STUDY

OF THE

INTEGRAL INVARIANTS

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Extracted from *Rendiconti de Circolo mathematico* in Palermo, Tome XV, 1901

Presented on 17 March 1901

PARIS

GAUTHIER-VILLARS, PRINTER-BOOKSTORE

FOR THE BUREAU OF LONGITUDES OF L'ÉCOLE POLYTECHNIQUE

Quai des Grands-Augustin, 55

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1901

To my friend

H. DIEUDONNÉ

PREFACE

H. POINCARÉ introduced *integral invariants* into analysis while studying the three-body problem (*). Few geometers seem to be interested in that theory, although it is quite original and fruitful.

KOENIGS published two important notes in the Comptes-Rendus of the Paris Academy of Sciences, which are notes that will frequently be in question in this treatise. HADAMARD used integral invariants in one of his studies, and P. APPELL gave an elementary presentation of the theory in his *Cours de Mécanique rationelle* (tome II).

That is a summary of the bibliography of integral invariants.

In that study, we have attempted to present, in a systematic manner, all of the notions that were acquired in that theory thanks to H. POINCARÉ and KOENIGS. In addition, we have added some results that are due to our own personal research. In order for persons that are already knowledgeable in the theory of integral invariant to avoid reading the entire treatise, I shall list some of those contributions:

Nos. **11**, **12**, **14**, and **15**: Theorems relating to first-order invariants.

Nos. **16**, **17**, **21**, **25** (cont.), **30**, **34**, **35**, **38**, **39**, and **40**: Study of the solutions by variations of or order one or arbitrary order. In order for the equations:

$$\sum_i \frac{\partial \theta}{\partial x_i} X_i + \frac{\partial \theta}{\partial t} = 0 ,$$

$$\sum_i \frac{\partial \theta}{\partial x_i} \xi_i = 0$$

to form a Jacobian system, it is necessary and sufficient that (ξ_i) should be a solution of the variational equations:

$$\frac{\delta x_i}{X_i} = \delta t .$$

Nos. **20**, **35**, **37**, and **41**. Introducing the theory of infinitesimal transformations into that of integral invariants.

Nos. **23**, **24**, and **25**. Integral invariants of order $n - 1$.

No. **28**. Generalization of a theorem of KOENIGS.

No. **29**. Theorem.

Nos. **31**, **32**, and **33**. Integral invariants of order p.

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Nos. **47**, **48**, **49**, **50**. Application to vortices.

(*) H. POINCARÉ, *Les Méthodes nouvelles de la Mécanique céleste*, 3 vols., Paris.

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CHAPTER I

DEFINITION OF AN INTEGRAL INVARIANTS

1. – Consider the system of n differential equations:

$$\frac{\delta x_i}{X_i} = \delta t_i \qquad (i = 1, 2, \ldots, n) , \tag{1}$$

in which $X_1, X_2, \ldots, X_n$ are given uniform, analytic functions of $x_1, x_2, \ldots, x_n$, and t. Suppose that t represents time and that $x_1, x_2, \ldots, x_n$ are the n coordinates of a point M that displaces in n-dimensional hyperspace. If (x_i) represents the n coordinates of M at the instant t then the moving point M will occupy the position $(x_i + X_i\ \delta t)$ at the instant $t + \delta t$. We say that equations (1) completely determine the variation of M.

If equations (1) are satisfied when one sets:

$$x_i = f_i\,(t) \qquad (i = 1, 2, \ldots, n) \tag{2}$$

then one says that equations (2) define a particular *solution* of equations (1).

In order for $F\,(x_1, x_2, \ldots, x_n, t)$ to be an *integral* of equations (1), it is necessary and sufficient that the variation of F should be identically zero when the variation is taken to conform to equations (1). Here is what we mean by that: Give an increment δt to t, so the x_i will become $x_i + \delta x_i$ or $x_i + X_i\,\delta t$, and the function F will submit to the variation:

$$\sum_k \frac{\partial F}{\partial x_k}\delta x_k + \frac{\partial F}{\partial t}\delta t$$

or

$$\left(\sum_k \frac{\partial F}{\partial x_k} X_k + \frac{\partial F}{\partial t}\right)\delta t\,.$$

The quantity in parentheses must be identically zero, which is to say, it must reduce to zero, even before one replaces the x_i with a solution (2).

Therefore, in order for $F\,(x_i, t)$ to be an integral of equations (2), it is necessary and sufficient that:

$$\sum_k \frac{\partial F}{\partial x_k} X_k + \frac{\partial F}{\partial t} = 0\,.$$

We write simply:

$$\delta F = 0\,,$$

but the reader should not forget that the variations of the x_i are always determined by equations (1).

F has zero variation, so it preserves the same value, but is otherwise arbitrary. Therefore:

$$F(x_1, \ldots, x_n, t) = C$$

will imply a relation between the x and t. (C is an arbitrary constant.)

2. – Let:

$$F_i(x_1, \ldots, x_n, t) = C_i \qquad (i = 1, 2, \ldots, n)$$

be the general integral of equations (1). The presence of n arbitrary constants $C_1, \ldots, C_n$ permits us to place the moving point at an arbitrary point (x_i^0) in n-dimensional hyperspace at the arbitrary instant t_0. Among that infinitude of positions, take a set of them that forms a manifold V, of order p, for example. That manifold is continuous, but it has an arbitrary form. It will be represented n equations such as:

$$x_i = \theta_i(\lambda_1, \lambda_2, \ldots, \lambda_p) \qquad (i = 1, \ldots, n), \tag{3}$$

in which the θ_i are *arbitrary* functions that are finite and continuous in the p independent variables $\lambda_1, \ldots, \lambda_p$. Those functions cannot include t explicitly, because otherwise they would no longer be arbitrary. We can always suppose that the functional determinants of p of the n functions θ are never annulled simultaneously when the point $(\lambda_1, \ldots, \lambda_p)$ describes a certain manifold V' in p-dimensional hyperspace, so the point (x_i) will describe the manifold V in such a way the manifolds V and V' correspond uniformly. One will once more have a p-dimensional manifold V if one combines equations (3) with a certain number of inequalities such as:

$$\psi(\lambda_1, \lambda_2, \ldots, \lambda_p) > 0\,. \tag{4}$$

Those inequalities serve to limit the region in which one forms the manifold V.

3. – Now extend the integral:

$$I_p = \int^{p-\text{fold}} \sum M_{\alpha_1 \cdots \alpha_p} dx_{\alpha_1} \cdots dx_{\alpha_p}$$

over the manifold V that we just formed. The differentials $dx_{\alpha_1}, \ldots, dx_{\alpha_p}$ are p differentials that are chosen arbitrarily from among the n differentials $dx_1, \ldots, dx_n$. The $M_{\alpha_1 \cdots \alpha_p}$ are given functions of $x_1, \ldots, x_n$, and t. They are assumed to be finite and continuous, along with their first-order partial derivatives in the domain considered. There are as many of them as there are combinations of n letters taken p at a time.

Thanks to equations (3) for the manifold V, we can transform I_p into an *ordinary* integral of order p. We knows that we will then have:

$$I_p = \int^{p-\text{fold}} \sum M_{\alpha_1 \cdots \alpha_p} \frac{\partial(x_{\alpha_1}, \ldots, x_{\alpha_p})}{\partial(\lambda_1, \ldots, \lambda_p)} d\lambda_1 \cdots d\lambda_p\,.$$

We must perform the integration with respect to the p variables λ in succession, and the limits of integration are defined by (4). In other words, we now extend the integral I_p , no longer over V, but over the manifold V', which has the remarkable property that it remains *fixed* in p-dimensional space; the same thing will be true for the limits of integration or the boundary of V': The equations of that boundary are obtained by successively annulling inequalities such as (4). Therefore, if we would like to look for the variation of I_p then we must no longer preoccupy ourselves with the limits of integration, if, as we have done, we previously transform it into an *ordinary* integral of order p. We shall indicate how one can calculate the variation of that integral.

4. – From what was said, we will have:

$$\delta I_p = \int \sum \left[\frac{\partial (x_{\alpha_1}, \ldots, x_{\alpha_p})}{\partial (\lambda_1, \ldots, \lambda_p)} \delta M_{\alpha_1 \cdots \alpha_p} + M_{\alpha_1 \cdots \alpha_p} \delta \frac{\partial (x_{\alpha_1}, \ldots, x_{\alpha_p})}{\partial (\lambda_1, \ldots, \lambda_p)} \right] d\lambda_1 \cdots d\lambda_p$$

$$\delta M_{\alpha_1 \cdots \alpha_p} = \left(\sum_k \frac{\partial M_{\alpha_1 \cdots \alpha_p}}{\partial x_k} X_k + \frac{\partial M_{\alpha_1 \cdots \alpha_p}}{\partial t} \right) \delta t \ .$$

How do we calculate the variation of $\dfrac{\partial (x_{\alpha_1}, \ldots, x_{\alpha_p})}{\partial (\lambda_1, \ldots, \lambda_p)}$? That question reduces to the following one: For example, what does the variation $\partial x_i / \partial \lambda_1$ of equal? Recall that at the arbitrary instant t_0 (or more simply t), the points of the manifold V have the coordinates:

(3) $$x_i = \theta_i (\lambda_1, \ldots, \lambda_p) \ .$$

At the instant $t + \delta t$, those points will have the coordinates:

$$x_i + \delta x_i = \theta_i (\lambda_1, \ldots, \lambda_p) + X_i (\theta_1, \ldots, \theta_n, t) \, \delta t \ .$$

Therefore, during the time δt, the manifold V will deform, along with its boundary, but that deformation is completely-determined, in other words, each of the point of V will describe a small trajectory that is determined perfectly by equations (1). Let $V + \delta V$ denote what the manifold V will become at the instant $t + \delta t$.

Give another value λ_1' to λ_1 , while preserving the previously-given values for the other λ. At the instant t, we will have:

$$x_1' = \theta_i (\lambda_1', \lambda_2, \ldots, \lambda_p) \ ,$$

and at the instant $t + \delta t$:

$$x_1' + \delta x_1' = \theta_i (\lambda_1', \lambda_2, \ldots, \lambda_p) + X_i (\theta_1', \theta_2', \ldots, \theta_n' \, t) \delta t \ .$$

The primes on the θ in X_i signify that the first of the λ in those functions has the value λ_1' .

Let $\lambda_1' = \lambda_1 + d\lambda_1$. One will then have:

$$x_i' = x_i + \frac{\partial x_i}{\partial \lambda_1} d\lambda_1 = \theta_i(\lambda_1 + d\lambda_1, \lambda_2, \ldots, \lambda_p),$$

$$x_i' + \delta x_i' = x_i + \frac{\partial x_i}{\partial \lambda_1} d\lambda_1 + \delta x_i + \delta \frac{\partial x_i}{\partial \lambda_1} d\lambda_1$$

$$= \theta_i(\lambda_1 + d\lambda_1, \lambda_2, \ldots, \lambda_p) + X_i\left(x_k + \frac{\partial x_k}{\partial \lambda_1} d\lambda_1, t\right)\delta t,$$

so

$$\delta \frac{\partial x_i}{\partial \lambda_1} d\lambda_1 = \sum_k \frac{\partial X_i}{\partial x_k} \frac{\partial x_k}{\partial \lambda_i} d\lambda_1 \delta t$$

or

(5) $$\frac{\delta}{\delta t} \frac{\partial x_i}{\partial \lambda_1} = \sum_k \frac{\delta X_i}{\delta x_k} \frac{\partial x_k}{\partial \lambda_i}.$$

That formula is fundamental in the theory of integral invariants. It can also be written:

$$\frac{\delta}{\delta t} \frac{\partial}{\partial \lambda_1} x_i = \frac{\partial}{\partial \lambda_1} \frac{\delta}{\delta t} x_i .$$

The derivatives $\partial / \partial\lambda_1$ and $\delta / \delta t$ then commute when one is dealing with the coordinates of the moving point M. Indeed, one has:

$$\frac{\partial}{\partial \lambda_1} \frac{\delta}{\delta t} x_i = \frac{\partial}{\partial \lambda_1} X_i = \sum_k \frac{\partial X_i}{\partial x_k} \frac{\partial x_k}{\partial \lambda_i} \text{ , which finally } = \frac{\delta}{\delta t} \frac{\partial x_i}{\partial \lambda_1},$$

by virtue of (5).

One easily concludes from the values of x_i and $x_i + \delta x_i$, along with those of x_i' and $x_i' + \delta x_i'$, that $\delta\lambda_1 = 0$, …, $\delta\lambda_p = 0$, and $d\delta\lambda_1 = 0$, …, $d\delta\lambda_p = 0$.

We are now in a position to calculate the value of I_p . If that variation is *identically* zero, for any manifold V, then I_p will be an *integral invariant* of order p of equations (1). The reader might have immediately glimpsed the analogy that exists between an integral of equations (1) and an integral invariant of equations (1).

The variation of I_p must be zero, no matter what the size or form of the manifold V, to the same thing must also be true for an arbitrary element of V, i.e., the variation of:

$$\sum M_{\alpha_1 \cdots \alpha_p} \frac{\partial(x_{\alpha_1}, \ldots, x_{\alpha_p})}{\partial(\lambda_1, \ldots, \lambda_p)} d\lambda_1 \cdots d\lambda_p$$

must be identically zero.

Since the instant t was chosen arbitrarily, I_p will preserve not only the same value while V deforms into $V + \delta V$, but it will also always preserve its initial value, which was absolutely arbitrary, moreover. In no. **1**, we have described the relation:

$$F = \text{arbitrary constant } C ,$$

for the integral $F(x_1, \ldots, x_n, t)$, and similarly, we will have:

$$I_p = \text{arbitrary constant } C$$

for the integral invariant I_p .

5. – In summation, imagine an arbitrary manifold V of order p that is located in n-dimensional space at the arbitrary instant t. V will become $V + \delta V$ at the instant $t + \delta t$. Extend the integral I_p over V and $V + \delta V$, respectively. If the two values thus-found are the same then I_p will be an invariant integral.

Later on, we shall give the necessary and sufficient conditions for I_p to be an integral invariant.

CHAPTER II

ANALOGIES

6. – Here are the main reasons why I use the symbol δ from the calculus of variations in equations (1).

At the instant t, the manifold will have the equation:

$$x_i = \theta_i (\lambda_1 , \ldots, \lambda_p) . \tag{3}$$

At the instant $t + \delta t$, the manifold $V + dV$ will have the equation:

$$x_i = \theta_i (\lambda_1 , \ldots, \lambda_p) + X_i (\theta_1 , \ldots, \theta_p) \, \delta t .$$

We then add an infinitely-small quantity that is represents the variation of θ_i as a result of the variation δt of t to θ_i , and meanwhile t does not enter into those functions θ. The reader will recall that the same thing is true in the calculus of variations, and it is even characteristic of that calculus. The main problems that the calculus of variations attempts to solve also demands that one must annul the variation of an integral that is extended over a manifold V. In those problems, the variations δ are arbitrary or compatible with the conditions on the problem, and the differentials d enter into some differential equations that serve to determine the desired functions. They will be made known by, e.g., the manifold V over which one must extend the proposed integral in order for its first variation to be zero. In addition, it is often necessary that a variation of even order must be non-zero, while all of the variations of lower order are zero. The same thing is not true for the variation of I_p : The variations δ are completely determined by equations (1). The differentials d that denote the displacements on V are *arbitrary*, since V is arbitrary. Finally, $\delta I_p = \delta^2 I_p = \ldots = 0$.

7. – Upon solving the following problem, one will then be led to annul the variation of an integral that is extended over a manifold V:

What are the necessary and sufficient conditions for the integral:

$$J = \int^{p-\text{fold}} \sum M_{\alpha_1 \cdots \alpha_p} dx_{\alpha_1} \cdots dx_{\alpha_p} ,$$

when extended over a manifold V of order p that is bounded by a *fixed* boundary of order $p - 1$, to keep the same value, no matter what the manifold V might be, while all of those manifolds are subject to only that they must pass through and be limited by those fixed boundaries?

In that problem, the differentials d and the variations δ are arbitrary. The variations of the integrals, when extended over the boundary manifolds are zero, since they are fixed, by hypothesis.

Let $p = 1$. In that case, the necessary and sufficient conditions for one to have:

$$\delta \int \sum_i M_i \, dx_i = 0$$

are

$$\frac{\partial M_i}{\partial x_j} - \frac{\partial M_j}{\partial x_i} = 0 \, ,$$

in which i and j are $n\,(n-1)\,/\,2$ combinations of the indices 1, 2, …, n taken two at a time. (*Traité d'Analyse* by É. PICARD, time I, page 76).

Let $p = 2$, $n = 3$. The necessary and sufficient conditions will become:

$$\frac{\partial M_{12}}{\partial x_3} + \frac{\partial M_{23}}{\partial x_1} + \frac{\partial M_{31}}{\partial x_2} = 0$$

in this case, if the surface integral is:

$$\iint M_{12} \, dx_1 \, dx_2 + M_{23} \, dx_2 \, dx_3 + M_{31} \, dx_3 \, dx_1 \; .$$

(Same *Traité*, tome I, page 114). PICARD made use of the calculus of variations in those proofs; he said that himself on page 74: "In order to find that condition, we shall have recourse to a method *of extreme generality* in mathematics that one calls *the method of variations*."

The reader will soon see that the problem in question here is intimately linked with the theory of integral invariants.

POINCARÉ gave the general formula for those conditions in his article "Sur les résidues des intégrales doubles" that was included in tome IX of the Acta Mathematica.

The case in which $p = 2$ and n is arbitrary is treated completely there. Upon writing (α_1, …, α_p), instead of $M_{\alpha_1 \cdots \alpha_p}$, and $[\alpha_p]$, instead of x_{α_p}, those conditions will become:

$$(6) \qquad \begin{cases} N_{\alpha_{p+1}} \quad \text{or} \\ \dfrac{\partial(\alpha_1, \alpha_2, \ldots, \alpha_p)}{\partial[\alpha_{p+1}]} \pm \dfrac{\partial(\alpha_2, \alpha_3, \ldots, \alpha_{p+1})}{\partial[\alpha_1]} \pm \cdots \pm \dfrac{\partial(\alpha_{p+1}, \alpha_1, \ldots, \alpha_{p-1})}{\partial[\alpha_p]} = 0. \end{cases}$$

One can always take the sign to be + if p is even, and alternatively, take the signs to be + and – if p is odd. There are as many condition equations as there are combinations of n letters taken $p + 1$ at a time. One must take care to write the indices in M and x as was indicated in (6) (cyclic permutations). We shall return to that important point.

