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Einsteinian gravity

Six lectures given at l'Institut Henri Poincaré

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These pages refer to a succinct presentation of the six lectures that I had the honor of giving at l'Institut Henri Poincaré in November and December of 1929. Those lectures were on the following topics:

I. The massive gravitational field.
II. The electromagnetic gravitational field.
III. Application to wave mechanics.
IV. Electrodynamics of moving bodies.
V. Electromagnetostriction and relativistic thermodynamics.
VI. Synthesis.

I was forced to show that general relativity provided an instrument that was adapted perfectly to the study of those problems. Contrary to an opinion that is very widespread, Einsteinian gravity does not need to be modified: When it is conceived in a sufficiently broad sense, it will still remain in harmony with the most modern theories of wave mechanics.

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Einsteinian gravity

1. The fundamental equations of the gravitational field. – Consider a function $\mathcal{M}^g$ that depends upon only $g^{\alpha\beta}$, $g^{\alpha\beta,i}$, … Its variance under an arbitrary change of variables x_1, x_2, x_3, x_4 is that of a multiplier or density factor. We call $\mathcal{M}^g$ the *characteristic function of gravity*.

In addition, consider a function $\mathcal{M}$ that has the same variance, but which can depend upon other functions such as u^α, A_α, etc., independently of the $g^{\alpha\beta}$, $g^{\alpha\beta,i}$. That function will be made explicit later on; it will be given the name of *phenomenological characteristic function.*

The fundamental *variational* principle of gravity consists of equating the tenvariational derivatives with respect to $g^{\alpha\beta}$ to zero. One will then obtain the *ten fundamental equations* of the gravity, namely:

$$\frac{\delta(\mathcal{M}^g + \mathcal{M})}{\delta g^{\alpha\beta}} = 0. \tag{1}$$

The variational derivative $\delta / \delta g^{\alpha\beta}$ is written:

$$\frac{\delta}{\delta g^{\alpha\beta}} \equiv \frac{\partial}{\partial g^{\alpha\beta}} - \frac{d}{dx_j}\left(\frac{\partial}{\partial g^{\alpha\beta,j}}\right) + \frac{d^2}{dx_j dx_k}\left(\frac{\partial}{\partial g^{\alpha\beta,jk}}\right) - \dots \tag{2}$$

Set:

$$\mathcal{T}^g_{\alpha\beta} \equiv \frac{\delta \mathcal{M}^g}{\delta g^{\alpha\beta}} \quad \text{and} \quad \mathcal{T}_{\alpha\beta} \equiv -\frac{\delta \mathcal{M}}{\delta g^{\alpha\beta}}; \tag{3}$$

hence:

$$\mathcal{T}^g_{\alpha\beta} = \mathcal{T}_{\alpha\beta}. \tag{4}$$

We call $\mathcal{T}^g_{\alpha\beta}$ the *symmetric covariant gravitational tensor* and $\mathcal{T}_{\alpha\beta}$, the *symmetric covariant phenomenological tensor*, or more simply, the *symmetric tensor*.

Let C denote the *curvature invariant*, and let a and b denote *universal* constants. Take the value:

$$\mathcal{M}^g = (a + b\, C)\sqrt{-g} \tag{5}$$

for $\mathcal{M}^g$.

Upon performing the indicated operations in (9.II), one will obtain:

$$-\tfrac{1}{2}(a + b\, C)\, g_{\alpha\beta} + b\, C_{\alpha\beta} = T_{\alpha\beta}, \tag{6}$$

in which:

$$T_{\alpha\beta} = \frac{\mathcal{T}_{\alpha\beta}}{\sqrt{-g}}, \tag{7}$$

and $C_{\alpha\beta}$ are the components of the well-known Riemann tensor.

Remark. – The variational principle that we just presented obviously amounts to a generalization of *Hamilton's principle;* i.e., that one must annul the variation:

$$\delta\int_{\Omega}(\mathcal{M}^g+\mathcal{M})\,dx_1\,dx_2\,dx_3\,dx_4=0, \tag{8}$$

in which Ω is a portion of space-time on whose boundary the variations must be annulled.

2. The gravitational identities. – Upon applying the gravity identities ([1]) to the function $\mathcal{M}^g$, one will get:

$$\frac{\partial}{\partial x_i}g^{ij}\frac{\delta\mathcal{M}^g}{\delta g^{\alpha j}}+\frac{1}{2}g^{ij,\alpha}\frac{\delta\mathcal{M}^g}{\delta g^{ij}}\equiv 0, \tag{9}$$

or even, by virtue of (3):

$$\frac{\partial\mathcal{T}_{\alpha}^{gi}}{\partial x_i}+\tfrac{1}{2}g^{ij,\alpha}\mathcal{T}_{ij}^{g}=0, \tag{10}$$

in which:

$$\mathcal{T}_{\alpha}^{gi}=g^{ij}\mathcal{T}_{\alpha j}^{g}. \tag{11}$$

3. Energy-impulse theorem. – The ten equations (4), when combined with the four identities (10), will immediately give the four *equations:*

$$\mathcal{F}_{\alpha}\equiv\mathcal{T}_{\alpha;i}^{i}\equiv\frac{\partial\mathcal{T}_{\alpha}^{i}}{\partial x_i}+\tfrac{1}{2}g^{ij,\alpha}\mathcal{T}_{ij}=0, \tag{12}$$

in which we have set, as in (24.II):

$$\mathcal{T}_{\alpha}^{i}=g^{ij}\,\mathcal{T}_{\alpha j}. \tag{13}$$

We say that the four equations (12) express the *energy-impulse* theorem. One can also express that theorem by saying that the generalized force $\mathcal{F}_{\alpha}$ is zero.

4. Gravity waves and rays. – One chooses the new variables x_1, x_2, x_3, x_4 in such a way that the new $g_{\alpha\beta}$ satisfy the four *complementary* equations ([2]):

[1] Th. DE DONDER, *Théorie invariantive du Calcul des variations*, Bull. Acad. Roy. Belg.; cl. des Sciences (5) **15** (1929), see § 12.

[2] Th. DE DONDER, *La Gravifique einsteinienne*, Annales de l'Observatoire Royale de Belgique, 1921. (or Gauthier-Villars, Paris). See § 29.

(14) $$g^{\alpha\beta}\begin{bmatrix}\alpha\ \beta\\ \sigma\end{bmatrix} = 0,$$

or – what amounts to the same thing:

(15) $$g^{\alpha\beta}\begin{Bmatrix}\alpha\ \beta\\ \sigma\end{Bmatrix} = 0.$$

Upon using those new variables, the fundamental equations of the gravitational field will become:

(16) $$g^{\alpha\beta}\, g_{\sigma\tau,\,\alpha\beta} = (\sigma,\ \tau),$$

in which the right-hand sides $(\sigma,\ \tau)$ *do not* contain any second derivatives of the Einsteinian potentials. We further remark that each of the left-hand sides of (16) contains only second derivatives of *just one* Einsteinian potential.

With HADAMARD and VESSIOT, we say that the solutions $f = f(x_1, x_2, x_3, x_4)$ of:

(17) $$G \equiv g^{\alpha\beta}\,\frac{\partial f}{\partial x^\alpha}\frac{\partial f}{\partial x^\beta} = 0$$

yield *characteristic manifolds* or (elementary) *gravity waves*:

(18) $$f = 0.$$

Now consider the (CAUCHY) characteristics of $G = 0$; one will then have the *bicharacteristics* or *gravity rays.* VESSIOT ([1]) showed that they are *null-length geodesics* of the gravity field.

5. Massive gravity field. – In the case of a gravity field that is due to some masses, we set:

(19) $$\mathcal{M} \equiv -\, g^{\alpha\beta}\,(\mathcal{N}\, u_\alpha\, u_\beta + \mathcal{P}_{\alpha\beta}),$$

in which $\mathcal{N}$ is the generalized mass density, u_α are the covariant components of the velocity, and $\mathcal{P}_{\alpha\beta}$ are the massive stresses.

Upon using (3) and (13), one will obtain the tensor:

(20) $$\mathcal{T}_\alpha^\beta = \mathcal{N}\, u_\alpha\, u^\beta + \mathcal{P}_\alpha^\beta\ .$$

The *energy-impulse theorem* (12) then becomes:

(21) $$\mathcal{F}_\alpha = \mathcal{N}_\alpha + \mathcal{P}_\alpha = 0,$$

([1]) Th. DE DONDER, *La Gravifique einsteinienne*, Annales de l’Observatoire Royale de Belgique, 1921. (or Gauthier-Villars, Paris). See § 29.

in which:

$$\mathcal{N}_\alpha \equiv \mathcal{N} A_\alpha + u_\alpha \frac{\partial}{\partial x_\beta}(\mathcal{N} u^\beta) \tag{22}$$

and

$$\mathcal{P}_\alpha \equiv \mathcal{P}^\beta_{\alpha;\beta} = \frac{\partial \mathcal{P}^\beta_\alpha}{\partial x_\beta} - \tfrac{1}{2} g^{\beta\gamma} g_{\varepsilon\gamma,\alpha} \mathcal{P}^\varepsilon_\beta . \tag{23}$$

Upon multiplying (21) by u^α and summing, one will obtain the *equation of continuity:*

$$\frac{\partial}{\partial x_\alpha}(\mathcal{N} u^\alpha) + \mathcal{P}_\alpha u^\alpha = 0. \tag{24}$$

6. Electromagnetic and massive gravitational field. – Now consider the case in which the gravity field is due to electric charges and mass.

To that effect, we introduce the characteristic function:

$$\mathcal{M} \equiv g^{\alpha\beta}\left[-\mathcal{N} u_\alpha u_\beta - \mathcal{P}_{\alpha\beta} + \frac{\sqrt{-g}}{2} g^{ij} H_{\alpha i} H_{\beta j}\right]. \tag{25}$$

The functions $H_{\alpha i}$ are the components of the electromagnetic force. They are antisymmetric; i.e.:

$$H_{\alpha i} = - H_{i\alpha}. \tag{26}$$

Thanks to (3) and (13), one will have the tensor:

$$\mathcal{T}^\beta_\alpha \equiv \mathcal{N} u_\alpha u^\alpha + \mathcal{P}^\beta_\alpha + \frac{1}{4}\varepsilon^\beta_\alpha \sqrt{-g} H_{ij} H^{ij} + \sqrt{-g} H^i_\alpha H^\beta_i . \tag{27}$$

The *energy-impulse theorem* results immediately from that:

$$\mathcal{J}_\alpha \equiv \mathcal{N}_\alpha + \mathcal{P}_\alpha + \mathcal{J}^{(e)}_\alpha = 0, \tag{28}$$

in which $\mathcal{N}_\alpha$ and $\mathcal{P}_\alpha$ were defined before (22 and 23), and:

$$\mathcal{J}^{(e)}_\alpha \equiv \left[\sqrt{-g}\, H^{\overline{\alpha j}} \frac{\partial H_{\overline{ij}}}{\partial x_i} - H_{\alpha j} \frac{\partial(\sqrt{-g}\, H^{ij})}{\partial x_i}\right]. \tag{29}$$

The notation $H_{\overline{ij}}$ signifies that one must adorn H with two indices that form, along with ij, an even permutation $ij\overline{ij}$ of the indices 1234. Upon multiplying (28) by u^α and summing over α, one will get the *continuity equation* ([1]):

$$\frac{\partial(\mathcal{N}\,u^\alpha)}{\partial x_\alpha} + \left(\mathcal{P}_\alpha + \mathcal{J}_\alpha^{(e)}\right)u^\alpha = 0. \tag{30}$$

Upon multiplying (28) by A^α and summing over α, one will obtain the relation ([1]):

$$\mathcal{N} = \frac{-A^\beta(\mathcal{J}_\beta^{(e)} + \mathcal{P}_\beta)}{A_\alpha A^\alpha}. \tag{31}$$

7. Maxwell's equations. – Now, introduce *Maxwell's equations:*

$$\left\{\begin{aligned} &\frac{\partial(\sqrt{-g}\,H^{\alpha i})}{\partial x_i} = \sigma u^\alpha, && (32)\\ &\frac{\partial \mathcal{H}_*^{\alpha i}}{\partial x_i} = 0, && (33)\end{aligned}\right.$$

in which we have set:

$$\mathcal{H}_*^{\alpha i} = H_{\overline{\alpha i}}. \tag{34}$$

It will result immediately from (33) and (34) that:

$$H_{\alpha\beta} = \frac{\partial \Phi_\alpha}{\partial x_\beta} - \frac{\partial \Phi_\beta}{\partial x_\alpha}, \tag{35}$$

in which Φ is the electromagnetic potential.

