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ON THE

CANONICAL HAMILTON-VOLTERRA EQUATIONS

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TABLE OF CONTENTS

INTRODUCTION

Hamilton's canonical equations have been generalized in several very different ways. Volterra (*), while studying hyperspace functions, found a new generalization of those equations and their principal properties, as well as the theorem of Jacobi that relates to them. The results that were obtained by that profound analyst often supposed that the number of dependent variables x was three, and that of the independent variables t was two. I propose to call those equations the *canonical Hamilton-Volterra equations.*

Fréchet (**) extended most of Volterra's results to the case in which the number of variables, whether dependent or independent, was arbitrary. He used the parametric form systematically. Above all, what Fréchet had in mind was a generalization of Jacobi's theorem, so his contributions are more important.

It seems useful to me to make a *complete* presentation of the theory of the Hamilton-Volterra canonical equations and to establish the relationships that exist between Volterra's beautiful research and that of Lie and Poincaré that related to *invariants*. Please permit me to point out some of the new parts in this treatise. The generalized Lagrange equations, as well as the Hamilton-Volterra canonical equations, have been obtained by starting from the very simple *relation* ($d\delta$) (nos. **1** and **2**):

$$\sum_{\lambda=1}^{r} \frac{dj_\lambda}{dt_\lambda} = \delta W \,.$$

Thanks to the theory of integral invariants (***), one can one can immediately deduce a *generalized relative invariant* of the proposed equations from that (no. **6**), as well as the property that is pointed out in no. **7**. In order to make its statement precise and prove the fifth and sixth properties (nos. **11** and **12**), I have considered a system of equations that I have called *the special Volterra equations*. The seventh property (no. **13**) has been added to that group of properties. I have indicated (in no. **15**) a *generalization of the (finite) invariant*, a *covariant*, and a *theorem* that has been used frequently by S. Lie. That generalized theorem shows its utility in the proof of the direct Jacobi theorem (no. **19**). I have *extended* the converse of Jacobi's theorem, which was generalized by Volterra and Fréchet (no. **21**).

A glance at the *Table of Contents* will clearly indicate the plan of this treatise.

(*) V. VOLTERRA, "Sopra una estensione della Jacobi-Hamilton del calcolo delle variazione," Atti della R. Acc. Lincei, Roma, Rendiconti, (4) **6**, 1st semester (1890), pp. 127. See also the complete bibliography of Volterra's works on hyperspace functions or Volterra functions in my paper, "Sur les fonctions de Volterra et les invariants intégraux," Bull. de l'Acad. royale de Belgique, Classe des sciences, no. 6, 1906.

(**) M. FRÉCHET, "Sur une extension de la méthode de Jacobi-Hamilton, Annali di mat. (3) **11** (1905).

(***) TH. DE DONDER, "Étude sur les invariants intégraux (deuxième mémoire)," Rendiconti del Circ. mat. Palermo **16** (1902), See no. 60 in that study.

CHAPTER ONE

Establishing the canonical Hamilton-Volterra equations.

1. Generalized Lagrange equations. – Consider n functions $x_1, \ldots, x_n$ of $r \leq n$ independent variables $t_1, \ldots, t_r$ and several arbitrary parameters. The infinitely-small increases that are given to the independent variables $t_1, \ldots, t_r$ will be denoted by $dt_1, \ldots, dt_r$, resp., and the increases in the $x_1, \ldots, x_n$ that result will be denoted by:

$$\frac{dx_i}{dt_\lambda} dt_\lambda \equiv x_i^\lambda \, dt_\lambda \qquad \begin{array}{l} i = 1, \ldots, n \\ \lambda = 1, \ldots, r. \end{array}$$

The infinitesimal variations of $x_1, \ldots, x_n$ that are due to the variations of one of the arbitrary parameters will be denoted by $\delta x_1, \ldots, \delta x_n$.

Consider the functions N_i^λ and W of t_λ, x_i, x_i^λ, and set:

$$j_\lambda \equiv \sum_{i=1}^n N_i^\lambda \, \delta x_i \,, \qquad \lambda = 1, \ldots, r.$$

Look for the necessary and sufficient conditions for one to have the identity:

$$\sum_{\lambda=1}^r \frac{dj_\lambda}{dt_\lambda} = \delta W .$$

Upon noting that $\delta t_\lambda = 0$, we will find that:

$$\sum_{\lambda=1}^r \sum_{i=1}^n \left(\frac{dN_i^\lambda}{dt_\lambda} \delta x_i + N_i^\lambda \, \delta x_i^\lambda \right) = \sum_{i=1}^n \frac{\partial W}{\partial x_i} \delta x_i + \sum_{i=1}^n \sum_{\lambda=1}^r \frac{\partial W}{\partial x_i^\lambda} \delta x_i^\lambda .$$

Upon identifying, we will obtain the necessary and sufficient conditions:

$$\left\{ \begin{aligned} \frac{\partial W}{\partial x_i} &= \sum_{\lambda=1}^r \frac{dN_i^\lambda}{dt_\lambda}, \\ \frac{\partial W}{\partial x_i^\lambda} &= N_i^\lambda . \end{aligned} \right.$$

We will then have:

$$j_\lambda \equiv \sum_{i=1}^n \frac{\partial W}{\partial x_i^\lambda} \delta x_i \,,$$

and we say that those r linear forms in $\delta x_1, \ldots, \delta x_n$ constitute a (generalized) *relative invariant* (*) of the equations:

$$\left\{\begin{aligned} & x_i^\lambda = \frac{dx_i}{dt_\lambda}, \\ & \frac{\partial W}{\partial x_i} - \sum_{\lambda=1}^{r} \frac{d}{dt_\lambda}\left(\frac{\partial W}{\partial x_i^\lambda}\right) = 0. \end{aligned}\right.$$

Now suppose, with Volterra (**), that W depends upon the x_i^λ only by way of the *determinants of order r*, such as:

$$\begin{vmatrix} x_{i_1}^1 & \cdots & x_{i_1}^r \\ \vdots & \ddots & \vdots \\ x_{i_r}^1 & \cdots & x_{i_r}^r \end{vmatrix} \equiv \frac{d\,(x_{i_1}, \ldots, x_{i_r})}{d\,(t_1, \ldots, t_r)} \equiv \xi_{i_1,\ldots,i_r},$$

in which $i_1, \ldots, i_r$ represent any combination of the first n whole numbers taken r at a time. That number of combinations will be denoted by C_n^r in what follows.

Now calculate the $\partial W / \partial x_i^\lambda$; one will have:

$$\frac{\partial W}{\partial x_i^\lambda} = \sum_{k_1,\ldots,k_r} \frac{\partial W}{\partial \xi_{k_1\cdots k_r}} \frac{\partial \xi_{k_1\cdots k_r}}{\partial x_i^\lambda} = \sum_{k_1,\ldots,k_{r-1}} \frac{\partial W}{\partial \xi_{k_1\cdots k_{r-1}}} \frac{\partial W}{\partial x_i^\lambda}\, \xi_{k_1\cdots k_{r-1}}^{(\lambda)},$$

in which

$$\xi_{k_1\cdots k_{r-1}}^{(\lambda)} \equiv \begin{vmatrix} x_{k_1}^1 & \cdots & x_{k_1}^{\lambda-1} & x_{k_1}^{\lambda+1} & \cdots & x_{k_1}^r \\ \vdots & & & & & \vdots \\ x_{k_{r-1}}^1 & \cdots & x_{k_{r-1}}^{\lambda-1} & x_{k_{r-1}}^{\lambda+1} & \cdots & x_{i_r}^r \end{vmatrix} \equiv \frac{d\,(x_{k_1}, \ldots, x_{k_{r-1}})}{d\,(t_1, \ldots, t_{\lambda-1}, t_{\lambda+1}, \ldots, t_r)},$$

and in which $\sum_{k_1,\ldots,k_r}$ denotes a summation over all *combinations* of the n indices $1, \ldots, n$ taken r at a time.

The relative invariant can then be written:

$$j_\lambda \equiv (-1)^{\lambda+1} \sum_{i=1}^{n} \sum_{k_1\cdots k_{r-1}} \frac{\partial W}{\partial \xi_{i\,k_1\cdots k_{r-1}}}\, \xi_{k_1\cdots k_{r-1}}^{(\lambda)} = 0 .$$

However, one easily verifies the two identities:

(*) See no. 60 in cited paper: "Étude sur les invariants intégraux."
(**) See the cited paper by Volterra: "Sopra una estensione…"

$$\sum_{\lambda=1}^{r}(-1)^{\lambda+1}\,\xi^{(\lambda)}_{k_1\cdots k_{r-1}}\sum_{k_1\cdots k_{r-1}}\frac{d}{dt_\lambda}\left(\frac{\partial W}{\partial \xi_{i k_1\cdots k_{r-1}}}\right) \equiv \sum_{k_1\cdots k_{r-1}}\frac{d}{d(t_1,\ldots,t_r)}\left(\frac{\partial W}{\partial \xi_{i k_1\cdots k_{r-1}}}, x_{k_1},\ldots,x_{k_{r-1}}\right),$$

$$\sum_{\lambda=1}^{r}(-1)^{\lambda+1}\frac{d}{dt_\lambda}\xi^{(\lambda)}_{k_1\cdots k_{r-1}} \equiv 0\,.$$

The equations that correspond to the invariant j_λ can then be put into the form:

$$\begin{cases} \xi_{k_1\cdots k_{r-1}} = \dfrac{d(x_{k_1},\ldots,x_{k_r})}{d(t_1,\ldots,t_r)}, \\[2ex] \dfrac{\partial W}{\partial x_i} = \displaystyle\sum_{k_1\cdots k_{r-1}}\frac{d}{d(t_1,\ldots,t_r)}\left(\frac{\partial W}{\partial \xi_{i k_1\cdots k_{r-1}}}, x_{k_1},\ldots,x_{k_{r-1}}\right). \end{cases}$$

Those are the *generalized Lagrange equations.*

One will get the classical Lagrange equation upon setting $r = 1$. Those generalized equations were obtained by Volterra and Fréchet in succession, upon starting from a problem in the calculus of variations. Fréchet reduced it to the case in which W is a homogeneous function of degree one in $\xi_{k_1\cdots k_{r-1}}$ (see below, no. **22**).

