# MODERN METHODS

**IN THE**

# RESISTANCE OF MATERIALS

**BY**

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PROFESSOR AT L'ÉCOLE CENTRALE DES ARTS ET MANUFACTURES

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# FOREWORD

**1. –** The goal of the modern methods of the resistance of materials is to answer two of the most important questions of that branch of applied mechanics: The determination of the elastic and caloric deformations, and the calculation of the constraint forces in the pieces and hyperstatic systems of pieces.

These methods, which are based upon the theorems of *vis viva* and virtual work, are entirely general. They apply uniformly to all cases that might present themselves. They offer the advantages over the old geometric and kinematic methods of being a more rapid application and that they do not necessitate the introduction of auxiliary unknowns into the calculations whose elimination is often laborious.

Meanwhile, they are not very widely known in France, where, among the numerous didactic works on the resistance of materials, there are, to our knowledge, only four that mention the modern methods with developments that are thorough to varying degrees.

The treatise on *Statique graphique* by Maurice Levy (Part IV, 1886, Note I) contains an outline of the method of General Menabrea and that of Mohr for calculating the efforts in articulated systems with redundant bars.

The French translation (1901) by Hahn of Föppl's *Résistance des matériaux* presents Castigliano's general method for determining elastic displacements and forces of redundant constraints in the pieces and systems of pieces with mean fibers.

The book by Ernst Flamard that is entitled *Calcul des systèmes élastiques de la Construction* (1918), reviews Castigliano's method and makes numerous and varied applications to straight beams, arches, and articulated systems.

Finally, the *Cours de Mécanique professé à l'École Polytechnique* by Léon Lecornu (t. III, 1918) gives the general equation for the elasticity of constructions, as well as Castigliano's method, with applications to various hyperstatic systems.

Furthermore, it seems useful to us to present a discussion of the diverse collection of modern methods here, which would be a discussion that we are forced to make as simple as possible while striving to highlight the close links that unite those methods. In the name of this second order of business, we shall show that taking caloric deformations into account, which has been accomplished only by means of the theorem of virtual work up to now, can also be achieved by applying the *vis viva* theorem. We shall likewise establish that the latter theorem can permit one to prove the beautiful theorem of Betti, Boussinesq, and Maurice Levy just as well, and more simply.

It seems interesting to me to appeal to the mathematical theory of elasticity in order to shed some light upon the path along which we believe that the research of scholars in regard to that theory must be directed, in view of permitting the extension of the modern methods to system of isotropic bodies and to thus liberate the calculation of constructions from the hypotheses of the resistance of materials. However, one cannot hide the fact that such research is extremely tedious, due to some difficulties that are presented by the integration of the partial differential equations of the mathematical theory of elasticity.

The methods that are based upon the *vis viva* theorem have their origins in the paper by Clapeyron on the work done by elastic forces (1858) ([1]) and in that of General Menabrea that was entitled "Principe général pour déterminer les pressions et les tensions dans un système élastique," (1868) ([2]). The first application of the theorem of virtual work to the study of elastic deformations was made by Mohr in the context of articulated systems with the title "Beitrag zur Theorie des Bogenfachwerksträger (1874) ([3]).

Since then, these methods have been the goal of the research of numerous scholars and engineers. One will find a very complete history of that, accompanied by detailed bibliographic references, in the doctoral thesis that Ernest Flamard presented to the Science Faculty at Nancy in 1914 ([4]).

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([1]) Comptes rendus de l'Académie des Sciences, t. XLVI, pp. 208. – *See* also LAMÉ, *Leçons sur la Théorie mathématique de l'Élasticité des corps solides* (1866): Théorème de Clapeyron, pp. 80.

([2]) *See* also a note that General Menabrea read at the session of the Academy of Sciences on 31 May 1858 (Comptes rendus, t. XLVI, pp. 1056).

([3]) Zeitschrift der Architekten- und Ingenieur-Vereins zu Hannover (1874), pp. 223.

([4]) Ernest FLAMARD, Inspector of metallic constructions for the railroad company of Orléans, *Étude sur les Méthodes nouvelles de la Statique des constructions.*

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## CHAPTER I

# REVIEW OF SOME NOTIONS FROM THE MATHEMATICAL THEORY OF ELASTICITY AND THE RESISTANCE OF MATERIALS

### 2. – Elastic forces in the mathematical theory of elasticity

Consider an elastic solid body that is in equilibrium under the action of a system of external forces.

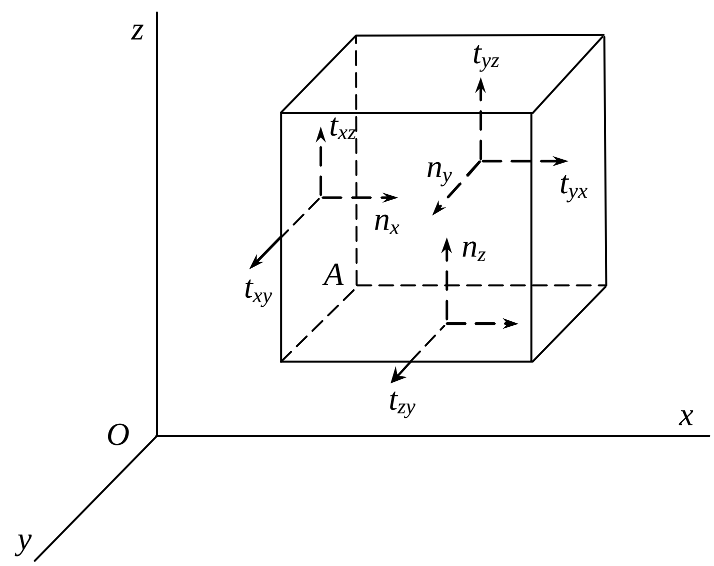


Figure 1.

Let (Fig. 1):

*A* be an arbitrary point of the body

$x$, $y$, $z$ be the rectangular coordinates of that point before deforming the body

$dx$, $dy$, $dz$ be the lengths of the edges (measured before deformation) of an infinitely-small parallelepiped that is taken in the body with *A* for one of its summits.

$n_x$ , $t_{xy}$ , $t_{xz}$ , $t_{yx}$ , $n_y$ , $t_{yz}$ , $t_{zx}$ , $t_{zy}$ , $n_z$

be the components of the elastic forces per unit area that are parallel to the axes on the three faces of the parallelepiped that have the point *A* for their common summit and are normal to *Ox*, *Oy*, *Oz*, respectively.

In order to specify the signs of these components, we consider the elastic forces to be the actions that are exerted by the parts of the body that are situated *outside* the parallelepiped upon the ones that are situated *inside* the body, and we agree to measure them as positive when they have the positive sense of the axes.

Following the convention that has been adopted by engineers, we say that $n$ and $t$ are the *normal fatigue* and *tangential fatigue,* respectively, on three planar elements that are mutually rectangular before deformation and drawn through the point $A$. From that convention, an element will be subject to compression or traction according to whether the normal fatigue on that element is positive or negative, resp.

