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# On relative integral invariants and their applications to mathematical physics 

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## Introduction

Upon studying the modern theories of electricity, I have been led to write this new article on relative integral invariants.

By adopting Goursat's notations, I shall establish (no. 1) a relationship between the operations $D$ and $E$, which is a relationship from which I will immediately deduce ( ${ }^{*}$ ) an extension of a theorem by Hargreaves.

The main goal of this article is to find some procedures that will allow one to deduce from an absolute or relative integral invariant $J_{p}$ for the system:

$$
\frac{d x_{i}}{X_{i}}=d t, \quad i=1, \ldots, n
$$

an absolute or relative invariant $J_{p}^{\prime}$ for the transformed systems:

$$
\frac{d z_{i}}{Z_{i}}=d u
$$

in which $z_{1}, \ldots, z_{n}$ represent the new dependent variables and $u$ is the new independent variable. The solution to that problem is given completely in nos. $\mathbf{2}$ and $\mathbf{3}$. A first application to the equations of the ether and electric charge is developed in no. 4. I have added (no. 5) some bibliographic citations concerning my recent research into the generalized canonical equations in no. $\mathbf{2}$ of MI. Finally, after pointing out a theorem that relates to second-order differential equations (no. 6), I shall apply the preceding developments to the equations of motion of a corpuscle in an electromagnetic field.

[^0]
## 1. - The operations $D$ and $E$.

Definitions. - Consider the system of $n$ differential equations:

$$
\frac{d z_{i}}{Z_{i}} \equiv d u, \quad i=1, \ldots, n,
$$

and a $p$-uple integral form:

$$
J_{p} \equiv \mathrm{~S}_{i_{1} \cdots i_{p}} N_{i_{1} \cdots i_{p}} \delta z_{i_{1}} \cdots \delta z_{i_{p}} \quad p \leq n
$$

in which the $N_{i_{1} \cdots i_{p}}$ are functions of $z_{1}, \ldots, z_{n}$, and $u$.
The operation $D$, when applied to $J_{p}$, consists of symbolically differentiating $J_{p}$ in such a manner as to deduce a $(p+1)$-uple integral form that is equal to:

$$
\begin{aligned}
D J_{p} & \equiv \underset{i_{1} \cdots i_{p}}{\mathrm{~S}} \mathrm{~S}_{p_{p+1}} \frac{\partial N_{i_{1} \cdots i_{p}}}{\partial z_{i_{p+1}}} \delta z_{i_{1}} \cdots \delta z_{i_{i_{p}}} \delta z_{i_{i_{p+1}}} \\
& \equiv \underset{i_{1} \cdots i_{p+1}}{\mathrm{~S}}\left(\frac{\partial N_{i_{1} \cdots i_{p}}}{\partial z_{i_{p+1}}}-\frac{\partial N_{i_{p+1} i_{2} \cdots i_{p}}}{\partial z_{i_{1}}}-\cdots-\frac{\partial N_{i_{1} \cdots i_{p-1} i_{p+1}}}{\partial z_{i_{p}}}\right) \delta z_{i_{1}} \cdots \delta z_{i_{p+1}} .
\end{aligned}
$$

We recall that $\underset{i_{1} \cdots i_{p}}{S}$ indicates a summation that extends over all combinations of $1, \ldots, n$ taken $p$ at a time. It is good to remark that one sets $\delta u \equiv 0$ in the operation $D$, i.e., that the variation $\delta$ of the independent variable is zero. In order to avoid any confusion in this topic, we shall often write $D^{\alpha} J_{p}$, instead of $D J_{p}$, in which the exponent $\alpha$ serves to remind us that we are dealing with the system $(\alpha)$, i.e., a system with the independent variable $u$.

The operation $E$, when applied to $J_{p}$, consists of replacing the $\delta_{p} z_{i_{1}}, \ldots, \delta_{p} z_{i_{p}}$ that appear in the last row of each of the determinants:

$$
J_{p} \equiv \mathbf{S}_{i_{1} \cdots i_{p}} N_{i_{1} \cdots i_{p}}\left|\begin{array}{ccc}
\delta_{1} z_{i} & \cdots & \delta_{1} z_{i_{p}} \\
\vdots & \ddots & \vdots \\
\delta_{p} z_{i_{1}} & \cdots & \delta_{p} z_{i_{p}}
\end{array}\right|
$$

with $Z_{i_{1}}, \ldots, Z_{i_{p}}$, respectively. One will then obtain a $(p-1)$-uple integral form that is equal to:

$$
E J_{p} \equiv \underset{i_{1}, \cdots i_{p-1} i_{p}}{\mathbf{S}} \mathbf{S}_{i_{i_{1} \cdots i_{p}}} Z_{i_{p}} \delta z_{i_{1}} \cdots \delta z_{i_{p-1}}
$$

We often write $E^{\alpha} J_{p}$, instead of $E J_{p}$.

Fundamental relation. - If the operation $D$ is applied to $J_{p}$ and followed by the application of the operation $E$ to $D J_{p}$ then that will give a $p$-uple integral form that we will denote by $E D J_{p}$. If we permute those two operations then we will generally get another integral form that we will denote by $D E J_{p}$. We verify that if the coefficients of $J_{p}$ do not include $u$ explicitly then we will have the fundamental formula:

$$
E D J_{p}+D E J_{p}=\frac{d}{d u} J_{p}
$$

in which:

$$
\frac{d}{d u} \equiv \frac{\partial}{\partial t}+\sum_{i=1}^{n} Z_{i} \frac{\partial}{\partial z_{i}}
$$

(see no. 1 of MI).