It is important to note that MAXWELL's equations (32) can be derived from a fundamental function:

$$\mathcal{D}^{(e)} = \sigma u^\alpha \Phi_\alpha + \frac{\sqrt{-g}}{4} g^{\alpha\beta} g^{ij} H_{\alpha i} H_{\beta j} \tag{36}$$

upon taking the variational derivatives with respect to Φ_α and supposing that the $H_{\alpha\beta}$ that enter into it have the form (35). The symbol σ is an electric density factor.

It results immediately from (32) that one will have:

([1]) Th. DE DONDER, *Théorie des Champs graviques*, Mémorial des Sciences mathématiques, fasc XIV. Paris 1926) (See chapters III and VI).

(37) $$\frac{\partial}{\partial x_\alpha}(\sigma u^\alpha) = 0,$$

which expresses the *conservation of electric charge in motion.*

Upon accounting for equations (32) and (33) in the value (29) for $\mathcal{J}_\alpha^{(e)}$, one will find that:

(38) $$\mathcal{J}_\alpha^{(e)} \equiv + \sigma u^i H_{\alpha i}\,.$$

Upon multiplying $\mathcal{J}_\alpha^{(e)}$ by u^α and summing, one will get:

(39) $$\mathcal{J}_\alpha^{(e)} u^\alpha \equiv 0.$$

Introduce the value $\mathcal{J}_\alpha^{(e)}$ in (31); we then obtain a relation between the density factors $\mathcal{N}$ and σ, namely ([1]):

(40) $$\mathcal{N} = \frac{-A^\beta(\sigma u^i H_{\beta i} + \mathcal{P}_\beta)}{A_\alpha A^\alpha}.$$

By definition, one calls the following relation between the electromagnetic potentials MAXWELL's *complementary equation:*

(41) $$\Psi = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\alpha}(\sqrt{-g}\,\Phi^\alpha) = 0,$$

in which we have written $\Phi^\alpha = g^{\alpha\beta}\,\Phi_\beta$. MAXWELL's electromagnetic equations can then be simplified. After some calculations, one will obtain:

(42) $$\frac{\sigma u^\alpha}{\sqrt{-g}} = K_\alpha + g^{ij}\,\frac{\partial^2 \Phi_\alpha}{\partial x_i\,\partial x_j},$$

in which K_α does not contain any second derivatives of Φ_1, Φ_2, Φ_3, Φ_4 .

8. Lagrangian form and canonical form of the energy-impulse theorem. – Consider the case of an incoherent mass – i.e., one for which *the* $\mathcal{P}_{\alpha\beta}$ *are zero.* If one takes (38) into account then the energy-impulse theorem (28) can be written:

(43) $$\mathcal{J}_\alpha = \mathcal{N}\left[\frac{d}{ds}\left(\frac{\partial(\frac{1}{2}W^2)}{\partial u^\alpha}\right) - \left(\frac{\partial(\frac{1}{2}W^2)}{\partial x_\alpha}\right)\right] + \sigma\left[\frac{d}{ds}\left(\frac{\partial U}{\partial u^\alpha}\right) - \left(\frac{\partial U}{\partial x_\alpha}\right)\right] = 0,$$

([1]) *Loc. cit.*, eq. (341).

in which one has set:

(44) $$W^2 = g_{\alpha\beta}\, u^\alpha\, u^\beta,$$

(45) $$U \;= u^\alpha\, \Phi_\alpha\,.$$

Set ([1]):

(46) $$\delta\tau^{(m)} \equiv \mathcal{N}\, \delta x_1\, \delta x_2\, \delta x_3\, \delta x_4\,, \qquad \delta\tau^{(e)} \equiv \sigma\, \delta x_1\, \delta x_2\, \delta x_3\, \delta x_4\,.$$

When one takes (39) and the fact that $\mathcal{P}_{\alpha\beta} = 0$ into account, it will then result from the continuity equations (30) and (37) that:

(47) $$\frac{d}{ds}\int \delta\tau^{(m)} = 0, \qquad \frac{d}{ds}\int \delta\tau^{(e)} = 0.$$

Upon multiplying the last two members of (43) by a volume element and integrating, one will get:

(48) $$\int \delta\tau^{(m)} \left[\frac{d}{ds}\left(\frac{\partial(\frac{1}{2}W^2)}{\partial u^\alpha}\right) - \left(\frac{\partial(\frac{1}{2}W^2)}{\partial x_\alpha}\right)\right] + \int \delta\tau^{(e)} \left[\frac{d}{ds}\left(\frac{\partial U}{\partial u^\alpha}\right) - \left(\frac{\partial U}{\partial x_\alpha}\right)\right] = 0.$$

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([1]) *Loc. cit.*, equations (188) and (184).

9. Relativistic mechanics of point-like charges. – In order to study the dynamics of point-like charges in space-time, based upon (46), we introduce the two *constants* $\tau^{(m)}$ and $\tau^{(e)}$, which characterize the particle from standpoint of mass and charge, resp.

We then write equation (48) in the form:

(49) $$\frac{d}{ds}\left(\frac{\partial L}{\partial u^{\alpha}}\right)-\left(\frac{\partial L}{\partial x_{\alpha}}\right)=0,$$

in which we have set:

(50) $$L = \tfrac{1}{2} W^2 + U^{\varepsilon}$$

and

(51) $$\varepsilon = \frac{\tau^{(m)}}{\tau^{(e)}}.$$

Introduce the *canonical variables:*

(52) $$p_{\alpha} \equiv \frac{\partial L}{\partial u^{\alpha}} = u_{\alpha} + \varepsilon \Phi_{\alpha}$$

and the *Hamiltonian function:*

(53) $$H = - L + p_{\alpha} u^{\alpha}.$$

It is easy to calculate the value of that function, by means of (52). One will find that:

(54) $$H = \tfrac{1}{2}.$$

Equations (49), when combined with (52), will then be equivalent to the system:

(55) $$\frac{dx_{\alpha}}{ds} = \frac{\partial H}{\partial p_{\alpha}}, \qquad \frac{dp_{\alpha}}{ds} = -\frac{\partial H}{\partial x_{\alpha}}.$$

The Hamiltonian (53) can be expressed as functions of the canonical variables p_{α}, thanks to (52); one will get:

(56) $$H \equiv \tfrac{1}{2} g^{\alpha\beta} (p_{\alpha} - \varepsilon \Phi_{\alpha})(p_{\beta} - \varepsilon \Phi_{\beta}).$$

We now propose to find the JACOBI system (55); to that effect, set:

(57) $$p_{\alpha} \equiv \frac{\partial S}{\partial x_{\alpha}}.$$

Hence, by virtue of (56), the *Jacobian equation:*

(58) $$\frac{\partial S}{\partial s} + \tfrac{1}{2} g^{\alpha\beta} \left(\frac{\partial S}{\partial x_\alpha} - \varepsilon\,\Phi_\alpha \right) \left(\frac{\partial S}{\partial x_\beta} - \varepsilon\,\Phi_\beta \right) = 0,$$

in which S is the JACOBI function. On the other hand, the classical theory of JACOBI teaches us that:

(59) $$\frac{\partial S}{\partial s} = - H = - \tfrac{1}{2}.$$

Upon substituting that into (58), one will get:

(60) $$g^{\alpha\beta} \left(\frac{\partial S}{\partial x_\alpha} - \varepsilon\,\Phi_\alpha \right) \left(\frac{\partial S}{\partial x_\beta} - \varepsilon\,\Phi_\beta \right) - 1 = 0.$$

We finally remark that, thanks to (59), the JACOBI function can be written:

(61) $$S = - s + S_0\,(x_1, x_2, x_3, x_4).$$

10. Relativistic equation of wave mechanics. – Set:

(62) $$k\,S = \log \Psi,$$

in which k is a universal constant whose value will be given later on. One will then have:

(63) $$\Psi = - \frac{2}{k} \frac{\partial \Psi}{\partial s}$$

and

(64) $$\frac{\partial S}{\partial x_\alpha} = - \frac{1}{2} \frac{\partial \Psi / \partial x_\alpha}{\partial \Psi / \partial s}.$$

We then represent the Jacobian equation (60). It can then be written:

(65) $$J \equiv g^{\alpha\beta} \left(\frac{1}{2} \frac{\partial \Psi}{\partial x_\alpha} + \varepsilon\,\Phi_\alpha \frac{\partial \Psi}{\partial s} \right) \left(\frac{1}{2} \frac{\partial \Psi}{\partial x_\beta} + \varepsilon\,\Phi_\beta \frac{\partial \Psi}{\partial s} \right) - \left(\frac{\partial \Psi}{\partial s} \right)^2 = 0.$$

In order to obtain the relativistic equation of wave mechanics, we propose to extremize the expression (65); we then write:

(66) $$\frac{\delta (J \sqrt{-g})}{\delta \Psi} = 0,$$

in which symbol $\delta / \delta\Psi$ signifies that one must take the *variational derivative* of $J\sqrt{-g}$ with respect to Ψ, namely:

$$\frac{\delta}{\delta\Psi} = \frac{\partial}{\partial\Psi} - \sum_{\sigma=0}^{4} \frac{d}{dx_\sigma}\left(\frac{\partial}{\partial \frac{\partial\Psi}{\partial x_\sigma}}\right) \qquad \left\{\begin{array}{l} x^\sigma = s, x_1, x_2, x_3, x_4, \\ \sigma = 0,1,2,3,4. \end{array}\right. \tag{67}$$

Equation (66) explicitly gives the *generalized* DE BROGLIE-SCHRÖDINGER *equation* ([1]):

$$\boxed{\Box\Psi - 2k\varepsilon \sum_\alpha \Phi^\alpha \frac{\partial\Psi}{\partial x_\alpha} - k\varepsilon D\Psi + (k\varepsilon)^2\left(F - \frac{1}{\varepsilon^2}\right)\Psi = 0,} \tag{68}$$

in which one has set:

$$\Box\Psi = \frac{1}{\sqrt{-g}} \sum_\alpha \frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\sum_\beta g^{\alpha\beta}\frac{\partial\Psi}{\partial x_\beta}\right) \qquad \alpha, \beta = 1, 2, 3, 4, \tag{69}$$

$$D \equiv \frac{1}{\sqrt{-g}} \sum_\alpha \frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\sum_\beta g^{\alpha\beta}\Phi_\beta\right), \tag{70}$$

$$F \equiv \sum_\beta \Phi_\alpha \Phi^\alpha . \tag{71}$$

The universal constant k is given by:

$$-k = 2i\pi \frac{m_0 c^2}{hc}, \tag{72}$$

in which m_0 is the *rest* mass of the electron. By virtue of (72), one will have:

$$\frac{2i\pi}{-k} = 2.42 \times 10^{-2} \text{ Å}. \tag{73}$$

In the right-hand side of (73) one sees the *wave length that corresponds to the transformation of the energetic content $m_0 c^2$ of an electron into a light quantum.*

Minkowski field. – Suppose that the gravity field is that of MINKOWSKI, and that the components Φ_1, Φ_2, Φ_3 are *zero*. In addition, set:

([1]) Th. DE DONDER, Bull. Acad. Roy. Belgique, cl. de Sc. (5) **8** (1927); Session in February 5.

(74) $$\Phi_4 = c\,V, \qquad \text{so} \qquad \Phi^4 = \frac{1}{c} V.$$

Since the system considered is stationary, with O. KLEIN, we set:

(75) $$\Psi \equiv \Theta\, e^{-\frac{2i\pi}{h} E\cdot t},$$

in which Θ no longer depends upon t, and E is a constant.

Substitute this in equation (68); hence, by virtue of (74):

(76) $$\Delta\Psi + \frac{4\pi^2}{c^2 h^2}\left[(\mathcal{E} - e_0\,V)^2 + 2(\mathcal{E} - e_0\,V)\,m_0\,c^2\right]\Psi = 0,$$

in which one has set:

(77) $$\mathcal{E} = E - m_0\,c^2.$$

The preceding equation can be written *approximately:*

(78) $$\Delta\Psi + \frac{8\pi^2}{c^2 h^2}\,m_0\,c^2\,(\mathcal{E} - e_0\,V)]\,\Psi = 0,$$

in which:

(79) $$\Delta\Psi = \sum_i \frac{\partial^2\Psi}{\partial x_i^2}.$$

The latter equation is the *ordinary* DE BROGLIE-SCHRÖDINGER *equation.*

Electrodynamics of moving bodies

11. Gravitational equations. – We have seen that the ten fundamental equations of the gravity field can be deduced from the variational principle (1). Here, we set the function $\mathcal{M}$ equal to:

(80) $$\boxed{\mathcal{M} = -\mathcal{N} W^2 + \mathcal{M}_*^{(m)} + \mathcal{M}_*^{(m,e)} + \mathcal{M}^{(e)},}$$

in which $\mathcal{N}$ is the tensorial mass factor:

(81) $$W^2 \equiv g_{\alpha\beta}\, u^\alpha\, u^\beta = 1,$$

and in which $\mathcal{M}_*^{(m)}$, $\mathcal{M}_*^{(m,e)}$, $\mathcal{M}_*^{(e)}$ represent the characteristic functions of the *massive, mass-electromagnetic,* and *purely electromagnetic* phenomena, respectively. We also say that $\mathcal{M}_*^{(m,e)}$ characterizes the phenomenon of electromagnetostriction, or more simply, *striction.*

12. Electromagnetic equations. – We write the equations of the generalized Maxwell field in the form:

(82) $$\frac{d\mathcal{K}^{\alpha\beta}}{dx_\beta} = \mathcal{C}^\alpha,$$

(83) $$\frac{d\mathcal{K}_*^{\alpha\beta}}{dx_\beta} = \mathcal{C}_*^\alpha,$$

in which:

(84) $$\mathcal{C}^\alpha \equiv \sigma_{(e)}\, u^\alpha + \mathcal{L}_{(e)}^\alpha,$$

(85) $$\mathcal{C}_*^\alpha \equiv \sigma_{(\mu)}\, u^\alpha + \mathcal{L}_{(\mu)}^\alpha,$$

in which the index (e) signifies *electric* and the index (μ) signifies *magnetic.* The expressions $\sigma\, u^\alpha$ and $\mathcal{L}^\alpha$, represent the components of the convection currents and generalized conduction currents, respectively.

