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# WAVES AND MOTIONS

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# PREFACE

In this little volume, I shall present to the scientific public the present form of some new concepts concerning the relationship of mechanics to optics that I have previously developed in various papers and articles, and which I first presented in summary form in my doctoral thesis.

For the reader who is engaged in the reading of this work, it is convenient to first point out what the points are that I have assumed without proof. In the first place, I assume and suppose known the entire theory of relativity, whether in its original form, which is called "special" today, or in its general form. In particular, I have made constant use of relativistic dynamics, and I suppose that its fundamental formulas are quite familiar to the mind of the reader. I have often employed the tensor calculus and the summation convention for indices. Nevertheless, the tensor calculus does not occupy a significant place here, and I have intentionally avoided the complicated formulas, especially in the chapter on gravitational fields.

I must point out the very special manner by which I have introduced the famous "wave equation" in the course of presentation. In the past, that equation was deduced from the properties of elastic media and extended to the hypothetical ether that one has charged with the task of transmitting light vibrations, and whose singular properties are not, all things considered, much stranger than those of our modern atoms. Much later, Maxwell come onto the scene, and the wave equation become a consequence of the properties of electricity, condensed into a compact form to which the name of that great English scholar is attached. These days, now that theoreticians have recognized the fundamental importance of the "Lorentz group," there has been a certain tendency to consider the wave equation to be a more general postulate than the formulas of electromagnetism. In particular, that viewpoint was evident in the way that M. v. Laue introduced the Lorentz group at the beginning of his treatise on relativity. I have sacrificed that new tendency, by introducing the equation of propagation, out of hand, in the first chapter.

In the second part of the book, I assume the existence of light quanta, and I seek to show that this idea is not as incompatible with the old concepts as one might believe. Today, I admit that I like a great many experimenters have a tendency to believe that light quanta constitute an experimental reality.

Finally, in the third part, I have rapidly reviewed all of statistical thermodynamics and followed the lead of Einstein and Planck by assuming, as a definition, that the entropy of a state is proportional to the logarithm of the number of different ways that the state can be realized.

Those are the main points that I assume without proof, while borrowing them from theories that have already become more or less classical. The reader who wishes to know immediately the interpretation of quanta that I have sought to construct upon that basis would do well to refer to the final chapter, in which the entire book is rapidly summarized.

It is indeed a pleasant task for me to acknowledge my respect and gratitude to Marcel Brillouin for having accepted this book into the present collection of mathematical physics.

Paris, 20 February 1926

# TABLE OF CONTENTS

Page

Page



END OF THE TABLE OF CONTENTS

# WAVES AND MOTIONS

# PART ONE

## THE DYNAMICS OF QUANTA

### CHAPTER I

## THE MATERIAL POINT CONCEIVED AS A STATIONARY WAVE



**1. Atomicity of matter and energy. –** The progress of physics over the last half-century has shown the atomic nature of matter as clearly as possible. At the present state of our knowledge, hardly any physicist would contest the idea that matter is composed by combining some primordial elements (probably just two of them: the proton and electron), and that the simple objects of chemistry differ from each other because their atoms are different assemblages of those primordial elements. The volume that is occupied by those elementary quantities of matter (to the extent that such a volume can be defined with the aid of our usual conceptions) is certainly very small with respect to our scale of measurement, so it would seem justified for us to consider the electron and the proton to be the physical realization of the classical concept of a material point in mechanics ([1]).

On the other hand, the theory of relativity presents us with a notion of great importance: Namely, that of the identity of mass and energy, or if one prefers, since mass is the essential characteristic of matter, the identity of matter and energy. That identification implies the very seductive consequence that the conservation of matter coincides with the conservation of energy, but it also poses a grave problem. Indeed, the theory of electromagnetism has accustomed us to consider an electric charge to be surrounded by an electromagnetic field in which its energy resides. Besides those fields that are coupled to the elements of matter and participate in their atomic character, electromagnetism also predicts the existence of free fields with a continuous structure.

([1]) See FRENKEL, Zeit. Phys. **32** (1925), pp. 518.

They constitute the various radiations of all frequencies. However, if matter and energy are synonymous terms then radiation will be a continuous form of matter, and that is why it can possess a quantity of motion, as Henri Poincaré showed. We will then have two types of matter to distinguish: A discontinuous one, which will be matter in the usual sense of the word, and another continuous one, which will be radiation. Is that distinction truly justified? Would it not be simpler and more satisfying to the mind to assume that all forms of matter have an atomic structure? We know that for some twenty years now, the evolution of physics has been leading us towards the corpuscular theory of light. Furthermore, electromagnetic theory cannot be integrally conserved, since that would lead to the absurd conclusion that matter is unstable and tends to transform into radiation completely.

Today, we believe that it is necessary to attribute an atomic structure to all forms of energy and to assume that they are organized around certain singular points or material points (proton, electron, light corpuscle).

Certainly, the concept of an isolated material point is only an abstraction. It is nonetheless legitimate and necessary to first study that simple case.

**2. The quantum relation.** – Consider an isolated material point then, which is an element of matter in the usual sense or a quantum of radiation. It will be characterized by a quantity $m_0$ – namely, its *proper mass* – which will be equal to the mass that is measured by an observer that is coupled with it. The theory of relativity teaches us that the total energy that is linked with the element and measured by the aforementioned observer is $m_0 c^2$, in which $c$ is the universal constant that is called the speed of light. We suppose that our quantum of energy is found in a Euclidian region of space-time and that it is not accelerated. Under those conditions, it will be either at rest or in uniform, rectilinear motion for any Galilean system. In particular, imagine two Galilean systems, one of which is linked with the moving body, and the other of which possesses a velocity $\upsilon = \beta c$. If the energy is $m_0 c^2$ in the first system then it will be $\frac{m_0 c^2}{\sqrt{1-\beta^2}}$ in the second one. The difference between those two expressions will be the kinetic energy of the point in the second system, and if $\beta$ is small then it will reduce to the classical value:

$$\tfrac{1}{2} m_0 c^2.$$

Having said that, we shall seek to introduce a peculiarity into the constitution of our element of energy that will prepare us for the solution of the enigmas that are posed by quantum theory. Now, the idea that first presented itself to the mind of Planck was the intuition of genius that he had when he introduced the notion of a "quantum" into science, namely, that mechanical energy should be proportional to the frequency of radiation. Indeed, one also recovers that idea under the hypothesis of a quantum of light and in the photo-electric formula that one derives from it. It is also contained in Bohr's law of frequencies in a more disguised form. If we attribute a completely general significance to that idea then we will take the following proposition to be a basic postulate:

*Whenever a material element, in the most general sense, possesses an energy W in a reference system, there will exist a periodic phenomenon in that system that possesses a frequency ν that is defined by the quantum relation W = hν, in which h is Planck's constant, which is equal to* $6.55 \times 10^{-27}$ erg-s *in CGS units.*

Let us first apply that statement to the system that is linked with the element. The postulated frequency will be $\nu_0 = m_0 c^2 / h$, and it will belong to a periodic phenomenon that prevails all around the material point, and which constitutes a singularity, just as the electron does for an electrostatic field. Since the frequency is unique and the material point is in a permanent state, the periodic phenomenon must be analogous to a stationary wave, and we can represent it by an expression of the form:

$$f(x_0, y_0, z_0) \sin 2\pi\nu_0 t_0 ,$$

in which $x_0$, $y_0$, $z_0$, $t_0$ are the space-time coordinates of the proper system considered, and $f(x_0, y_0, z_0)$ is the amplitude of the phenomenon at each point.

How does that periodic phenomenon appear to a Galilean observer that passes the moving body with a velocity $\beta c$? With no loss of generality, we can suppose that the $x$ and $y$ axes of that observer are parallel to the $x_0$ and $y_0$, and that his $z$-axis coincides with both the $z_0$-axis and the direction of the relative velocity. Under those conditions, the periodic phenomenon will appear to have the following form to him:

$$f\left(x, y, \frac{z - \upsilon t}{\sqrt{1-\beta^2}}\right) \sin 2\pi \frac{\nu_0}{\sqrt{1-\beta^2}} \left(t - \frac{\beta z}{c}\right),$$

as a Lorentz transformation will show. Set:

$$\nu = \frac{\nu_0}{\sqrt{1-\beta^2}}, \quad V = \frac{c}{\beta}.$$

The preceding expression will become:

$$f\left(x, y, \frac{z - \upsilon t}{\sqrt{1-\beta^2}}\right) \sin 2\pi\nu \left(t - \frac{z}{V}\right).$$

It represents a wave of frequency $\nu$ whose amplitude and phase displace in the $z$-direction with the velocities $\upsilon$ and $V$, respectively. On the other hand, since the energy $W$ of the moving body in the $x$, $y$, $z$ system is $m_0 c^2 / \sqrt{1-\beta^2}$, one will indeed have:

$$W = \frac{h\nu_0}{\sqrt{1-\beta^2}} = h\nu.$$

The quantum relation is then preserved.

Before going further, we must point out the existence of a frequency $\nu_1$ that one must be careful not to confuse with $\nu$. It is the frequency by which the periodic phenomenon appears to vary for the $x$, $y$, $z$ observer when his eyes follow a well-defined point in the proper system. Indeed, that point-location of a periodic phenomenon is a "clock" that is subject to Einstein's dilatation, so its frequency, as it appears to the passing observer, will be:

$$\nu_1 = \nu_0\sqrt{1-\beta^2}\,.$$

If the clock passes the point $z = 0$ at time $t = 0$ then it will be found at the point $z = \beta ct$ at time $t$, and the phase will change by:

$$\nu_1 t = \nu_0\sqrt{1-\beta^2}\,\frac{z}{\beta c}.$$

Now, when one varies time from 0 to $t$ and the $z$-coordinate from 0 to $\beta ct$, the phase of the wave whose frequency is $\nu$ and whose velocity is $V$ will change by:

$$\nu\left(t-\frac{\beta z}{c}\right) = \frac{\nu_0}{\sqrt{1-\beta^2}}\left(\frac{z}{\beta c}-\frac{\beta z}{c}\right) = \nu_0\sqrt{1-\beta^2}\,\frac{z}{\beta c}.$$

The clock will then remain in phase with the wave, which explains the results that were obtained above.

Two remarkable relations exist between the two velocities $\upsilon$ and $V$. The first of them is deduced immediately from the preceding formulas; it is:

$$\upsilon V = c^2.$$

The other is obtained by considering $\upsilon$ and $V$ to be functions of $\beta$. One will find them by varying $\beta$ slightly:

$$d\nu = \frac{\nu_0\beta}{(1-\beta^2)^{3/2}}\,d\beta, \qquad d\left(\frac{1}{V}\right) = d\beta.$$

Form the combination $\dfrac{d\left(\dfrac{\nu}{V}\right)}{d\nu} = \dfrac{1}{V}+\nu\dfrac{d\left(\dfrac{1}{V}\right)}{d\nu}$. One will get:

$$\frac{d\left(\dfrac{\nu}{V}\right)}{d\nu} = \frac{\beta}{c}+\frac{\nu_0}{\sqrt{1-\beta^2}}\frac{(1-\beta^2)^{3/2}}{\nu_0\beta c} = \frac{\beta}{c}+\frac{1-\beta^2}{\beta c} = \frac{1}{\beta c} = \frac{1}{\upsilon}.$$

One will then have the relation:

$$\frac{1}{\upsilon} = \frac{d\left(\frac{\nu}{V}\right)}{d\nu},$$

which is entirely analogous to the relationship between group velocity and phase velocity that Lord Rayleigh introduced into the theory of dispersive media.

We finally remark that none of the preceding considerations depend upon the numerical value of the mass $m_0$ in any way. We can take it be as small as we wish, so we can make it tend to zero. That is a point that we shall discuss again in the context of radiation.

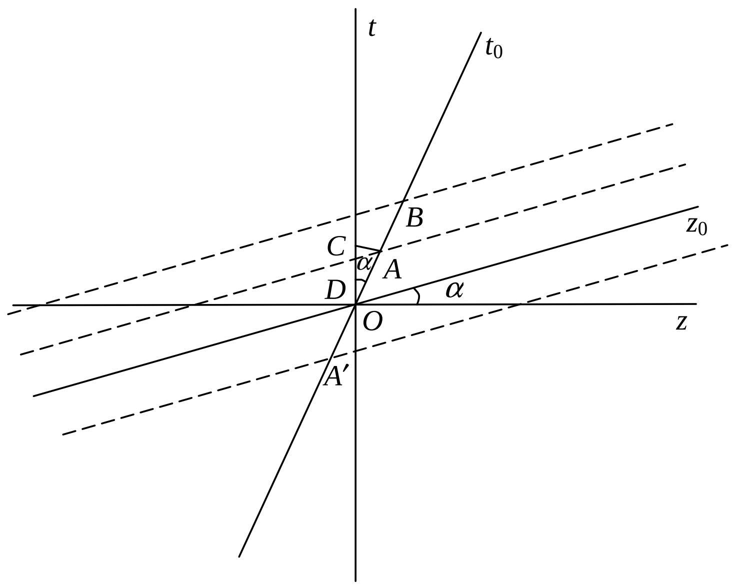


Figure 1.

**3. Space-time interpretation.** – We would like to interpret all of those results upon adopting the viewpoint of the geometric interpretation of space-time. Here, we can content ourselves with a plane interpretation whose axis of the abscissas corresponds to the *z*-coordinate of the point in space, and whose axis of ordinates corresponds to time, multiplied by the constant *c*. Here, we shall postulate the absence of any gravitational field, so the plane $ct - z$ will have a hyperbolic pseudo-Euclidian whose $ds^2$ is equal to $c^2 dt^2 - dz^2$. As we learn from the theory of relativity, the world-line of a moving body that is animated with a velocity $\beta c$ is a line that makes an angle of $\alpha$ with the $Ot$ axis (Fig. 1) such that $\tan \alpha = \beta < 1$, and is thus less than $45^{\mathrm{o}}$. That world-line is also the "proper" time axis of the moving body, and the locus of successive points in space-time that are simultaneous for the comoving observer, and which consequently constitute his "proper" spaces, will be planes whose traces on the plane of the figure will the lines (parallel to $Oz_0$) that make the angle $\alpha$ with $Oz$ that was defined above, and are thus symmetric to $Ot_0$ with respect to the bisector of the z and $Ot$ axes.

The space that surrounds the observer that is linked with the element of energy is, we have supposed, the site of a stationary periodic phenomenon of frequency $\nu_0$ . At time intervals that are equal to $T_0 = 1 / \nu_0$ , the phase of the phenomenon will become the same again, as will the state of the proper space. Mark out a series of points $A'$, $O$, $A$, $B$, … along $Ot_0$ that are separated from each other by a distance $cT_0$ . The lines parallel to $Oz_0$ that are drawn through those points represent a series of equiphase proper spaces. The intersections of those lines with $Oz$ represent planes in the space $x$, $y$, $z$ that are perpendicular to $Oz$ and on which the phase of the periodic phenomenon will be the same at the instant $t_0 = 0$; those are the wave planes of the phenomenon.

As time evolves, the corresponding observer will see the moving body displace with the velocity $\beta c$. The state of a point that is linked to the moving body will appear to become the same to him whenever a time $T_1$ has elapsed, such that $\overline{OC}$, which is the projection of $\overline{OA}$ onto the $t$-axis, will be equal to $cT_1$ . On the triangle $OAC$, one will have:

$$\overline{OA}^2 = \overline{OC}^2 - \overline{AC}^2 = \overline{OC}^2 (1 - \tan^2 \alpha) = \overline{OC}^2 (1 - \beta^2),$$

and as a result, $T_1 = T_0 / \sqrt{1-\beta^2}$ . If one returns to frequencies then that will be written:

$$\nu_1 = \nu_0 \sqrt{1-\beta^2} ,$$

which is a relation that was obtained before.

It is more important then to recover the frequency $\nu$ and velocity $V$. The frequency $\nu$ is inverse to time $T$ that it takes for the phase to reproduce itself at a given point on the $Oz$-axis – at $O$, for example. Hence, $cT = \overline{OD}$. Now, the angle $\widehat{ADC}$ is obviously equal to $\pi / 2 - \alpha$, and one will have $\cot \widehat{ADC} = \tan \alpha = \beta = \overline{CD} / \overline{AC}$. Hence:

$$\overline{CD} = \beta \overline{AC} = \beta^2 \overline{OC} ,$$

and as a result, $\overline{OD} = \overline{OC} - \overline{CD} = (1 - \beta^2)\overline{OC}$. The period $T$ will then be equal to:

$$T = \tfrac{1}{2}\overline{OD} = (1 - \beta^2)\frac{\overline{OC}}{c} = (1 - \beta^2)\, T_1 = T_0\sqrt{1-\beta^2} ,$$

or, in terms of frequencies:

$$\nu = \frac{\nu_0}{\sqrt{1-\beta^2}} .$$

It remains for us to calculate $V$. That is the speed at which the phase advances along $Oz$.

Now, the world-lines of the various phases are, we have seen, the parallels to $Oz_0$ . Those phases then progress in the space $Oz$ with a velocity $\beta' c$ such that $\beta' = \tan \widehat{z_0 O t} = \cot \alpha$.

Hence, $\beta' = 1 / \beta$, and the velocity $V$ is equal to:

$$V = \frac{c}{\beta} = \frac{c^2}{\upsilon}.$$

Having thus verified all of our preceding results, we shall examine the distribution of phases in the space of $x$, $y$, $z$ at a given instant $t$. The moving body occupies a certain point of its rectilinear trajectory at that moment, and the phases of the periodic phenomenon that accompanies it will be the same as those of the wave planes that are perpendicular to that trajectory.

Upon considering Fig. 1, one will see that the phases of the wave planes that are placed *before* the moving body will correspond to the *past* state of the moving body, while the wave planes that are placed *behind* the moving body will correspond to the *future* states of the moving body. As time evolves, the moving body will displace along $Oz$, but less rapidly than the phases of its wave ($\upsilon < V$). It will then be unceasingly subjected to new phase states, which have been "yet to come," up to now. One can say, more picturesquely: *The moving body is preceded by its past, which extends from it without end, and is followed by its future, which unceasingly rejoins it.*

**4. Rectilinear motion and plane waves.** – A plane wave can be represented by the formula:

$$f(x, y, z - \upsilon t) \sin 2\pi\nu\left(t - \frac{z}{V}\right),$$

in which the $z$-direction is normal to the wave planes, while $\upsilon$ and $V$ are the speeds of the energy and phase, resp. The phase factor $\nu t - \nu z / V$ is, by hypothesis, independent of the reference system considered, and since it has the form of a scalar product in the pseudo-Euclidian space-time $zt$, we conclude that $\nu / c$ and $\nu / V$ are the time and space components of a certain world-vector. If one knows that "world-wave" vector then the frequency and phase velocity of the wave will be well-defined in all Galilean systems.

On the other hand, relativistic dynamics characterizes the motion of a material point by the four-dimensional "world-velocity" vector. That vector is a vector that is equal to unity and points along the tangent to the world-line in space-time. Its component along an arbitrary $x^i$-axis will obviously be:

$$u^i = \frac{dx^i}{dt}.$$

In the present case, the world-velocity is constant in magnitude and direction. Its time and space components are:

$$u^4 = u_t = \frac{d(ct)}{ds} = \frac{d(ct)}{\sqrt{c^2dt^2 - dz^2}} = \frac{1}{\sqrt{1-\beta^2}},$$

$$u^3 = u_z = \frac{dz}{ds} = \frac{\beta}{\sqrt{1-\beta^2}}, \qquad u^1 = u_x = 0, \qquad u^2 = u_y = 0,$$

respectively.

Upon multiplying the "world-velocity" vector by the scalar quantity $m_0\, c$, one will get the world-impulse of the moving body. One confirms immediately that this vector will have the quantity of motion for its spatial component and the energy, divided by $c$, for its temporal component.

Let $x^1$, $x^2$, $x^3$ be the spatial coordinates $x$, $y$, $z$, and let $x^4$ denote the coordinate $ct$.

Let **O** and **I** be the world-wave vector and the world-impulse vector, resp., of the plane wave and the moving body that is linked with it. One will have:

$$O^1 = 0, \qquad O^2 = 0, \qquad O^3 = \frac{\nu}{V}, \qquad O^4 = \frac{\nu}{c},$$

$$I^1 = 0, \qquad I^2 = 0, \qquad I^3 = \frac{m_0 \beta c}{\sqrt{1-\beta^2}}, \qquad I^4 = \frac{m_0 c}{\sqrt{1-\beta^2}}.$$

Well, well! All of the results that were obtained up to now can be summarized in the following vectorial formula:

$$\mathbf{I} = h\,\mathbf{O} \qquad (h = \text{Planck's constant}).$$

When applied to the index 4, it will give:

$$\frac{m_0 c^2}{\sqrt{1-\beta^2}} = h\nu,$$

which is the quantum relation.

When applied to the index 3, it will give:

$$\frac{m_0 \beta c}{\sqrt{1-\beta^2}} = h\frac{\nu}{V}, \qquad \text{or} \qquad \frac{m_0 c^2}{\sqrt{1-\beta^2}} = h\nu \times \frac{c}{\beta V},$$

which is a relation that is indeed verified if $V = c / \beta$, and which expresses the idea that the quantity of motion of the moving body is equal to $h$ times the inverse of the wave length $\nu / V$ (or wave number) of the plane wave. That will give us the following expression for the wave length in question, moreover:

$$\lambda = \frac{V}{\nu} = \frac{h}{m_0 \nu}\sqrt{1-\beta^2}.$$

As we remarked at the beginning of this paragraph, the phase is invariant. The same thing will be true for its differential ([2]):

$$d\varphi = 2\pi\nu = 2\pi\,[O^4\, d\,(ct) - O^3\, dz] = 2\pi\, O_i\, dx^i\,.$$

If we take the equation **I** = $h$ **O** into account then we will also have:

$$d\varphi = \frac{2\pi}{h}(W\,dt - G\,dz),$$

in which $W$ and $G$ are the energy and quantity of motion of the material point. Now, the expression in parentheses is nothing but the element of Hamiltonian action of the moving body with its sign changed. Indeed, we will have:

$$W\,dt - G\,dz = \frac{m_0 c^2}{\sqrt{1-\beta^2}}\,dt - \frac{m_0 \beta c}{\sqrt{1-\beta^2}}\,dz = m_0\, c^2 \sqrt{1-\beta^2}\;dt,$$

and that is, in fact, the action for a free material point in Einstein's dynamics (up to sign). Of course, for small values of $\beta$, the function $m_0\, c^2 \sqrt{1-\beta^2}$ will differ from the kinetic energy by only a sign, and that kinetic energy will coincide with the Hamiltonian action in this case. One will then have:

$$d\varphi = -\frac{2\pi}{h}dA.$$

The variations that are experienced by the phase of the plane wave will be proportional to the action of the material point when one displaces in space-time.

It results from this very important consequence, to which we will return much later, that the principle of least action, which fixes the rectilinear form of the trajectories here, is not distinct from Fermat's principle, which determines the form of the rays of the wave. In particular, if we consider two points $A$ and $B$ on the trajectory of the material point then the principle of least action, in its de Maupertuis form, will be $\delta\int_A^B \frac{m_0\, \upsilon}{\sqrt{1-\beta^2}}\,dz = 0$ (when we take the variation of mass with velocity into account) or $\delta\int_A^B G\,dz = 0$. If we replace $G$ with $h\nu / V$ then we will find that:

$$\delta\int_A^B \frac{h\nu}{V}\,dz = 0,$$

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([2]) Indeed, one will have the relations:

$$o_1 = -\,o^1, \qquad o_2 = -\,o^2, \qquad o_3 = -\,o^3, \qquad o_4 = +\,o^4$$

between the covariant and contravariant components of a vector in a Cartesian system $ds^2 = c^2\, dt^{\,2} - dx^2 - dy^2 - dz^2$.

and according to Fermat's principle, that will show us that the trajectory of the moving body is necessarily a ray of the wave, which was predicted *a priori*.

Although the considerations of the present paragraph apply to only the case of uniform, rectilinear motion, thanks to their invariant form, they are susceptible to some important generalizations that will be goal of the following chapters.

**5. The material point envisioned as a point-source.** – In the description of the periodic phenomena that we have attached to every material center, up to now, we have preoccupied with phase above all else. We have said only this much on the subject of amplitude: In the Galilean system that is coupled to the material point, the periodic phenomenon will have an amplitude $f(x_0, y_0, z_0)$ for which the origin, which is the location of the material point, is a singular point. Moreover, it is obvious that when the amplitude is envisioned in the $x, y, z$ system, it will displace *en masse* along the $z$-axis with a velocity $v$.

As an attempt to specify the form of the function $f$, we shall envision a material point with *spherical symmetry*, in particular. In that way, we shall exclude the atom of radiation – viz., the "light quantum" – from the present considerations, because the phenomenon of polarization shows that such an element cannot possess spherical symmetry, and we shall reserve the study of that particular case to the last chapter.

Everything leads us to believe that the electron (and undoubtedly, the proton, as well) possesses spherical symmetry. Therefore, what follows will then apply to the electron.

If a material point possesses spherical symmetry in its proper system then the amplitude at a point whose distance from the material point is $r_0$ will be:

$$f(r_0) = f\left(\sqrt{x_0^2 + y_0^2 + z_0^2}\right)$$

in that proper system, and the surfaces of equal amplitude will be concentric spheres. In the $x, y, z$ system, the amplitude will be given by the function:

$$f\left(\sqrt{x^2 + y^2 + \frac{(z - vt)^2}{1 - \beta^2}}\right),$$

as a Lorentz transformation will show, and the surfaces of equal amplitude will be ellipsoids of revolutions that are flattened around the $z$-axis and move *en masse* along that axis with velocity $v$.

In order to determine the function $f(r_0)$, we shall introduce a hypothesis here that is strongly suggested by the classical theories and by the primordial role that is played by the constant $c$ in all of those questions. We suppose that the function $\varphi(x, y, z, t)$, which is an expression for the periodic phenomena in an arbitrary Galilean system, satisfies the wave equation $\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}$. The constant $c$ also keeps its essential character of being the "propagation velocity of perturbations in Euclidian space-time."

Since the wave equation is invariant under the Lorentz transformation, in order to determine $f$, it will suffice to write down the fact that the periodic phenomena will satisfy that equation in an arbitrary Galilean system. Choose the system $x_0, y_0, z_0$ that is coupled with the moving body. The periodic phenomenon will then have the expression:

$$\varphi(x_0, y_0, z_0, t_0) = f(r_0) \sin 2\pi \nu_0 (t_0 - \tau_0),$$

in which $\tau_0$ is a constant that depends upon the origin of time. One will then have:

$$\Delta\varphi = \sin 2\pi \nu_0 (t_0 - \tau_0)\, \Delta f = \sin 2\pi \nu_0 (t_0 - \tau_0) \left[\frac{d^2 f}{dr_0^2} + \frac{2}{r_0}\frac{df}{dr_0}\right],$$

$$\frac{\partial^2 \varphi}{\partial t_0^2} = -4\pi^2 \nu_0^2 \varphi,$$

and as a result, the wave equation will give us:

$$\frac{d^2 f}{dr_0^2} + \frac{2}{r_0}\frac{df}{dr_0} = -\frac{4\pi^2\nu_0^2}{c^2} f.$$

One will easily verify that the general integral of that second-order differential equation is:

$$f(r_0) = \frac{A}{r_0} \sin\left[2\pi \frac{\nu_0}{c} r_0 + \alpha\right],$$

in which $A$ and $\alpha$ are integration constants. The periodic phenomenon will have the complete expression in the system $x_0, y_0, z_0$:

$$\varphi(r_0, t_0) = \frac{A}{r_0} \sin\left[2\pi \frac{\nu_0 r_0}{c} + \alpha\right] \sin 2\pi\nu_0 (t_0 - \tau_0),$$

and

$$\varphi(x, y, z, t)$$

$$= \frac{h}{\sqrt{x^2 + y^2 + \dfrac{(z - \upsilon t)^2}{1-\beta^2}}} \sin\left[2\pi \frac{\nu_0}{c} \sqrt{x^2 + y^2 + \frac{(z-\upsilon t)^2}{1-\beta^2}} + \alpha\right] \sin \frac{2\pi\nu_0}{\sqrt{1-\beta^2}}\left(t - \tau_0 - \frac{\beta z}{c}\right)$$

in the $x, y, z$-system.