One knows that the tangential fatigues on two mutually-rectangular elements, which are directed normally to the intersections of those elements, are equal to each other; i.e., one has:

$$t_{zy} = t_{yz}\,, \qquad t_{xz} = t_{zx}\,, \qquad t_{yx} = t_{xy}\,,$$

which will reduce the number of unknown fatigues on the three rectangular elements considered from nine to six.

### 3. – Elastic deformation parameters of an isotropic body

Letting $u$, $v$, $w$ be the components parallel to the axes of the elastic displacement of the point $A$ ($x$, $y$, $z$). The six quantities:

$$\varepsilon_x = -\frac{\partial u}{\partial x}\,, \qquad \varepsilon_y = -\frac{\partial v}{\partial y}\,, \qquad \varepsilon_z = -\frac{\partial w}{\partial z}\,,$$

$$\gamma_{yz} = -\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \qquad \gamma_{yx} = -\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \qquad \gamma_{xy} = -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

define the elastic deformation of the parallelepiped. We shall call them the *elastic deformation parameters.*

$\varepsilon_x$ , $\varepsilon_y$ , $\varepsilon_z$ are the contractions that are felt by the edges $dx$, $dy$, $dz$, resp., of the parallelepiped, when referred to the initial lengths of those edges. According to whether the value of $\varepsilon$ that corresponds to an edge is positive or negative, there will actually be a contraction or elongation, resp., of that edge.

$\gamma_{yz}$ , $\gamma_{zx}$ , $\gamma_{xy}$ are the increases that are felt by the right angles that the edges $dy$ and $dz$, $dz$ and $dx$, $dx$ and $dy$, resp., defined with each other before deformation. According to whether the value of $\gamma$ that corresponds to one of those angles is positive or negative, there will actually be an increase or decrease, resp., in the angle considered. The $\gamma$ bear the name of *shears* or *distortions.*

The six normal and tangential fatigues are expressed as functions of the six elastic deformation parameters by way of the formulas:

(1)
$$\begin{cases} n_x = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2\mu\varepsilon_x, & t_{yz} = \mu\gamma_{yz}, \\ n_y = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2\mu\varepsilon_y, & t_{zx} = \mu\gamma_{zx}, \\ n_z = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2\mu\varepsilon_z, & t_{xy} = \mu\gamma_{xy}, \end{cases}$$

in which $\lambda$ and $\mu$ denote two physical constants of the body considered. Those constants are linked with the longitudinal and transverse elastic moduli that are considered in the resistance of materials by the relations:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad G = \mu.$$

Conversely, the expressions for the parameters of the elastic deformation as functions of the normal and tangential fatigues are:

(2)
$$\begin{cases} \varepsilon_x = \dfrac{1}{2\mu(3\lambda + 2\mu)}[2(\lambda + \mu)\,n_x - \lambda(n_y + n_z)], & \gamma_{yz} = \dfrac{t_{yz}}{\mu}, \\ \varepsilon_y = \dfrac{1}{2\mu(3\lambda + 2\mu)}[2(\lambda + \mu)\,n_y - \lambda(n_z + n_x)], & \gamma_{zx} = \dfrac{t_{zx}}{\mu}, \\ \varepsilon_z = \dfrac{1}{2\mu(3\lambda + 2\mu)}[2(\lambda + \mu)\,n_z - \lambda(n_x + n_y)], & \gamma_{xy} = \dfrac{t_{xy}}{\mu}. \end{cases}$$

### 4. – Work done by elastic forces. Internal potential of an isotropic body

By means of the expressions above for normal and tangential fatigues and the elastic deformation parameters, one easily calculates the work done by the elastic forces that act on the six faces of the parallelepiped while it passes from its natural state to a state of deformation that is defined by either the values of the fatigues or those of the parameters. Let $\varpi$ be the quotient of that infinitely-small work by the volume $dx\,dy\,dz$ of the parallelepiped; that will be the work done by elastic forces per unit volume at the point $A(x, y, z)$. One will find that:

(3)
$$\varpi = \frac{1}{2\mu(3\lambda + 2\mu)}[(\lambda + \mu)(n_x^2 + n_y^2 + n_z^2) - \lambda(n_x\,n_y + n_y\,n_z + n_z\,n_x)] + \frac{1}{2\mu}(t_{yz}^2 + t_{zx}^2 + t_{xy}^2),$$

(4)
$$\varpi = \frac{\lambda}{2}(\varepsilon_x + \varepsilon_y + \varepsilon_z) + \mu(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + \frac{\mu}{2}(\gamma_{yz}^2 + \gamma_{zx}^2 + \gamma_{xy}^2).$$

The work done by elastic forces for the parallelepiped considered is $\varpi\,dx\,dy\,dz$. If one then decomposes the body into an infinitude of elementary parallelepipeds by means of planes that are normal to the coordinate axes then one will see that the total work Π that is done by the elastic forces for the entire body is:

$$\Pi = \iiint \varpi\, dx\, dy\, dz\,, \tag{5}$$

where the triple integral is taken over the entire volume that is occupied by the body.

It is clear that this expression for the total force done by elastic forces likewise applies to a system of bodies that are coupled to each other in an arbitrary manner and deformed by external forces in equilibrium when the integral is taken over the total volume that is occupied by the system of bodies.

One will immediately verify that if one derives the expression (4) for $\varpi$ with respect to $\varepsilon_x$ , …, $\gamma_{xy}$ , in succession, then one will get back to the expressions (1) for $n_x$ …, $t_{xy}$ , in such a way:

$$\frac{\partial \varpi}{\partial \varepsilon_x} = n_x\,, \qquad \ldots, \qquad \frac{\partial \varpi}{\partial \gamma_{xy}} = t_{xy}\,. \tag{6}$$

The partial derivatives of the function $\varpi$ with respect to the six elastic deformation parameters of the parallelepiped are then equal to the normal and tangential fatigues at the point $A$ ($x$, $y$, $z$). One then concludes that the elastic forces that are applied to the faces of the parallelepiped depend upon a force function and that this function is $\varpi\, dx\, dy\, dz$. As a result, the elastic forces depend upon the force function Π for the entire body (or for a system of bodies.

The elastic deformation of an arbitrary parallelepiped gives rise to molecular forces inside of it. Those forces are unknown, but it is easy to evaluate the work that they do during deformation. Indeed, the system of material points that this parallelepiped is composed of is at rest before and after the deformation, so the algebraic sum of the works done by external and internal forces that act upon that system will be zero, by virtue of the *vis viva* theorem. As a result, the work done by molecular forces will be equal and opposite to the work done by elastic forces. Its value will then be $-\varpi\, dx\, dy\, dz$ for the parallelepiped considered and – Π for the entire body (or system of bodies). Consequently, as far as the evaluation of the work done by molecular forces is concerned, everything happens as if it depended upon the force function – Π, which amounts to the same thing as saying that the molecular forces are derived from a potential Π. As a result, we say that the function Π is the molecular force potential or the *internal potential* of the deformed body (or system of bodies), to abbreviate, provided that if changing the sign of that function indeed represents the work done by molecular forces during the deformation then it cannot be used to calculate the values of the forces that remain unknown.