In the most general case, we define the *electromagnetic force* by the antisymmetric tensor:

(86) $$\mathcal{K}^{\alpha\beta} \equiv \mathcal{H}^{\alpha\beta} - \mathcal{P}_{(e)}^{\alpha\beta},$$

and the *adjoint electromagnetic force* by the antisymmetric tensor:

(87) $$\mathcal{K}_*^{\alpha\beta} = \mathcal{H}_*^{\alpha\beta} - \mathcal{P}_{(\mu)}^{\alpha\beta},$$

in which:

(88) $$\mathcal{H}_*^{\alpha\beta} = H_{\overline{\alpha\beta}} .$$

With EINSTEIN, we now write the *polarization force* that is defined by the six components in the form:

(89) $$\mathcal{P}_{(e)}^{\alpha\beta} \equiv \mathcal{P}_{(e)}^{\alpha} u^{\beta} - \mathcal{P}_{(e)}^{\beta} u^{\alpha} ,$$

in which $\mathcal{P}_{(e)}^{\alpha}$ are the four contravariant tensorial components of the *intensity of electric polarization.* Recall that $u^{\alpha} = dx_a / ds$.

We likewise express the *magnetic polarization force* by means of the *magnetic polarization intensity:*

(90) $$\mathcal{P}_{(\mu)}^{\alpha\beta} \equiv \mathcal{P}_{(\mu)}^{\alpha} u^{\beta} - \mathcal{P}_{(\mu)}^{\beta} u^{\alpha} .$$

One immediately infers from (82) and (83) that:

(91) $$\frac{d\mathcal{C}^{\alpha}}{dx_{\alpha}} = 0, \qquad \frac{d\mathcal{C}_*^{\alpha}}{dx_{\alpha}} = 0.$$

In order to be able to easily give the physical interpretation of the terms above, suppose that x_1, x_2, x_3 represent the right-handed rectangular coordinates, and that x_4 represents the time t. Instead of x_1, x_2, x_3, we also employ the notation x, y, z. It is then convenient to employ the usual notations of electromagnetism, upon setting:

(92) $$\left\{\begin{array}{ll} \left\{\begin{array}{l} \mathcal{K}^{23} = c\mathcal{H}_x, \\ \mathcal{K}^{31} = c\mathcal{H}_y, \\ \mathcal{K}^{12} = c\mathcal{H}_z, \end{array}\right. & \left\{\begin{array}{l} \mathcal{K}_*^{23} = cH_x^*, \\ \mathcal{K}_*^{31} = cH_y^*, \\ \mathcal{K}_*^{12} = cH_z^*, \end{array}\right. \\ \left\{\begin{array}{l} \mathcal{K}^{14} = -B_x, \\ \mathcal{K}^{24} = -B_y, \\ \mathcal{K}^{34} = -B_z, \end{array}\right. & \left\{\begin{array}{l} \mathcal{K}_*^{14} = \mathcal{B}_x, \\ \mathcal{K}_*^{24} = \mathcal{B}_y, \\ \mathcal{K}_*^{34} = \mathcal{B}_z. \end{array}\right. \end{array}\right.$$

The symbols in this table have the following physical significance:

$(\mathcal{H}_x, \mathcal{H}_y, \mathcal{H}_z)$ = components of the magnetic force,

$(\mathcal{B}_x, \mathcal{B}_y, \mathcal{B}_z)$ = components of the magnetic induction.

On the other hand, one has set:

(93) $$H_x^* \equiv H_x - H_x^a , \qquad H_y^* \equiv H_y - H_y^a , \qquad H_z^* \equiv H_z - H_z^a ,$$

in which:

(H_x, H_y, H_z) = components of the electric force,

(H_x^a, H_y^a, H_z^a) = components of the applied (Ger.: *eingeprägt*) electric force.

Finally:

(B_x, B_y, B_z) = components of the electric induction.

It is almost pointless to add that all of those expressions must be taken in a *generalized* sense.

Thanks to the notations (92), the electromagnetic equations (82) and (83) for moving bodies keep the Maxwellian form:

(94) $$\text{div } \mathbf{B} = \sigma_{(e)} u^4 + \mathcal{L}_{(e)}^4, \qquad \text{rot } \boldsymbol{\mathcal{H}} = \frac{1}{c}\left(\frac{\partial B}{\partial t} + \sigma_{(e)} u + \mathcal{L}_{(e)}\right),$$

(95) $$\text{div } \boldsymbol{\mathcal{B}} = \sigma_{(\mu)} u^4 + \mathcal{L}_{(\mu)}^4, \qquad \text{rot } \mathbf{H}^* = \frac{1}{c}\left(-\frac{\partial \mathcal{B}}{\partial t} + \sigma_{(\mu)} u + \mathcal{L}_{(\mu)}\right).$$

The rectangular components of the vector $\sigma_{(e)} u + \mathcal{L}_{(e)}$ are $\sigma_{(e)} u^i + \mathcal{L}_{(e)}^i$ (i = 1, 2, 3). The same thing will be true for the vector $\sigma_{(\mu)} u + \mathcal{L}_{(\mu)}$.

13. Return to the gravity equations. – Take the variational derivative of the function (80) with respect to $g_{\alpha\beta}$ and set:

(96) $$\mathcal{T}_{\alpha\beta}^{(m)} \equiv -\frac{\delta \mathcal{M}_*^{(m)}}{\delta g^{\alpha\beta}}, \qquad \mathcal{T}_{\alpha\beta}^{(m,e)} \equiv -\frac{\delta \mathcal{M}_*^{(m,e)}}{\delta g^{\alpha\beta}}, \qquad \mathcal{T}_{\alpha\beta}^{(e)} \equiv -\frac{\delta \mathcal{M}_*^{(e)}}{\delta g^{\alpha\beta}}.$$

We will then have:

(97) $$\mathcal{T}_{\alpha\beta} = \mathcal{N} u_\alpha u_\beta + \mathcal{T}_{\alpha\beta}^{(m)} + \mathcal{T}_{\alpha\beta}^{(m,e)} + \mathcal{T}_{\alpha\beta}^{(e)}.$$

In the case of an arbitrary electromagnetic field, we set ([1]):

(98) $$\boxed{\mathcal{M}_*^{(e)} \equiv \tfrac{1}{2}\sqrt{-g}\, g^{\alpha\beta} g^{ij} K_{\alpha i} K_*^{\overline{\beta j}}.}$$

It will then result that:

(99) $$\mathcal{T}_{\alpha\beta}^{(e)} \equiv -\frac{\sqrt{-g}}{2} g^{i\gamma}\left(K_{\alpha i}\mathcal{K}_*^{\overline{\beta\gamma}} + K_{\beta i}\mathcal{K}_*^{\overline{\alpha\gamma}}\right) + \frac{\sqrt{-g}}{4} g_{\alpha\beta} g^{kl} g^{i\gamma} K_{ki} \mathcal{K}_*^{l\gamma}.$$

We then set:

(100) $$P_{\alpha\beta} = T_{\alpha\beta}^{(m)} + T_{\alpha\beta}^{(m,e)};$$

([1]) Th. DE DONDER, *The mathematical Theory of Relativity*, Massachusetts Institute of Technology, Cambridge, MA, USA, 1927. See page 77.

hence:

(101) $$T_{\alpha\beta} = N\, u_\alpha u_\beta + P_{\alpha\beta} + T^{(e)}_{\alpha\beta}\,.$$

It is easy to deduce the values of the mixed components $\mathcal{T}^{\beta(e)}_{\alpha}$, P^{β}_{α}, T^{β}_{α} from (99), (100), (101), resp. The *energy-impulse theorem* (12) becomes:

(102) $$\mathcal{J}_\alpha \equiv \mathcal{T}^{\beta}_{\alpha;\beta} \equiv \mathcal{N}_\alpha + \mathcal{P}_\alpha + \mathcal{T}^{\beta(e)}_{\alpha}$$

here, in which $\mathcal{N}_\alpha$ and $\mathcal{P}_\alpha$ were defined in (22) and (23), while taking (100) into account. In addition, one will have:

(103) $$\mathcal{T}^{(e)}_{\alpha} \equiv \mathcal{T}^{\beta(e)}_{\alpha;\beta}$$
$$\equiv -\frac{1}{2}\frac{d}{dx_i}\left[\sqrt{-g}\, g^{\varepsilon\nu} g^{\tau i}\left(K_{\alpha i}\mathcal{K}^{\overline{\tau\nu}}_{*} + K_{\tau i}\mathcal{K}^{\overline{\alpha\nu}}_{*}\right) + \tfrac{1}{4}\sqrt{-g}\, g^{kl} g^{\tau\nu}\frac{d}{dx_\alpha}\left(K_{ki}\mathcal{K}^{\overline{l\nu}}_{*}\right)\right].$$

14. Electromagnetic hysteresis. – By definition, the *electromagnetic hysteresis* is a quadri-vector $\mathcal{H}^{(e)}_{\alpha}$ (α = 1, 2, 3, 4) that is given by ([1]):

(104) $$\boxed{\mathcal{H}^{(e)}_{\alpha} \equiv \mathcal{L}_\alpha - \mathcal{J}^{(e)}_{\alpha},}$$

in which one has set:

(105) $$\mathcal{L}_\alpha \equiv \mathcal{K}^{\overline{\alpha\beta}}_{*}\frac{d\mathcal{K}^{\alpha\beta}}{dx_\gamma} - \mathcal{K}^{\overline{\alpha\beta}}\frac{d\mathcal{K}^{\beta\gamma}_{*}}{dx_\gamma},$$

or, by virtue of (82) and (83):

(106) $$\boxed{\mathcal{L}_\alpha \equiv \mathcal{K}^{\overline{\alpha\beta}}_{*}\mathcal{C}^{\beta} - \mathcal{K}^{\overline{\alpha\beta}}\mathcal{C}^{\beta}_{*}.}$$

One sees that the expression (105) is identical to (29) in systems that are devoid of electric and magnetic polarization, in such a way that in that case $\mathcal{H}^{(e)}_{\alpha} \equiv 0$. It then results that *hysteresis*, as we have defined it, *is essentially due to those polarizations.*

Upon introducing the notations (92) into (106), it is easy to see that the first three components of (106) can be put into vectorial form; namely:

(107) $$\left.\begin{matrix}\mathcal{L}_1\\ \mathcal{L}_2\\ \mathcal{L}_3\end{matrix}\right\}\quad \mathcal{L} = [\mathcal{C}\cdot\mathbf{B}] + c\,\mathbf{H}^{*}\,\mathcal{C}^{4} + [\mathcal{C}_{*}\,\mathbf{B}] + c\mathcal{H}\,\mathcal{C}^{4}_{*}\,.$$

That expression generalizes the classical expression for the LORENTZ *force* (multiplied by *c*).

([1]) Th. DE DONDER, C. R. Acad. des Sc. de Paris, 2 July 1928.

The fourth component can be written:

(108) $$\mathcal{L}_4 = -\,c\,(\mathbf{H}^* \cdot \mathcal{C}) + c\,(\boldsymbol{\mathcal{H}} \cdot \mathcal{C}_*),$$

which is the general expression for the JOULE *effect*.