One can infer some remarkable consequences from that relation.
If $J_{p}$ is an absolute invariant of $(\alpha)$ then one will recover a relation that Goursat encountered $\left(^{*}\right)$. If $J_{p}$ is a relative integral invariant of $(\alpha)$ such that:

$$
\frac{d}{d u} J_{p}=D W_{p-1},
$$

in which $W_{p-1}$ is a $(p-1)$-uple integral form, and if the $Z_{i}$ do not include $n$ explicitly, in addition, then one will have:

$$
D\left(E J_{p}-W_{p-1}\right)=-E D J_{p} .
$$

Now, H. Poincaré's theory of integral invariants teaches us that $D J_{p}$, as well as $E D J_{p}$, are absolute invariants of $(\alpha)$ and that consequently:

$$
E J_{p}-W_{p-1}
$$

will be a $(p-1)$-uple relative integral invariant of $(\alpha)$. We verify, in addition, that by virtue of equations $(\alpha)$, we will have:

$$
\frac{d}{d u}\left(E J_{p}-W_{p-1}\right)=-D E W_{p-1}
$$

This important result is equivalent to the following generalization of Hargreaves's theorem:

## Extension of Hargreaves's theorem:

Consider the system:
(I)

$$
\frac{d x_{i}}{X_{i}}=d t, \quad i=1, \ldots, n,
$$

[^1]in which $X_{i}$ are functions of $x_{1}, \ldots, x_{n}$, and the independent variable $t$.
Suppose that:
$$
J_{p}^{\mathrm{I}} \equiv \underset{i_{1} \cdots i_{p}}{ } N_{i_{1} \cdots i_{p}} \delta z_{i_{1}} \cdots \delta z_{i_{p}} \quad p \leq n
$$
is a relative invariant of (I) whose coefficients can include texplicitly, and suppose that one has:
by virtue of equations (I).
It results from those hypotheses that:
$$
J_{p+1}^{\mathrm{II}} \equiv \underset{i_{1} \cdots i_{p}}{\mathbf{S}} N_{i_{1} \cdots i_{p}} \delta x_{i_{1}} \cdots \delta x_{i_{p}} \delta t
$$
will be a $(p+1)$-uple relative invariant of the system:
\[

$$
\begin{equation*}
\frac{d x_{i}}{X_{i}}=\frac{d t}{1}=d \tau \tag{II}
\end{equation*}
$$

\]

in which $\tau$ is the independent variable $(\delta \tau=0)$, and one will have:

$$
\frac{d}{d t} J_{p+1}^{\mathrm{II}} \equiv-D^{\mathrm{II}} \underset{i_{1} \cdots i_{p-1}}{\mathrm{~S}} W_{i_{1} \cdots i_{p-1}} \delta x_{i_{1}} \cdots \delta x_{i_{p-1}} \delta t
$$

by virtue of equations (II).
It will then result that:

$$
J_{p}^{\mathrm{II}} \equiv E^{\mathrm{I}} J_{p+1}^{\mathrm{II}}+\underset{i_{1} \cdots i_{p-1}}{\mathrm{~S}} W_{i_{1} \cdots i_{p-1}} \delta x_{i_{1}} \cdots \delta x_{i_{p-1}} \delta t
$$

is a relative invariant of (II) and that:

$$
\frac{d}{d \tau} J_{p}^{\mathrm{II}} \equiv D^{\mathrm{II}} E^{\mathrm{II}} \underset{i_{1} \cdots i_{p-1}}{\mathrm{~S}} W_{i_{1} \cdots i_{p-1}} \delta x_{i_{1}} \cdots \delta x_{i_{p_{-1}}} \delta t
$$

That is Hargreaves's theorem. One notes the analogy between this proof and the one in no. 3 of MI that relates to absolute integral invariants.

## 2. - Change of dependent variables.

Notations. - In equations (I), we replace the $n$ dependent variables $x$ with $n$ new dependent variables $y$ that are defined by the equations:

$$
y_{i}=y_{i}\left(x_{1}, \ldots, x_{n}, t\right) .
$$

When those equations are solved for $x$, that will give:

$$
x_{i}=x_{i}\left(y_{1}, \ldots, y_{n}, t\right) .
$$

The system (I) becomes:

$$
\begin{equation*}
\frac{d y_{i}}{Y_{i}}=d t, \quad i=1, \ldots, n, \tag{I'}
\end{equation*}
$$

in which:

$$
Y_{i} \equiv\left[\sum_{k=1}^{n} \frac{\partial y_{i}}{\partial x_{k}} X_{k}+\frac{\partial y_{i}}{\partial t}\right]
$$

The brackets always serve as a reminder that one has replaced the $x$ as functions of the $y$ and $t$ here. Thus:

$$
\left[x_{i}\right] \equiv x_{i}\left(y_{1}, \ldots, y_{n}, t\right),
$$

and similarly:

$$
[\varphi(x, t)] \equiv \psi(y, t) .
$$

Recall that in the theory of integral invariants, one has ( ${ }^{*}$ ):

$$
\frac{d}{d t} \delta x_{i}=\delta \frac{d x_{i}}{d t}
$$

Finally, we set:

$$
\begin{aligned}
X f & \equiv \frac{\partial f}{\partial t}+\sum_{i=1}^{n} \frac{\partial f}{\partial x_{i}} X_{i}, \\
Y g & \equiv \frac{\partial g}{\partial t}+\sum_{i=1}^{n} \frac{\partial g}{\partial y_{i}} Y_{i} .
\end{aligned}
$$