It results from the preceding that the molecular forces form a *conservative system*. The work that they do depends upon only the final state of deformation and not upon the intermediate states. It is exclusively a function of the final values of the deformation parameters or the final values of the fatigues.

The function Π admits another interpretation, which is:

Formula (4) shows that $\varpi$ is essentially positive and that it will be annulled when the body is in its natural state; the same thing is true for the total potential $\Pi$. The latter is then a minimum when the body is in its natural state, which proves (and this should be obvious, moreover) that this state is the state of stable equilibrium of the molecules of the body. If one then imagines that the deformed body returns to its natural state as a result of suppressing the external forces that determined its deformation then the potential of its molecular forces will pass from the value $\Pi$ to the value zero. It will then do positive work equal to $\Pi$ when the molecules of the body return to their stable equilibrium position. Now, as one knows ([1]), the potential energy of a material system that occupies an arbitrary position is equal to the essentially-positive work done by the internal forces of that system when it passes from that position to its stable equilibrium. It is then the internal potential energy of the deformed body (or system of bodies).

## 5. – Deformation parameters for an isotropic body when it is both elastic and caloric

Let a free body be subject to the action of an arbitrary system of external forces in equilibrium and suppose that its temperature goes up by $\tau$ degrees. Let $\alpha$ be its coefficient of linear dilatation. The deformation that an arbitrary parallelepiped in the body submits to can be considered to be the resultant of the superposition of the purely elastic deformation that is due to external forces and the caloric dilatation. Now, that dilatation does not modify the angles that the edges of the parallelepiped define with each other, which simply submit to elongations that are equal to $\alpha\tau$ per unit length. As a result, $\varepsilon_x$ , …, $\gamma_{xy}$ denote the six purely-elastic deformation parameters, as before, while the elastic and caloric deformation parameters are:

$$\varepsilon_x - \alpha\tau,\ \varepsilon_y - \alpha\tau,\ \varepsilon_z - \alpha\tau, \qquad \gamma_{xy}\ ,\ \gamma_{xy}\ ,\ \gamma_{xy}\ .$$

These expressions remain valid in the case of a decrease in temperature, with the condition that $\tau$ must be counted negatively in that case.

They are likewise valid for the bodies that are not free, since those bodies can be considered to be free under the action of forces that are applied to them directly and corresponding constraint forces.

We remark that the internal potential obviously depends upon the elastic deformation. As a result, if the deformation is both elastic and caloric then the expressions for the internal potential will remain the ones that were given in no. **4**.

## 6. – Remark

Everything that was just said about isotropic bodies applies immediately to the pieces with mean fibers that are considered in the resistance of materials if one adopts the very clear and precise viewpoint of General Menabrea and considers those pieces to be semi-rigid bodies. That viewpoint permits one to pass from the formulas of the mathematical

(1) Maurice LEVY, *Sur le principe de l'énergie*, 1888, pp. 9.

theory of elasticity to the corresponding ones in the resistance of materials, but it is simpler to establish the latter ones directly.

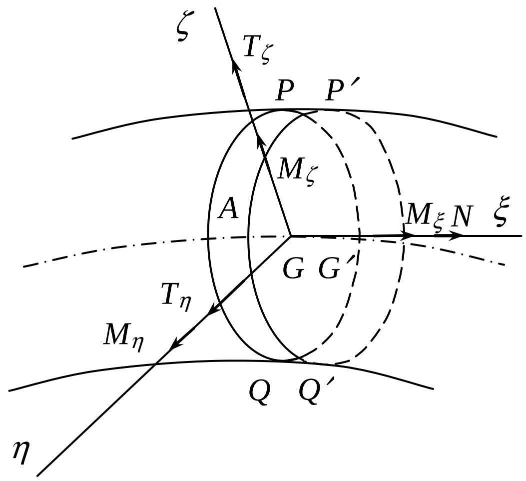


Figure 2.

**7. – Elastic forces in the resistance of materials. Elements of their reduction.**

Suppose that one has (*see* Fig. 2) a body with planar or skew mean fiber, and that:

| | |
|---|---|
| $G$ and $G'$ | are their centers of gravity, with curvilinear abscissas $s$ and $s + ds$, when measured from the origin that is taken on the mean fiber to the left of $G$ |
| $G\xi$ | is the positive tangent to the mean fiber at $G$ |
| $G\eta$ and $G\zeta$ | are the principal axes of inertia of the section $PQ$ |
| $n$, $t_\eta$ , $t_\zeta$ | are the components parallel to the axes $G\xi$, $G\eta$, $G\zeta$ of the elastic force per unit area at an arbitrary point $A$ ($\eta$, $\zeta$) of the section $PQ$. |

In order to specify the signs of those components, we shall consider elastic forces in the section $PQ$ to be the actions that are exerted by the part of the body that is situated to the *left* of that section on the part that is situated to the *right*, and we agree to count them as positive when they have the sense of the positive coordinate axes.

$n$ is the normal fatigue at the point $A$, and $t_\eta$ and $t_\zeta$ are the tangential fatigues. From the convention that was just made, there will be compression or traction at the point $A$ according to whether $n$ is positive or negative, resp.

The elastic forces in the section $PQ$ define a system that is equivalent to the external forces that are applied to the left of that section. At the center of gravity of the section $PQ$, one of those two systems, like the other, will be reducible to:

The *normal effort* $N$, which is directed along $G\xi$

The *shearing efforts* $T_\eta$ and $T_\zeta$, which are directed along $G\eta$ and $G\zeta$

The *torsion couple* $M_\xi$ along the axis $G\xi$

The *flexure couples* $M_\eta$ and $M_\zeta$ along the axes $G\eta$ and $G\zeta$, resp.

The couples $M_\xi$, $M_\eta$, $M_\zeta$ will be considered to be positive when they tend to turn their lever-arms in the sense of rotation that one must impose upon the positive parts $G\eta$, $G\zeta$ and $G\xi$, resp., of the three coordinate axes in order to bring them into coincidence with $G\zeta$, $G\xi$, and $G\eta$, resp.

### 8. – Elastic deformation parameters of a body with a mean fiber

Under elastic deformation, the section $PQ$ will tend to the infinitely-close section $P'Q'$ by an infinitely-small relative displacement that is equivalent to a translation that is equal to the relative displacement of its center of gravity $G$ and a rotation around an axis that passes through the center of gravity.