15. Equation of continuity. – That equation is obtained by multiplying the components $\mathcal{J}_\alpha^{(e)}$ by u^α and summing. One finds that:

(109) $$\frac{d(\mathcal{N}u^\alpha)}{dx_\alpha} + \left(\mathcal{P}_\alpha + \mathcal{J}_\alpha^{(e)}\right)u^\alpha = 0$$

or furthermore:

(110) $$\frac{d}{ds}\,[\mathcal{N}\,\delta(x^1, \ldots, x^4)] = -\left(\mathcal{P}_\alpha + \mathcal{J}_\alpha^{(e)}\right)u^\alpha\,\delta(x^1, \ldots, x^4).$$

It is easy to transform (109) and to put it into the form:

(111) $$\frac{d(\mathcal{N}u^\alpha)}{dx_\alpha} + \left(\mathcal{T}_{\alpha;\beta}^{\beta(m)} + \mathcal{T}_{\alpha;\beta}^{\beta(m,e)} + \mathcal{T}_{\alpha;\beta}^{\beta(e)}\right)u^\alpha = 0.$$

The expressions in the parentheses can be integrated by parts; for example, one will have:

(112) $$\mathcal{T}_{\alpha;\beta}^{\beta(m.e)}u^\alpha = \frac{d\left(\mathcal{T}_\alpha^{\beta(m.e)}u^\alpha\right)}{dx_\beta} + \mathcal{K}^{(m,\,e)},$$

in which one has set:

(113) $$\mathcal{K}^{(m,\,e)} \equiv -\tfrac{1}{4}\left(\mathcal{T}_\alpha^{\beta(m.e)} + \mathcal{T}_\beta^{\alpha(m.e)}\right)\left(\frac{du^\alpha}{dx_\beta} + \frac{du^\beta}{dx_\alpha}\right)$$

$$-\tfrac{1}{4}\left(\mathcal{T}_\alpha^{\beta(m.e)} - \mathcal{T}_\beta^{\alpha(m.e)}\right)\left(\frac{du^\alpha}{dx_\beta} - \frac{du^\beta}{dx_\alpha}\right) - \frac{1}{2}\frac{dg_{\alpha\beta}}{ds}\,\mathcal{T}_{(m,e)}^{\alpha\beta}.$$

One will then have analogous expressions for $\mathcal{K}^{(m)}$ and $\mathcal{K}^{(e)}$.

Thanks to the preceding formulas, the equation of continuity can then be finally written:

(114) $$\frac{d\left[\left(\mathcal{N} + \mathcal{T}_\beta^{\alpha(m)} + \mathcal{T}_\beta^{\alpha(m,e)} + \mathcal{T}_\beta^{\alpha(e)}\right)u^\beta\right]}{dx^\alpha} + \mathcal{K} = 0,$$

in which:

(115) $$\mathcal{K} \equiv \mathcal{K}^{(m)} + \mathcal{K}^{(m,\,e)} + \mathcal{K}^{(e)}.$$

16. Fundamental principle of electromagnetostriction. – We say that $\mathcal{K}^{(m,e)}$, as defined by (113), is the *power* (per unit volume) *of the striction*. Upon generalizing a hypothesis in the classical theory of electromagnetostriction, we assume ([1]) that we have:

(116) $$\boxed{\mathcal{K}^{(m,e)} = \mathcal{H}_\alpha^{(e)} u^\alpha.}$$

In other words: *The power of the striction is equal to the power of the hysteresis.*

The principle (116) can again be written:

(117) $$\mathcal{K}^{(m,e)} = \left(\mathcal{H}_i^{(e)} v^i + \mathcal{H}_4^{(e)}\right) u^4, \qquad v^i \equiv \frac{dx_i}{dt} \qquad (i = 1, 2, 3).$$

Upon assimilating u^4 to $1 / c$, one will have:

(118) $$K^{(m,e)} = \frac{1}{c}\left(H_i^{(e)} v^i + H_4^{(e)}\right),$$

in the first approximation.

On the other hand, thanks to the notations (92), one can write $H_4^{(e)}$, which is defined by (104), in the form:

(119) $$H_4^{(e)} \equiv \frac{1}{2}\left[\left(\mathbf{H}^* \frac{\partial \mathbf{B}}{\partial t}\right) - \left(\mathbf{B}\frac{\partial \mathbf{H}^*}{\partial t}\right) + \left(\boldsymbol{\mathcal{H}}\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t}\right) - \left(\boldsymbol{\mathcal{B}}\frac{\partial \boldsymbol{\mathcal{H}}}{\partial t}\right)\right] + \frac{c}{2}\,\text{div}\cdot([\mathbf{H}^*\cdot\mathcal{H}] - [\mathbf{B}\cdot\mathcal{B}])$$

.

In the case of oscillating deformations of the body considered, we assume that one has the *balance principle:*

(120) $$H_i^{(e)} v^i + \frac{c}{2}\,\text{div}\cdot([\mathbf{H}^*\cdot\boldsymbol{\mathcal{H}}] - [\mathbf{B}\cdot\boldsymbol{\mathcal{B}}]) = 0.$$

We can then finally write the fundamental principle (116) in the form:

(121) $$K^{(m,e)} = \frac{1}{2c}\left[\left(\mathbf{H}^* \frac{\partial \mathbf{B}}{\partial t}\right) - \left(\mathbf{B}\frac{\partial \mathbf{H}^*}{\partial t}\right) + \left(\boldsymbol{\mathcal{H}}\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t}\right) - \left(\boldsymbol{\mathcal{B}}\frac{\partial \boldsymbol{\mathcal{H}}}{\partial t}\right)\right].$$

17. Calculation of $K^{(m,e)}$ as a function of the deformations. – Set:

([1]) *Loc. cit.*, above.

(122) $$x_i = x_i^0 + \lambda_i(x_1^0, x_2^0, x_3^0, t) \qquad i = 1, 2, 3,$$

in which λ_4 represents the infinitely-small displacement that starts with the initial point x_i^0 (i = 1, 2, 3); for t = 0, one will have $x_i = x_i^0$. The body considered *collectively* is then *at rest* with respect to the trihedron (x_1, x_2, x_3).

It then results that:

(123) $$v^i = \frac{\partial \lambda_i}{\partial t}.$$

Set:

$$\frac{d\lambda_i}{dt} \equiv \dot{\lambda}_i; \qquad \text{hence,} \qquad v^i \equiv \dot{\lambda}_i, \qquad i = 1, 2, 3.$$

We also adopt the classical notations:

(124) $$\left\{\begin{aligned} &x_x = \frac{\partial \lambda_1}{\partial x^0}, \quad y_y = \frac{\partial \lambda_2}{\partial y^0}, \quad z_z = \frac{\partial \lambda_3}{\partial x^0}, \\ &x_y = y_x = \frac{\partial \lambda_1}{\partial y^0} + \frac{\partial \lambda_2}{\partial x^0}, \quad x_z = z_x = \frac{\partial \lambda_1}{\partial z^0} + \frac{\partial \lambda_3}{\partial x^0}, \quad y_z = z_y = \frac{\partial \lambda_2}{\partial z^0} + \frac{\partial \lambda_3}{\partial y^0}, \\ &x_t = \frac{\partial \lambda_1}{\partial t}, \quad y_t = \frac{\partial \lambda_2}{\partial t}, \quad z_t = \frac{\partial \lambda_3}{\partial t}, \end{aligned}\right.$$

and

(125) $$\omega^1 = \omega_x = \frac{\partial \lambda_3}{\partial y} - \frac{\partial \lambda_2}{\partial z}, \qquad \omega^2 = \omega_y = \frac{\partial \lambda_1}{\partial z} - \frac{\partial \lambda_2}{\partial x}, \qquad \omega^3 = \omega_z = \frac{\partial \lambda_2}{\partial x} - \frac{\partial \lambda_1}{\partial y}.$$

It is good to recall that one can identify x_i^0 with x_i (i = 1, 2, 3), up to an infinitesimal. We can replace the x_i^0 in (80) with the x_i. It will then result that the expressions that appear in (113) will become:

(126) $$\left\{\begin{aligned} &\frac{\partial v^1}{\partial x_1} = \frac{\partial \dot{\lambda}_1}{\partial x} = \dot{x}_x, \quad \frac{\partial v^2}{\partial x_2} = \dot{y}_x, \quad \frac{\partial v^3}{\partial x_3} = \dot{z}_z, \\ &\frac{\partial v^1}{\partial x_2} + \frac{\partial v^2}{\partial x_1} = \dot{x}_y, \quad \frac{\partial v^1}{\partial x_3} + \frac{\partial v^3}{\partial x_1} = \dot{x}_z, \quad \frac{\partial v^2}{\partial x_3} + \frac{\partial v^3}{\partial x_2} = \dot{y}_z, \end{aligned}\right.$$

and

(127) $$\dot{\omega}_x = \frac{\partial v^3}{\partial y} - \frac{\partial v^2}{\partial z}, \qquad \dot{\omega}_y = \frac{\partial v^1}{\partial z} - \frac{\partial v^3}{\partial x}, \qquad \dot{\omega}_z = \frac{\partial v^2}{\partial x} - \frac{\partial v^1}{\partial y},$$

where the dot over the x_x, y_y, … indicates a partial derivative with respect to t.

The *power of striction* $K^{(m,e)}$ can then be written (113), in the same approximation ($u^4 \approx 1 / c$).

(128) $$K^{(m,e)} \equiv$$

$$\equiv -\frac{1}{c}\left[\begin{array}{l} T_1^{1(m,e)}\dot{x}_x + T_2^{2(m,e)}\dot{y}_y + T_3^{3(m,e)}\dot{z}_z \\ +\frac{1}{2}(T_1^{2(m,e)} + T_2^{1(m,e)})\,\dot{x}_y + \frac{1}{2}(T_2^{3(m,e)} + T_3^{2(m,e)})\,\dot{y}_z + \frac{1}{2}(T_3^{1(m,e)} + T_1^{3(m,e)})\,\dot{z}_x \\ -\frac{1}{2}(T_1^{2(m,e)} - T_2^{1(m,e)})\,\dot{\omega}_z - \frac{1}{2}(T_2^{3(m,e)} - T_3^{2(m,e)})\,\dot{\omega}_x - \frac{1}{2}(T_3^{1(m,e)} - T_1^{3(m,e)})\,\dot{\omega}_y \\ +T_1^{4(m,e)}\dot{x}_t + T_2^{4(m,e)}\dot{y}_t + T_3^{4(m,e)}\dot{z}_t \end{array}\right].$$

18. Calculation of the right-hand side of (121). – We return to the right-hand side of (121) and assume that we have:

(129) $$H_i = \sum_j \varepsilon'_{ij}\, B_j\,, \qquad \mathcal{H}_i = \sum_j \mu'_{ij}\, \mathcal{B}_j\,.$$

Set:

(130) $$W = T_4^{4(e)}.$$

Hence, W represents the density of electromagnetic energy that is localized in the volume v:

(131) $$\int_v W\,\delta v \equiv \tfrac{1}{2}\int_v [(\mathbf{H}\cdot\mathbf{B}) + (\boldsymbol{\mathcal{H}}\cdot\boldsymbol{\mathcal{B}})]\,\delta v = \tfrac{1}{2}\int_v \sum_i \sum_j (\varepsilon'_{ij} B_i B_j + \mu'_{ij}\mathcal{B}_i\mathcal{B}_j)\,\delta v\,.$$

The right-hand side of (121) can then be written:

(132) $$-\frac{1}{c}\left(\frac{\partial W}{\partial t}\right)_{\mathbf{B}\boldsymbol{\mathcal{B}}} \delta v\,,$$

in which the indices **B**, $\mathcal{B}$ serve to remind one that the partial derivative with respect to t is performed while keeping the B_i and $\mathcal{B}_i$ (i = 1, 2, 3) constant; one will then have:

(133) $$-\frac{1}{c}\int\left(\frac{\partial W}{\partial t}\right)_{\mathbf{B}\boldsymbol{\mathcal{B}}} \delta v = -\frac{1}{2c}\int_v \sum_i \sum_j \left(\frac{\partial \varepsilon'_{ij}}{\partial t} B_i B_j + \frac{\partial \mu'_{ij}}{\partial t}\mathcal{B}_i\mathcal{B}_j\right)\delta v\,.$$

Suppose that the coefficients ε'_{ij} and μ'_{ij} are functions of the *linear angular deformations* $x_x, \ldots, z_z$, the *rotations* ω^i, and the velocities v^i. In other words, the density W of the electromagnetic energy depends upon:

(134) $$\left\{\begin{array}{ll} x_x, y_y, z_z & x_y, y_z, z_x \\ \omega_x, \omega_y, \omega_z & x_t, y_t, z_t \end{array}\right.$$

explicitly.