Suppose that the coefficients M of the integral J satisfy the conditions (6) identically. We then say, by analogy with the terminology that is applied to simple integrals, that the integral J is an *integral of an exact differential* (*Méthodes Nouvelles de la Mécanique céleste* by POINCARÉ, time III, page 14).

One will obtain an integral of an exact differential upon applying the generalized STOKES's theorem to transform an integral that is extended over an arbitrary *closed* manifold of order p into an integral of order $p + 1$ that is extended over an arbitrary non-closed manifold of order $p + 1$ that is bounded by the closed manifold. One therefore transforms the integral J_p and obtains:

$$J_{p+1} = \int^{(p+1)-\text{fold}} \sum N_{\alpha_{p+1}} \, dx_{\alpha_1} \cdots dx_{\alpha_p} dx_{\alpha_{p+1}} \, .$$

One sees immediately that if J_p is an integral of an exact differential then it will be identically zero when one extends it over a closed manifold. Indeed, one will then have $J_p = J_{p+1} = 0$ [formulas (6)].

Suppose the J_p is not an integral of exact differential and that one extends it over a closed manifold. The integral J_{p+1} that one deduces will be non-zero, and it will be an integral of an exact differential. The formulas (6) permit one to verify that last point. J_{p+1} will then be zero when one extends it over a closed manifold.

The converse of STOKES's theorem that we just recalled is true, and it will be greatly useful to us on what follows: The integral J_{p+1} *of the exact differential* is reducible to the integral J_p of order p .

Remark. – In the preceding, we were often concerned with *closed* manifolds. Here is how POINCARÉ defined those manifolds in his paper "Analysis Situs," which was published in the Journal de l'École Polytechnique in 1895: If a manifold is at the same time finite, continuous, and unlimited then it will be called closed.

CHAPTER III

INTEGRAL INVARIANTS OF ORDER ONE

8. – Let us look for the necessary and sufficient conditions for:

$$I_1 = \int \sum_i M_i \, dx_i \qquad\qquad (i = 1, 2, \ldots, n)$$

to be an integral invariant of the equations:

(1) $$\frac{\delta x_i}{X_i} = \delta t \, .$$

In Chapter I, we showed how that comes down to annulling the variation of I, or more simply, the arbitrary element:

$$\sum_i M_i \frac{\partial x_i}{\partial \lambda} d\lambda \, .$$

(In that case, $p = 1$, so we write λ instead of λ_1.) Recall that the variation of (x_i) is determined by (1), the (x_i) are arbitrary functions of λ, the curve along which we extend the integral I_1 is arbitrary at the initial instant t, and that finally we have:

(5) $$\frac{\delta}{\delta t}\frac{\partial x_i}{\partial \lambda} = \sum_k \frac{\partial X_i}{\partial x_k}\frac{\partial x_k}{\partial \lambda} \, ,$$

$$\delta\left(\sum_i M_i \frac{\partial x_i}{\partial \lambda}\right) = 0 \, ,$$

which will then become:

$$\sum_i \sum_k \frac{\partial M_i}{\partial x_k}\delta x_k \frac{\partial x_k}{\partial \lambda} + \sum_i \frac{\partial M_i}{\partial t}\delta t \frac{\partial x_k}{\partial \lambda} + \sum_k M_i \, \delta \frac{\partial x_k}{\partial \lambda} = 0 \qquad (k = 1, 2, \ldots, n) \, ,$$

or

$$\sum_i \sum_k \left(\frac{\partial M_i}{\partial x_k} X_k \frac{\partial x_k}{\partial \lambda} + \frac{\partial M_i}{\partial t}\frac{\partial x_k}{\partial \lambda} + M_i \frac{\partial X_i}{\partial x_k}\frac{\partial x_k}{\partial \lambda} \right) = 0$$

or

$$\sum_i \frac{\partial x_i}{\partial \lambda} \sum_k \left(\frac{\partial M_i}{\partial x_k} X_k + \frac{\partial M_i}{\partial t} + M_k \frac{\partial X_k}{\partial x_i} \right) = 0 \, .$$

Since the curve:

(3) $$x_i = \theta_i(\lambda)$$

is arbitrary, the $\frac{dx_i}{d\lambda}$ or $\frac{d\theta_i(\lambda)}{d\lambda}$ will also be arbitrary. The coefficients of the $\frac{\partial x_i}{\partial \lambda}$ in the latter expression that was obtained must all be identically zero then.

Although no doubt can persist on that subject, let us insist upon a little. Suppose that $\frac{\partial x_1}{\partial \lambda} \neq 0$, and set $x_2 = x_3 = \ldots = x_n = 0$. Therefore:

$$\frac{\partial x_2}{\partial \lambda} = \frac{\partial x_3}{\partial \lambda} = \ldots = \frac{\partial x_n}{\partial \lambda} = 0 .$$

The condition becomes:

$$\sum_k \left(\frac{\partial M_1}{\partial x_k} X_k + \frac{\partial M_1}{\partial t} + M_k \frac{\partial X_k}{\partial x_1} \right) = 0 .$$

What we have done for x_1, we can also do for the other x. We must then have the n identities:

(7) $$\sum_k \left(\frac{\partial M_1}{\partial x_k} X_k + \frac{\partial M_1}{\partial t} + M_k \frac{\partial X_k}{\partial x_1} \right) = 0 \qquad (i = 1, 2, \ldots, n) .$$

Those conditions are obviously sufficient. They are n first-order linear partial differential equations in M_i . Recall that:

$$\sum_k \frac{\partial M_i}{\partial x_k} X_k + \frac{\partial M_i}{\partial t} = \frac{\delta}{\delta t} M_i ,$$

in which the variations of x_i are determined by equations (1). Consequently, equations (7) can be replaced with the $2n$ ordinary differential equations:

(8) $$\frac{\delta x_i}{X_i} = \frac{\delta M_i}{-\sum_k M_k \frac{\partial X_k}{\partial x_i}} = \delta t .$$

9. – If the X_i do not refer to t explicitly then the invariant I_1 will yield the integral $\sum_i M_i X_i$. (POINCARÉ)

Proof. – Add corresponding sides of the n equations (7) after having multiplied by X_1 , X_2 , …, X_n , respectively. The expression thus-obtained shows that $\delta \sum_i M_i X_i \equiv 0$.

10. – In order for the $2n$ equations:

$$\frac{\delta x_i}{X_i} = \frac{\delta x_i}{X_i} = \delta t \qquad (i = 1, 2, \ldots, n)$$

to admit an invariant of the form $I_1 = \int \sum_i y_i \, dx_i$, it is *necessary* that they must be canonical. (Note by KOENIGS, Comptes-Rendus, December 1895)

Proof. – The conditions (7) will become:

$$Y_i + \sum_k y_k \frac{\partial X_k}{\partial x_i} = 0 ,$$
$$\sum_k y_k \frac{\partial X_k}{\partial y_i} = 0$$

in this case, or upon setting $H = \sum_k y_k \, dx_k$:

$$Y_i = -\frac{\partial H}{\partial x_i} ,$$
$$X_i = \frac{\partial H}{\partial y_i} . \qquad \text{Q.E.D.}$$

One will then have $H = \sum_k y_k \frac{\partial H}{\partial y_k}$, which proves that H is homogeneous and of degree one with respect to the y.

If $I_1 = \int \sum_i y_i \, dx_i - dx$ is an integral invariant of the $2n + 1$ equations:

$$\frac{\delta x_i}{X_i} = \frac{\delta y_i}{Y_i} = \frac{\delta x}{X} = \delta t$$

then those equations will certainly have the form:

$$\frac{\delta x_i}{\frac{\partial H}{\partial y_i}} = \frac{\delta y_i}{-\frac{\partial H}{\partial x_i}} = \delta t = \frac{\delta x}{\sum_k y_k \frac{\partial H}{\partial y_k} - H} ,$$

in which $H = \sum_k y_k X_k - X$. (Same note by KOENIGS)

Proof. – Formulas (7), or a direct calculation, will give:

$$Y_i + \sum_k y_k \frac{\partial X_k}{\partial x_i} - \frac{\partial X}{\partial x_i} = 0 ,$$
$$\sum_k y_k \frac{\partial X_k}{\partial y_i} - \frac{\partial X}{\partial y_i} = 0$$

$$\sum_k y_k \frac{\partial X_k}{\partial x} - \frac{\partial X}{\partial x} = 0 ,$$

or

$$Y_i = -\frac{\partial H}{\partial x_i} ,$$

$$X_i = \frac{\partial H}{\partial y_i} ,$$

$$\frac{\partial H}{\partial x} = 0 . \qquad \text{Q.E.D.}$$

The last identity shows that H is independent of x.

11. – In order for $\int \sum_i M_i \, dx_i + N_i \, dy_i$ to be an integral invariant of the canonical system:

$$\frac{\delta x_i}{\dfrac{\partial H}{\partial y_i}} = \frac{\delta y_i}{-\dfrac{\partial H}{\partial x_i}} = \delta t ,$$

it is necessary and sufficient that the following $2n$ conditions should be satisfied identically:

$$\frac{\delta M_i}{\delta t} + \sum_k \left(M_k \frac{\partial^2 H}{\partial y_k \, \partial x_i} - N_k \frac{\partial^2 H}{\partial x_k \, \partial x_i} \right) = 0 ,$$

$$\frac{\delta N_i}{\delta t} + \sum_k \left(M_k \frac{\partial^2 H}{\partial y_k \, \partial y_i} - N_k \frac{\partial^2 H}{\partial x_k \, \partial y_i} \right) = 0 ,$$

in which:

$$\frac{\delta M_i}{\delta t} \equiv \sum_k \left(\frac{\partial M_i}{\partial x_k} \frac{\partial H}{\partial y_k} - \frac{\partial M_i}{\partial y_k} \frac{\partial H}{\partial x_k} \right) + \frac{\partial M_i}{\partial t} \equiv (M_i , H) + \frac{\partial M_i}{\partial t} .$$

I say that one can deduce the integral $\sum_i \left(\dfrac{\partial M_i}{\partial x_i} - \dfrac{\partial N_i}{\partial x_i} \right)$ from the invariant $\int \sum_i M_i \, dx_i + N_i \, dy_i$.

Proof. – Differentiate the $2n$ preceding identities with respect to $y_1 , \ldots, y_n ; x_1 , \ldots, x_n$, and then add corresponding sides of the $2n$ identities thus-obtained after changing the signs in the last n terms.

12. – The n second-order differential equations:

$$\frac{\delta^2 x_i}{X_i} = \delta t^2$$

are reducible to a system of 2*n* first-order equations:

$$\frac{\delta x_i}{x_i'} = \frac{\delta x_i'}{X_i} = \delta t \ .$$

Such a system will never admit an invariant of the form $\int \sum_i M_i \, dx_i$.

Proof. – *n* of the conditions (7) will become:

$$M_i \equiv 0 \ .$$

In this case, it is more practical to calculate the variation of $\sum_i M_i \, dx_i$ directly and then annul it.

13. – If *F* is an integral of equations (1) then one will have the invariant:

$$I_1 = \int dF = \int \sum_k \frac{\partial F}{\partial x_k} dx_k \ .$$

Proof. – $\delta F \equiv 0$, by hypothesis, or:

$$\sum_k \frac{\partial F}{\partial x_k} X_k + \frac{\partial F}{\partial t} \equiv 0 \ .$$

Upon differentiating this with respect to x_i, one will deduce that:

$$\sum_k \left(\frac{\partial \frac{\partial F}{\partial x_i}}{\partial x_k} X_k + \frac{\partial \frac{\partial F}{\partial x_i}}{\partial t} + \frac{\partial F}{\partial x_k} \frac{\partial X_k}{\partial x_i} \right) \equiv 0 \ .$$

Therefore, the $\partial F / \partial x_i$ are the coefficients of a first-order invariant (7).

Example. – The canonical equations in no. **11** will admit the integral *H* when that function does not refer to *t* explicitly because one would then have $\delta H \equiv 0$. If that is the case then one will have the invariant:

$$I_1 = \int dH = \int \sum_k \frac{\partial H}{\partial x_k} dx_k + \frac{\partial H}{\partial y_k} dy_k \ .$$

14. – Let us find the necessary and sufficient condition for:

$$I_1 = \int \sqrt{\sum_i \sum_k A_{ik}\, dx_i\, dx_k} \qquad \begin{pmatrix} i = 1, 2, \ldots, n \\ k = 1, 2, \ldots, n \end{pmatrix}$$

to be a first-order integral invariant of equations (1).

One supposes that $A_{ik} \equiv A_{ki}$ and $A_{ii} \neq 0$, in general.

One will have, in succession:

$$\delta \sqrt{\sum_i \sum_k A_{ik} \frac{dx_i}{d\lambda} \frac{dx_k}{d\lambda}} = 0 ,$$

$$\sum_i \sum_k \sum_l \left(\frac{\partial A_{ik}}{\partial x_l} X_l \frac{dx_i}{d\lambda} \frac{dx_k}{d\lambda} + \frac{\partial A_{ik}}{\partial t} \frac{dx_i}{d\lambda} \frac{dx_k}{d\lambda} + A_{ik} \frac{\partial X_i}{\partial x_l} \frac{dx_l}{d\lambda} \frac{dx_k}{d\lambda} + A_{ik} \frac{\partial X_k}{\partial x_l} \frac{dx_l}{d\lambda} \frac{dx_i}{d\lambda} \right) = 0 .$$

Let us focus on $\frac{dx_i}{d\lambda} \frac{dx_k}{d\lambda}$ and annul its coefficient:

$$\sum_l \left(\frac{\partial A_{ik}}{\partial x_l} X_l + \frac{\partial A_{ik}}{\partial t} + A_{lk} \frac{\partial X_l}{\partial x_i} + A_{il} \frac{\partial X_l}{\partial x_k} \right) = 0 .$$

Those are the desired conditions. They can be put into the form of ordinary differential equations:

$$\frac{\delta x_i}{X_i} = \frac{\delta A_{ik}}{-\sum_l \left(A_{lk} \frac{\partial X_l}{\partial x_i} + A_{il} \frac{\partial X_l}{\partial x_k} \right)} = \delta t .$$

15. – Further consider an invariant of the form:

$$I_1 = \int \sqrt[n]{A\, dx_1\, dx_2 \cdots dx_n} .$$

A calculation that is analogous to the one no. **14** will give the conditions:

$$\frac{\delta A}{\delta t} + A \sum_k \frac{\partial X_k}{\partial x_k} = 0 , \qquad \frac{\partial X_k}{\partial x_j} = 0 \qquad (j = 1, 2, \ldots, k-1, k+1, \ldots, n) .$$

16. – In the preceding calculations, we made constant use of the formula (5):

$$\frac{\delta}{\delta t} \frac{\partial x_i}{\partial \lambda} = \sum_k \frac{\partial X_i}{\partial x_k} \frac{\partial x_k}{\partial \lambda} .$$

In no. **8**, we focused on $\partial x_i / \partial\lambda$ and annulled its coefficients, which gave the conditions (7); however, the $\partial x_i / \partial\lambda$ do not enter into them. It is obvious that if one replaces the $\frac{\partial x_i}{\partial\lambda}$ in $\sum_i M_i \frac{\partial x_i}{\partial\lambda}$ with functions ξ_i of $x_1, \ldots, x_n, t$ whose variations are *identical* to those of the $\frac{\partial x_i}{\partial\lambda}$, and if one then calculates the variation of $\sum_i M_i \xi_i$ then that variation will be identically zero. Indeed, that variation will present itself in the same form as that of $\sum_i M_i \frac{\partial x_i}{\partial\lambda}$, but with the single difference that the $\frac{\partial x_i}{\partial\lambda}$ have been replaced with the ξ_i. The coefficients of the $\frac{\partial x_i}{\partial\lambda}$ or the ξ_i will be the same, but those coefficients will be zero by virtue of (7), therefore, etc.

Set:

$$\frac{\partial x_i}{\partial\lambda} \Leftrightarrow \xi_i \qquad\qquad (i = 1, 2, \ldots, n),$$

in which the symbol $\Leftrightarrow$ signifies that the $\frac{\partial x_i}{\partial\lambda}$ can be replaced by the functions ξ_i of x and t, and *vice versa*, and those functions will have variations that verify the equations:

$$(5') \qquad \frac{\delta}{\delta t}\xi_i = \sum_k \frac{\partial X_i}{\partial x_k}\xi_k$$

identically. However, by virtue of equations (1), one has, on the other hand:

$$\frac{\delta}{\delta t}\xi_i = \sum_k \frac{\partial \xi_i}{\partial x_k} X_k + \frac{\partial \xi_i}{\partial t},$$

so

$$(9) \qquad \sum_k \left(\frac{\partial \xi_i}{\partial x_k} X_k + \frac{\partial \xi_i}{\partial t} - \frac{\partial X_i}{\partial x_k}\xi_k \right) = 0 .$$

Those are n first-order linear partial differential equations that serve to determine the functions ξ_1, $\xi_2, \ldots, \xi_n$. The set of those n functions constitutes a *solution by variations* and will be denoted simply by (ξ_i) or (ξ_i'). The prime serves to remind us that the solution is intimately coupled with the first-order integral invariants. That is why we sometimes say that (ξ_i') is a first-order solution by variations.

One will then have $\delta\sum_i M_i \xi_i' \equiv 0$ or $\sum_i M_i \xi_i'$ = an integral of equations (1) if the M_i are the coefficients of a first-order integral invariant and the ξ_i' form a first-order solution by variations.

For example, if the X_i do not include t explicitly then equations (9) admit the integral $\xi_i = X_i$. Thus $\sum_i M_i X_i$ = an integral of equations (1). That is the result that was obtained in no. **8**.

17. – The preceding arguments extend immediately to the invariants of nos. **14** and **15**. Set:

$$\frac{\partial x_i}{\partial \lambda}\frac{\partial x_k}{\partial \lambda} \Leftrightarrow \xi'_{ik} \ .$$

The necessary and sufficient conditions that those ξ must satisfy are:

$$\sum_i \left(\frac{\partial \xi'_{ik}}{\partial x_l} X_l + \frac{\partial \xi'_{ik}}{\partial t} - \frac{\partial X_i}{\partial x_l} \xi'_{lk} - \frac{\partial X_k}{\partial x_l} \xi'_{il} \right) = 0 \ . \tag{10}$$

In Chapter VIII, we shall make a deeper study of the solutions by variations.