Identities. - Regardless of the function $\varphi$ of the $x$ and $t$, one will have the following identities:

$$
\begin{aligned}
{[X \varphi] } & \equiv Y[\varphi], \\
{[\delta \varphi] } & \equiv Y[\varphi], \\
{[\delta X \varphi] } & \equiv \delta Y[\varphi] .
\end{aligned}
$$

In particular, one will have:

$$
\begin{aligned}
{\left[X x_{i}\right] } & \equiv Y\left[x_{i}\right], \\
{\left[\delta x_{i}\right] } & \equiv Y\left[x_{i}\right], \\
{\left[\delta X x_{i}\right] } & \equiv \delta Y\left[x_{i}\right] .
\end{aligned}
$$

Those identities can be deduced from the formula from differential calculus:

[^2]$$
\sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_{i}} D x_{i}+\frac{\partial \varphi}{\partial t} D t=\sum_{i=1}^{n} \frac{\partial[\varphi]}{\partial y_{i}} D y_{i}+\frac{\partial[\varphi]}{\partial t} D t
$$
in which $D x_{i}, D t$ are arbitrary infinitely-small increments that are given to $x_{i}$ and $t$.

## Theorem:

If $J_{p}$ and $W_{p}$ are two integral forms such that one has:

$$
X J_{p}=W_{p}
$$

then one will have:

$$
Y\left[J_{p}\right]=\left[W_{p}\right] .
$$

That theorem results from the preceding identities.
In particular, if $J_{p}$ is a relative invariant of (I) such that one has:

$$
X J_{p}=D^{1} W_{p-1}
$$

then one will have:

$$
Y\left[J_{p}\right]=\left[D^{1} W_{p-1}\right]=D^{1^{\prime}}\left[W_{p-1}\right] .
$$

Point transformation that is generated by equations (I). - Let:

$$
\varphi_{i}(x, t)=\varphi_{i}(y, u), \quad i=1, \ldots, n,
$$

in which the right-hand side is deduced from the left-hand side by replacing $x$ with $y$ and $t$ with $u$. We then infer that:

$$
y_{i}=\psi_{i}(x, t, u) .
$$

We say that this point transformation is generated by equations (I). If we consider $u$ to be constant, i.e., if $d u / d t \equiv 0$, then the $y_{i}$ will be invariants of (I). Indeed, from the system:

$$
\varphi_{i}(y, u)=\varphi_{i},
$$

we infer that:

$$
y_{i}=\theta_{i}(\varphi, u) .
$$

Q. E. D.

The transformed equations ( $\mathrm{I}^{\prime}$ ) will then become:

$$
\frac{d y_{i}}{0}=d t, \quad i=1, \ldots, n
$$

To fix ideas, consider a relative invariant:

$$
J_{1}=\sum_{i} N_{i} \delta x_{i}
$$

such that $X J_{1}=D K$. From the change of variables, $Y\left[J_{1}\right]=D[K]$, and if we set $\left[J_{1}\right]=\sum_{i} N_{i}^{\prime} \delta y_{i}$ then we will find that:

$$
\frac{\partial N_{i}}{\partial t} \equiv \frac{\partial[K]}{\partial y_{i}}
$$

In addition to those relations, one will have the following relation from the calculus of variations:

$$
\sum_{i} N_{i}(x, t) \delta x_{i}-\sum_{i} N_{i}(y, t) \delta y_{i}=\delta \int_{u}^{t} K(x, t) d t
$$

The integral that appears on the right-hand side is extended over a trajectory of (I). Those trajectories are defined by the equations:

$$
\varphi_{i}(x, y)=\varphi_{i}(y, u),
$$

when they are solved for the $x$.

## 3. - Change of the independent variable.

Systems of differential equations. - Once more, consider the proposed system:
(I)

$$
\frac{d x_{i}}{X_{i}}=d t, \quad i=1, \ldots, n
$$

Since the independent variable is $t$, one will have $\delta t=0$.
After adding another variable $t$, that system will become:
(II)

$$
\frac{d x_{i}}{X_{i}}=\frac{d t}{1}=d \tau
$$

The independent variable here is $\tau$; one will then have $\delta \tau=0$.
Replace the variables $x$ and $t$ with the new variables $y$ and $u$. The system will become:

$$
\begin{equation*}
\frac{d y_{i}}{Y_{i}}=\frac{d u}{U}=d \tau \tag{III}
\end{equation*}
$$

The independent variable is once more $\tau$; one will again have $\delta \tau=0$.
Divide both sides of (III) by $U$ and introduce a new independent variable $\sigma$. One will have:

$$
\begin{equation*}
\frac{d y_{i}}{Y_{i} / U}=\frac{d u}{1}=d \sigma \tag{IV}
\end{equation*}
$$

Since the independent variable is $\sigma$, one will again have $\delta \sigma=0$.
Finally, by analogy with the system (I), consider the system:
(V)

$$
\frac{d y_{i}}{Y_{i} / U}=d u
$$

Since the independent variable is $u$, one will again have $\delta u=0$.

Problem. - If one knows an absolute or relative integral invariant $J_{p}^{1}$ of the system (I) then deduce some $p$-uple relative invariants of the system (II), (III), (V).