When viewed in the proper system, the phenomenon then appears to be spherical stationary wave, and it can further be conceived to be the superposition of a converging spherical wave and a diverging one, because one can write:

$$\varphi(r_0, t_0) = \frac{A}{r_0}\left\{\cos\left[2\pi\nu_0\left(t-\frac{r_0}{c}\right)+c_1\right]-\cos\left[2\pi\nu_0\left(t+\frac{r_0}{c}\right)+c_2\right]\right\}.$$

If only the first term exists in the preceding formula then one can say: The material point can be compared to a source that unceasingly emits a perturbation of frequency $\nu_0 = m_0\, c^2 / h$ into the surrounding medium. However, there is also a converging wave whose interpretation can indeed attract great interest, philosophically, and its intervention seems to be necessary in order to assure the stability of the material point, moreover. That converging wave corresponds to the creation of a perturbation in the medium by the point-source that obeys, not the classical law of retarded potentials, but the law of advanced potentials. In effect, the phase of a wave at a point is determined by the state of the point-source at the future instant when the converging wave will be absorbed by it. The divergent wave unceasingly carries the past phases of the point-source in space with the speed of light, while the converging wave seems to unceasingly bring its future phases, and likewise with the speed $c$.

Without stopping here to discuss the philosophical consequences that the systematic introduction of advanced potentials into physics would entail, we can state the following conclusion (viz., the point-source theorem):

*In a uniform, rectilinear motion of a material point with spherical symmetry, the periodic phenomenon whose existence we have devised can be calculated by considering the material point to be a source of* ***proper*** *frequency $\nu_0$ and superposing the advanced and retarded actions that emanate from that source.*

As a simple application of this principle, we calculate the periodic phenomenon at a point on the $z$-axis in our $x$, $y$, $z$ system. The frequency of the point-source must naturally be taken to be equal to $\nu_1 = \nu_0\sqrt{1-\beta^2}$ .

The given general form $\varphi(x, y, z, t)$ reduces to:

$$\varphi(0, 0, z, t) = \frac{\sqrt{1-\beta^2}}{z-\upsilon t}A\sin\left[2\pi\frac{\nu_0}{c\sqrt{1-\beta^2}}(z-\upsilon t)+\alpha\right]\sin\frac{2\pi\nu_0}{c\sqrt{1-\beta^2}}\left(t-\tau_0-\frac{\beta z}{c}\right),$$

here. We must now derive that formula.

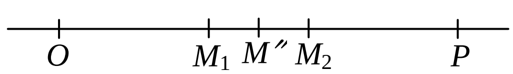


Figure 2.

In order to do that (Fig. 2), we mark out the point $P$ along the $z$-axis with coordinate origin $O$ whose abscissa is $z$ and which is where we would like to determine the phenomenon, and finally the point $M$ whose abscissa is $vt$, which is the position of the moving point at the instant $t$.

The point $P$ at the instant $t$ is the site of two waves: A diverging wave that is emitted by the moving body at the point $M_1$ and the instant $t - \tau_1$ and a converging wave that will be absorbed by it at the point $M_2$ at the time $t + \tau_2$. Upon appealing to Liénard's formulas, which are classical in the theory of electrons, one will see that those waves have the values:

$$\frac{A}{2\overline{PM_1}(1-\beta)} \cos [2\pi\nu_0 (t - \tau_1) + c_1]$$

and

$$\frac{A}{2\overline{PM_2}(1+\beta)} \cos [2\pi\nu_1 (t + \tau_2) + c_2],$$

respectively.

We calculate $\overline{PM_1}$, $\overline{PM_2}$, $\tau_1$, and $\tau_2$:

$$\overline{MP} = \overline{OP} - \overline{OM} = z - \upsilon t = \overline{M_1P} - \overline{M_1M} = c\,\tau_1 - \upsilon\,\tau_2 ,$$

so

$$\tau_1 = \frac{z-\upsilon t}{c(1-\beta)}, \qquad \overline{PM_1} = c\tau_1 = \frac{z-\upsilon t}{1-\beta}.$$

One similarly finds that:

$$\tau_2 = \frac{z-\upsilon t}{c(1+\beta)}, \qquad \overline{PM_2} = c\tau_1 = \frac{z-\upsilon t}{1+\beta}.$$

Upon superimposing these two waves, one will deduce from this that:

$$\varphi_1 (0, 0, z, t) = \frac{A}{z-vt}\sin\left[2\pi\nu_1\frac{\tau_1+\tau_2}{2}+\alpha\right]\sin 2\pi\nu_1\left[t-\frac{\tau_1-\tau_2}{2}-\tau_0\right]$$

$$= \frac{A}{z-vt}\sin\left[2\pi\frac{\nu_1}{1-\beta^2}\frac{z-\upsilon t}{c}+\alpha\right]\sin 2\pi\frac{\nu_1}{1-\beta^2}\left[t-\frac{\beta z}{2}-\tau_0\right].$$

upon denoting the phase constants by $\alpha$ and $\tau_0$.

Finally, upon replacing $\nu_1$ with $\nu_0 (1 - \beta^2)^{1/2}$, we will get the expected formula, except that it will be lacking the factor of $\sqrt{1-\beta^2}$ in the numerator of the amplitude. In Chapter VII, we will see that this deviation is explained by the tensorial character of the quantity $\varphi$, for which $\varphi_1$ is only the time component, by definition.

A calculation that is a bit long, but still simple, can be performed for a point that is not situated on the trajectory, and it will lead to the same conclusions. It is important to remark that the amplitudes of the converging and diverging perturbations are equal in any case. That fact, which was proved here, will be postulated in the following chapter in the course of an analogous calculation.

**6. Summary. –** In summary, the material point now appears to us to be a system of stationary waves that surround a central point-source. When viewed in a Galilean reference system that is in motion relative to it, that set of stationary waves will be manifested by an inhomogeneous plane wave. By that, I mean a wave whose surfaces of equal phase are planes normal to the direction of relative motion and whose surfaces of equal amplitude displace with the same velocity as the moving body. It will suffice to direct one's attention to the expression for the function $\varphi(x, y, z, t)$ that was given above in order to account for the following fact: The surfaces of equal amplitude are flattened ellipsoids of revolution that are centered on the moving body, so they are, in summary, spheres that have been subjected to the Lorentz contraction. It is customary (at least, among physicists) to consider only very simple solutions of the wave equation. It is very interesting to see that upon imagining a somewhat-more-complicated solution that presents a singularity, one will arrive at a representation of the material point that is, in some sense, wave-like.

We have had to distinguish three velocities in the preceding study:

1. The velocity of propagation $c$,
2. The phase velocity $V$,
3. The velocity of energy $\upsilon$.

Those three quantities are coupled by a very simple relation:

$$\upsilon V = c^2.$$

If one confines oneself to imagining the displacement of the phase then one can represent this by saying that space presents an "index" to the waves that are associated with the moving body of velocity $\beta c$, which is defined by the relation:

$$n = \frac{c}{V} = \beta.$$

If the moving body is characterized by a proper frequency $\nu_0$ then each value of $\beta$ will correspond to a value of the frequency $\nu$ of the associated wave that is given by:

$$\nu = \frac{\nu_0}{\sqrt{1-\beta^2}}.$$

One can then also say that the dispersion of the waves that are characterized by a certain value of $\nu_0$ will obey the equation:

$$n^2 = 1 - \frac{\nu_0^2}{\nu^2}.$$

We shall recover some analogous viewpoints upon studying the uniform, non-rectilinear motions, for which we shall now seek to generalize the results that were obtained up to now.

## CHAPTER II

# THE MATERIAL POINT IN A CONSTANT GRAVITATIONAL FIELD

1. Nature of the gravitational field. – 2. The dynamics of a material point in a gravitational field. – 3. The wave associated with motion in a gravitational field. – 4. The point-source theorem. – 5. Phase agreement between the moving body and its wave.

**1. Nature of the gravitational field.** – When the motion of a material point is not uniform and rectilinear, one says that it is found in a force field. Einstein has shown us the equivalence of the forces of inertia and the forces of gravitation, so one can say, without prejudicing the future progress of science, that we know two and only two types of force fields: The gravitational field and the electromagnetic field. A fundamental difference exists between these two field categories, since the trajectory of a material point in a constant gravitational field is determined completely by the position and velocity (both its magnitude and direction) of the point at a given instant. It does not depend at all upon the nature of the material point – i.e., its mass $m_0$ . As we shall see in the following chapter, things are quite different for the case of a constant electromagnetic field, in which the trajectory of a moving point depends upon not only the initial conditions of position and velocity, but also upon its nature, or more precisely, upon the ratio of its charge to its mass.

The fact that the trajectory of a material point in a constant gravitational field does not depend upon its mass permitted Einstein to formulate his magnificent interpretation of the gravitational force. For him, in a gravitational field, as in the absence of any field, the motion of a material point is always represented by a geodesic in space-time, which is a geodesic that is determined completely when one knows a point and the tangent at that point. However, space-time is not Euclidian in a gravitational field, so geodesics and the trajectories, which are the spatial projections of those geodesics, will no longer be lines.

Today, it is well-known to physicists that the nature of space-time is characterized by the expression for the square of the element of length – viz., by its $ds^2$. The form of the geodesic that passes through two points $P$ and $Q$ in space-time is fixed by the condition of minimum length: The integral $\int_P^Q ds$, which is taken along the curve, must be stationary. Here, as in a Euclidian space-time, one can introduce a "world-velocity" vector that is tangent to the world-line considered at each point and has length unity. Its contravariant components are defined by the relations:

$$u^i = \frac{dx^i}{ds},$$

and its covariant components by:

$$u_k = g_{ik}\, u^i ,$$

in which the $g_{ik}$ are the quantities that are classical in the theory of relativity today and are such that $ds^2 = g_{ik}\, dx^i\, dx^k$ . One verifies directly that the length of the vector $u_i\, u^i$ is indeed equal to unity.

The geodesic equation is then written:

$$\delta \int u_i\, dx^i = \delta \int g_{ik} \frac{dx_k}{ds} dx^i = 0.$$

In the preceding chapter, we saw that the product of the differential form $u_i\, dx^i$ with $m_0\, c$ is equal to the differential of the Hamiltonian action of the moving body, up to sign. If we transpose that definition into the present case by considering the quantities $m_0\, cu_i$ to be the covariant components of the world-impulse of the moving body then we will see that it amounts to the same thing to say that the motion is represented by a geodesic in space-time or that it obeys Hamilton's principle, except that, in reality, the mass $m_0$ does not enter into the determination of the motion at any point.

In the case of a Euclidian space-time, the $ds^2$ can be put into the form:

$$ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2 = c^2\, dt^2 - dl^2,$$

in which $dl$ is the unit of length in space. That being the case, the rectilinear motion of velocity will correspond to geodesics of null length. In a non-Euclidian space-time, one can define a velocity $\gamma$ *at each point* that is analogous to $c$ and corresponds to an element of a null-length geodesic traversing a neighborhood of a point.

**2. The dynamics of a material point in a gravitational field.** – In any case that physics can present, it must be possible to find a form of $ds^2$ such that it is possible to separate time from space. We shall then envision only the forms for $ds^2$ that do not contain rectangular terms in which time appears. $ds^2$ will then have the form:

$$ds^2 = g_{44}\, (dx^4)^2 + g_{ik}\, dx^i\, dx^k \quad (i, k = 1, 2, 3),$$

with $dx^4 = d\,(ct)$, as in the Euclidian case. The form $g_{ik}\, dx^i\, dx^k$, in which only the spatial variables appear, is equal to the square of the spatial element of length, when given a sign. That is, in fact, the form that $ds^2$ presents in the case of a gravitational field with central symmetry (e.g., the Schwarzschild formula).

We have defined the speed $\gamma$ of light at a point in space-time above. With the form for $ds^2$ that was adopted here, it must be given by the relation:

$$g_{44}\, c^2\, dt^2 - dl^2 = 0,$$
$$\frac{dl}{dt} \equiv \gamma = c\sqrt{g_{44}}\ .$$

If the gravitational field is assumed to be constant then $g_{44}$ will be a function of $x^1$, $x^2$, $x^3$, in general. The same thing will then be true for $\gamma$.

With that value for $\gamma$, one can write $ds^2 = \gamma^2\, dt^2 - dl^2$, which is an expression that is analogous to the one in Euclidian space-time, except that $dl^2$ is not forced to be Euclidian here, and $\gamma$ will vary with the point in space, moreover. Although $dl^2$ cannot be Euclidian, the velocity of a moving body will always be defined to be the quotient of the distance covered by the time that it took to cover it:

$$\upsilon = \frac{dl}{dt}.$$

As a result, if one sets $\beta = \upsilon / \gamma$, instead of $\beta = \upsilon / c$ then one will have:

$$ds = \gamma\, dt \sqrt{1-\beta^2},$$

and one will find the following familiar appearance for the action of the material point of mass $m_0$ :

$$dA = m_0\, c\, \gamma \sqrt{1-\beta^2}\, dt.$$

We have seen that the element $ds$ can be written $u_i\, dx^i$. Now, one will have:

$$u_i\, dx^i = g_{44} \frac{dx^4}{ds} dx^4 + g_{ik} \frac{dx^k}{ds} dx^i$$
$$= \gamma^2 \frac{dt}{ds} dt - \frac{dl}{ds} dl,$$

here.

One will easily infer from the expression itself for $ds^2$ that:

$$\frac{dt}{ds} = \frac{1}{\gamma\sqrt{1-\beta^2}},$$
$$\frac{dl}{ds} = \frac{\beta}{\gamma\sqrt{1-\beta^2}},$$

so

$$ds = u_i\, dx^i = \frac{\gamma}{\sqrt{1-\beta^2}} dt - \frac{\beta}{\sqrt{1-\beta^2}} dl.$$

If we multiply this by $m_0\, c$ then we will obtain the element of action in the form:

$$dA = \frac{m_0 c\, \gamma}{\sqrt{1-\beta^2}} dt - \frac{m_0 \beta c}{\sqrt{1-\beta^2}} dl.$$

From the invariance of that expression, we conclude that $\dfrac{m_0 c\gamma}{\sqrt{1-\beta^2}}$ and $\dfrac{m_0 \beta c}{\sqrt{1-\beta^2}}$ are the time and space (i.e., tangent to *dl*) components of a quadri-dimensional vector, namely, the world-impulse. We once more call the quantity:

$$W = \frac{m_0 c\gamma}{\sqrt{1-\beta^2}} = \frac{m_0}{\sqrt{g_{44}}}\frac{\gamma^2}{\sqrt{1-\beta^2}}$$

the *energy* and the vector that is tangent to *dl* and equal to:

$$\mathbf{G} = \frac{m_0 \beta c}{\sqrt{1-\beta^2}} = \frac{m_0}{\sqrt{g_{44}}}\frac{\beta\gamma}{\sqrt{1-\beta^2}} = \frac{m_0}{\sqrt{g_{44}}}\frac{\upsilon}{\sqrt{1-\beta^2}}$$

the quantity of motion.

We remark that these expressions have the same form as in the Euclidian case, in which $\gamma$ plays the role of the speed of light and the mass is divided by $\sqrt{g_{44}}$.

The equations of motion are deduced from the condition:

$$\delta\int ds = \delta\int_{t_0}^{t_1} L\,dt = 0,$$

in which $L = m_0\, c\gamma \sqrt{1-\beta^2}$. As the calculus of variations teaches us, that condition implies that:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} \qquad (i = 1, 2, 3) \qquad \text{(Lagrange equation)},$$

in which $\dot{x}^i = dx^i / dt$. $L$ depends upon $x^i$ and $\dot{x}^i$, but not upon time explicitly, which will permit one to verify the constancy of the function $\sum_{1,2,3} \dot{x}^i \dfrac{\partial L}{\partial \dot{x}^i} - L$ in time by a classical calculation.

Measure one of the coordinates along the trajectory: The corresponding Lagrange equation will be:

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) = \frac{d}{dt}\left(\frac{m_0\beta c}{\sqrt{1-\beta^2}}\right) = -\frac{\partial L}{\partial l}.$$

The quantity $p_l = -\,\partial L / \partial\dot{l}$ is the quantity of motion, and we recover a fundamental equation of dynamics. To bring things back to the case of classical mechanics, suppose that the velocity is very small, so we can neglect $\beta^2$ in comparison to unity. We will get:

$$\frac{m_0 c}{\gamma}\frac{d\upsilon}{dt} = -\, m_0\, c\, \frac{d\upsilon}{dt},$$

or

$$\frac{d\upsilon}{dt} = -\,\gamma\frac{d\upsilon}{dl} = -\,\frac{d}{dl}\left(\frac{c^2 g_{44}}{2}\right).$$

The quantity $\gamma^2 / 2 = c^2\, g_{44} / 2$ then pays the role of the gravitational potential, which is a classical result of Einstein's theory.

Finally, the motion admits the first integral:

$$L - \dot{l}\,\frac{\partial L}{\partial \dot{l}} = m_0\, c\gamma\sqrt{1-\beta^2} + \upsilon\frac{m_0 \beta c}{\sqrt{1-\beta^2}} = \frac{m_0 c\,\gamma}{\sqrt{1-\beta^2}}\,.$$

That is the conservation of energy.

Having thus sketched out the broad features of the dynamics of the material point in a constant gravitational field, we shall attempt to introduce some wave-like elements into it.

**3. The wave associated with motion in a gravitational field.** – As we did for uniform, rectilinear motion, we shall now seek to associate the displacement of a material point in a constant gravitational field with the propagation of a certain wave.

First, consider the set of trajectories that correspond to a certain given value $W$ for energy. One of those trajectories that is tangent to a given direction passes through each point in space, and for any direction considered, moreover, so the speed of the moving body when it passes through the given point on the trajectory will be the same, because the speed is determined by the relation:

$$\frac{m_0 c\,\gamma}{\sqrt{1-\beta^2}} = W,$$

and $\gamma$ depends upon only the spatial coordinates. Any point $(x^1,\ x^2,\ x^3)$ will then correspond to a value $\upsilon\,(x^1, x^2, x^3)$ of the speed on the trajectories of energy $W$. When the velocity is known, one can deduce the quantity of motion by the equation:

$$G = \frac{m_0 \beta c}{\sqrt{1-\beta^2}}\,,$$

which is also a function of $W$ and $x^1$, $x^2$, $x^3$.

The form of the trajectory of energy $W$ that connects two points $A$ and $B$ is obtained from de Maupertuis's principle, which is deduced from that of Hamilton by a classical argument, and is stated thus:

*The trajectory of energy W that goes from A to B is such that the variation of the integral* $\int_A^B G\,dl$ *shall be zero when one deforms it infinitesimally without displacing the extremities and without varying the value of the energy.*

On the other hand, upon generalizing a result that was proved in Chapter I, one will easily see that the propagation of a wave along a spatial line can always be represented by a "world-wave" vector that has frequency for its temporal component at each point and whose spatial component is a vector whose length is equal to the wave number $1 / \lambda = \nu / V$ and is carried along the tangent to the line of propagation or ray of the wave.

If we would like to associate the motion of a material point along a trajectory with the displacement of a "phase" along that trajectory, when it is considered to be a ray of a wave, then we must establish a relationship between the value of the world-impulse of the moving body along that curve and that of the corresponding wave-vector.

We will then be led to the following statement by a natural generalization of the results that were obtained for rectilinear motion:

*When a material point describes a certain trajectory T of energy W in a constant gravitational field, the gravitating medium will be site of a periodic phenomenon that presents the form at each point* ***of the trajectory*** *of a wave whose frequency and speed of propagation are defined by the relation* $\mathbf{I} = h\,\mathbf{O}$. *In other words, the world-wave vector will be proportional to the world-impulse vector along the trajectory.*

The trajectory $T$ will then be a ray of the associated wave, and one can remark that here, as in the case of rectilinear motion, the de Maupertuis integral $\int_A^B G\,dl$ will be proportional to the Fermat integral $\int_A^B dl/\lambda = \int_A^B \nu\,dl/V$.

The two relations $W = h\nu$, $G = h\nu / V$ show us that:

$$V = \frac{W}{G} = \frac{\gamma}{\beta}$$

along the trajectory. One then concludes that the product $\upsilon\,V$ is equal to $\gamma^2$ ; that is a relation that is analogous to the one that was obtained for rectilinear motion. $V$ will then be a function of the coordinates along the orbit. If one considers the various trajectories through a point that have a given energy $W$ and the propagation of waves that are associated with the description of those trajectories by the moving body then one can say that the gravitating medium presents an index $n = \gamma / V$ for that propagation that is defined by the equation:

$$n = \beta = \sqrt{1 - \frac{m_0^2\,c^2\gamma^2}{h^2\,\nu^2}} = \sqrt{1 - \frac{\nu_0^2}{\nu^2}}\,,$$

upon setting $\nu_0 = m_0\,c\,\gamma / h$. One sees that this dispersion formula has the same character as in the absence of a field, but the proper frequency $\nu_0$ will vary with the position of the moving body. The orbits of energy $W$ coincide with the rays of the waves of frequency $\nu$ in a medium of index $n$.

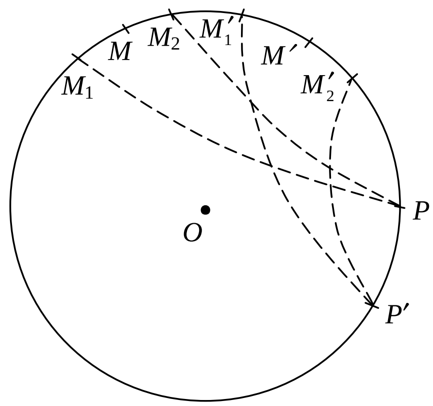


Figure 3.

**4. The point-source theorem.** – The preceding considerations rest upon some inductions, and as a result, they will be more hypothetical than the ones in the first chapter. It is tempting to once more adopt the viewpoint here that allowed us to specify the constitution of the wave that is associated with uniform, rectilinear motion. One must then consider the moving body to be a point-source of variable frequency $\nu_1$ and suppose that perturbations propagate in the gravitating medium along null-length geodesics. One must recover the statements of the preceding paragraph upon superimposing the advanced and retarded actions, and at the same stroke, determine the wave that is associated with a given motion completely.

I shall not study the question in a general fashion, but I can point out a few aspects of an important special case: viz., uniform, circular motion in a field with central symmetry (see Fig. 3).

Imagine a circular trajectory. Let $P$ be a point on that curve, and let $M$ be the position of the moving body at the instant $t$. At that moment, the point $P$ will be the site of a diverging wave that was emitted by the point-source when it was at $M_1$ at the instant $t - \tau_1$ and a converging wave that will be absorbed by the source $M_2$ at the instant $t + \tau_2$ . If we assume (and this must be proved rigorously) that the amplitude of the two perturbations is the same then the perturbation that results at $P$ at the time $t$ will be proportional to:

$$\sin 2\pi\nu_1 (t - \tau_1) + \sin 2\pi\nu_1 (t - \tau_1) = \cos 2\pi\nu_1 \frac{\tau_1 + \tau_2}{2} \cos 2\pi\nu_1 \left( t - \frac{\tau_1 - \tau_2}{2} \right).$$

At the instant $t + dt$, the moving body will go to $M'$, such that:

$$\overline{MM'} = \upsilon\, dt = \omega R\, dt,$$

That point will be subject to the waves that are emitted by the points $M_1'$, $M_2'$, such that:
in which $R$ is the radius of the trajectory. Consider the point $P$ that is defined by:

$$\overline{PP'} = dl = \overline{MM'}.$$

That point will be subject to the waves that are emitted by the points $M_1'$, $M_2'$, such that:

$$\overline{M'M_1'} = \upsilon\tau_1', \qquad \overline{M_2'M'} = \upsilon\tau_2'.$$

However, by reason of the central symmetry, no matter what path is followed by the perturbations, one must have:

$$\tau_1 = \tau_1', \qquad \tau_2 = \tau_2'.$$

As a result, upon passing from the point $P$ at time $t$ to the point $P'$ at time $t + dt$, one cannot vary the amplitude factor of the cosine, whereas the phase factor in the sine will vary by:

$$2\pi\nu_1\, dt = 2\pi\nu\,(1 - \beta^2)\, dt = 2\pi\nu\left(dt - \frac{\beta}{\gamma_R}\,dl\right).$$

That shows us that the amplitude and phase propagates along the trajectory with the speeds $\upsilon$ and $V = \gamma_R / \beta$. Our inductions are then found to be verified.

However, we can go further and consider a circle that is concentric to the trajectory and has a radius of $\rho$. An argument that is analogous to the preceding one will show that the speeds of the amplitude and phase on that circle will be $\gamma_R / \beta$ and $\gamma_R\,\rho / (\beta R)$, respectively. It is easy to conclude from this that periodic phenomenon that is coupled with the moving body can be expressed in cylindrical coordinates $\rho$, $\theta$, $z$ by the function:

$$\varphi(\rho, \theta, z, t) = f(\rho, z, \theta - \omega t)\,\sin 2\pi\nu\left(t - \frac{\beta}{\gamma_R}R\theta - \chi\right)$$

if the circular trajectory of radius $R$ is situated in the plane $z = 0$, and the phase $\chi$ can depend upon $\rho$ and $z$, moreover.

**5. Phase agreement between the moving body and its wave. –** The hypothesis that the material point is the source of its associated wave imposes the continual agreement between its phase and that of the wave. In the absence of a general proof of the point-source theorem, that will permit us to say:

*If one assumes that the associated wave possesses the constant frequency* $\nu = \nu_1 / (1 - \beta^2)$ *along the trajectory the phase velocity at any point M on that trajectory will be:*

$$V_R = \frac{\gamma_M}{\beta_M}.$$

Indeed, the hypothesis that was made for the associated wave gives the value for its phase factor at the points on the trajectory:

$$\sin 2\pi\nu\,[t - f\,(l)].$$

Now, the vibration of the point-source is proportional to:

$$\sin 2\pi \int_0^t \nu_1(t)\,dt\,,$$

and we have seen that $\nu_1$ is linked with $\beta$ by the relation:

$$\nu_1\,(t) = \nu\,[1 - \beta^2\,(t)],$$

so let $M_1$ be the point whose curvilinear abscissa is $l_1$ that is occupied at the time $t_1$ , and suppose that phase matching exists. One will then have:

$$\nu\,[t_1 - f\,(l_1)] = \int_0^{t_1} \nu_1(t)\,dt\,.$$

At the time $t_1 + dt_1$ , the moving body has moved to $M_2$ in such a way that:

$$\overline{M_1\,M_2} = dl_1 = \upsilon_{M_1} dt_1\,.$$

In order for phase matching to persist, one must have:

$$\nu\left[dt_1 - \left(\frac{df}{dt}\right)_{M_1} dl_1\right] = \nu_1\,(t)\,dt_1$$

or

$$\upsilon_{M_1}\left(\frac{df}{dt}\right)_{M_1} = \beta_{M_1}^2\,.$$

Now, the derivative $\left(\frac{df}{dt}\right)_{M_1}$ is obviously the inverse of the phase velocity of the wave at the point $M_1$ , and one will indeed get:

$$V_{M_1} = \frac{\upsilon_{M_1}}{\beta_{M_1}^2} = \frac{\gamma_{M_1}}{\beta_{M_1}}\,.$$

It is quite obvious that the argument can be pursued step-by-step, and that the relation $V = \gamma / \beta$ must be valid for an arbitrary point of the orbit. That would again seem to confirm the possibility of extending the results of the first chapter to the case of a static gravitational field; nonetheless, a rigorous, general proof would still be desirable.

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## CHAPTER III

# THE CHARGED MATERIAL POINT IN A CONSTANT ELECTROMAGNETIC FIELD

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1. Nature of the electromagnetic field. – 2. Dynamics of the point-like charge. – 3. The wave associated with motion in an electromagnetic field. – 4. The point-source theorem.

**1. Nature of the electromagnetic field.** – In order to study the second type of field that we have been led to distinguish, we envision a constant electromagnetic field that prevails in a space in which no appreciable gravitational field exists. Up to now, we have assumed that space-time is Euclidian in such a field, in such a way that $ds^2$ has the form $c^2\, dt^2 - dl^2$. Meanwhile, a point-like charge that displaces in a field will not describe a line, and one can no longer say that the world-line of an arbitrary material point is a geodesic. The world-line of a moving body of mass $m_0$ and charge $e$ that passes through two points $P$ and $Q$ in space-time will no longer be given by the condition $\delta\int_P^Q ds = 0$, but, in fact, by a condition with the new form:

$$\delta\int_P^Q \left(1+\frac{e}{m_0 c}\varphi_s\right) ds = 0,$$

in which $\varphi_s$ is the projection onto the trajectory of the world-line of a certain four-dimensional vector that defines the electromagnetic field at each point in space-time – viz., the world-potential vector. Since one obviously has $\varphi_s = \varphi_i\, u^i$, one can write:

$$\delta\int_P^Q (m_0 c\, u_i + e\,\varphi_i)\, dx^i = 0,$$

upon multiplying the relation above by $m_0\, c$, or rather, by a simple transformation (upon setting $\upsilon^i = dx^i / dt$ for $i = 1, 2, 3$):

$$\delta\int_P^Q (m_0 c^2\sqrt{1-\beta^2} + ec\,\varphi_4 + e\,\varphi_1\,\upsilon^1 + e\,\varphi_2\,\upsilon^2 + e\,\varphi_3\,\upsilon^3)\, dt = 0.$$

Later on, we shall see that this formula does indeed give the dynamics of the electron, and we shall deduce the components of the world-potential from it.