Let:

1. $$\varepsilon\, ds, \quad \gamma_\eta\, ds, \quad \gamma_\zeta\, ds$$

be the components of the translation along the three coordinate axes:

$$G\xi, \quad G\eta, \quad G\zeta,$$

which are components that are counted as positive when they have the sense of the positive axes.

Let:

2. $$\theta_\xi\, ds, \quad \theta_\eta\, ds, \quad \theta_\zeta\, ds$$

be the components of the rotation around those same axes when counted as positive or negative under the same condition as the torsion and flexure couples.

The six quantities $\varepsilon$, $\gamma_\eta$, $\gamma_\zeta$, $\theta_\xi$, $\theta_\eta$, $\theta_\zeta$ define the deformation of the *slice* of the body that is found between the two sections $PQ$ and $P'Q'$. They are the *elastic deformation parameters* of that slice:

$\varepsilon$ is the *shortening* per unit length – or *unit shortening* – of the element of the mean fiber $GG' = ds$. Depending upon whether it is positive or negative, there will be true shortening or, in the contrary case, elongation, resp.

$\gamma_\eta$, $\gamma_\zeta$ are the *unit shears* in the section $PQ$ along the two directions $G\eta$ and $G\zeta$

$\theta_\xi$ is the *unit angle of torsion*

$\theta_\eta$ and $\theta_\zeta$ are the *unit angles of flexure* around $G\eta$ and $G\zeta$, resp.

Let:

$\Omega$ be the area of the section $PQ$

$I_\xi$ be its polar moment of inertia with respect to its center of gravity

$I_\eta$ and $I_\zeta$ be its moments of inertia with respect to $G\eta$ and $G\zeta$, resp. – i.e., its principal moments of inertia

$E$ and $G$ be the longitudinal and transverse elastic moduli

The six deformation parameters of the slice $PQP'Q'$ are coupled with the elements of the reduction of the elastic forces in the section $PQ$ (or the external forces that are applied to the left of that section) by the relations:

$$\left\{ \begin{array}{lll} N = E\,\Omega\varepsilon, & T_\eta = G\,\Omega\gamma_\eta, & T_\zeta = G\,\Omega\gamma_\zeta, \\ M_\xi = G\,I_\xi\theta_\xi, & M_\eta = G\,I_\eta\theta_\eta, & M_\zeta = G\,I_\zeta\theta_\zeta. \end{array} \right. \tag{7}$$

The normal and tangential fatigues at an arbitrary point $A$ $(\eta, \zeta)$ of the section $PQ$ are given by the formulas:

$$\left\{ \begin{array}{l} n = \dfrac{N}{\Omega} + \dfrac{\zeta M_\eta}{I_\eta} + \dfrac{\eta M_\zeta}{I_\zeta}, \\ t_\eta = \dfrac{T_\eta}{\Omega} - \dfrac{\zeta M_\xi}{I_\xi}, \\ t_\zeta = \dfrac{T_\zeta}{\Omega} + \dfrac{\zeta M_\xi}{I_\xi}. \end{array} \right. \tag{8}$$

## 9. – Work done by elastic forces and internal potential of a body with a mean fiber

When the slice $PQP'Q'$ passes from its natural state to an arbitrary state of deformation, the elastic forces that are applied to the two sections $PQ$ and $P'Q'$ that bound that slice accomplish a certain amount of work. Let $\varpi$ be the quotient of that infinitely-small work by the element $GG' = ds$ of the mean fiber that is found between those two sections. It is the work done by elastic forces at the point $G$ per unit length of the mean fiber.

$\varpi$ is expressed as a function of the elements of the reduction of the elastic forces that are developed in the section $PQ$ by the deformation by the formula:

(9)
$$\varpi = \frac{1}{2}\left(\frac{N^2}{E\Omega} + \frac{T_\eta^2}{G\Omega} + \frac{T_\zeta^2}{G\Omega} + \frac{M_\xi^2}{G I_\xi} + \frac{M_\eta^2}{G I_\eta} + \frac{M_\zeta^2}{G I_\zeta}\right),$$

and as a function of the elastic deformation parameters of the slice by the formula:

(10)
$$\varpi = \tfrac{1}{2}(E\,\Omega\,\varepsilon^2 + G\,\Omega\,\gamma_\eta^2 + G\,\Omega\,\gamma_\zeta^2 + G\,I_\xi\theta_\xi + E\,I_\eta\theta_\eta + E\,I_\zeta\theta_\zeta)\,.$$

The work done by the elastic forces for the slice $PQP'Q'$ is $\varpi\,ds$, and as a result, it will be:

(11)
$$\Pi = \int \varpi\,ds$$

for the entire body, in which the integral is taken over the total length of the mean fiber. That formula is obviously applicable to a system of bodies that are coupled to each other in an arbitrary manner on the condition that one agrees that the integral is extended along the total lengths of the mean fibers of all the bodies that comprise the system.

There is *planar flexure* when:

1. The mean fiber is a planar curve.

2. The body has a symmetric structure with respect to the plane of the mean fiber.

3. The external forces are applied in that plane.

In that case, if one supposes that $G\xi\zeta$ is the plane of the mean fiber then $T_\eta$, $M_\xi$, $M_\zeta$, $\gamma_\eta$, $\theta_\xi$, $\theta_\zeta$ will be zero, and as a result, one will have simply:

(9′)
$$\varpi = \frac{1}{2}\left(\frac{N^2}{E\Omega} + \frac{T^2}{G\Omega} + \frac{M^2}{E\,I}\right),$$

(10′)
$$\varpi = \tfrac{1}{2}(E\,\Omega\,\varepsilon^2 + G\,\Omega\,\gamma^2 + E\,I\,\theta^2),$$

when one suppresses the indices $\zeta$ and $\eta$ that the letters $T$, $M$, $I$, $\gamma$, and $\theta$ were endowed with in formulas (9) and (10).

The elastic forces depend upon a force function, just as in the mathematical theory of elasticity. That function will be $\varpi\,ds$ for an arbitrary slice $PQP'Q'$. That will result from the fact that if one forms the partial derivatives of the expression (10) for $\varpi$ with respect to the six deformation parameters then one will get back to the expressions (7) for the six elements of the reduction of the elastic force. The force function is $\Pi$ for the entire body (or for a system of bodies). As before (no. **4**), one can conclude from this that the function $\Pi$ is the molecular force potential or *internal potential of the deformed body* (or the system of bodies) and that it likewise represents the *potential energy* of that body or system of bodies.

## 10. – Deformation parameters of a body with a mean fiber when that deformation is both elastic and caloric

One will see in the same way as before (no. **5**) that the *elastic* and *caloric* deformation parameters of an arbitrary slice are:

$$\varepsilon - \alpha\tau, \qquad \gamma_\eta, \quad \gamma_\zeta, \quad \theta_\xi, \quad \theta_\eta, \quad \theta_\zeta,$$

in which $\varepsilon, \ldots, \theta_\zeta$ denote the purely elastic deformation parameters, $\alpha$ is the coefficient of linear dilatation, and $\tau$ is the variation of the temperature, which will be positive in the case of an increase and negative in the case of a decrease.