The right-hand side of (121) can then be written:

$$(135)\quad -\frac{1}{c}\int_v \left(\frac{\partial W}{\partial t}\right)_{\mathbf{B},\mathcal{B}} \delta v \equiv \int_v \left[\begin{array}{c} \left(\frac{\partial W}{\partial x_x}\right)_{\mathbf{B},\mathcal{B}} \dot{x}_x + \left(\frac{\partial W}{\partial y_y}\right)_{\mathbf{B},\mathcal{B}} \dot{y}_y + \left(\frac{\partial W}{\partial z_z}\right)_{\mathbf{B},\mathcal{B}} \dot{z}_z \\ + \left(\frac{\partial W}{\partial x_x}\right)_{\mathbf{B},\mathcal{B}} \dot{x}_y + \left(\frac{\partial W}{\partial y_z}\right)_{\mathbf{B},\mathcal{B}} \dot{y}_z + \left(\frac{\partial W}{\partial z_z}\right)_{\mathbf{B},\mathcal{B}} \dot{z}_x \\ + \left(\frac{\partial W}{\partial \omega_x}\right)_{\mathbf{B},\mathcal{B}} \dot{\omega}_x + \left(\frac{\partial W}{\partial \omega_z}\right)_{\mathbf{B},\mathcal{B}} \dot{\omega}_y + \left(\frac{\partial W}{\partial \omega_z}\right)_{\mathbf{B},\mathcal{B}} \dot{\omega}_x \\ + \left(\frac{\partial W}{\partial x_t}\right)_{\mathbf{B},\mathcal{B}} \dot{x}_t + \left(\frac{\partial W}{\partial y_t}\right)_{\mathbf{B},\mathcal{B}} \dot{y}_t + \left(\frac{\partial W}{\partial z_t}\right)_{\mathbf{B},\mathcal{B}} \dot{z}_t \end{array}\right] \delta v\,.$$

19. Electrostriction tensor.

$$(136)\quad \begin{cases} T_1^{1(m,e)} = \left(\frac{\partial W}{\partial x_x}\right)_{\mathbf{B},\mathcal{B}}, \\ T_2^{2(m,e)} = \left(\frac{\partial W}{\partial y_y}\right)_{\mathbf{B},\mathcal{B}}, \\ T_3^{3(m,e)} = \left(\frac{\partial W}{\partial z_z}\right)_{\mathbf{B},\mathcal{B}}, \end{cases}$$

$$(137)\quad \begin{cases} \tfrac{1}{2}(T_1^{2(m,e)} + T_2^{1(m,e)}) = \left(\frac{\partial W}{\partial x_y}\right)_{\mathbf{B},\mathcal{B}}, \\ \tfrac{1}{2}(T_2^{3(m,e)} + T_3^{2(m,e)}) = \left(\frac{\partial W}{\partial y_z}\right)_{\mathbf{B},\mathcal{B}}, \\ \tfrac{1}{2}(T_3^{1(m,e)} + T_1^{3(m,e)}) = \left(\frac{\partial W}{\partial z_x}\right)_{\mathbf{B},\mathcal{B}}, \end{cases}$$

$$(138)\qquad \begin{cases} \tfrac{1}{2}(T_1^{2(m,e)} - T_2^{1(m,e)}) = -\left(\dfrac{\partial W}{\partial \omega_z}\right)_{\mathbf{B},\mathcal{B}}, \\ \tfrac{1}{2}(T_2^{3(m,e)} - T_3^{2(m,e)}) = -\left(\dfrac{\partial W}{\partial \omega_x}\right)_{\mathbf{B},\mathcal{B}}, \\ \tfrac{1}{2}(T_3^{1(m,e)} - T_1^{3(m,e)}) = -\left(\dfrac{\partial W}{\partial \omega_y}\right)_{\mathbf{B},\mathcal{B}}, \end{cases}$$

$$(139)\qquad \begin{cases} T_1^{4(m,e)} = \left(\dfrac{\partial W}{\partial x_t}\right)_{\mathbf{B},\mathcal{B}}, \\ T_2^{4(m,e)} = \left(\dfrac{\partial W}{\partial y_t}\right)_{\mathbf{B},\mathcal{B}}, \\ T_3^{4(m,e)} = \left(\dfrac{\partial W}{\partial z_t}\right)_{\mathbf{B},\mathcal{B}}. \end{cases}$$

In these tables:

$$(140)\qquad T_4^{1(m,e)},\ T_4^{2(m,e)},\ T_4^{3(m,e)},\ T_4^{4(m,e)}$$

no longer occur.

Meanwhile, we remark that one has, in a general:

$$(141)\qquad T_{(m,e)}^{\alpha\beta} \equiv T_{(m,e)}^{\alpha\beta} \equiv \sum_\gamma g^{\gamma\beta} T_\gamma^{\alpha(m,e)} \equiv \sum_\gamma g^{\gamma\alpha} T_\gamma^{\beta(m,e)}.$$

In a MINKOWSKI field, those relations will become:

$$(142)\qquad \begin{cases} T_{(m,e)}^{ab} \equiv T_{(m,e)}^{ba} \equiv -T_b^{a(m,e)} = -T_b^{a(m,e)}, \qquad a, b, = 1, 2, 3, \\ T_{(m,e)}^{a4} \equiv T_{(m,e)}^{4a} \equiv -\dfrac{1}{c^2} T_4^{a(m,e)} = -T_a^{4(m,e)}, \\ T_{(m,e)}^{44} \equiv \dfrac{1}{c^2} T_4^{4(m,e)}. \end{cases}$$

It results from (142) that the $T_a^{b(m,e)}$ are symmetric here (a, b = 1, 2, 3); hence, by virtue of (138), the function W will not refer to ω_x, ω_y, ω_z explicitly. It results from (142) that the $T_4^{a(m,e)}$ differ from the $T_a^{4(m,e)}$ that are provided by (139) only by the factor ($-c^2$).

We have thus calculated all of the components of the tensor $T_\alpha^{\beta(m,e)}$ (α, β = 1, 2, 3, 4), except for $T_4^{4(m,e)}$. One knows that $T_4^{4(m,e)}$ has the dimension of energy per unit volume; there is then good reason to equate it to the elastic energy density (of deformation). One can also reproduce all of these theories in space-time by using the perfect bodies that are defined in our *Théorie des champs gravifiques* (*loc. cit.*, eq. 328).

20. Radiation stresses. – With Léon BRILLOUIN ([1]), imagine the case of an *electromagnetic wave* that propagates along the x-axis; let H_y (or H_y^*) be the electric field, and let $\mathcal{H}_z$ be the magnetic field. By virtue of (130) and the definition of an electromagnetic wave, one will have:

$$-W = \mathbf{B}\cdot\mathbf{H} = \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\mathcal{H}}. \tag{143}$$

The wave velocity will then be:

$$V = \frac{c}{\sqrt{\varepsilon\mu}}, \tag{144}$$

in which ε and μ are the specific inductive power and magnetic permeability of the system considered, respectively. One has:

$$\frac{\partial \log D}{\partial t} = -\frac{\partial}{\partial t}(x_x + y_y + z_z), \tag{145}$$

in which D represents the mass density. Hence, for an *isotropic body:*

$$\left(\frac{\partial W}{\partial x_x}\right)_{\mathbf{B},\mathcal{B}} = \left(\frac{\partial W}{\partial y_y}\right)_{\mathbf{B},\mathcal{B}} = \left(\frac{\partial W}{\partial z_z}\right)_{\mathbf{B},\mathcal{B}} = \left(\frac{\partial W}{\partial (x_x + y_y + z_z)}\right)_{\mathbf{B},\mathcal{B}} = -\left(\frac{\partial W}{\partial \log D}\right)_{\mathbf{B},\mathcal{B}}. \tag{146}$$

By virtue of (143) and (144), one will easily get that the latter also:

$$= -W\left(\frac{\partial \log V}{\partial \log D}\right)_{\mathbf{B},\mathcal{B}}. \tag{147}$$

Upon forming the tensor that relates to both electromagnetism and striction:

$$\| T^{\alpha\beta(e)} + T^{\alpha\beta(m,e)} \|, \tag{148}$$

by means of the formulas (136 and 139) and formulas (142), one will finally obtain L. BRILLOUIN's *complete tensor:*

([1]) Léon BRILLOUIN, "Les tensions de Radiation; leur interpretation en Mécanique classique et en relativité," J. de Phys. (6) **6** no. 11 (1925), 337-353.

$$
(149) \qquad \begin{vmatrix}
W - W\dfrac{\partial \log V}{\partial \log D} & 0 & 0 & -\dfrac{VW}{c^2}\left[\dfrac{1+(c/V)^2}{2}\right] + \left(\dfrac{\partial W}{\partial x_t}\right)_{\mathbf{B},\boldsymbol{\mathcal{B}}} \\
0 & -W\dfrac{\partial \log V}{\partial \log D} & 0 & \left(\dfrac{\partial W}{\partial y_t}\right)_{\mathbf{B},\boldsymbol{\mathcal{B}}} \\
0 & 0 & -W\dfrac{\partial \log V}{\partial \log D} & \left(\dfrac{\partial W}{\partial z_t}\right)_{\mathbf{B},\boldsymbol{\mathcal{B}}} \\
-\dfrac{VW}{c^2}\left[\dfrac{1+(c/V)^2}{2}\right] + \left(\dfrac{\partial W}{\partial x_t}\right)_{\mathbf{B},\boldsymbol{\mathcal{B}}} & \left(\dfrac{\partial W}{\partial y_t}\right)_{\mathbf{B},\boldsymbol{\mathcal{B}}} & \left(\dfrac{\partial W}{\partial z_t}\right)_{\mathbf{B},\boldsymbol{\mathcal{B}}} & -\dfrac{1}{c^2}[W + P^{44(m,e)}
\end{vmatrix} .
$$

21. Helmholtz-Lippman formula (MINKOWSKI field). – Consider the case of the *perfect, isotropic* massive body *at rest*, and suppose that the *variation of the mass density in time is negligible,* as well as the MAXWELL *stresses*. On the other hand, let $f(D)$ be the expression for the pressure as a function of the density. Upon taking into account the formulas that we found for the electrostriction tensor, we will get very easily:

$$
(150) \qquad \boxed{\left(\frac{\partial f(D)}{\partial D}\right)_0 (D - D_0) = W_e \frac{\partial \log \varepsilon}{\partial \log D} + W_\mu \frac{\partial \log \mu}{\partial \log D},}
$$

in which:

$$
(151) \qquad W_e = \tfrac{1}{2}\mathbf{B}\cdot\mathbf{H}, \quad \text{and} \quad W_\mu = \tfrac{1}{2}\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\mathcal{H}} .
$$

____

Relativistic thermodynamics

22. First law of thermodynamics. – We write the first law in the form [1]:

$$\frac{d}{ds}[\mathcal{U}\,\delta(x_1, \ldots, x_4)] = \mathcal{Q}\,\delta(x_1, \ldots, x_4) - \mathcal{K}\,\delta(x_1, \ldots, x_4), \tag{152}$$

or in the equivalent form:

$$\sum_\alpha \frac{\partial(\mathcal{U}u^\alpha)}{\partial x^\alpha} = \mathcal{Q} - \mathcal{K}. \tag{153}$$

The symbol $\mathcal{U}$ is the factor of *internal* energy density of the system; $\mathcal{Q}$ is the factor of the caloric input to the system. An element of space-time is represented by $\delta(x_1, \ldots, x_4)$. The symbol $\mathcal{K}$ was defined in (115). Here are the dimensions of those symbols in Cartesian coordinates:

$$\mathcal{Q}\,\delta x\,\delta y\,\delta z\,\delta t \equiv \text{energy},$$

$$\mathcal{K}\,\delta x\,\delta y\,\delta z\,\delta t \equiv \text{energy},$$

$$\mathcal{U}\,\delta x\,\delta y\,\delta z\,\delta t \equiv \text{energy} \times \text{length}.$$

We use *right-hand rectangular coordinates.* The relation (153) is written:

$$\frac{\partial(\mathcal{U}V^{-1}v^\alpha)}{\partial x_\alpha} = \mathcal{Q} - \mathcal{K}, \tag{154}$$

in which:

$$v^\alpha = \frac{dx^\alpha}{dt} \qquad \text{and} \qquad V \equiv \frac{ds}{dt}. \tag{155}$$

Multiply the two sides of (154) by dv, which is defined by $dv = \delta x\,\delta y\,\delta z$. Hence, by virtue of the theory of integral invariants:

$$\frac{d}{dt}\int_v \mathcal{U}V^{-1}\delta v = \int_v \mathcal{Q}\,\delta v - \int_v \mathcal{K}\,\delta v. \tag{156}$$

Set:

$$\overline{V^{-1}}\int_v \mathcal{U}\,\delta v \equiv \int_v \mathcal{U}V^{-1}\,\delta v. \tag{157}$$

in which $\overline{V^{-1}}$ represents the mean of V^{-1} taken over the entire system in question.

Similarly, set:

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[1] Th. DE DONDER, Comptes rendus de l'Acad. des Sc. de Paris **186** (1928), 1599-1601; *ibid.* **187** (1928), 28-30.

$$(158)\qquad \begin{cases} U \equiv \dfrac{1}{c}\displaystyle\int_v \mathcal{U}\,\delta v, \\ \dfrac{dQ}{dt} \equiv \displaystyle\int_v \dfrac{V}{c}\mathcal{Q}\,\delta v, \quad \text{or} \quad \dfrac{\bar{V}}{c}\displaystyle\int_v \mathcal{Q}\,\delta v, \end{cases}$$

upon representing the mean of V over the system considered by $\bar{V}$.