2. Canonical Hamilton-Volterra equations. – Set:

$$\frac{\partial W}{\partial \xi_{k_1\cdots k_{r-1}}} \equiv p_{k_1\cdots k_{r-1}}\,,$$

and with Volterra we suppose, first of all, that we can solve those C_n^r equations for the $\xi_{k_1\cdots k_{r-1}}$, when they are considered to be C_n^r *distinct* variables (*). After that change of variables, W will become a function of t_λ, x_i, and $p_{k_1\cdots k_r}$; we let W_1 represent that function. We will then have:

$$\sum_{\lambda=1}^{r}\frac{dj_\lambda}{dt_\lambda} = \delta W_1\,,$$

or

(*) Those $\xi_{k_1\cdots k_{r-1}}$ satisfy the identities:

$$\sum_{\mu=1}^{r+1}(-1)^\mu\,\xi_{i_1\cdots i_{\mu-1} i_{\mu+1}\cdots i_{r+1}}\,\xi_{i_\mu k_1\cdots i_{r-1}} \equiv 0\,.$$

$$\sum_{i=1}^{n} \left[\sum_{\lambda=1}^{r} \frac{d}{dt_\lambda} (-1)^{\lambda+1} \sum_{k_1,\dots,k_{r-1}} p_{i k_1,\dots,k_{r-1}} \, \xi^{(\lambda)}_{k_1,\dots,k_{r-1}} \right] \delta x_i + \sum_{\lambda=1}^{r} (-1)^{\lambda+1} \sum_{i=1}^{n} \sum_{k_1,\dots,k_{r-1}} p_{i k_1,\dots,k_{r-1}} \, \xi^{(\lambda)}_{k_1,\dots,k_{r-1}} \, \delta x_i^{\lambda} = \delta W_1 .$$

Now, from the calculation that was performed at the end of no. 1 on an entirely-analogous expression, the expression in *brackets* can be written:

$$\sum_{k_1,\dots,k_{r-1}} \frac{d\,(p_{i k_1,\dots,k_{r-1}}, x_{k_1}, \dots, x_{k_{r-1}})}{d\,(t_1, \dots, t_r)} .$$

One then concludes that (*):

$$\sum_{i=1}^{n} \sum_{k_1,\dots,k_{r-1}} \frac{d\,(p_{i k_1 \cdots k_{r-1}}, x_{k_1}, \dots, x_{k_{r-1}})}{d\,(t_1, \dots, t_r)} \delta x_i + \sum_{k_1,\dots,k_r} p_{i k_1 \cdots k_r} \, \delta \xi_{k_1 \cdots k_r} = \delta W_1 ,$$

or finally:

$$\sum_{i=1}^{n} \sum_{k_1,\dots,k_{r-1}} \frac{d\,(p_{i k_1 \cdots k_{r-1}}, x_{k_1}, \dots, x_{k_{r-1}})}{d\,(t_1, \dots, t_r)} \delta x_i - \sum_{k_1,\dots,k_r} \xi_{k_1 \cdots k_r} \, \delta p_{i k_1 \cdots k_r} = \delta \left[W_1 - \sum_{k_1,\dots,k_r} p_{i k_1 \cdots k_r} \, \xi_{k_1 \cdots k_r} \right].$$

Set:

$$H \equiv W_1 - \sum_{k_1,\dots,k_r} p_{i k_1 \cdots k_r} \, \xi_{k_1 \cdots k_r} ,$$

so

$$\begin{cases} \displaystyle\sum_{k_1,\dots,k_{r-1}} \frac{d\,(p_{i k_1 \cdots k_{r-1}}, x_{k_1}, \dots, x_{k_{r-1}})}{d\,(t_1, \dots, t_r)} = \frac{\partial H}{\partial x_i}, \\[2ex] \displaystyle\frac{d\,(x_{k_1}, \dots, x_{k_1})}{d\,(t_1, \dots, t_r)} = -\frac{\partial H}{\partial p_{k_1 \cdots k_r}}. \end{cases}$$

Those are the *canonical Hamilton-Volterra equations.*

The function H depends upon $t_1, \dots, t_r$, the $x_1, \dots, x_n$, and the C_n^r variables $p_{k_1 \cdots k_r}$. It should be pointed out that if one would like to give the $k_1, \dots, k_r$ all of the values from $1, \dots, n$, respectively, then one must set:

$$p_{k_1 \cdots k_r} \equiv \pm p_{k_1' \cdots k_r'}$$

according to whether $k_1, \dots, k_r$ and $k_1', \dots, k_r'$ differ by an even or odd number of inversions; here, I shall represent the *same* combination by $k_1, \dots, k_r$ and $k_1', \dots, k_r'$.

If $r = 1$ then one will get the canonical Hamilton equations.

(*) The same verification will be presented again in no. **7** of this treatise.

They were generalized by Volterra for $r = 2$. Fréchet deduced the parametric form for them for an arbitrary r (see no. **22**).

Scholia. – Once more, consider the function:

$$H \equiv W - \sum_{k_1,\ldots,k_r} p_{i\,k_1\cdots k_r}\, \xi_{k_1\cdots k_r} .$$

Upon utilizing the relations:

$$\frac{\partial W}{\partial \xi_{k_1\cdots k_r}} = p_{k_1\cdots k_r} ,$$

one will immediately find that:

$$\delta H \equiv \sum_{i=1}^{n} \frac{\partial W}{\partial x_i} \delta x_i - \sum_{k_1,\ldots,k_r} \xi_{k_1\cdots k_r}\, \delta p_{i\,k_1\cdots k_r} .$$

One will then have:

$$\begin{cases} \dfrac{\partial H}{\partial x_i} = \dfrac{\partial W}{\partial x_i} , \\ -\dfrac{\partial H}{\partial p_{k_1\cdots k_r}} = \xi_{k_1\cdots k_r} . \end{cases}$$

Upon referring to the generalized Lagrange equations, those equations will take the form of the canonical Hamilton-Volterra equations.

3. Another generalization. – Volterra (*) gave another generalization of Hamilton's canonical equations that presents a certain analogy with the preceding one. That is why I will pause for a moment to discuss it.

Once more, consider the relative invariant:

$$j_\lambda \equiv \sum_{i=1}^{n} \frac{\partial W}{\partial x_i^\lambda} \delta x_i \qquad\qquad \lambda = 1, \ldots, r .$$

of the equations:

$$\begin{cases} x_i^\lambda = \dfrac{dx_i}{dt_\lambda} , \\ \dfrac{\partial W}{\partial x_i} - \displaystyle\sum_{\lambda=1}^{r} \frac{d}{dt_\lambda}\left(\frac{\partial W}{\partial x_i^\lambda}\right) = 0. \end{cases}$$

Now set:

(*) V. VOLTERRA, "Sulle equazioni differentiali che provengono da questioni di calcolo delle variazione," Rendiconti Acc. R. Lincei, Roma (4), **6**, 1st semester (1890), pp. 43.

$$\frac{\partial W}{\partial x_i^\lambda} \equiv p_i^\lambda ,$$

and suppose, with Volterra, that those equations can be solved for the x_i^λ. W will then become a function of the t_λ, x_i, and p_i^λ; let W_2 denote that function. One will then have:

$$\sum_{\lambda=1}^{r} \frac{d}{dt_\lambda} \sum_{i=1}^{n} p_i^\lambda \, \delta x_i = \delta W_2$$

or

$$\sum_{\lambda=1}^{r} \frac{d}{dt_\lambda} \sum_{i=1}^{n} p_i^\lambda \, \delta x_i - \sum_{\lambda=1}^{r} \frac{dx_i}{dt_\lambda} \delta p_i^\lambda = \delta \left(W_2 - \sum_{\lambda=1}^{r} p_i^\lambda \, x_i^\lambda \right).$$

Set:

$$\mathcal{H} \equiv W_2 - \sum_{\lambda=1}^{r} p_i^\lambda \, x_i^\lambda ,$$

so one will have the equations:

$$\begin{cases} \displaystyle\sum_{\lambda=1}^{r} \frac{dp_i^\lambda}{dt_\lambda} = \frac{\partial \mathcal{H}}{\partial x_i}, \\[2ex] \displaystyle\frac{dx_i}{dt_\lambda} = -\frac{\partial \mathcal{H}}{\partial p_i^\lambda}. \end{cases}$$

Volterra found some interesting properties of those equations.

4. Attempt at a further generalization. – One can generalize the equations that were obtained in nos. **1** and **2** by starting from the relative invariant:

$$\sum_{\lambda=1}^{r} \frac{dj_\lambda}{dt_\lambda} = \delta W ,$$

in which:

$$j_\lambda \equiv \sum_{i=1}^{n} N_i^\lambda \delta x_i + \sum_{\mu=1}^{r} \sum_{i=1}^{n} p_i^{\lambda\mu} \, \delta x_i^\mu ,$$

$$x_i^\mu \equiv \frac{dx_i}{dt_\mu} ,$$

and one supposes that W is a function of the functional determinants of the x_i and x_i^μ (of order r) with respect to $t_1, \ldots, t_r$. However, the canonical equations thus-obtained do not seem to enjoy any remarkable properties.

5. Problem of the calculus of variations. – I have shown (*) that any *relative invariant* will correspond to a problem in the calculus of variations. Therefore, recall the relative invariant of nos. 1 and 2:

$$j_\lambda \equiv (-1)^{\lambda+1} \sum_{i=1}^{n} \sum_{k_1,\dots,k_{r-1}} p_{i k_1 \cdots k_{r-1}} \, \xi^{(\lambda)}_{k_1 \cdots k_{r-1}} \, \delta x_i \qquad \lambda = 1, \dots, r,$$

with

$$\sum_{\lambda=1}^{r} \frac{dj_\lambda}{dt_\lambda} = \delta W_1 = \delta \left[H - \sum_{k_1,\dots,k_{r-1}} p_{k_1 \cdots k_{r-1}} \frac{\partial H}{\partial p_{k_1 \cdots k_r}} \right].$$

One immediately deduces that on any r-dimensional multiplicity M_r that satisfies the canonical Hamilton-Volterra equations, one will have:

$$\delta \overset{r-\text{fold}}{\int_{M_r}} W_1 \, dt_1 \dots dt_r = \overset{r-\text{fold}}{\int_{M_r}} \left(\sum_{\lambda=1}^{r} \frac{dj_\lambda}{dt_\lambda} \right) dt_1 \dots dt_r .$$

Let T_r be the r-dimensional domain (or multiplicity, and taken in the space of t) in which we vary the t. Let T_{r-1} be the complete boundary. That boundary will not be subject to any variation under the variation δ that just calculated. The δt_λ are all *zero*, by hypothesis, so it will result that the points $(x_i, p_{k_1 \cdots k_r})$ and $(x_i + \delta x_i, p_{k_1 \cdots k_r} + \delta p_{k_1 \cdots k_r})$ *correspond* to the *same* values of $t_1, \dots, t_r$ in M_r .

Let M_{r-1} denote the complete boundary of M_r . From the fact that:

$$\overset{r-\text{fold}}{\int_{M_r}} \left(\sum_{\lambda=1}^{r} \frac{dj_\lambda}{dt_\lambda} \right) dt_1 \dots dt_r$$

is an r-fold exact differential (in the Poincaré sense), one can transform it into an $(r-1)$-fold integral (*):

$$\begin{aligned} &\int_{M_r} \left(\sum_{\lambda=1}^{r} \frac{dj_\lambda}{dt_\lambda} \right) dt_1 \dots dt_r \\ &\qquad = \int_{M_{r-1}} \sum_{\lambda=1}^{r} (-1)^{\lambda+1} j_\lambda \, dt_1 \dots dt_{\lambda-1} \, dt_{\lambda+1} \dots dt_r \\ &\qquad = \int_{M_{r-1}} \sum_{i=1}^{n} \sum_{k_1 \dots k_{r-1}} p_{i k_1 \dots k_{r-1}} \, \delta x_i \sum_{\lambda=1}^{r} \xi^{(\lambda)}_{k_1 \dots k_r} \, dt_1 \dots dt_{\lambda-1} \, dt_{\lambda+1} \dots dt_r \end{aligned}$$

(*) No. 60 of my cited paper: "Étude sur les invariants intégraux."