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## CHAPTER II

# FIRST METHOD BASED UPON THE *VIS VIVA* THEOREM

### 11. – Fundamental principle of the methods deduced from the *vis viva* theorem. Clapeyron's equation. Extension of that equation.

Consider an elastic body that is isotropic or has a mean fiber or a system such bodies that satisfies the following properties:

That body (or those systems) possesses a certain number of *simple supports* in the form of *balls* or *anchors*. It will then be restricted to those constraints that are called *support constraints* or *external constraints.* The bodies that comprise the system are coupled with each other in an arbitrary manner. They will then be restricted to mutual constraints, which are called *internal system constraints.*

The external constraint forces are ordinarily called *support reactions.*

One says that a body or system of them is *isostatic* or *hyperstatic* according to whether the constraint forces can or cannot be calculated by means of pure statics, resp.; i.e., by means of the six universal conditions of equilibrium.

A hyperstatic body (or system of bodies) can always be made isostatic by suppressing some of its constraints without perturbing its equilibrium state, with the reservation that one must apply the corresponding constraint forces to the suppressed constraints. The constraints that remain will then be called *static constraints* and the ones that were suppressed will be called *redundant constraints.*

A system of bodies is *externally* or *internally* hyperstatic according to whether its redundant constraints are external or internal, resp. Furthermore, a system can be both externally and internally hyperstatic.

One realizes the suppression of the external redundant constraints by either completely suppressing certain supports or replacing a ball support with simple supports, or finally replacing the anchor support with a ball support or a simple support.

The suppression of the redundant internal constraints is achieved by either suppressing a certain number of contacts that exist between the various bodies of the system or by some modifications that relate to how those contacts happen.

According to the number and nature of the constraints on a hyperstatic body (or system of bodies), there will exist only one way of making it isostatic or several of them. Hence, there will be no other way of rendering an arch that rests upon two ball supports isostatic than to replace one of those two supports with a simple support. By contrast, one can render an anchored arch with its two extremities isostatic by either purely and simply suppressing one of the two built-in supports or by substituting a ball support and a simple support for them.

**Remark.** – In what follows, whenever the direction of a constraint force is unknown *a priori* (which will be the case, for example, for a constraint that is realized by a ball support), we shall take the term "constraint force" to mean, not the force itself, but each of its components along two arbitrarily-chosen directions. Indeed, it is important to make only forces with directions that are known in advance enter into the calculation of constraint forces.

**12.** – Apply a system of external forces and couples that are initially zero to a body or system of bodies under consideration (whether it is isostatic or hyperstatic is of little importance) and then slowly increase them to their final values $F$ and $C$. An elastic deformation will then result. The point of application of any one of the forces $F$ will experience a certain absolute elastic displacement, and the line $aa'$ that joins the points of application of the two forces that form a couple $C$ will experience a certain absolute elastic rotation.

Let:

$\lambda$ be the projection of the elastic displacement of a point of application of the force $F$ onto the direction of that force, which is a projection that will be regarded as positive when it has the same sense as the force $F$ and negative when it has the opposite sense.

$\varphi$ be the projection of the elastic rotation of the line $aa'$ onto the axis of the couple $C$ (here, and likewise in what follows in the present note, one takes the term "projection of the rotation" to mean the "projection of the vector that represents the rotation," and that projection will be regarded as positive or negative, moreover, according to whether it has the same sense an the vector that represents the couple or the opposite sense, resp.).

$\mathcal{T}$ be the work done by external forces and couples during the deformation

The work done by constraint forces is essentially zero.

The work done by internal or molecular forces is equal to – $\Pi$ (no. **4**), where $\Pi$ denotes the internal potential of the deformed body or system of bodies.

By virtue of the *vis viva* theorem, one will have:

$$\mathcal{T} = \Pi, \tag{12}$$

where the material points of the body (or system of them) that are initially at rest will arrive at a new state of rest.

Hence:

*The work done by external forces and couples that are applied directly is equal to the internal potential of the deformed body (or system of bodies).*

Consequently, that work will depend upon only the final state of deformation. It is independent of the intermediate states, and as a result, of the way that one varies the external forces and couples ([1]). In order to calculate them, one can fix the mode of variation of those forces and couples that one deems to be suitable. Furthermore, assume that they constantly keep the same ratios with each other during the deformation, in such a way that an arbitrary intermediate state of deformation can be represented by $F\rho$ and $C\rho$, where $\rho$ denotes a positive number that varies from 0 to 1 during the complete deformation. With that same state of deformation, from the principle of the superposition of the elastic effects of the force, the projection of the elastic displacement of a point of application of the force $F\rho$ onto the direction of that force will be $\lambda\rho$, and the projection of the elastic rotation of the line $aa'$ onto the axis of the couple $C\rho$ will be $\varphi\rho$, where $\lambda$ and $\varphi$ denote the values of those projected displacements at the end of the deformation.

During the passage from that intermediate state of deformation to the following state that is infinitely-close to it, the forces and couples will increase by $d\,(F\rho)$ and $d\,(C\rho)$, resp., and the corresponding projected displacements by $d\,(\lambda\rho)$ and $d\,(\varphi\rho)$, resp. As a result, those forces and couples will do an elementary work $d\mathcal{T}$ between those two states, which will have a value of:

$$d\mathcal{T}=\sum F\rho\,d(\lambda\rho)+\sum C\rho\,d(\varphi\rho)=\left(\sum F\lambda+\sum C\varphi\right)\rho\,d\rho,$$

up to second-order infinitesimals, in which the sum extends over all forces $F$ and all couples $C$.

The total work done for the entire deformation is then:

$$\mathcal{T}=\left(\sum F\lambda+\sum C\varphi\right)\int_{\rho=0}^{\rho=1}\rho\,d\rho=\frac{1}{2}\left(\sum F\lambda+\sum C\varphi\right).$$

Upon substituting this expression for $\mathcal{T}$ in the relation (12) that is deduced from the *vis viva* theorem, one will obtain the fundamental equation:

$$\frac{1}{2}\left(\sum F\lambda+\sum C\varphi\right)=\Pi, \tag{13}$$

which is nothing by Clapeyron's equation ([2]), in a different form. The proof that was just given is, as one sees, extremely simple.

**13.** – We propose to extend Clapeyron's equation to the case in which the deformation is both elastic and caloric, which we believe has not been done yet. To that effect, consider a system of isotropic bodies, whether isostatic or hyperstatic. Subject it to the action of arbitrary external forces and couples and an elevation of the temperature

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([1]) In order for that to be true, it is necessary that a given system of external forces and couples can correspond to only one state of deformation of the body, which can be considered to be obvious from the physical viewpoint and has been proved analytically by Betti, Kirchhoff, and Cosserat (APPELL, *Traité de Mécanique rationelle,* t. III, 1903, pp. 515).