Equation (156) can then be written:

$$(159)\qquad \frac{d}{dt}\left[c\overline{V^{-1}}U\right] = \frac{dQ}{dt}\frac{c}{V} - \int_v \mathcal{K}\,\delta v.$$

We say that U is the internal energy of the body at the instant t, and that dQ is the heat that is received by the body during dt.

In the case of a perfect, massive body, when one takes (115) into account, it will be easy to write (159) in the form:

$$(160)\qquad \frac{d}{dt}\left[c\overline{V^{-1}}U\right] = \frac{dQ}{dt}\frac{c}{V} - \bar{p}\,c\overline{V^{-1}}\oint_\sigma v_\nu \delta\sigma + cp\frac{\partial u^4}{\partial t}\delta v,$$

in which v_ν represents the component along the exterior semi-normal ν to the body at a point whose element $\delta\sigma$ is taken on the closed surface that bounds the body.

Now, one will obviously have:

$$(161)\qquad \oint_\sigma v_\nu \delta\sigma \equiv \frac{dv}{dt}.$$

We then recover the *first law of classical thermodynamics,* in the first approximation:

$$(162)\qquad \boxed{d\mathcal{U} = dQ - \bar{p}\,\delta v.}$$

23. Second law of thermodynamics. – In general relativity, we write the second law as follows:

$$(164)\qquad \frac{d}{ds}\left[\mathcal{S}\,\delta(x_1, \ldots, x_4)\right] = \frac{\mathcal{Q} + \mathcal{Q}^*}{T^*}\delta(x_1, \ldots, x_4),$$

in which $\mathcal{S}$ is the factor of *entropic* density, $\mathcal{Q}^*$ is the (positive) factor of *uncompensated* heat input (or of physico-chemical viscosity).

One deduces from (164) that:

$$(165)\qquad \frac{\partial(\mathcal{S}u^\alpha)}{\partial x_\alpha} = \frac{\mathcal{Q} + \mathcal{Q}^*}{T^*}.$$

Return to the Euclidian image; as in (156), one will have:

(166) $$\frac{d}{dt}\int \mathcal{S} V^{-1}\delta v = \frac{1}{\overline{\overline{T^*}}}\int_v (\mathcal{Q}+\mathcal{Q}^*)\,\delta v\,,$$

in which $\overline{T^*}$ is the mean of T^* over the system considered.

Now, set:

(167) $$S = \int_v \mathcal{S} V^{-1}\delta v\,,$$

and, as in (158):

(168) $$\frac{dQ^*}{dt} \equiv \frac{V^{-1}}{c}\int_v \mathcal{Q}^*\,\delta v\,.$$

Equation (166) can then become:

(169) $$\frac{dS}{dt} = \frac{c}{\overline{\overline{T^*}}\overline{V}}\left(\frac{dQ}{dt}+\frac{dQ^*}{dt}\right).$$

Set:

(170) $$T \equiv T^*\,\frac{\overline{V}}{c}\,,$$

by which the differential equation will express the *second law* of *classical* thermodynamics:

(171) $$\boxed{dS = \frac{dQ + dQ^*}{T}.}$$

24. Thermodynamics of electromagnetic systems endowed with hysteresis and animated with an arbitrary motion. – We use the first law of thermodynamics by substituting (115) into (152); hence ([1]):

(172) $$\frac{d}{ds}[\mathcal{U}\,\delta(x_1,\ldots,x_4)] = \mathcal{Q}\,\delta(x_1,\ldots,x_4) - [\mathcal{K}^{(m)} + \mathcal{H}^{(e)}_\alpha u^\alpha + \mathcal{K}^{(e)}]\,\delta(x_1,\ldots,x_4).$$

Now use the second law of thermodynamics by substituting (172) in (164); hence:

(173) $$\frac{d}{ds}[\mathcal{U}\,\delta(x_1,\ldots,x_4)] - T^*\frac{d}{ds}[\mathcal{S}\,\delta(x_1,\ldots,x_4)] =$$
$$- [\mathcal{K}^{(m)} + \mathcal{H}^{(e)}_\alpha u^\alpha + \mathcal{K}^{(e)}]\,\delta(x_1,\ldots,x_4) - \mathcal{Q}^*\,\delta(x_1,\ldots,x_4).$$

([1]) *Addenda.* In order to complete this energetic theory, there is good reason to suppose that:

$$\mathcal{U} = \mathcal{U}^{(m)} + \mathcal{U}^{(m,e)} + \mathcal{U}^{(e)},$$

and to assume that:

$$\mathcal{U}^{(e)} = \mathcal{T}^{(e)}_{\alpha\beta}\,u^\alpha u^\beta.$$

One can make the function $\mathcal{U} - T^{*}\mathcal{S}$, which generalizes the *free energy*, appear in this relation.

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Generalized Dirac wave mechanics

25. Gravitational and electronic equations. – We shall now present the study of some electromagnetic systems that are more general than the ones that were considered at the beginning of our present work in a form that is entirely synthetic. To that effect, first recall the *ten gravity equations;* namely:

$$(174)\qquad \frac{\delta(\mathcal{M}^g - \mathcal{N}W^2 + \mathcal{M}_*^{(m)} + \mathcal{M}_*^{(m,e)} + \mathcal{M}_*^{(e)})}{\delta g^{\alpha\beta}} = 0,$$

in which the characteristic functions $\mathcal{M}^g$, etc., that are written above have been defined previously.

We now pass on to the four *electronic* – or *Maxwellian* – equations, whose generalization we indicated in (82) and (83), and suppose that there is no magnetic current; i.e., that $C_*^\alpha = 0$. In that case, one can write:

$$(175)\qquad \mathcal{K}_*^{\overline{\alpha\beta}} = \Phi_{\alpha,\beta} - \Phi_{\beta,\alpha} .$$

In other words, those quantities derive from an electromagnetic vector potential Φ_1, …, Φ_4 . We shall also write these Maxwellian equations in the form of variational derivatives, namely:

$$(176)\qquad \frac{\delta(\mathcal{M}^g - \mathcal{N}W^2 + \mathcal{M}_*^{(m)} + \mathcal{M}_*^{(m,e)} + \mathcal{M}_*^{(e)})}{\delta \Phi_\alpha} = 0.$$

We shall specify the manner by which the variational derivatives with respect to Φ_α must be taken in (176) in order for one to obtain the aforementioned equations (82) and (83). In order to do that, let us turn to $\mathcal{M}_*^{(e)}$, whose value is given explicitly in (98). In that expression, we consider the $K_{\alpha i}$ to be functions *of only* x_1, …, x_4 (but not the Φ_α). The variations of those functions with respect to the Φ_α will then be zero.

Return to (176). We know that $\mathcal{M}^g$, $\mathcal{N}W^2$, $\mathcal{M}_*^{(m)}$ do not depend upon the Φ_α.

Set:

$$(177)\qquad \frac{\delta \mathcal{M}^{(m,e)}}{\delta \Phi_\alpha} = C_{(e)}^\alpha ,$$

in which $C_{(e)}^\alpha$ represents the (total) electric current.

The ten *gravity* equations (174) can then be written explicitly:

$$(178)\qquad -\tfrac{1}{2}\,(a + b\,C)\,g_{\alpha\beta} + b\,\mathcal{G}_{\alpha\beta} = N\,u_\alpha\,u_\beta + P_{\alpha\beta} + T_{\alpha\beta}^{(e)} ,$$

and the four *Maxwellian* equations are written explicitly by means of (177):

(179) $$\frac{\partial \mathcal{K}^{\alpha\beta}}{\partial x_\beta} = C^{\alpha}_{(e)}.$$

26. Photonic equations. – We assume that the equations of the *photonic field* have the same form as the electronic (i.e., Maxwell) equations that are given by (82) and (83). We then set:

(180) $$\frac{\partial \mathcal{U}^{\alpha\beta}}{\partial x_\beta} = C^{\alpha}_{(ph)},$$

(181) $$\frac{\partial \mathcal{U}^{\alpha\beta}_{*}}{\partial x_\beta} = C^{\alpha}_{\times(ph)}$$

as those photonic equations ([1]), in which $\mathcal{U}^{\alpha\beta}$ represents *the photon force,* and $\mathcal{U}^{\alpha\beta}_{*}$, the *dual photonic force.* Those antisymmetric tensors are then the analogues of $\mathcal{K}^{\alpha\beta}$ and $\mathcal{K}^{\alpha\beta}_{*}$. The symbols $C^{\alpha}_{(ph)}$ and $C^{\alpha}_{\times(ph)}$ are the *total* currents in the photonic field considered.

All of the quantities that enter into this paragraph will be regarded as being *complex;* they each then involve both a real part and a pure imaginary part.

In order to establish *a link* between the photon field and (Maxwell's) electromagnetic field, set (cf., previous footnote):

(182) $$C^{\alpha}_{(ph)} \equiv x\,\Phi_\beta\,\mathcal{U}^{\alpha\beta} + \mathcal{A}^{\alpha},$$

(183) $$C^{\alpha}_{\times(ph)} \equiv x\,\Phi_\beta\,\mathcal{U}^{\alpha\beta}_{*} + \mathcal{A}^{\alpha}_{\times}.$$

Therefore, the $\mathcal{A}^{\alpha}$ and $\mathcal{A}^{\alpha}_{\times}$ define the *diminished* photonic currents of the interaction currents between the electric and photonic field.

Upon substituting (182) and (183) into (180) and (181), one will get a new form for the photonic equations:

(184) $$\left[\frac{d\mathcal{U}^{\alpha\beta}}{dx_\beta}\right] = \mathcal{A}^{\alpha},$$

(185) $$\left[\frac{d\mathcal{U}^{\alpha\beta}_{*}}{dx_\beta}\right] = \mathcal{A}^{\alpha}_{\times},$$

in which one has set:

(186) $$\left[\frac{d}{dx_\beta}\right] \equiv \frac{d}{dx_\beta} - \kappa\Phi_\beta.$$

([1]) Th. DE DONDER, "Le champ photonique," Bull. Acad. Roy. Belgique (2 June 1928), 307-312.

We have set:

(187) $$\kappa = \frac{2i\pi ec}{h}.$$

By analogy with the way that one expresses electric and magnetic forces in terms of electromagnetic potentials Φ_α, set:

(188) $$\chi\, U_{\mu\nu} = [P_{\mu\nu}] + [Q^*_{\mu\nu}],$$

(189) $$c\; U^*_{\mu\nu} = [P^*_{\mu\nu}] - [Q_{\mu\nu}],$$

in which:

(190) $$[P_{\mu\nu}] = \left[\frac{dP_\nu}{dx_\mu} - \frac{dP_\mu}{dx_\nu}\right], \qquad [Q_{\mu\nu}] = \left[\frac{dQ_\nu}{dx_\mu} - \frac{dQ_\mu}{dx_\nu}\right],$$

in which the *brackets* that appear in the right-hand sides of (190) must be applied to each of the terms that they contain; they have the same significance as in (186). The symbol $Q^*_{\mu\nu}$ signifies that one must take the dual of $Q_{\mu\nu}$. On the other hand, the P_α and Q_α are the *photonic potentials* ([1]).

We have set:

(191) $$\chi \equiv \frac{2i\pi mc}{h}.$$

We write down the complementary photonic equations, always by analogy with the Maxwellian electronic field, namely:

(192) $$\left[\frac{d\mathcal{P}^\alpha}{dx_\alpha}\right] = \chi\,\mathcal{S},$$

(193) $$\left[\frac{d\mathcal{Q}^\alpha}{dx_\alpha}\right] = \chi\,\mathcal{B},$$

in which $\mathcal{S}$ and $\mathcal{B}$ are the *ether potentials* ([2]).

Now *reduce* the photonic currents $\mathcal{A}^\alpha$ and $\mathcal{A}^\alpha_\times$ of the (contravariant) gradients of those potentials S and B of the ether; hence:

(194) $$\mathcal{D}^\alpha \equiv \mathcal{A}^\alpha - \sqrt{-g}\, g^{\alpha\beta}\left[\frac{dS}{dx_\beta}\right],$$

(194) $$\mathcal{D}^\alpha_\times \equiv \mathcal{A}^\alpha_\times - \sqrt{-g}\, g^{\alpha\beta}\left[\frac{dB}{dx_\beta}\right].$$

([1]) J. M. WHITTAKER, Proc. Roy. Soc. **788** (1928), pp. 543.

([2]) Th. DE DONDER, *loc. cit.* (2 June 1928).