(*) Since the x and p are expressed as functions of the t, those integrals will become $(r-1)$-fold integrals that are extended over T_{r-1} .

$$= \int_{M_{r-1}} \sum_{i=1}^{n} \sum_{k_1 \dots k_{r-1}} p_{i\,k_1 \dots k_{r-1}}\, \delta x_i \sum_{\lambda=1}^{r} \frac{d\,(x_{k_1}, \dots, x_{k_1})}{d\,(t_1, \dots t_{\lambda-1}, t_{\lambda+1}, \dots, t_r)}\, dt_1 \dots dt_{\lambda-1}\, dt_{\lambda+1} \dots dt_r$$

$$= \int_{M_{r-1}} \sum_{i=1}^{n} \delta x_1 \sum_{k_1 \dots k_{r-1}} p_{i\,k_1 \dots k_{r-1}}\, dx_{k_1}, \dots, dx_{k_{r-1}} .$$

If the δx_i are zero on M_{r-1} (in other words, if the boundary M_{r-1} is *fixed*) then one will have:

$$\delta \int_{M_r} W_1\, dt_1 \dots dt_r \equiv 0 ,$$

so M_r *will be an r-dimensional extremal that is provided by the canonical Hamilton-Volterra equations.*

If the $p_{k_1 \dots k_r}$ can be determined as functions of x then we can replace M_r and M_{r-1} with a multiplicity X_r and its complete boundary X_{r-1} , which is taken in the space of the x, and no longer in the space of the x and p . It is obvious that X_r will again be an *extremal* of the problem that consists of finding the manifolds on which one will have:

$$\delta \int_{M_r} W_1\, dt_1 \dots dt_r \equiv 0 ,$$

when the boundary of that manifold is supposed to be fixed. That is the problem that was the starting point for the research of Volterra and the generalizations of Fréchet. We shall return to that important problem again (nos. **7**, **20**, and **21**).

CHAPTER II

Properties of the canonical Hamilton-Volterra equations.

6. First property. – *The canonical Hamilton-Volterra equations:*

$$\begin{cases} (A) \qquad \sum_{k_1,\ldots,k_{r-1}} \dfrac{d\,(p_{k_1,\ldots,k_{r-1}}, x_{k_1}, \ldots, x_{k_{r-1}})}{d\,(t_1,\ldots,t_r)} = \dfrac{\partial H}{\partial x_i}, \\ (B) \qquad \dfrac{d\,(x_{k_1},\ldots,x_{k_r})}{d\,(t_1,\ldots,t_r)} = -\dfrac{\partial H}{\partial p_{k_1,\ldots,k_{r-1}}} \end{cases}$$

possess the *relative invariant:*

$$j_\lambda \equiv (-1)^{\lambda+1} \sum_{i=1}^{n} \sum_{k_1,\ldots,k_{r-1}} p_{i\,k_1\ldots k_{r-1}} \frac{d\,(x_{k_1},\ldots,x_{k_r})}{d\,(t_1,\ldots,t_{\lambda-1},t_{\lambda+1},\ldots,t_r)}\,\delta x_i\,,$$

and one will have:

$$\sum_{\lambda=1}^{r} \frac{dj_\lambda}{dt_\lambda} = \delta\left[H - \sum_{k_1,\ldots,k_r} p_{k_1,\ldots,k_r} \frac{\partial H}{\partial p_{k_1,\ldots,k_r}} \right].$$

Conversely, any system of equations that possesses a relative invariant j_λ can be put into the form of the canonical Hamilton-Volterra equations.

The proof of that first property results immediately from nos. **1** and **2**.

7. Second property. – *Any solution to the canonical Hamilton-Volterra equations will produce an r-dimensional extremal to the problem of the calculus of variations that consists of annulling the variation:*

$$\delta \int \left(H - \sum_{k_1,\ldots,k_r} p_{k_1,\ldots,k_r} \frac{\partial H}{\partial p_{k_1,\ldots,k_r}} \right) dt_1 \cdots dt_r = 0\,.$$

Conversely, any extremal of the problem in the calculus of variations that consists of annulling the variation:

$$\delta \int \left[H + \sum_{k_1,\ldots,k_r} p_{k_1,\ldots,k_r} \frac{d\,(x_{k_1},\ldots,x_{k_r})}{d\,(t_1,\cdots,t_r)} \right] dt_1 \cdots dt_r = 0$$

will be a solution to the canonical Hamilton-Volterra equations.

The direct property was proved in no. **5**. One can verify it as follows: Since the boundary is supposed to be fixed and the δt_λ are zero, one will have:

$$\delta \int \left(H - \sum_{k_1,\ldots,k_r} p_{k_1,\ldots,k_r} \frac{\partial H}{\partial p_{k_1,\ldots,k_r}} \right) dt_1 \cdots dt_r$$

$$= \int \left[\sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \delta x_i + \sum_{k_1,\ldots,k_r} p_{k_1,\ldots,k_r} \delta \frac{d(x_{k_1},\ldots,x_{k_r})}{d(t_1,\cdots,t_r)} \right] dt_1 \cdots dt_r$$

$$= \int \left[\sum_{i=1}^{n} \delta x_i \sum_{k_1,\ldots,k_r} \frac{d(p_{i\,k_1,\ldots,k_{r-1}}, x_{k_1},\ldots,x_{k_{r-1}})}{d(t_1,\cdots,t_r)} + \sum_{k_1,\ldots,k_r} p_{k_1,\ldots,k_r} \delta \frac{d(x_{k_1},\ldots,x_{k_r})}{d(t_1,\cdots,t_r)} \right] dt_1 \cdots dt_r \;.$$

One *verifies* that the last integral can be written:

$$\int \left[\sum_{\lambda=1}^{r} \frac{dj_\lambda}{dt_\lambda} \right] dt_1 \ldots dt_r \;,$$

in which j_λ has the same significance as in no. **6**. In order to do that, one verifies the identity:

$$\sum_{k_1,\ldots,k_r} p_{k_1,\ldots,k_r} \delta \frac{d(x_{k_1},\ldots,x_{k_r})}{d(t_1,\cdots,t_r)} \equiv \sum_{\lambda=1}^{r} (-1)^{\lambda+1} \sum_{i=1}^{n} \sum_{k_1,\ldots,k_{r-1}} p_{i\,k_1,\ldots,k_{r-1}} \frac{d(x_{k_1},\ldots,x_{k_{r-1}})}{d(t_1,\cdots,t_{\lambda-1},t_{\lambda+1},\ldots,t_r)} \frac{d\,\delta x_i}{dt_\lambda} \;.$$

Since the boundary is fixed, one will have (see no. **5**):

$$\int \left[\sum_{\lambda=1}^{r} \frac{dj_\lambda}{dt_\lambda} \right] dt_1 \ldots dt_r = 0 \;. \qquad \text{Q. E. D.}$$

Let us move on to the converse property; it was proved in nos. **2** and **5**. When we continue the verification, we will have:

$$\delta \int \left[H + \sum_{k_1,\ldots,k_r} p_{k_1,\ldots,k_r} \frac{d(x_{k_1},\ldots,x_{k_r})}{d(t_1,\cdots,t_r)} \right] dt_1 \cdots dt_r$$

$$= \int \left[\sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \delta x_i + \sum_{k_1,\ldots,k_r} \left(\frac{\partial H}{\partial p_{k_1,\ldots,k_r}} \delta p_{k_1,\ldots,k_r} + \frac{d(x_{k_1},\ldots,x_{k_{r-1}})}{d(t_1,\cdots,t_r)} \delta p_{k_1,\ldots,k_r} + p_{k_1,\ldots,k_r} \delta \frac{d(x_{k_1},\ldots,x_{k_r})}{d(t_1,\cdots,t_r)} \right) \right] dt_1 \cdots dt_r$$

in succession.

In the verification that we just carried out, we saw that this integral is:

$$= \int \left[\sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \delta x_i + \sum_{k_1,\ldots,k_r} \left(\frac{\partial H}{\partial p_{k_1,\ldots,k_r}} + \frac{d(x_{k_1},\ldots,x_{k_{r-1}})}{d(t_1,\cdots,t_r)} \right) \delta p_{k_1,\ldots,k_r} - \sum_{i=1}^{n} \delta x_i \sum_{k_1,\ldots,k_r} \frac{d(p_{i,k_1,\ldots,k_{r-1}}, x_{k_1},\ldots,x_{k_{r-1}})}{d(t_1,\cdots,t_r)} \right] dt_1 \cdots dt_r \,.$$

One will obtain the canonical Hamilton-Volterra equations upon annulling the coefficients of δx_i and $\delta p_{k_1,\ldots,k_r}$. Q. E. D.

8. Third property. – *If the function H is independent of* $t_1, \ldots, t_r$ *then any solution* $(x_i, p_{k_1,\ldots,k_r})$ *of the canonical Hamilton-Volterra equations* (*A*) and (*B*) (no. **6**) *will give a constant as a result when it is substituted in H.*

We must show that the derivative of *H* with respect to t_1 (for example) will be *zero* when one replaces the *x* and *p* in *H* with the solution considered:

$$\frac{dH}{dt_i} \equiv \frac{\partial H}{\partial t_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \frac{dx_i}{dt_1} + \sum_{k_1,\ldots,k_r} \frac{\partial H}{\partial x_i} \frac{dp_{k_1,\ldots,k_r}}{dt_1}$$

$$\equiv \frac{\partial H}{\partial t_i} + \sum_{i=1}^{n} \frac{dx_i}{dt_1} \sum_{k_1,\ldots,k_r} \frac{d\,(p_{k_1,\ldots,k_r}, x_{k_1}, \ldots, x_{k_r})}{d\,(t_1, \ldots, t_r)} - \sum_{k_1,\ldots,k_r} \frac{dp_{k_1,\ldots,k_r}}{dt_1} \frac{d\,(x_{k_1}, \ldots, x_{k_r})}{d\,(t_1, \ldots, t_r)} .$$

On the right-hand side, one first looks for the coefficient of $\dfrac{dp_{k_1,\ldots,k_r}}{dt_1}$, and one sees that it is zero. One then seeks the coefficients of $\dfrac{dp_{k_1,\ldots,k_r}}{dt_\mu}$ $(\mu > 1)$, and one will again find zero.

Therefore:

$$\frac{dH}{dt_\mu} = \frac{\partial H}{\partial t_\mu} \qquad \lambda = 1, \ldots, r .$$

If the function *H* is independent of $t_1, \ldots, t_r$ then one will have:

$$\frac{dH}{dt_\mu} = 0 . \qquad \text{Q. E. D.}$$

9. Fourth property. – *If there exist* C_n^r *functions* $p_{i_1,\ldots,i_r}$ *of* $t_1, \ldots, t_r, x_1, \ldots, x_n$ *that satisfy:*

$$H\,(t, x, p) = 0 ,$$

along with the conditions (*):

(*) These are the necessary and sufficient conditions for:

$$\sum_{i_1,\ldots,i_r} p_{i_1,\ldots,i_r}\, \delta x_{i_1} \cdots \delta x_{i_r} \qquad i_1, \ldots, i_r = 1, \ldots, n$$

to be an *r*-fold *exact* differential (in the Poincaré sense).

$$\sum_{\nu=1}^{r+1}(-1)^{\nu}\,\frac{\partial p_{i_1,\dots,i_{\nu-1},i_{\nu+1},\dots,i_{r+1}}}{\partial x_{i_\nu}} = 0\;,$$

and if, in addition, there exist n functions $x_1, \dots, x_n$ *of* $t_1, \dots, t_r$ *that satisfy equations* (*B*) *then I say that the* $p_{i_1,\dots,i_r}$ *will satisfy equations* (*A*) *when one has replaced the x with that solution to* (*B*).