The form that was given above for the principle of least action shows that the trajectory of a moving body will depend upon the ratio $k = e / m_0$, in addition to the initial conditions of position and velocity. We shall call it the "electromagnetic sensitivity of the moving body" because the action of an electromagnetic field on the moving body will be an increasing function of that quantity.

Whereas the nature of gravitation has been at least interpreted, if not explained, by the curvature of space-time, the nature of the electromagnetic field remains more mysterious. Indeed, Weyl's theory interpreted Maxwell's equations in an ingenious way. However, along with the fact that the theory has been rejected by many scholars (including Einstein himself!), it cannot interpret either the mechanical action that the field exerts upon a charge or the role that is played by the electromagnetic sensibility $e / m_0$ , at least to my knowledge.

In the absence of any field, the speed of propagation that figures in the wave equation will be defined, as we have seen, by the null-length geodesics and will be equal to $c$. At first glance, it seems that the same thing must be true here, since $ds^2$ has the same form. Later on, we shall see that there can be other speeds of propagation in electromagnetic media than the speed $c$.

**2. Dynamics of the point-like charge.** – In order to obtain the equations of motion in a constant electromagnetic field, we start with Hamilton's principle:

$$\delta\int_P^Q L\,ds = 0,$$

in which:

$$L = m_0c^2\sqrt{1-\beta^2} + e\,[c\,\varphi_4 + \upsilon^1\,\varphi_1 + \upsilon^2\,\varphi_2 + \upsilon^3\,\varphi_3].$$

The Lagrange equations resolve the issue. They are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \upsilon^i}\right) = \frac{\partial L}{\partial x^i},$$

and two analogous ones. The Lagrange momenta $p_i = \partial L / \partial \upsilon^i$ ($i$ = 1, 2, 3) are:

$$p_i = -\frac{m_0\,\upsilon^i}{\sqrt{1-\beta^2}} + e\,\varphi_i\,.$$

The first Lagrange equation will become:

$$-\frac{d}{dt}\left(\frac{m_0\,\upsilon^1}{\sqrt{1-\beta^2}}\right) + e\frac{\partial\varphi_1}{\partial x^i}\upsilon^i = e\left[c\frac{\partial\varphi_4}{\partial x^1} + \upsilon^1\frac{\partial\varphi_1}{\partial x^1} + \upsilon^2\frac{\partial\varphi_2}{\partial x^1} + \upsilon^3\frac{\partial\varphi_3}{\partial x^1}\right]$$

or

$$\frac{d}{dt}\left[\frac{m_0\,\upsilon^1}{\sqrt{1-\beta^2}}\right] = e\left[c\frac{\partial\varphi_4}{\partial x^1} - \upsilon^2\left(\frac{\partial\varphi_2}{\partial x^1} - \frac{\partial\varphi_1}{\partial x^2}\right) + \upsilon^3\left(\frac{\partial\varphi_1}{\partial x^3} - \frac{\partial\varphi_3}{\partial x^1}\right)\right].$$

Upon recalling that the electric force is the gradient of a scalar potential (in a constant field) and that the magnetic force is the rotation of the vector potential, one will see that the right-hand side gives the Lorentz force ([3]) if one sets:

$$\varphi_4 = \varphi^4 = \frac{\psi}{c}, \qquad \varphi_1 = -\varphi^1 = -\frac{a^1}{c}, \qquad \varphi_2 = -\varphi^2 = -\frac{a^2}{c}, \qquad \varphi_3 = -\varphi^3 = -\frac{a^3}{c},$$

in which $\psi$ is the scalar potential, and $a^i$ are the components of the vector potential. The world-potential is thus found to be defined in that way. The quantities:

$$p^1 = -p_1 = \frac{m_0\,\upsilon^1}{\sqrt{1-\beta^2}} + e\frac{a^1}{c}, \quad p^2 = -p_2 = \frac{m_0\,\upsilon^2}{\sqrt{1-\beta^2}} + e\frac{a^2}{c}, \quad p^3 = -p_3 = \frac{m_0\,\upsilon^3}{\sqrt{1-\beta^2}} + e\frac{a^3}{c}$$

are the components of the quantity of motion, which contains an electromagnetic term that depends upon the vector potential, along with the ordinary mechanical term. We remark that the quantity of motion *is no longer tangent to the trajectory* here. The Lagrange equation admits the first integral.

$$L = -\,\upsilon^i\,\frac{\partial L}{\partial \upsilon^i} \qquad\qquad (i = 1, 2, 3),$$

which is equal to the energy:

$$W = c\,p_4 = c\,p^4 = \frac{m_0\,c^2}{\sqrt{1-\beta^2}} + e\,\psi.$$

Here, as well, aside from the properly mechanical term, an electromagnetic term that depends upon a scalar potential will appear. The world-impulse vector **I** will *no longer be tangent to the world-line of the moving body.* That is what essentially distinguishes this type of motion from the ones that were studied previously.

The expressions for the components of the energy-quantity of motion tensor show that the action integral will always have the canonical form:

$$\int (W\,dt - G\,dt)$$

here, in which $W$ and $G$ have their usual meanings.

**3. The wave associated with motion in an electromagnetic field.** – We can repeat what we said in paragraph **3** of the preceding chapter word-for-word here, because

([3]) Recall that this form is given by the formula:

$$F = e\left(\operatorname{grad}\psi + \frac{1}{c}[\mathbf{v}\operatorname{rot}\mathbf{A}]\right).$$

knowing the world-potential (which does not vary in time) at each point of space will permit us to deduce the values of the velocities $\upsilon$ and $V$ from each value $W$ of the energy.

We once more associate any motion of the moving body with a propagation by assuming inductively that the impulse vector and the wave vector are proportional *along the trajectory* [$\mathbf{I} = h\,\mathbf{O}$].

We will then be led to some conclusions that are identical to the ones in the last chapter. In particular, the trajectory of the moving body will always be one of the rays of its wave. However, a new situation presents itself here: Since the quantity of motion is not tangent to the trajectory, in general, the phase velocity will not be directed along the ray. Along the orbit, the electromagnetic medium will then behave like an *anisotropic* medium for the propagation of the wave that is associated with the motion.

We seek to determine a dispersion formula for the present case that is analogous to the one that was obtained before for the gravitational media, and whose significance is identical to it.

The propagation will be tangential to the amplitude at a point $M$ of a trajectory of energy $E$ where the vector potential is $\mathbf{a}$, and one will have the relations:

$$h\nu = W = \frac{m_0\,c^2}{\sqrt{1-\beta^2}} + e\,\psi, \qquad \frac{h}{V}\boldsymbol{v} = \mathbf{G}$$

for the determination of the phase.

The vector $\mathbf{G}$ is the geometric sum of the vector $\mathbf{g} = \dfrac{m_0\,c^2}{\sqrt{1-\beta^2}}$, which is tangent to the ray, and the vector $(e / c)\,\mathbf{a}$ (see Fig. 4).

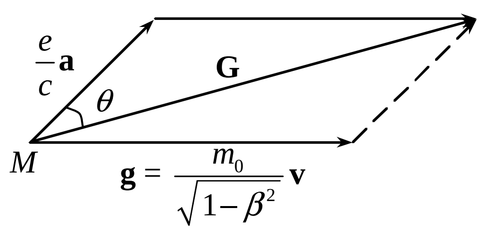


Figure 4.

Let $\theta$ denote the angle that the direction of propagation of the phase (i.e., the direction of $\mathbf{G}$) makes with the direction of the vector potential. One easily gets that:

$$G = \frac{ea}{c}\cos\theta + \sqrt{g^2 - \frac{e^2a^2}{c^2}\sin^2\theta}\,.$$

Now, with the aid of energy, one can eliminate $\beta$ from the expression for $g$, and one will easily find that:

$$G = \frac{ea}{c}\cos\theta + \sqrt{\left(\frac{h\nu - e\psi}{c}\right)^2 - m_0^2c^2 - \frac{e^2a^2}{c^2}\sin^2\theta}\,.$$

One will deduce the index from this by means of the defining relation:

$$n = \frac{c}{V} = \frac{eG}{h\nu}.$$

That index is then a function of frequency, the position of the point $M$, and the angle between the direction of propagation of the phase and the vector potential. In the simple case of the electrostatic field (viz., $\mathbf{a} = 0$), one will get the simple formula:

$$n = \sqrt{\left(1 - \frac{e\psi}{h\nu}\right)^2 - \frac{m_0^2 c^4}{h^2\nu^2}}.$$

It is easy to generalize an important result of the first chapter. Indeed, consider the passage of the moving body from a point $M$ on its trajectory, let $\mathbf{G}$ be the quantity of motion at that point, and let $\alpha$ denote the angle between that vector and the velocity $\mathbf{v}$ of the moving body. The phase propagates along $G$, so its speed of propagation $V_r$ along the ray will be equal to $V / \cos\alpha$, while the projection $G_r$ of $\mathbf{G}$ onto the tangent to the ray will be $G \cos\alpha$. Hence, from the fact that:

$$G = \frac{h\nu}{V},$$

one will deduce that:

$$G_r = \frac{h\nu}{V_r}.$$

Now, if one chooses the coordinates in such a fashion that one of them is measured along the trajectory then the corresponding Hamilton equation will be:

$$\upsilon = \frac{\partial W}{\partial G_r} = \frac{\partial \nu}{\partial\left(\dfrac{\nu}{V_r}\right)}.$$

Once more, Rayleigh's formula appears here as a general consequence of the canonical equations. The derivative of the right-hand side is a partial derivative because $V_r$ is a function of the coordinates, along with the frequency.

We shall now pose a question whose significance we shall see later on. We have found that in the absence of any field, such as a gravitating field, the geometric mean of the velocities $\upsilon$ and $V$ will be equal to the fundamental velocity of the propagation of perturbations. Can that theorem be transposed over to the theory of a constant electromagnetic field? In order to see that it can, one must calculate $\upsilon\, V_r$, which is the product of the radial phase velocity with the velocity of the moving body:

$$V_r = \frac{W}{G_r} = \frac{\dfrac{m_0 c^2}{\sqrt{1-\beta^2}} + e\psi}{\dfrac{m_0 \upsilon}{\sqrt{1-\beta^2}} + \dfrac{e}{c} a_r} = \frac{c^2}{\upsilon} \frac{1 - \dfrac{e\,a_r}{cG_r}}{1 - \dfrac{e\psi}{h\nu}}, \qquad \upsilon\, V_r = c^2 \frac{1 - \dfrac{e\,a_r}{cG_r}}{1 - \dfrac{e\psi}{h\nu}}.$$

The product of the velocities is not equal to the square of the velocity $c$ then; it is equal to the square of a velocity $c'$ that is given by the equation:

$$c' = c \sqrt{\frac{1 - \dfrac{e\,a_r}{cG_r}}{1 - \dfrac{e\psi}{h\nu}}}.$$

We shall recover that velocity $c'$ in a moment.

**4. The point-source theorem.** – Here, we shall naturally recall the idea of considering a moving body to be a source of advanced and retarded actions whose superposition constitutes an associated wave. However, we will find ourselves confronted with a new difficulty whose importance is undoubtedly quite great. In order to understand it, recall the argument in the last paragraph of the preceding chapter by which we showed that the permanence of the phase agreement between the point-source and its wave seemed to imply the validity of the relation **I** = $h$ **O** along the orbit. We shall assume that the associated wave is represented along its trajectory (as far as its phase is concerned) by sin $2\pi\nu\,[t - f\,(l)]$. On the other hand, for the observer that coupled to a constant electromagnetic field, the vibration of the point-source will have a variable frequency $\nu_1\,(t)$, and can be written:

$$\sin \int_0^t 2\pi \nu_1(t)\, dt.$$

Having said that, the persistence of the phase agreement will imply the relation:

$$\nu_1\,(t) = \nu \left(1 - \frac{df}{dt}\upsilon\right),$$

and as a result, we will have:

$$\nu_1 = \nu \left(1 - \frac{\upsilon}{V_r}\right)$$

at each instant, in which $V_r$ is the radial phase velocity at the point that is occupied by the moving body. If we now introduce the velocity $c'$ that was defined above then we will get:

$$\nu_1 = \nu \left(1 - \frac{\upsilon^2}{c'^2}\right).$$

The permanence of phase then demands that one must make the quantity $c'$ play the role of the velocity of perturbations at the point where the moving body is found, instead of the constant $c$.

An examination of circular motion will lead to an analogous conclusion. We must then recall the argument that was given for the case of gravitation. If the constant velocity of the moving body is $\upsilon$ then variation that the phase of the associated wave will experience when one varies time by $dt$ and one simultaneously displaces the point of observation through a length $\upsilon\, dt$ along the circle will be:

$$2\pi \nu_1\, dt.$$

If the hypothesis **I** = $h$ **O** is exact then that variation must be equal to:

$$2\pi \nu \left(dt - \frac{dl}{V_r}\right) = 2\pi \nu \left(1 - \frac{\upsilon}{V_r}\right) dt,$$

and one must once again have:

$$\nu_1 = \nu \left(1 - \frac{\upsilon}{V_r}\right) = \nu \left(1 - \frac{\upsilon^2}{c'^2}\right).$$

On first glance, the preceding results seem quite surprising. Since space-time was assumed to be Euclidian, the form itself of $ds^2$ invites one to take the constant $c$ to be the velocity of propagation of perturbations. One will not understand why one must substitute $c'$ for $c$, especially since the quantity $c'$ is a function of the nature of the moving body, or more exactly, its electromagnetic sensitivity.

Upon reflection, I believe that the significance of this difficulty must be very profound. Space-time is Euclidian in an electromagnetic field, and meanwhile the motion of a point-like charge is not described by a geodesic, but depends upon both its electromagnetic sensitivity and the world-potential. That is the essential difference that separates electromagnetic dynamics from gravitational dynamics, and I believe that it is basically that same difference that we must recover in a different form when we are obliged to replace $c$ by $c'$ in our formulas. Perhaps it is by reflecting upon that enigma that we will begin to understand the true nature of the electromagnetic field.

I shall add one last word for the sake of the following chapter. The preceding considerations and the analogy with the gravitational case make the following conclusion quite reasonable: In a central electromagnetic field, the wave that is associated with the motion of an electron on a circular trajectory of radius $R$ will be given by the function:

$$\varphi(\rho, \theta, z, t) = f(\rho, z, \theta, -\omega t)\, \sin 2\pi \nu \left(t - \frac{G}{W} R\theta - \chi\right),$$

in which $G$ is the quantity of motion of the moving body, $W$ is its total energy, and $\chi$ is a function of only $\rho$ and $z$.

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## CHAPTER IV

# STABILITY OF PERIODIC MOTIONS

1. Uniform, circular motion. – 2. General case of strictly-periodic motions. – 3. Quasi-periodic trajectories. – 4. Bohr's correspondence theorem.

**1. Uniform, circular motion. –** In order to better understand the meaning of the stability conditions that are introduced by the theory of quanta, we shall first imagine uniform, circular motions in a centrally-symmetric field. We have seen that one can then very probably represent the associated wave by the form:

$$\varphi(\rho, z, \theta, t) = f(\rho, z, \theta - \omega t) \sin 2\pi\nu \left[ t - \frac{G}{W} R\,\theta - \chi(\rho, z) \right].$$

However, this expression will represent a stable state only if the sine factor takes on the same value again when one increases $\theta$ by a whole number times $2\pi$, while the $t$, $z$, and $\rho$ remain fixed. In other words, it is necessary that the wave must be in resonance on all circles that are coaxial to the trajectory. That condition gives us:

$$2\pi\nu \frac{G}{W} 2\pi R = 2\pi \frac{G}{h} R\, 2\pi = n\, 2\pi \qquad (n \text{ integer})$$

or

$$G\, 2\pi R = nh.$$

One can say that the moment of the quantity of motion of the moving body with respect to the attractive center must be an integer multiple of $h / 2\pi$. That is in fact the form that the condition for the stability of the circular trajectories in the hydrogen atom took on for Bohr in his initial research.

**2. General case of strictly-periodic motions. –** Whenever the motion is strictly periodic, the orbit will be a closed curve. The postulates that were assumed in the last chapter show us that the phase of the associated wave will vary when one displaces it by the quantity:

$$2\pi \left[ \frac{W}{h} dt - \frac{G_l}{h} dl \right]$$

by $dl$ along the trajectory during a time $dt$, in which $W$ is the energy of motion and $G_l$ is the projection of the quantity of motion of a point $l$ onto the tangent to the trajectory. Therefore, in order for there to be resonance along the ray here, it is necessary that one must have:

$$\frac{2\pi}{h}\int_0 G_l\,dl = n\,2\pi \qquad (n \text{ integer})$$

or

$$\int_0 G_l\,dl = n\,h,$$

which can be further written:

$$\int_0 \sum_{i=1}^{3} p_i\,dx^i = n\,h$$

with the aid of the Lagrange momenta.

That is indeed the general condition for the stability of the closed trajectories that would result from developing the theory of quanta. Of course, it includes the formula that relates to circular trajectories as a special case. The success of our interpretation rests essentially upon the identification of the de Maupertuisian action integral, divided by $h$, with a phase difference.

We cannot prove here that the resonance is found to be realized in the entire inter-atomic domain, as we did in the case of circular trajectories; nevertheless, that situation is very probable ([4]).

**3. Quasi-periodic trajectories. –** One knows that the quantum conditions for stability have been extended from periodic motions to quasi-periodic motions. The work of a great number of authors of the first rank, among whom one must include Sommerfeld and Bohr, have permitted us to specify the general of the stability conditions and their number. The periodic motions then appear to be a degenerate case of the quasi-periodic motions. Conversely, in my doctoral thesis, I showed how the case of quasi-periodic motion can be reduced to the case of periodic motions by considering the time intervals to be infinite in number and mutually incommensurable, after which, the moving body will have returned to a distance from its starting point that is less than a very small quantity that has been chosen in advance. From that standpoint, the multiple conditions of Sommerfeld-Bohr are consequences of the single condition by which one expresses the stability of closed orbits. I shall not repeat the proof that was given in my thesis here, because it demands that some precautions must be taken in regard to the language, and as a result, it is very long. Furthermore, one can ameliorate that by

([4]) Much has been said in recent times about the semi-quanta that are introduced into the theory of the Zeeman Effect and into the theory of rotational spectra. It seems that in certain cases the condition for the stability of a periodic motion must be written as:

$$\int_0 \sum_{i=1}^{3} p_i\,dq^i = \frac{2n\pm1}{2}h \qquad (n \text{ integer}).$$

In our way of thinking, the semi-quanta can be interpreted if, in certain situations, one must add a supplementary difference of $\pm\pi$ that is analogous to the one that is produced when one passes a focus in Gouy's theory to the phase difference that is expressed by the de Maupertuis integral. One will then get the resonance condition:

$$\frac{2\pi}{h}\int_0 \sum_{i=1}^{3} p_i\,dq^i \pm\pi = n\,2\pi,$$

from which, one will infer the formula for semi-quanta immediately.

introducing the notion of angular variables and showing how the degree of degeneracy of the motion limits the number of independent conditions.

None of that will introduce any essential difficulties, but it would demand some digressions that would go beyond the scope of this book. We confine ourselves to stating the following conclusion:

*The interpretation of the quantum condition for the stability of the closed trajectories as something that expresses the resonance of the wave that is associated with the moving body implies the existence of the Sommerfeld-Bohr multiple conditions for the stability of quasi-periodic motions.*

**4. Bohr's correspondence theorem.** – Bohr's correspondence principle was suggested to its author by the proof of an important theorem on the frequencies that were emitted by his model of the atom. I would like to consider the simple case of circular trajectories and show how one can interpret Bohr's theorem from our viewpoint.

Let $C$ and $C'$ be two circular trajectories in the atom that are stable in the sense of the theory of quanta and correspond to the whole numbers $n$ and $n'$. From the law of emission, when an electron passes from the stable trajectory $C'$ of energy $W'$ to the stable trajectory $C$ of energy $W < W'$, a quantum:

$$h\nu = \partial W = W' - W$$

will be radiated.

Imagine two electrons $E$ and $E'$, one of which describes a circle $C$ of length $L = 2\pi R$, while the other describes a circle $C'$ of length $L' = 2\pi R$. If $G$ and $G'$ denote the quantities of motion then one must have:

$$GL = nh, \qquad G'L' = n'h.$$

On a circle $\gamma$ of radius $\rho$ that is concentric to the preceding ones, the distribution of phases that are associated with $E$ is represented by the function:

$$\sin 2\pi\left[\frac{W}{h}t - \frac{G}{h}\frac{R}{\rho}t\right],$$

and the distribution of phases that are associated with $E'$ is given by the function:

$$\sin 2\pi\left[\frac{W'}{h}t - \frac{G'}{h}\frac{R'}{\rho}t\right].$$

Upon adding these phase factors, we will obtain a "beat":

$$2\cos 2\pi\left[\frac{\partial W}{2h}t - \frac{l}{2h\rho}(G'R' - GR)\right]\ \sin 2\pi\left[\left(W + \frac{\partial W}{2}\right)t - \frac{l}{2h\rho}(G'R' + GR)\right].$$

Examine the amplitude factor; upon setting:

$$n' - n = \partial n \qquad \text{and } 2\pi\rho = L,$$

it can be written:

$$\cos 2\pi \left[ \frac{\partial W}{2h} t - \frac{l}{2L} \partial n \right].$$

Along the circle $\gamma$, there are $\partial n$ maxima that displace with the velocity $\frac{L}{h}\frac{\partial W}{\partial n}$. The frequency of rotation $\omega$ of one of the maxima of the beat along $\gamma$ will then be:

$$\omega = \frac{\partial W}{h\,\partial n} = \frac{\nu}{\partial n}.$$

Hence, one has the following theorem:

*The superposition of the phases that are associated with the motions of E and E′ gives rise to a beat that turns with a frequency $\omega$ on all of the circles that are concentric to C and C′ in such a way that the frequency of the radiation that is emitted when an electron passes from C′ to C will be equal to the harmonic of $\omega$ of order $\partial n$.*

In order to recover Bohr's theorem, it will suffice to apply the preceding result to the circle $C'$ upon supposing that $n$ and $n'$ are very large and that $\partial n$ is small with respect to them. Under those conditions, the frequencies $W / h$ and $W' / h$ will be very close, and by reason of the general significance of Rayleigh's formula, the beat on $C'$ will possess the same velocity as the electron $E'$. It will then have the same frequency $\omega$ of mechanical rotation as the beat, and as a result, one will get the following statement by Bohr in this limiting case:

*The frequency of radiation when one passes from C to C′ is the harmonic of order $\partial n$ of the frequency of mechanical rotation of the electron on C′.*

On first glance, it seems that one has thus encountered a physical interpretation of the role of harmonics in Bohr's correspondence theory, but there is an objection: In general, the circles $C$ and $C'$ are not occupied simultaneously. Suppose that $C'$ is occupied, but $C$ is not. What does it mean to speak of the phase that is associated with an electron $E$ that does not exist? Perhaps one might answer this with: The electron $E$ does not exist at the instant considered, but it has existed or it will exist in the sequence of successive states of the atom, and by the effects of advanced and retarded actions, it will suffice for one to continually have a phase of frequency $W / h$ along the entire circle $\gamma$.

Briefly, the preceding argument will probably open up a small fissure through which one can penetrate the mysteries of the intra-atomic region.

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## CHAPTER V

# THE DYNAMICS OF SYSTEMS OF MATERIAL POINTS

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**1. The energy of systems of material points. –** Up to now, we have envisioned only constant fields. In fact, a field is always produced by the action of a certain number of centers that are themselves material points and are subject to the reaction of the attracting point. It is then only in the case where the attractive (or repulsive) centers have an extremely large mass that one can consider them to be fixed and determine a constant field. In all other cases, one will always be dealing with a set of material points that are all in motion under the influence of their mutual actions.

Therefore, let a certain number of material points be given that will be affected with the indices *a*, *b*, *c*, etc. Each of them is in motion under the influence of gravitational or electromagnetic forces that emanate from the others. Those forces obviously depend upon the spatial coordinates only by way of combinations that express the mutual distances between the various points. I shall say that the motion of one of the moving bodies – the moving body *a*, for example – is given by the following principle:

*If $p_{ia}$ denotes the Lagrange momentum of the moving body a with respect to the spatial coordinate $x^i$, and if $W_a$ is its energy (i.e., the temporal component of its world-impulse), expressed as a function of the coordinates of the moving body and time, then the world-line of the moving body a that passes through the points P and Q of space-time is such that the integral $\int_P^Q \left( W_a dt - \sum_{i=1}^{3} p_{ia}\, dx_a^i \right)$ is stationary.*

$W_a$ depends upon time, since it depends upon the coordinates of the other material points, which are themselves in motion.

In other form, one can say: The motion of the moving body is determined by Hamilton's principle, for which the Lagrangian function includes time and is equal to:

$$L_a = W_a - \sum_{i=1}^{3} p_{ia}\, dx_a^i \,.$$

Hamilton's principle leads to the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L_a}{\partial \dot{x}_a^i}\right) = \frac{\partial L_a}{\partial x_a^i} \qquad \left(\dot{x}_a^i = \frac{dx_a^i}{dt}\right)$$

by a known method.

For the other points, one will obtain some analogous equations in which $L_a$ is replaced by $L_b$, $L_c$, …, in turn.

The motion of each material point then obeys Hamilton's principle when one supposes that the motion of all the other points is known. The problem of the dynamics of systems consists of finding the motions that all of the points of the system must possess in order to make the least-action principle applicable to each of them. One will solve that problem by looking for the solutions of the Lagrange equations, when they are considered to be a system of simultaneous differential equations.

Classical mechanics then poses the following problem: Can one choose a function $L$ of the coordinates of all points of the systems and their velocities such that the Hamilton integral $\int_{t_0}^{t_1} L\,dt$ is stationary for the collective motion of the system?

Here is the manner by which that problem is solved by the classical methods: The function $L$ must be such that:

$$\frac{d}{dt}\left(\frac{\partial L_a}{\partial \dot{x}_a^i}\right) = \frac{\partial L_a}{\partial x_a^i}$$

can be written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_a^i}\right) = \frac{\partial L}{\partial x_a^i}.$$

Terms enter into the function $L_a$ that depend upon only the velocity, along with terms that also depend upon the coordinates by way of the intermediary of the distances from the point $a$ to the other points $b$, $c$, … Let $L_a'$ and $L_a''$ be those two parts of $L$.

Furthermore, the principle of action and reaction teaches us that the term in $L_a''$ that involves the distance from $a$ to $b$, should be recovered in $L_b''$. On the contrary, $L_a'$ and $L_b'$ have no term in common. The sum $\sum_{a,b,\dots} L_a$ will then contain the terms that depend upon only the velocities just one time, but the terms that depend upon the distances twice. That sum of the individual Lagrange functions cannot be taken to be a global Lagrange function $L$, because if one has:

$$\frac{\partial L_a'}{\partial \dot{x}_a^i} = \frac{\partial \sum L_a'}{\partial \dot{x}_a^i}$$

then one will find, by contrast:

$$\frac{\partial L_a''}{\partial \dot{x}_a^i} = \frac{1}{2}\frac{\partial \sum L_a''}{\partial \dot{x}_a^i}, \qquad \frac{\partial L_a'}{\partial x_a^i} = \frac{1}{2}\frac{\partial L_a''}{\partial x_a^i}.$$

In order to preserve the form of the Lagrange equation, one must obviously set:

$$L = \sum L_a' + \tfrac{1}{2}\sum L_a''.$$

Thanks to this choice of function $L$, one will easily find that the equations of motion admit the first integral:

$$H = L - \sum_{a,b,c,} \sum_{i=1,2,3} \dot{x}_a^i \frac{\partial L_a}{\partial \dot{x}_a^i}.$$

The constant quantity $H$ is the energy of the system, and the least-action principle will become:

$$\delta \int_{t_0, x_0^i}^{t_0, x_1^i} \left( H\, dt + \sum_{a,b,c,} \sum_{i=1,2,3} \frac{\partial L_a}{\partial \dot{x}_a^i} d\dot{x}_a^i \right) = 0.$$

The $\partial L / \partial \dot{x}_a^i$ plays the role of impulse, and $H$ plays that of energy in the individual Hamilton principles. However (and this is essential!), *$H$ is not* the sum of the individual energies, because that function contains the mutual interaction terms only once. It is, perhaps, even regrettable that one employs the same word to refer to the function $W$ and the function $H$.

That is the classical viewpoint. How does one accommodate the relativity principle? There is no problem as far as the motion of a system of point-like charges in an electromagnetic field is concerned: Here the individual Lagrange functions easily divided into two parts, one of which depends upon only the velocities (viz., properly-mechanical terms) and the other of which contains the mutual distances (viz., electromagnetic terms that depend upon the scalar vector potentials). One will easily define the function $L$ and the global energy $H$ by conforming to the usual notions.