CHAPTER IV

INTEGRAL INVARIANTS OF ORDER *n*

18. – Let us look for the necessary and sufficient conditions for:

$$I_n = \int^{p-\text{fold}} M\, dx_1\, dx_2 \cdots dx_n$$

to be an invariant integral of order n of equations (1). M is a function of x and t.

It is necessary and sufficient that one must have:

$$\delta M \frac{\partial(x_1,\ldots,x_n)}{\partial(\lambda_1,\ldots,\lambda_n)} = 0 .$$

Thanks to formula (5), one will easily find that:

$$\frac{\delta}{\delta t}\frac{\partial(x_1,\ldots,x_n)}{\partial(\lambda_1,\ldots,\lambda_n)} = \frac{\partial(x_1,\ldots,x_n)}{\partial(\lambda_1,\ldots,\lambda_n)}\sum_k \frac{\partial X_k}{\partial x_k} .$$

(In the following chapter, the reader will find a calculation of the same type that is done completely.)

Thus:

$$\frac{\delta M}{\delta t} + M\sum_k \frac{\partial X_k}{\partial x_k} = 0 , \tag{11}$$

or

$$\sum_k \left(\frac{\partial M}{\partial x_k} X_k + \frac{\partial M}{\partial t} + M\frac{\partial X_k}{\partial x_k} \right) = 0 ,$$

or

$$\sum_k \frac{\partial M\, X_k}{\partial x_k} + \frac{\partial M}{\partial t} = 0 . \tag{12}$$

One deduces the following theorem from (12), which is due to POINCARÉ (*Méthodes Nouvelles*, t. III, pp. 41):

In order for $\int M\, dx_1 \cdots dx_n$ *to be an invariant of order n of equations* (1), *it is necessary and sufficient that M should be a* ***multiplier*** *of equations* (1) (in the JACOBI sense).

19. – Make a change of variables that affects the x. Let Δ be the functional determinant of the x with respect to the new variables $y_1, \ldots, y_n$.

I_n will become $I'_n = \int M \Delta dy_1 \cdots dy_n$.

Therefore, the multiplier will become $M \Delta$ with the new variables y_i . That is the property of the invariance of the last multiplier.

Change the independent variable. Suppose that one has $\delta t / Z = \delta t_1$, in which t_1 is the new independent variable and Z is a given function of x and t.

Equations (1) become:

$$(1') \qquad \frac{\delta x_i}{Z X_i} = \delta t_1 .$$

If M is a multiplier of equations (1) that does not include t explicitly then one will have:

$$\sum_k \frac{\partial M X_k}{\partial x_k} \quad \text{or} \quad \sum_k \frac{\partial \left(\frac{M}{Z} \cdot Z X_k \right)}{\partial x_k} = 0 .$$

That indicates that M / Z will be a multiplier of equations (1′) if Z does not include t explicitly. (*Méthodes Nouvelles*, t. III, pp. 30)

Recall the first change of variables. Let:

$$(13) \qquad y_i = \varphi_i (x_1 , \ldots, x_n , t) \qquad (i = 1, \ldots, n)$$

be n given relations between the old and new variables. We suppose that they are soluble for x_1 , …, x_n .

Take the variation of y_i in agreement with equations (1). Hence:

$$(14) \qquad \frac{\delta y_i}{\delta t} = \sum_k \frac{\partial \varphi_i}{\partial x_k} X_k + \frac{\partial \varphi}{\partial t} .$$

After replacing the x with their values as functions of the y and t, one will have the transformed equations (1). Let $\psi (y_1 , \ldots, y_n , t)$ be an integral of equations (14). It will also be an integral of equations (1) when one replaces the y_i with their values (13).

Now suppose that the first p functions φ are distinct integrals of equations (1) and that last $n - p$ of them reduce to x_{p+1} , …, x_n . Equations (14) become:

$$(15) \qquad \frac{\delta y_1}{0} = \ldots = \frac{\delta y_p}{0} = \frac{\delta y_{p+1}}{(X_{p+1})} = \ldots = \frac{\delta y_n}{(X_n)} = \delta t .$$

The parentheses around the X_{p+1} , …, X_n indicates that one has replaced x_1 , …, x_p with their values that are inferred from the first p equations (13) as functions of the x_{p+1} , …, x_p , t, and y_1 , …, y_p . The last p of them are considered to be constants. I say that any multiplier of the last $n - p$ equations in (15) is also a multiplier of the n equations (15). Indeed, the condition (12) reduces to the same expression when one applies it to those two systems, respectively. That remark will be useful later on.

Once more, consider the change of variables that is defined by the equations:

$$y_i = \varphi_i (x_1 , \ldots, x_n , t) + \psi_i (t) ,$$

in which φ_i are distinct integrals of equations (1) and ψ_i are given functions of t. Equations (14) become:

$$\frac{\delta y_i}{\delta t} = \psi_i'(t) , \tag{16}$$

and the condition (11) will reduce to $\delta M = 0$. Therefore, any multiplier of (16) will be, at the same time, an integral, and consequently any multiplier of (16) will be an integral of equations (1). In addition, we just saw that if M' is a multiplier of (16) then $M \equiv M' \dfrac{\partial(\varphi_1,\ldots,\varphi_n)}{\partial(x_1,\ldots,x_n)}$ will be a multiplier of equations (1). One will then deduce that $\dfrac{\partial(\varphi_1,\ldots,\varphi_n)}{\partial(x_1,\ldots,x_n)}$ is a multiplier of equations (1). Indeed, it is easy to show that the quotient of the two multipliers of the same system will be an integral of that system. Let μ_1 and μ_2 be two multipliers of equations (1). In no. **18** we saw that:

$$\delta\mu \, \frac{\partial(\varphi_1,\ldots,\varphi_n)}{\partial(x_1,\ldots,x_n)} = 0 .$$

Let ∇ represent the determinant that enters into that identity. One will have:

$$\delta\left(\frac{\mu_1}{\mu_2}\right) = \delta\left(\frac{\mu_1 \nabla}{\mu_2 \nabla}\right) = \frac{\mu_2 \nabla \, \delta(\mu_1 \nabla) - \mu_1 \nabla \, \delta(\mu_2 \nabla)}{(\mu_2 \nabla)^2} = 0 . \qquad \text{Q.E.D.}$$

20. – Replace the n^2 elements $\dfrac{\partial x_i}{\partial \lambda_1}, \ldots, \dfrac{\partial x_i}{\partial \lambda_n}$ with n solutions to the first-order variations $(\xi_i)_1$, $\ldots, (\xi_i)_n$, which are assumed to be distinct. Let ∇' represent what ∇ becomes with that substitution. One will again have:

$$\delta(M \nabla') = 0 ,$$

or

$$M \nabla' = \text{an integral of equations (1)} ,$$

or

$$\nabla' = \text{inverse of a multiplier of equations (1)} .$$

Now suppose that one knows p integrals of equations (1) and $n - p$ solutions to the distinct first-order variations of the last $n - p$ equations in (15). Thanks to those solutions, one will find the inverse of a multiplier for those $n - p$ equations or the n equations (15). (See the remark in no. **19**.)

Let M' be that multiplier. Therefore:

$$M' = \frac{1}{\nabla'} ,$$

if ∇' represents the determinant that is formed by means of the $n - p$ known solutions.

However, one can deduce a multiplier M of equations (1) from M'. One will have:

$$M = M' \frac{\partial(\varphi_1, \ldots, \varphi_p, x_{p+1}, \ldots, x_n)}{\partial(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n)} ,$$

or

$$M = \frac{\dfrac{\partial(\varphi_1, \ldots, \varphi_p)}{\partial(x_1, \ldots, x_p)}}{\nabla'} .$$

Remarks. – The result $M' = 1 / \nabla'$ is obtained from the theory of infinitesimal transformations, such as on page 87 of tome III of C. JORDAN's *Cours d'Analyse*, 1896.

We will soon recover the other results that relate to infinitesimal transformations that were presented in that treatise.

21. – Set:

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(\lambda_1, \ldots, \lambda_n)} \Leftrightarrow \xi^n .$$

The significance of the symbol $\Leftrightarrow$ was given in no. **16**. ξ^n is an nth-order solution by variation.

It is necessary and sufficient that one should have:

$$\frac{\delta \xi^n}{\delta t} = \xi^n \sum_k \frac{\partial X_k}{\partial x_k}$$

(see no. **18**), or:

$$\sum_k \left(\frac{\partial \xi^n}{\partial x_k} X_k + \frac{\partial \xi^n}{\partial t} - \xi^n \frac{\partial X_k}{\partial x_k} \right) = 0 . \tag{17}$$

If ξ^n is an nth-order solution by variation and M is a multiplier then $M\,\xi^n$ will be an integral of equations (1).

22. – For example, let $\sum_k \frac{\partial X_k}{\partial x_k} = 0$. That will give rise to the canonical equations. One will then have $M = 1$, $\xi^n = 1$.

If equations (1) admit the multiplier μ, which is independent of t, then the equations:

$$\frac{\delta x_i}{\mu X_i} = \delta t_1$$

will admit the multiplier 1, in other words, all of the multipliers of that system will be integrals of it.

If $\sum_k \frac{\partial X_k}{\partial x_k} = 1$ then one will have $M = e^{-t}$ and $\xi^n = e'$.

If the X_i do not include t explicitly then one can deduce some other multipliers from M :

$$\frac{\partial M}{\partial t}, \quad \frac{\partial^2 M}{\partial t^2}, \ \ldots,$$

and so on, up to the moment when one of those partial derivatives becomes zero. The same thing will be true for the solution ξ^n .

If X_i depends upon only x_i and t then the multiplier M will give:

$$I_1 = \int^n \sqrt{M\, dx_1 \cdots dx_n} \, .$$

(no. **15**)

CHAPTER V

INTEGRAL INVARIANTS OF ORDER $n - 1$

23. – Let us look for the necessary and sufficient conditions for:

$$I_{n-1} = \int \sum_i M_i \, dx_{i+1} \, dx_{i+2} \cdots dx_n \, dx_1 \cdots dx_{i-1}$$

to be an integral invariant of order $n - 1$ of equations (1) ($i = 1, 2 \ldots, n$).

One first observes the order in which we have written the differentials. If $n = 5$ then one will have:

$$1234, \quad 2345, \quad 3451, \quad 4512, \quad 5123 \, .$$

That is why we shall always suppose that the differentials of an invariant of order $n - 1$ are written that way. The reader will soon see that it is important to indicate the way that those differentials are organized.

Set:

$$\Delta_i \equiv \frac{\partial (x_{i+1}, x_{i+2} \ldots, x_n, x_1, \ldots, x_{i-1})}{\partial (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})} \, .$$

One will have, in succession:

$$\frac{\delta}{\delta t} \sum_i M^i \, \Delta_i = \sum_i \left(\Delta_i \frac{\delta M^i}{\delta t} + M^i \frac{\delta \Delta_i}{\delta t} \right) = 0 \, ,$$

$$\frac{\delta M^i}{\delta t} = \sum_k \frac{\partial M^i}{\partial x_k} X_k + \frac{\partial M^i}{\partial t} \, ,$$

(5)
$$\frac{\delta}{\delta t} \frac{\partial x_{i+1}}{\partial \lambda_1} = \sum_k \frac{\partial X_{i+1}}{\partial x_k} \frac{\partial x_k}{\partial \lambda_1} \, ,$$

$$\frac{\delta}{\delta t} \frac{\partial x_{i+1}}{\partial \lambda_2} = \sum_k \frac{\partial X_{i+1}}{\partial x_k} \frac{\partial x_k}{\partial \lambda_2} \, ,$$

…..……..,

so

$$\frac{\delta \Delta_i}{\delta t} = \begin{vmatrix} \frac{\partial X_{i+1}}{\partial x_{i+1}}\frac{\partial x_{i+1}}{\partial \lambda_1} + \frac{\partial X_{i+1}}{\partial x_i}\frac{\partial x_i}{\partial \lambda_1} & \frac{\partial x_{i+2}}{\partial \lambda_1} & \cdots & \frac{\partial x_{i-1}}{\partial \lambda_1} \\ \frac{\partial X_{i+1}}{\partial x_{i+1}}\frac{\partial x_{i+1}}{\partial \lambda_2} + \frac{\partial X_{i+1}}{\partial x_i}\frac{\partial x_i}{\partial \lambda_2} & \frac{\partial x_{i+2}}{\partial \lambda_2} & \cdots & \frac{\partial x_{i-1}}{\partial \lambda_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial X_{i+1}}{\partial x_{i+1}}\frac{\partial x_{i+1}}{\partial \lambda_{n-1}} + \frac{\partial X_{i+1}}{\partial x_i}\frac{\partial x_i}{\partial \lambda_{n-1}} & \frac{\partial x_{i+2}}{\partial \lambda_{n-1}} & \cdots & \frac{\partial x_{i-1}}{\partial \lambda_{n-1}} \end{vmatrix}$$

+ some determinants that are analogous to the previous one, but which relate to the 2nd, 3rd, …, (n – 1)th column of Δ_i , respectively. One will then have:

$$\frac{\delta \Delta_i}{\delta t} = \Delta_i \sum_l \frac{\partial X_{i+l}}{\partial x_{i+l}} + \sum_l \frac{\partial X_{i+l}}{\partial x_i}(\Delta_{i+l}) \ ,$$

in which $l = 1, 2, \ldots, n - 1$. If $i + l > n$ then one subtracts n from $i + l$. For example, $i + n - 1$ has the same significance as $i - 1$. The symbol Δ_{i+l} is a determinant that is analogous to Δ_i , but the elements $\frac{\partial x_{i+l}}{\partial \lambda_1}$, …, $\frac{\partial x_{i+1}}{\partial \lambda_{n-1}}$ do not enter into it. The parentheses that are placed around Δ_{i+l} signify that one must once more permute the columns of that determinant in such a manner that the indices on the x are placed into the order that was indicated above. If $i + l \leq n$ then the indices on the x in (Δ_{i+l}) are arranged as follows:

$$i + 1 + 1 , \quad i + l + 2 , \quad \ldots, \quad n , \quad 1 , \quad 2 , \quad \ldots, \quad i , \quad \ldots, i + l - 1 .$$

An easy calculation will show that this requires $n + l + 1 + (n + l)(l + 1)$ permutations, which will be an odd number of permutations when n is odd and $l + 1$ permutations when n is even.

If $i + l > n$ then the indices on the x in (Δ_{i+l}) will be arranged as follows:

$$i + 1 , \quad i + 2 , \quad \ldots, \quad n , \quad i + l - 1 - n , \quad i , \quad \ldots, \quad i + l + 1 - n , \quad \ldots, i - 1 .$$

From our conventions, it is necessary that this order should become:

$$i + 1 + 1 - n , \quad i + l + 2 - n , \quad \ldots, \quad i , \quad n , \quad 1 , \quad \ldots, \quad i + l - 1 - n .$$

That requires the same number of permutations as in the first case (one can always neglect an even number of permutations). Therefore, if n is odd:

$$(\Delta_{i+l}) = - \Delta_{i+l} .$$

If n is even:

$$(\Delta_{i+l}) = (- 1)^{l+1} \Delta_{i+l} .$$

Case 1: n odd.

$$\sum_i \sum_l \left(\Delta_i \frac{\delta M^i}{\delta t} + M^i \Delta_i \frac{\partial X_{l+i}}{\partial x_{l+i}} - M^i \frac{\partial X_{l+i}}{\partial x_i} \Delta_{i+l} \right) = 0 .$$

One can set $l = n$ in that formula, because that would amount to adding and subtracting $M^i \Delta_i \dfrac{\partial X_i}{\partial x_i}$. Set $i + l = k$ ($k = 1, 2, \ldots, n$):

$$\sum_i \sum_k \left(\Delta_i \frac{\delta M^i}{\delta t} + M^i \Delta_i \frac{\partial X_k}{\partial x_k} - M^i \frac{\partial X_k}{\partial x_i} \Delta_k \right) = 0 .$$

The i and k play the same role, so they can be permuted. We can then focus on Δ_i . If we annul its coefficient then:

(18) $$\sum_k \left(\frac{\delta M^i}{\delta t} + M^i \frac{\partial X_k}{\partial x_k} - M^k \frac{\partial X_i}{\partial x_k} \right) = 0 .$$

Case 2: n is even.

One argues as in the first case. One sets $l + i = k$, so $l = k - i$ ($k = 1, 2, \ldots, n$).
One gets:

$$\sum_i \sum_k \left[\Delta_i \frac{\delta M^i}{\delta t} + M^i \Delta_i \frac{\partial X_k}{\partial x_k} - (-1)^{k+1} \frac{\partial X_k}{\partial x_i} \Delta_k M^i \right] = 0 ,$$

and finally, after multiplying the coefficient of Δ_i by $(-1)^i$:

(19) $$\sum_k \left(\frac{\delta \,|-M\,|^i}{\delta t} + |-M\,|^i \frac{\partial X_k}{\partial x_k} - |-M\,|^k \frac{\partial X_i}{\partial x_k} \right) = 0 ,$$

in which $|-M\,|^i$ is written for $(-1)^i M^i$.

Remark. – The formula (19) pertains to invariants of order n – 1 of any parity for n if one agrees to write those invariants as follows:

$$I'_{n-1} = \int \sum_i M^i \, dx_1 \, dx_2 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_n \ .$$

That is because if n is even then:

$$I'_{n-1} = \int \sum_i M^i \, dx_{i+1} \cdots dx_n \, dx_1 \cdots dx_{i-1} ,$$

and if n is odd then:

$$I'_{n-1} = - \int \sum_i |-M\,|^i \, dx_{i+1} \cdots dx_{i-1} \ .$$

I conclude from this that the formula that appears in the note by KOENIGS includes a sign error. (Comptes-Rendus, page 25, year 1896) That error is repeated on page 463 of time II of PAUL APPELL's *Traité de Mécanique rationelle* (1896). KOENIGS wrote M^i, while one must have $|-M|^i$.

