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On Poisson’s theorem and Lie’s differential invariants

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I. – In a previous note ([1]), I showed that the differential parameters in the integral form $\sum_i \sum_k N_{ik}\, \delta_1 x_i\, \delta_2 x_k$ are invariants of the generalized canonical system:

$$\text{(1)} \qquad \frac{dx_i}{\sum_k \frac{\nu_{ik}}{\nu} \frac{\partial H}{\partial x_k}} = dt \qquad (i, k = 1, \ldots, 2m),$$

if the functions that appear in those differential parameters are invariants of (1). I might add that H can include t explicitly.

If one remarks that $\sum_i \sum_k N_{ik}\, \delta_1 x_i\, \delta_2 x_k$ is an absolute integral invariant of the system (1) then one can deduce the following theorem from that generalization of Poisson’s theorem:

II. – *Any differential invariant or parameter, in the Lie sense:*

$$I\left(f_1, \ldots, \frac{\partial f_1}{\partial x_i}, \ldots, a_{ij\ldots}, \ldots, \frac{\partial a_{ij\ldots}}{\partial x_i}, \ldots \right)$$

of a multilinear form $F \equiv \sum_i \sum_j \cdots a_{ij\ldots}\, \delta_1 x_i\, \delta_2 x_j \cdots$ *is an invariant of the system:*

$$\text{(2)} \qquad \frac{dx_i}{X_i} = dt \qquad (i = 1, \ldots, n),$$

is $F, f_1, \ldots, a_{ij\ldots}$ *are invariants of* (2).

([1]) **Th. De Donder**, “Généralisation du théorème de Poisson,” C. R. Acad. Sci. Paris, 8 March 1909.

The $X_i, f_1, \ldots, a_{ij\ldots}$ are functions of x and t. The point transformations of that infinite group can include t explicitly. One always sets $\delta t = 0$.

III. – Upon considering the group of linear transformations:

$$\xi_i = \sum_{i=1}^{n} a_{ij}\, \xi'_k \,,$$

one will get an extension of a theorem that is due to H. Poincaré ([1]):

Any invariant or covariant form of weight p:

$$J\,(M_{ij\ldots}, \xi^{(1)}, \xi^{(2)}, \ldots)$$

of the multilinear form in $\xi^{(1)}$, $\xi^{(2)}$, …:

$$F = \sum_i \sum_j \cdots M_{ij\ldots}\, \xi^{(1)}\, \xi^{(2)} \cdots$$

will provide a finite differential or integral invariant $J\,\mu^{-p}$ *of the system* (2) *if* $\sum_i \sum_j \cdots M_{ij\ldots}\, \xi^{(1)}\, \xi^{(2)} \cdots$ *is a differential or integral invariant and m is a multiplier of* (2).

In the covariant J, one replaces the $\xi^{(1)}$, $\xi^{(2)}$, … with $\delta_1 x_i$, $\delta_2 x_i$, … or with one or more solutions to the variations of (2).

IV. – *Any differential invariant or parameter* $I\left(\varphi_1, \ldots, \dfrac{d\varphi_1}{dx_i}, \ldots\right)$ *of an infinite group G is an invariant of* (2) *if* φ_1, … *are invariants of* (2) *and the* X_i *satisfy the defining equations* ([2]) *for G.*

V. – In order to give an application of the preceding theorem, consider the group of special contact transformations in x, y. One will have: Any differential invariant parameter of the group of special contact transformations in x, y will be an invariant of the canonical equations:

$$\frac{dx_i}{\dfrac{\partial H}{\partial y_i}} = \frac{dy_i}{-\dfrac{\partial H}{\partial y_i}} = dt \qquad (i = 1, \ldots, n)$$

([1]) **H. Poincaré**, *Méthodes Nouvelles de la Mécanique céleste*, t. III, pp. 36.
([2]) **S. Lie**, “Ueber Differentialinvarianten,” Math. Ann. **24** (1884).

if the functions φ_1, … are invariants of that system.

In the case where H is independent of t, one will obtain a recent theorem of H. Vergne (C. R. Acad. Sci. Paris, 26 April 1910). We point out, following Lie (*loc. cit.*, pp. 576), that the differential invariants of the group considered can all be obtained by repeating the Poisson-Hamilton operation: (φ_1, φ_2), $[(\varphi_1, \varphi_2), \varphi_3]$, etc.