The case of the motion of a system of points that are subject to their mutual gravitational action is much more delicate. As we have seen, the Lagrange function for the point $a$ will then be:

$$L_a = m_0\, c\, \gamma_a \sqrt{1 - \beta^2} \qquad \left( \beta_a = \frac{v_a}{\gamma_a} \right).$$

$\gamma_a$ is the speed of light at the point that is occupied by the moving body $a$, and is expressed as a function of the potential $(g_{44})_a$ at that point by the relation:

$$\gamma_a = c \sqrt{(g_{44})_a}\,.$$

Let $\pi_a$ denote the gravitational potential (in the usual sense) to which $a$ is subjected. One will have:

$$(g_{44})_a = 1 + \frac{2\pi_a}{c^2} \qquad \text{(see Chap. II)}.$$

If $\pi_a / c^2$ is very small then the series development in $\sqrt{(g_{44})_a}$ will be $1 + \pi_a / c^2$, and the function $L_a$ will decompose into $L'_a$ and $L''_a$. However, if that development is not legitimate then one cannot see how to define $L$ and $H$.

At the end of the tale, it then seems proper that the most general method for the solution of the problems in the dynamics of systems is to apply Hamilton's principle to each point, as we explained at the beginning of this paragraph.

**2. Uniform, circular motion of two interacting material points.** – We shall begin with a deeper study of a very simple case: viz., the case of two interacting material centers – for example, two point-like charges with opposite signs that describe circular orbits around their common center of gravity. That problem has great historical importance, because it led Bohr to make some predictions that were plainly confirmed by comparing with the spectra of hydrogen and ionized helium. In that particular case, we can study the motion of two moving bodies completely, along with the propagation of the associated waves, which will permit us to then enter into the general case.

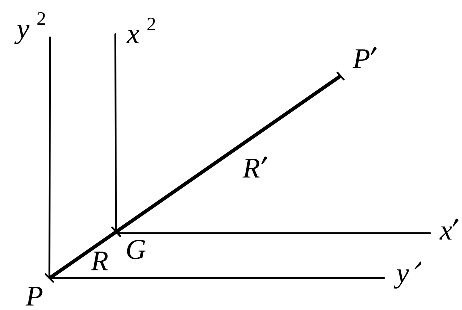


Figure 5.

Therefore, let $P$ and $P'$ be two moving bodies of masses $M_0$ and $M_0'$, resp., between which are exerted, for example, electromagnetic actions. One will easily see that the equations of motion are verified if the points $P$ and $P'$ rotate with a collective motion that has a suitable angular velocity $\omega$ around a point $G$ on the line $PP'$. The point $G$ determines two segments $\overline{MG} = R$ and $\overline{M'G} = R'$ such that:

$$\frac{M}{R'} = \frac{M'}{R} = \frac{M+M'}{R+R'},$$

in which $M$ and $M'$ are the masses of the moving bodies here, namely:

$$M = \frac{M_0}{\sqrt{1-\beta^2}}, \qquad M' = \frac{M_0'}{\sqrt{1-\beta'^2}},$$

in which $\beta$ and $\beta'$ are constants that are equal to $\omega R / c$ and $\omega R' / c$, resp. (see Fig. 5). Consider the system of axes $x^1$, $x^2$ that is linked with the point $G$: It is a Galilean system. On the contrary, a system of parallel axes that is linked with $P$ is not Galilean. If $y^1$, $y^2$, $y^3$ are the rectangular coordinates of the latter system then they can be deduced from the $x^i$ by the obvious formulas:

$$y^1 = x^1 + R\cos\omega t, \qquad y^2 = x^2 + R\sin\omega t, \qquad y^3 = x^3, \qquad y^4 = x^4 = c\,t.$$

One will then deduce that:

$$\begin{aligned} ds^2 &= (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \\ &= \left(1 - \frac{\omega^2 R^2}{c^2}\right)(dy^4)^2 - (dy^1)^2 - (dy^2)^2 - (dy^3)^2 - 2\frac{\omega R}{c}\sin\omega t\, dy^1\, dy^4 \\ &\qquad + 2\frac{\omega R}{c}\cos\omega t\, dy^2\, dy^4. \end{aligned}$$

In the $y^i$ system of axes, the electromagnetic field will be constant, and the trajectory of $P$ will be a circle of center $P'$ and radius $R + R'$. That will conform to the results that were obtained for the constant electrostatic field if one supposes that the circular trajectory is a ray of the wave that is associated with $P'$, and that if one displaces along that ray at constant time then the phase will vary by $p_1\, dx^1 + p_2\, dx^2$.

The components $p_i$ of the quantity of motion are given by the general formulas of relativity:

$$p_i = M_0'\, c\, u_i + e\,\varphi_i = M_0'\, c\, g_{ij}\, u^j + e\,\varphi_i \qquad \left(u^i = \frac{dx^i}{ds}\right).$$

Thanks to those formulas and the obvious relations:

$$v^1 = \frac{dy^1}{dt} = -\,\omega(R + R')\sin\omega t, \qquad v^2 = \frac{dy^2}{dt} = \omega(R + R')\cos\omega t,$$

one will get:

$$p_1 = \frac{M_0'}{\sqrt{1-\beta'^2}}\frac{dy^1}{dt}\frac{R'}{R+R'}, \qquad p_2 = \frac{M_0'}{\sqrt{1-\beta'^2}}\frac{dy^2}{dt}\frac{R'}{R+R'},$$

$$= \frac{M\,M'}{M+M'}\frac{dy^1}{dt}, \qquad p_2 = \frac{M\,M'}{M+M'}\frac{dy^2}{dt}$$

by some easy calculations.

Everything happens as if the mass of the moving body $P'$ were equal to $\frac{M\,M'}{M+M'}$, instead of $M'$. That is a classical result of rational mechanics, but here it has been generalized somewhat by the substitution of the moving masses for the proper masses. Those values $p_1$ and $p_2$ permit one to calculate the small variations of the Rydberg constant that were predicted by Bohr and confirmed quite well by observation.

Nothing in what was just said will prevent us from inverting the roles of the material points $P$ and $P'$. The resonance of the wave that is associated with $P'$ will be expressed by the relation:

$$\frac{M\,M'}{M+M'}\sin 2\pi\,\omega(R + R')^2 = nh$$

in the system that is linked to $P$. The resonance of the wave that is associated with $P$ will be expressed in the same way in the system that is linked with $P'$, by reason of the symmetry in the formula that was obtained. The Bohr condition then expresses the resonances of both the waves associated with $P$ and $P'$.

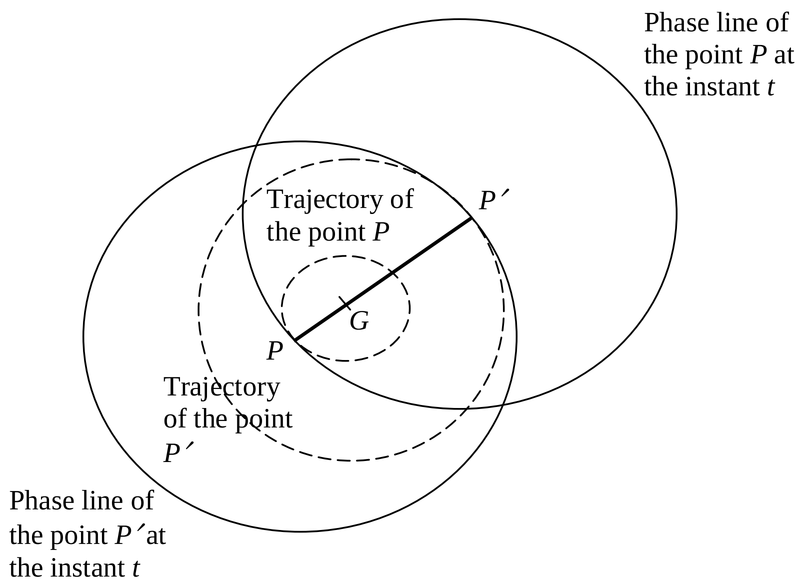


Figure 6.

It is instructive to trace out the rays at the instant $t$ of the two associated waves that pass through $P$ and $P'$ in the system of axes that is linked with the center of gravity, along with the trajectories that are described by the two moving bodies in the course of time (Fig. 6). One will then arrive at a representation of how each moving body describes its trajectory with a velocity that is tangent at each instant to the ray of the associated wave if one continues to call the envelopes of the phase velocities at a given instant "rays." However, it seems preferable to reserve the word "ray" for the trajectories of the energy and to give the name of "phase lines" to the envelopes of the phase velocities. We then perceive a fundamental fact that is true for not just the moving body itself, but all parts of its wave: The rays do not coincide with phase lines that displace in space. That recalls a known conclusion in hydrodynamics by which the streamlines, which are envelopes of the velocities, are the trajectories of the fluid particles only when their form is invariable – in other words, if the motion is permanent.

**3. Associated waves in the dynamics of systems.** – We just gave an account of the nature of associated waves for a very special case of a system of two material points. We need to generalize our results, but unfortunately that undertaking presents some grave difficulties. In the general case where more than two material points are present, it is not possible to determine a reference system in which one of the moving bodies displaces in a constant field. The gimmick that was employed in paragraph **2** will break down, and we will no longer see any way to determine the phase lines. Obviously, one can appeal to the point-source theorem for that determination, but its proof and application will again be more difficult than it was in the case of constant fields. If the ideas that are presented in this book are correct then it will be necessary to answer those questions if one is to know how to apply the quantum conditions in the general $n$-body problem. As one knows, that point is very important for the progress and experimental verification of quantum theory, especially since the methods that are based upon perturbation theory do not seem to lead to exact results ([5]).

Nothing further can be said about those questions at the moment, so I shall confine myself to making a few general remarks whose correctness does not seem to be in doubt.

First of all, the phase lines of the waves that are associated with the various material points generally move in space and do not coincide with the rays. In the case of circular motion that was just studied, the trajectory is, at any instant, tangent to the phase line that corresponds to the point at which the moving body is found. That is a general fact, as an examination of the elliptic solutions of the two-body problem has already shown. Moreover, the energy of each center will not remain constant in the course of motion, in general, and one can no longer attribute a well-defined value to the frequency $\nu$ of the associated wave. Meanwhile, one thing seems certain: At a point where one of the moving bodies is found, the propagation of the phase that is associated with that location along its trajectory will conform to the law **I** = $h$ **O**.

Of course, that reduction is valid here only at the instant when the moving body passes the point considered. We have seen that the world-line of a moving body that passes through two point $P$ and $Q$ in space-time is *always* determined by Hamilton's principle:

$$\delta\int_P^Q (W\,dt - G_l\,dl) = 0,$$

in which $W$ is the (variable) energy and $G_l$ is the projection of the quantity of motion onto the tangent to the trajectory. Therefore, the ray of the associated wave on which the material point moves will be determined by the relation:

$$\delta\int_P^Q \left(\nu\,dt - \frac{\nu}{V}\,dl\right) = 0,$$

in which $V_l$ is the radial phase velocity. The preceding equation expresses the idea that the variation of the phase along the world-ray is a minimum; that is a generalization of Fermat's principle to the case of variable frequencies. We saw that de Maupertuis's principle coincided with the usual Fermat principle in the case of constant fields in the same way that Hamilton's principle coincides here with the generalized form of that great law of optics.

If one supposes that the points $P$ and $Q$ are extremely close, and if one varies $\upsilon$ without deforming the element $PQ$ then one will recover the Rayleigh relation $\frac{dl}{dt} = \frac{\partial \nu}{\partial\left(\frac{\nu}{V_l}\right)}$.

([5]) Regarding that point, see the last part of the book by Born that was cited before.

In summary, the determination of the associated waves in the general case of the *n*-body problem can be very difficult in practice, but it does not seem to encounter any objections, in principle.

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# PART TWO

## THE OPTICS OF QUANTA

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### CHAPTER VI

## THE ATOM OF RADIATION

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1. Hypothesis of a quantum of light. – 2. The quantum of light and its rectilinear propagation. – 3. Doppler effects. – 4. Radiation pressure.

**1. Hypothesis of a quantum of light.** – The idea that light is composed of corpuscles that displace with a large velocity is quite old; notably, one can find it in *De natura rerum* by Lucrèce. It has Newton to bolster its authority, and one cannot deny that it expresses the phenomena of rectilinear propagation and reflection with a marvelous ease. Similarly, it interprets the phenomenon of radiation pressure, which was discovered around thirty years ago, in a striking manner. Nonetheless, that theory was not in favor throughout the Nineteenth Century. Indeed, Fresnel and his disciples showed that if one attributes the nature of a vibration – whether elastic or electromagnetic – to light then one will succeed in explaining the phenomena of rectilinear propagation, reflection, and radiation pressure, as well as the entire set of phenomena of diffraction, interference, diffusion, dispersion, and crystalline optics, which is where the corpuscular theory had run aground. Furthermore, in the year 1900, the physicists declared that there could no longer be any doubt as to the wave-like nature of light radiation. They undoubtedly had their reasons for that, but in their minds, the correctness of wave theory would have to imply the falsity of the corpuscular theory, and the latter conclusion seems much more doubtful today.

At the beginning of our century, the development of the theory of quanta, which was necessitated by the interpretation of the laws of blackbody radiation and study of photoelectric phenomena, redirected attention to the corpuscular theory, and in 1905, Einstein proposed that light energy should be considered to be grains of value $h\nu$ that would be proportional to their frequency. He deduced his photoelectric equation from that, which was first verified in the light domain, properly speaking, and was subsequently confirmed more precisely in the domains of X-rays and $\gamma$ rays. Einstein also showed that fluctuations of energy in blackbody radiation involved the quantum $h\nu$, and Bohr (1912) made us aware that atoms always absorb and emit quanta. Nevertheless, the new corpuscular theory encountered strong resistance on the part of many physicists because, just like the old theory, it seemed incapable of accounting for the phenomena of wave optics, which are quite numerous and quite precise.

The discovery by A. H. Compton of the change of wave length by diffusion, which is a phenomenon that was immediately contested, but which is now, I believe, beyond question, and the quantitative interpretation of the Compton Effect by the simultaneous theories of Debye and Compton himself seemed to impose that one should adopt the hypothesis of quanta of light without it also being possible to abandon the fundamental idea of oscillations. Upon assuming, in line with the suggestions of the present book, that radiant energy, like all other forms of matter, possesses an atomic structure and that the motion of any unit of energy is associated with the propagation of a wave, we will foresee the possibility of that synthesis becoming necessary. We must now examine the extent to which that synthesis is currently realizable.

**2. The quantum of light and its rectilinear propagation. –** In Chapter I, we were led to associate the uniform, rectilinear motion with a velocity $\upsilon = \beta c$ of a moving body of mass $m_0$ with the propagation of a wave in the same direction whose amplitude possesses the velocity $\upsilon$ of the moving body and whose phase possesses the velocity $V = c^2 / \upsilon$.

Having recalled that, suppose that our moving body has an extraordinarily small proper mass. Since its internal energy is very weak, it will possess an appreciable total energy that is due to its motion entirely only if its velocity is very close to the value $c$.

That will become obvious if one remembers the formula that gives the total energy:

$$W = \frac{m_0 c^2}{\sqrt{1-\beta^2}} .$$

It will then seem quite natural to compare the quantum of light to a moving body of very small mass $m_0$. Obviously, the velocity of that moving body will be a function of its energy, but if $m_0$ is very small then its energy will have a detectable value only if its velocity is indiscernible from the velocity $c$. On the other hand, as we have seen many times, the wave is associated with a frequency $h\nu$ such that:

$$W = h\nu,$$

and we will then obtain a satisfactory representation of the quantum of light. We further add that the quantity of motion:

$$G = \frac{m_0 \upsilon}{\sqrt{1-\beta^2}} = h\nu \frac{\upsilon}{c^2}$$

will be very close to $h\nu / c$, and that is, in fact, the value that is attributed to it by Einstein's theory, as well as the theory of the Compton effect. Meanwhile, an objection persists that was first communicated to me orally by Langevin, and was then discussed in a series of notes by W. Anderson (Phil. Mag., May 1924): The velocity $\beta c$ of the quantum is equal to $\sqrt{1-\frac{m_0^2\,c^4}{h^2\nu^2}}$, and since one supposes that $m_0$ is small, it will have a frequency, as above, that will make propagation impossible (viz., $\beta$ will be imaginary). In a different form, one can say that upon diminishing the frequency $\nu$ sufficiently, one must conclude by discarding the predictions of the classical theory ($\upsilon = c$) and also in the domain in which one must expect to find those predictions best realized (viz., the correspondence principle). That objection is certainly very embarrassing, although it is not, perhaps, definitive.

One can obviously escape it by taking a radical step and setting:

$$m_0 = 0, \qquad \beta = 1,$$

but one must then imagine a very delicate passage to the limit such that one simultaneously makes $m_0$ tend to 0 and $\beta$ tend to 1, while the quotient $\frac{m_0\,c^2}{\sqrt{1-\beta^2}}$ keeps the value $h\nu$.

I shall leave open the delicate question of the proper mass of the quantum of light, and I shall content myself by saying: The dynamical properties of the atom of radiation are deduced from the properties of the material point of finite mass by making the proper mass tend to zero.

The two velocities $\upsilon$ and $V$ are then equal to $c$, so the wave and the moving body will be transported in unison.

**3. Doppler effects.** – We have already pointed out that the phenomenon of reflection can be interpreted immediately under the hypothesis of light quanta. We shall show that the same thing is true for the two main types of Doppler Effect.

*a. Doppler Effect due to the relative motion of the source and the observer.* – Consider a source of light in motion in the direction of an observer that is supposed to be fixed. In a reference system that is coupled with the source, the atoms of emitted radiation have an energy $W = h\nu$ and a quantity of motion $G = h\nu / c$. The transformation formulas for a quadri-vector teach us that the energy $W$ of a corpuscle is:

$$W = \frac{W + \upsilon G}{\sqrt{1-\beta^2}}$$

for the observer, and since $W' = h\nu'$, where $\nu'$ is the observed frequency, one will have:

$$\nu' = \nu\,\frac{1+\beta}{\sqrt{1-\beta^2}} = \nu\sqrt{\frac{1+\beta}{1-\beta}}\,.$$

It is also quite simple to find the ratio of the emitted intensities in the two reference systems. In that of the source, a number $n$ of atoms of light will be emitted per unit time per unit area. The density of the sheaf will then be $nh\nu / c$, and its intensity will be $I = nh\nu$.

On the contrary, for the observer, those $n$ atoms are emitted during a time that is equal to $1/\sqrt{1-\beta^2}$, and they fill up a volume $c\dfrac{1-\beta}{\sqrt{1-\beta^2}}$. For him, the density of the sheaf will then be $\dfrac{nh\nu'}{c}\sqrt{\dfrac{1+\beta}{1-\beta}}$, and the intensity will be $I' = nh\nu'\sqrt{\dfrac{1+\beta}{1-\beta}}$.

Hence:

$$\frac{I'}{I} = \frac{\nu'}{\nu}\sqrt{\frac{1+\beta}{1-\beta}} = \left(\frac{\nu'}{\nu}\right)^2 .$$

We have recovered the well-known formulas, and the case in which the source displaces in an arbitrary direction with respect to the observer is also dealt with quite simply.

*b. Reflection by a moving mirror.* – Once more, consider the reflection of light corpuscles by a perfectly-reflecting planar mirror that displaces with the velocity $\beta c$ normally to the surface.

For an observer that is coupled with the mirror, it is fixed, and the light corpuscles have the same energy $W$ and the same frequency before and after reflection.

For the fixed observer, the incident corpuscles have energy $W_1'$ and frequency $\nu_1'$, while the reflected corpuscles have energy $W_2'$ and frequency $\nu_2'$. The tensor transformation formulas will give:

$$W = \frac{W_1' + \upsilon G_1'}{\sqrt{1-\beta^2}}, \qquad W = \frac{W_2' - \upsilon G_2'}{\sqrt{1-\beta^2}},$$

so

$$h\nu_1'\,(1+\beta) = h\nu_2'\,(1-\beta),$$

$$\frac{\nu'}{\nu} = \frac{1+\beta}{1-\beta}.$$

If $\beta$ is small then one will revert to the classical formula:

$$\frac{T_2}{T_1} = 1 - 2\frac{\upsilon}{c}.$$

Let $n$ denote the number of corpuscles that are reflected during a given time interval.

The total energy of the $n$ corpuscles after reflection $E_2'$ has a ratio of $\frac{nh\nu_2'}{nh\nu_1} = \frac{\nu_2'}{\nu_1}$ with their total energy before reflection $E_1'$. That formula can also be obtained by the old theories, but here it is entirely self-evident.

If the $n$ corpuscles occupy a volume $V_1$ before reflection then they will occupy a volume $V_2 = V_1\frac{1-\beta}{1+\beta}$ after reflection, as a simple argument will show. The intensities $I_1'$ and $I_2'$ before and after reflection, resp., will then have the ratio:

$$\frac{I_2'}{I_1'} = \frac{nh\nu_2'}{nh\nu_1'}\frac{1-\beta}{1+\beta} = \left(\frac{\nu_2'}{\nu_1'}\right)^2.$$

One can also treat the case of oblique incidence quite easily.

**4. Radiation pressure.** – The hypothesis of light corpuscles gives a correct explanation of radiation pressure, and it is even probable that if Newton had known of that phenomenon then he would have considered it to be a particularly striking proof of the correctness of his concepts.

Consider a cylindrical sheaf of light of frequency $\nu$ that falls upon a plane mirror with an angle of incidence $\theta$. Let $\rho = nh\nu$ be the energy density of that sheaf, and let $S$ be the area of the mirror that is illuminated. During the time $dt$, the mirror will receive $n \times S \cos\theta \times c\, dt$ particles and reflect them. Before and after reflection, each quantum will possess the quantity of motion $h\nu / c$, but although the rebound from the mirror will not modify the tangential component of that quantity of motion, it will reverse the sense of the normal component. The impulse that is exerted upon the mirror by the reflection of the $n$ corpuscles is then:

$$2n\, S \cos\theta\, c\, dt\, \frac{h\nu}{c} \cos\theta = F\, dt,$$

in which $F$ is the total force that is applied to the surface $S$ of the mirror. Hence:

$$F = 2\rho\, S \cos^2\theta.$$

The pressure $p$, or force per unit area, is found to be:

$$p = 2\rho \cos^2\theta.$$

If one now changes the notation and lets $\rho$ denote the energy density in the region in which the incident and reflected sheaves are superimposed then the new $\rho$ will be equal to twice the old one, and one will have:

$$p = \rho \cos^2\theta.$$

Instead of one sheaf, consider an infinitude of sheaves of the same density such that all of the angles of incidence are represented equally, so one will then have:

$$p = \overline{\rho \cos^2 \theta} = \tfrac{1}{3}\rho ,$$

in which $\rho$ is the density of radiation in the vicinity of the mirror. The formula thus-obtained is exact, not only for monochromatic, isotropic radiation, but also quite clearly for arbitrary polychromatic radiation (for example, blackbody radiation). The formulas that we will then obtain are indeed those of the electromagnetic theory.

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## CHAPTER VII

# THE ASSOCIATED WAVE FOR THE ATOM OF RADIATION

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1. Return to some results of Chapter I. – 2. The associated wave for the atom of radiation.

**1. Return to some results of Chapter I.** – In electromagnetic theory, one shows that the quantities that are called the "scalar potential" and the "vector potential," from which, the fields are deduced by derivation, propagate in space while obeying the equations:

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\rho,$$

$$\Delta\mathbf{a} - \frac{1}{c^2}\frac{\partial^2\mathbf{a}}{\partial t^2} = -\rho\,\frac{\mathbf{v}}{c}.$$

As before, $\psi$ and **a** continue to denote the two potentials, while $\rho$ is the density of electricity at a point in space, and **v** is its velocity at the same point.

Kirchhoff showed that the equations that were written above admit the solutions:

$$\psi_R = \iiint \frac{1}{4\pi r}[\rho]_{t-r/c}\,dv,$$

$$\mathbf{a}_R = \iiint \frac{1}{4\pi c r}[\rho\mathbf{v}]_{t-r/c}\,dv,$$

in which $r$ denotes the distance from the point at which one calculates the potential to the element $dv$. The quantities in brackets denote the values of $\rho$ and $\rho\,\mathbf{v}$ in the element $dv$ at the instant $t - r / c$. That instant is the one at which the action was emitted that is due to the volume element and is felt by the point of the potential at time $t$. The potentials thus-determined are known by the names of "retarded potentials."

It is easy to obtain another solution. Indeed, it suffices to remark that the equations of propagation contain the square of the velocity $c$ and that if one has obtained a solution that contains $c$ then one can obtain another one by changing $c$ into $-c$. One can then describe the new solutions as:

$$\psi_A = \iiint \frac{1}{4\pi r}[\rho]_{t+r/c}\,dv,$$

$$\mathbf{a}_A = \iiint \frac{1}{4\pi c r}[\rho\mathbf{v}]_{t+r/c}\,dv,$$

These are the "advanced potentials." The instant $t + r / v$ is the one at which the action that is felt by the potential point at time $t$ will be absorbed by the element $dv$. Classical theory neglects the advanced solution in order to keep only the retarded solution. For the moment, assume that this is wrong, and look for the form that the two solutions must combine into. In order to do that, imagine an electron of charge $e$ at rest. From the definition itself of the quantity of electricity and upon remarking that the units that are employed here are those of Lorentz-Heaviside, we will be obliged to attribute the value:

$$\psi = \frac{e}{4\pi r}$$

to the scalar potential at the distance $r$ from the charge.

Since the charge of the electron is assumed to be constant, one will immediately obtain that result upon considering only the retarded potential and setting $\psi = \psi_R$ ; that is what the classical theory does. If one would like to take the advanced potential into account then one will have:

$$\psi = \alpha\, \psi_A + \beta\, \psi_R ,$$

in which $\alpha$ and $\beta$ are constants. However, the symmetry of phenomena in time, or – if one prefers – the equivalence of the past and the future, seems to impose the condition $\alpha = \beta$ upon us. Now, the formulas that were given above show that one has $\psi_A = \psi_B = e / 4\pi r$ in all space. One must then set:

$$\psi = \tfrac{1}{2} [\psi_A + \psi_B].$$

Therefore, if electromagnetic theory wished to take the advanced potential into account then it would seem necessary for one to choose the solution of the wave equation to be one-half the sum of the advanced and retarded solutions.

Electromagnetic theory imposes the conservation of electricity, which is a fact that experiment always confirms. Meanwhile, nothing would prevent us from imagining functions $\psi$ and **a** that obey the equations of propagation for the potentials without the entity such that $\rho$ represents its density and **v** represents its velocity being constrained to be conserved. Indeed, the conservation of electricity results, not from the equations of propagation of the potentials, but from Maxwell's equations; i.e., from certain relations between the derivatives of $\psi$ and **a**.

Logically, nothing prevents us from supposing that those Maxwell equations are not satisfied and that the charges, which are sources of the potentials, have oscillating values, except that the functions $\psi$, **a**, $e$, and $\rho$ would not longer represent the usual electric quantities.

Hence, consider a charge with oscillating value:

$$e = e_0 \sin 2\pi \nu_0 (t_0 - \tau_0),$$

in which $t_0$ is the time in the system that is coupled to the charge. In that system, the scalar potential at the distance $r_0$ from the charge will be:

$$\psi_0 = \tfrac{1}{2}[\psi_A + \psi_B] = \frac{1}{8\pi r_0}\left([e]_{t_0+r_0/c} + [e]_{t_0-r_0/c}\right)$$

$$= \frac{e_0}{8\pi r_0}\left[\sin 2\pi\nu_0\left(t_0 - \frac{r_0}{c} - \tau_0\right) + \sin 2\pi\nu_0\left(t_0 + \frac{r_0}{c} - \tau_0\right)\right],$$

when one takes the advanced solution into account. Naturally, the vector potential is zero.

In paragraph **5** of Chapter I, we found that the periodic phenomenon that is associated with a material point with spherical symmetry can be represented in the "proper" system by the function:

$$\varphi(r_0, t_0) = \frac{A}{2r_0}\left\{\cos\left[2\pi\nu_0\left(t_0 - \frac{r_0}{c}\right) + c_1\right] - \cos\left[2\pi\nu_0\left(t_0 + \frac{r_0}{c}\right) + c_2\right]\right\}.$$

The two expressions for $\psi$ and $\varphi$ can obviously be identified by making a convenient choice of phase constants and setting:

$$e_0 = 4\pi A.$$

The periodic quantity that is coupled with a material point with spherical symmetry can then be likened to a potential (viz., one-half the sum of the advanced and retarded solutions) that will be produced around that of the material point; it will carry an oscillating charge of amplitude $4\pi A$ and a frequency that is determined by the quantum relation. Much later, we will have to examine whether one must suppose that there exists an arbitrary ratio between that oscillating charge and the electric charge that is carried by the material point.

Assume (and we shall return to this point) that the amplitude $e_0$ of the oscillating charge presents the same invariance character as the electric charge in the usual sense of the word. When one changes the Galilean coordinate system, the phase factors in the expression for the potentials that are due to the oscillating charge will necessarily be invariants, while the amplitude factors will transform like electromagnetic potentials – i.e., like the components of a world-tensor.