We shall suppose that $r = 2$ for simplicity of notation. We then substitute the p_{ik} ($i, k = 1, \dots, n$) in H . By hypothesis, we will have an identity in t and x :

$$H \equiv 0\;,$$

so

$$\frac{\partial H}{\partial x_i} + \sum_{l,k}\frac{\partial H}{\partial p_{lk}}\frac{\partial p_{lk}}{\partial x_i} \equiv 0 \qquad\qquad i, l, k = 1, \dots, n\;.$$

However, by hypothesis, the p_{ik} satisfy the conditions:

$$\frac{\partial p_{ik}}{\partial x_l} - \frac{\partial p_{il}}{\partial x_k} + \frac{\partial p_{kl}}{\partial x_i} \equiv 0\;,$$

and since:

$$p_{ki} \equiv -\,p_{ik}\;,$$

one will deduce that:

$$\frac{\partial H}{\partial x_i} + \sum_{l,k}\frac{\partial H}{\partial p_{lk}}\left(\frac{\partial p_{ik}}{\partial x_l} - \frac{\partial p_{il}}{\partial x_k}\right) \equiv 0\;.$$

One has:

$$\frac{\partial H}{\partial p_{lk}} \equiv -\,\frac{\partial H}{\partial p_{kl}}\;,$$

so

$$\frac{\partial H}{\partial x_i} + \sum_{l,k}\frac{\partial H}{\partial p_{lk}}\frac{\partial p_{ik}}{\partial x_l} \equiv 0\;.$$

Replace the x_i with a solution of equations (*B*) for t_1, t_2 :

$$\frac{\partial H}{\partial x_i} - \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{d\,(x_i, x_k)}{d\,(t_1, t_2)}\frac{\partial p_{ik}}{\partial x_l} \equiv 0\;,$$

which can be written:

$$\frac{\partial H}{\partial x_i} - \sum_{k=1}^{n}\frac{d\,(p_{ik}, x_k)}{d\,(t_1, t_2)} \equiv 0\;. \qquad\qquad \text{Q. E. D.}$$

10. Special Volterra equations. – Suppose that H is independent of $t_1, \dots, t_r$ and that the $p_{i_1,\dots,i_r}$ are functions of $x_1, \dots, x_n$. The system (A), (B) can be written:

$$\begin{cases} (A') \quad -\sum_{i_1}\sum_{i_2,\dots,i_r} \dfrac{\partial H}{\partial p_{i_1,\dots,i_r}} = \dfrac{\partial H}{\partial x_i}, \\ (B) \qquad \dfrac{d\,(x_{i_r},\dots,x_{i_r})}{d\,(t_1,\dots,t_r)} = -\dfrac{\partial H}{\partial p_{i_1,\dots,i_r}}. \end{cases}$$

Let us make a third hypothesis: Suppose that the $p_{i_1,\dots,i_r}$ satisfy equations (A') *before* replacing the x_i with a solution of equations (B) in terms of t . When those three conditions are fulfilled, I will say that the system (A'), (B) is a *system of special Volterra equations.*

In the following section, we will see that equations (A') can be put into a form that is more advantageous for discovering certain properties of those special equations.

11. Fifth property. – *If* $r = n - 1$ *and there exist n functions* $p_{i_1,\dots,i_{n-1}}$ *of* $x_1, \dots, x_n$ *, as well as n functions* x_i *of* $t_1, \dots, t_r$ *and two arbitrary constants that satisfy the special Volterra equations* (A'), (B), *then I say that the n functions* $p_{i_1,\dots,i_r}$ *of* $x_1, \dots, x_n$ *satisfy the condition* (*):

$$\sum_{\nu=1}^{n} (-1)^{\nu} \frac{\partial p_{i_1,\dots,i_{\nu-1},i_{\nu+1},\dots,i_n}}{\partial x_{i_\nu}} = 0 .$$

We once more suppose that $r = 2$, to simplify the notation. Equations (A') can be written:

$$\frac{\partial H}{\partial x_i} + \sum_{l,k} \frac{\partial H}{\partial p_{lk}} \frac{\partial p_{lk}}{\partial x_i} + \sum_{l,k} \frac{\partial H}{\partial p_{lk}} \left(\frac{\partial p_{ik}}{\partial x_l} + \frac{\partial p_{kl}}{\partial x_i} + \frac{\partial p_{li}}{\partial x_k} \right) = 0 .$$

The solution (x_i) has the form:

$$x_i = x_i\,(t_1, t_2, C_1, C_2) \qquad\qquad i = 1, 2, 3,$$

so

$$p_{ik} = p_{ik}\,(t_1, t_2, C_1, C_2) .$$

By virtue of the third property, one will have:

$$H\,(x, p) \equiv \varphi\,(C_1, C_2) \equiv h ,$$

(*) In other words: The $p_{i_1,\dots,i_{n-1}}$ are the coefficients of the (n – 1)-fold exact differential:

$$\sum_{i_1,\dots,i_{n-1}} p_{i_1,\dots,i_{n-1}}\,\delta x_{i_1} \cdots \delta x_{i_{n-1}} .$$

so

$$C_2 = \psi(h, C_1) .$$

Eliminate C_2 in order to introduce the constant h :

$$\begin{cases} x_i = x_i(t_1, t_2, C_1, h), \\ p_{ik} = p_{ik}(t_1, t_2, C_1, h). \end{cases}$$

Suppose that $\frac{\partial(x_1, x_2, x_3)}{\partial(C_1, t_1, t_2)} \neq 0$ and solve the first three equations for C_1, t_1, t_2 ; thus:

$$\begin{cases} p_{ik} = p_{ik}(x_1, x_2, x_3, h), \\ H' \equiv H'(x_1, x_2, x_3, h). \end{cases}$$

If we replace the x_i in H' with their values as functions of t_1, t_2, C_1, h then we will have:

$$H' \equiv h ,$$

so the result obtained will be *independent* of t_1, t_2, C_1 ; hence:

$$\sum_{i=1}^{3} \frac{\partial H'}{\partial x_i} \frac{\partial x_i}{\partial t_1} \equiv 0 ,$$

$$\sum_{i=1}^{3} \frac{\partial H'}{\partial x_i} \frac{\partial x_i}{\partial t_2} \equiv 0 ,$$

$$\sum_{i=1}^{3} \frac{\partial H'}{\partial x_i} \frac{\partial x_i}{\partial C_1} \equiv 0 .$$

Those three equations, which are homogeneous in $\frac{\partial H'}{\partial x_i}$ (i = 1, 2, 3), imply that:

$$\frac{\partial H'}{\partial x_i} \equiv 0 ,$$

so H is independent of the x_i, i.e., the x will disappear when one replaces the p_{ik} in H as functions of x. Therefore, *before* one replaces the x with a solution of equations (B) in terms of t, one will have:

$$\frac{\partial H}{\partial x_i} + \sum_{l,k} \frac{\partial H}{\partial p_{lk}} \frac{\partial p_{lk}}{\partial x_i} \equiv 0 .$$

The special equations (A') will then become:

$$\frac{\partial p_{ik}}{\partial x_l}+\frac{\partial p_{kl}}{\partial x_i}+\frac{\partial p_{li}}{\partial x_k}\equiv 0\,. \qquad \text{Q. E. D.}$$

Remark I. – An exact differential 2-form appears under the integral sign of $\iint p_{12}\,\delta x_1\,\delta x_2 + p_{13}\,\delta x_1\,\delta x_3 + p_{23}\,\delta x_2\,\delta x_3$. One can then transform that integral 2-form into a curvilinear integral. Set:

$$V \equiv \int p_1\,\delta x_1 + p_2\,\delta x_2 + p_3\,\delta x_3 \ ,$$

so

$$p_{ik}\equiv \frac{\partial p_i}{\partial x_k}-\frac{\partial p_k}{\partial x_i} \qquad i, k = 1, 2, 3.$$

Volterra wrote:

$$p_{ik}\equiv \frac{\partial V}{\partial (x_i, x_k)}\ ,$$

in which V is a *Volterra function*, of the first degree of simplicity. The p_{ik} are then *generalized partial derivatives.*

Due to the fact that:

$$H' \equiv h\,,$$

one will have:

$$H\left(x_1, x_2, x_3, \frac{\partial V}{\partial (x_1, x_2)}, \frac{\partial V}{\partial (x_1, x_3)}, \frac{\partial V}{\partial (x_2, x_3)}\right) = h\,.$$

Remark II. – It would be pointless to consider the special Volterra equations if one is content to prove that one has the identities:

$$\sum_{\nu=1}^{n}(-1)^{\nu}\,\frac{\partial p_{i_1,\ldots,i_{\nu-1},i_{\nu+1},\ldots,i_n}}{\partial x_{i_\nu}} = 0$$

in $t_1, \ldots, t_{n-1}$, i.e., *after* having replaced the x with functions of t.

Remark III. – The fifth property easily extends to the case of $r = n$. One then supposes that the solution x_i depends upon only one arbitrary constant. Remarks I and II also extend to that case.

12. Sixth property. – *If* $r = n - 1$ *and there exist n functions* $p_{i_1,\ldots,i_{n-1}}$ *of* $x_1, \ldots, x_n$ *that satisfy:*

$$H\,(x, p) = 0\,,$$

along with n functions x_i of $t_1, \ldots, t_{n-1}$ that satisfy the special Volterra equations (A'), (B), *then I say that the n functions* $p_{i_1,\ldots,i_{n-1}}$ *satisfy the condition:*

$$\sum_{\nu=1}^{n}(-1)^{\nu}\frac{\partial p_{i_1,\ldots,i_{\nu-1},i_{\nu+1},\ldots,i_n}}{\partial x_{i_\nu}} = 0 .$$

That property is closely-related to the previous one. One will again have the identities in x :

$$\frac{\partial H}{\partial x_i}+\sum_{l,k}\frac{\partial H}{\partial p_{lk}}\frac{\partial p_{lk}}{\partial x_i}+\sum_{l,k}\frac{\partial H}{\partial p_{lk}}\left(\frac{\partial p_{ik}}{\partial x_l}+\frac{\partial p_{kl}}{\partial x_i}+\frac{\partial p_{li}}{\partial x_k}\right)\equiv 0 .$$

However, the hypothesis:

$$H(x, p) = 0$$

implies the identities in x :

$$\frac{\partial H}{\partial x_i}+\sum_{l,k}\frac{\partial H}{\partial p_{lk}}\frac{\partial p_{lk}}{\partial x_i}\equiv 0 .$$

One immediately deduces the condition to be proved from the three identities in x :

$$\sum_{l,k}\frac{\partial H}{\partial p_{lk}}\left(\frac{\partial p_{ik}}{\partial x_l}+\frac{\partial p_{kl}}{\partial x_i}+\frac{\partial p_{li}}{\partial x_k}\right)\equiv 0 .$$

13. Seventh property. – *If $r = n - 1$ and if there exist n functions* $p_{i_1,\ldots,i_{n-1}}$ *of $x_1, \ldots, x_n$ that satisfy:*

$$H(x, p) = 0 ,$$

along with the condition:

$$\sum_{\nu=1}^{n}(-1)^{\nu}\frac{\partial p_{i_1,\ldots,i_{\nu-1},i_{\nu+1},\ldots,i_n}}{\partial x_{i_\nu}} = 0 ,$$

and if, in addition, there exist n functions x_i of $t_1, \ldots, t_{n-1}$ that will yield a solution to equations (A) *when they are substituted in the* $p_{i_1,\ldots,i_{n-1}}$ *then I say that those functions x_i are a solution of* (B).