In no. 1 of this article, we showed how we can deduce a relative invariant $J_{p}^{\mathrm{II}}$ for (II) from $J_{p}^{\mathrm{I}}$. In no. 2, we then showed how we could deduce a relative invariant $J_{p}^{\text {III }}$ for (III) from $J_{p}^{\text {II }}$. It remains for us to solve the last part of the problem that was posed. We proceed as follows: Deduce $J_{p+1}^{\mathrm{II}}$ from $J_{p}^{\mathrm{I}}$, in the way that was explained at the and of no. 1. Then deduce $J_{p+1}^{\mathrm{II}}$ from $J_{p+1}^{\mathrm{II}}$ in the way that was explained in no. 2. Suppose that we have:

$$
\frac{d}{d u_{\alpha}^{\mathrm{II}}} J_{p+1}^{\mathrm{II}}=D^{\mathrm{III}} V_{p}^{\mathrm{II}}
$$

by virtue of equations (III), in which the index III on $d u$ serves to remind us that we are dealing with the system (III) here, and $V_{p}^{\text {III }}$ represents a $p$-uple integral form. We will then get the relative invariant $J_{p}^{\mathrm{V}}$ of the system $(\mathrm{V})$ by setting:

$$
J_{p}^{\mathrm{V}} \equiv\left(E^{\mathrm{II}} J_{p+1}^{\mathrm{III}}-V_{p}^{\mathrm{III}}\right)_{\delta u=0} .
$$

The subscript $\delta u=0$ signifies that one must replace the $\delta u$ in the $p$-uple integral form that enters into the parentheses with zero. In addition, the reader will verify that:

$$
\frac{d}{d u_{\mathrm{V}}} J_{p}^{\mathrm{V}}=-D^{\mathrm{V}}\left(E^{\mathrm{IV}} V_{p}^{\mathrm{III}}\right)_{\delta u=0}
$$

In the left-hand side, the subscript V on $d u$ signifies that the derivative $d / d u$ must be taken with respect to the system $(\mathrm{V})$. In the right-hand side, one must perform the operation $D^{\mathrm{V}}$ on the integral form that is obtained by replacing $\delta u$ with zero in $E^{\mathrm{IV}} V_{p}^{\mathrm{III}}$. The problem that was posed is then solved completely.

## 4. - Equations of the ether and electric charge.

Equations. - When one adopts the notations of H.-A. Lorentz ( ${ }^{*}$ ), the equations of the ether and electric charge can be written:
(A)
(B)
(C)
(D)

$$
\begin{aligned}
& \operatorname{div} \mathbf{d}=\rho, \\
& \operatorname{div} \mathbf{h}=0, \\
& \operatorname{rot} \mathbf{h}=\frac{1}{c}(\mathbf{d}+\rho \mathbf{v}), \\
& \operatorname{rot} \mathbf{d}=\frac{1}{c} \mathbf{h} .
\end{aligned}
$$

The coordinate axes $x, y, z$ are rectangular and right-handed. The vector $\mathbf{d}$ is called the electric displacement. The vector $\mathbf{h}$ represents the magnetic force. The scalar quantity $\rho$ is the volume density of electricity. The vector $\mathbf{v}$ represents the velocity of the point $x, y, z$ at the instant $t$. The speed of light in the ether (with no electric charge) is denoted by $c$. The three components of a vector will be denoted by the letter that represents the vector when it is endowed with a subscript $x, y$, or $z$. Finally, one sets:

$$
\operatorname{div} \mathbf{d} \equiv \frac{\partial d_{x}}{\partial x}+\frac{\partial d_{y}}{\partial y}+\frac{\partial d_{z}}{\partial z}
$$

and similarly for $\operatorname{div} \mathbf{h}$. The vector rot $\mathbf{h}$ has the components:

$$
\frac{\partial h_{z}}{\partial y}-\frac{\partial h_{y}}{\partial z}, \quad \frac{\partial h_{x}}{\partial z}-\frac{\partial h_{z}}{\partial x}, \quad \frac{\partial h_{y}}{\partial x}-\frac{\partial h_{x}}{\partial y}
$$

and similarly for rot $\mathbf{d}$. Finally, $\dot{\mathbf{d}}$ is a vector whose components are:

$$
\frac{\partial d_{x}}{\partial t}, \frac{\partial d_{y}}{\partial t}, \frac{\partial d_{z}}{\partial t},
$$

and similarly for $\dot{\mathbf{h}}$.

[^3]The two integral forms of Hargreaves. - In my MI, I drew attention to the two integral forms that were first studied by Hargreaves. Here, those forms will be written:

$$
\begin{aligned}
J_{2} & \equiv \mathrm{~S}\left(d_{x} \delta y \delta z+c h_{x} \delta t \delta x\right), \\
\Omega_{2} & \equiv \mathrm{~S}\left(h_{x} \delta y \delta z-c d_{x} \delta t \delta x\right),
\end{aligned}
$$

in which the symbol $S$ will serve to remind us that we are dealing with integral forms and will indicate, in addition, that we must perform a summation that extends over all cyclic permutations of $x, y, z$.

Those integrals forms enjoy the following property:

The necessary and sufficient conditions for the symbolic differential to be:

$$
D J_{2}=\rho d x d y d z-\mathrm{S} v_{x} d y d z d t
$$

are nothing but equations (A) and (C).
The necessary and sufficient conditions for the symbolic differential to be:

$$
D \Omega_{2}=0
$$

are nothing but equations (B) and (D).
That property was utilized to good effect by Bateman (*) in order to study the changes of variables $x, y, z$, and $t$ that leave the form of equations (A), (B), (C), and (D) invariant. I propose to recall that question later on when I appeal to differential parameters.