If, as in Chapter I, we pass from the system $x_0, y_0, z_0$, in which the material point is are rest, to a system $x, y, z$ in which it moves along the $Oz$-axis with velocity $\beta c$ then the expression $\varphi(x, y, z, t)$ that is obtained by replacing the quantities $r_0$ and $t_0$ in the $\varphi(r_0, t_0)$ with their values as functions of $x, y, z, t$ will represent the magnitude of the tensor potential, but not the scalar potential, which is only its temporal component. Conforming to the transformation formulas of the tensorial components, the scalar potential will then be given by the formula:

$$\psi(x, y, z, t) = \frac{1}{\sqrt{1-\beta^2}}\,\varphi(x, y, z, t).$$

That is in fact the result that we obtained upon applying Liénard's formula to a point on the $z$-axis. In order to complete it, we must then add a vector potential to that scalar potential that is parallel to the $z$-axis and defined by:

$$a_x = 0, \qquad a_y = 0, \qquad a_z = \frac{\beta}{\sqrt{1-\beta^2}}\,\varphi(x, y, z, t).$$

We will then be dealing with a longitudinal wave for the vector potential.

**2. The associated wave for the atom of radiation. –** The solution that we studied in Chapter I and that we just re-examined pertains to material points with spherical symmetry and associates their uniform, rectilinear motion with a longitudinal wave. It does not pertain to the quantum of light then, since light is necessarily a transverse vibration, which has been proven by polarization phenomena especially.

In order to avoid a very delicate passage to the limit, suppose that the proper mass of the quantum of light is extremely small, but nonetheless finite. For a given value of its energy $h\nu$, under the condition that this value of not extraordinarily small, the quantum will be animated with a velocity that is slightly less than $c$.

The associated periodic phenomenon will then be represented by a vector that is normal to the direction of propagation. If we pass to the system that is coupled with the moving body then that vectorial quantity will not suffer any modification. In the proper system, we will then have a periodic phenomenon that is represented at every point in space by a vector that has the same direction everywhere. It is entirely reasonable to attribute the nature of a vector potential to it. One will get a picture for the creation of that vector potential by equating the atom of radiation, when imagined in its proper system, with a dipole of electromagnetic theory: The two equal and opposite charges are assumed to remain close enough to each other that no appreciable scalar potential will be produced in space, but they will cause only a vector potential to appear in all of space that is parallel to the direction of their relative vibration, which conforms to the vectorial equation:

$$\Delta\mathbf{a} - \frac{1}{c^2}\frac{\partial^2\mathbf{a}}{\partial t^2} = -\,\rho\frac{\mathbf{v}}{c}.$$

One must suppose that the charges are constant and that the vibration takes place with a frequency $\nu_0$ that is defined by the quantum relation:

$$W_0 = h\nu_0 .$$

Perhaps we should be a bit wary of the simplicity of the preceding picture: Indeed, we saw that in our theory, the electron cannot be represent in manner that conforms to classical electromagnetic theory (viz., the impossibility of an oscillating charge that is implied by Maxwell's equations) and that in this case, the frequency $\nu_0$ cannot be compared to a simple mechanical frequency. It is doubtful that anything would be different for the quantum of light, and it would be more prudent to say only that:

*Whereas in the case of a material point with spherical symmetry, the periodic phenomenon in the proper system is a scalar potential that is excited by a tensor (viz., an analogue of the world-current) whose spatial component is zero, in the case of the atom* *of radiation, it is, on the contrary, a vector potential that is excited by a tensor whose temporal component is zero.*

Let **A** denote the vector to which the tensor that produced the potential reduces and always add the advanced and retarded solutions; we will have:

$$\mathbf{a}\,(x_0, y_0, z_0, t_0) = \frac{\mathbf{A}}{2r_0}\left[\sin 2\pi\nu_0\left(t_0-\tau_0-\frac{r_0}{c}\right)+\sin 2\pi\nu_0\left(t_0-\tau_0+\frac{r_0}{c}\right)\right]$$

$$= \frac{\mathbf{A}}{r_0}\sin 2\pi\nu_0(t_0-\tau_0)\cos 2\pi\frac{\nu_0}{c}r_0\,.$$

That is the most likely form for the function that represents the periodic phenomenon that is associated with the quantum of light.

For some unknown reason, the motion of the quantum always takes place normally to the vector **a** and is therefore not affected by the value of that vector. As a result, in a system $x, y, z, t$ for which the luminous atom displaces with velocity $\beta c$ along the $z$-axis, one will have:

$$\mathbf{a}\,(x_0, y_0, z_0, t_0)$$

$$= \frac{\mathbf{A}}{\sqrt{x^2+y^2+\dfrac{(z-\upsilon t)^2}{1-\beta^2}}}\cos\frac{2\pi\nu_0}{c}\sqrt{x^2+y^2+\frac{(z-\upsilon t)^2}{1-\beta^2}}\,\sin\frac{2\pi\nu_0}{\sqrt{1-\beta^2}}\left(t-\frac{\beta z}{c}-\tau\right).$$

Since $\nu_0 = m_0\,c^2 / h$ must be assumed to be extremely small, in order for $\nu_0/\sqrt{1-\beta^2}$ to have a finite value $\nu$, it is necessary that $\upsilon$ must be close to $c$. Then set $\beta = 1-\varepsilon$ and neglect the terms in $\varepsilon^2$ ; one will get:

$$\mathbf{a}\,(x, y, z, t)$$

$$= \frac{\mathbf{A}}{\sqrt{x^2+y^2+\dfrac{(z-\upsilon t)^2}{2\varepsilon}}}\cos\frac{2\pi\nu_0}{c}\sqrt{x^2+y^2+\frac{(z-\upsilon t)^2}{2\varepsilon}}\,\sin 2\pi\nu\left(t-\frac{\beta z}{c}-\tau\right).$$

The first two factors represent the amplitude, animated with the velocity $\upsilon = (1-\varepsilon)\,c$; the last one gives the phase, whose velocity is $V = c/\beta = (1+\varepsilon)\,c$.

Let us examine the distribution of the amplitudes at the instant $t$. At the point $M\,(x, y, z)$ whose distance from the rectilinear trajectory of the quantum is $\rho = \sqrt{x^2+y^2}$ and whose distance to the wave plane that contains the quantum is $d = z - \upsilon t$, one will have the amplitude:

$$\frac{\mathbf{A}}{\sqrt{\rho^2+\frac{d^2}{2\varepsilon}}}\cos\frac{2\pi\nu_0}{c}\sqrt{\rho^2+\frac{d^2}{2\varepsilon}}\,.$$

If $M$ is in the wave plane that contains the quantum then $d = 0$, and the amplitude will vary with the distance $\rho$ like $\frac{A}{\rho}\cos\frac{2\pi\nu_0\rho}{c}$; the cosine term will be less noticeable at small distances, due to the smallness of $\nu_0$ , moreover.

On the contrary, if $M$ is not in the immediate neighborhood of the plane of the quantum then $\rho^2$ will be negligible in comparison to $d^2 / 2\varepsilon$, and the amplitude will vary with $d$ like $\frac{A\sqrt{2\varepsilon}}{d}\cos\frac{2\pi\nu d}{c}$. At equal distances from the atom of radiation, the amplitude will then be much larger in the wave plane that contains it than it is outside of it.

That seems to be the probable constitution of the potential of the wave that is associated with the quantum of light. One can try to pass to the limit by setting $\nu_0 = 0$ and $\upsilon = c$, but we shall see that this passage is delicate for the present example.

One must first suppose that the limit of $\nu_0 / \sqrt{2\varepsilon}$ is equal to $\nu$ when $\nu_0$ and $\varepsilon$ tend to zero simultaneously. Having said that one will get:

$$\begin{aligned}\mathbf{a}\,(x, y, z, t) &= \frac{\mathbf{A}}{\sqrt{x^2+y^2}}\sin 2\pi\nu\left[t-\frac{z}{c}-\tau\right]\\ &= \frac{\mathbf{A}}{\rho}\sin 2\pi\nu\left[t-\frac{z}{c}-\tau\right]\end{aligned}$$

for the wave plane of the quantum, and:

$$\begin{aligned}\mathbf{a}\,(x, y, z, t) &= \frac{\mathbf{A}\sqrt{2\varepsilon}}{z-ct}\cos\frac{2\pi\nu}{c}(z-ct)\sin 2\pi\nu\left[t-\frac{z}{c}-\tau\right]\\ &= \frac{\mathbf{A}\sqrt{2\varepsilon}}{d}\cos\frac{2\pi\nu}{c}d\sin 2\pi\nu\left[t-\frac{z}{c}-\tau\right]\end{aligned}$$

outside of the plane $z = ct$.

However, a difficulty presents itself here: If $\mathbf{A}$ has a finite value then the potential $\mathbf{a}$ will be zero everywhere outside of the plane $z = ct$, whereas if $\mathbf{A}\sqrt{2\varepsilon}$ has a finite non-zero limit then $\mathbf{a}$ will be infinite in the plane $z = ct$. Either alternative is difficult to accept. It would then seem, at the very least, more convenient to not perform the passage to the limit in order to study the properties of the associated wave.

Finally, we point out an interesting situation. Let an ensemble of light quanta with the same polarization displace with the same energy in the same direction. Is it possible that the associated waves can form a single wave *from the standpoint of phases?* Yes, that will be produced if the quanta are all synchronous in the system in which they are at relative rest. Indeed, in that system the periodic potential has the same phase:

$$\sin 2\pi \nu_0 (t_0 - \tau_0)$$

in space. As for the amplitude, at each point, it will depend upon the distances from that point to the various point-sources. Return to the first reference system. The phase will then be given by:

$$\sin 2\pi \nu \left( t - \frac{z}{c} - \tau \right)$$

everywhere, and if that plane wave is transported through an enormous number of quanta, it will show a marked analogy with the homogeneous plane waves that are often studied by the classical theories.

To conclude, we once more insist upon an essential point. We just attempted to determine the periodic quantities that are associated with the various kinds of material points, and we have often referred to them as "potentials." Indeed, they have just the tensorial character as the potentials of electromagnetic theory, and they propagate according to the same law. However, they must not be confused with the latter potentials. We shall seek to examine that point more closely by studying the Maxwell-Lorentz equations.

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# CHAPTER VIII

# THE LINK WITH ELECTROMAGNETISM

1. The Maxwell-Lorentz equations. – 2. The value of electromagnetic theory. – 3. Attitude of electromagnetic theory toward the problem of radiation.

**1. The Maxwell-Lorentz equations. –** In our research, we have come to understand that the equations of propagation for potentials are more general than the group of relations that are universally known by the name of Maxwell's equations. It would therefore be interesting to invert the order of deductions that are generally followed in classical works and look for the conditions under which the Maxwell-Lorentz equations can be derived from the equations of propagation.

Therefore, consider two quantities $\psi$ and $\mathbf{a}$ that define the temporal and spatial components, resp., of a world-tensor, and suppose that they are restricted to ones that verify the relations:

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\rho,$$

$$\Delta\mathbf{a} - \frac{1}{c^2}\frac{\partial^2\mathbf{a}}{\partial t^2} = -\frac{\mathbf{C}}{c},$$

in which the functions $\rho$ and $\mathbf{C}$ are given in all space-time and are also the components of a world-tensor.

Having said that, we define two vectors by the relation:

$$\mathbf{h} = -\operatorname{grad}\psi - \frac{1}{c}\frac{\partial\mathbf{a}}{\partial t},$$

$$\mathbf{H} = \operatorname{rot}\mathbf{a}.$$

We effortlessly obtain:

$$\text{(I)} \qquad \operatorname{div}\mathbf{H} = 0,$$

$$\text{(II)} \qquad \frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} = \frac{1}{c}\operatorname{rot}\left(\frac{\partial\mathbf{a}}{\partial t}\right) = -\operatorname{rot}\mathbf{h}.$$

These are two equations of Maxwellian form, which are written with the Lorentz-Heaviside convention for the suppression of the factor $4\pi$.

In order to go further, we let $L(x, y, z, t)$ denote the function $\frac{1}{c}\frac{\partial\psi}{\partial t} + \operatorname{div}\mathbf{a}$.

One will easily find that:

$$\operatorname{div}\mathbf{h} = -\Delta\psi - \frac{1}{c}\frac{\partial}{\partial t}(\operatorname{div}\mathbf{a}),$$

$$= -\Delta\psi - \frac{1}{c}\frac{\partial L}{\partial t} + \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2},$$

$$\frac{\partial\mathbf{h}}{\partial t} = -\operatorname{grad}\left(\frac{\partial\psi}{\partial t}\right) - \frac{1}{c}\frac{\partial^2\mathbf{a}}{\partial t^2}$$

$$= c\operatorname{grad}\operatorname{div}\mathbf{a} - c\operatorname{grad}L - \frac{1}{c}\frac{\partial^2\mathbf{a}}{\partial t^2}.$$

If we recall that grad div $\mathbf{a} = \Delta\mathbf{a}$ + rot rot $\mathbf{a}$, and take the equations of propagation into account then our point of departure will be:

(III) $$\operatorname{div}\mathbf{h} = \rho - \frac{1}{c}\frac{\partial L}{\partial t},$$

(IV) $$\frac{\partial\mathbf{h}}{\partial t} = \operatorname{rot}\mathbf{H} - \frac{\mathbf{C}}{c} - \operatorname{grad}L.$$

Finally, we form the combination $\frac{1}{c}\frac{\partial}{\partial t}$(III) – div (IV):

$$0 = \frac{1}{c}\frac{\partial\rho}{\partial t} - \frac{1}{c^2}\frac{\partial^2 L}{\partial t^2} + \operatorname{div}\frac{\mathbf{C}}{c} + \Delta L$$

or

(V) $$\frac{\partial\rho}{\partial t} + \operatorname{div}\mathbf{C} = -c\left(\Delta L - \frac{1}{c^2}\frac{\partial^2 L}{\partial t^2}\right) = -c\,\Box L.$$

If we suppose that $L$ is identically zero then equations (III) and (IV) will become the last two Lorentz equations upon setting $\mathbf{C} = \rho\mathbf{v}$. Equation (V) will then express the conservation of electricity.

Briefly, it is the hypothesis that $L = 0$ that permits one to deduce the group of Maxwell-Lorentz relations upon starting with the equations of propagation; that is what imposes the conservation of electricity.

The deeper reason that we are prevented from equating the potentials that were studied in the last chapter with true electromagnetic potentials is the non-zero value of the function $L$ that corresponds to them. For example, in the system that is associated with an electron, we found a zero vector potential and a periodic scalar potential. The function $L$ that reduces to $\partial\psi/\partial t$ will not be zero then, and $\psi$ and will not be an electromagnetic potential that satisfies the Maxwell equations.

To conclude this rapid review of electromagnetic theory, recall that it is further based upon a sixth relation that is independent of the preceding one.

That would be the expression for the Lorentz force that is exerted by an electromagnetic field on a charge:

$$\text{(VI)} \qquad \mathbf{F} = e\left(\mathbf{h} + \frac{1}{c}[\mathbf{v}\,\mathbf{H}]\right).$$

Upon combining that equation with the preceding one, one will obtain the energetics of the electromagnetic field by a well-known calculation. One determines the energy density $\rho\,(w)$ and that of the quantity of motion $\rho\,(\mathbf{G})$, as well as the energy flux or Poynting vector $\mathbf{P}$ by the relations:

$$\rho\,(w) = \tfrac{1}{2}(h^2 + H^2),$$

$$\rho\,(\mathbf{G}) = \frac{1}{c}[\mathbf{h}\;\mathbf{H}],$$

$$\mathbf{P} = c\,[\mathbf{h}\;\mathbf{H}] = c^2\,\rho\,(\mathbf{G}).$$

**2. The value of electromagnetic theory. –** Electromagnetic theory, when completed by the introduction of the atomic structure of electricity, has known great successes and can be legitimately considered to be the expression of reality itself. However, the development of our knowledge of radiation and its relationship to matter (emission, absorption, diffusion, etc.) has undermined that conviction and has shown that electromagnetism is insufficient, at least for the interpretation of those phenomena. Nevertheless, the predictions of the theory of electrons will be verified whenever one is dealing with phenomena in which radiation is not involved. Hence, they will be true for all of the usual electric phenomena, and in particular, those of electrical engineering. In the Bohr atom, forces are given quite correctly by the Lorentz relation, and that verification is quite remarkable. It then seems precise for to say:

*Electromagnetic theory is well-verified for the purely-material phenomena in which radiation is not involved.*

It seems that we can even liberate the theory of electrons from one of the grievances that has weighed upon it. That grievance is the prediction that energy will be emitted by an electron in accelerated motion in the form of a spherical wave with a continuous distribution. That conclusion is in absolute contradiction with the stability of atoms, and furthermore, experiments never seem to reveal either emission by an accelerated electron or the existence of waves with a continuous distribution of energy. However, it is easy to make the acceleration wave disappear from the theory of electrons. It will suffice to introduce advanced potentials.

Indeed, recall the reaction force that is due to radiation, as Lorentz calculated it in his classical book on the theory of electrons ([6]): If we take one-half the sum of the advanced and retarded potentials instead of the retarded potentials, for reasons that were explained

([6]) *Theory of Electrons*, 2nd English edition, pp. 252.

before, then we will find a zero value for the radiation reaction, and we will conclude that the accelerated electron does not radiate. That case of radiation acceleration is very instructive, moreover! Indeed, suppose that an electron is in accelerated motion. Classical theory tells us that it will emit a wave due to the "retardation" of the potential. Suppose that the direction of time has been reversed at a given moment. We shall review the phases of motion of the electron as they proceed in the opposite sense, and the electron *absorbs* the wave that it has emitted. How is a physicist that is not aware of the inversion of time to describe the phenomenon? He can do that only by neglecting the retarded solutions and keeping only the advanced solution. Only then can he interpret the absorption of the wave by the electron. However, if he assumes that only the retarded solutions should be counted then our physicist will perceive that the sense of time has been reversed. Time will then have a privileged direction, and I must say that this conclusion seems rather shocking to me because it implies a sort of fundamental irreversibility in natural phenomena. Naturally, if one forms one-half the sum of the advanced and retarded solutions then reversibility will be re-established by the suppression of the radiation that is due to acceleration ([7]).

**3. Attitude of electromagnetic theory regarding the problem of radiation. –** If the ideas that we have developed are exact then the classical theory will be found in the impossibility of interpreting radiation phenomena in detail, and that is because the oscillating potentials that are necessary for their true description do not satisfy the Lorentz condition $L = 0$ and are, in turn, excluded from the framework of Maxwell's equations. One will then be constrained to look for solutions to Maxwell's equations that represent waves whose speed of propagation has the value $c$ that experiments reveal, and one will have then recovered the solutions of the elastic theory, and notably the plane wave and spherical wave whose amplitudes are constant on each wave surface. However, those solutions are, so to speak, fictitious, and the alleged electric and magnetic field of the light wave are not true fields. In particular, the action of light on a charge cannot be calculated by the Lorentz formula upon starting from those pseudo-fields, and that is indeed what experiments seem to verify. Finally, if those quantities are electromagnetic fields then the energy $\frac{1}{2}(h^2 + W^2)$ will be distributed in a continuous fashion, and it will also be inexact.

Meanwhile, the concept of homogeneous waves, such as one utilizes in the classical theory, has permitted the interpretation of phenomena that have been observed very exactly and described in physics books under the rubric of wave optics. That concept of homogeneous waves is not without value, since it does represent something, and the main task of the optics of quanta is to establish a link between that concept of homogeneous waves and the discontinuity in the structure of radiant energy.

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([7]) PAGE, Phys. Rev. **24** (1925), pp. 296.

**CHAPTER IX**

# HOMOGENEOUS WAVES AND RECTILINEAR MOTION.

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1. Homogeneous waves and rectilinear motion. – 2. Homogeneous waves and curved trajectories. – 3. Diffraction and interference. – 4. Motion of an isolated quantum. – 5. Interactions between quanta.

**1. Homogeneous waves and rectilinear motion. –** In order to better understand how homogeneous waves can play a role in the optics of quanta, we shall go back a bit. In the study of uniform, rectilinear motion, we saw that space behaves as if it possessed an index of refraction $n = \beta$ for the associated wave because the phase velocity is $c / \beta$. That index is a function of the frequency of the wave, and more precisely, in such a fashion that speed of the energy is given by Rayleigh's formula.

On the other hand, we showed at the end of Chapter VII that a set of atoms of radiation that displace with the same velocity and the same direction and are synchronous in their proper system will have associated waves whose phases coincide at every point, which is a result that is exact for any type of moving body. Therefore, consider a set of material points that are animated with the same velocity in the same direction and are synchronous in the proper system. One can consider them to be associated with the same wave whose phase is given by an expression for the form:

$$\sin 2\pi\nu\left(t - \frac{z}{V} - \tau\right),$$

with $V = c / \beta$. As for the amplitude of the wave, it propagates with the velocity $\upsilon = \beta c$. However, at each instant, its value at any point will depend upon the manner by which the various material points are distributed throughout the wave.

Now imagine a homogeneous plane wave:

$$A \sin 2\pi\nu\left(t - \frac{z}{V} - \tau\right),$$

in which the constant quantity $A$ is such that its square is proportional to the mean number of atoms per unit volume, or if one prefers, the density of energy in the wave. The propagation of that wave in the medium of index $n$ that constitutes the space will give us a statistical picture of the motion of our ensemble of material points because we know that energy will displace with the Rayleigh velocity in such a wave.

In the case where our material points are light quanta, the velocities $\upsilon$ and $V$ are both indiscernible from $c$, and the homogeneous wave that represents the motion globally is:

$$A \sin 2\pi\nu\left(t - \frac{z}{V} - \tau\right).$$

We can then recognize the statistical significance of the homogeneous plane wave in classical theories.

**2. Homogeneous waves and curved trajectories. –** In Chapters II and III, while studying fields of constant force, we further defined an index of refraction whose value we calculated at each point. Furthermore, we have also seen that the velocity of the moving body can be calculated by Rayleigh's formula.

Having recalled that, consider the constant force field to be a medium of index $n$ that is defined by the formulas in Chapters II and III. Imagine that a wave propagates in that medium that has no singularity of the classical kind and possesses the frequency $\nu$. The rays of that wave will be certain trajectories of energy $W = h\nu$ in the field of constant force; that will result from the identity of Fermat's and de Maupertuis's principles. Furthermore, it is easy to see that the propagation that is imagined represents the motion of an ensemble of a large number of material points that are affected by the field, but *do not interact with each other,* with the reservation that the following two conditions are realized:

1. The amplitude $A$ of the wave must be proportional to the square root of the density of the moving body at each point of the field. If $dn$ denotes the number of material points that occupy a volume $d\tau$ in space then one must have:

$$dn = K\, A_\tau^2 d\tau,$$

in which $A_\tau$ is the amplitude of the continuous wave in the element $d\tau$. If that condition is verified everywhere at the initial instant then it will always be verified because the trajectory tubes coincide with the ray tubes and the speed of the moving body is the Rayleigh speed.

2. The moving bodies must present a certain coherence between them because each of them must be found to be in phase with the continuous wave when it is envisioned as a vibrating source.

The reader must reflect upon the fundamental difference that exists between the continuous wave, the statistical representation of the motion of numerous, coherent, material points, and the wave phenomena that are associated with each of the material points, whose character of involving a singularity is essential.

**3. Diffraction and interference. –** Let an ensemble of coherent light quanta have the same frequency. Their propagation in free space can be represented globally by the homogeneous wave:

$$A \sin 2\pi\nu \left(t - \frac{z}{c} - \tau\right).$$

If that plane wave penetrates a region of space that is studded with *fixed* obstacles then it will produce diffraction and interference phenomena. From numerous extremely precise experiments, we have learned since the end of the Eighteenth Century that the *statistical* distribution of light energy in the regions of interference is predicted very accurately by the wave theory when one supposes that the incident wave is homogeneous. Therefore, the calculation of the wave:

$$A \sin 2\pi\nu \left(t - \frac{z}{c} - \tau\right)$$

in the region that is studded with obstacles by the classical interference methods will always give an exact statistical image of the displacement of quanta in that domain. In particular, the trajectories of the atoms of radiation must coincide with the rays of the classical theory, the square of the amplitude at each point must give the density of the quanta, and the phases of the wave along each ray must correspond with those of the wave that is associated with the quantum that follows it.

If one assumes those ideas then all of the phenomena of diffraction and interference can be interpreted by the theory of quanta of light. The same thing will then be true for the experiments, like those of Taylor in which interference was obtained with extraordinarily weak light intensities as a result of the extremely long time intervals that were used. Indeed, the quanta of the same wave did not interact with each other, so the form of the possible trajectories in a given interference experiment could not depend upon the intensity, and as a result, the distribution of the impressions on a photographic plate did not depend upon the time interval of the experiment. If the plate received the energy that was necessary for its impression very rapidly as a result of a very intense irradiation (that is, it is subject to a burst of quanta) then it will, if I may say, exhibit spatial statistics. On the contrary, if it receives the same amount of energy very slowly then it will exhibit temporal statistics. However, the distribution of quanta between the various incident rays is supposed to be ruled by chance, so the results must necessarily be the same.

In Maxwell's theory, the energy density of a wave will have the mean value of $\frac{1}{2}(\overline{h^2} + \overline{H^2})$ at each point. It seems that this expression must be considered to represent the density of the quanta. Moreover, we learn from Poynting's theorem that the vector $c[\mathbf{h}\ \mathbf{H}]$ represents the energy flux per unit time and area in magnitude and direction. In the interference phenomena that are due to fixed obstacles, the direction of the radiant vector is constant at each point and its envelopes, which will be the rays, are likewise fixed.

By contrast, the magnitude of the vector is variable, and it seems that one must consider a mean value in order to obtain the velocity of energy. In order to do that, imagine a very thin tube whose walls are composed of rays (i.e., a tube of trajectories), and let $\delta\sigma$ be a cross-section to which the quantities $h$ and $H$ are referred. Finally, let $\delta t$ denote a time interval that is very small, but still long with respect to the period of the wave, and let $\delta\tau$ be the small cylindrical volume that is composed of the segment of the tube with base $\delta\sigma$ and height $\upsilon\,\delta t$. The energy flux across $\delta\sigma$ during the time $\delta t$ is equal to the energy that is contained in the volume $\delta\tau$. One will then have the equality:

$$c\,\overline{[\mathbf{h}\,\mathbf{H}]}\,\delta\sigma\,\delta t = \tfrac{1}{2}[\overline{h^2} + \overline{H^2}]\,\upsilon\,\delta\sigma\,\delta t.$$

Finally, the speed of the energy at any point must be taken to be:

$$\upsilon = c\,\frac{2\overline{[\mathbf{h}\,\mathbf{H}]}}{\overline{h^2} + \overline{H^2}}.$$

We first adopt the electromagnetic viewpoint. If we displace all of the points of $\delta\tau$ along the corresponding rays by $\upsilon\,\delta t$ then we will pass from the volume $\delta\tau$ at the instant $t$ to the volume $\delta\tau'$ at the instant $t + \delta t$, and conservation of energy will give:

$$\int_{\delta\tau}\tfrac{1}{2}(\overline{h^2} + \overline{H^2})\,d\tau = \int_{\delta\tau'}\tfrac{1}{2}(\overline{h^2} + \overline{H^2})\,d\tau.$$

From the corpuscular viewpoint, we say that the $n$ quanta that occupy $\delta\tau$ at the instant $t$ will occupy $\delta\tau'$ at the instant $t + \delta t$. One easily deduces from this that if the density of quanta is originally proportional to $\tfrac{1}{2}[\overline{h^2} + \overline{H^2}]$ at any point of the wave then that will be continually true. That is the necessary condition for our statistical representation to be acceptable.

**4. Motion of an isolated quantum.** – For the sake of dynamics, the study of the motion of a material point in a constant field shows us that it is possible to represent the various motions of a given energy globally by considering homogeneous waves. In other words, we start from the study of the motions of the individual moving bodies, and we are then led to a statistical representation. The situation is the opposite for radiation: Experiments have taught us that the motion of quanta is represented statistically quite well by the theory of continuous waves. We must then return to the individual viewpoint and study the motion of the isolated quantum and the propagation of its associated wave in the region of interference.

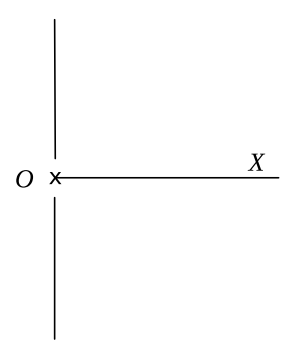


Figure 7.

One should find that the quantum always follows one of the rays of the classical wave with undoubtedly the speed $\upsilon$ that was defined above.