Once more, suppose that $r = 2$. Substitute the p_{ik} in H. We have an identity in x :

$$H \equiv 0 ,$$

so

$$\frac{\partial H}{\partial x_i}+\sum_{l,k}\frac{\partial H}{\partial p_{lk}}\frac{\partial p_{lk}}{\partial x_i}\equiv 0 .$$

Upon arguing as in no. **9**, one will then deduce that:

$$\frac{\partial H}{\partial x_i} + \sum_l \sum_k \frac{\partial H}{\partial p_{lk}} \frac{\partial p_{ik}}{\partial x_l} \equiv 0 .$$

Replace x_i with a solution of equations (A):

$$\sum_{l,k} \left[\frac{d(x_l, x_k)}{d(t_1, t_2)} + \frac{\partial H}{\partial p_{lk}} \right] \frac{\partial p_{kl}}{\partial x_i} \equiv 0 .$$

It will then result that:

$$\frac{d(x_l, x_k)}{d(t_1, t_2)} = - \frac{\partial H}{\partial p_{lk}} . \qquad \text{Q. E. D.}$$

Remark. – If $r = n$ then the preceding property will be modified as follows: If there exists a function p of $x_1, \ldots, x_n$ that satisfies:

$$H(x, p) = 0 ,$$

and if, in addition there exist n functions $x_1, \ldots, x_n$ of $t_1, \ldots, t_n$ that yield a solution to (A) when they are substituted in the p then I will say that those n functions $x_1, \ldots, x_n$ yield a solution to (B).

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CHAPTER III

Generalization of Jacobi's theorem.

14. Review of Jacobi's theorem. – *If* $V(t, x_1, \ldots, x_n, a_1, \ldots, a_n) + a_{n+1}$ *is a complete integral* (*) *of the partial differential equation:*

$$\frac{\partial V}{\partial t} - H\left(t, x_1, \ldots, x_n, \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right) = 0$$

then:

$$\begin{cases} \dfrac{\partial V}{\partial x_i} = p_i, \\ \dfrac{\partial V}{\partial a_i} = b_i, \end{cases} \qquad i = 1, \ldots, n,$$

in which a_i, b_i *are* $2n$ *arbitrary constants, will define a general integral of Hamilton's canonical system:*

$$\begin{cases} \dfrac{dx_i}{dt} = -\dfrac{\partial H(t, x, p)}{\partial p_i}, \\ \dfrac{dp_i}{dt} = \dfrac{\partial H(t, x, p)}{\partial x_i}. \end{cases}$$

Conversely, if:

$$\begin{cases} x_i = x_i(t, t^0, x_1^0, \ldots, x_n^0, p_1^0, \ldots, p_n^0), \\ p_i = p_i(t, t^0, x_1^0, \ldots, x_n^0, p_1^0, \ldots, p_n^0), \end{cases} \quad \text{with} \quad \begin{cases} x_1^0 \equiv x_i(t^0, t^0, x_1^0, \ldots, x_n^0, p_1^0, \ldots, p_n^0), \\ p_1^0 \equiv p_i(t^0, t^0, x_1^0, \ldots, x_n^0, p_1^0, \ldots, p_n^0), \end{cases}$$

is the general integral of Hamilton's canonical system that was written above then (**):

$$V(t, t^0, x, x^0) \equiv \int_{t_0}^{t_1} \left[H - \sum_{i=1}^{n} p_i \frac{\partial H(t, x, p)}{\partial p_i} \right] dt + a$$

will be a complete integral to the partial differential equation:

(*) One supposes that $\dfrac{\partial\left(\dfrac{\partial V}{\partial x_1}, \ldots, \dfrac{\partial V}{\partial x_n}\right)}{\partial(a_1, \ldots, a_n)}$ is non-zero.

(**) The expression in brackets was first expressed as a function of t, t^0, x^0, p^0, and then one integrates over t from t^0 to t. The additive constant is represented by a here.

$$\frac{\partial V}{\partial t} - H\left(t, x_1, \ldots, x_n, \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right) = 0 \; .$$

The proof of that *converse* theorem results essentially from the fact that:

$$\frac{\partial V}{\partial t} = H\,(t, x, p) \; ,$$
$$\frac{\partial V}{\partial x} = p \; .$$

One should point out that in view of what one will learn in no. **21**, one will also have:

$$-\frac{\partial V}{\partial t^0} = H\,(t^0, x^0, p^0) \; ,$$
$$\frac{\partial V}{\partial x^0} = p^0 \; ,$$

and consequently:

$$-\frac{\partial V}{\partial t^0} - H\left(t^0, x^0, -\frac{\partial V}{\partial x^0}\right) = 0 \; .$$

The role of arbitrary constants is played by $t, x_1, \ldots, x_n$, here.

If one considers a point P in the space of t and x that has the coordinates (t, x_i) and a point P^0 that has the coordinates (t^0, x_1^0) then one can say that *the function V of the point P* satisfies the first partial differential equation (viz., the Jacobi equation) and that *the function V of* P^0 satisfies the second partial differential equation.

Those two equations will become identical in form when one sets:

$$\left\{\begin{aligned} \frac{\partial V}{\partial t} &\equiv \frac{\partial V[P]}{\partial t} , \\ \frac{\partial V}{\partial x} &\equiv \frac{\partial V[P]}{\partial x} , \end{aligned}\right. \qquad \left\{\begin{aligned} -\frac{\partial V}{\partial t^0} &\equiv \frac{\partial V[P^0]}{\partial t^0} , \\ -\frac{\partial V}{\partial x^0} &\equiv \frac{\partial V[P^0]}{\partial x^0} . \end{aligned}\right.$$

That is how one must interpret the generalized derivatives and the generalized Jacobi equation in no. **21**.

15. Generalized invariant. – Consider an $(r-1)$-fold integral:

$$U_{r-1} \equiv \int^{r-\text{fold}} \sum_{i_1,\ldots,i_{r-1}} M_{i_1,\ldots,i_{r-1}}\, \delta x_{i_1} \cdots \delta x_{i_{r-1}}$$

in which:

$$\frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})} \equiv \sum_{\nu=1}^{r}(-1)^{\nu}\frac{\partial M_{i_1,\ldots,i_{\nu-1},i_{\nu+1},\ldots,i_r}}{\partial x_{i_\nu}}.$$

I will say that the C_n^{r-1} functions $M_{i_1,\ldots,i_{r-1}}$ form a *generalized invariant* of the equations:

$$\frac{d(x_{i_1},\ldots,x_{i_r})}{d(t_1,\ldots,t_r)} = X_{i_1,\ldots,i_r},$$

if one has the identities in x :

$$\sum_{i_1,\ldots,i_r}\frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})}X_{i_1,\ldots,i_r} \equiv 0 .$$

The $X_{i_1,\ldots,i_r}$ are functions of $x_1, \ldots, x_n$.

When $r = 1$, that definition will coincide with that of (finite) *invariant* of a system of differential equations in one independent variable. We might have supposed that the $X_{i_1,\ldots,i_r}$ include $t_1, \ldots, t_r$. However, here we must suppose that the $\frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})}$ are independent of $t_1, \ldots, t_r$ if $r > 1$.

What advantage can one derive from a generalized invariant? Suppose that one knows a solution (x_i) of the equations:

$$\frac{d(x_{i_1},\ldots,x_{i_r})}{d(t_1,\ldots,t_r)} = X_{i_1,\ldots,i_r}.$$

Denote it by:

$$x_i = \varphi_i(t_1, \ldots, t_r) \qquad\qquad i = 1, \ldots, n .$$

Those equations define an r-dimensional multiplicity in *the space of* x. Let X_{r-1} and X'_{r-1} be the complete boundary of X_r . Extend the r-fold integral:

$$\int_{X_r}\sum_{i_1,\ldots,i_r}\frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})}\,dx_{i_1}\cdots dx_{i_r}$$

over that multiplicity X_r . The result obtained will be *zero*, since that integral can be written:

$$\int\sum_{i_1,\ldots,i_r}\frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})}\frac{d(x_{i_1},\ldots,x_{i_r})}{d(t_1,\cdots,t_r)}dt_1\cdots dt_r ,$$

or

$$\int\sum_{i_1,\ldots,i_r}\frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})}X_{i_1,\ldots,i_r}\,dt_1\cdots dt_r .$$

Now, the expression under the radical sign is *zero* by virtue of the definition of a generalized invariant. (That expression is zero *before* replacing the x with the solution considered.)

The r-fold integral that we just studied is the symbolic differential d of an $(r - 1)$-fold integral. One will then have:

$$\int_{X_r} \sum_{i_1,\dots,i_r} \frac{\partial U}{\partial(x_{i_1},\dots,x_{i_r})}\, dx_{i_1}\cdots dx_{i_r} = \int_{X_{r-1}} \sum_{i_1,\dots,i_{r-1}} M_{i_1,\dots,i_{r-1}}\, dx_{i_1}\cdots dx_{i_{r-1}} - \int_{X'_{r-1}} \sum_{i_1,\dots,i_{r-1}} M_{i_1,\dots,i_{r-1}}\, dx_{i_1}\cdots dx_{i_{r-1}} = 0\,.$$

One then has the *remarkable equality:*

$$\sum_{i_1,\dots,i_{r-1}} M_{i_1,\dots,i_{r-1}}\, dx_{i_1}\cdots dx_{i_{r-1}} = \int_{X'_{r-1}} \sum_{i_1,\dots,i_{r-1}} M_{i_1,\dots,i_{r-1}}\, dx_{i_1}\cdots dx_{i_{r-1}}\,. \tag{C}$$

Those $(r - 1)$-fold integrals are taken in *one sense* or *one direction* such that the equality will be valid, from the standpoint of its sign. The sense or direction of a hyperspace was neatly defined by Volterra (*).

The equality (C) expresses the *generalization of a theorem* that is frequently employed in the theories of S. Lie. Indeed, if $r = 1$ and if $x_i = \varphi_i(t)$ represents a solution of:

$$\frac{dx_i}{dt} = X_i(x_1, \dots, x_n) \qquad\qquad i = 1, \dots, n$$

then any *invariant* $U(x_1, \dots, x_n)$ of those equations will lead to the remarkable equality:

$$U(x_1, \dots, x_n) = U(x_1^0, \dots, x_n^0)\ ,$$

in which $x_i^0 = \varphi_i(t^0)$. Here, the complete boundary that was in question above is composed of the two points X and X^0 that have the coordinates (x_i) and (x_i^0) in the space of x.