Change of variables $x, y, z$, and $t$. - The equations:

$$
\begin{equation*}
\frac{d x}{v_{x}}=\frac{d y}{v_{y}}=\frac{d z}{v_{z}}=d t \tag{I}
\end{equation*}
$$

define the trajectories. It results from equations (A), $\ldots,(\mathrm{D})\left({ }^{* *}\right)$ that equations (I) will possess the 3-uple absolute integral invariant:

$$
I_{3}^{1} \equiv \mathrm{~S} \rho \delta x \delta y \delta z
$$

The system (II) of no. $\mathbf{3}$ will become:

[^4]\[

$$
\begin{equation*}
\frac{d x}{v_{x}}=\frac{d y}{v_{y}}=\frac{d z}{v_{z}}=\frac{d t}{1}=d \tau . \tag{II}
\end{equation*}
$$

\]

Those equations define the trajectories in space-time. They admit the absolute invariants:

$$
\begin{aligned}
& I_{4}^{\mathrm{II}} \equiv \mathrm{~S} \rho \delta x \delta y \delta z \delta t \\
& I_{5}^{\mathrm{II}} \equiv E^{\mathrm{II}} I_{4}^{\mathrm{II}}=\rho \delta x \delta y \delta z-\mathrm{S} v_{x} \delta y \delta z \delta t
\end{aligned}
$$

They will also admit the relative invariant $J_{2}$, which we will denote by $J_{2}^{\text {II }}$ here. We have seen that $D^{\mathrm{II}} J_{2}^{\mathrm{II}}=I_{3}^{\mathrm{II}}$. It will then result that $I_{3}^{\mathrm{II}}$ is invariant, it is an exact differential, and $E^{\mathrm{II}} I_{3}^{\mathrm{II}}=0$. In addition, we know that:

$$
\frac{d J_{2}^{\mathrm{II}}}{d \tau}=D^{\mathrm{II}} S\left(c h_{x}+v_{z} d_{x}-v_{z} d_{x}\right)\left(v_{x} \delta t-\delta x\right)
$$

Now, perform a change of the variables $x, y, z$, and $t$; call the new variables $x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}$. No. $\mathbf{2}$ of this article showed use how to use the invariants of (II) in order to deduce the invariants of a new system that we write as follows:

$$
\begin{equation*}
\frac{d x^{\prime}}{v_{x^{\prime}}}=\frac{d y^{\prime}}{v_{y^{\prime}}}=\frac{d z^{\prime}}{v_{z^{\prime}}}=\frac{d t^{\prime}}{v_{t^{\prime}}}=d \tau . \tag{III}
\end{equation*}
$$

Finally, if we would like to make $t^{\prime}$ play a role that is analogous to that of $t$ then we will have to consider the system:

$$
\begin{equation*}
\frac{d x^{\prime}}{v_{x^{\prime}} / v_{t^{\prime}}}=\frac{d y^{\prime}}{v_{y^{\prime}} / v_{t^{\prime}}}=\frac{d z^{\prime}}{v_{z^{\prime}} / v_{t^{\prime}}}=d t^{\prime}, \tag{V}
\end{equation*}
$$

whose absolute integral invariant will be, by virtue of no. 3:

$$
I_{3}^{\mathrm{v}} \equiv \mathrm{~S} \rho \frac{\partial(x, y, z, t)}{\partial\left(x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}\right)} v_{t^{\prime}} \delta x^{\prime} \delta y^{\prime} \delta z^{\prime}
$$

If the $x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}$ are defined by the famous Voigt-H.-A. Lorentz transformation ( ${ }^{*}$ ):

[^5]\[

\left\{$$
\begin{array}{l}
x^{\prime}=k l(x-w t), \\
y^{\prime}=l y, \\
z^{\prime}=l z \\
t^{\prime}=k l\left(t-\frac{w}{c^{2}} x\right)
\end{array}
$$\right.
\]

then we will find that:

$$
I_{3}^{\mathrm{v}} \equiv \frac{k}{l^{3}} \mathrm{~S} \rho\left(1-\frac{w}{c^{2}} v_{x}\right) \delta x^{\prime} \delta y^{\prime} \delta z^{\prime}
$$

which will lead us to set:

$$
\rho^{\mathrm{v}} \equiv \rho\left(1-\frac{w}{c^{2}} v_{x}\right) \frac{k}{l^{3}},
$$

which is a result that agrees perfectly with that of H. Poincaré (*). One knows that Lorentz had set $\rho^{\prime} \equiv \rho / k l^{3}$ (loc. cit., pp. 197). One sees that $\rho^{\prime}$ is not a multiplier of (V), but of the system (III). Some other divergences can be explained in the same way by means of the systems (I), (II), (III), and (V).

## 5. - Generalized canonical equations.

Complements. - In my article MI, I showed that when one starts from a relative integral invariant of a given form, one will immediately find a system of differential equations that can be considered to be a generalization of Hamilton's canonical equations. Since then, I have advanced that study: In two recent notes $\left(^{* *}\right)$, I have especially studied the generalized canonical equations:

$$
\frac{d x_{i}}{\sum_{k} \frac{v_{k i}}{v} \frac{\partial H}{\partial x_{k}}}=d t \quad i, k=1, \ldots, 2 m
$$

that are deduced from the relative invariant $J \equiv \sum_{i=1}^{2 m} N_{i} \delta x_{i}$, and I have shown how one can deduce some other invariants from an invariant of a system of differential equations when one knows a differential parameter that is attached to a multilinear form and is invariant with respect to those differential equations. That generalization points to the deeper meaning of Poisson's theorem. Vergne (C. R. Acad. Sci. Paris, 25 April 1910) has partially outlined it thanks to the theory of contact transformations ( ${ }^{* * *}$ ).