I will try to make the problem more precise in a particular case. Consider a screen that is pierced with a hole (Fig. 7). Classical theory teaches us how a homogeneous plane wave that falls on the screen normally will be diffracted when it crosses the opening. If one imagines the ray *XO*, which falls normally on a point *O* in the opening, then Poynting's theorem will permit us to prolong that ray to the left of the screen, at least, in principle. It goes without saying that this prolongation will be a curved line, in general, because that is precisely what diffraction consists of. Now, here is how that problem must be solved: Consider a quantum that advances along *XO* with its train of inhomogeneous waves. The oscillating (non-electromagnetic) vector potential **a** propagates according to the wave equation:

$$\Delta \mathbf{a} - \frac{1}{c^2}\frac{\partial^2 \mathbf{a}}{\partial t^2} = 0.$$

When the quantum is far to the right of the screen, the wave that surrounds it will be the one that is associated with a uniform, rectilinear motion:

$$\mathbf{a} = \frac{1}{\sqrt{x^2 + y^2 + \dfrac{(z-\upsilon t)^2}{1-\beta^2}}} \cos \frac{2\pi \nu_0}{c} \sqrt{x^2 + y^2 + \frac{(z-\upsilon t)^2}{1-\beta^2}} \sin 2\pi\nu\left(t - z\frac{\beta}{c} - \tau\right),$$

in which $\nu_0$ is very small and $\beta$ is close to 1.

Can one use continuity and Kirchhoff's theorem to predict how that inhomogeneous wave will be diffracted when it crossed the opening, and what will the trajectory of the moving singularity be beyond the point *O*? One should be able to show that (no matter what the position of the point *O* in the opening) the trajectory will coincide with the prolongation of the ray *XO* that is predicted by the classical theory. It will indeed give the trajectory of each isolated quantum and will represent the collective motion of a large number of quanta statistically.

That is what one must prove in order to substantiate the postulate that was assumed. That proof seems difficult to me (I certainly do not know how to do it), but I have no doubt in my mind that a proof of that type might be possible.

We can make a few interesting remarks. First of all, the motion of the quantum does not generally take place along a straight line, so the principle of inertia is no longer valid and the world-line is not a space-time geodesic.

Everything happens then as if the screen exerted a force on the moving body, and that remark was already shocking to Newton. However, that force has a very special nature, since it is generated by the diffraction of the wave itself. If the screen is kept fixed then the diffraction cannot modify the frequency. The atom of radiation will then conserve energy. Is the quantity of motion also conserved? No, because once the quantum leaves the region of interference its quantity of motion will indeed be equal in magnitude to its initial value on *XO*, but the direction of motion will not generally be the same, and there will be no "vectorial" conservation. In classical dynamics, when a moving body describes a motion around a fixed attractive center, its quantity of motion will not generally remain constant, and from that fact, an impulse will be transmitted by the intermediary of the force field of the attractive center; now, it is the force of that fixed center that equilibrates that impulse. By analogy, it would seem probable that when the isolated quantum of light crosses the region of diffraction, it will transmit a certain impulse to the screen by the intermediary of its associated wave, and that will put it into motion if it is maintained by a foreign force. Of course, if a sheaf of very many quanta arrives at the opening of the screen then the partial impulses will cancel in the mean, and the screen will not be subject to any resultant force.

If a homogeneous plane wave falls upon a fixed, perfectly-reflecting mirror at a certain angle of incidence then classical theory teaches us that the incident wave and the reflected wave will interfere, and a simple calculation will show that the light rays will break before they arrive at the mirror and recombine behind it without having touched the surface. We must then suppose that the incident quanta do not really "bump into" the mirror, but that their path will turn back without having touched it.

That in no way invalidates the argument that permitted us to obtain the radiation pressure because the variation in the quantity of motion of the quantum must be transmitted to the mirror by the wave in the form of impulse.

**5. Interactions of quanta. –** What we just studied was, in summary, the effects that are exerted upon a quantum of light by the presence of certain obstacles, but the quanta that are associated with the same wave, which are the only kind that we have considered up to now, exerted no influence on each other; each of them pursued its route independently of the other ones. Things are surely different for the quanta that are associated with different waves.

We shall always proceed by analogy with the theory of homogeneous waves. If two homogeneous plane waves meet then they will interfere, and the paths of the energy will be modified by the action of the interference. Two sheaves of quanta that are associated with two different waves must then undoubtedly react to each other in such a way that the modified trajectories will coincide with those of the classical theory. Nevertheless, that conclusion can be precise here only if every wave carries a very large number of quanta. Indeed, it can no longer be a question of assuming the independence of the motions of a quantum in regard to the presence of the other quanta, because we are dealing with mutual interactions, now.

That sort of mutual interaction can hardly arise in the usual optical experiments, since one always employs coherent sheaves in them, but they do play an important role in the theory of blackbody radiation and the energy fluctuations in that kind of radiation. Furthermore, no matter what the nature of the interactions between atoms of radiation might be, they will certainly be accompanied by conservation of energy and conservation of quantity of motion. Even when one takes that into account, the proof that we gave for radiation pressure will still persist for blackbody radiation.

## CHAPTER X

# DIFFUSION AND DISPERSION

1. Diffusion of charged particles. – 2. Diffusion of radiation. – 3. Dispersion. – 4. Motion of a quantum in a refringent medium.

**1. Diffusion of charged particles. –** If an electrified particle whose trajectory is initially rectilinear passes close to a charge *C*, which is assumed to be fixed, then it will be subjected to an attractive or repulsive action, and its motion will experience a deviation. If one knows the distance from the center *C* to the initial direction in which the particle moves then one can predict the total deviation that the particle experiences. The calculation is very simple due to the form of Coulomb's law (at least, if one neglects relativistic corrections). If one considers an entire ensemble of particles that approach *C* along parallel rectilinear trajectories then one will obtain a sheaf of trajectories, such as in Fig. 8 (in which, the force is assumed to be repulsive). Some simple statistical considerations will give the angular distribution of the deviated trajectories. The formulas thus-obtained have been verified by Sir. E. Rutherford and his collaborators in their beautiful experiments on the passage of $\alpha$ rays through matter.

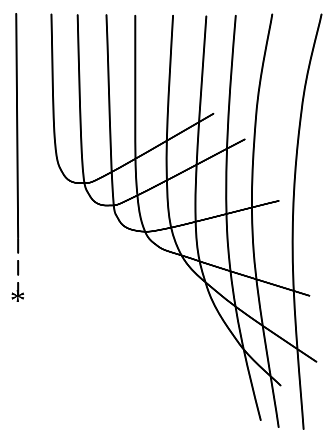

Figure 8.

We can envision that problem from a different angle from the standpoint that is of interest to us. Indeed, we saw that the trajectories of energy *W* for a charge of value *e* and mass $m_0$ will coincide with the rays of a homogeneous wave of classical type that propagates in a medium whose index is defined at each point by the law:

$$n^2 = \left(1 - \frac{e\psi}{h\nu}\right)^2 - \frac{m_0^2\, c^4}{h^2\nu^2},$$

in which $\psi$ is the electrostatic potential and $\nu$ is the frequency $W / h$. If the potential is created by a charge $C$ then it will be inversely proportional to the distance from $C$ at any point. In order to obtain the scattering of a sheaf of electrified particles under the influence of a charge $C$, it will be necessary to study the deformation of a homogeneous wave that is initially planar when it propagates in a medium whose index varies according to the law above. The rays of that wave will be the trajectories of the deviated particles. If the moving particles are "associated with the same wave" originally, with the meaning that we gave to that expression, then the refracted wave will provide us with the distribution of the phases of the associated waves along the various rays.

The deviation of particles is produced in a small space surrounding $C$, while the trajectories at a great distance will appear to be straight lines that pass $C$. The diffusing charge will then appear to be the center of the rectilinear rays, and the classical wave that represents the distribution of the particles at a distance will be very approximately a spherical wave. The distribution of the amplitudes over the spherical wave will depend upon the statistical law that Rutherford studied, since the square of that amplitude will represent the mean density of the deviated particles. In summary, if one confines oneself to imagining the homogeneous wave that represents the motion statistically, and if one attaches it to only the description of phenomena that are distant from the center then one can say: Under the action of the incident wave, the center will emit a secondary spherical wave, over which, the amplitude is not distributed in a uniform fashion, moreover. That statement reveals to us a deep relationship between the diffusion of rays ($\alpha$ or $\beta$, for example) and the diffusion of light, as it is conceived according to classical ideas.

**2. Diffusion of radiation**. – If we adopt the corpuscular hypothesis then the diffusion phenomena that have such great importance in optics and the domain of X-rays will oblige us to assume that when a quantum of radiation passes close to an electric charge or a set of such charges (e.g., an atom or molecule), it will be subjected to a certain action whose result will be a curving of the trajectory. If we imagine a sheaf of quanta of the same frequency whose initial velocities are parallel and which are associated with the same wave then that sheaf will be scattered – i.e., diffused – when it passes close to the charge center. The homogeneous wave that statistically represents the motion of the quanta will propagate as if there existed an index of refraction around the diffusing center. However, that homogeneous wave is nothing but the wave of the classical theory, and from the very nature of the validity that we have attributed to those theories, we can think that they will once more give us an exact global representation of that diffusion here.

One knows what the image of the phenomena would be that the theory of electrons would provide us with. If a light wave is assumed to be planar and homogeneous, and if the light variable is identified with an electric field then a particle that is placed in that wave must be subjected to a periodic force and enter into vibration. The amplitude and phase of that vibration will depend upon the incident wave, the charge and mass of the particle, the frequency, and finally, the proper frequency of vibration that is determined by the constraint forces. The calculation is simple and too well-known for it to be necessary to reproduce it here. Thanks to the use of retarded potentials, one can then predict the emission of a spherical wave by the vibrating charge, and the energy that is necessary to sustain it will be drawn from the incident wave. A scattering of the radiant energy will result, and that will represent the phenomenon of diffusion.

The fact that this description of the phenomenon is inexact will seem almost certain. First of all, the granular structure that we are almost forced to attribute to light nowadays will not permit us to suppose that real light waves are homogeneous, nor, it seems, to consider the oscillating magnitude of those waves to be a true electric field that is capable of exerting the Lorentz force on an electric charge. The radiation from an accelerating charge might seem very problematic then, since such a radiation process would involve a very strange irreversibility, as I have remarked before. Finally, the quantities that the theory of electrons introduces in order to explain diffusion and dispersion as if they were the mechanical resonant frequencies for the vibrating particles indeed seem to have a more complex character. Now, the dispersion formulas that have been verified in not only the optical domain, but also in the Röntgen domain of high frequencies (Siegbahn, Bergen Davis, etc.) have shown that those alleged resonant frequencies are determined by the intra-atomic energy levels, and from that point onward, it will be difficult to assimilate them to true mechanical frequencies that are due to constraint forces with no further assumptions.

Nevertheless, the classical theory of diffusion has achieved some admirable successes. To cite only two examples, recall the explanation for the blue color of the sky by Rayleigh and J.-J. Thomson's formula for the diffusion of X-rays. It will then seem certain that the emission of a secondary wave under the action of the primary wave will represent the statistics of the motion of the diffused quanta in a fashion that is less approximate. However, one must pass from that statistical viewpoint to the individual dynamics of each quantum. When applied to a sheaf of quanta that are in phase with each other, those dynamics must recover the conclusions of the theory of electrons from the viewpoint of the number of diffused quanta, their distribution in space, and the phases that are associated along the trajectories. The action that is exerted by a diffusing center on an atom of radiation belongs to a very different type from the one that we encountered in the dynamics of material points. Indeed, it will depend upon the energy of motion (i.e., its $h\nu$) and the "resonant" frequencies that belong to that diffusing center. The study of that kind of dynamics will undoubtedly open up new perspectives. Charges that are subject to radiation will not be put into motion under its influence, so they will not radiate in the classical way, and yet, *from the statistical viewpoint*, everything happens as if they do. That is a very mysterious problem, and we sense its kinship with the correspondence principle and the "virtual oscillators" of Bohr, Kramers, and their collaborators. Without speaking of the phenomena of emission and absorption in the atom, which raise some very considerable difficulties in their own right, it is still in the realm of diffusion and dispersion that the theories that are presented here will agree with those of the Copenhagen school.

Right now, one thing is certain: The global representation that is provided by the theory of electrons is valid only if the center of diffusion can be supposed to be immobile. In order for that to be true, except for the case in which an exterior force is maintained fixed, its mass must be much greater than the mass $h\nu / c^2$ of the moving quantum. Otherwise, the principle of action and reaction, whose validity seems essentially absolute, would teach us that the center would be subjected to an impulse under each individual diffusion, and at the end of that process, the quantum would have lost a fraction of its energy. The relationship between the frequency and the energy would then lead us to predict a reduction of the frequency by diffusion. That is precisely the beautiful phenomenon whose existence now seems beyond question and which is associated with the name of A.-H. Compton. We shall return to the Compton effect in the next chapter, but we shall insist upon this point: The same relationship exists between diffusion without a change in wave length and Compton diffusion as the one that exists between the motion of an electron around a fixed nucleus and the motion of that electron with "dragging of the nucleus." The two phenomena must take place within the scope of one and the same theory.

**3. Dispersion**. – As one knows, the classical theory of dispersion is deduced from that of diffusion. A homogeneous dispersive medium is considered to contain an enormous number of diffusing particles. If a wave propagates in the medium then the particles will emit secondary waves and the superposition of all the wavelets from the primary wave will explain the manner by which the resultant wave will propagate and the existence of an index that varies with frequency. Without a doubt, one can deduce the macroscopic phenomena of dispersion in the same manner by which one gets an interpretation for the microscopic phenomenon of the diffusion of quanta by a center.

For the moment, I will confine myself to the macroscopic viewpoint. In a medium of index $n$ that is a function of frequency, the classical theory envisions the propagation of homogeneous waves with a phase velocity of $V = c / n$. Moreover, Rayleigh's profound insights have taught us that energy propagates (at least, outside of absorption zones) with a velocity of $\upsilon = d\nu / d(\nu / V)$. In line with our guiding ideas, we must attribute the following significance to the waves of the classical theory: Their phases will give the phases of the waves that are associated with the quanta along the trajectories, and their amplitudes will be proportional to the square root of the mean density of the quanta at each point. The quantum relation, in its quadri-dimensional form:

$$\mathbf{I} = h\,\mathbf{O},$$

will then permit us to deduce the energy and quantity of motion for the light quantum from the classical wave. When it enters the refringent medium, its energy $h\nu$ will not naturally experience any variation, while its quantity of motion will vary from $h\nu / c$ to $h\nu / V = hn\nu / c$. As for the velocity that equals $c$ *in vacuo*, from Hamilton's equations, it must equal $c\, d\nu / d(n\nu)$, in agreement with the theory of group velocity.

The adherents of Fresnel's ideas believe that they have demolished Newton's corpuscular theory by showing experimentally that light propagates less quickly in water than it does in air. Here is their argument:

Consider a light trajectory that goes between a point $A$ that is situated in a medium of uniform index $n_1$ and a point $B$ that is situated in a medium of uniform index $n_2$ . That trajectory is composed of two lines that agree at a point on the separation surface, where they make angles of $\alpha_1$ and $\alpha_2$ with the normal. In the wave conception, one must express Fermat's principle $\delta\int_A^B n\,dt = 0$, which will lead to Descartes's equation $n_1 \sin \alpha_1 = n_2 \sin \alpha_2$ . On the contrary, in the corpuscular theory, one must employ de Maupertuis's principle $\delta \int_A^B g\, dt = 0$. Now, the quantity of motion $g$ is the product of the constant mass of the corpuscle of light with its velocity $\upsilon = c / n$. One infers from the condition $\delta \int_A^B dl / n = 0$ that:

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2 .$$

If medium 1 is a vacuum or air, with index unity, and medium 2 is water then experiments will show that $\alpha_1 > \alpha_2$ .

Therefore, $n_2 > 1$ for the wave theory, and light propagates slower in water than it does in air; the opposite is true for the corpuscular theory. Now, as we have said, experiments favor the former situation.

The flaw in the argument seems obvious. First of all, in principle, it is $V$ that is equal to $c / n$, and not $\upsilon$. It is true that one can consider water to be a less dispersive medium for which $\upsilon = V$, approximately, in the visible spectrum. However, there is another objection: We have no reason to assume that the quantity of motion of the light corpuscle is the product of the same constant by its velocity in both media. Indeed, in vacuo, it has the value $W \upsilon / c^2$, with $\upsilon$ equal or close to $c$. From our fundamental postulates, in a medium of index $n$, it can preserve the same value only if one has:

$$\frac{W}{c^2}\upsilon = \frac{W}{c}\frac{d\nu}{d(n\nu)} = \frac{h}{c}(n\nu), \qquad W = h\nu,$$

which implies the form:

$$n^2 = 1 + \frac{A}{\nu^2}$$

for the dispersion equation.

The integration constant $A$ must be negative, moreover, in order for the velocity $\upsilon$ to be less than $c$, and as a result, that dispersion law will imply a phase velocity that is greater than $c$. In general, it is not applicable under the usual conditions, and in particular, it is not applicable to water in the visible spectrum, because the index will then be greater than unity.

In summary, the only correct general form for de Maupertuis's principle seems to be:

$$\delta \int_A^B \frac{h\nu}{c} n\, dl = 0.$$

That cannot be in conflict with wave optics, since it is identical to Fermat's principle.

**4. Motion of a quantum in a refringent medium.** – Suppose that a refringent medium is homogeneous and isotropic, and its index $n$ varies as a function of frequency according to a known law. The motion of a quantum of light in that medium is rectilinear and uniform. From our hypothesis, the energy, quantity of motion, and velocity are given by the three relations:

$$W = h\nu, \qquad \mathbf{g} = \frac{h\nu}{c}, \qquad \upsilon = c\,\frac{d\nu}{d(n\nu)}.$$

As we just saw in the preceding paragraph, one will not have:

$$\mathbf{g} = \frac{W}{c^2}\mathbf{v},$$

in general, which would be true *in vacuo*.

One can compare the motion of the quantum in the refringent medium to that of an electron along the axis of an electrically-charged hollow cylinder. Indeed, in that case, the electron will possess a uniform rectilinear motion, by reason of symmetry, but its energy and quantity of motion, rather than being related by the relation:

$$\mathbf{g} = \frac{W}{c^2}\mathbf{v},$$

as they are *in vacuo*, will now be related by the relation:

$$\mathbf{g} = \frac{W - P}{c^2}\mathbf{v},$$

in which $P$ represents the potential of the action that the charge of the cylinder exerts upon the electron. If that charge is positive then it will attract the electron, and $P$ will be negative; otherwise, $P$ will be positive.

Return to the quantum in the refringent medium. It will be subjected to actions of an unknown nature as a result of the material particles of the medium, which will be in equilibrium, in the mean, due to the homogeneity of the medium. The motion must then be uniform and rectilinear, but the moving body must be subject to a sort of mean potential $\bar{P}$ that is due to the action of material particles. One might then be tempted to write:

$$\mathbf{g} = \frac{W - \bar{P}}{c^2}\mathbf{v},$$

which amounts to defining $\bar{P}$ by the relation:

$$\bar{P} = W - \frac{c^2 g}{\upsilon}.$$

Now, the magnitudes $W$, $g$, and $\upsilon$ are defined as functions of the wave quantities, and one will get:

$$\bar{P} = h\nu\left(1 - \frac{c^2}{\upsilon V}\right) = h\nu\left(1 - \frac{cn}{\upsilon}\right)$$

$$= h\nu\left(1 - n\frac{d(n\nu)}{d\nu}\right) = h\nu\left(1 - \frac{d(n^2\nu^2)}{d\nu^2}\right).$$

Suppose that the dispersion of the medium obeys the Lorentz relation:

$$n^2 = \sum_i \frac{a_i}{\nu_i^2 - \nu^2} + 1,$$

in which the $a_i$ are constants, and the $\nu_i$ are the resonant frequencies of the material elements. One will easily find that:

$$\frac{d(n^2\nu^2)}{d\nu^2} = 1 + \sum_i \frac{a_i\,\nu_i^2}{(\nu_i^2 - \nu^2)^2},$$

and as a result:

$$\bar{P} = -\,h\nu \sum_i \frac{a_i\,\nu_i^2}{(\nu_i^2 - \nu^2)^2},$$

so $\bar{P}$ will be proportional to $h\nu$ (which is quite natural) and essentially negative. Furthermore, that potential will grow considerably when $\nu$ approaches the critical frequencies $\nu_i$, although one cannot nonetheless say that it will become infinite when $\nu = \nu_i$, because the Lorentz formula, in the form that was written above, would no longer be exact then.

One can then say that, in a certain sense, the quantum is subject to an attractive action that is due to the material particles and whose value is especially large in the neighborhood of the absorption zones. That is one very interesting way of envisioning the question of refringent media ([8]).

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([8]) On that subject, see also, L. BRILLOUIN, "Les tensions de radiations, leur interprétation en Mécanique classique et en Relativité," J. de Phys. (1925), pp. 337.

## CHAPTER XI

# THE COMPTON EFFECT

1. Collision of an electron with a quantum of light. – 2. Collisions between atoms of radiation.

**1. The collision of an electron with a quantum of light.** – We have seen how the Compton Effect can be attached to diffusion with no change in wave length. When the diffusing center responds to the reaction that it exerts upon the quantum, one can say that the two dynamical elements collide, in the broader sense of the word. We do not also know the manner by which we can represent the mutual interaction of those elements in detail, but we do have a guide that has, in short, always proved itself: viz., the principles of conservation of energy and quantity of motion.

In order to obtain a formula that is more general than the one that H. A. Compton originally found, I will consider the collision of a quantum with a moving electron. Take the $x$-axis to be the original direction of motion of a quantum whose initial frequency is $\nu_1$, and let the $y$ and $z$ axes be perpendicular to each other in a plane that is normal to $Ox$ and passes through the point at which the collision occurs. The direction of the velocity $\beta_1 c$ of the electron before the collision is defined by the direction cosines $a_1$, $b_1$, $c_1$, and we let $\theta_1$ denote the angle that it makes with $Ox$ in such a way that $a_1 = \cos\theta_1$. After the collision, the quantum of diffused radiation of frequency $\nu_2$ propagates in a direction whose direction cosines $p$, $q$, $r$ make an angle of $\varphi$ with the initial velocity of the electron:

$$(\cos\varphi = a_1\,p + b_1\,q + c_1\,r),$$

and the angle $\theta$ with $Ox$ ($p = \cos\theta$). Finally, the electron will possess a final velocity $\beta_2 c$ with direction cosines $a_2$, $b_2$, $c_2$.

The conservation principle gives us four equations, namely:

$$h\nu_1 + \frac{m_0\,c^2}{\sqrt{1-\beta_1^2}} = h\nu_2 + \frac{m_0\,c^2}{\sqrt{1-\beta_2^2}},$$

$$\frac{h\nu_1}{c} + \frac{m_0\,\beta_1\,c}{\sqrt{1-\beta_1^2}}\,a_1 = \frac{h\nu_2}{c}\,p + \frac{m_0\,\beta_2\,c}{\sqrt{1-\beta_2^2}}\,a_2,$$

$$\frac{m_0\,\beta_1\,c}{\sqrt{1-\beta_1^2}}\,b_1 = \frac{h\nu_2}{c}\,q + \frac{m_0\,\beta_2\,c}{\sqrt{1-\beta_2^2}}\,b_2,$$

$$\frac{m_0\,\beta_1\,c}{\sqrt{1-\beta_1^2}}\,c_1 \;=\; \frac{h\nu_2}{c}\,r + \frac{m_0\,\beta_2\,c}{\sqrt{1-\beta_2^2}}\,c_2\,.$$

Eliminate $a_2$, $b_2$, $c_2$, and $b_1$; with Compton, set $\frac{h\nu_1}{m_0\,c^2} = \alpha$. One will get:

$$\nu_2 = \nu_1\,\frac{1-\beta_1\cos\theta_1}{1-\beta_1\cos\varphi + 2\alpha\sqrt{1-\beta_1^2}\,\sin^2\dfrac{\theta}{2}}\,.$$

If the electron is found to be at rest at the moment of collision then one will get the usual Compton formula:

$$\nu_2 = \nu_1\,\frac{1}{1+2\alpha\sin^2\dfrac{\theta}{2}}\,,$$

which was confirmed by experiment brilliantly. However, the general formula presents a special interest because it contains both the Compton Effect and the Doppler Effect. In order to see that, set $\alpha = 0$, which amounts to neglecting the Compton Effect. What will remain is:

$$\nu_2 = \nu_1\,\frac{1-\beta_1\cos\theta_1}{1-\beta_1\cos\varphi}\,.$$

It is easy to verify that the change of frequency that is given by the formula is indeed the one that is predicted by the classical theory by considering the electron to be a moving resonator with a constant velocity $\beta_1\,c$. We could have expected that because if the Compton Effect is neglected then the diffusion will take place in a system that is linked with the electron under the action of a *fixed* center, and as a result, with no change in wave length.

One can pose the following question:

*Can there be a collision between a quantum and an electron such that there will be no diffused quantum after the collision, since the atom of radiation has been absorbed by the electron in some way?*

The preceding formulas show directly that this is impossible. Indeed, in the contrary case, one would have to be able to verify the conservation equations after setting $\nu_2 = 0$. Now, from the formula that was obtained, that would imply the condition that $\beta_1\cos\theta_1 = 1$, which is an unrealizable condition, since $\beta_1$ is necessarily less than unity.

The same conclusion can be reached along a different path. Indeed, we can write the conservation relations in an arbitrary Galilean system. We choose a system in which the electron will be at rest after the collision, and write down that there is conservation of energy:

$$h\nu + \frac{m_0 c^2}{\sqrt{1-\beta^2}} = m_0 c^2.$$

Now, that relation cannot be satisfied for any value of $\nu$ and $\beta$, since the left-hand side us necessarily larger than the right-hand side.

Experiments have shown that for a given exciting frequency, the Compton Effect will be less noticeable as the diffusing element becomes heavier. That raises the difficult question of knowing the moment at which the binding energy of an intra-atomic electron is small enough in comparison to that of the incident quantum that one can regard the electron as free. Our knowledge of the nature of intra-atomic coupling does permit us to answer that question.

Along the same line of ideas, one finds an explanation that H. A. Compton gave for an auxiliary phenomenon that was pointed out by Duane and his collaborators (who have rejected the existence itself of the Compton Effect for a long time); that explanation has no place here, since the Duane effect remains highly doubtful.

**2. Collisions between atoms of radiation. –** Let us examine another very curious question: Can two quanta of light exchange energy under a collision; in other words, can their frequencies change as a result of a collision? Experiments have shown nothing similar to that, and such a phenomenon is entirely foreign to classical theories. In the last part of this book, I will show that as a result of the work of Bose and Einstein, as well as my own, it is legitimate to consider radiation to be a gas composed of atoms of light. Some statistical arguments, when applied to our wave concepts, will lead us to Planck's law, but they will not show us how the mechanism by which that distribution of quanta over the various values of energy is found to be realized and maintained. It seems very tempting to suppose that equilibrium results from the exchanges of energy and quantity of motion between quanta due to their mutual interaction, viz., their collisions, in the most general sense of the word.

Perhaps it would be interesting to study that type of collision a little.

It will be easy to solve the general problem of the collision of two quanta of different frequencies as we did for the Compton Effect.

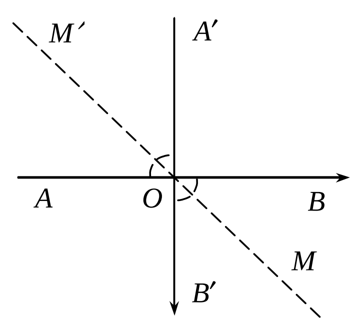


Figure 9.

In order to not increase the number of formulas, I shall content myself to envisioning a very simple case. Suppose that one makes two monochromatic sheaves of the same frequency cross at a right angle with the aid of collimators, although there is not requirement that they should be coherent. Let us represent those two sheaves schematically by straight lines $AB$ and $A'B'$ (Fig. 9) that cross at $O$.

Observe the diffused light at the point $O$ in the direction $OM$ that bisects the angle $\widehat{BOB'}$, if it exists. If one quantum is diffused along $OM$ as a result of a collision then the other quantum will be diffused along $OM'$. Let $\nu$ be the initial frequency, while $\nu_1$ and $\nu_2$ are the frequencies of the quanta that are diffused towards $M$ and $M'$, resp. The conservations equations are written:

$$h\nu_1 + h\nu_2 = 2h\nu,$$

$$\frac{h\nu_1}{c} - \frac{h\nu_2}{c} = 2\frac{h\nu}{c}\cos 45^{\circ};$$

hence:

$$\nu_1 = \nu\,(1 + \cos 45^{\circ}) = 1.7\ \nu,$$
$$\nu_2 = 0.3\ \nu.$$

If the wave length of the sheaf of light that is employed is $\lambda = 0.68\ \mu$ then the observed radiation at $M$ will correspond to $\lambda_1 = 0.68 / 1.7 = 0.4\ \mu$. Upon viewing the point at which two red sheaves cross, one will see violet light. That would be a beautiful phenomenon! I do not know if it exists or if it is observable in this case, but if is detected someday then its place in the set of new theoretical viewpoints on radiation will be quite distinguished.