We say that $\mathcal{D}^\alpha$ and $\mathcal{D}_\times^\alpha$ are the *photonic currents, properly speaking.*

We remark that by virtue of (180) and (181), one will have the laws of *photonic conservation:*

(196) $$\frac{d\mathcal{C}^\alpha_{(ph)}}{dx_\alpha} = 0,$$

(197) $$\frac{d\mathcal{C}^\alpha_{\times(ph)}}{dx_\alpha} = 0.$$

27. Correspondence principle for wave systems. – Now introduce the characteristic function:

(198) $$\mathcal{M}^g + \underset{*}{\mathcal{M}}^{(e)} + \mathcal{M}^{(ph)}.$$

If we compare that function to the one that figures in (174) then we will see that we have replaced all of the massive terms $[-\mathcal{N}W^2 + \underset{*}{\mathcal{M}}^{(m)} + \underset{*}{\mathcal{M}}^{(m,e)}]$ in the latter with just the one photonic term $\mathcal{M}^{(ph)}$. We shall now define the function in terms of the photonic potentials P_μ and Q_μ and the electromagnetic potentials Φ_α. We write, to simplify:

(199) $$\xi_\mu = \kappa \Phi_\mu .$$

With J. M. Whittaker, set:

(200) $$\mathcal{M}^{(ph)} \equiv \sqrt{-g}\{U^{\mu\nu}\bar{U}_{\mu\nu} + 2(S\bar{S} - B\bar{B}) - 2(P^\mu \overline{P_\mu} - Q^\mu \overline{Q_\mu})\},$$

where the overbars that enter into (200) indicate that one must take the imaginary *conjugate* of the expressions that are affected with them.

Take the variational derivatives of $\mathcal{M}^g + \underset{*}{\mathcal{M}}^{(e)} + \mathcal{M}^{(ph)}$ with respect to $g^{\alpha\beta}$, Φ_α, and the photonic potentials P_μ, Q_μ, $\overline{P_\mu}$, $\overline{Q_\mu}$. We will then have the *gravity equations of wave mechanics:*

(201) $$\frac{\delta(\mathcal{M}^g + \underset{*}{\mathcal{M}}^{(e)} + \mathcal{M}^{(ph)})}{\delta g^{\alpha\beta}} = 0,$$

the electronic (or Maxwell) equations of wave mechanics:

(202) $$\frac{\delta(\mathcal{M}^g + \underset{*}{\mathcal{M}}^{(e)} + \mathcal{M}^{(ph)})}{\delta \Phi_\alpha} = 0,$$

and finally, the *photonic equations of wave mechanics:*

(203) $$\frac{\delta(\mathcal{M}^g + \underset{*}{\mathcal{M}}^{(e)} + \mathcal{M}^{(ph)})}{\delta P_\mu} = 0, \qquad \frac{\delta(\mathcal{M}^g + \underset{*}{\mathcal{M}}^{(e)} + \mathcal{M}^{(ph)})}{\delta Q_\mu} = 0,$$

(204) $$\frac{\delta(\mathcal{M}^g + \mathcal{M}_*^{(e)} + \mathcal{M}^{(ph)})}{\delta \overline{P_\mu}} = 0, \qquad \frac{\delta(\mathcal{M}^g + \mathcal{M}_*^{(e)} + \mathcal{M}^{(ph)})}{\delta \overline{Q_\mu}} = 0.$$

Upon performing the indicated calculations in (201), we will obtain explicitly:

(205) $$-\tfrac{1}{2}(a + b\,C)\,g^{\alpha\beta} + b\,\mathcal{G}^{\alpha\beta} = T_{(e)}^{\alpha\beta} + M^{\alpha\beta},$$

in which $T_{(e)}^{\alpha\beta}$ is the contravariant expression of (9), and in which we have set:

(206) $$M^{\alpha\beta} \equiv -2L^{\alpha\beta} + \left(\tfrac{1}{2}L - P^\mu \overline{P_\mu} + Q^\mu \overline{Q_\mu}\right) g^{\alpha\beta} + 2N^{\alpha\beta} - 2O^{\alpha\beta} + 2P^\alpha \overline{P^\beta} - 2Q^\alpha \overline{Q^\beta}\,.$$

In this expression, the symbols $L^{\mu\nu}$, L, $N^{\mu\nu}$, $O^{\mu\nu}$ are defined by the following relations:

(207) $$g\,L^{\kappa\alpha\lambda\beta} \equiv \frac{h^2}{4\pi^2 m^2 c^4}\left(-[P_{\overline{\alpha\beta}}][\overline{P}_{\overline{\kappa\lambda}}] + [\overline{Q_{\alpha\beta}}][Q_{\overline{\kappa\lambda}}]\right),$$

(208) $$L^{\alpha\beta} = g_{\gamma\delta}\,L^{\alpha\beta\gamma\delta}, \qquad L = g_{\alpha\beta}\,L^{\alpha\beta},$$

(209) $$N^{\alpha\beta} \equiv \left(\tfrac{1}{2}\overline{P^\mu} g^{\alpha\beta} - \overline{P^\beta} g^{\alpha\mu}\right)\left(\frac{1}{\chi}\left[\frac{dS}{dx_\mu}\right]\right) + \text{conjugate} - \tfrac{1}{2}S\overline{S}\,g^{\alpha\beta},$$

(210) $$O^{\alpha\beta} \equiv \left(\tfrac{1}{2}\overline{Q^\mu} g^{\alpha\beta} - \overline{Q^\beta} g^{\alpha\mu}\right)\left(\frac{1}{\chi}\left[\frac{dB}{dx_\mu}\right]\right) + \text{conjugate} - \tfrac{1}{2}B\overline{B}\,g^{\alpha\beta}\,.$$

Upon performing the indicated calculations in (202), we will get explicitly:

(211) $$\frac{d\mathcal{K}^{\alpha\beta}}{dx_\beta} = -\frac{e\sqrt{-g}}{mc^2}\left(U^{\alpha\beta}\overline{P_\beta} + U_*^{\alpha\beta}\overline{Q_\beta} + S\overline{P^\alpha} - B\overline{Q^\alpha} + \text{conjugate}\right).$$

Finally, upon performing the indicated calculations in (203) and (204), we will get the following relations, along with their conjugates:

(212) $$\left[\frac{d\mathcal{U}^{\alpha\beta}}{dx_\beta}\right] = \sqrt{-g}\,g^{\alpha\beta}\left[\frac{dS}{dx_\beta}\right] - \chi\,\mathcal{P}^\alpha,$$

(213) $$\left[\frac{d\mathcal{U}_*^{\alpha\beta}}{dx_\beta}\right] = \sqrt{-g}\,g^{\alpha\beta}\left[\frac{dB}{dx_\beta}\right] - \chi\,\mathcal{Q}^\alpha.$$

Upon *identifying* ([1]) *equations* (178) and (205), (179), and (211), (184), with equations (212), (185), and (213), one will get equations that express the *correspondence principle* in wave mechanics.

Equations (178) and (205) give:

$$N u^{\alpha} u^{\beta} + P^{\alpha\beta} = M^{\alpha\beta}, \tag{214}$$

in which $M^{\alpha\beta}$ is given by (206).

Equations (179) and (211) give:

$$\mathcal{C}^{\alpha}_{(e)} = -\frac{e\sqrt{-g}}{mc^2}\left(U^{\alpha\beta}\overline{P_{\beta}} + U_{*}^{\alpha\beta}\overline{Q_{\beta}} + S\overline{P^{\alpha}} - B\overline{Q^{\alpha}} + \text{conjugate}\right). \tag{215}$$

Finally, when one takes (194) and (195) into account, equations (179) and (211), as well as (184) and (212), will give:

$$\mathcal{D}^{\alpha} = -\chi\mathcal{P}^{\alpha}, \qquad \mathcal{D}^{\alpha}_{\times} = -\chi\mathcal{Q}^{\alpha}. \tag{216}$$

One then sees that it is possible to express the *matter tensor*, the *electric current*, and the *photonic currents*, properly speaking, as functions of Whittaker's *photonic potentials*.

28. Energy-impulse theorem and the law of conservation of electricity, expressed in terms of photonic potentials. – Return to the energy-impulse theorem (102) and replace the massive part $N u^{\alpha} u^{\beta} + P^{\alpha\beta}$ of the tensor $T^{\alpha\beta}$ with its value $M^{\alpha\beta}$ that is given in (206). One will then have:

$$\left(M^{\alpha\beta} + T^{\alpha\beta}_{(e)}\right)_{,\beta} = 0, \tag{217}$$

which expresses the *photonic energy-impulse theorem.*

Likewise, return to the law conservation of electricity (91) and replace the $\mathcal{C}^{\alpha}_{(e)}$ in the equation:

$$\frac{d\mathcal{C}^{\alpha}_{(e)}}{dx_{\alpha}} = 0 \tag{218}$$

with its value (211).

One will then have the *law of conservation of electricity* (expressed by means of photonic potentials).

([1]) I have indicated that method of identification in order to find the correspondence principle in my prior papers. [See my note that appeared in Bull. Acad. Roy. des Belg, Cl. des Sciences (5) **13** (1927).]

29. D'Alembertian equation or the equation of propagation of the photonic potentials. – Upon replacing the $\mathcal{U}^{\alpha\beta}$ and $\mathcal{U}_*^{\alpha\beta}$ in (212) and (213) with their values (188) and (189), resp., one will get second-order equations:

(219) $$g^{\sigma\tau}(P^{\mu})_{\sigma\tau} + \mathcal{G}^{\mu\nu} P_\nu - \frac{4\pi iec}{h}(P^{\mu})_\nu + \frac{4\pi^2 m^2 c^4}{h^2}\left(1 - \frac{e^2}{m^2c^2}\Phi_\nu\Phi^\nu\right)P^{\mu} - \frac{2\pi iec}{h}\left\{H^{\mu\nu}P_\nu - H^{\overline{\mu\nu}}Q_\nu\right\} = 0$$

and

(220) $$g^{\sigma\tau}(Q^{\mu})_{\sigma\tau} + \mathcal{G}^{\mu\nu} Q_\nu - \frac{4\pi iec}{h}(Q^{\mu})_\nu + \frac{4\pi^2 m^2 c^4}{h^2}\left(1 - \frac{e^2}{m^2c^2}\Phi_\nu\Phi^\nu\right)Q^{\mu} - \frac{2\pi iec}{h}\left\{H^{\mu\nu}P_\nu - H^{\overline{\mu\nu}}Q_\nu\right\} = 0,$$

as well as conjugate equations.

30. Dirac wave equations. – We shall apply the general equations above in the particular case of a Minkowski field, and we choose the variables in such a fashion that the ds^2 that they define have the form:

(221) $$ds^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2).$$

Instead of the vector potential Φ_α that we have used above, with Whittaker, we shall use the potential Φ'_α that is coupled to the preceding one by the relation:

(222) $$\Phi'_\alpha = \frac{1}{c^2}\Phi_\alpha.$$

Upon using the notations that we have defined before, we will have:

(223) $$\Phi'_1 = -\frac{1}{c^2}A_x, \quad \Phi'_2 = -\frac{1}{c^2}A_x, \quad \Phi'_3 = -\frac{1}{c^2}A_z, \quad \Phi'_4 = c\,V.$$

We likewise set:

(224) $$X_{\mu\nu} = \frac{\partial\Phi'_\nu}{\partial x_\mu} - \frac{\partial\Phi'_\mu}{\partial x_\nu},$$

which will give us:

(225) $$X_{\mu\nu} = \frac{1}{c^2}\left(\frac{\partial\Phi_\nu}{\partial x_\mu} - \frac{\partial\Phi_\mu}{\partial x_\nu}\right) = -\frac{1}{c^2}\mathcal{K}_*^{\overline{\mu\nu}},$$

by virtue of (175).

In this paragraph, we will suppose that there is neither electric nor magnetic polarization, in such a way that we can write:

(226) $$X_{\mu\nu} = -\frac{1}{c^2}\ H_{\mu\nu}\,.$$

We remark that for Whittaker and his school, the indices μ, ν, etc., vary from 0 to 3, instead of 1 to 4. For more uniformity, we use that manner of proceeding here and choose 0, 1, 2, 3 to be the fundamental permutation.

Thanks to (221), we can deduce the components $H^{\mu\nu}$ from (226) very easily; namely:

(227) $$X^{\mu\nu} = -\,c^2\,H^{\mu\nu}\,.$$

As before, we write:

(228) $$\left\{\begin{aligned} X_{01} &= \frac{1}{c^2}H_{14} = \frac{1}{c}H_x, \\ X_{02} &= \frac{1}{c^2}H_{24} = \frac{1}{c}H_y, \\ X_{03} &= \frac{1}{c^2}H_{34} = \frac{1}{c}H_z, \end{aligned}\right. \qquad \left\{\begin{aligned} X_{23} &= -\frac{1}{c^2}\mathcal{H}_{23} = -\frac{1}{c}\mathcal{H}_x, \\ X_{31} &= -\frac{1}{c^2}\mathcal{H}_{31} = -\frac{1}{c}\mathcal{H}_y, \\ X_{12} &= -\frac{1}{c^2}\mathcal{H}_{12} = -\frac{1}{c}\mathcal{H}_z. \end{aligned}\right.$$

It results from (227) that:

(229) $$\left\{\begin{aligned} X^{01} &= -cH_x, \qquad & X^{23} &= \mathcal{H}_x, \\ X^{02} &= -cH_y, \qquad & X^{31} &= \mathcal{H}_y, \\ X^{03} &= -cH_z, \qquad & X^{12} &= \mathcal{H}_z. \end{aligned}\right.$$

Recall that **H** and $\boldsymbol{\mathcal{H}}$ are the electric and magnetic forces, respectively.