([2]) LAMÉ, *Leçons sur la théorie mathématique de l'Élasticité des corps solides,* 1866, pp. 80.

that is initially zero and increases slowly up to the final values $F$, $C$, and $\tau$. ($\tau$ is measured by starting from the temperature at which the constraints on the system are realized.) Let:

1. $\lambda$ be the projection of the elastic and caloric displacement of the point of application of any one $F$ of the external forces along the direction of that force.

2. $\varphi$ be the projection onto the axis of any one $C$ of the external couples of the elastic and caloric rotation of the line that joins the points of application of the two forces that define that couple.

3. $$\varepsilon_x - \alpha\,\tau, \qquad \varepsilon_y - \alpha\,\tau, \qquad \varepsilon_z - \alpha\,\tau, \qquad \gamma_{yz}, \qquad \gamma_{zx}, \qquad \gamma_{xy}$$

   be the elastic and caloric deformation of an elementary parallelepiped that is cut by planes that are normal to the coordinate axes at an arbitrary ($x$, $y$, $z$) (no. **5**).

4. $$n_x, \qquad n_y, \qquad n_z$$

   be the normal fatigues on the three faces of that parallelepiped that have the point ($x$, $y$, $z$) as its common summit (no. **2**).

5. $$t_{yz}, t_{xz}; \qquad t_{yx}, t_{yz}; \qquad t_{zx}, t_{zy}$$

   be the tangential fatigues on the same faces.

6. $\varpi\,dx\,dy\,dz$ and $\Pi$ be the internal potentials of that deformed parallelepiped and the system of bodies, respectively (no. **4**).

These various notations relate to the final state of deformation of the system. We shall preserve them for an arbitrary intermediate state, but with a prime.

Consider the infinitely-small deformation that takes the system of bodies from one of the two intermediate deformation states to the other, the first of which corresponds to the values $F'$, $C'$, and $\tau'$ of the external forces and couples and the variation of temperature, while the second one corresponds to the values $F' + dF'$, $C' + dC'$, and $\tau' + d\tau'$ of those same quantities. During that deformation, the work done by external forces and couples will be equal to the work done by elastic forces, by virtue of the *vis viva* theorem. We shall now calculate each of those two works:

**Work done by external forces and couples.** – That work is:

$$\sum F'\,d\lambda' + \sum C'\,d\varphi',$$

up to second-order infinitesimals.

**Work done by elastic forces.** – During the infinitesimal deformation considered, the elastic and caloric deformation parameters of an arbitrary parallelepiped vary by:

$$d\varepsilon'_x - \alpha\, d\tau', \quad d\varepsilon'_y - \alpha\, d\tau', \quad d\varepsilon'_z - \alpha\, d\tau', \quad d\gamma'_{yz}, \quad d\gamma'_{yx}, \quad d\gamma'_{xy},$$

and the potential of that parallelepiped increases by:

$$d\,(\varpi' dx\, dy\, dz) = d\varpi' dx\, dy\, dz.$$

The work done by elastic forces, when calculated for the purely elastic deformation of the parallelepiped that is defined by the variations $d\varepsilon'$, …, $d\gamma'_{xy}$ of the six elastic deformation parameters, is equal to the increase $d\varpi' dx\, dy\, dz$ in internal energy. The work done by those same forces, when calculated for the purely caloric deformation that is defined by the variation $-\alpha\, d\tau'$ of the single caloric deformation parameter, is:

$$\begin{gathered} -(n'_x\, dy\, dz)\, \alpha\, d\tau' dx - (n'_y\, dy\, dz)\, \alpha\, d\tau' dy - (n'_z\, dy\, dz)\, \alpha\, d\tau' dz \\ = -\alpha\, (n'_x + n'_y + n'_z)\ d\tau' dx\, dy\, dz, \end{gathered}$$

up to higher-order infinitesimals, since that variation corresponds to elongations:

$$\alpha\, d\tau' dx, \qquad \alpha\, d\tau' dy, \qquad \alpha\, d\tau' dz$$

of the edges of the parallelepiped and that the work done by the tangential elastic is zero, while the purely caloric deformation takes place without distortions.

Consequently, the work done by elastic and caloric deformation is:

$$d\omega' dx\, dy\, dz - \alpha\, (n'_x + n'_y + n'_z)\ d\tau' dx\, dy\, dz$$

for the parallelepiped, and:

$$d\,\Pi' - \iiint \alpha\, (n'_x + n'_y + n'_z)\, d\tau' dx\, dy\, dz$$

for the entire system, where the triple integral is extended aver the entire volume that the system occupies.

The equation that expresses the equality of the works done by the external forces and couples and the work done by elastic forces during the infinitely-small deformation considered of the system of bodies is then:

$$\sum F' d\lambda' + \sum C' d\varphi' = d\,\Pi' - \iiint \alpha\, (n'_x + n'_y + n'_z)\, d\tau' dx\, dy\, dz\,. \tag{14}$$

The equality of those two works is likewise true for the complete deformation of the system, no matter what the mode of increase in the external forces and couples and the temperature might be, moreover. We then adopt the hypothesis that one has:

$$F' = F\rho, \qquad C' = C\rho, \qquad \tau' = \tau\rho$$

for an arbitrary intermediate state of the deformation, where $\rho$ denotes a positive number that increases from 0 to 1 during the deformation, and as a result (and this can be considered to be obvious from the physical viewpoint) that:

$$\lambda' = \lambda\,\rho, \qquad \varphi' = \varphi\,\rho, \qquad n'_x = n_x\,\rho, \qquad n'_y = n_y\,\rho, \qquad n'_z = n_z\,\rho.$$

Furthermore, under that hypothesis, equation (14) will be written:

$$\left(\sum F\,\lambda + \sum C\,\varphi\right)\,\rho\,d\,\rho = d\,\Pi' - \alpha\tau\left[\iiint (n_x + n_y + n_z)\,dx\,dy\,dz\right]\,\rho\,d\,\rho,$$

in which one assumes that the elevation in temperature $\tau$ is the same at all points of the system of bodies.

Upon integrating between the limits $\rho = 0$ and $\rho = 1$, which correspond to the start and finish of the deformation, respectively, and upon remarking that the integral of $d\,\Pi'$, when taken over the complete deformation, is equal to $\Pi$ (namely, the internal potential of the deformed system), one will get:

(15) $$\frac{1}{2}\left(\sum F\,\lambda + \sum C\,\varphi\right) = \Pi - \frac{1}{2}\,\alpha\tau\iiint (n_x + n_y + n_z)\,dx\,dy\,dz\,.$$

That is Clapeyron's equation, when it is extended to the case in which the deformation is both elastic and caloric. Although it was established for a particular mode of variation of the external forces and couples and the temperature, it will be true for any final values $F$, $C$, and $\tau$ of those quantities, whereas the final state of deformation of the system is independent of the intermediate states. However, its left-hand and right-hand sides represent the work done by external forces and couples and work done by elastic forces only in the case of the particular mode of variation in question. On the contrary, the two sides of Clapeyron's equation (13) represent those two works no matter what the mode of variation of the external forces and couples and the temperature.