24. – The conditions (18) and (19) can also be written:

$$(18') \qquad \sum_k \left(\frac{\partial M^i}{\partial x_k} X_k + \frac{\partial M^i}{\partial t} + M^i \frac{\partial X_k}{\partial x_k} - M^k \frac{\partial X_i}{\partial x_k} \right) = 0 ,$$

$$(19') \qquad \sum_k \left(\frac{\partial |-M|^i}{\partial x_k} X_k + \frac{\partial |-M|^i}{\partial t} + |-M|^i \frac{\partial X_k}{\partial x_k} - |-M|^k \frac{\partial X_i}{\partial x_k} \right) = 0 .$$

Differentiate equations (18′) with respect to x_1, x_2, …, x_n and add corresponding sides of the identities thus-obtained. That sum can be written:

$$\sum_k \frac{\partial}{\partial x_k} \left(X_k \sum_i \frac{\partial M^i}{\partial x_i} \right) + \frac{\partial}{\partial t} \sum_i \frac{\partial M^i}{\partial x_i} = 0 .$$

Thus, $\sum_i \frac{\partial M^i}{\partial x_i}$ is a multiplier.

If n is even then $\sum \frac{\partial |-M|^i}{\partial x_i}$ will be a multiplier. We shall recover those results later on thanks to a theorem by POINCARÉ.

25. – In the note that was mentioned above, KOENIGS said:

> "In order to construct the integral invariant ($n - 1$) of the most general form, one seeks an equation $C(\theta) = 0$ that will form a Jacobian system with $\sum_k X_k \frac{\partial \theta}{\partial x_k} = 0$. If $C(\theta) = \sum_k \Xi_k \frac{\partial \theta}{\partial x_k}$ and μ denotes a multiplier of the system (1) then the general expression for the coefficients will be:
>
> $$M^i = \mu \Xi_i .\text{"}$$

I shall not repeat the proof that was given in that note here, because we shall recover that result in a much simpler manner in Chap. VIII. In addition, I will show that one must write $|-M|^i = \mu \Xi_i$, the M^i can include t explicitly, the equation $\sum_k \frac{\partial \theta}{\partial x_k} X_k = 0$ must become $\sum_k \frac{\partial \theta}{\partial x_k} X_k + \frac{\partial \theta}{\partial t}$

= 0 when θ includes t explicitly, and finally that it is necessary and sufficient that the Ξ_i must form a first-order *solution by variations* of equations (1).

25. (cont.). – Set:

$$\Delta_i \quad \text{or} \quad \Leftrightarrow \qquad \frac{\partial(x_{i+1},\ldots,x_{i-1})}{\partial(\lambda_1,\ldots,\lambda_{n-1})} \Leftrightarrow \xi_i^{n-1} \qquad (i = 1, 2, \ldots, n) .$$

(ξ_i^{n-1}) will then be an $(n-1)^{\text{th}}$-order solution by variations of equations (1) when those n functions of $x_1, \ldots, x_n$, and t verify the n equations:

$$\frac{\delta\xi_i^{n-1}}{\delta t} + \sum_k \left(\frac{\partial X_k}{\partial x_i} \xi_k^{n-1} - \xi_i^{n-1} \frac{\partial X_k}{\partial x_k} \right) = 0 \tag{20}$$

identically when n is odd, and:

$$\frac{\delta |-\xi|_i^{n-1}}{\delta t} + \sum_k \left(\frac{\partial X_k}{\partial x_i} |-\xi|_k^{n-1} - |-\xi|_i^{n-1} \frac{\partial X_k}{\partial x_k} \right) = 0 \tag{21}$$

when n is even.

In those equations, $\delta\xi_i^{n-1}$ represents the variation of ξ_i^{n-1} when taken in conformity with equations (1), and it will then equal $\left(\sum_k \frac{\partial \xi_k^{n-1}}{\partial x_i} X_k + \frac{\partial \xi_i^{n-1}}{\partial t} \right) \delta t$. On the other hand, one can write $|-\xi|_i^{n-1}$ for $(-1)^i \xi_i^{n-1}$.

Set $\xi_i^{n-1} = \nu^i / M$, in which M is a multiplier of equations (1). The ν^i must satisfy the equations:

$$\frac{\delta \nu^i}{\delta t} + \sum_k \frac{\partial X_k}{\partial x_i} \nu^k = 0 \qquad (n \text{ odd}) , \tag{20$'$}$$

$$\frac{\delta |-\nu|^i}{\delta t} + \sum_k \frac{\partial X_k}{\partial x_i} |-\nu|^k = 0 \qquad (n \text{ even}) . \tag{21$'$}$$

In no. **16**, one had:

$$\delta \sum_i M_i \xi_i' = 0 .$$

One will likewise have:

$$\delta \sum_i M_i \xi_i^{n-1} = 0 ,$$

or

$$\frac{\sum_i M^i M_i}{M} = \text{an integral of equations (1)} \qquad (n \text{ odd}),$$

or

$$\frac{\sum_i |-M|^i M_i}{M} = \text{an integral of equations (1)} \quad (n \text{ even}).$$

Particular case. – Let φ be an integral of equations (1). Hence (no. **13**): $M_i = \dfrac{\partial \varphi}{\partial x_i}$ and $\sum_i M^i \dfrac{\partial \varphi}{\partial x_i}$ = a multiplier if n is odd, etc. That last result is found in the note by KOENIGS that was cited before. (See no. **25**).

CHAPTER VI

RELATIONS BETWEEN INTEGRAL INVARIANTS OF DIFFERENT ORDERS

26. Theorem:

If one knows I_p and I_q then one can deduce I_{p+q} . Meanwhile, if $p = q$ is an odd number then I_{2p} will be identically zero.

[I_p , I_q , I_{p+q} represent invariants of order p, q, $p + q$, respectively. We shall not recall the meanings of those symbols, nor the ones that were employed previously. Therefore, M will always represent a multiplier of equations (1), etc.] That theorem is due to POINCARÉ (*Méthodes Nouvelles*, t. III, pp. 21).

Proof. – Let D be a determinant of order n. By virtue of a theorem by LAPLACE, one will have:

$$D = \sum (-1)^m \Delta \Delta' \ .$$

One knows that the rows in D are divided into two groups that are composed of p and $n - p$ rows, respectively. Δ is a partial determinant that is formed from rows in the first group and p arbitrary columns of D. Δ' is the complement of Δ . Finally, m equals the sum of the ranks of the rows and columns of D that appear in Δ . $\sum$ includes $n\,(n-1)\,\ldots\,(n-p+1)\,/\,p\,!$ terms.

Let $n = 5$, $p = 2$. Set $\dfrac{\partial x_i}{\partial \lambda_1} \equiv i_1$, etc.

$$I_2 = \int \sum_{i,j} M_{ij}\, dx_i\, dx_j \ ,$$

so:

$$\delta \sum_{i,j} M_{ij} \begin{vmatrix} i_1 & i_2 \\ j_1 & j_2 \end{vmatrix} \qquad \text{or} \qquad \delta \sum_{i,j} M_{ij} \begin{vmatrix} i_3 & i_4 \\ j_3 & j_4 \end{vmatrix} \equiv 0 \ .$$

$$I_3 = \int \sum_{k,l,m} N_{klm}\, dx_k\, dx_l\, dx_m \ ,$$

so:

$$\delta \sum_{k,l,m} N_{klm} \frac{\partial (klm)}{\partial (123)} \equiv 0 \ .$$

One must show that:

$$I_5 = \int \sum_{ijklm} M_{ij}\, N_{klm}\, dx_i\, dx_j\, dx_k\, dx_l\, dx_m \ ,$$

or that:

$$\delta \sum M_{ij}\, N_{klm} \frac{\partial (ijklm)}{\partial (12345)} \equiv 0 \ .$$

Now, by virtue of the theorem that was just recalled, one has:

$$\frac{\partial(ijklm)}{\partial(12345)} = \begin{pmatrix} i & j \\ 1 & 2 \end{pmatrix}\begin{pmatrix} k & l & m \\ 3 & 4 & 5 \end{pmatrix} - \begin{pmatrix} i & j \\ 1 & 3 \end{pmatrix}\begin{pmatrix} k & l & m \\ 2 & 4 & 5 \end{pmatrix} + \begin{pmatrix} i & j \\ 1 & 4 \end{pmatrix}\begin{pmatrix} k & l & m \\ 2 & 3 & 5 \end{pmatrix} - \text{etc.},$$

in which $\begin{pmatrix} i & j \\ 1 & 2 \end{pmatrix}$ is written for $\dfrac{\partial(x_i, x_j)}{\partial(\lambda_1, \lambda_2)}$.

When one substitutes that development in the formula to be proved, it will become:

$$\delta\left[\sum_{i,j} M_{ij}\begin{pmatrix} i & j \\ 1 & 2 \end{pmatrix}\sum_{klm}\begin{pmatrix} k & l & m \\ 3 & 4 & 5 \end{pmatrix} - \text{etc.}\right] = 0 .$$

That is now obvious, since the variation of each of the factors $\sum_{ij}$ and $\sum_{klm}$ is identically zero, by hypothesis.

It remains to show that if $q = p$ then I_{2p} will be zero if p is odd and non-zero if p is even. In order to do that, it suffices to point out that the sum of the indices 1, 2, …, $2p$ of the λ is $2p\,(2p + 1) / 2$. If p is odd then that sum will be odd, so if the sum of p of those indices is even then the sum of the other p will be odd. The same thing will not be true when p is even. $2p\,(2p + 1) / 2$ will then be even. Therefore, if the sum of p of the indices of λ is even then the sum of p others will also be even or, more generally, have the same parity as the other sum. Now, it is the parity of the sums of those indices that decides the signs, so (etc.). For example:

$$I_2 = \int \sum_{i,j} M_{ij}\, dx_i\, dx_j .$$

The element of I_4 that one deduces is:

$$\sum M_{ij}\, M_{kl}\, \frac{\partial(ijkl)}{\partial(1234)}$$

$$= \sum M_{ij} M_{kl}\left[\begin{pmatrix} i & j \\ 1 & 2 \end{pmatrix}\begin{pmatrix} k & l \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} i & j \\ 1 & 3 \end{pmatrix}\begin{pmatrix} k & l \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} i & j \\ 1 & 4 \end{pmatrix}\begin{pmatrix} k & l \\ 2 & 3 \end{pmatrix}\right.$$
$$\left. + \begin{pmatrix} i & j \\ 2 & 3 \end{pmatrix}\begin{pmatrix} k & l \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} i & j \\ 2 & 4 \end{pmatrix}\begin{pmatrix} k & l \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} i & j \\ 3 & 4 \end{pmatrix}\begin{pmatrix} k & l \\ 1 & 2 \end{pmatrix}\right]$$

$$= 2\sum M_{ij} M_{kl}\left[\begin{pmatrix} i & j \\ 1 & 2 \end{pmatrix}\begin{pmatrix} k & l \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} i & j \\ 1 & 3 \end{pmatrix}\begin{pmatrix} k & l \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} i & j \\ 1 & 4 \end{pmatrix}\begin{pmatrix} k & l \\ 2 & 3 \end{pmatrix}\right] .$$

Thus, $I_4 \neq 0$.

Once more, take a very simple example:

$$I_1 = \int M_1\, dx_1 + M_2\, dx_2 .$$

One then deduces that:

$$I_2 = \iint M_1 M_2 \, dx_1 \, dx_2 + M_2 M_1 \, dx_2 \, dx_1 \ .$$
$$= \iint (M_1 M_2 - M_2 M_1) \, dx_1 \, dx_2 \equiv 0 \ .$$

That example should serve to show how important it is to write the differentials that appear under the integration sign in a suitable *order*. When an odd permutation is performed on the *dx*, that will always be equivalent to *a change of sign.* The indices of the coefficients of I_p are the same and are placed in the same order as those of the differentials *dx* that multiply those coefficients. For example, in the foregoing, we wrote $M_{ij} \, dx_i \, dx_j$, and not $M_{ji} \, dx_j \, dx_i$. We must also suppose that:

$$M_{ij} \equiv - M_{ji} \ ,$$

since otherwise we would not have:

$$M_{ij} \, dx_i \, dx_j \equiv M_{ji} \, dx_j \, dx_i \ .$$

Therefore, an odd permutation of the indices of a coefficient of an invariant is also equivalent to *a change of sign.*

Corollary. – The converse to that theorem is not true, in general.
If p is even then one can deduce I_{2p} , I_{4p} , etc., from I_p .

27. Theorem:

If one knows I_p then one can deduce I_{p+1} . (*Méthodes nouvelles*, t. III, pp. 14).

Extend I_p over an arbitrary *closed* manifold of order p. Thanks to the generalized STOKES theorem (see no. **7**), we can deduce I_{p+1}, which we can extend over an arbitrary *open* manifold that is bounded by the closed manifold of order p. In that same section **7**, we saw that if I_p is an integral of an exact differential then I_{p+1} will be identically zero. We also know that when I_{p+1} is deduced from I_p , it will be an integral of an exact differential, so if one can deduce I_{p+2} from I_{p+1} by means of the same procedure the one will have $I_{p+2} \equiv 0$.

We extended I_p over an arbitrary *closed* manifold of order p. Hence, if J_p is an integral invariant only when it is extended over an arbitrary *closed* manifold of order p then one can once more deduce I_{p+1} . POINCARÉ called J_p a *relative integral invariant.* The preceding theorem can be stated more generally:

If one knows I_p or J_p then one can deduce I_{p+1} .

28. – Here are some consequences of the two preceding theorems:

If one knows I_p and I_{n-p} then one will find a multiplier M. Indeed, while preserving the notations, one will have:

$$I_n = \int \sum_i M_i M^i dx_i \, dx_{i+1} \cdots dx_{i-1}$$

if $p = 1$, for example. In order to pass from the ordering $i, i + 1, \ldots, n, 1, \ldots, i - 1$ to $1, 2, \ldots, n$, one will require $(i - 1)(n - i + 1)$ permutations, so if n is even then one will have (no. **25**):

$$M = \sum_i M_i \,|-M\,|^i \,.$$

If one knows p integrals that are distinct from (1) and I_{n-p} then one can find a multiplier M. Each of those integrals will yield a first-order invariant (no. **13**). Those p invariants will yield one of order p (no. **26**). One has thus come back to the preceding case. It is easy to see that this invariant of order p will be:

$$\int \sum_{\alpha_1,\ldots,\alpha_p} \frac{\partial(\varphi_1,\ldots,\varphi_p)}{\partial(x_{\alpha_1},\ldots,x_{\alpha_p})} dx_{\alpha_1} \cdots dx_{\alpha_p} \,,$$

if $\varphi_1, \ldots, \varphi_p$ are the p known integrals. If $p = 1$ then one will recover KOENIGS's theorem (no. **25**).

If one knows I_1 then one can, in general, deduce one and only one multiplier by means of the theorems in nos. **26** and **27**. The proof of that proposition is long, but easy, so I shall not give it.

Thanks to the theorem in no. **27**, one can recover the result of no. **24**.

29. – It is obvious that if I add an integral E_p of an *exact differential* of order p to an invariant integral I_p then I will get a *relative* integral invariant J_p of order p. However, the converse is not obvious. It is nonetheless true, and can be stated in the form: Any relative integral invariant J_p is the sum of an integral of an exact differential E_p and an (absolute) integral invariant I_p . (*Méthodes Nouvelles*, t. III, pp. 14).

One can show that $J_p = I_p + E_p$, which is an equality in which only J_p is known.

Let $p = 1$:

$$J_1 = \int \sum_i N_i \, dx_i \,,$$

$$\frac{\delta J_1}{\delta t} = \int \sum_i \left[\sum_k \left(\frac{\partial N_i}{\partial x_k} + \frac{\partial N_i}{\partial t} + \frac{\partial X_k}{\partial x_k} N_k \right) \right] dx_i \,.$$

By definition, δJ_1 is identically zero when one extends J_1 over a *closed* curve. One concludes from this that the integral that is equal to δJ_1 is an integral of an exact differential, because it is easy to show that the converse of the following proposition is true (no. **7**): An integral of an exact differential is identically zero when one extends it over a closed manifold. (The functions that we consider are uniform.) One can then write:

$$\frac{\delta J_1}{\delta t} = \int \sum_i \frac{\partial R}{\partial x_i} dx_i = \int dR \,.$$

In addition, one will have:

$$\frac{\delta J_1}{\delta t} = \frac{\delta E_1}{\delta t} = \int d \frac{\delta U}{\delta t}$$

if one sets:

$$E_1 = \int dU \ .$$

Hence:

$$R = \frac{\delta U}{\delta t} = \sum_k \frac{\partial U}{\partial x_k} X_k + \frac{\partial U}{\partial t} \ .$$

If U is an integral of that equation then one will have:

$$I_1 = \int \sum_i \left(N_i - \frac{\partial U}{\partial x_i} \right) dx_i \ .$$

Let $p = 2$.

$$J_2 = \int \sum_{ij} N_{ij} \, dx_i \, dx_j \, ,$$

$$\frac{\delta J_2}{\delta t} = \int \sum_{ij} \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) dx_i \, dx_j \, ,$$

because $\delta J_2 \ / \ \delta t$ is an integral of an exact differential, and we know that such an integral is reducible to the form that was indicated above. We deduce the converse of STOKES's theorem from this, which we stated at the end of no. **7**. (*Traité d'Analyse* by E. PICARD, t. I, page 117). Therefore:

$$\frac{\delta J_2}{\delta t} = \int \sum_k A_k \, dx_k \, ,$$

$$= \frac{\delta}{\delta t} E_2 \, .$$

E_2 is also reducible to a first-order integral. Set:

$$E_2 = \int \sum_k B_k \, dx_k \ .$$

Hence:

$$\int \sum_k A_k \, dx_k = \int \sum_k \left(\frac{\partial B_k}{\partial x_l} X_l + \sum_l \frac{\partial X_l}{\partial x_k} B_l \right) dx_k \ .$$

That will be true when one satisfies the following n equations:

$$A_k = \sum_l \left(\frac{\partial B_k}{\partial x_l} X_l + \frac{\partial B_k}{\partial t} + \sum_l \frac{\partial X_l}{\partial x_k} B_l \right) .$$

Those equations do not represent *necessary* condition. Indeed, it is sufficient to satisfy the conditions:

(24) $$A_k + \frac{\partial F}{\partial x_k} = \sum_l \left(\frac{\partial B_k}{\partial x_l} X_l + \frac{\partial B_k}{\partial t} + \sum_l \frac{\partial X_l}{\partial x_k} B_l \right) ,$$

in which F is an arbitrary function of $x_1, \ldots, x_n$, and t.

One will proceed similarly for the case in which p has an arbitrary value.

Remark. – The presence of an arbitrary function F in the conditions (24) proves that any relative integral invariant can be decomposed into a sum of an integral of an exact differential and an (absolute) integral invariant *in an infinitude of ways*.