# PART THREE

## QUANTUM STATISTICS

### CHAPTER XII

## KINETIC THEORY OF GASES

1. Some formulas from statistical thermodynamics. – 2. Maxwell's law. – 3. Free energy and entropy of a perfect gas. – 4. The light gas.

**1. Some formulas from statistical thermodynamics.** – Boltzmann was the first to show that the entropy of a gas in a well-defined state is the product of the logarithm of the probability of that state with a constant $k$ – namely, Boltzmann's constant – that depends upon the choice of temperature scale. As one knows, one will arrive at that conclusion by analyzing the collisions of atoms under the hypothesis of a completely-disorganized state of agitation. Today, as a result of the work of Planck and Einstein, one prefers to consider the relation $S = k \log P$ to be the definition itself of the entropy in the system. However, one must make the meaning of $P$ more precise. The mathematical probability of a given macroscopic state for a system is, by definition, the quotient of the number of different microscopic states that correspond to that macroscopic state by the total number of possible states. It will then be a fractional number, and one cannot identify it with $P$, since entropy is an essentially positive quantity. One agrees to identify $P$ with the numerator of the preceding fraction, and that definition of the "thermodynamic probability," which still contains some degree of arbitrariness, is justified by the exactness of these consequences. We adopt it and infer the analytical expressions for some thermodynamics quantities by a proof that has the advantage of being just as valid when the sequence of possible states is discontinuous as its is in the opposite case. Statistical thermodynamics, for which we shall give the formula, corresponds to the classical viewpoint, moreover, and we shall have to develop another one in the following chapter that is more general.

Consider $\mathcal{N}$ objects that one can distribute arbitrarily among $m$ "cells," whose probabilities are assumed to be all equal *a priori*. A certain macroscopic state of the system is realized by placing $n_1$ objects in cell number 1, $n_2$ objects in cell number 2, etc. If one permutes a certain number of objects that belong to different cells then one will get a macroscopic state that is identical to the previous one, but microscopically different. It will then result that the thermodynamic probability of the global state that is defined by the numbers $n_1$, $n_2$, … is:

$$P = \frac{\mathcal{N}}{n_1!\, n_2! \cdots}.$$

If $\mathcal{N}$ and $n_i$ are large numbers then Stirling's formula will give the entropy as:

$$S = k \log P = k\,\mathcal{N} \log \mathcal{N} - k \sum_{i=1}^{m} n_i \log n_i \,.$$

Each cell corresponds to a value of a certain function $\varepsilon$ that we will call the *energy* of an object that is placed in that cell. Imagine a modification of the distribution of objects among the cells that is restricted by the condition that it must leave the sum of the energy invariant. That modification will vary the entropy by:

$$\delta S = -k\,\delta \left[ \sum_{i=1}^{m} n_i \log n_i \right] = -k\,\delta \sum_{i=1}^{m} n_i - k \sum_{i=1}^{m} \log n_i \,\delta n_i \,,$$

with the adjoint conditions:

$$\sum_{i=1}^{m} \delta n_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{m} \varepsilon_i \,\delta n_i = 0.$$

The condition $\delta S = 0$ will be verified when the entropy is a maximum. By reason of the conditions to which the $\delta n_i$ are subject, one must then have:

$$\sum_{i=1}^{m} [\log n_i + \eta + \beta\,\varepsilon_i]\,\delta n_i = 0,$$

for any $\delta n_i$ . The coefficients $\eta$ and $\beta$ are undetermined coefficients.

The most probable distribution, which is the only one that is observable in practice, is then governed by the law:

$$n_i = \alpha\, e^{-\beta \varepsilon_i}.$$

The entropy of the system that corresponds to that most-probable distribution, which coincides with its thermodynamic entropy, is given by:

$$S = k\,\mathcal{N} \log \mathcal{N} - \sum_{i=1}^{m} [k\alpha\, e^{-\beta\varepsilon_i} (\log n_i - \beta\,\varepsilon_i)],$$

or, since:

$$\sum_{i=1}^{m} n_i = \mathcal{N}, \qquad \sum_{i=1}^{m} \varepsilon_i \, n_i = \text{total energy } E.$$

$$S = k\,\mathcal{N} \log \frac{\mathcal{N}}{\alpha} + k\,\beta E = k\,\mathcal{N} \log \sum_{i=1}^{m} e^{-\beta\varepsilon_i} + k\,\beta E.$$

In order to identify $\beta$ with a thermodynamic quantity, we will employ the classical relation $\frac{1}{T} = \frac{\partial S}{\partial E}$. Hence:

$$\frac{1}{T} = \frac{\partial S}{\partial \beta}\frac{\partial \beta}{\partial E} = -\,k\mathcal{N}\,\frac{\sum_{i=1}^{m}\varepsilon_i\,e^{-\beta\varepsilon_i}}{\sum_{i=1}^{m}e^{-\beta\varepsilon_i}}\frac{d\beta}{dE} + kE\frac{d\beta}{dE} + k\beta = k\beta,$$

$$\beta = \frac{1}{kT}.$$

It is interesting to also calculate the free energy, which is a function whose importance is well-known. One will find that:

$$F = E - TS = -\,k\mathcal{N}\,T\log\sum_{i=1}^{m}e^{-\beta\varepsilon_i}.$$

The mean value of the free energy per object is then:

$$\overline{F} = -\,k\,T\log\sum_{i=1}^{m}e^{-\beta\varepsilon_i}.$$

Now that we are endowed with these general formulas, we can study some special cases.

**2. Maxwell's law.** – Let us study the case of a perfect gas that is enclosed within a container with fixed walls that is removed from all external influences. The $\mathcal{N}$ molecules of the gas will play the role of the objects in the general theory, and one must examine the most probable distribution of them over cells of equal probability. But, how does one choose those cells?

The dynamical state of each molecule is determined by its coordinates $x$, $y$, $z$ and Lagrange momenta $p$, $q$, $r$. Upon choosing the six preceding quantities to be rectangular coordinates, one will construct a space that has been given the name of *extension-in-phase* ever since Gibbs. Liouville's theorem, which is deduced from Hamilton's equations in relativistic mechanics, as in classical mechanics, will permit one to take the elements of the extension-in-phase that are equal to $dx$, $dy$, $dz$, $dp$, $dq$, $dr$ to be the cells of equal probability, with the aid of the ergodic hypothesis. We further remark that these equal elements must not be considered to be infinitely small in the strict sense, since otherwise the $n_i$ would not be very large, and Stirling's formula would not be applicable. Those elements are then very small in comparison to the quantities that are accessible to measurement, but nonetheless contain a very considerable number of molecules.

Consider one of the cells that is determined in that way. Let $w$ denote the *kinetic* energy of a molecule of the cell, let $v_i = \beta_i\, c$ be its velocity, and let $g_i$ be its quantity of motion. One will have:

$$\varepsilon_i = \frac{m_0 c^2}{\sqrt{1-\beta_i^2}} \equiv m_0\, c^2 + w, \qquad g_i = \frac{m_0 v_i}{\sqrt{1-\beta_i^2}}\,.$$

Let $\mathcal{V}$ denote the total volume that is occupied by the gas and look for the number of cells for which the kinetic energy is found between two very close limits $w$ and $w + dw$. That will obviously be:

$$\mathcal{V}\int_w^{w+dw} dp\,dq\,dr = \mathcal{V}\int_w^{w+dw} 4\pi g^2 dg = \mathcal{V}4\pi\, m_0^2\, c\,(1+\alpha)\sqrt{\alpha(\alpha+2)}\,dw$$

up to a constant factor, when one sets $\alpha = w / m_0\, c^2$.

The general formula for the canonical distribution will then show that the number of molecules that are contained in these cells is:

$$n_w\, dw = \text{const.}\, 4\pi\, m_0^2\, c\,(1+\alpha)\sqrt{\alpha(\alpha+2)}\,\mathcal{V}\,dw\,e^{-w/kT}\,.$$

In all of the usual cases, that formula will simplify, since the kinetic energy of the molecules will always be small with respect to the internal energy $m_0\, c^2$, and one can neglect $a$ in comparison to unity. What will then remain will be:

$$n_w\, dw = \text{const.}\, 4\pi\, m_0^{3/2}\sqrt{2w}\;e^{-w/kT}\,\mathcal{V}\,dw\,.$$

The constant is determined upon writing down that the total number of molecules is $\mathcal{N}$. One will then get Maxwell's celebrated law:

$$n_w\, dw = \frac{4\pi\mathcal{N}}{(2\pi kT)^{3/2}}\,e^{-w/kT}\sqrt{2w}\,dw\,.$$

**3. Free energy and entropy of a perfect gas.** – The general formula:

$$F = -\,k\,\mathcal{N}\,T\log\sum_{i=1}^{m} e^{-\varepsilon_i/kT}$$

will, in principle, permit one to calculate the free energy when the energy and entropy are deduced from the classical relations:

$$S = -\,\frac{\partial F}{\partial T}\,, \qquad E = F - TS.$$

However, the calculation of the sum Σ demands certain precautions that would be pointless in the deduction of Maxwell's law. First of all, one must introduce the finite value $\omega$ of the elementary domains in the extension-in-phase by taking the expression:

$$\frac{\mathcal{V}}{\omega}\int_{w}^{w+dw} dp\,dq\,dr$$

for the number of energy cells that are found between $w$ and $w + dw$.

However, that will not suffice. One will be tempted to set:

$$\sum_{i=1}^{m} e^{-\varepsilon_i/kT} = \int_{-\infty}^{+\infty} e^{-\varepsilon/kT}\,\frac{\mathcal{V}}{\omega}\,dp\,dq\,dr\ .$$

Now, as Planck has shown, that result would not be exact. In order to arrive at the correct formulas, one must appeal to an argument that has since been criticized by Ehrenfest. Here are the broad points of that argument:

Imagine $N$ exemplars of the same gas, and represent the state of one of the exemplars by a point in a $6\mathcal{N}$-dimensional space. The thermodynamic free energy of the perfect must be the mean value of the free energies of the $N$ gases – i.e., it is given by the formula:

$$\bar{F} = -\,k\,T\log\sum_{i=1}^{m} e^{-\varepsilon_i/kT}\ ,$$

when applied to the set of $N$ gases.

The integral of $e^{-\varepsilon_i/kT}$, when it is extended over all of $6\mathcal{N}$-dimensional space, will be equal to the product of $\mathcal{N}$ sextuples of integrals in six-dimensional space. If one confines oneself to writing:

$$\sum_{i=1}^{m} e^{-\varepsilon_i/kT} = \left[\frac{1}{\omega}\int_{-\infty}^{+\infty} e^{-\varepsilon/kT}\,dx\,dy\,dz\,dp\,dq\,dr\right]^{\mathcal{N}}$$

then one will get back to the imprecise formula. However (and this is a very delicate point in Planck's argument), if one exchanges the coordinates $x, y, z, p, q, r$ of the two gas molecules then one will get a new representative point in $6\mathcal{N}$-dimensional space, whereas by reason of the absolute identity of the molecules, the two states will be indiscernible, even from the microscopic viewpoint. Since the number of possible permutations of the molecules is $\mathcal{N}!$, the result that is obtained upon extending the integration over the entire $6\mathcal{N}$-dimensional space must be divided by $\mathcal{N}!$, and Planck thus obtained:

$$\sum_{i=1}^{m} e^{-\varepsilon_i/kT} = \frac{1}{\mathcal{N}\,!}\left[\int_{-\infty}^{+\infty} e^{-\varepsilon/kT}\,\frac{dx\,dy\,dz\,dp\,dq\,dr}{\omega}\right]^{\mathcal{N}}$$

$$= \left[ \frac{e}{\mathcal{N}} \int_{-\infty}^{+\infty} e^{-\varepsilon/kT} \, \frac{dx\,dy\,dz\,dp\,dq\,dr}{\omega} \right]^{\mathcal{N}} .$$

Planck's argument is not entirely convincing, but the result is surely exact. We shall adopt that result for the moment, although in the following chapter we shall demand to know whether a simpler interpretation does not exist. We effortlessly obtain:

$$F = -\,k\mathcal{N}T \log \left[ \frac{e}{\mathcal{N}} \int_{-\infty}^{+\infty} e^{-\varepsilon/kT} \, \frac{dx\,dy\,dz\,dp\,dq\,dr}{\omega} \right]^{\mathcal{N}}$$

$$= \mathcal{N} m_0\, c^2 - k\,\mathcal{N}T \log \left[ \frac{e\mathcal{V}}{\mathcal{N}\omega} (2\pi m_0 kT)^{3/2} \right],$$

$$S = -\,\frac{\partial F}{\partial T} = k\,\mathcal{N} \log \left[ \frac{e^{5/2}\mathcal{V}}{\mathcal{N}\omega} (2\pi m_0 kT)^{3/2} \right],$$

$$E = F + TS = \mathcal{N} m_0\, c^2 + \frac{3}{2}\mathcal{N}kT.$$

These expressions contain no undetermined constants, except for the quantity $\omega$, whose dimensions are those of the cube of an action. Adopting the simplest hypothesis, Planck set $\omega = h^3$. The formulas that are obtained in that way have been confirmed by measuring the constants that enter into the equilibrium of a gas in its condensed phase (i.e., chemically constant).

**4. The gas of light. –** The corpuscular theory of radiation will necessarily lead one to consider blackbody radiation in equilibrium at a temperature $T$ to be a true gas of atoms of light. We have already seen that this hypothesis explains the radiation pressure that the enclosure experiences quantitatively.

According to our usual custom, we attribute an extraordinarily small proper mass to the quantum of light, since the atoms of blackbody radiation will have speeds that are extremely close to $c$. Nevertheless, the interval of speeds $c - \varepsilon$ to $c$ will correspond to all values of energy from zero to essentially infinity. In order to obtain the distribution of energies, we take the general formula that gives $n_w\,dw$ and neglect unity in comparison to $\alpha$, which is very large. Finally, we remark that the kinetic energy $w$ of an atom of radiation is approximately equal to its total energy $h\nu$. Therefore, one will have:

$$n_w\,dw = n_\nu\,d\nu = \text{const.}\ \frac{4\pi}{c^3} h^2\nu^2 e^{-h\nu/kT}\,\mathcal{V}\,h\,d\nu.$$

The quantity of energy per unit volume that corresponds to the interval of frequencies $\nu$, $\nu + d\nu$ will then be:

$$n_\nu\, d\nu = \frac{n_\nu d\nu}{\mathcal{V}} h\nu = \text{const.}\ \frac{4\pi}{c^2} h^4 \nu^3 e^{-h\nu/kT} d\nu\ .$$

One will recognize immediately that we have arrived at Wien's law of radiation here, which has been verified by experiment for large values of $h\nu / kT$.

The calculation of the free energy will provide us with the exact value of the numerical coefficient. But that brings with it a new minor complication! Unlike the ideal molecule in the theory of gases, the atom of radiation is not isotropic; one must take its polarization into account. The oscillating vector that is coupled with the quantum is always normal to its velocity, but its orientation around the direction of motion is arbitrary. We can draw a vertical plane $\pi$ through the position of each quantum that contains its velocity and another one $\pi'$ that also contain it and is normal to $\pi$. If we suppose that each of $\mathcal{N}$ atoms of light is polarized in the plane $\pi$ then we will establish a particular property that favors the vertical directions, which is incompatible with the isotropy of blackbody radiation, whereas we will get a *statistically* exact representation upon supposing that $\mathcal{N} / 2$ are polarized in their $\pi$-plane, while $\mathcal{N} / 2$ are polarized in their $\pi'$-plane. That device, whose meaning will become clearer in the next chapter, will permit us to treat each of the two groups of $\mathcal{N} / 2$ atoms as if they were molecules with spherical symmetry, because the dynamical state of each of their constituents is determined uniquely by its coordinates and momenta. Since free energies are additive quantities, one will get:

$$\begin{aligned} F &= -\,2k\,T \log \frac{1}{(\mathcal{N}/2)!}\left[\int e^{-\varepsilon/kT}\, \frac{dx\,dy\,dz\,dp\,dq\,dr}{\omega}\right]^{\mathcal{N}/2} \\ &= -\,\mathcal{N}\,k\,T \log \left[\frac{2e}{\mathcal{N}}\int e^{-\varepsilon/kT}\, \frac{dx\,dy\,dz\,dp\,dq\,dr}{\omega}\right] \\ &= -\,\mathcal{N}\,k\,T \log \left[\frac{8\pi e h^3}{\mathcal{N}\omega c^3}\int e^{-h\nu/kT}\mathcal{V}\nu^2 d\nu\right] \\ &= -\,\mathcal{N}\,k\,T \log \left[\frac{8\pi e \mathcal{V}}{\mathcal{N}\omega c^3}\, 2k^3 T^3\right]. \end{aligned}$$

Now, it will suffice to cast one's eyes on the expression for $n_w\, dw$ in order to see that $\mathcal{N}$ is necessarily proportional to $T^3$. Then set:

$$\mathcal{N} = A\, k^3\, T^3.$$

On the other hand, one will also see that the total energy:

$$E = \int_0^{+\infty} \mathcal{V} u_\nu\, d\nu \ = \mathcal{N}\, 3kT = 3A\, k^4\, T^4.$$

One will then find the Stefan-Boltzmann law, which one can write $E = \sigma T^4 \mathcal{V}$, upon setting $\sigma = 3A\,k^4 / \mathcal{V}$. A purely-classical thermodynamic argument leads to the well-known formulas:

$$\log\left[\frac{16\pi e\mathcal{V}}{\mathcal{N}\omega c^3}k^3T^3\right] = 1$$

or

$$\mathcal{N} = \frac{16\pi}{\omega c^3}k^3\,T^3\,\mathcal{V}.$$

That determines the constant in the expression for $n_w\,dw$. One will easily find that:

$$n_w\,dw = \frac{8\pi h^3\mathcal{V}}{c^3\,\omega}e^{-h\nu/kT}\nu^2 d\nu\,,$$

and as a result, one will find:

$$n_\nu\,d\nu = \frac{8\pi h^4}{c^3\,\omega}e^{-h\nu/kT}\nu^2 d\nu$$

for the energy density.

It suffices to set $\omega = h^3$ to get Wien's law in its entirety.

We have arrived at some interesting results by purely dynamical methods then. However, if we wish to explain the properties of radiation as we would in the theory of gases, properly speaking, then Planck's hypothesis would be justified only by its success. On the other hand, we have only been able to prove Wien's law, and not Planck's law in its entirely. The introduction of wave concepts will make those difficulties disappear.

## CHAPTER XIII

# THE NEW THEORY OF GASES

1. Introduction of waves associated with atoms. – 2. New statistical formulas. – 3. Passing from Wien's law to Rayleigh's law.

**1. Introduction of waves associated with atoms. –** We shall seek to complete the theory of gases by introducing our fundamental notion that the motion of any atom is associated with the propagation of a wave. If that were true then the vessel that contains the gas could be considered to be traversed in every direction by plane waves of all frequencies, and those plane waves can even be considered to be homogeneous because we have seen that if one confines oneself to studying an ensemble of material points without being preoccupied with the structure of the waves that are associated with each of them then it will suffice to keep the phase factor. Hence, both a material gas and blackbody radiation can be imagined to be ensembles of corpuscles in totally uncoordinated motion or ensembles of waves of all frequencies with an isotropic distribution. However, upon choosing the wave picture, we shall be able to specify and complete the results that the corpuscular picture provided us with in the preceding chapter.

First of all, as in the Jeans theory of radiation, we will be led quite naturally to consider the waves that are in resonance with the dimensions of the container to be the only stable ones, and therefore, the only physically-observable ones. We will recover an idea here that is completely analogous to the one from which we inferred an interpretation for the stability conditions. Here, as in the Bohr atom, there is a discontinuous sequence of possible motions, while the other ones were eliminated by a mechanism that is impossible to specify. In the three-dimensional extension-in-momenta, not all positions are therefore possible as representative points for the molecules. Furthermore, the possible positions form a very dense set of isolated points.

Thanks to that "arithmeticization" of the extension-in-phase, the problem of determining the cells can be solved immediately: The representative points must be distributed among the various permissible positions. In other words, the Jeans stationary waves play the role of the cells among which the atoms are distributed.

As in the usual kinetic theory, we must above all preoccupy ourselves with the number of cells that correspond to the values of energy that are found between $\varepsilon$ and $\varepsilon + d\varepsilon$, which represents a very small quantity – i.e., a physical infinitesimal. The total energy $\varepsilon$ of an atom is related to the frequency of the associated wave by the quantum relation, and the energy interval considered will correspond to the frequency interval $\varepsilon / h$, $\varepsilon / h + d\varepsilon / h$. A well-known argument, which Jeans was the first to propose, gives the following value for the number $\varepsilon_\nu \, d\nu$ of stationary waves that are found between $\nu$ and $\nu + d\nu$ and are in resonance with a container of volume $\mathcal{V}$:

$$z_\nu \, d\nu = \gamma \frac{4\pi}{UV^2} \nu^2 \, d\nu \, \mathcal{V}.$$

$\gamma$ is equal to 1 if the waves are longitudinal and 2 if they are transverse. As always, $U$ and $V$ are the group and phase velocities, resp. For the proof of that formula, the reader would do well to refer to pages 33 *et seq.* of the book on quantum theory by Léon Brillouin.

Let us see what the Jeans formula will give when one applies it to the motion of atoms. For an atom in rectilinear motion with a velocity $\beta c$, one will have:

$$U = \beta c, \qquad V = \frac{c}{\beta}, \qquad \varepsilon = h\nu = m_0 c^2 + w = m_0 c^2 (1 + \alpha) \qquad \left(\alpha = \frac{w}{m_0 c^2}\right)$$

so

$$z_\varepsilon \, d\varepsilon = z_w \, dw = z_\nu \, d\nu = \gamma \frac{4\pi}{h^3} m_0^2 \, c \, (1+\alpha)\sqrt{\alpha(\alpha+2)} \, dw \, \mathcal{V} .$$

Now, in the last chapter, when we calculated the portion of the extension-in-phase that corresponded to the interval $dw$, we found:

$$\mathcal{V} \, 4\pi m_0^2 \, c \, (1+\alpha)\sqrt{\alpha(\alpha+2)} \, dw \ .$$

We conclude the following relation from that:

$$z_\varepsilon \, d\varepsilon = z_w \, dw = \frac{\gamma}{h^3} \int_w^{w+dw} dx \, dy \, dz \, dp \, dq \, dr \ .$$

For the spherically-symmetric atoms of an ideal gas, the associated wave will be longitudinal, and $\gamma$ must be equal to 1. One will then see that upon assuming the Gibbs viewpoint, everything will happen as if the extension-in-phase were divided into cells of magnitude $h^3$. That is Planck's postulate, and it has now been justified.

**2. New statistical formulas.** – In my doctoral thesis, I showed how my ideas would lead one to recover Planck's law for a gas of light. Almost simultaneously, as a result of the work of Bose ([9]) and Einstein ([10]), a new statistics has been founded that is in profound agreement with my concepts. I would like to follow the path that Einstein indicated.

When we previously evaluated the probability of a distribution of objects among cells, we assumed that the number of objects that were contained in any of those cells could be considered to be very large. That is what allowed us to use Stirling's formula.

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([9]) S. N. BOSE, Zeit. Phys. **27** (1924), pp. 384.
([10]) A. EINSTEIN, Berl. Acad. (1924), pp. 261; *ibid.* (1925), 3.

Now, nothing justifies that hypothesis. In reality, it leads to some difficulties, and because of them, Planck had to support his calculations of the free energy – which were exact, moreover – with a fragile argument.

Here, we shall confine ourselves to supposing that the numbers $z_\varepsilon\, d\varepsilon$ and $n_\varepsilon\, d\varepsilon$ of cells and atoms that correspond to a very small interval $d\varepsilon$, resp., are very large, without assuming in the process that the number of atoms per cell is very large.

Before all else, one must calculate the number of possible distributions of $n_\varepsilon\, d\varepsilon$ atoms among $z_\varepsilon\, d\varepsilon$ cells. It is given by a formula that was employed by Planck in his original theory of blackbody radiation and is equal to:

$$\frac{(n_\varepsilon d\varepsilon + z_\varepsilon d\varepsilon)!}{(n_\varepsilon d\varepsilon)!(z_\varepsilon d\varepsilon)!}.$$

The total number $P$ of microscopic states of the gas such that there are $n_\varepsilon\, d\varepsilon$ atoms in the interval $\varepsilon$, $\varepsilon + d\varepsilon$ will then be:

$$P = \Pi \frac{(n_\varepsilon d\varepsilon + z_\varepsilon d\varepsilon)!}{(n_\varepsilon d\varepsilon)!(z_\varepsilon d\varepsilon)!}.$$

By definition, the entropy of the corresponding global state is:

$$S = k \log P = k\int_0^{+\infty} [(n_\varepsilon + z_\varepsilon)\log(n_\varepsilon + z_\varepsilon) - n_\varepsilon \log n_\varepsilon - z_\varepsilon \log z_\varepsilon)]\, dz .$$

The most probable distribution is always obtained by writing $\delta S = 0$ and taking into account the two conditions:

$$(A) \qquad \int_0^{+\infty} n_\varepsilon d\varepsilon = \mathcal{N}, \qquad \int_0^{+\infty} \varepsilon\, n_\varepsilon d\varepsilon = E,$$

in which $\mathcal{N}$ is the total number of atoms in the gas and $E$ is its total energy, which are quantities that must remain constant. One finds the definition of the most probable distribution by the usual methods:

$$n_\varepsilon = \frac{z_\varepsilon}{e^{a+bz} - 1},$$

in which $a$ and $b$ are constants that are determined by the conditions ($A$).

When that distribution is realized, the usual thermodynamic entropy will be consequently equal to:

$$\begin{aligned} S &= k\int_0^{+\infty} [(n_\varepsilon + z_\varepsilon)\log(n_\varepsilon\, e^{a+bz}) - n_\varepsilon \log n_\varepsilon - z_\varepsilon \log z_\varepsilon)]\, dz \\ &= k\int_0^{+\infty} [(a+bz)n_\varepsilon - z_\varepsilon \log(1 - e^{-(a+bz)})]\, dz \\ &= k\left[\mathcal{N}a + bE - \int_0^{+\infty} z_\varepsilon \log(1 - e^{-(a+bz)})]\, d\varepsilon\right]. \end{aligned}$$

It is easy for us to relate the constant $b$ to the temperature. It always suffices to write:

$$\frac{1}{T} = \frac{\partial S}{\partial b}\frac{db}{dE} + \frac{\partial S}{\partial E} = kE\frac{db}{dE} + kb - k\frac{db}{dE}\int_0^{+\infty}\frac{z_\varepsilon \varepsilon}{1-e^{-(a+bz)}}e^{-(a+bz)}\,d\varepsilon = kb,$$

or

$$b = \frac{1}{kT}.$$

The main applications of our formulas must be, once more, the case of the perfect gas, in the usual sense, and that of the gas of light.

For the usual gas, the total energy of the molecule $\varepsilon = m_0\,c^2 + w$ is always very large with respect to $k\,T$. The number of atoms whose kinetic energy is found between $w$ and $w + dw$ is then:

$$n_w\,dw = \text{const. } z_w\;e^{-w/kT}\,dw,$$

and one will easily recover Maxwell's law upon replacing $z_w$ with its value.

In order to get the free energy, start with its definition:

$$F = E - TS = -\,a\,\mathcal{N}kT + kT\int_0^{+\infty} z_\varepsilon \log[1-e^{-(a+b\varepsilon)}]\,d\varepsilon\,.$$

We have a quantity that is very close to unity under the log sign, which will give us the authority to develop it into a series. That will give:

$$F = -\;a\,\mathcal{N}kT - kT\int_0^{+\infty} z_\varepsilon e^{-(a+b\varepsilon)}\,d\varepsilon = -\,\mathcal{N}kT\,(a+1).$$

Now, the constant $a$ satisfies the condition:

$$\mathcal{N} = \int_0^{+\infty} z_\varepsilon e^{-(a+b\varepsilon)}\,d\varepsilon\,,$$

so

$$a = \log\left[\frac{1}{\mathcal{N}}\int_0^{+\infty} z_\varepsilon e^{-b\varepsilon}\,d\varepsilon\right]$$

and

$$F = -\,\mathcal{N}\,k\,T\,(1+a) = -\,\mathcal{N}\,k\,T\log\left[\frac{e}{\mathcal{N}}\int_0^{+\infty} z_\varepsilon e^{-b\varepsilon}\,d\varepsilon\right]$$

$$= -\,\mathcal{N}\,k\,T\log\left[\frac{e}{\mathcal{N}}\int_{-\infty}^{+\infty} e^{-b\varepsilon}\,\frac{dx\,dy\,dz\,dp\,dq\,dr}{h^3}\right].$$

The expression for the free energy that was given by Planck is also obtained with no difficulty.