If the system is subject to a decrease in temperature then it will suffice to measure $\tau$ negatively in formula (15).

When the preceding proof is applied to a system of bodies with mean fibers, that will give:

(16) $$\frac{1}{2}\left(\sum F\,\lambda + \sum C\,\varphi\right) = \Pi - \frac{1}{2}\,\alpha\tau\int N\,ds\,,$$

where $N$ denotes the normal effort on an arbitrary transverse section of any of the bodies, and $ds$ denotes the element of the mean fiber that is found between that section and the infinitely-close section, and the integral is taken along the mean fibers of the bodies of the system.

One can pass directly from equation (15) to equation (16) by establishing that one has:

(17) $$\iiint (n_x + n_y + n_z)\,dx\,dy\,dz = \int N\,ds$$

for any system of bodies with mean fibers.

Here is the proof of that formula that we will use later on.

We evaluate the triple integral, first for an infinitely-thin slice between the two sections ($S$) and ($S'$) of an any one of the bodies that are made at two points $G$ and $G'$ of the mean fiber that are separated by a distance of $GG' = ds$. To that effect, consider the volume of that slice to be composed of an infinitely-large number of elementary parallelepipeds of volume $dx\,dy\,dz$ such that one of their two faces $dy\,dz$ is placed in the section ($S$) and the four edges $dx$ are perpendicular to that section. Since the bodies with mean fibers are, by hypothesis, considered to be rigid in the transverse sense of those fibers, the fatigues $n_y$ and $n_z$ will be zero. On the other hand, up to second-order infinitesimals and the convergence of the two sections ($S$) and ($S'$) (which is always quite weak in the bodies that are considered in the resistance of materials), one will have:

$$dx = ds.$$

As a result, for the slice considered:

$$\iiint (n_x + n_y + n_z)\,dx\,dy\,dz = ds \iint n_x\,dy\,dz ,$$

in which $n_x\,dy\,dz$ represents the elastic force on the surface element $dy\,dz$ of the section ($S$), and the double integral is the algebraic sum of those forces for all of that section, which is a sum that is the normal effort $N$, by definition. Hence, and always for the slice considered:

$$\iiint (n_x + n_y + n_z)\,dx\,dy\,dz = N\,ds.$$

Consequently, one will have:

$$\iiint (n_x + n_y + n_z)\,dx\,dy\,dz = \int N\,ds$$

for the whole system of bodies, where the latter integral is taken along the entire mean fibers of the all bodies in the system Q.E.D.

## 14. – Castigliano's theorem on the derivatives of work

The isostatic or hyperstatic body or system of bodies that was considered before (no. **12**) assumed its elastic equilibrium state under the action of a system of external forces and couples $F$ and $C$ when those forces and couples are given arbitrary, infinitely-small increments $dF$ and $dC$.

Let:

$d\lambda$ and $d\varphi$ be the corresponding (positive or negative) increments in the projected displacements $\lambda$ and $\varphi$.

$d\mathcal{T}$ be the corresponding increment in the work done by external forces and couples

$d\Pi$ be the corresponding increment in the internal potential.

$d\mathcal{T}$ represents the elementary work that is performed by the forces and couples $F$ and $C$ when they increase by $dF$ and $dC$, resp., which are correlated with the projected displacements $d\lambda$ and $d\varphi$, resp. It will then have the expression:

$$d\mathcal{T} = \sum F\, d\lambda + \sum C\, d\varphi,$$

and substituting that expression in formula (12) (no. **12**), when differentiated, will give:

$$d\,\Pi = \sum F\, d\lambda + \sum C\, d\varphi.$$

On the other hand, upon differentiating the fundamental equation (13) (no. **12**), one will have:

$$\sum F\, d\lambda + \sum \lambda\, dF + \sum C\, d\varphi + \sum \varphi\, dC = 2\, d\,\Pi.$$

Upon adding corresponding sides of the last two equations and developing the total differential $d\,\Pi$, one will get the relation:

$$\sum \lambda\, dF + \sum \varphi\, dC = \sum \frac{\partial \Pi}{\partial F} dF + \sum \frac{\partial \Pi}{\partial C} dC,$$

which can be satisfied only if one has:

$$\lambda = \frac{\partial \Pi}{\partial F}, \qquad \varphi = \frac{\partial \Pi}{\partial C} \tag{18}$$

separately, since the increments $dF$ and $dC$ are arbitrary. Hence:

**Theorem**. – *If an elastic body, whether isotropic or with a mean fiber, or an (isostatic or hyperstatic) system of such bodies is subjected to an arbitrary system of external forces and couples then:*

1. *The projection of the elastic displacement of the point of application of any of the forces onto the direction of that force will be equal to the partial derivative of the internal potential of the body (or system of bodies) with respect to that force.*

2. *The projection of the elastic rotation of the line that joins the points of application of two forces of any couple onto the axis of that couple will be equal to the partial derivative of the internal potential of the deformed body (or system of bodies) with respect to that couple.*

That is the theorem of Castigliano ([1]) that that engineer called the *theorem of the derivatives of work,* which is a term that is justified by the fact that, from formula (12) (no. **12**), the internal potential is equal to the work done by external forces during the deformation.

**15. Corollary I.** – *If two F among the external forces are equal and opposite then the elastic increment* $\Delta l$ *in the distance* $l = AB$ *between their points of application A and B is equal to the partial derivative* $\partial \Pi / \partial F$ *of the internal potential with respect to F.*

Indeed, first suppose that the two forces are unequal. Let $F'$ denote the one that is applied to *A*, and let $F''$ denote the one that is applied to *B*. One can always set:

$$F' = k' F, \qquad F'' = k'' F, \tag{a}$$

in which $k'$ and $k''$ denote two arbitrary positive numbers.

Let:

$\Delta' l$ be the elastic increment in the distance $AB = l$, which is an increment that differs from $\Delta l$, but becomes equal to it in the special case where $k' = k'' = 1$.

$\lambda'$ and $\lambda''$ be the elastic displacements of the two points *A* and *B*, which are estimated along the common direction *AB* of the two force $F'$ and $F''$ and regarded as positive when the former has the same sense as $F'$ and the latter has the same sense as $F''$.

One will obviously have:

$$\Delta' l = \lambda' + \lambda''.$$

However, from Castigliano's theorem:

$$\lambda' = \frac{\partial \Pi'}{\partial F'}, \qquad \lambda'' = \frac{\partial \Pi'}{\partial F''},$$

in which $\Pi'$ denotes the internal potential, which differs from $\Pi$, but will become equal to $\Pi$ in the special case where $F' = F'' = F$.