[^6]In another article (*), I have used the integral invariant:

$$
j_{\lambda}=\sum_{i=1}^{n} N_{i}^{\lambda} \delta x_{i}, \quad l=1, \ldots, r \leq n,
$$

in which $N_{i}^{\lambda}$ are some functions of $r$ independent variables $t_{1}, \ldots, t_{r}$, the $n$ dependent variables $x_{1}$, $\ldots, x_{n}$, and the partial derivatives $x_{i}^{\lambda} \equiv d x_{i} / d t \lambda$, in order to deduce the Hamilton-Volterra canonical equations, and I have made a deep study of those equations.

Let me make one last remark: One knows that the differential form in the $2 m$ variables $x_{1}, \ldots$, $x_{2 m}$ can be identified with another differential form $\sum_{k=1}^{m} y_{k} \delta z_{k}$, which includes at least $m$ terms, in general. The $y_{k}$ and $z_{k}$ are then $2 m$ distinct functions of the $x$.

That reduction bears the name of the Pfaff problem. An analogous problem presents itself for integral forms. One knows that the relative invariant $\sum_{k=1}^{m} y_{k} \delta z_{k}$ that has the reduced or canonical Pfaff form plays an essential role in the classical theory of canonical equations. The same thing will be true for the $p$-uple relative invariant when it is put into the reduced form as it relates to the generalization of the canonical equations. That will be the subject of another article.

## 6. - Second-order differential equations.

Notations. - The $n$ second-order differential equations:

$$
\frac{d^{2} x_{i}}{d t^{2}}=\xi_{i}\left(x, \frac{d x}{d t}, t\right) \quad i=1, \ldots, n
$$

are equivalent to $2 n$ first-order equations:

$$
\begin{equation*}
\frac{d x_{i}}{x_{i}^{\prime}}=\frac{d x_{i}^{\prime}}{\xi_{i}\left(x, x^{\prime}, t\right)}=d t, \quad i=1, \ldots, n \tag{VI}
\end{equation*}
$$

Now consider a system of $n$ first-order equations:

$$
\begin{equation*}
\frac{d x_{i}}{x_{i}^{\prime}(x, t)}=d t \tag{VII}
\end{equation*}
$$

[^7]in which the $x_{i}^{\prime}(x, t)$ are functions of the $x$ and $t$, such that one will have, by virtue of equations (VI):
$$
\frac{d x_{i}^{\prime}(x, t)}{d t} \equiv \xi_{i}\left(x, x^{\prime}, t\right)
$$
after having replaced the $x^{\prime}$ with their expressions in terms of $x$ and $t$ in the right-hand side. Finally, set:
\[

$$
\begin{aligned}
\xi f & \equiv \sum_{i=1}^{n} \frac{\partial f}{\partial x_{i}} x_{i}^{\prime}+\sum_{i=1}^{n} \frac{\partial f}{\partial x_{i}^{\prime}} \xi_{i}+\frac{\partial f}{\partial t} \\
\eta g & \equiv \sum_{i=1}^{n} \frac{\partial g}{\partial x_{i}} x_{i}^{\prime}(x, t)+\frac{\partial g}{\partial t}
\end{aligned}
$$
\]

which represent the infinitesimal transformations (VI) and (VII).
The operation consists of replacing the $x_{i}^{\prime}$ with functions of $x$ and $t$ that satisfy the preceding relations $d x_{i}^{\prime} / d t=\xi_{i}$, and it will be denoted by $\}$. Thus:

$$
\left\{x_{i}^{\prime}\right\} \equiv x_{i}^{\prime}(x, t)
$$

and

$$
\left\{\varphi\left(x, x^{\prime}, t\right)\right\} \equiv \varphi\left(x, x^{\prime}(x, t), t\right) .
$$

Identities. - Regardless of the function $\varphi\left(x, x^{\prime}, t\right)$, one will have the identities:

$$
\begin{aligned}
\{\xi \varphi\} & \equiv \eta\{\varphi\} \\
\delta\{\varphi\} & \equiv\{\delta \varphi\} \\
\{\delta \xi \varphi\} & \equiv \delta \eta\{\varphi\}
\end{aligned}
$$

In the second identity, one has, explicitly:

$$
\sum_{i} \frac{\partial\{\varphi\}}{\partial x_{i}} \delta x_{i} \equiv\left\{\sum_{i} \frac{\partial \varphi}{\partial x_{i}} \delta x_{i}+\sum_{i} \frac{\partial \varphi}{\partial x_{i}^{\prime}} \delta x_{i}^{\prime}\right\} .
$$

That identity will persist for any functions $x_{i}^{\prime}(x, t)$.