29. (cont.). – In order for $J_1 = \int \sum_i X_i \, dx_i$ to be a relative invariant of equations (1), it is necessary and sufficient that $\delta X_i / \frac{\partial H}{\partial x_i} = \delta t$, in which H is an arbitrary function of the x and t. [The variations δ are always defined by equations (1).]

Proof:

$$\frac{\delta J_1}{\delta t} = \int_\gamma \sum_i \left(\frac{\delta X_i}{\delta t} dx_i + X_i \, dX_i \right) = \int_\gamma \sum_i \frac{\delta X_i}{\delta t} dx_i + \int_\gamma d \left(\sum \frac{X_i^2}{2} \right) .$$

The variation of J_1 must be zero when the curve γ is closed, so one sees that it is necessary and sufficient that the $\delta X_i / \delta t$ should be the partial derivatives of the same function H.

30. Theorem:

If one knows q distinct solutions by first-order variations and I_p then one can deduce I_{p-q}. (One supposes that $q \leq p$.)

One has (let $p = 3$, $q = 1$):

$$\delta \sum_{ijk} M_{ijk} \frac{\partial (x_i, x_j, x_k)}{\partial (\lambda_1, \lambda_2, \lambda_3)} \equiv 0 ,$$

$$\frac{\partial x_i}{\partial \lambda_2} \Leftrightarrow \xi_i ,$$

$$\frac{\partial x_j}{\partial \lambda_2} \Leftrightarrow \xi_j ,$$

$$\frac{\partial x_k}{\partial \lambda_2} \Leftrightarrow \xi_k .$$

Thus:

$$\delta \sum_{ijk} M_{ijk} \begin{vmatrix} \xi_i & \dfrac{\partial x_i}{\partial \lambda_2} & \dfrac{\partial x_i}{\partial \lambda_3} \\ \xi_j & \dfrac{\partial x_j}{\partial \lambda_2} & \dfrac{\partial x_j}{\partial \lambda_3} \\ \xi_k & \dfrac{\partial x_k}{\partial \lambda_2} & \dfrac{\partial x_k}{\partial \lambda_3} \end{vmatrix} \equiv 0 \ .$$

Upon developing that determinant, one will finally get:

$$I_2 = \int \sum_{ijk} M_{ijk} (\xi_i \, dx_j \, dx_k + \xi_j \, dx_k \, dx_i + \xi_k \, dx_i \, dx_j) \ .$$

If one knows yet another solution by variations then can deduce the invariant I_1 from I_2 , etc.

Generalization. – Knowing a solution by variations of order q will permit one to deduce the invariant I_{p-q} from I_p .

Recall the preceding example. One has:

$$\delta \sum_{ijk} M_{ijk} \left[\frac{\partial x_k}{\partial x_3} \frac{\partial (x_i, x_j)}{\partial (\lambda_1, \lambda_2)} + \cdots \right] \equiv 0 \ .$$

Replace $\dfrac{\partial (x_i, x_j)}{\partial (\lambda_1, \lambda_2)}$ with ξ_{ij}^2 , and the other two determinants (which we have not written out) with ξ_{ki}^2 and ξ_{jk}^2 . We deduce the invariant I_1 from the identity thus-obtained.

Example. – If the X_i in equations (1) do not refer to t explicitly then one can deduce I_{p-1} from I_p (no. **16**).

CHAPTER VII

INTEGRAL INVARIANTS OF ORDER *p*

31. – We have already studied the invariants of order one, n, and $n-1$. If we would like to study the necessary and sufficient conditions that the coefficients of an invariant of order 2, 3, …, or $n-2$ must satisfy directly then we would be obliged to make some very bothersome calculations. Here is how one can avoid those lengths calculations.

Let $p = 2$. One deduces from $J_1 = \int \sum_i N_i \, dx_i$ (no. **27**) that:

$$I_2 = \int \sum_{ij} \left(\frac{\partial N_i}{\partial x_j} - \frac{\partial N_j}{\partial x_i} \right) dx_i \, dx_j \, ,$$

$$\frac{\delta J_1}{\delta t} = \int \sum_i \left[\sum_k \left(\frac{\partial N_i}{\partial x_j} X_k + \frac{\partial N_i}{\partial t} + \frac{\partial X_i}{\partial x_j} N_k \right) \right] dx_i \, .$$

We have seen that the latter integral is an integral of an exact differential, so:

$$\frac{\partial P_j}{\partial x_i} - \frac{\partial P_i}{\partial x_j} = 0 \, , \tag{25}$$

in which P_i denotes the coefficient of dx_i in that integral. The conditions (25) can be written:

$$\sum_k \left[X_k \frac{\partial}{\partial x_k} \left(\frac{N_j}{i} - \frac{N_i}{j} \right) + \frac{\partial}{\partial t} \left(\frac{N_j}{i} - \frac{N_i}{j} \right) + \frac{X_k}{j} \left(\frac{N_k}{i} - \frac{N_i}{k} \right) + \frac{X_k}{i} \left(\frac{N_j}{k} - \frac{N_k}{j} \right) \right] = 0 \, , \tag{25'}$$

in which N_j / i, X_k / j, … are written for $\frac{\partial N_j}{\partial x_i}$, $\frac{\partial X_k}{\partial x_j}$, …

(25′) represents the necessary and sufficient conditions for:

$$I_2 = \int \sum_{i,j} \left(\frac{N_j}{i} - \frac{N_i}{j} \right) dx_i \, dx_j$$

to be an (absolute) integral invariant. In order to convince oneself of that, it suffices to remark that instead of calculating δI_2 from I_2 itself, one can first calculate δJ_2 and then transform the result obtained into a second-order integral; it will be $\delta I_2 / \delta t$. It is necessary and sufficient that the variation δI_2 should be identically zero, which gives (25′).

Set:

$$\frac{N_j}{i} - \frac{N_i}{j} \Leftrightarrow M_{ij} .$$

(25′) becomes:

(26) $$\sum_k \left(X_k \frac{\partial M_{ij}}{\partial x_k} + \frac{\partial M_{ij}}{\partial t} + M_{ij} \frac{\partial X_k}{\partial x_j} + M_{kj} \frac{\partial X_k}{\partial x_i} \right) = 0 .$$

It is curious that formulas (26) represent not only the necessary and sufficient conditions for $\int \sum_{i,j} \left(\frac{N_j}{i} - \frac{N_i}{j} \right) dx_i \, dx_j$ to be an integral invariant, but also for $\int \sum_{ij} M_{ij} \, dx_i \, dx_j$ to be a second-order integral invariant that is no longer an exact differential integral. Indeed, suppose that we have calculated the variation of $\sum_{i,j} \left(\frac{N_j}{i} - \frac{N_i}{j} \right) \frac{\partial (x_i, x_j)}{\partial (\lambda_1, \lambda_2)}$ directly. One sees immediately that the particular form of the coefficients $\frac{N_j}{i} - \frac{N_i}{j}$ has no effect on the final result, in other words, the conditions (26) will not be modified when one supposes that the M_{ij} do not have the form $\frac{\partial N_i}{\partial x_j} - \frac{\partial N_j}{\partial x_i}$.

Thanks to that process, it would be quite easy to find the conditions in the case of $p = 3$. One deduces from $J_2 = \int \sum_{ij} N_{ij} \, dx_i \, dx_j$ that:

$$I_3 = \int \sum_{i,j,l} \left(\frac{N_{ij}}{l} + \frac{N_{jl}}{i} + \frac{N_{li}}{j} \right) dx_i \, dx_j .$$

Instead of annulling the variation of I_3 , we can first calculate δJ_2 and then put the result obtained into the form of a third-order integral. It will be equal to δI_3 , so it must be identically zero. If we proceed in that manner then we will have, in succession:

$$\frac{\delta J_2}{\delta t} = \int \sum_{ij} P_{ij} \, dx_i \, dx_j ,$$

in which P_{ij} represents the left-hand side of (26), and then:

$$\frac{\delta J_2}{\delta t} = \int \sum_{ijl} \left(\frac{P_{ij}}{l} + \frac{P_{jl}}{i} + \frac{P_{li}}{j} \right) dx_l \, dx_i \, dx_j \equiv 0 .$$

If we perform the indicated calculations and replace:

$$\frac{N_{ij}}{l} + \frac{N_{jl}}{i} + \frac{N_{li}}{j} \qquad \text{with} \quad M_{lij} \quad \text{or} \quad M_{ijl}$$

then we will obtain the desired conditions:

$$\frac{\delta M_{ijl}}{\delta t} + \sum_k \left(\frac{\partial X_k}{\partial x_k} M_{ijk} + \frac{\partial X_k}{\partial x_i} M_{jlk} + \frac{\partial X_k}{\partial x_j} M_{lik} \right) = 0 \ . \tag{27}$$

One can deduce the formulas that are appropriate to the case of $p = 4$ from those formulas, and so on.

32. – One can prove the following propositions by means of formulas (26) or (27): In order for $\iint \sum_i M_i \, dx_i \, dy_i$ to be an integral invariant of the canonical system in no. **11**, it is necessary and sufficient that the M_i should be *the same* integral of that system.

Corollary. – The canonical equations of no. **11** admit the integral invariant $I_2 = \int \sum_i dx_i \, dy_i$.

In order for $I_2 = \int \sum_i dx_i \, dy_i$ to be an invariant of:

$$\frac{\delta x_i}{X_i} = \frac{\delta y_i}{Y_i} = \delta t \ ,$$

it is necessary and sufficient that one should have:

$$\left\{ \begin{aligned} & \frac{\partial X_i}{\partial x_k} + \frac{\partial Y_k}{\partial y_i} = 0 , \\ & \frac{\partial Y_i}{\partial x_k} = \frac{\partial Y_k}{\partial x_i} , \\ & \frac{\partial X_i}{\partial y_k} = \frac{\partial X_k}{\partial y_i} . \end{aligned} \right. \tag{28}$$

Corollary. – Those are also necessary and sufficient conditions for the proposed system to admit the relative invariant $J_2 = \int \sum_i y_i \, dx_i$. (Compare that with no. **10**.) In order for that same system to admit the invariant $I_1 = \int \sum_i (l+1) \, y_i \, dx_i + l \, x_i \, dy_i$, it is *necessary* that it should be canonical. Set:

$$H = \sum_k [(l+1) \, y_k \, X_k + l \, x_k \, Y_k] \ .$$

It is then necessary that one must have:

$$X_i = \frac{\partial H}{\partial y_i} \ ,$$

$$Y_i = -\frac{\partial H}{\partial x_i},$$

or that:

$$X_i = -\frac{\sum_k \left[(l+1)\, y_k \frac{\partial X_k}{\partial y_i} + l\, x_k \frac{\partial Y_k}{\partial y_i} \right]}{l},$$

$$Y_i = -\frac{\sum_k \left[(l+1)\, y_k \frac{\partial X_k}{\partial x_i} + l\, x_k \frac{\partial Y_k}{\partial x_i} \right]}{l+1}.$$

Now, those are precisely the necessary and sufficient conditions for the variation of I_1 to be identically zero (7).

It is again necessary that:

$$H = \sum_k \left[(l+1)\, y_k \frac{\partial H}{\partial y_k} - l\, x_k \frac{\partial H}{\partial x_k} \right],$$

which one can easily interpret (no. **10**).

One deduces $I_2 = \int \sum_i dx_i\, dy_i$ from I_1 . The conditions (28) are not sufficient for I_1 . That apparent contradiction will disappear when one remarks that I_2 can also be deduced from a relative invariant of the same form as I_1 . It can also be written:

$$\int \sum y\, dx + l \int d \sum xy .$$

33. – In order for the system:

$$\frac{\delta x_i}{X_i} = \frac{\delta y_i}{Y_i} = \frac{\delta z_i}{Z_i} = \delta t$$

to admit the invariant $I_3 = \int \sum dx\, dy\, dz$, it is necessary and sufficient that one should have:

$$\frac{\partial X_i}{\partial x_i} + \frac{\partial Y_i}{\partial y_i} + \frac{\partial Z_i}{\partial z_i} = 0 \tag{29}$$

identically.

The proof of that proposition is lengthy. Here is how one proceeds: Since there are $3n$ dependent variables, one gives the values 1, 2, …, n to the indices i, j, l, or k . However, one sets:

$$x_{i+n} = y_i\,, \qquad x_{i+2n} = z_i\,, \qquad X_{i+n} = Y_i\,, \qquad X_{i+2n} = Z_i\,.$$

One will then have only the M whose indices have the form $(i, i + n, i + 2n)$ which will be different from zero. In addition, one must not write $\delta M_{k,k+n,k+2n}$, because that variation is identically zero.

Corollary. – In order for the proposed system to admit the relative invariant $J_2 = \int \sum (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$, it is necessary and sufficient that the conditions (29) should be satisfied.

34. – Equations (26) or (27), or analogous ones for the cases when $p > 3$, permit us to write out immediately the equation that serves to determine the solutions by variation of arbitrary order.

Set:

$$\frac{\partial(x_i, x_j)}{\partial(\lambda_1, \lambda_2)} \qquad \text{or} \qquad (ij) \Leftrightarrow \xi_{ij} .$$

One has:

$$\delta \sum_{ij} M_{ij}(i\ j) = 0 ,$$

or

$$\sum \frac{\delta M_{ij}}{\delta t}(i\ j) + \sum M_{ij} \frac{\delta(i\ j)}{\delta t} = 0 .$$

One must then exhibit $(i\,j)$. That operation will turn the M_{ij} into M_{ik} and M_{kj} (26). Before permuting the indices i, j, and k, one will have (26):

$$\sum M_{ij} \frac{\delta(i\ j)}{\delta t} = \sum_i \sum_j \sum_k M_{ij} \left[\frac{\partial X_j}{\partial x_k}(i\,k) + \frac{\partial X_i}{\partial x_k}(k\ j) \right] ,$$

so:

$$(26') \qquad \frac{\delta \xi_{ij}}{\delta t} = \sum_k \left(\frac{\partial X_j}{\partial x_k} \xi_{ik} + \frac{\partial X_i}{\partial x_k} \xi_{kj} \right) .$$

One deduces from (27) that:

$$(27') \qquad \frac{\delta \xi_{ijl}}{\delta t} = \sum_k \left(\frac{\partial X_l}{\partial x_k} \xi_{ijk} + \frac{\partial X_i}{\partial x_k} \xi_{jlk} + \frac{\partial X_j}{\partial x_k} \xi_{lik} \right) .$$

One recalls that:

$$\frac{\delta \xi_{ij}}{\delta t} = \sum_k \frac{\partial \xi_{ij}}{\partial x_k} X_k + \frac{\partial \xi_{ij}}{\partial t}$$

by virtue of equations (1).

CHAPTER VIII

SOLUTION BY VARIATION

35. Theorem: In order for the two equations:

$$(30)\qquad \begin{cases} \sum_k \dfrac{\partial\theta}{\partial x_k} X_k + \dfrac{\partial\theta}{\partial t} = 0, \\ \sum_k \dfrac{\partial\theta}{\partial x_k} \xi_k = 0, \end{cases} \qquad (k = 1, 2, \ldots, n)$$

form a *Jacobian* system, it is necessary and sufficient that (ξ_k) should be a solution by variations of equations (1):

$$(1)\qquad \frac{\delta x_i}{X_i} = \delta t\,.$$

Proof. – Write the two equations (30) in the form:

$$(30')\qquad \begin{cases} A(\theta) = 0, \\ C(\theta) = 0. \end{cases}$$

In order for the system to be Jacobian, it is necessary and sufficient that (*):

$$A\,[C\,(\theta)] - C\,[A\,(\theta)] \equiv 0\,,$$

or that:

$$(31)\qquad \sum_k \left(\frac{\partial \xi_i}{\partial x_k} X_k + \frac{\partial \xi_i}{\partial t} - \frac{\partial X_i}{\partial x_k} \xi_k \right) = 0\,.$$

Those are precisely the n equation (9) of no. **16**. Q.E.D.

Theorem:

If φ is an integral of equations (1) *then* $\sum_i \dfrac{\partial\varphi}{\partial x_i}\xi_i$ *will once more be an integral of those equations if* (ξ_i) *is a solution by variations of those equations* (1).

Similarly, if ψ is an integral of:

(*) *Cours d'Analyse* by C. JORDAN, t. III, 1899, pp. 70-79.

$$\sum_k \frac{\partial \varphi}{\partial x_k} \xi_k = 0$$

then $A(\psi)$ will also be an integral of that equation.

Indeed, since φ is an integral of equations (1), one will have:

$$A(\varphi) \equiv 0 ,$$

so

$$A[C(\varphi)] \equiv 0 ,$$

or

$$C(\varphi) = \text{an integral of } A(\theta) = 0 .$$

Remark. – Equations (30) form a Jacobian system, so they will admit $n - 1$ distinct *common* integrals. If $C(\varphi) \equiv 0$ or if $A(\psi) \equiv 0$ then one will conclude that φ or ψ is a common integral to the two equations (30).

The second equation of the system (30) admits the integral t ; it is not an integral of the system. The arbitrary function $F(t)$ is not an integral that is *distinct* from it.

Theorem:

If n is odd then one will have:

$$\xi_i = \frac{M^i}{M} .$$

If n is even then one has:

$$\xi_i = \frac{|-M|^i}{M} \qquad (i = 1, \ldots, n) .$$

Proof. – Set $\xi_i = \nu_i / M$, in which M is a multiplier. I say that the ν_i are the coefficients of an invariant of order $n - 1$, or one with the sign changed (n even):

$$\frac{\delta}{\delta t}\frac{\nu_i}{M} = \sum_k \frac{\partial X_i}{\partial x_k}\frac{\nu_i}{M}$$

$$= \frac{M\dfrac{\delta \nu_i}{\delta t} - \nu_i \dfrac{\delta M}{\delta t}}{M^2} \tag{9}$$

$$= \frac{M\dfrac{\delta \nu_i}{\delta t} + \nu_i M \displaystyle\sum_k \frac{\partial X_k}{\partial x_k}}{M^2} .$$

It is therefore necessary and sufficient that:

$$\frac{\delta \nu_i}{\delta t} + \nu_i \sum_k \frac{\partial X_k}{\partial x_k} - \sum_k \frac{\partial X_k}{\partial x_k} \nu_k = 0 .$$

[Compare those conditions to (18) and (19) in no **23**.]

Remark. – That theorem gives the true significance of the one by KOENIGS (no. **25**).