If remains for us to apply the general formula to the gas of light. However, radiation in equilibrium with matter can be compared to a gas in the presence of its condensed phase, and the number $\mathcal{N}$ of atoms present is not present *a priori*. That results in the necessity of suppressing the first equation ($A$) and setting $a = 0$.

The number of atoms of radiation whose frequency is found between $\nu$ and $\nu + d\nu$ is:

$$n_\nu \, d\nu = \frac{1}{e^{h\nu/kT} - 1} \times z_\nu \, d\nu = \frac{8\pi}{c^3} \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1} \mathcal{V},$$

because the waves are transverse here, and $\gamma = 2$.

The density of the radiant energy then obeys the Planck law:

$$u_\nu \, d\nu = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu,$$

and the synthesis of the theory of blackbody radiation with the theory of gases is found to have been realized completely.

It would be easy to calculate some thermodynamics quantities for blackbody radiation, but it would pointless to increase the number of formulas.

**3. Passing from Wien's law to Rayleigh's law.** – Planck's general formula is, as one knows, capable of taking on two particularly interesting degenerate forms. If the quotient $h\nu / kT$ is very large then one will get the law that was proposed by Wien, which will then be valid in a certain domain:

$$u_\nu \, d\nu = \frac{8\pi h \nu^3}{c^3} e^{-h\nu/kT} \, d\nu .$$

On the contrary, if $h\nu / kT$ is very small then one will revert to the formula that Lord Rayleigh deduced from classical concepts:

$$u_\nu \, d\nu = \frac{8\pi k \nu^2 T}{c^3} \, d\nu .$$

Now, Wien's law is nothing but another form of Maxwell's law, as well as Rayleigh's, when one appeals to the notion of homogeneous waves. Planck's formula then implies both the corpuscular and wave nature of radiation. Since the day when it was discovered, a synthesis of dynamics and optics has become necessary.

Statistical equilibrium of the gas of light is realized by the interaction of atoms of radiation. When the number of atoms that is carried by each stationary wave is small (i.e., $h\nu / kT$ is large), the theory of homogeneous waves cannot give an exact representation of those interactions; the older kinetic viewpoint will, however, and one will have Wien's law. On the contrary, if one has a very large number of atoms on each stationary wave in the mean then the theory of homogeneous waves will indeed represent the interactions, and one will get Rayleigh's law.

The study of energy fluctuations leads to the same conclusions. Consider a small element of volume $\tau$ in the vessel that is filled with radiation. The mean value of the energy that is contained in $\tau$ and corresponds to the frequency interval $d\nu$ is obviously $\tau u_\nu d\nu$. However, at each instant, the real value of that energy is subtracted from the mean value of that variable quantity $\varepsilon$. A very general thermodynamic argument will show that the mean-square of $\varepsilon$ will be equal to:

$$\overline{\varepsilon^2} = k\,T^2 \frac{\partial}{\partial T}(u_\nu\,\tau\,d\nu).$$

In the domain of validity of Wien's law, one will get:

$$\overline{\varepsilon^2} = h\nu\,(u_\nu\,\tau\,d\nu).$$

Now, that formula can be recovered by assuming the classical kinetic viewpoint uniquely. Indeed, if an ensemble of points is distributed randomly in a volume that is divided into small elements that are equal to $\tau$ in such a way that the mean number of points per element is $n$ then the calculus of probabilities will show that the mean value of the square of the deviation $\delta n$ will be:

$$\overline{\delta n^2} = n.$$

Here, one has:

$$n = \frac{\tau\,u_\nu\,d\nu}{h\nu}, \qquad \varepsilon^2 = \delta n^2\,(h\,\nu)^2,$$

and as a result:

$$\overline{\varepsilon^2} = \overline{\delta n^2}\,h^2\nu^2 = h\nu\,(u_\nu\,\tau\,d\nu).$$

If one passes to the other limiting case, namely, Rayleigh's law then one will find that:

$$\overline{\varepsilon^2} = \frac{8\pi k^2}{c^3}\,\nu^2\,T^2\,\tau\,d\nu = \frac{c^3}{8\pi\nu^2 d\nu}\,\frac{(\tau\,u_\nu\,d\nu)^2}{\tau}.$$

Calculating the interference between the homogeneous electromagnetic waves of the old theory will likewise lead to that formula.

Those results confirm our conclusions and clearly show the following result: As a result of the interactions of quanta, the hypothesis of the incoherence of atomic motions that one assumes to be the basis for the old kinetic theory will cease to be exact when several quanta are associated with the same wave. It was that viewpoint that I developed in the presentation of my thesis.

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**CHAPTER XIV**

# COLLISION PROBABILITIES IN A GAS

1. Einstein's proof of Planck's law. – 2. Collision probabilities.

**1. Einstein's proof of Planck's law** (1917) ([11]). – The work that I summarized in the last chapter is all attached to a fundamental paper by Einstein in which one finds a proof of Planck's law that is founded upon only some general statistical considerations, combined with Bohr's frequency relations.

I shall first recall the principle of that proof. Atoms are systems that are capable of taking on a discontinuous sequence of energetic states, and passing from one state to another always happens by way of the absorption or emission of radiation. Naturally, the energy emitted or absorbed will be difference between the energy that is contained in the atom before and after that transformation, and the corresponding frequency of that radiation is equal to that energy divided by $h$. In other words, any radiation that is emitted or absorbed is equal to $h\nu$ as a quantity of energy, which is indeed consistent with our current corpuscular theories. With Einstein, consider an ensemble of atoms of the same type that is found in a container that is maintained at a uniform temperature $T$, and as a result, it is full of blackbody radiation that is consistent with that temperature. Among those atoms, at each instant, there exists a very large number $N_i$ that are found in a state of total energy $\varepsilon_i$ , and another very large number $N_j$ that are found in a state of total energy $\varepsilon_j$ .

During each unit of time, a certain number of atoms passes from the state $\varepsilon_i$ to the state $\varepsilon_j$ and conversely. One can say that there is a certain probability for the passage from the state $i$ to the state $j$ per unit time, and the problem is now to evaluate those probabilities.

Suppose that $\varepsilon_i < \varepsilon_j$ . The passage $i \to j$ can be produced only with the absorption of the frequency $\nu = (\varepsilon_j - \varepsilon_i) / h$, so it is natural to make the corresponding probability proportional to the density $\rho(\nu)$ of surrounding blackbody radiation. By contrast, the passage $j \to i$ can be produced spontaneously, and the corresponding probability can reduce to a constant that is characteristic of the atom itself: Einstein added a term that was proportional to $\rho(\nu)$ to that constant, which is a term that he described as negative absorption, and we shall see how that term can be interpreted later on.

In order to obtain Planck's law, it will suffice to write down that the number of transitions $i \to j$ is equal to that of the transitions $j \to i$, which will give:

$$N_i\, B_1\, \rho(\nu) = N_j\, [A + B\, \rho(\nu)] \, .$$

If one assumes the law of the canonical distribution then one must have:

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([11]) Phys. Zeit. **18** (1917), pp. 121.

$$\frac{N_i}{N_j} = \frac{e^{-\varepsilon_i/kT}}{e^{-\varepsilon_j/kT}} = e^{h\nu/kT}.$$

Now, that exponential will coincide with unity for a very large value of $T$, and on the other hand, the term $B\,\rho(\nu)$ will become much more important than $A$. The condition of equilibrium itself then demands the equality of the constants $B$ and $B_i$ . Hence, one has the relation:

$$\rho(\nu) = \frac{A/B}{e^{h\nu/kT}-1}.$$

In order to obtain the desired result, it will suffice to set $\frac{A}{B} = \frac{8\pi h\nu^3}{c^3}$.

It is important to make a remark at this point: As several authors have pointed out – and notably Lewis – the way that one has written down the equation of equilibrium implies a new sort of principle. Indeed, the usual notion of thermodynamic equilibrium does not demand that the transitions $i \rightarrow j$ during any unit of time must be equal in number to the transitions $j \rightarrow i$. It demands only that the numbers $N_i$ and $N_j$ should be constant, and thanks to a cyclic process that involves other states than the states $i$ and $j$, one can satisfy the second condition without satisfying the first one. One can materialize that idea by means of a concrete example. Let an electrical conductor be traversed by an electric current, and attribute the phenomenon of the current to the displacement of free electrons with respect to the atoms, which are assumed to be fixed. Two portions of the conductor will exchange electrons, with preference given to the exchange in a certain direction (since there is a current), and meanwhile the number of electrons in each portion will remain constant since the conductor is not electrostatically charged.

For my own part, I am tempted to believe, with Lewis, that there is good reason to adopt this new equilibrium principle for every reversible process, in particular, and to then specify the notion of thermodynamic equilibrium. I do not see the one objection to Einstein's argument.

The term "negative absorption" has been interpreted by the correspondence principle.

When one studies the motion of a Planck resonator that is embedded in blackbody radiation in a classical fashion, one will find that certain waves of that radiation will accelerate the motion of the resonator (i.e., positive absorption), while others will retard it (i.e., negative absorption). The sign of the absorption depends upon the phase difference between the motion of the resonator and the wave considered. In total, the positive absorption will prevail over the negative, and there will be a small resultant absorption that equilibrates the radiation by acceleration exactly. That will justify not only the introduction of the term in $B\,\rho(\nu)$, but also the value $8\pi h\nu^3/c^3$ that one attributes to $A$ / $B$. Those results, which have been extended in various ways, are very interesting, and must not be lost from view. Nevertheless, since they are not further attached to the concepts of the present study, and the true meaning of the correspondence principle seems even more obscure to me, moreover, I shall not insist upon them.

**2. Collision probabilities.** – After the discovery of the Compton Effect, Pauli, Jr. ([12]) posed the following question: If one imagines a vessel that is filled with electrons and blackbody radiation then what must the probability of collision between quanta and electrons be in order that those collisions should not perturb the thermodynamic equilibrium?

He arrived at a very unpredicted result: The probability of a Compton collision must depend, to a certain degree, upon the result of that collision, or more precisely, it must depend upon the density of quanta whose energy is equal to that of the quantum once it has been diffused. Einstein and Ehrenfest ([13]) showed the provenance of that statement with the argument that was analyzed above that led to Planck's law. Finally, quite recently, Jordan ([14]) generalized the result of Pauli, Jr., by attaching it to the new statistics of Einstein and the author.

Without following the methods of the preceding work in detail, I shall only make a summary presentation that will allow one to rapidly understand the nature of the question.

Suppose one has two gases in the most general sense of the word; whether they are material gases or blackbody radiation is of no importance. Each gas corresponds to a three-dimensional extension-in-momenta, and the possible positions of the representative points of an atom will define a network of isolated points in each of those extensions.

Let $P_1$, $Q_1$, … denote the point-cells of the first gas, and let $n_{P_1}$, $n_{Q_1}$, … and $\varepsilon_{P_1}$, $\varepsilon_{Q_1}$, … denote the numbers of atoms and the values of their corresponding energies, resp. Use similar notations for the second gas, but substitute the index 2 for the index 1.

We say *collisions of the direct kind* to mean the ones that make a representative point of the first gas pass from the cell $P_1$ to the cell $Q_1$, and simultaneously make a representative point of the second gas pass from the cell $P_2$ to the cell $Q_2$. Since collisions conserve total energy, one must have:

$$\varepsilon_{P_1} + \varepsilon_{P_2} = \varepsilon_{Q_1} + \varepsilon_{Q_2} .$$

The number of collisions between the $n_{P_1}$ atoms in the cell $P_1$ and the $n_{P_2}$ atoms in the cell $P_2$ that are produced per unit time is proportional to the product $n_{P_1} n_{P_2}$, and among them, a certain proportion of them $\alpha \, n_{P_1} n_{P_2}$ will belong to the direct kind. Similarly, the number of collisions of the opposite kind per second is $\alpha' n_{Q_1} n_{Q_2}$. With the same reservations as in paragraph **1**, equilibrium will be expressed by the relation:

$$\alpha \, n_{P_1} n_{P_2} = \alpha' n_{Q_1} n_{Q_2} .$$

The older statistics sets:

$$n_{P_1} = C_1 e^{-\varepsilon_{P_1}/kT} , \qquad n_{P_2} = C_2 \, e^{-\varepsilon_{P_2}/kT} , \qquad \ldots$$

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([12]) W. PAULI, Jr., Zeit. Phys. **18** (1923), pp. 272; *ibid.* **22** (1924), pp. 261.
([13]) A. EINSTEIN and P. EHRENFEST, Zeit. Phys. **19** (1923), pp. 301.
([14]) P. JORDAN, Zeit. Phys. **33** (1925), pp. 649.

Furthermore, due to the conservation of energy, equilibrium will be realized when $\alpha = \alpha'$. That is precisely the result of Boltzmann's classical analysis of collisions in kinetic theory.

Things will no longer be like that when one adopts Einstein's statistics. One must then set:

$$n_{P_1} = \frac{1}{e^{\alpha_1 + \varepsilon_{P_1}/kT} - 1}, \qquad n_{P_2} = \frac{1}{e^{\alpha_2 + \varepsilon_{P_2}/kT} - 1}, \qquad \dots$$

The equation of equilibrium will then imply the condition:

$$\frac{\alpha'}{\alpha} = \frac{n_{P_1}\, n_{P_2}}{n_{Q_1}\, n_{Q_2}} = \frac{(1+n_{P_1})(1+n_{P_2})}{(1+n_{Q_1})(1+n_{Q_2})}.$$

It is easy to account for the fact that the only acceptable way to satisfy that relation is to write:

$$\alpha = C(1+n_{Q_1})(1+n_{Q_2}), \qquad \alpha' = C\,(1+n_{P_1})(1+n_{P_2}).$$

That is Jordan's general result for the case of two gases, and its extension to the case of more than two gases is immediate.

We shall show the expression that was obtained for the collision probability will account for the earlier results of Einstein and Pauli, Jr.

In the case of a material gas, the numbers $n$ will be small compared to unity, which must say that the number of cells that are occupied at a given instant is small (at least, under the usual conditions). Therefore, if gas 1 is a material gas of atoms or electrons and gas 2 is blackbody radiation then the equilibrium will reduce to:

$$n_{P_1} n_{P_2} (1+n_{Q_2}) = n_{Q_1} n_{Q_2} (1+n_{P_2}).$$

Now, for blackbody radiation one will have:

$$n_{P_2} = \frac{1}{e^{h\nu/kT} - 1}, \qquad n_{Q_2} = \frac{1}{e^{h\nu'/kT} - 1}, \qquad \dots,$$

if $\nu$ and $\nu'$ denote the frequencies of the quantum before and after the collision of the first kind, resp. Upon multiplying the two sides of the equilibrium equation by the same factor, one will recover the expression for the direct collision probability:

$$\text{const. } n_{P_1}\, \rho(\nu) \left[\frac{8\pi h \nu'^2}{c^3} + \rho(\nu')\right].$$

That is the formula of Pauli, Jr.

If one assumes that the quantum is absorbed during the collision then a slight modification of the preceding argument will easily give:

$$n_{P_1} n_{P_2} = n_{Q_1}(1 + n_{P_2}),$$

or, upon multiplying this by $8\pi\eta\nu^3 / c^3$:

$$n_{P_1}\,\rho(\nu) = n_{Q_1}\left[\frac{8\pi h\nu^3}{c^3} + \rho(\nu)\right].$$

That is the equilibrium formula that Einstein wrote out, when one uses our present notations.

Now that we have obtained these results, what meaning can we attribute to them? That is still quite mysterious.

Pauli has remarked that his probability formula has the classical theory as its limit.

However, the agreement between the classical theory and the new theory in one particular case is not sufficient to explain the meaning of the general formulas. When one can neglect the $n_i$ in comparison to unity (i.e., the case of material gases and Wien radiation), one will recover the probability of the old corpuscular doctrines, but in the general case, the collision probability will depend upon the number of atoms that belong to the kind that results from the collision. Certainly that is very difficult to imagine. Is the question perhaps ill-posed? Is it perhaps a fact that our astonishment comes from our habitual refusal to consider the influence of the future upon the present, which is an influence that is implied already in the use of the advanced potentials? We shall undoubtedly focus upon that point in the very near future.

# SUMMARY AND OVERVIEW

We shall attempt to summarize the main stages that we went through in the book.

From my previous work on quanta, I assumed that all forms of energy had an atomic structure, and I sought to link each atom with a periodic phenomenon whose frequency is determined by the quantum relation.

Here, I can make that idea more precise by considering the material point to be the center of a system of stationary waves. From the mathematical standpoint, that amounts to the same thing as envisioning the material point as an oscillating source of retarded and advanced actions whose superposition will give a stationary state. When studied by an observer in uniform, rectilinear motion with respect to the atom-source, the stationary wave will take on a different aspect: Its amplitude will displace with the velocity of motion, and always with the same value on the surface of a flattened ellipsoid of revolution that is centered on the moving body, which is explained simply by the Lorentz contraction.

The distribution of phases is much more curious and unexpected: They form wave planes that displace with a speed that is greater than that of light and becomes larger as the speed of the moving body becomes smaller. That is, one could say, the main point of my theory. It is derived directly from the Lorentz transformation and the relativity of time. The study of the wave in motion leads one to generalize the quantum relation by giving it a tensorial form that couples the energy-quantity of motion tensor of the moving body to the two fundamental quantities of the wave: namely, its frequency and phase velocity. As I emphasized in the Preface, one can remark that in all of this theory, the wave equation takes the form of a sort of postulate.

Since the case of rectilinear motion of a moving body can be studied completely, one must attack the variation of its motion in a force field, and one agrees to commence with the simplest case: viz., that of the constant field. Now, two kinds of fields are known today: gravitational fields and electromagnetic ones.

The essential character of the gravitational field is that the form of the trajectories does not depend upon the nature of the moving body – i.e., its mass. That is why Einstein was able to interpret the force of gravitation by means of the curvature of space-time, that is, by means of a non-Euclidian form for $ds^2$.

On the contrary, in an electromagnetic field, the motion of the charged point depends upon the nature of that point, or more precisely, the ratio of its charge to its mass. To my knowledge, there exists no satisfactory explanation for the electromagnetic force in terms of the Lorentz formula, and its nature remains mysterious. Be that as it may, my ideas lead naturally to the association of the varied motion of the moving body with the propagation of a wave whose frequency is given by the quantum relation. It would also be quite natural to try to make that propagation more precise by extending the tensorial relation that was discovered for uniform motion. I did that already in my thesis, but here I attempted to go a bit further. Upon recalling the idea that the material point is a source, one can look for a solution of the wave equation that is valid in the medium that would correspond to the solution that was employed in the case of rectilinear motion.

That mathematical search is hampered in the case of the electromagnetic field by our ignorance of the true nature of that field. However, in the gravitational case, things seem much easier in theory, and although I have not solved the problem in full generality, I hope that more talented geometers might arrive at that solution.

I sensed the possibility of finding a solution that would be applicable to the simple case of uniform, circular motion with no difficulty. I believe it to be exact, and it gives a concise idea as to what the general solution might be. It seems quite probable to me that one will arrive at a proof of the tensorial quantum relation for the points of the trajectory along that route. An examination of the manner by which one maintains phase agreement between the moving body and its wave will support that way of looking at things, and it is also confirmed by the simple interpretation that one deduces for the stability conditions, which I shall recall.

Indeed, the development of atomic theories has proved that among the motions that are predicted for the material point by classical dynamics, some of them possess a very special stability character, whereas the others are unstable enough that they can be considered to be physically unrealizable. For closed, periodic trajectories, the stability condition is obtained by equating the cyclic integral of de Maupertuisian action to an integer multiple of Planck's constant, and the multiple conditions that relate to quasi-periodic motions are deduced from the single condition of periodic motion; it will then suffice to interpret them. Now, the tensorial quantum relation that is supposed to be valid at all points along a trajectory shows that the de Maupertuisian action is proportional to the phase difference in the wave that is associated with the motion, and that difference is taken along the trajectory. It then results directly from this that one can equate the stability condition to a resonance conditions. That result already appears in my thesis, but here I was able to make it more precise in the case of circular motion the mathematical form that I spoke of above as an expression for the associated wave. One will then see that resonance does not take place solely along the trajectory, but also in the entire force field. I think that the same thing will be true in the general case.

All of the preceding was solely concerned with constant fields. If one would like to extend the same consideration to variable fields and the dynamics of systems of material points then one would first encounter a difficulty that was studied in the beginning of Chapter V. It deserved extra special mention because it is concerned with the development of the dynamics of relativity, independently of all my ideas on quanta. That difficulty, which is usually tacitly ignored, is the following one: What one calls the total energy of a system of material points is not the sum of the energies of each point when one defines them to be the temporal components of the energy-quantity of motion tensor at each point, and that is due to the intervention of potential energies. Hamilton's principle seems to be, in essence, a property of the motion of each material point, and it was only by a gimmick that classical mechanics was able to extend to systems of points. That defines an essential, delicate distinction of my theory that deserves more attention.

The study of waves that are associated with the motions of a set of material points would be very interesting and would perhaps bring with it some clarifications on the subject of the obstacles that quantum theory meets up with. I have been able to indicate the form of such waves only in the simple case that I studied in my thesis.

Upon passing from the dynamics of quanta to the optics of quanta, one will see some difficulties being created. For my own part, I see no reason to attribute only a statistical value to the energy principle. It seems to me that the photoelectric effect and the

Compton Effect bring with them some very strong experimental justifications in favor of the existence of the quanta that Einstein imagined, following Newton.

That hypothesis of granular radiation easily accounts for reflection, the Doppler Effect, and radiation pressure. But then, how does one reconcile that with wave optics – i.e., how does one explain interference, diffraction, and diffusion? I have no pretensions of having solved that enigma, but I have been able to introduce a new idea that does not figure explicitly in my thesis and now seems essential. Here is its basic principle: Upon studying the waves that are associated with motions in a constant field, one will perceive the following fact: One can attribute an index of refraction to each point of the medium such that the Fermat paths that are deduced from that index coincide with the possible dynamical trajectories. When the moving body describes one of those possible trajectories, the motion of the phase that is associated with it will take place along the orbit with a velocity that corresponds to that phase. Now, if one imagines an infinitude of moving bodies that simultaneously describe all of the possible trajectories without exerting any influence on each other then one will see that the phase distribution can be represented by propagation according to the indicated law with the index of a continuous wave with no singularity of the classical type. We will then arrive at the following conclusion, which seems fundamental: Whereas the wave phenomenon that is associated with the material point involves an essential singularity, the motion of an ensemble of material points can be represented statistically by a continuous wave. I have indicated how one can appeal to that idea in order to reconcile the optics of quanta with that of homogeneous waves. My attempt was hardly satisfactory, but I do believe that it has pointed to an interesting avenue to pursue.

The theory of associated waves that explains the intervention of quanta in dynamics must also explain their intervention in statistical mechanics. That point was verified in my thesis, but I had to employ the classical statistical method and assume a formula that was due to Planck and whose correctness I sensed to be true while doubting the argument that he appealed to. Here, I made use of a new statistical method that was inaugurated by Einstein, as a result of the work of Bose. That permitted me to make the arguments more rigorous and to arrive at a Planck's formula without appealing to his argument. The essential result of that study was the following one: The introduction of quanta into the theory of gases that leads to an exact prediction of the chemical constant is justified by the particular character of the stability of atomic motions whose associated waves are in resonance with the dimensions of the vessel. The last chapter related to collisions between units of energy and their probabilities; above all, it has the goal of showing how everything else could be understood in that context.

To summarize, we add some general considerations.

In the course of this book, we have glimpsed a new dynamics in which the notion of waves plays an important role, but that theory of dynamics cannot by any means be regarded as having been defined completely. For example, in the case of periodic motions, it will suffice for me to say: There is stability when a certain resonance condition is realized. One must also explain why the other motions are not stable.

It seems that the various portions of the associated wave superimpose upon the moving body exert a perturbing action that disturbs the motion that is predicted by ordinary dynamics. That action will be cancelled if all portions of the wave have the same phase.

The introduction of that new element that is distinct from the forces in the classical sense must undoubtedly lead to a theory of mechanics that is hereditarily discontinuous and is analogous to the one that Marcel Brillouin glimpsed some years ago. When a quantum of light follows a curved trajectory in a field of diffraction or interference, it will not be subject to a force of classical type, but if one would like to continue to attribute the curvature of the trajectory to a certain action that is exerted upon the moving body, and that action undoubtedly belongs to the type that we have just imagined. One will then perceive the possibility of a general doctrine that brings about the synthesis of the laws of dynamics and optics, which is a doctrine in which the laws of dynamics appear to be properties of waves with moving singularities.

We now turn to the subatomic domain and envision the simplest case of an atom with one electron, while neglecting the motion of the nucleus.

If we abstract from the internal state of the nucleus then it will first seem that the state of the atom is defined entirely by the motion of the electron. The manner by which we interpreted Bohr's correspondence theorem (Chap. IV.4) seems to indicate the contrary conclusion: The wave state of the atom does not depend solely upon the wave that is associated with the *present* motion of the moving body, but its expression will also contain oscillating terms of all emission frequencies that are defined by Bohr's rule. That idea seems to conform quite well to the present development of atomic theory and notably presents a certain analogy with the starting point for the recent work of Heisenberg, Born, and Jordan. Indeed, according to those scholars, the state of a system that is quantified with one degree of freedom cannot be expressed with the aid of two coordinates, namely, a position and a velocity. In order to define it, one must envision matrices or arrays of oscillating quantities whose frequencies are the emission frequencies of the system.

Experiments provide us with a monotonically-increasing number of facts about the subatomic world, and yet it still remains mysterious. It is only by means of very bold ideas that can perhaps see them clearly, and that is the excuse for the audacious attempts that were contained in this book. Today, as before, following Newton's beautiful simile, we are like children playing on the beach while the ocean of truth extends before us completely unexplored.

# NOTE

## ON

## THE RECENT WORK OF SCHRÖDINGER ([15]).

Since the time when the present work was composed, some very interesting articles by Schrödinger ([16]) in Zürich have appeared, and I shall attempt to summarize the essential idea in them.

Conforming to the general notions that were developed here, Schrödinger assumed that the study of the trajectories of a material point in a constant field must be deduced from the study of the sinusoidal solutions of the equation:

$$\Delta\Psi - \frac{1}{V^2}\frac{\partial^2\Psi}{\partial t^2} = \Delta\Psi - \frac{4\pi^2\nu^2}{c^2}n^2\,\Psi = 0,$$

in which $n$ is the index of refraction that was introduced in Chapters I , II, and III. We ignore the case in which a magnetic field is present, and expressing $n$ as a function of the spatial coordinates and energy $W$. We easily find that:

$$(A) \qquad \Delta\Psi + \frac{4\pi^2}{n^2\,c^2}\{[W - F(x, y, z)]^2 - m_0^2c^4\}\Psi = 0,$$

in which the function $F(x, y, z)$ is the potential energy of the material point in the field at the point $x, y, z$, and $m_0$ is its proper mass. If one is content with the approximation that is realized in classical Newtonian mechanics then equation ($A$) will take the simplest form:

$$(A') \qquad \Delta\Psi + \frac{8\pi^2}{h^2}m_0(E - F)\Psi = 0.$$

Here, $E$ denotes the energy in the classical mechanical sense – i.e., the sum $\frac{1}{2}m_0\,v^2 + F$ of the kinetic and potential energy.

The study of sinusoidal solutions of equations ($A$) and ($A'$) can be carried out in all cases that are envisioned by the usual mechanics and celestial mechanics by employing the approximation procedures of geometrical optics and one will then arrive at the least-action principle and the Lagrange equations. The considerations of Part I in the present book will then be valid, and the quantum stability condition for closed orbits will present itself in the form $\int p_i\,dq^i = nh$.

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([15]) Note added in proof.

([16]) Ann. Phys. (Leipzig) **79** (1926), pp. 361 and pp. 489.

However, for motions at the atomic scale (which are precisely the only ones for which the quantum conditions are actually interesting), the use of the procedures of geometrical optics for the study of the solutions of ($A$) and ($A'$) will no longer seem justified, and the equations of mechanics must no longer be valid, moreover. That was the essential remark that Schrödinger made. In order to solve the dynamical problem, one must then study the solutions ($A$) and ($A'$) *directly*. In particular, the quantum conditions must be deduced from that direct study, and no longer from the condition $\int p_i \, dq^i = nh$. Schrödinger, with the aid of some delicate mathematical considerations, showed that one will thus be led to some results that are in good agreement with experiments. We must confine ourselves to referring the reader to the original papers for more details.

Therefore, a new theory of mechanics seems to have been constructed for which dynamical phenomena have an essentially wave-like nature. Predicting them by means of the Lagrange-Hamilton equations is an approximate process that has exactly the same value and limits as the prediction of optical phenomena by means of Fermat's principle. The new mechanics is to the old mechanics (including Einstein's) what wave optics is to geometrical optics.