One can go even further. Suppose that one has:

$$\sum_{i_1,\dots,i_r} \frac{\partial U}{\partial(x_{i_1},\dots,x_{i_r})}\, X_{i_1,\dots,i_r} = \lambda\,,$$

in which λ is a non-zero constant. I say that:

$$U_{r-1} = \int \sum_{i_1,\dots,i_{r-1}} M_{i_1,\dots,i_{r-1}}\, \delta x_{i_1}\cdots\delta x_{i_{r-1}}$$

(*) V.VOLTERRA, "Delle variabili complesse negli iperspazii," Rendicont Acc. R. Lincei, Roma, 1889. See page 158. See also no. 25 of the present treatise.

is a *generalized covariant* of the equations:

$$\frac{d(x_{i_1}, \ldots, x_{i_r})}{d(t_1, \ldots, t_r)} = X_{i_1, \ldots, i_r} .$$

Upon reasoning as before and preserving the same notations, we will find that:

$$(D) \qquad \int_{X_{r-1}} \sum_{i_1, \ldots, i_{r-1}} M_{i_1, \ldots, i_{r-1}} \, dx_{i_1} \cdots dx_{i_{r-1}} - \int_{X'_{r-1}} \sum_{i_1, \ldots, i_{r-1}} M_{i_1, \ldots, i_{r-1}} \, dx_{i_1} \cdots dx_{i_{r-1}} = \lambda \int_{T_r} dt_1 \cdots dt_r \, ,$$

in which T_r represents the r-dimensional multiplicity in the space of t, which is a multiplicity that corresponds to the multiplicity X_r in the space of x, thanks to the equations:

$$x_i = \varphi_i (t_1, \ldots, t_r) .$$

16. Lemma. – *If the* $\dfrac{\partial U}{\partial (x_{i_1}, \ldots, x_{i_r})}$ *that are deduced from* (*):

$$U_{r-1} \equiv \int \sum_{i_1, \ldots, i_{r-1}} M_{i_1, \ldots, i_{r-1}} \, \delta x_{i_1} \cdots \delta x_{i_{r-1}}$$

depend upon an arbitrary constant a and verify the generalized partial differential equation:

$$H \left(\ldots, x_i, \ldots, \ldots, \frac{\partial U}{\partial (x_{i_1}, \ldots, x_{i_r})}, \ldots \right) = 0$$

for any a then the $\partial U_{r-1} / \partial a$ *, i.e., the* C_n^{r-1} *functions* $\partial M_{i_1, \ldots, i_{r-1}} / \partial a$*, will form a generalized invariant of the equations:*

$$\frac{d(x_{i_1}, \ldots, x_{i_r})}{d(t_1, \ldots, t_r)} = \frac{\partial H(x, p)}{\partial p_{i_1, \ldots, i_r}} ,$$

which are equations in which one has replaced the $p_{i_1, \ldots, i_r}$ *with the* $\dfrac{\partial U}{\partial (x_{i_1}, \ldots, x_{i_r})}$.

Proof. – Substitute the known solution that depends upon the arbitrary constant a in the equation:

(*) One knows that any hyperspace function of the first degree of simplicity can be put into the form a multiple integral.

$$H\left(\ldots, x_i, \ldots, \ldots, \frac{\partial U}{\partial(x_{i_1}, \ldots, x_{i_r})}, \ldots\right) = 0 .$$

One will obtain an identity in $x_1, \ldots, x_n$, and a. Differentiate it with respect to a :

$$\sum_{i_1,\ldots,i_r} \frac{\partial H(x, p)}{\partial p_{i_1,\ldots,i_r}} \frac{\partial}{\partial a}\left(\frac{\partial U}{\partial(x_{i_1}, \ldots, x_{i_r})}\right) \equiv 0 ,$$

in which $p_{i_1,\ldots,i_r}$ is substituted for $\dfrac{\partial U}{\partial(x_{i_1}, \ldots, x_{i_r})}$. However, that identity can be written:

$$\sum_{i_1,\ldots,i_r} \frac{\partial H(x, p)}{\partial p_{i_1,\ldots,i_r}} \frac{\partial\left(\dfrac{\partial U}{\partial a}\right)}{\partial(x_{i_1}, \ldots, x_{i_r})} \equiv 0 . \qquad \text{Q. E. D.}$$

Corollary. – Upon preserving the notations of the preceding section, one will have:

$$\int_{X_{r-1}} \sum_{i_1,\ldots,i_{r-1}} \frac{dM_{i_1,\ldots,i_{r-1}}}{da}\, dx_{i_1} \cdots dx_{i_{r-1}} = \int_{X'_{r-1}} \sum_{i_1,\ldots,i_{r-1}} \frac{\partial M_{i_1,\ldots,i_{r-1}}}{\partial a}\, dx_{i_1} \cdots dx_{i_{r-1}} .$$

17. Lemma. – *If the* $\dfrac{\partial U}{\partial(x_{i_1}, \ldots, x_{i_r})}$ *that are deduced from:*

$$U_{r-1} \equiv \int \sum_{i_1,\ldots,i_{r-1}} M_{i_1,\ldots,i_{r-1}}\, \delta x_{i_1} \cdots \delta x_{i_{r-1}}$$

and depend upon the arbitrary constant h verify the generalized partial differential equation:

$$H\left(\ldots, x_i, \ldots, \ldots, \frac{\partial U}{\partial(x_{i_1}, \ldots, x_{i_r})}, \ldots\right) - h = 0$$

then the $\partial U_{r-1} / \partial h$ *, i.e., the* C_n^{r-1} *functions* $\partial M_{i_1,\ldots,i_{r-1}} / \partial h$*, will form a generalized covariant of the equations:*

$$\frac{d(x_{i_1}, \ldots, x_{i_r})}{d(t_1, \ldots, t_r)} = \frac{\partial H(x, p)}{\partial p_{i_1,\ldots,i_r}} ,$$

which are equations in which one has replaced the $p_{i_1,\dots,i_r}$ *with the* $\dfrac{\partial U}{\partial(x_{i_1},\dots,x_{i_r})}$.

Proof. – Upon proceeding as in the previous proof, one will find that:

$$\sum_{i_1,\dots,i_r} \frac{\partial H(x,p)}{\partial p_{i_1,\dots,i_r}} \frac{\partial\left(\dfrac{\partial U}{\partial h}\right)}{\partial(x_{i_1},\dots,x_{i_r})} \equiv 0 . \qquad \text{Q. E. D.}$$

Corollary. – Upon always preserving the same notation, one will have:

$$\int_{X_{r-1}} \sum_{i_1,\dots,i_{r-1}} M_{i_1,\dots,i_{r-1}}\, dx_{i_1}\cdots dx_{i_{r-1}} - \int_{X'_{r-1}} \sum_{i_1,\dots,i_{r-1}} M_{i_1,\dots,i_{r-1}}\, dx_{i_1}\cdots dx_{i_{r-1}} = \int_{T_r} dt_1\cdots dt_r ,$$

by virtue of (*D*) in no. **15**.

18. Complete integral. – We say that:

$$U_{r-1} \equiv \int^{r-\text{fold}} \sum_{i_1,\dots,i_{r-1}} M_{i_1,\dots,i_{r-1}}\, \delta x_{i_1}\cdots\delta x_{i_{r-1}}$$

is a *complete integral* of:

$$H\left(\dots,x_i,\dots,\dots,\frac{\partial U}{\partial(x_{i_1},\dots,x_{i_r})},\dots\right) = 0$$

if the functions $M_{i_1,\dots,i_{r-1}}$ depend upon the C_n^r arbitrary constants $a_{i_1,\dots,i_{r-1}}$, and the functional determinant of the C_n^r functions $\dfrac{\partial U}{\partial(x_{i_1},\dots,x_{i_r})}$ with respect to the arbitrary constants is *zero*, whereas one of the minors of order at least $C_n^r - 1$ is non-zero.

19. Generalization of Jacobi's direct theorem for Volterra functions of the first degree of simplicity. – *If:*

$$U_{r-1} \equiv \int \sum_{i_1,\dots,i_{r-1}} M_{i_1,\dots,i_{r-1}}\, \delta x_{i_1}\cdots\delta x_{i_{r-1}}$$

is a complete integral of the partial differential equation:

$$H\left(\ldots, x_i, \ldots, \ldots, \frac{\partial U}{\partial(x_{i_1}, \ldots, x_{i_r})}, \ldots\right) = 0$$

then:

$$\begin{cases} \dfrac{\partial U}{\partial(x_{i_1}, \ldots, x_{i_r})} = p_{i_1, \ldots, i_r}, \\ \dfrac{\partial U_{r-1}}{\partial a_\nu} = b_\nu, \end{cases} \qquad \nu = 1, \ldots, C_n^r,$$

in which a_ν, b_ν are arbitrary constants that define a system of integrals to the canonical Hamilton-Volterra equations (A), (B).

The significance of $\partial U_{r-1} / \partial a_\nu = b_\nu$ was given in the lemma in no. **16**.

Proof. – Consider two multiplicities X_{r-1} and X'_{r-1} that have the parametric equations:

$$x_i = \psi_i(\theta_1, \ldots, \theta_{r-1}, \theta_r),$$

in which θ_r represents an arbitrary constant that will have the values θ_r or θ'_r according to whether one generates X_{r-1} or X'_{r-1}, resp.

Those multiplicities X_{r-1} and X'_{r-1} have been chosen (*) in such a manner that:

$$\frac{\partial U_{r-1}}{\partial a_\nu} = b_\nu,$$

i.e., such that:

$$\int_{X_{r-1}} \sum_{i_1, \ldots, i_{r-1}} \frac{dM_{i_1, \ldots, i_{r-1}}}{da} dx_{i_1} \cdots dx_{i_{r-1}} = \int_{X'_{r-1}} \sum_{i_1, \ldots, i_{r-1}} \frac{\partial M_{i_1, \ldots, i_{r-1}}}{\partial a} dx_{i_1} \cdots dx_{i_{r-1}}.$$

Now suppose that θ_r or θ'_r differ by an infinitely-small quantity $d\theta_r$. The multiplicities X_{r-1} and X'_{r-1} are infinitely close, and upon repeating that operation an infinitude of times, we will generate a finite r-dimensional multiplicity.