Corollary. – Let n be odd. One deduces from $M\ \xi_i = M_i$ that $\sum_i \frac{\partial M\ \xi_i}{\partial x_i}$ is a multiplier. Therefore, $\sum_i \left(\frac{\partial \xi_i}{\partial x_i} + \xi_i \frac{\partial}{\partial x_i} \log M \right)$ is an integral of equations (1). (JORDAN's *Cours d'Analyse*, t. III, pp. 84).

36. Theorem:

The canonical system of no. **11** *admits the solution by variations:*

$$\begin{aligned} \xi_i &= \frac{\partial \varphi}{\partial y_i}, \\ \eta_i &= -\frac{\partial \varphi}{\partial x_i}, \end{aligned} \qquad (i = 1, 2, \ldots, n)$$

when φ is a solution to that system. (*Méthodes Nouvelles*, by H. POINCARÉ, t. I, page 166.)

The first n conditions (31) can be written:

$$(\xi_i\ , H) + \frac{\partial \xi_i}{\partial t} = \sum_k \frac{\partial^2 H}{\partial y_i\, \partial x_k} \xi_k + \frac{\partial^2 H}{\partial y_i\, \partial y_k} \eta_k\ .$$

One can prove that there are satisfied identically for the solution that was just indicated, i.e., that one has:

$$\left(\frac{\partial \varphi}{\partial y_i}, H \right) + \frac{\partial \frac{\partial \varphi}{\partial y_i}}{\partial t} \equiv \left(\frac{\partial H}{\partial y_i}, \varphi \right) .$$

That is what will happen. Since φ is an integral:

$$\delta \varphi \equiv 0 ,$$

or

$$(\varphi, H) + \frac{\partial \varphi}{\partial t} \equiv 0 .$$

If we differentiate that identity with respect to ψ_i then we will get the preceding identity that was to be established.

Corollary. (POISSON's theorem):

If φ_1 and φ_2 are two integrals of the canonical system (no. **11**) *then* (φ_1, φ_2) *will also be an integral of that system.*

That will result immediately from the second theorem in the preceding section. One will have that:

$$\sum_i \left(\frac{\partial \varphi_1}{\partial x_i} \frac{\partial \varphi_2}{\partial y_i} - \frac{\partial \varphi_1}{\partial y_i} \frac{\partial \varphi_2}{\partial x_i} \right)$$

is an integral.

N. B. – The theorem that we just cited can be considered to be a generalization of POISSON's theorem. It certainly allows one to better understand the true sense and scope or power of the latter than the proof that one usually gives, which is a proof that based upon the POISSON identity.

37. – Let us write the system (30) as follows:

$$(30) \qquad \left\{ \begin{array}{l} \sum_l \frac{\partial \theta}{\partial x_l} X_l = 0, \\ \sum_l \frac{\partial \theta}{\partial x_l} \xi_l = 0, \end{array} \right. \qquad \left(\begin{array}{l} l \quad = 1, 2, \ldots, n+1, \\ X_{n+1} \equiv 1, \\ x_{n+1} \equiv 1, \\ \xi_{n+1} \equiv 0 \end{array} \right)$$

Set:

$$\frac{\partial \theta}{\partial x_l} = p_i ,$$

so

$$\frac{\partial p_l}{\partial x_q} = \frac{\partial p_q}{\partial x_l} \qquad (q = 1, \ldots, n + 1) ,$$

$$A\,[C\,(\theta)] - C\,[A\,(\theta)] \equiv \sum_q \sum_l \left(X_l \frac{\partial \xi_q}{\partial x_l} - \xi_q \frac{\partial X_l}{\partial x_l} \right) p_q .$$

The latter expression is identical to the generalized POISSON bracket:

$$(A\,(\theta), C\,(\theta)) ,$$

in which x_l, p_l are the conjugate variables.

The necessary and sufficient conditions (31) can then be written:

(31′) $$(A\,\theta, C\,\theta) = 0\ .$$

Of course, that identity must be true for any θ. Each of the coefficient of $\partial\theta / \partial x_l$ or p_l in that expression must be identically zero. For example, if φ is an integral of (1), i.e., if $A\,\varphi \equiv 0$ then one should not conclude from this that the condition (31′) is satisfied identically. By definition, one would then have $(0, C\,\varphi) = A\,C\,\varphi$.

The θ, and as a result, the p_l , remain indeterminate.

Theorem:

If ξ_i and ξ'_i are two solutions by variation of equations (1) *then one will deduce the solution:*

$$\sum_k \left(\frac{\partial \xi'_i}{\partial x_k} \xi_k - \frac{\partial \xi_i}{\partial x_k} \xi'_k \right) .$$

Proof. – Set $C'_\theta = \sum_l \frac{\partial \theta}{\partial x_l} \xi'_l$, and recall the POISSON identity:

$$(A_\theta\,(C_\theta, C'_\theta)) + (C_\theta\,(C'_\theta, A_\theta)) + (C'_\theta\,(A_\theta, C_\theta)) \equiv 0\ .$$

The last two terms on the left-hand side are identically zero for any θ; hence:

$$(A_\theta\,(C_\theta, C'_\theta)) = 0\ .$$

Therefore, $(C_\theta, C'_\theta) = 0$ will form a Jacobian system with the first of the two equations (30):

$$(C_\theta, C'_\theta) = C\,C'_\theta - C'\,C_\theta = \sum_l \sum_q \left(\frac{\partial \xi'_l}{\partial x_q} \xi_q - \frac{\partial \xi_l}{\partial x_q} \xi'_q \right) \frac{\partial \theta}{\partial x_i}\ .$$

One remarks that the coefficient of $\partial\theta / \partial x_{n+1}$ is identically zero, since:

$$\xi_{n+1} \equiv 0\ , \qquad \xi'_{n+1} \equiv 0\ . \qquad \text{Q.E.D.}$$

Remark. – One can also deduce that theorem from (31) and similar identities in which ξ_i has been replaced with ξ'_i . One differentiates the n identities (31) with respect to x_j and multiplies the results obtained by ξ'_j . One then takes the sum of corresponding sides of the n identities that are deduced from (31); one does that for the other n identities, as well. One subtracts corresponding sides of the two identities thus-obtained. The new identity will permit one to show that:

$$\sum_k \left(\frac{\partial \xi'_i}{\partial x_k} \xi_k - \frac{\partial \xi_i}{\partial x_k} \xi'_k \right)$$

satisfies (31) identically.

(Compare that with no. **71** in JORDAN's treatise, t. III.)

Corollary:

If M^i *and* M'^i *are the coefficients of an invariant of order n* – 1 *(one supposes that n is odd, for example), and if M is a multiplier* $\left(e.g.\, M = \sum_i \frac{\partial M^i}{\partial x_i} \right)$ *then I will say that:*

$$\sum_i \sum_j \frac{\partial}{\partial x_i} \left(\frac{\frac{\partial M^j}{\partial x_j} M'^i - \frac{\partial M'^j}{\partial x_j} M^i}{M} \right)$$

is also a multiplier.

Proof. – One deduces ξ''_i from $\xi_i = M^i / M$, $\xi'_i = M'^i / M$. One knows that $M\,\xi''_i = M''^i$.

One deduces that $\sum_i \frac{\partial M''^i}{\partial x_i}$ is a multiplier from M''^i . After some obvious reductions, that multiplier can be put into the form that was indicated by the corollary.

38. – In order for $\delta x_i / X_i = \delta y_i / Y_i = \delta t$ to admit the solution in no. **36**, it is necessary and sufficient that the system should admit the invariant $I_2 = \int \sum dx\,dy$ (no. **32**).

Proof. – One deduces $\delta\,\partial\varphi / \partial y_i$ or $\delta\xi_i$ from $\delta\varphi \equiv 0$. One equates that value to the one that was given in (31). Hence, one gets two condition equations upon annulling the coefficients of $\partial\varphi / \partial x_k$ and $\partial\varphi / \partial y_k$. One performs the same calculations for η_i .

39. – LAPLACE's theorem (no. **26**) gives:

$$\sum_i (-1)^i \xi_i\, \xi_i^{n-1} = \xi^n ,$$

$$\sum_i (-1)^{i+j} \xi_{ij}^2\, \xi_{ij}^{n-2} = \xi^n \text{, etc.,}$$

respectively, if we set:

$$\xi_i \quad \Leftrightarrow \frac{\partial x_i}{\partial \lambda_1} ,$$

$$\xi_i^{n-1} \Leftrightarrow \frac{\partial (x_1, \ldots, x_{i-1}, x_{i-1}, \ldots, x_n)}{\partial (\lambda_2, \ldots, \lambda_n)} ,$$

$$\xi_{ij}^{2} \quad \Leftrightarrow \frac{\partial(x_i, x_j)}{\partial(\lambda_1, \lambda_2)} \ , \qquad\qquad (i < j) \ ,$$

$$\xi_{ij}^{n-2} \ \Leftrightarrow \frac{\partial(x_1, \ldots, x_{i-1}, x_{i-1}, \ldots, x_{j-1}, x_{j-1}, \ldots, x_n)}{\partial(\lambda_3, \ldots, \lambda_n)} \ ,$$

$$\xi^{n} \quad \Leftrightarrow \pm \frac{\partial(x_1, \ldots, x_n)}{\partial(\lambda_1, \ldots, \lambda_n)} \ .$$

If one adopts that notation then, from no. **25**, one will have:

$$(-1)^i \xi_i^{n-1} = \frac{M_i}{M} \ .$$

From no. **21**:

$$\xi^n = \frac{M_0}{M} \ ,$$

in which M_0 denotes an integral of equations (1); from no. **35**:

$$(-1)^i \xi_i' = \frac{M_i}{M} \ .$$

One likewise has:

(32) $$(-1)^{i+j} \xi_i^{n-2} = \frac{M_{ij}}{M} \ .$$

Indeed, let $I_n = \int M \, dx_1 \cdots dx_n$.

One then deduces:

$$I_n = \int \sum_{ij} (-1)^{i+j} M \, \xi_{ij}^{n-2} dx_i \, dx_j \qquad\qquad (i < j) \ ,$$

which shows that $(-1)^{i+j} M \, \xi_{ij}^{n-2}$ is a coefficient M_{ij} of a second-order integral invariant. Conversely, $(-1)^{i+j} M_{ij} / M$ can be considered to be a solution by variation of order $n - 2$. Indeed, one will have that:

(33) $$\sum_{ij} M^{ij} \, \xi_{ij}^{n-2} \text{ is an integral}$$

if one agrees to write:

$$I_{n-2} = \int \sum_{ij} M^{ij} \, dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_{j-1} \, dx_{j+1} \cdots dx_n \ .$$

I say that one will have:

(33′) $$\sum_{ij} M^{ij}\,\frac{(-1)^{i+j}M_{ij}}{M} \text{ is an integral} \qquad (i<j)\,.$$

One deduces I_n from I_2 and I_{n-2} , and thus a multiplier. It is divided by another multiplier, which gives an integral. Q.E.D.

In what follows, we write (ξ_{ij}) for $(-1)^{i+j}\,\xi_{ij}^{n-2}$. $(i<j)$.

By virtue of formula (32), we will have:

(34) $$\frac{\delta}{\delta t}(\xi_{ij}) = \sum_k \left[(\xi_{ij})\frac{\partial X_k}{\partial x_k} - \frac{\partial X_k}{\partial x_i}(\xi_{kj}) - \frac{\partial X_k}{\partial x_j}(\xi_{ik}) \right] \qquad (k = 1, 2, \ldots, n)\,.$$

If k becomes $> j$ in those formulas then one must replace (ξ_{kj}) with $-(\xi_{kj})$, because in the foregoing, one has assumed that the first index is the smaller of the two. One must perform a change of sign because:

$$(\xi_{kj}) = \frac{M_{kj}}{M} = -\frac{M_{jk}}{M} = -(\xi_{kj})\,.$$

One makes an analogous change when k is $< i$.

One deduces immediately from (34) (no. **34**) that:

(35) $$\frac{\delta(M^{ij})}{\delta t} + \sum_k \left[\frac{\partial X_k}{\partial x_k}(M^{ij}) - \frac{\partial X_i}{\partial x_k}(M^{kj}) - \frac{\partial X_j}{\partial x_k}(M^{ik}) \right] = 0\,,$$

in which (M^{ij}) is written for $(-1)^{i+j}(M^{ij})$.

Conclusion: Thanks to the solutions by variation of higher order, we can find the conditions that the coefficients of I_{n-p} must satisfy when we know the ones that relate to the coefficients of I_p (no. **31**).

40. – Recall the change of variables that was defined by equations (13) that were given in no. **19**.

One will have:

$$\frac{\partial y_i}{\partial \lambda} = \sum_k \frac{\partial y_i}{\partial x_k}\frac{\partial x_k}{\partial \lambda}\,.$$

Once more, set:

$$\frac{\partial x_k}{\partial \lambda} \Leftrightarrow \xi_k\,,$$
$$\frac{\partial y_i}{\partial \lambda} \Leftrightarrow \eta_k\,.$$

Hence:

$$\eta_k = \sum_k \frac{\partial y_i}{\partial x_k} \xi_k \ .$$

One will likewise find that:

(38) $$\xi_i = \sum_k \frac{\partial x_i}{\partial y_k} \eta_k \ .$$

(η_i) will be a solution by variation of (14). I say that the η_i will be *integrals* of equations (1) if the change of variables is such that the transformed equations take the form of equations (16).

Indeed, one will deduce the identity $\sum_k \frac{\partial \eta_i}{\partial x_k} X_k + \frac{\partial \eta_i}{\partial t} \equiv 0$, in which η_i must be replaced with its previously-given value, from the facts that $\frac{\delta y_i}{\delta t}$ or $\sum_k \frac{\partial y_i}{\partial x_k} X_k + \frac{\partial y_i}{\partial t} \equiv F_i(t)$ and that (ξ_i) satisfies (9) identically.

One also shows that $\delta\eta_i$ is identically zero when the variation is taken in conformity with equations (16). Since the η_i are integrals of the latter equations, one concludes that they will also be integrals of the original equations (1). (See no. **19**.)

In addition, it is necessary and sufficient that the coefficients of an integral invariant of arbitrary order for equations (16) should be *integrals* of the latter equations, or (what amounts to the same thing) equations (1). Indeed, the conditions that those coefficients must satisfy will become:

$$\delta M_{ijk\ldots} \equiv 0$$

for equation (16).

Remark. – That change of variables, although impractical in general, takes on very great importance in POINCARÉ's treatise (*Méthodes Nouvelles*, t. III, pp. 7).

Here is one application:

Suppose that Φ represents an algebraic *form* with respect to the ξ_i , and that the form is an integral of the equations:

$$\frac{\delta x_i}{X_i} = \frac{\delta \xi_i}{\sum_k \frac{\partial X_i}{\partial x_k} \xi_k} = \delta t \ .$$

For example, if one has:

$$I_1 = \int \left(\sum A_{\alpha_1 \cdots \alpha_p} dx_{\alpha_1} \cdots dx_{\alpha_p} \right)^{1/p}$$

then it will suffice to replace the dx in the expression under the integral sign with a first-order solution by variation in order to obtain a form such as Φ.

Equations (38) define a linear substitution whose modulus is $\frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)}$ or $\frac{1}{M}$, in which M denotes a multiplier of equations (1). Let Φ' be the transformed form (38). Let i denote an invariant that is supposed to be known for the form Φ an let i' denote the same invariant when it is taken for the transformed form Φ'. Since the transformed invariant i' is a function of the coefficients of

η_i that enter into Φ', it will be an *integral* of equations (1). Indeed, one can deduce the integral invariant:

$$I_1' = \int \left(\sum A'_{\alpha_1 \cdots \alpha_p} dy_{\alpha_1} \cdots dy_{\alpha_p} \right)^{1/p}$$

from Φ', and we have seen that the coefficients of that integral invariant are integrals of equations (1).

One then has:

$$i' \text{ is an integral of (1).}$$

On the other hand, one has:

$$i' = i \left(\frac{1}{M} \right)^q,$$

since $1 / M$ is the modulus of the linear substitution (38); however, that modulus is generally unknown. Let μ be a known multiplier, so:

$$i' \left(\frac{M}{\mu} \right)^q = i\, \mu^{-q} \qquad \text{is an integral of (1).}$$

For example, suppose that we know n distinct integral invariants of the form $I_1 = \int \sum_i M_i \, dx_i$. The determinant that is formed by means of the n^2 coefficients M_i is an invariant that is common to those n forms in ξ. In this example, $q = 1$. The determinant will then be equal to a multiplier.

If we replace the M_i in that determinant with $M\, \xi_i^{n-1}$ then we will see that the determinant that is formed by means of n solutions by variation of order $n - 1$ is equal to a multiplier raised to the $(1 - n)^{\text{th}}$ power.

Now let c be a covariant of Φ of weight p and degree q with respect to the ξ. As before, we will have:

$$c' = c \left(\frac{1}{M} \right)^p .$$

The coefficients of the η in c' are functions of the coefficients of the η in Φ, which are then integrals of equations (1).

One will deduce from:

$$c' \left(\frac{M}{\mu} \right)^p = \frac{c}{\mu^p}$$

that:

$$I_i' = \int \sqrt[q]{c' \left(\frac{M}{\mu} \right)^p} ,$$

after one has the replaced the η_i with the dy_i . Hence, one finally has:

$$I_1 = \int \sqrt[q]{c\,\mu^{-p}}\ .$$

(The ξ_i were previously replaced with dx_i in c.) (*Méthodes Nouvelles*, t. III, pp. 36).

One can extend the foregoing to solutions ξ^{n-1} of order $n-1$. If we perform the latter change of variables that was just now in question then we will get (n odd):

(38′) $$\xi_i^{n-1} = \sum_k \frac{\partial(x_{i+1},\ldots,x_{i-1})}{\partial(y_{k+1},\ldots,y_{k-1})}\eta_k^{n-1}\ .$$

Equations (38′) also define a linear substitution with respect to the ξ^{n-1}. The modulus of that substitution is precisely the adjoint or inverse determinant to $\frac{\partial(x_1,\ldots,x_n)}{\partial(y_1,\ldots,y_n)}$; hence, its modulus will be equal to $\left(\frac{1}{M}\right)^{n-1}$.

While preserving the previous notations, we will again have that:

$$i' = i\left(\frac{1}{M}\right)^{p(n-1)} \text{ is an integral of (1),}$$

so:

$$i = \mu^{p(n-1)}.$$

The determinant that is formed by means of the n^2 coefficients of the n distinct integral invariants of order $n-1$ is a multiplier with exponent $n-1$. Hence (upon replacing M^i with $M\,\xi_i'$), we will deduce that $\xi^n = \mu^{-1}$ (no. **20**). The case in which n is even will be treated in the same manner.