With Whittaker, one further sets:

(230) $$P_0 \equiv P_t\,, \qquad P_1 \equiv -\frac{1}{c}P_x\,, \quad P_0 \equiv -\frac{1}{c}P_y\,, \quad P_0 \equiv -\frac{1}{c}P_z\,,$$

and

(231) $$Q_0 \equiv Q_t\,, \qquad Q_1 \equiv -\frac{1}{c}Q_x\,, \quad Q_0 \equiv -\frac{1}{c}Q_y\,, \quad Q_0 \equiv -\frac{1}{c}Q_z\,.$$

Thanks to (221), we will have:

(232) $$P^0 \equiv P_t\,, \qquad P^1 \equiv -\,cP_x\,, \quad P^2 \equiv -\,cP_y\,, \quad P^4 \equiv -\,cP_z\,,$$

(233) $$Q^0 \equiv Q_t\,, \qquad Q^1 \equiv -\,cQ_x\,, \quad Q^2 \equiv -\,cQ_y\,, \quad Q^4 \equiv -\,cQ_z\,.$$

It finally remains for us to address the $U_{\mu\nu}$. We set:

(234)
$$\begin{cases} U_{01} = \frac{1}{c} s_x, \\ U_{02} = \frac{1}{c} s_y, \\ U_{03} = \frac{1}{c} s_z, \end{cases} \qquad \begin{cases} U_{23} = \frac{1}{c^2} b_x, \\ U_{31} = \frac{1}{c^2} b_y, \\ U_{12} = \frac{1}{c^2} b_z, \end{cases}$$

which will give immediately:

(235)
$$\begin{cases} U^{01} = -c s_x, \\ U^{02} = -c s_y, \\ U^{03} = -c s_z, \end{cases} \qquad \begin{cases} U^{23} = c^2 b_x, \\ U^{31} = c^2 b_y, \\ U^{12} = c^2 b_z. \end{cases}$$

Introduce the notations (223) and (236) into equations (212), (213). We get:

(236)
$$\begin{cases} -\frac{1}{c}\left[\frac{\partial \mathbf{s}}{\partial t}\right] - [\text{rot}\,\mathbf{b}] = \frac{2mi}{h} mc\,\mathbf{P} + [\text{grad}\,S], \\ [\text{div}\,\mathbf{s}] = \frac{2mi}{h} mc\,P_t - \frac{1}{c}\left[\frac{\partial S}{\partial t}\right], \\ -\frac{1}{c}\left[\frac{\partial \mathbf{b}}{\partial t}\right] + [\text{rot}\,\mathbf{s}] = \frac{2mi}{h} mc\,\mathbf{Q} - [\text{grad}\,B], \\ [\text{div}\,\mathbf{b}] = \frac{2mi}{h} mc\,Q_t - \frac{1}{c}\left[\frac{\partial B}{\partial t}\right]. \end{cases}$$

The brackets that enter into these equations have the same significance as before.

In the equations above, the symbols **P**, **Q** denote the ordinary vectors whose components are (P_x, P_y, P_z), (Q_x, Q_y, Q_z), resp. The scalars S and B, and the vectors $\mathbf{b} = (b_x, b_y, b_z)$ and $\mathbf{s} = (s_x, s_y, s_z)$ are defined by the relations (188) and (189), which become:

(237)
$$\begin{cases} \frac{2\pi i}{h} mc\,\mathbf{b} = -[\text{rot}\,\mathbf{P}] - \frac{1}{c}\left[\frac{\partial \mathbf{Q}}{\partial t}\right] - [\text{grad}\,Q_t], \\ \frac{2\pi i}{h} mc\,\mathbf{s} = [\text{rot}\,\mathbf{Q}] - \frac{1}{c}\left[\frac{\partial \mathbf{P}}{\partial t}\right] - [\text{grad}\,P_t], \\ \frac{2\pi i}{h} mc\,S = [\text{rot}\,\mathbf{P}] + \frac{1}{c}\left[\frac{\partial P_t}{\partial t}\right], \\ \frac{2\pi i}{h} mc\,B = [\text{rot}\,\mathbf{Q}] + \frac{1}{c}\left[\frac{\partial Q_t}{\partial t}\right], \end{cases}$$

here.

By way of example, we show what the first vectorial equation (237) will become when one performs the operations that are indicated by the brackets; we get:

(238) $$-\frac{1}{c}\frac{\partial \mathbf{s}}{\partial t} - \text{rot}\,\mathbf{b} - \text{grad}\,S + \frac{2\pi ie}{hc}\{V\,\mathbf{s} - [\mathbf{A}\cdot\mathbf{b}] - \mathbf{A}\cdot\mathbf{S}\} = \frac{2\pi i}{h}mc\,\mathbf{P}.$$

Equations (236), when specified in that way, can be considered to be wave equations, while equations (237) give the values of the vectors **b** and **s**, and the scalars *B* and *S*. Introduce these values into equations (236). We obtain:

(239) $$\begin{cases} D\mathbf{P} + \frac{2\pi ie}{hc}\{-[\boldsymbol{\mathcal{H}}\cdot\mathbf{P}] + \mathbf{H}\cdot P_t - [\mathbf{H}\cdot\mathbf{Q}] - \mathbf{H}\cdot Q_t\} = 0, \\ DP_t + \frac{2\pi ie}{hc}\{(\mathbf{H}\cdot\mathbf{P}) - (\boldsymbol{\mathcal{H}}\cdot\mathbf{Q})\} = 0, \\ D\mathbf{Q} + \frac{2\pi ie}{hc}\{-[\boldsymbol{\mathcal{H}}\cdot\mathbf{Q}] + \mathbf{H}\cdot Q_t + [\mathbf{H}\cdot\mathbf{P}] + \boldsymbol{\mathcal{H}}\cdot P_t\} = 0, \\ DQ_t + \frac{2\pi ie}{hc}\{(\mathbf{H}\cdot\mathbf{Q}) + (\boldsymbol{\mathcal{H}}\cdot\mathbf{P})\} = 0, \end{cases}$$

in which *D* is the operator:

(240) $$D = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{4\pi ie}{hc}\left[V\frac{1}{c}\frac{\partial}{\partial t} + A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right] - \frac{4\pi^2 m^2 c^2}{h^2 c^2}(V^2 - A_x^2 - A_y^2 - A_z^2).$$

Set:

(241) $$\begin{cases} \psi_1 = P_z + iQ_t, \quad \psi_2 = P_x + iP_y, \quad \psi_3 = -P_t - iQ_z, \quad \psi_4 = Q_y - iQ_x, \\ \omega_1 = -P_x + iP_y, \quad \omega_2 = P_z - iQ_t, \quad \omega_3 = Q_y + iQ_x, \quad \omega_4 = P_t - iQ_z. \end{cases}$$

With those notations, equations (239) will be equivalent to the system:

(242) $$\begin{cases} D\psi_1 + \frac{2\pi ie}{hc}(i\mathcal{H}_x\psi_2 + i\mathcal{H}_y\psi_2 + i\mathcal{H}_z\psi_1 - H_x\psi_4 + iH_y\psi_4 - H_z\psi_3) = 0, \\ D\psi_2 + \frac{2\pi ie}{hc}(i\mathcal{H}_x\psi_1 - i\mathcal{H}_y\psi_1 - i\mathcal{H}_z\psi_2 - H_x\psi_3 - iH_y\psi_3 + H_z\psi_4) = 0, \\ D\psi_3 + \frac{2\pi ie}{hc}(i\mathcal{H}_x\psi_3 - i\mathcal{H}_y\psi_3 - i\mathcal{H}_z\psi_4 - H_x\psi_2 + iH_y\psi_2 - H_z\psi_2) = 0, \\ D\psi_4 + \frac{2\pi ie}{hc}(i\mathcal{H}_x\psi_3 - i\mathcal{H}_y\psi_3 - i\mathcal{H}_z\psi_4 - H_x\psi_1 - iH_y\psi_1 + H_z\psi_2) = 0, \end{cases}$$

to which one must add a system that is equivalent, but in which one has replaced the ψ with the ω. Equations (242) are second-order equations that the Dirac functions ψ_1, ψ_2, ψ_3, ψ_4 must satisfy. If one sets:

(243)
$$\begin{cases} a_1 = -s_z + iB, & a_2 = -s_x - is_y, & a_3 = -ib_z + S, & a_4 = b_y - ib_x, \\ \beta_1 = s_x - is_y, & \beta_2 = -s_z - iB, & \beta_3 = ib_x + b_y, & \beta_4 = -ib_z - S \end{cases}$$

then equations (236) and (237) are equivalent to four systems of four equations; the first one has the following form:

(244)
$$\begin{cases} p_0 \psi_1 + (p_1 - i\, p_2)\psi_4 + p_3 \psi_3 = -mc\,\alpha_1, \\ p_0 \psi_2 + (p_1 + i\, p_2)\psi_2 - p_3 \psi_4 = -mc\,\alpha_2, \\ p_0 \psi_3 + (p_1 - i\, p_2)\psi_2 + p_3 \psi_1 = mc\,\alpha_3, \\ p_0 \psi_4 + (p_1 + i\, p_2)\psi_1 - p_3 \psi_2 = mc\,\alpha_4, \end{cases}$$

in which:

(245)
$$p_0 = -\frac{h}{2\pi i}\frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{c}V, \qquad p_1 = \frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c}A_x, \qquad \text{etc.}$$

In order to obtain the second system, it will suffice to permute ψ and α; namely:

(246)
$$\begin{cases} p_0\,\alpha_1 + (p_1 - ip_2)\,\alpha_4 + p_3\,\alpha_3 = -mc\,\psi_1, \\ \text{etc.}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

In order to obtain the third system, it will suffice to replace ψ with ω and α with β in (244), and finally in order to obtain the fourth system, one must permute ω and β in the third one.

In will then result that if ψ_1, ψ_2, ψ_3, ψ_4 is a solution of the Dirac equations then:

(247)
$$\alpha_\mu = \beta_\mu = \omega_\mu = \psi_\mu, \qquad \mu = 1, 2, 3, 4$$

will constitute a solution of (236) and (237).

We now examine what the components of the electric current (215):

$$C^{\alpha}_{(e)} = \frac{1}{\sqrt{-g}}\mathfrak{C}^{\alpha}_{(e)} \qquad (\alpha = 0, 1, 2, 3)$$

will become here. We set:

(248)
$$C^4_{(e)} = \rho, \qquad C^1_{(e)} \equiv j_x, \qquad C^2_{(e)} \equiv j_y, \qquad C^3_{(e)} \equiv j_z.$$

Upon introducing the notations (223) to (235) into (215), one will get:

$$(249)\begin{cases} \rho = -\frac{1}{2}(\psi_1\overline{\alpha_1} + \psi_2\overline{\alpha_2} + \psi_3\overline{\alpha_3} + \psi_4\overline{\alpha_4} + \omega_1\overline{\beta_1} + \omega_2\overline{\beta_2} + \omega_3\overline{\beta_3} + \omega_4\overline{\beta_4} + \text{conjugate}), \\ j_x = -\frac{c}{2}(\psi_1\overline{\alpha_4} + \psi_2\overline{\alpha_3} + \psi_3\overline{\alpha_2} + \psi_4\overline{\alpha_1} + \omega_1\overline{\beta_4} + \omega_2\overline{\beta_3} + \omega_3\overline{\beta_2} + \omega_4\overline{\beta_1} + \text{conjugate}), \\ j_y = \dots\dots, \\ j_z = \dots\dots, \end{cases}$$

in such a way that for a solution of the form (247), one will get:

$$(250)\qquad \begin{cases} \rho = -2(\psi_1\overline{\psi_1} + \psi_2\overline{\psi_2} + \psi_3\overline{\psi_3} + \psi_4\overline{\psi_4}), \\ j_x = 2c\,(\psi_1\overline{\psi_4} + \psi_2\overline{\psi_3} + \psi_3\overline{\psi_2} + \psi_4\overline{\psi_1}), \\ etc.\dots\dots, \end{cases}$$

which are precisely the expressions that Darwin found ([1]) that conform to Dirac's theory.

([1]) DARWIN, Proc. Roy. Soc. A**118**, no. 780, pp. 660, eq. (3.3).