I say that this multiplicity satisfies equations (B) (after a change of parameters $\theta_1, \ldots, \theta_r$). Indeed, if $\theta'_r = \theta_r + d\theta_r$ then the preceding C_n^r conditions:

$$\int_{X_{r-1}} - \int_{X'_{r-1}} = 0$$

(*) We shall not seek to find how one can determine the X_{r-1} and X'_{r-1} that satisfy those conditions here. The lemma in no. **16** shows us that those multiplicities exist, and that will suffice.

can be written (*):

$$\sum_{i_1,\ldots,i_r} \frac{\partial}{\partial a_\nu}\left[\frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})}\right]\frac{d(x_{i_1},\ldots,x_{i_r})}{d(\theta_1,\ldots,\theta_r)} = 0 .$$

Now, from the lemma of no. **16**, one will have the C_n^r identities:

$$\sum_{i_1,\ldots,i_r} \frac{\partial}{\partial a_\nu}\left[\frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})}\right]\frac{\partial H(x,p)}{\partial p_{i_1,\ldots,i_r}} = 0 .$$

Therefore, by virtue of the definition of the complete integral (no. **18**), one will have:

$$\frac{d(x_{i_1},\ldots,x_{i_r})}{d(\theta_1,\ldots,\theta_r)} \equiv \rho(\theta_1,\ldots,\theta_r)\,\frac{\partial H(x,p)}{\partial p_{i_1,\ldots,i_r}} .$$

Perform a change of parameters $\theta_1, \ldots, \theta_r$ such that one will have:

$$\frac{d(t_1,\ldots,t_r)}{d(\theta_1,\ldots,\theta_r)} = -\rho(\theta_1,\ldots,\theta_r) ,$$

so

(*B*)
$$\frac{d(x_{i_1},\ldots,x_{i_r})}{d(t_1,\ldots,t_r)} \equiv -\frac{\partial H(x,p)}{\partial p_{i_1,\ldots,i_r}} ,$$

with

$$p_{i_1,\ldots,i_r} = \frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})} .$$

The latter functions will satisfy the conditions that were imposed upon the $p_{i_1,\ldots,i_r}$ in the fourth property (no. **9**). Therefore, the $p_{i_1,\ldots,i_r}$ in which one has replaced the x_i with the solution of (*B*), will satisfy the equations (*A*). Q. E. D.

20. Hypotheses. – Suppose, with Volterra, that the canonical equations (*A*), (*B*) are such that a solution $(x_i, p_{i_1,\ldots,i_r})$ are *defined* when one is given the values of the x_i on the complete boundary T_{r-1} of the domain T_r in which one varies the variables $t_1, \ldots, t_r$ *arbitrarily*. The values of $t_1, \ldots, t_r$ that are taken on T_{r-1} correspond to values $(x_1, \ldots, x_n)$ that belong to the space of x and give a boundary-multiplicity X_{r-1} to an r-dimensional multiplicity X_r . Since they belong to the spaces of

(*) One considers the r-fold integral that is extended over the r-dimensional infinitesimal multiplicity that is found between X_{r-1} and X'_{r-1} or is generated by the infinitesimal displacement of X_{r-1} .

t and x, the multiplicities T_r and X_r, resp. will give a multiplicity that we denote by σ_r, and T_{r-1} and X_{r-1} will give the boundary σ_{r-1} of σ_r. *Any solutions* $(x_i, p_{i_1,\ldots,i_r})$ *to the equations* (A), (B) *can be considered to be functions of* $t_1, \ldots, t_r$ *and* σ_{r-1}. If we substitute those functions in:

$$\int_{T_r}\left[H - \sum_{i_1,\ldots,i_r} p_{i_1,\ldots,i_r}\frac{\partial H}{\partial p_{i_1,\ldots,i_r}}\right]dt_1\cdots dt_r$$

then we will obtain a function of σ_{r-1}, which is a function that we shall denote by $V[\sigma_{r-1}]$. *That Volterra function* $V[\sigma_{r-1}]$ *is not, in general, of the first degree of simplicity.* Therefore, it should not be confused with an r-fold integral that acts upon known or well-defined functions of points.

21. Generalization of the converse to Jacobi's theorem. – *The function that we just defined* (no. **20**):

$$V[\sigma_{r-1}] \equiv \int_{T_r}\left[H - \sum_{i_1,\ldots,i_r} p_{i_1,\ldots,i_r}\frac{\partial H}{\partial p_{i_1,\ldots,i_r}}\right]dt_1\cdots dt_r$$

satisfies the generalized Jacobi equation:

$$\frac{\partial V[\sigma_{r-1}]}{\partial(t_1,\ldots,t_r)} - H\left(\ldots, x_i, \ldots, \frac{\partial V[\sigma_{r-1}]}{\partial(x_{i_1},\ldots,x_{i_r})}, \ldots, t_1, \ldots, t_r\right) = 0 .$$

The generalized partial derivatives that appear in the that equation will be defined in the course of the proof. In no. **14**, we already explained the sense that one can give to them for $r = 1$. We suppose that H is not a homogeneous function of degree one with respect to the $p_{i_1,\ldots,i_r}$ (*).

Proof. – We have made the multiplicities X_r and X_{r-1} correspond to the multiplicities T_r and T_{r-1}. By hypothesis, we must make T_r and T_{r-1} correspond to other multiplicities $X_r + \delta(X_r)$ and $X_{r-1} + \delta(X_{r-1})$, which are infinitely-close to the preceding ones. Therefore, the multiple integral:

$$\int_{T_r}\left(H - \sum_{i_1,\ldots,i_r} p_{i_1,\ldots,i_r}\frac{\partial H}{\partial p_{i_1,\ldots,i_r}}\right)dt_1\cdots dt_r$$

will be subjected to a variation δ that we calculated in no. **5**. We will have:

(*) If $r = 1$ and if H is a homogeneous function of degree one with respect to p_i then see no. 230 of *Leçons sur calcul des variations*, which was based upon lectures that were taught by J. Hadamard and collected by Fréchet. (Tome I, which is the only one that appeared, 1910, Paris, Hermann et fils.)

$$\delta \int_{T_r \text{ or } X_r} \left(H - \sum p \frac{\partial H}{\partial p} \right) dt_1 \cdots dt_r = \int_{X_{r-1}} \sum_{i=1}^{n} \delta x_i \sum_{i_1,\ldots,i_{r-1}} p_{i\,i_1,\ldots,i_{r-1}} \, dx_{i_1} \cdots dx_{i_{r-1}}$$

$$= \int_{\delta(X_r)} \sum_{i_1,\ldots,i_{r-1}} p_{i\,i_1,\ldots,i_{r-1}} \frac{\partial(x_{i_1},\ldots,x_{i_r})}{\partial(\tau_1,\ldots,\tau_r)} d\tau_1 \cdots d\tau_r \, ,$$

in which the multiplicity $\delta(X_r)$ that is generated by the variation or displacement of X_{r-1} is supposed to be expressed as a function of the parameters $\tau_1, \ldots, \tau_r$.

By hypothesis, we must consider the multiplicities $T_r + d(T_r)$ and $T_{r-1} + d(T_{r-1})$, which are infinitely close to T_r and T_{r-1}, resp. We could make those new multiplicities correspond to X_r and X_{r-1} , respectively. In that way, the multiple integral:

$$\int_{T_r} \left(H - \sum p \frac{\partial H}{\partial p} \right) dt_1 \cdots dt_r$$

will be subject to an increment *d* that is given by:

$$d \int_{T_r} \left[H - \sum_{i_1,\ldots,i_r} p_{i_1,\ldots,i_r} \frac{d(x_{i_1},\ldots,x_{i_r})}{d(t_1,\cdots,t_r)} \right] dt_1 \cdots dt_r$$

$$= d \int_{T_r} H \, dt_1 \cdots dt_r - d \int_{X_r} \sum_{i_1,\ldots,i_r} p_{i_1,\ldots,i_r} \, dx_{i_1} \cdots dx_{i_r}$$

$$= d \int_{T_r} H \, dt_1 \cdots dt_r$$

$$= \int_{d(T_r)} H \, dt_1 \cdots dt_r \, .$$

By hypothesis, we can perform the two operations *simultaneously.* It will then result that we also have:

$$\begin{cases} \dfrac{\partial V[\sigma_{r-1}]}{\partial(t_1,\ldots,t_r)} = H, \\[2ex] \dfrac{\partial V[\sigma_{r-1}]}{\partial(x_{i_1},\ldots,x_{i_r})} = p_{i_1,\ldots,i_r} \end{cases}$$

in the space of x and t.

The partial derivatives of $V[\sigma_{r-1}]$, thus-interpreted, then satisfy the generalized Jacobi equation:

$$\frac{\partial V[\sigma_{r-1}]}{\partial(t_1,\ldots,t_r)} - H\left(\ldots, x_i, \ldots, \frac{\partial V[\sigma_{r-1}]}{\partial(x_{i_1},\ldots,x_{i_r})}, \ldots, t_1, \ldots, t_r \right) = 0 \, .$$

Remark. – **Volterra** gave a very beautiful proof of the converse to Jacobi's theorem in the case where $r = n - 1 = 2$. (*loc. cit.*, pp. 129). **Fréchet** generalized that theorem by using the notion of a *transverse extremal* (*), when extended to hyperspace. The Jacobi equation that he obtained has the form:

$$H\left(\ldots, x_i, \ldots, \frac{\partial V[\sigma_{r-1}]}{\partial(x_{i_1}, \ldots, x_{i_r})}, \ldots\right) = 0 .$$

In addition, **Fréchet** remarked that *the generalized derivatives of* $V[\sigma_{r-1}]$ *satisfy some other equations.* A first system of those equations is deduced immediately from (B), from $\frac{\partial V[\sigma_{r-1}]}{\partial(x_{i_1}, \ldots, x_{i_r})} = p_{i_1, \ldots, i_r}$, and from the identities that were written out in the footnote in no. **2**. Hence:

$$\sum_{\mu=1}^{r+1} (-1)^{\mu} \frac{\partial H(x, p)}{\partial p_{i_1, \ldots, i_{\mu-1}, i_{\mu+1}, \ldots, i_{r+1}}} \frac{\partial H(x, p)}{\partial p_{i_\mu, k_1, \ldots, k_{r-1}}} = 0 .$$

A second system of equations that are satisfied by the generalized derivatives of $V[\sigma_{r-1}]$ is obtained by eliminating $\frac{dx_2}{dx_1}, \ldots, \frac{dx_n}{dx_1}$ from equations (B) and the values of the direction cosines (see below, no. **25**) of an element that is taken on X_{r-1} .

In all of those proofs of the generalized Jacobi theorem, the role of the variables x_i is paramount. That return to the variables that Jacobi himself used is worthy of note.

(*) Fréchet's proof is the generalization of the one that is found in no. **145** in Hadamard's treatise that we just cited.

CHAPTER IV

Parametric form.

22. Parametric form. – If one expresses the $x_1, \ldots, x_n$ in the r-fold integral:

$$I \equiv \int^{r-\text{fold}} f\left(x_{r+1}, \ldots, x_n, x_1, \ldots, x_r, \frac{\partial x_{r+1}}{\partial x_1}, \ldots, \frac{\partial x_n}{\partial x_r}\right) dx_1 \cdots dx_r$$

as functions of r distinct parameters $t_1, \ldots, t_r$ then one will have:

$$\frac{\partial x_{r+k}}{\partial x_i} = \frac{d\,(x_1, \ldots, x_{i-1}, x_{r+k}, x_{i+1}, \ldots, x_r)}{d\,(t_1, \ldots, t_r)} : \frac{d\,(x_1, \ldots, x_r)}{d\,(t_1, \ldots, t_r)}.$$

Upon substituting that in:

$$W \equiv f\,\frac{d\,(x_1, \ldots, x_r)}{d\,(t_1, \ldots, t_r)},$$

one will obtain a function W that will be *homogeneous of degree one* with respect to the determinants $\dfrac{d\,(x_1, \ldots, x_r)}{d\,(t_1, \ldots, t_r)}$, in which $i_1, \ldots, i_r$ is any of the combinations of the first n numbers taken r at a time.