41. – Let us consider the case in which there are several *independent* variables.

In order to avoid pointless complications, we suppose that there are only two independent variables t_1 and t_2 . We let:

(39) $$\delta x_i = X_i^1\,\delta t_1 + X_i^2\,\delta t_2 \qquad (i = 1, 2, \ldots, n)$$

be a system of n completely-integrable total differential equations. We will then have:

$$\delta_1 X_i^2 = \delta_2 X_i^1$$

upon setting:

$$\frac{\delta\Phi}{\delta t_1} \equiv \delta_1\,\Phi \equiv \sum_i X_i^1 \frac{\partial\Phi}{\partial x_i} + \frac{\partial\Phi}{\partial t_1}\,,$$

$$\frac{\delta\,\Phi}{\delta t_2} \equiv \delta_2\,\Phi \equiv \sum_i X_i^{\,2}\,\frac{\partial\Phi}{\partial x_i} + \frac{\partial\Phi}{\partial t_2}\,.$$

Sometimes we can also set:

$$\begin{aligned} \delta_1\Phi &\equiv A_1\Phi\,, \\ \delta_2\Phi &\equiv A_2\Phi\,, \end{aligned} \qquad \text{(see no. } \mathbf{37}\text{).}$$

In order for φ to be an integral of (39), it is necessary and sufficient that:

$$\delta_1\,\varphi \equiv 0\,,$$

$$\delta_2\,\varphi \equiv 0\,.$$

Analogously, in order for $I_1 = \int \sum_i M_i\,dx_i$ to be an integral invariant of (39), it is necessary and sufficient that:

$$\delta_1 M_i + \sum_k M_k\,\frac{\partial X_k^{\,1}}{\partial x_i} = 0\,,$$

$$\delta_2 M_i + \sum_k M_k\,\frac{\partial X_k^{\,2}}{\partial x_i} = 0\,.$$

In order for (ξ_i) to be a solution by variation of (39), it is necessary and sufficient that one should have:

(40)
$$\begin{cases} \delta_1\,\xi_i - \sum_k \xi_k\,\dfrac{\partial X_i^{\,1}}{\partial x_k} = 0, \\[2ex] \delta_2\,\xi_i - \sum_k \xi_k\,\dfrac{\partial X_i^{\,2}}{\partial x_k} = 0. \end{cases}$$

Set:

$$\sum_k \xi_k\,\frac{\partial\Phi}{\partial x_k} \equiv C\,\Phi\,.$$

The system (40) can be written:

(40′)
$$\begin{cases} A_1\,\xi_i = C\,X_i^{\,1}, \\ A_2\,\xi_i = C\,X_i^{\,2}. \end{cases}$$

Therefore, if (ξ_i) is a solution by variation of (39) then the system:

(41)
$$\begin{cases} A_1\Phi = 0, \\ A_2\Phi = 0, \\ C\,\Phi = 0 \end{cases}$$

will be a *Jacobian* system.

One will have:

$$(42)\qquad \begin{cases} A_1 A_2 \Phi = A_2 A_1 \Phi\,, \\ A_1\, C\, \Phi = C\, A_1\, \Phi\,, \\ A_2\, C\, \Phi = C\, A_2\, \Phi\,. \end{cases}$$

If φ is an integral of:

$$A_1\, \Phi = 0$$

then $A_2\, \varphi$ and $C\, \varphi$ will also be integrals of that equation, etc.

Upon using the POISSON brackets (no. **37**), the identities (42) can be written:

$$(A_1\, \Phi, A_2\, \Phi) \equiv 0\,,$$
$$(A_1\, \Phi, C\ \ \Phi) \equiv 0\,,$$
$$(A_2\, \Phi, C\ \ \Phi) \equiv 0\,.$$

One will deduce from this (no. **37**) that if (ξ_i) and (ξ_i') are two solutions by variation then $\sum_q \left(\frac{\partial \xi_l'}{\partial x_q} \xi_q - \frac{\partial \xi_l}{\partial x_q} \xi_q' \right)$ will also be a solution by variation of the first order.

All of what was said in no. **37** extends to the case in which there are several independent variables with the same facility.

CHAPTER IX

CHARACTERISTIC EXPONENTS

42. – The n functions ξ of $x_1, \ldots, x_n$, and t, which collectively constitute a first-order solution by variation of the equations:

(1) $$\frac{\delta x_i}{X_i} = \delta t ,$$

are determined by the n partial differential equations:

(31) $$\sum_k \left(\frac{\partial \xi_i}{\partial x_k} X_k + \frac{\partial \xi_i}{\partial t} - \frac{\partial X_i}{\partial x_k} \xi_k \right) = 0 .$$

From our conventions, the system (31) can be replaced with the $2n$ equations:

(43) $$\frac{\delta x_i}{X_i} = \frac{\delta \xi_i}{\sum_k \frac{\partial X_i}{\partial x_k} \xi_k} = \delta t .$$

The complete integration of the latter system will obviously be more difficult than that of the system (1). We shall therefore not attempt to perform that integration. POINCARÉ (*Méthodes Nouvelles*, t. I, page 162) substituted a *solution* $\varphi_i(t)$ of equations (1) for the x_i in the $\partial X_i / \partial x_k$. Thanks to that artifice, the system (43) will become:

(44) $$\frac{\delta \xi_i}{\delta t} = \sum_k \frac{\partial X_i}{\partial x_k} \xi_k .$$

That is a system of n linear equations in the x in which the coefficients $\partial X_i / \partial x_k$ are known functions that depend upon only t.

In order to make the study of the system (44) easier, POINCARÉ supposed that the X_i and the solution $\varphi_i(t)$ are *periodic* functions of t of period T. Equations (44) then become a system of linear equations in the ξ with periodic coefficients in t of period T.

See *Méthodes Nouvelles*, tome I, pp. 63-68, 162-201; tome III, pp. 48-63.

N. B. We have cited only the parts of POINCARÉ's admirable work that refer to the *theory* of characteristic exponents and integral invariants directly.

CHAPTER X

INTEGRAL COVARIANTS

43. – The notion of an *integral covariant* is an extension of that of an integral invariant. We saw in no. **4** that if the variation of an arbitrary element of an integral that is extended over a manifold of order p is identically zero then that integral of order p will be an integral invariant of order p. We likewise say: If the $(q + 1)^{\text{th}}$ variation of an arbitrary element of an integral that is extended over a manifold of order p is identically zero then that integral of order p is an *integral covariant* of order p and degree q. Let I_p denote an integral invariant of equations (1) in the foregoing. Similarly, let $C_{p,q}$ denote an integral covariant of order p and degree q of equations (1).

Thus:

$$C_{p,q} \equiv I_p \,,$$

$$\frac{\delta C_{p,q}}{\delta t} \equiv C_{p,q-1} \,, \quad \text{etc.}$$

44. – Consider the canonical system of no. **11**. If H is a homogeneous function of degree p with respect to the x then one will have:

$$C_{1,1} = \int \sum_i x_i \, dy_i \;.$$

(*Méthodes Nouvelles*, t. III, page 63)

Indeed:

$$\begin{aligned} \frac{\delta}{\delta t} \sum_i x_i \, dy_i &= \sum_i \left(\frac{\partial H}{\partial y_i} dy_i - x_i \, d \frac{\partial H}{\partial x_i} \right) \\ &= d \left(H - \sum_i x_i \frac{\partial H}{\partial x_i} \right) \\ &= (1 - p) \, dH \,, \end{aligned}$$

$$\delta^2 \sum_i x_i \, dy_i \equiv 0 \,. \qquad \text{Q.E.D.}$$

One deduces from $C_{1,1}$ that:

$$I_1 = C_{1,1} + (p-1) t \int dH \,.$$

If H is a homogeneous function of degree q with respect to the y then one will have:

$$C_{1,1} = \int \sum_i y_i \, dx_i \,,$$

$$I_1 = C_{1,1} + (1-q) t \int dH \,.$$

If H is a homogeneous function of degree p with respect to x and degree q with respect to the y then $\sum x\,y$ will be a co-integral of degree one, i.e., one will have $\delta^2 \sum x\,y \equiv 0$.

If $H \equiv H_p + H_q$, in which H_p represents a homogeneous function in the x of degree p that is independent of the y and H_q represents a homogeneous function in the y that is independent of the x then one will have:

$$C_{1,1} = \int \sum_i (q\,x_i\,dy_i - p\,y_i\,dx_i)\,,$$

$$I_1 = C_{1,1} + (p\,q - p - q)\,t \int dH\,.$$

Indeed:

$$\begin{aligned}\frac{\delta C_{1,1}}{\delta t} &= \int \sum_i \left(q\frac{\partial H}{\partial y_i}dy_i - p\,x_i\,d\frac{\partial H}{\partial x_i} + p\frac{\partial H}{\partial x_i}dx_i - p\,y_i\,d\frac{\partial H}{\partial y_i}\right)\\ &= \int \sum_i \left(q\,dH_q - q\,dx_i\frac{\partial H}{\partial x_i} + q\frac{\partial H}{\partial x_i}dx_i + p\,dH_p - p\,dy_i\frac{\partial H}{\partial y_i}\right)\\ &= \int (p+q-p\,q)\,dH\,.\end{aligned}$$

Example. – Let $H = \sum_i \frac{y_i^2}{2m_i} - U$, in which U is a function (of force) that is supposed to be homogeneous in x of degree p and independent of the y . That value for H presents itself in dynamics.

If one sets $q = 2$ in the foregoing then one will get:

$$I_1 = \int \sum (2x\,dy - p\,y\,dx) + (p-2)\,t \int \sum \left(\frac{y\,dy}{m} - \frac{\partial U}{\partial x}dx \right)\,.$$

(*Méthodes nouvelles*, t. I, page 171 and t. III, page 66)

Suppose, in addition, that $p = -2$ (which will be the case when an attraction is inversely proportional to the cube of the distance). One will then have:

$$C_{1,1} = \int d \sum x\,y\,.$$

In dynamics, $y = m\frac{\delta x}{\delta t}$, so:

$$C_{1,1} = \int d\left(\frac{\delta}{\delta t} \sum m\,x^2 \right)\,.$$

Therefore, $\int m\,x^2$ is a co-integral of degree two, i.e.,:

$$\frac{\delta^3 \sum m\,x^2}{\delta t^3} \equiv 0\,.$$

If one has:

$$C_{1,1} = \int \sum_i M_i \, dx_i + N_i \, dy_i$$

then $\sum_i \frac{\partial M_i}{\partial y_i} - \frac{\partial N_i}{\partial x_i}$ will be a co-integral of degree one (no. **11**. Analogous proof.)

Remark. – The theorems concerned with integral invariants thus find their generalization in the theory of integral covariants. Nonetheless, a certain prudence is necessary. For example, it is not precise to say that the quotient of two co-multipliers of degree one is a co-integral of degree one.

45. – Let:

$$C_{n,1} = \int M \, dx_1 \cdots dx_n \, ,$$

in which we suppose that M is independent of t. Thus, by virtue of no. **18**:

$$I_n = \int \left(\sum_i \frac{\partial X_i M}{\partial x_i} \right) dx_1 \cdots dx_n \, ,$$

and if the X_i do not include t explicitly then:

$$\frac{\partial \left(X_k \sum_i \frac{\partial X_i M}{\partial x_i} \right)}{\partial x_k} = 0 \, , \tag{45}$$

That is the equation that serves to determine the co-multipliers of degree one, which will be independent of t when the X_i do not include t explicitly.

If the equations are canonical (no. **11**) then one will the multiplier (M, H) from $C_{n,1}$. Hence, one concludes that in the case of the canonical equations, equation (45) can be written:

$$((M, H) \, H) = 0 \, .$$

M will then be a co-integral of degree one.

46. – The notions that were developed in Chap. VI extend easily to the integral covariants. No. **26** becomes: If one knows $C_{p,q}$ and $C_{p',q'}$ then one can deduce $C_{p+p',q+q'}$. Meanwhile, if $p = p'$ is an odd number then $C_{2p,q+q'}$ will be identically zero.

If $p + p' = n$ then one will obtain a co-multiplier of degree $q + q'$. That is why one can sometimes deduce a co-multiplier from several known co-integrals and covariants.

The theorem in no. **27** becomes: If one knows $C_{p,q}$ then one can deduce $C_{p+1,q}$.

If $G_{p,q}$ represents a relative integral covariant then one will once more have the theorem: If one knows $G_{p,q}$ then one can deduce $C_{p+1,q}$ from it.

The theorem in no. **29** generalizes as follows: Any relative integral covariant $G_{p,q}$ is the sum of an integral of an exact differential E_p and an (absolute) integral covariant $C_{p,q}$.

The theorem in no. **30** becomes: If one knows r distinct solutions by variation of order one and $C_{p,q}$ then one can deduce $C_{p-r,q}$ (when one supposes that $r \leq p$). If $r = p$ then one will get a co-integral of degree q.

Application. – Let n be odd. One deduces from:

$$C_{n-1,1} = \int \sum_i M^i \, dx_{i+1} \cdots dx_{i-1}$$

that

$$C_{n,1} = \int \sum_i \frac{\partial M^i}{\partial x_i} dx_1 \cdots dx_n \,,$$

so $\dfrac{\partial X_k \sum_i \frac{\partial M^i}{\partial x_i}}{\partial x_k}$ will be a multiplier if the M^i do not include t explicitly (no. **45**). If the equations are canonical (n even) then $\sum_i \dfrac{\partial \,|-M\,|^i}{\partial x_i}$ will be a co-integral of degree one (no. **45**).

One deduces from:

$$C_{n,1} = \int M \, dx_1 \cdots dx_n$$

that (n odd):

$$C_{n-1,1} = \int \sum_i M \, \xi^i \, dx_{i+1} \cdots dx_{i-1} \,,$$

in which ξ^i represents a solution by variation. $\sum_i \dfrac{\partial M \, \xi_i}{\partial x_i}$ will be a co-multiplier of degree one. $\sum_i \dfrac{\partial M \, X_i}{\partial x_i}$ will be multiplier when the X do not exclude t explicitly (no. **45**).

CHAPTER XI

APPLICATION TO THE THEORY OF VORTICES

47. Hypothesis:

> "The continuous fluid that we shall study is supposed to be absolutely devoid of diffusibility. If we trace out a closed surface in the fluid at an arbitrary instant then the part of the fluid that is located on one side of that surface will never mix with the rest of it. The parts of the fluid that define the surface will never cease to define the same continuous surface, which moves and deforms with the fluid. In particular, one can divide the entire volume into elements that always include the same parts of the fluid despite the changes in size and form that they might experience, and they move like material points." (*)

Following the notation of EULER, let u, v, w be three rectangular components of the present velocity of the fluid at x, y, z.

The *elements* of the fluid that was just in question are rectangular parallelepipeds that have volumes of $dx\,dy\,dz$. The mass that is enclosed in $dx\,dy\,dz$ will be $\rho\,dx\,dy\,dz$, if ρ represents the density of the fluid at x, y, z at the present instant t ; one has written ρ for $\rho\,(x, y, z, t)$. The element $dx\,dy\,dz$ displaces and deforms during the time interval that is found between t and $t + \delta t$. That displacement will be defined completely by the equations:

$$\frac{\delta x}{u} = \frac{\delta y}{v} = \frac{\delta z}{w} = \delta t \tag{46}$$

when u, v, w are functions of x, y, z, and t that are supposed to be known. From the assumed hypotheses, the matter $\rho\,dx\,dy\,dz$ that is contained in the element $dx\,dy\,dz$ will neither decrease nor increase during the motion (46); in other words, one assumes that:

$$\delta\,[\rho\,dx\,dy\,dz] \equiv 0\ ,$$

in which δ is a variation that is defined by equations (46).

The quantity in brackets must be replaced with:

$$\rho\,\frac{\partial(x, y, z)}{\partial(\lambda_1, \lambda_2, \lambda_3)}$$

in order for the limits or boundaries to be fixed (no. **3**).

Since the variation of that quantity is zero, it will be likewise obvious that an infinitude of fluid elements $dx\,dy\,dz$ will collectively define a three-dimensional manifold of arbitrary form.

(*) M. BRILLOUIN, *Recherches récentes sur diverses questions d'Hydrodynamique*, Paris, Gauthier-Villars, 1891, page 10.

Therefore:

$$I_3 = \iiint \rho \, dx \, dy \, dz$$

will be an integral invariant of order three of equations (46).

The function ρ will then be a multiplier for equations (46). Hence (no. **18**):

$$\frac{\partial \rho}{\partial t} + \sum \frac{\partial \rho u}{\partial x} = 0 ,$$

which shows that the preceding hypotheses amount to establishing a necessary and sufficient condition between the motion and the density of the fluid. If the functions u, v, w do not include t explicitly (permanent regime) then $\frac{\partial \rho}{\partial t}$, $\frac{\partial^2 \rho}{\partial t^2}$, etc., will also be multipliers. One will then deduce some integrals of equations (46).

One saw (no. **29**, *cont.*) that the necessary and sufficient conditions for (46) to admit the relative invariant:

$$J_1 = \int u \, dx + v \, dy + w \, dz$$

are:

(47) $$\frac{\delta u}{\frac{\partial H}{\partial x}} = \frac{\delta v}{\frac{\partial H}{\partial y}} = \frac{\delta w}{\frac{\partial H}{\partial z}} = \delta t ,$$

in which H is a function of x, y, z, t . *Assume* that the motion of an arbitrary point of the fluid considered satisfies the six equations (46) and (47), so J_1 will be a relative invariant of those six equations. The latter supposition does not contradict the preceding hypotheses because that fluid considered is devoid of internal friction that might be produced by diffusibility. In addition, we have seen (no. **12**) that J_1 can never be an absolute invariant. I say that J_1 can sometimes be a covariant of degree one, and indeed calculate δJ_1 .

$$\frac{\delta}{\delta t}(u \, dx + v \, dy + w \, dz) = d\left(H + \frac{u^2 + v^2 + w^2}{2} \right) .$$

Set $\mathsf{H} = H - \frac{u^2 + v^2 + w^2}{2}$.