## Theorem:

If $J_{p}$ is a p-uple integral form such that:

$$
\xi J_{p}=K_{p},
$$

in which $K_{p}$ is also an integral form, then one will have:

$$
\eta\left\{J_{p}\right\}=\left\{K_{p}\right\} .
$$

In particular, if $J_{p}$ is a relative invariant of $(\mathrm{VI})$ such that one has:

$$
\xi J_{p}=D^{\mathrm{VI}} K_{p-1}
$$

then one will have:

$$
\eta\left\{J_{p}\right\}=D^{\mathrm{VII}}\left\{K_{p-1}\right\}
$$

Conversely, if:

$$
J^{\mathrm{VIII}}=\sum_{i=1}^{n} \frac{\partial \psi\left(x^{\prime}, t\right)}{\partial x_{i}^{\prime}} \delta x_{i}
$$

is a relative invariant of:
(VIII)

$$
\frac{d x_{i}}{x_{i}^{\prime}(x, t)}=d t, \quad i=1, \ldots, n,
$$

in which $x_{i}^{\prime}$ are arbitrary functions of $x$ and $t$, and if one has:

$$
\frac{d J^{\mathrm{VIII}}}{d t}=\delta Q(x, t)
$$

then I say that:

$$
J^{\mathrm{IX}}=\sum_{i=1}^{n} \frac{\partial \psi\left(x^{\prime}, t\right)}{\partial x_{i}^{\prime}} \delta x_{i}
$$

is also a relative invariant of:

$$
\begin{equation*}
\frac{d x_{i}}{x_{i}^{\prime}}-\frac{d x_{i}^{\prime}}{\xi_{i}\left(x, x^{\prime}, t\right)}=d t, \quad i=1, \ldots, n \tag{IX}
\end{equation*}
$$

in which the $\xi_{i}\left(x, x^{\prime}, t\right)$ are inferred from:

$$
\sum_{i=1}^{n} \frac{\partial^{2} \psi\left(x^{\prime}, t\right)}{\partial x_{i}^{\prime} \partial x_{k}^{\prime}} \xi_{k}\left(x, x^{\prime}, t\right)+\frac{\partial^{2} \psi\left(x^{\prime}, t\right)}{\partial x_{i}^{\prime} \partial t}=\frac{\partial S(x, t)}{\partial x_{i}}
$$

in which:

$$
S(x, t) \equiv Q(x, t)-\psi\left(x^{\prime}, t\right),
$$

and one:

$$
x_{i}^{\prime} \equiv x_{i}^{\prime}(x, t)
$$

in only that function S. By virtue of equations (IX), one will have:

$$
\frac{d J^{\mathrm{IX}}}{d t}=\delta\left(S(x, t)+\psi\left(x^{\prime}, t\right)\right) .
$$

In particular, if:

$$
\psi=\frac{1}{2} \sum_{i=1}^{n} x_{i}^{\prime 2},
$$

one will have:

$$
J \equiv \sum_{i=1}^{n} x_{i}^{\prime} \delta x_{i}
$$

which is an integral invariant that plays a fundamental role in the theory of vortices. Thus:

$$
\xi_{i}=\frac{\partial S(x, t)}{\partial x_{i}}
$$

and

$$
\frac{d J}{d t}=\delta\left(S(x, t)+\frac{1}{2} \sum_{i} x_{i}^{\prime 2}\right)
$$

## 7. - Application to superimposed electric and magnetic fields.

Problem. - Find the equations (VI) that possess the relative invariant:

$$
J^{\mathrm{VI}}=\sum_{i} x_{i}^{\prime} \delta x_{i}-U \delta W,
$$

in which $U$ and $W$ are two functions of $x_{1}, \ldots, x_{n}$ that are invariant in such a way that one will have:

$$
\frac{d J^{\mathrm{VI}}}{d t}=\delta K
$$

by virtue of equations (VI).
Upon introducing the auxiliary function:

$$
S \equiv K+U \sum_{i} \frac{\partial W}{\partial x_{i}} x_{i}^{\prime}
$$

in the course of calculation, one will easily find that in equations (VI), one will have:

$$
\xi_{i}\left(x, x^{\prime}, t\right)=-\frac{\partial V(x, t)}{\partial x_{i}}-\sum_{k=1}^{n} \frac{\partial(U, W)}{\partial\left(x_{i}, x_{k}\right)} x_{k}^{\prime},
$$

and that:

$$
K=\frac{1}{2} \sum_{i} x_{i}^{\prime 2}-V-\sum_{i} U \frac{\partial W}{\partial x_{i}} x_{i}^{\prime},
$$

in which $V(x, t)$ is an arbitrary function.
Equations (VI) are equivalent to the system:

$$
\frac{d^{2} x_{i}}{d t^{2}}=-\frac{\partial V(x, t)}{\partial x_{i}}-\sum_{k=1}^{n} \frac{\partial(U, W)}{\partial\left(x_{i}, x_{j}\right)} \frac{d x_{k}}{d t}, \quad i=1, \ldots, n .
$$

Equations of motion of a corpuscle. - The motion of a corpuscle ( $x, y, z$ ) of mass $m$ and charge $e$ in an electric and magnetic field is governed by the equation ( ${ }^{*}$ ):

$$
m \frac{d^{2} x}{d t^{2}}=-e \frac{\partial V}{\partial x}+e\left(Y \frac{d z}{d t}-Z \frac{d y}{d t}\right),
$$

and two others that are deduced from it by cyclic permutation. The components of the magnetic force ( $X, Y, Z$ ) satisfy the condition:

$$
\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0 .
$$

C. Störmer ( ${ }^{* *)}$ has remarked that one can write the equations of motion of the corpuscle in the form ( $\mathrm{VI}^{\prime}$ ). Those equations will then admit the relative invariant:

$$
J^{\mathrm{v1}}=\sum_{i=1}^{n}\left(x_{i}^{\prime}-U \frac{\partial W}{\partial x_{i}}\right) \delta x_{i},
$$

and can be immediately written in the canonical form if one sets:

$$
y_{i}=x_{i}^{\prime}-U \frac{\partial W}{\partial x_{i}}
$$

while preserving the other $n$ dependent variables $x_{i}$. After that change of dependent variables (no. 2), the transform will again admit the relative invariant $J=\sum_{i=1}^{n} y_{i} \delta x_{i}$, with $d J / d t=\delta[K]$. Upon setting $H=\left[\sum_{i} y_{i} x_{i}^{\prime}-K\right]$, the equations of motion will become:

[^8]$$
\frac{d x_{i}}{\frac{\partial H}{\partial y_{i}}}=\frac{d x_{i}}{-\frac{\partial H}{\partial y_{i}}}=d t, \quad i=1, \ldots, n
$$

One will easily see that:

$$
H=\frac{1}{2} \sum_{i}\left(y_{i}+U \frac{\partial W}{\partial x_{i}}\right)^{2}+V .
$$

Störmer obtained that reduction in the last of his cited notes.
That result can be generalized by considering the equations that admit the relative invariant:

$$
J=\sum_{i} \frac{\partial \psi\left(x^{\prime}, t\right)}{\partial x_{i}^{\prime}} \delta x_{i}-\sum_{\lambda} U_{\lambda} \delta W_{\lambda} .
$$

In particular, if:

$$
\psi\left(x^{\prime}, t\right)=\sum_{i} \sum_{k} m_{i k} x_{i}^{\prime} x_{k}^{\prime},
$$

with:

$$
m_{i k}=m_{k i} \quad \text { and } \quad \lambda=1
$$

then one will find a canonical system for which the characteristic function is:

$$
H=\frac{1}{2} \sum_{i} \sum_{k} \mu_{i k}\left(y_{i}+U \frac{\partial W}{\partial x_{i}}\right)\left(y_{k}+U \frac{\partial W}{\partial x_{k}}\right)+V,
$$

in which $\mu_{i k}$ represents the algebraic minor $M_{i k}$ in the determinant $m$ of $m_{i k}$ when that minor $M_{i k}$ is divided by $m$. Störmer ( ${ }^{*}$ ) wrote $M_{i k}$ in the expression for $H$, but it should be $\mu_{i k}$.

No. 6 of this article applies immediately to the foregoing. The functions $x_{i}^{\prime}(x, t)$ define a vector field for the corpuscle. In the study of that field, the invariant $\{J\}$ plays the same role that the invariant $\sum_{i} x_{i}^{\prime} \delta x_{i}$ does in hydrodynamics. The vorticity vector will have components here that are one-half of:

$$
\frac{\partial x_{i}^{\prime}}{\partial x_{k}}-\frac{\partial x_{k}^{\prime}}{\partial x_{i}}-\frac{\partial(U, W)}{\partial\left(x_{k}, x_{i}\right)} .
$$

Meanwhile, there is one difference: In hydrodynamics, the fluid particles exist simultaneously, whereas in the velocity field of corpuscles, they form only lines, surfaces, or volumes whose existence is fictitious. As one knows, the same procedure is employed in the study of the electric field by means of the potential function.

2 December 1910.

[^9]
[^0]:    (*) A somewhat-less general extension was given in my first article "Sur les invariants intégraux relatifs," Bull. Acad. roy. Belgique, classe de sci. 1 (1909), 66-83. I shall denote that article by MI here.

[^1]:    (*) E. Goursat, "Sur les invariants intégraux," J. math. pures appl. (6) 4 (1908), 331-365, see esp. pp. 347.

[^2]:    (*) See no. 4 of my "Étude sur les invariants intégraux," Rend. circ. mat. di Palermo 15 (1901).

[^3]:    (*) H.-A. Lorentz, The theory of electrons and its applications to the phenomena of light and radiant heat, Leipzig, 1909. (See esp., pps. 3 and 12.)

[^4]:    (*) H. Bateman, "The transformation of the electrodynamical equations," Proc. London Math. Soc. (2) 8 (1910), 223-264.
    (**) H. A. Lorentz, loc. cit., see pp. 232.

[^5]:    (*) W. Voigt, "Ueber das Doppler'sche Princip," Gött. Nachr. (1887). - H.-A. Lorentz, loc. cit., (See pps. 197 and 198.)

[^6]:    (*) H. Poincaré, "Sur la dynamique de electron," Rend. circ. Palermo 21 (1906). (See pp. 133.)
    (**) Th. De Donder, "Généralisation du théorie de Poisson," C. R. Acad. Sci. Paris, 8 March 1909. - "Sur le théorie de Poisson et sur les invariants différentiels," C. R. Acad. Sci. Paris, 1 August 1910.
    $\left({ }^{* * *}\right)$ A more-extensive article by Vergne will appear soon in the Annales de l'École normale supérieure de Paris and will treat the theory of integral invariants, in addition.

[^7]:    (*) "Sur les équations canoniques de Hamilton-Volterra," Mémoires in $-4^{\circ}$ de l'Acad. roy. de Belgique [classe des sciences], t. III).

[^8]:    (*) P. Appell, Traité de mécanique rationelle, $3^{\text {rd }}$ ed., 1909. (See pp. 368).
    $\left(^{* *}\right)$ C. Störmer, C. R. Acad. Sci. Paris 12 and 26 September 1910.

[^9]:    $\left(^{*}\right)$ Störmer gave the meaning of $M_{i k}$ in which note on 2 March 1908. (C. R. Acad. Sci. Paris.)