Then set:

$$\xi_{i_1, \ldots, i_r} \equiv \frac{d\,(x_1, \ldots, x_r)}{d\,(t_1, \ldots, t_r)}.$$

The r-fold integral I will become:

$$I \equiv \int^{r-\text{fold}} W(x_1, \ldots, x_n, \ldots, \xi_{i_1, \ldots, i_r}, \ldots)\, dt_1 \cdots dt_r \quad .$$

The new form of I, which is equivalent to the given integral, is the analogue of *the parametric form* that Weierstrass employed in the case where $r = 1$. The extension to the case of a multiple integral was realized (*) in a more practical form by Hadamard in his course on the calculus of variations that was taught at Collège de France. (The first part of that course appeared recently and was cited in no. **21**.)

(*) See page 190 of the cited paper by Fréchet.

23. On the condition $H = 0$. – It results from the fact that W is a homogeneous function of degree one in $\xi_{i_1,\ldots,i_r}$ that the Hessian of W with respect to those $\xi_{i_1,\ldots,i_r}$ (when considered to be independent variables) is identically zero. Now *suppose* that an initial minor (of order $C_n^r - 1$) is non-zero. It will then result that we can once more write:

$$\frac{\partial W}{\partial \xi_{i_1,\ldots,i_r}} = p_{i_1,\ldots,i_r},$$

so there will exist *only one relation:*

$$H(x_1, \ldots, x_n, p_{i_1,\ldots,i_r}, \ldots) = 0$$

between the $p_{i_1,\ldots,i_r}$. The hypothesis that was made at the beginning of no. **2** has then been abandoned, since the relations:

$$\frac{\partial W}{\partial \xi_{i_1,\ldots,i_r}} = p_{i_1,\ldots,i_r}$$

cannot be solved for the $\xi_{i_1,\ldots,i_r}$, in general.

Euler's theorem for homogeneous functions gives us:

$$W - \sum_{i_1,\ldots,i_r} \frac{\partial W}{\partial \xi_{i_1,\ldots,i_r}} \xi_{i_1,\ldots,i_r} = 0 .$$

The *single* relation between the $p_{i_1,\ldots,i_r}$ can be deduced from:

$$W - \sum_{i_1,\ldots,i_r} p_{i_1,\ldots,i_r} \, \xi_{i_1,\ldots,i_r} = 0 .$$

One can then proceed as follows: First, calculate the $C_n^r - 1$ ratios (*) of the $\xi_{i_1,\ldots,i_r}$ to each other, and then substitute them in the preceding relation.

Upon referring to no. **2** and preserving the same symbols, one will see that it is necessary and sufficient that one must have:

$$\sum_{i=1}^{n} \sum_{k_1,\ldots,k_{r-1}} \frac{d(p_{i,k_1,\ldots,k_{r-1}}, x_{k_1}, \ldots, x_{k_{r-1}})}{d(t_1, \ldots, t_r)} \delta x_i - \sum_{k_1,\ldots,k_{r-1}} \xi_{k_1,\ldots,k_r} \, \delta p_{k_1,\ldots,k_r} = \rho \, \delta H ,$$

with

$$H = 0 ,$$

(*) The $\partial W / \partial \xi_{i_1,\ldots,i_r}$ are homogeneous functions of degree zero with respect to the $\xi_{i_1,\ldots,i_r}$.

in which $H = 0$ represents the result of eliminating the $\xi_{i_1,\dots,i_r}$ (or the unique relation between the $p_{i_1,\dots,i_r}$) and in which r is an arbitrary function of the x_i and the $p_{i_1,\dots,i_r}$.

The generalized Lagrange equations will then be equivalent to the canonical Hamilton-Volterra equations:

$$\begin{cases} \displaystyle\sum_{k_1,\dots,k_{r-1}} \frac{d(p_{i,k_1,\dots,k_{r-1}}, x_{k_1},\dots,x_{k_{r-1}})}{d(t_1,\dots,t_r)} = \rho \frac{\partial H}{\partial x_i}, \\ \displaystyle\frac{d(x_{k_1},\dots,x_{k_r})}{d(t_1,\dots,t_r)} = -\rho \frac{\partial H}{\partial p_{k_1,\dots,k_r}}, \end{cases}$$

to which one must add the condition that:

$$H(x_i, \dots, p_{i_1,\dots,i_r}, \dots) = 0 .$$

Since the function H is independent of the $t_1, \dots, t_r$, one can, by a change of variables, make ρ equal to 1 in some way. We shall suppose that ρ has that value in what follows.

The equation $H = 0$ is that of the *figuratrix* (*). In order to satisfy it, it will suffice to consider a system of (initial) values x_i^0, $p_{i_1,\dots,i_r}^0$ that correspond to $t_1^0, \dots, t_r^0$, and which satisfy $H(\dots, x_i^0, \dots, p_{i_1,\dots,i_r}^0) = 0$. That will result immediately from the fact that H is an invariant of the canonical Hamilton-Volterra equations (no. **8**).

The integral I can now be written:

$$I = \int^{r-\text{fold}} \sum_{i_1,\dots,i_r} p_{i_1,\dots,i_r} \frac{\partial H}{\partial p_{i_1,\dots,i_r}} dt_1 \cdots dt_r .$$

24. Example: minimal surfaces. – One knows that in order to find the minimal surfaces $z = z(x, y)$, one must annul the first variation of:

$$\iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

If one replaces the rectangular coordinates of x, y, z with:

$$\begin{cases} x = x(t_1, t_2), \\ y = y(t_1, t_2), \\ z = z(t_1, t_2) \end{cases}$$

(*) See no. 140, *et seq.*, of the cited treatise by Hadamard (case in which $r = 1$).

then one will find the parametric form:

$$\iint \sqrt{\xi_{12}^2 + \xi_{13}^2 + \xi_{23}^2}\, dt_1\, dt_2 \ .$$

One will then immediately deduce (no. **1**) the generalized Lagrange equations that the minimal surfaces must satisfy.

In addition, one will have:

$$W \equiv (\xi_{12}^2 + \xi_{13}^2 + \xi_{23}^2)^{1/2},$$

so

$$p_{12} = \frac{\xi_{12}}{(\xi_{12}^2 + \xi_{13}^2 + \xi_{23}^2)^{1/2}},$$

and one will have analogous expressions for p_{13} and p_{23} .

The elimination of ξ_{12} , ξ_{13} , ξ_{23} will lead to the single relation:

$$p_{12}^2 + p_{13}^2 + p_{23}^2 - 1 = 0 \ .$$

One can then set:

$$H \equiv p_{12}^2 + p_{13}^2 + p_{23}^2 - 1 \ .$$

Before going further, we must interpret an r-fold integral that was obtained in no. **21** geometrically.

25. Geometric interpretation. – Consider the integral:

$$\int_X \sum_{i_1,\ldots,i_r} p_{i_1,\ldots,i_r} \frac{\partial(x_{i_1},\ldots,x_{i_r})}{\partial(\tau_1,\ldots,\tau_r)} \partial\tau_1 \cdots \partial\tau_r \ ,$$

which is extended over the r-dimensional multiplicity X. The x_1, …., x_n are expressed as functions of the r distinct parameters τ_1, …., τ_r . Let Δ^2 represent the square of the matrix:

$$\left\| \begin{matrix} \dfrac{\partial x_1}{\partial \tau_1} & \cdots & \dfrac{\partial x_n}{\partial \tau_1} \\ \vdots & & \vdots \\ \dfrac{\partial x_1}{\partial \tau_r} & \cdots & \dfrac{\partial x_r}{\partial \tau_r} \end{matrix} \right\| .$$

One makes a choice of sign for Δ at an arbitrary point of X and preserves the sign when the point (x_i) is displaced continuously over X. When one chooses the sign of Δ, one says that the *direction* or *sense* of that multiplicity X is given.

Define an *element* $d\sigma$ that is taken on the multiplicity X by the relation:

$$d\sigma \equiv \Delta\, \partial\tau_1 \ldots \partial\tau_r \,.$$

Finally, we say that the *direction cosines* of $d\sigma$ are given by:

$$\cos(x_{i_1}, \ldots, x_{i_r}) \equiv \frac{\dfrac{\partial(x_{i_1}, \ldots, x_{i_r})}{\partial(\tau_1, \ldots, \tau_r)}}{\Delta} \,.$$

Those $\cos(x_{i_1}, \ldots, x_{i_r})$ satisfy the following identities:

$$\sum_{i_1,\ldots,i_r} \cos^2(x_{i_1}, \ldots, x_{i_r}) \equiv 1 \,,$$

$$\sum_{\mu=1}^{r+1} (-1)^{\mu} \cos(x_{i_1}, \ldots, x_{i_{\mu-1}}, x_{i_{\mu+1}}, \ldots, x_{i_{r+1}}) \cos(x_{i_\mu}, x_{k_1}, \ldots, x_{k_{r-1}}) \equiv 0 \,.$$

Now recall no. **21**:

$$\delta \int_{T_r \text{ or } X_r} \left(H - \sum p \frac{\partial H}{\partial p} \right) dt_1 \cdots dt_r = \int_{\delta(X_r)} \sum_{i_1,\ldots,i_r} p_{i_1,\ldots,i_r} \frac{\partial(x_{i_1}, \ldots, x_{i_r})}{\partial(\tau_1, \ldots, \tau_r)} \partial\tau_1 \cdots \partial\tau_r$$

$$= \int_{\delta(X_r)} \left[\sum_{i_1,\ldots,i_r} p_{i_1,\ldots,i_r} \cos(x_{i_1}, \ldots, x_{i_r}) \right] d\sigma \,.$$

26. Minimal surfaces (cont.). – Here, that formula will become:

$$\int_{\delta(X_2)} [p_{12} \cos(x_1, x_2) + p_{13} \cos(x_1, x_3) + p_{23} \cos(x_2, x_3)]\, d\sigma \,.$$

The p_{12} , … are the direction cosines of the normal to the minimal surface. The cos (x_1, x_2), … are the direction cosines of the normal to the surface that is described by the contour (X_1). If that boundary curve (X_1) is unknown, but subject to being found on a given surface then the condition p_{12} cos (x_1, x_2) + … = 0 will signify that the surface must be intersected *normally* by the desired minimum surface. (Gauss's theorem)

Volterra deduced the following theorem, which is due to **Ernesto Padova** (*), from the fifth property (no. **11** and *Remarks*): The orthogonal trajectories to a system of minimal surfaces form a system of filaments with constant orthogonal section. (A filament is a tube that is composed of the lines that start from the points of the contour of an infinitesimal area.) Volterra deduced the following converse to that theorem from the fourth property (no. **9**): If a system of filaments with constant orthogonal section admits orthogonal surfaces then they will be minimal surfaces.

December 1909.

(*) Rendiconti Acc. Lincei, Roma, (4) **4**, 2nd semester (1888), pps. 369 and 454, especially pp. 373.