frac{v_x\sqrt{1-\beta^2}}{1-(\beta/c)v_z}, \qquad v_y^0 = \frac{v_y\sqrt{1-\beta^2}}{1-(\beta/c)v_z}, \qquad v_z^0 = \frac{v_z-\beta c}{1-(\beta/c)v_z}.$$

Upon taking (3), (6), and (7) into account, one will easily find that:

(9) $$\begin{cases} M_{yz}^0 = \int_V m_0[(z-\beta ct)\,\rho v_y - y\rho(v_z-\beta c)]\dfrac{d\tau}{\sqrt{1-\dfrac{v^2}{c^2}}\sqrt{1-\beta^2}}, \\ M_{zx}^0 = \int_V [x\rho(v_z-\beta c)-(z-\beta ct)\rho v_x]\dfrac{d\tau}{\sqrt{1-\dfrac{v^2}{c^2}}\sqrt{1-\beta^2}}, \\ M_{xy}^0 = \int_V m_0[y\cdot\rho v_x - x\cdot\rho v_y]\dfrac{d\tau}{\sqrt{1-\dfrac{v^2}{c^2}}\sqrt{1-\beta^2}}. \end{cases}$$

Upon comparing this with (2), we will then find:

(10) $$M_{yz}^0 = \frac{M_{yz}-\beta M_{yt}}{\sqrt{1-\beta^2}}, \qquad M_{zx}^0 = \frac{M_{zx}+\beta M_{xt}}{\sqrt{1-\beta^2}}, \qquad M_{xy}^0 = M_{xy}.$$

These formulas indeed show that the components of the kinetic moment with respect to the coordinate origin transform like the components *yz*, *zx*, and *xy* of an antisymmetric tensor of rank two, but we shall see that they are not the quantities M_{yz}, M_{zx}, and M_{xy} that represent the proper moment of rotation of the comoving body in the system *Oxyz*.

Indeed, if the observer that is linked to the axes *xyz* wishes to describe the rotational motion of the body in motion around the point O' then he must imagine axes that are linked to O' – for example, ones that are parallel to $O'x_0\,y_0\,z_0$. Those axes coincide with *Oxyz*, but I shall now call then $O'\xi\eta\zeta$. The coordinates ξ, η, ζ of a point of the body for an observer that is linked to *O* are obviously:

(11) $$\xi = x, \quad \eta = y, \quad \zeta = z - ct.$$

Moreover, the components of the quantity of motion of a molecule of the body – *to the extent that it is due to a proper rotational motion around O* – will be related to the quantities p_x, p_y, and p_z by the relations:

(11, *cont.*) $$\begin{cases} p_\xi = p_x, \quad p_\eta = p_y, \\ p_\zeta = p_z - \text{translational quantity of motion} = p_z - \dfrac{m_0\beta c}{\sqrt{1-\dfrac{v^2}{c^2}}}, \end{cases}$$

since the molecule considered has a "mass of motion" that is equal to $m_0 / \sqrt{1 - v^2/c^2}$ and its velocity of convection is βc in the Oz direction.

Now, the observer that is linked to O quite naturally defines the components of the proper kinetic moment of the body in motion by the formulas:

$$(12) \qquad \begin{cases} S_x = \int_V \rho[\zeta\, p_\eta - \eta\, p_\xi]\, d\tau, \\ S_y = \int_V \rho[\xi\, p_\zeta - \zeta\, p_\xi]\, d\tau, \\ S_z = \int_V \rho[\eta\, p_\xi - \xi\, p_\eta]\, d\tau, \end{cases}$$

which leads to the values:

$$(13) \qquad \begin{cases} S_x = \int_V m_0 \left[(z - \beta ct) \dfrac{\rho\, v_y}{\sqrt{1 - \dfrac{v^2}{c^2}}} - y \dfrac{\rho\,(v_z - \beta c)}{\sqrt{1 - \dfrac{v^2}{c^2}}} \right] d\tau, \\ S_y = \int_V m_0 \left[x \dfrac{\rho\,(v_y - \beta c)}{\sqrt{1 - \dfrac{v^2}{c^2}}} - (z - \beta ct) \dfrac{\rho\, v_x}{\sqrt{1 - \dfrac{v^2}{c^2}}} \right] d\tau, \\ S_z = \int_V m_0 \left[y \dfrac{\rho\, v_x}{\sqrt{1 - \dfrac{v^2}{c^2}}} - x \dfrac{\rho\, v_y}{\sqrt{1 - \dfrac{v^2}{c^2}}} \right] d\tau. \end{cases}$$

Upon comparing this with equations (10), one will find that:

$$(14) \qquad S_x = M_{yz}^0 \sqrt{1 - \beta^2}, \qquad S_y = M_{zx}^0 \sqrt{1 - \beta^2}, \qquad S_z = M_{xy}^0 .$$

For $\beta = 0$, one will get the results:

$$(15) \qquad S_x^0 = M_{yz}^0, \qquad S_y^0 = M_{zx}^0, \qquad S_z^0 = M_{xy}^0,$$

which are obvious *a priori*.

One then deduces that:

$$(16) \qquad S_x = S_x^0 \sqrt{1 - \beta^2}, \qquad S_y = S_y^0 \sqrt{1 - \beta^2}, \qquad S_z = S_z^0,$$

which are formulas that show us how the components of the moment of proper rotation **S** transform when one performs a Lorentz transformation. If the velocity of O' tends to the limiting velocity c then β will tend to unity, and the vector **S** will lie along the direction of motion.

In order to now recover the result of Dirac's theory that we pointed out to begin with, we suppose that in any Galilean system we can define a vector density $\boldsymbol{\sigma}$ such that one has:

$$\int \boldsymbol{\sigma}\, d\tau = \mathbf{S}. \tag{17}$$

In the proper system of the body in motion, we will then have:

$$S_x^0 = \int_{V_0} \sigma_x^0 \, d\tau_0 , \qquad S_y^0 = \int_{V_0} \sigma_y^0 \, d\tau_0 , \qquad S_z^0 = \int_{V_0} \sigma_z^0 \, d\tau_0 . \tag{18}$$

If one would like to perform the integrations in the *xyz* system then one must replace $d\tau_0$ with $d\tau / \sqrt{1-\beta^2}$ this time. Upon taking (16) into account, one will then find:

$$\left\{ \begin{aligned} \frac{S_x}{\sqrt{1-\beta^2}} &= S_x^0 = \int_V \sigma_x^0 \frac{d\tau}{\sqrt{1-\beta^2}}, \\ \frac{S_y}{\sqrt{1-\beta^2}} &= S_y^0 = \int_V \sigma_y^0 \frac{d\tau}{\sqrt{1-\beta^2}}, \\ S_z &= S_z^0 = \int_V \sigma_z^0 \frac{d\tau}{\sqrt{1-\beta^2}}. \end{aligned} \right. \tag{19}$$

The definition (17) then gives:

$$\sigma_x = \sigma_x^0 , \qquad \sigma_y = \sigma_y^0 , \qquad \sigma_z = \frac{\sigma_z^0}{\sqrt{1-\beta^2}} . \tag{20}$$

The quantities σ_x , σ_y , and σ_z then transform like the three spatial rectangular components of a space-time quadri-vector whose fourth component is zero in the proper system of the rotating body. That is in fact the result that was obtained for the spin density in Dirac's theory. When β tends to 1, the spatial vector $\boldsymbol{\sigma}$ will lie along the direction of motion, which is a fact that plays an important role in the new theory of light that was proposed by the author.

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