Equations (46) and (47) become:

(48) $$\frac{\delta x}{-\frac{\partial \mathsf{H}}{\partial u}} = \frac{\delta y}{-\frac{\partial \mathsf{H}}{\partial v}} = \frac{\delta z}{-\frac{\partial \mathsf{H}}{\partial w}} = \frac{\delta u}{\frac{\partial \mathsf{H}}{\partial x}} = \frac{\delta v}{\frac{\partial \mathsf{H}}{\partial y}} = \frac{\delta w}{\frac{\partial \mathsf{H}}{\partial z}} = \delta t ,$$

in which the variables u, v, w play the same role as x, y, and z.

Equations (48) are canonical, so if H does not include t explicitly then H or $H - \dfrac{u^2+v^2+w^2}{2}$ will be an integral. In that case, J_1 will be a covariant of degree one if $u^2+v^2+w^2$ is an integral; in other words, if:

$$u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial y}+w\frac{\partial H}{\partial z} \equiv 0 . \tag{49}$$

That is what happens, for example, when the fluid rotates with a uniform motion around a fixed axis in the manner of a solid body. Under that motion, the velocity V of a molecule remains constantly the same during its displacement; hence:

$$\delta V^2 \equiv 0 ,$$

or

$$\delta(u^2+v^2+w^2) \equiv 0 ,$$

in which the variation is taken in conformity with equations (46).

N.B. – By hypothesis, u, v, w satisfy the other three equations (47). The condition (49) expresses the idea that the velocity of each fluid point is normal to the total acceleration of that point. Indeed, equations (46) and (47) can be written:

$$\left\{\begin{aligned} \frac{\delta^2 x}{\delta t^2} &= \frac{\partial H}{\partial x}, \\ \frac{\delta^2 y}{\delta t^2} &= \frac{\partial H}{\partial y}, \\ \frac{\delta^2 z}{\delta t^2} &= \frac{\partial H}{\partial z}. \end{aligned}\right. \tag{50}$$

One will deduce the following absolute invariant from the relative invariant J_1 :

$$I_2 = \iint \xi\, dy\, dz + \eta\, dz\, dx + \zeta\, dx\, dy$$

if one sets:

$$\begin{aligned} \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z} &= 2\,\xi , \\ \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x} &= 2\,\eta , \\ \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} &= 2\,\zeta . \end{aligned}$$

The vector with the components (ξ, η, ζ) is what HELMHOLTZ called the *vorticity*. One also has that (ξ, η, ζ) are the components of the *rotational velocity* of the parallelepiped element around its

center of gravity (x, y, z). That vorticity or rotation will be zero if the functions u, v, and w are the partial derivatives with respect to x, y, and z, respectively, of the same function, which one calls the *velocity potential*.

One will have $\delta I_2 \equiv 0$. One calculates that variation by means of the formula in no. **24**:

$$\frac{\delta\xi}{\delta t} = -\,\xi\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\xi\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial y}+\zeta\frac{\partial u}{\partial z}\text{, etc.}$$

Particular case. – The function ρ is an integral of (46) (*incompressible fluid* or *perfect fluid*): $\delta\rho \equiv 0$, so $\delta[dx\,dy\,dz] \equiv 0$ or:

$$I_3 = \iiint dx\,dy\,dz \ ,$$

which means that an arbitrary fluid element can deform, but not change in volume (incompressible fluid) during the motion that is determined by (46). Equations (46) admit the multiplier 1 in that case; it is necessary and sufficient that one should have:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \equiv 0 \ .$$

Hence:

$$\frac{\delta\xi}{\delta t} = \xi\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial y}+\zeta\frac{\partial u}{\partial z}\text{, etc.}$$

or

$$\frac{\delta\xi}{\delta t} = \xi\frac{\partial u}{\partial x}+\eta\frac{\partial v}{\partial x}+\zeta\frac{\partial w}{\partial x}\text{, etc.}$$

One deduces from I_3 that:

$$J_2 = \int z\,dx\,dy + x\,dy\,dz + y\,dz\,dx \ .$$

Since the system (48) is canonical, it will admit the invariants:

$$I_2 = \int du\,dx + dv\,dy + dw\,dz \ ,$$
$$I_6 = \int dx\,dy\,dz\,du\,dv\,dw \ .$$

In the case where H is homogeneous and of degree p in x, y, z, one will have (no. **44**):

$$C_{1,1} = \int \sum (2x\,du - p\,u\,dx) \ ,$$
$$I_1 = \int \sum (2x\,du - p\,u\,dx) + (p-2)\,t\int d\mathrm{H} \ .$$

48. Case of a permanent regime. – Suppose that the functions u, v, w that enter into equations (46) do not include t explicitly. One then supposes that when one is given the coordinates x, y, z of a fluid point, one knows the velocity (u, v, w) of that point for any value of t. In other words, all

of the elements that pass through (x, y, z) *in succession* will have *the same* velocity. That is what the permanent regime consists of. In that case, one will then have:

$$\frac{\partial u}{\partial t} = 0 , \qquad \text{etc.}$$

Therefore, (u, v, w) will be a solution by variations to equations (46).

Recall that $J_1 = \int u\,dx + v\,dy + w\,dz$.

We know that $J_1 = I_1 + E_1$ (no. **29**).

When we preserve the notation of that no., we will have:

$$R = H + \frac{u^2 + v^2 + w^2}{2} ,$$

$$E_1 = \int dU ,$$

$$\frac{\delta U}{\delta t} \quad \text{or} \quad \frac{\partial U}{\partial x} u + \frac{\partial U}{\partial y} v + \frac{\partial U}{\partial z} w + \frac{\partial U}{\partial t} = R ,$$

$$I_1 = \int \sum \left(u - \frac{\partial U}{\partial x} \right) dx$$

$$= \int u'\,dx + v'\,dy + w'\,dz$$

upon setting:

$$u = \frac{\partial U}{\partial x} + u' ,$$
$$v = \frac{\partial U}{\partial y} + v' ,$$
$$w = \frac{\partial U}{\partial z} + w' .$$

That amounts to decomposing the velocity V of an arbitrary element of the fluid into two other velocities, one of which derives from a velocity potential and is consequently incapable of producing vortices. One deduces from I_1 (permanent regime) that:

$$\sum u'u = \text{constant}$$

or

$$VV' \cos(VV') = \text{constant.}$$

One then has the **theorem**:

During the permanent motion of a fluid, the geometric product of the velocity of an element with the component of that velocity that produces vortices will be constant. (For the other component of velocity, see U.)

That decomposition of the velocity (u, v, w) can be accomplished in an infinitude of ways because U is determined by only its variation, so one can add an integral of (46) to U.

The preceding theorem can be interpreted geometrically by considering the *trajectory* that is described by one of the fluid elements.

One deduces from:

$$I_2 = \int \sum \xi \, dy \, dz$$

that:

$$I_1 = \int \sum (\eta \, w - v \, \zeta) \, dx \ .$$

One can deduce a second-order invariant from I_1 . When it is combined with I_1, that will give a multiplier of equations (46). Similarly, one can derive a second-order invariant from $I_3 = \int \rho \, dx \, dy \, dz$.

49. – The lines that have the vector that represents the velocity as their tangent at each point are called *streamlines*. Their differential equations are:

$$\frac{Dx}{u} = \frac{Dy}{v} = \frac{Dz}{w} .$$

We shall not write $\delta x / u = \ldots$ since *t is supposed to be constant*. We shall not write $dx / u = \ldots$ since we would like to preserve the symbol d for the differentials that are not subject to the differential equations. The streamlines are identical to the trajectories that are described by the fluid molecules when u, v, w do not include t explicitly.

The lines that have the vector that represents the vorticity as their tangent at each point are the *vortex lines*. Their differential equations are:

$$\frac{Dx}{\xi} = \frac{Dy}{\eta} = \frac{Dz}{\zeta} \qquad (t \text{ constant}).$$

A *vortex surface* is a surface such that the tangent plane passes through the vortex at each of its points. Therefore, if (l, m, n) are the direction cosines of the normal at the point (x, y, z) of that surface then one will have:

$$l \, \xi + m \, \eta + n \, \zeta = 0 \ .$$

Therefore:

$$I_2 = \int \xi \, dy \, dz + \eta \, dz \, dx + \zeta \, dx \, dy \ ,$$

or

$$\int d\omega \, (l \, \xi + m \eta + n \zeta)$$

will be identically zero when extended over that surface. One will also have:

$$I_2 = J_1 = \int_C u\,dx + v\,dy + w\,dz = 0\,,$$

in which J_1 is a curvilinear integral that is extended over the closed curve C that is traced on the vortex surface, such that the region that it bounds is supposed to be simply-connected, i.e., there are no holes in it. Conversely, if a surface is such that any J_1 that is extended over an entire closed curve that is traced on it and bounds a simply-connected region is zero then that surface will be a vortex surface.

Theorem:

In order for the vortex surface to be conserved in time it is sufficient that:

$$\frac{\delta\xi}{\delta t} + \xi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \xi\frac{\partial u}{\partial x} - \eta\frac{\partial u}{\partial y} - \zeta\frac{\partial u}{\partial z} \equiv 0\,, \quad \text{etc.}$$

Indeed, $I_2 = \iint \sum \xi\,dy\,dz$ will be an integral invariant. If the integral $I_2 \equiv 0$ at the instant t then that integral will still be $\equiv 0$ at the instant $t + \delta t$.

Those conditions are not necessary, because it is not necessary that $I_2 = \iint \sum \xi\,dy\,dz$ should be an invariant. The element $d\omega$, whose projections are $dx\,dy$, $dy\,dz$, $dz\,dx$, *is no longer arbitrary.* Let us look for the *necessary* conditions. In order to do that, we remark that I_2 is provided by:

$$J_1 = \int \sum u\,dx + \int d\psi\,,$$

in which ψ is an arbitrary uniform function of x, y, z, and t. J_1 is an integral that is taken along a *closed curve that is traced on a vortex surface* at the instant t :

$$\frac{\delta J_1}{\delta t} = \int \sum \frac{\delta u}{\delta t}dx + \int d\left(\sum \frac{u^2}{2}\right) + \int \frac{\delta}{\delta t}\left(\sum \frac{\partial \psi}{\partial x}\,dx\right)$$

$$= \int \sum \left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w\right)dx + \int \frac{\delta}{\delta t}d\psi\,.$$

Let us transform those curvilinear integrals into surface integrals. The latter will give an integral that is identically zero, because:

$$\int \frac{\delta}{\delta t}d\psi = \frac{\delta}{\delta t}\iint \sum \left(\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial^2\psi}{\partial y\,\partial x}\right)dx\,dy \equiv 0\,,$$

$$\frac{\delta J_1}{\delta t} = \iint d\omega \left[l\left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) + m\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) + n\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)\right] .$$

One has set:

$$A = \frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v + \frac{\partial u}{\partial z} w \ ,$$

$$B = \frac{\partial v}{\partial x} u + \frac{\partial v}{\partial y} v + \frac{\partial v}{\partial z} w \ ,$$

$$C = \frac{\partial w}{\partial x} u + \frac{\partial w}{\partial y} v + \frac{\partial w}{\partial z} w \ .$$

(l, m, n) are again the direction cosines of the normal at (x, y, z) to the vortex surface considered at the instant t. When quantity in brackets is equal to zero, that will give the necessary and sufficient condition for the vortex surface to be preserved.

Corollary.

When the vortex surfaces are preserved in time, the same thing will be true for the vortex lines. However, the converse is not true.

Indeed, the intersection of two vortex surfaces is obviously a vortex line. Let vortex surfaces S and S' pass a vortex line L (which is always possible).

When the time t has been subjected to the variation δt, the surfaces S and S' will occupy new positions, but they will still be vortex surfaces, by virtue of the preceding theorem. Their intersection will be a vortex line. However, it is nothing but L at the instant $t + \delta t$, therefore, etc.

Here is another proof of the conservation of vortex lines for the case in which $\iint \sum \xi \, dy \, dz$ is an integral invariant.

Set:

$$\frac{Dx}{\xi} = \frac{Dy}{\eta} = \frac{Dz}{\zeta} = \varepsilon$$

at the instant t . I say that the ratios Dx / ξ , Dy / η , Dz / ζ will again be equal to each other at the instant $t + \delta t$. In order to show that, calculate the variations of Dx / ξ , Dy / η , Dz / ζ , and show that those variations are equal to each other.

$$D\frac{\delta x}{\delta t} = Du = \varepsilon\left(\frac{\partial u}{\partial x}\xi + \frac{\partial u}{\partial y}\eta + \frac{\partial u}{\partial z}\zeta\right) ,$$

$$\frac{\delta}{\delta t} Dx = \frac{\delta}{\delta t}\xi\varepsilon = \varepsilon\left[-\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\xi + \frac{\partial u}{\partial y}\eta + \frac{\partial u}{\partial z}\zeta\right] + \varepsilon\frac{\delta\varepsilon}{\delta t} \ .$$

Since $D\frac{\delta x}{\delta t} = \frac{\delta}{\delta t}Dx$, one will have:

$$\varepsilon\left(\frac{\partial u}{\partial x}\xi + \frac{\partial u}{\partial y}\eta + \frac{\partial u}{\partial z}\zeta\right) = \frac{\delta\varepsilon}{\delta t}. \qquad \text{Q.E.D.}$$

Set:

$$\Delta = \frac{\partial(x, y, z)}{\partial(\lambda, \mu, \nu)}.$$

We know that (no. **18**):

$$\frac{\delta\Delta}{\delta t} = \Delta\sum\frac{\partial u}{\partial x}.$$

Thus:

$$\frac{\delta\varepsilon}{\varepsilon} = \frac{\delta\Delta}{\Delta},$$

or

$$\varepsilon = \Delta\,\Phi,$$

in which Φ represents an integral of equations (46).

Set:

$$\Phi \equiv \psi(\Delta\,\rho)^{-1},$$

in which ψ is an integral of equations (46). We know that $\delta(\Delta\,\rho) \equiv 0$. Thus:

$$\varepsilon = \frac{\psi}{\rho} = \frac{Dx}{\xi} = \frac{Dy}{\eta} = \frac{Dz}{\zeta}.$$

Theorem:

If $I_2 = \iint\sum\xi\,dy\,dz$ *is an integral invariant then the ratios* Dx/ξ, Dy/η, Dz/ζ *will submit to variations that are equal to each other and proportional to the variation of the specific volume.*

A *vortex tube* is a vortex surface that has a particular form. It is generated by the vortex lines that are drawn through the points of a *closed* curve.

Let C be a closed curve that is traced on the tube T and does not surround it. Therefore, C will bound a well-defined simply-connected region of the tube T. One will have $\int_C\sum u\,dx \equiv 0$, because that integral is equal to a surface integral that is extended over a simply-connected vortex surface.

Let C' be a closed curve that encircles the tube T. One will have:

$$\int_{C'}\sum u\,dx = \iint\sum\xi\,dy\,dz,$$

in which the latter integral is extended over a surface that is arbitrary, but simply connected and bounded by C'. Therefore, $\int_C \sum u\,dx$ will be zero only in exceptional cases; its variation is zero. Let C'' be another curve that encircles the tube *T*. One will have:

$$\int_{C'} \sum u\,dx = \int_{C''} \sum u\,dx \ .$$

Indeed, the latter integral will equal $\iint \sum \xi\,dy\,dz$ when it is extended over, e.g., the region of the tube *T* that is comprised of C', C'', and a simply-connected surface that is bounded by C'. The value of $\int_{C'} \sum u\,dx$ that relates to a vortex tube is called the *moment* of that tube. In the fluids that we have considered up to now, the tubes and their moments are *conserved* in time.

Let μ be the moment of an *infinitely-thin* tube. One will have:

$$\mu = \int_{C'} \sum u\,dx = 2\ \tau_n\,d\omega = 2\ \tau\,d\omega_n\ ,$$

in which τ_n represents the component of the vorticity τ or $(\xi,\ \eta,\ \zeta)$ that is taken normal to the surface element $d\omega$ (which is bounded by C'). $d\omega_n$ is the *cross* section of the tube.

50. Rectilinear vortices in a liquid. – In the theory of vortices, one encounters several integrals that present a strong analogy with the integral invariants and covariants. I shall give two examples:

Consider an indefinite liquid (or incompressible fluid) in which the velocity is parallel to the *xy*-plane and depends upon only *x* and *y*. The velocity is supposed to be zero at infinity, and is never infinite, by hypothesis. One will then have:

$$w = 0\ , \qquad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0\ ,$$

so

$$\xi = 0\ , \qquad \eta = 0\ , \qquad 2\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\ ,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{and} \quad \delta\zeta = 0 \quad \text{and} \quad I_2 = \iint dy\,dz\ .$$

Consider a center of gravity whose coordinates are:

$$x_0 = \frac{\iint x\,\zeta\,dy\,dz}{\iint \zeta\,dy\,dz}\ ,$$

$$y_0 = \frac{\iint y\,\zeta\,dy\,dz}{\iint \zeta\,dy\,dz}\,,$$

in which the integrals are not just extended over cross-sections of the vortices, but over all of the *xy*-plane. That will be permissible if we suppose that there are no vortices outside of those sections. In this case, the center (x_0, y_0) is fixed, in other words:

$$\delta \iint x\,\zeta\,dy\,dx \equiv 0\,,$$
$$\delta \iint y\,\zeta\,dx\,dy \equiv 0\,.$$

In order to show that, one replaces those double integrals with integrals that are extended over a closed curve that encircle the *xy*-plane. For example:

$$\iint u\,\zeta\,dx\,dy = \int (u^2 - v^2)\,dx + 2\,u\,v\,dy\,.$$

The latter integral is zero because the element is a second-order infinitesimal, while the curve along which one integrates is a first-order infinitesimal. (*Théorie des Tourbillons* by H. POINCARÉ).

One likewise proves that:

$$\iint (x^2 + y^2)\,\zeta\,dx\,dy \text{ is constant}$$

or that

$$\delta \iint (x^2 + y^2)\,\zeta\,dx\,dy \equiv 0$$

if the velocity at infinity is such that $u^2 x$, $u^2 y$, $v^2 x$, $v^2 y$, $u\ v\ x$, $u\ v\ y$ are second-order infinitesimals, because one will then have:

$$\int (u^2 - v^2)(x\,dx - y\,dy) + 2\,u\,v\,(y\,dx + x\,dy) \equiv 0\,.$$

One transforms that curvilinear integral into a double integral, and after some reductions, one will get the surface integral:

$$\iint (x\,u + y\,v)\,\zeta\,dx\,dy\,. \qquad \text{Q.E.D.}$$