As a result:

$$\Delta' l = \frac{\partial \Pi'}{\partial F'} + \frac{\partial \Pi''}{\partial F''},$$

and upon setting $F' = F'' = F$:

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([1]) CASTIGLIANO, "Nouvelle théorie de l'équilibre des systèmes articulés," Actes de l'Académie de Turin, (1875); *Théorie de l'équilibre des systèmes élastiques,* Turin, 1879.

$$(b) \qquad \Delta l = \left(\frac{\partial \Pi'}{\partial F'}\right)_{F'=F} + \left(\frac{\partial \Pi''}{\partial F''}\right)_{F''=F} .$$

Now $\Pi$, which is a function of $F'$ and $F''$, can also be considered to be a function of $F$, since, by hypothesis, $F'$ and $F''$ are coupled with $F$ by the relations ($a$), which will permit one to write:

$$\frac{\partial \Pi'}{\partial F} dF = \frac{\partial \Pi'}{\partial F'} dF' + \frac{\partial \Pi'}{\partial F''} dF'' = \frac{\partial \Pi'}{\partial F'} k' \, dF + \frac{\partial \Pi'}{\partial F''} k'' \, dF ,$$

or

$$\frac{\partial \Pi'}{\partial F} = \frac{\partial \Pi'}{\partial F'} k' + \frac{\partial \Pi'}{\partial F''} k'' ,$$

or rather, upon setting $k' = k'' = 1$, which would imply that $F' = F'' = F$ and $\Pi' = \Pi$:

$$\frac{\partial \Pi}{\partial F} = \left(\frac{\partial \Pi'}{\partial F'}\right)_{F'=F} + \left(\frac{\partial \Pi'}{\partial F''}\right)_{F''=F} .$$

As a result, the expression ($b$) for $\Delta l$ will become:

$$\Delta l = \frac{\partial \Pi}{\partial F} .$$

Q. E. D.

**Corollary II.** – *If four of the external forces are applied at four points a, a′, b, b′, which are situated in the same plane, then those forces will be likewise situated in that plane, and if the ones that are applied at a and a′ constitute a couple C and the ones that are applied at b and b′ constitute a couple that is equal and opposite to the preceding one then the line aa′ will submit to an elastic rotation relative to the line bb′ whose projection onto the axis of couple C will be equal to the partial derivative* $\partial \Pi / \partial C$ *of the internal potential with respect to C.*

That corollary drops out of the second part of Castigliano's theorem and is proved in the same manner as corollary I.

**Corollary III.** – *In the case of a body with mean fiber or a system of such bodies, if a couple C is applied to an arbitrary transverse section then the projection of the elastic rotation of that section onto the axis of that couple will be equal to the partial derivative of the internal potential of the deformed body (or system of bodies) with respect to that same couple. Furthermore, if there is planar flexure, which demands that the axis of the couple C must be normal to the plane of flexure, then that derivative will represent the rotation of the section.*

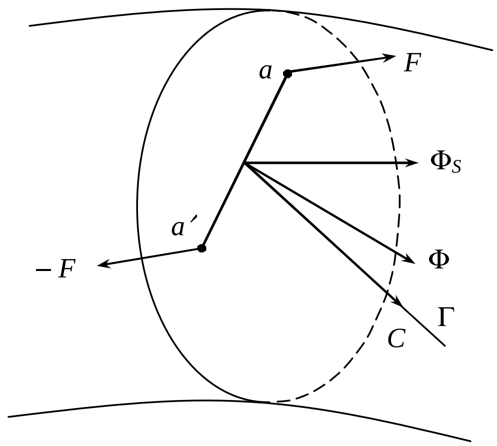


Figure 3

Let (*see* Fig. 3):

*a* and *a*´ be the points of application in the section considered ($S$) of the two forces $F$ and $-F$ constitute the couple $C$

Γ be the direction of the axis of that couple, which is normal to the plane of $F$ and $-F$, and in turn, to the line *aa*´ that is contained in that plane

$\Phi_S$ be the rotation of the section ($S$)

Φ be the rotation of the line *aa*´

$\varphi$ be the projection of the latter rotation onto the direction Γ.

From the second part of Castigliano's theorem, one will have:

$$\varphi = \frac{\partial \Pi}{\partial C}.$$

In order to establish the corollary, it will then suffice to show that the projection of the rotation $\Phi_S$ of the section ($S$) onto the direction Γ of the axis of the couple $C$ is equal to the projection $\varphi$ of the rotation Φ of the line *aa*´ onto that same direction.

Now, the rotation $\Phi_S$ can be decomposed into:

1. A rotation that is equipollent to Φ and takes the line *aa*´, which is situated in the plane of the section ($S$), from its initial position to its final position.

2. A rotation Ψ around *aa*´, in such a way that one has the equipollence:

$$\overline{\Phi_S} = \bar{\Phi} + \Psi.$$

Project that equipollence onto the direction Γ, where Ψ is normal Γ with a zero projection. Hence, the projection of the rotation $\Phi_S$ onto the direction Γ is equal to the projection $\varphi$ of the rotation Φ onto that direction.

Q. E. D.

## 16. – Applying Castigliano's theorem to the calculation of elastic displacements

In order to apply Castigliano's theorem to the calculation of elastic displacements in isotropic bodies or systems of bodies, it is necessary to define the expression for the internal potential as a function of the external forces and couples *F* and *C*. To that effect, one must calculate the six fatigues *n* and *t* (or the six parameter $\varepsilon$ and $\gamma$) (nos. **2** and **3**) as functions of those forces and couples and then substitute them in the general expression (3) (no. **4**) [or (4), same number] for the potential per unit volume. Now, in the present state of the mathematical theory of elasticity, that calculation is possible only in a very small number of particular cases. It will then follow that (at least, for the time being) Castigliano's theorem is generally impracticable insofar as the isotropic bodies and systems of bodies are concerned.

By contrast, it is immediately applicable to bodies and systems of bodies with mean fibers. Indeed, from formulas (9) and (11) (no. **9**), the general expression for their potential is:

$$\Pi = \frac{1}{2}\int\left(\frac{N^2}{E\,\Omega}+\frac{T_\eta^2}{G\,\Omega}+\frac{T_\zeta^2}{G\,\Omega}+\frac{M_\xi^2}{G\,I_\xi}+\frac{M_\eta^2}{E\,I_\eta}+\frac{M_\zeta^2}{E\,I_\zeta}\right)ds\,,$$

and if one substitutes that expression into the ones (18) (no. **14**) for $\lambda$ and $\varphi$ then one will get:

$$\lambda = \int\left(\frac{N}{E\,\Omega}\frac{\partial N}{\partial F}+\frac{T_\eta}{G\,\Omega}\frac{\partial T_\eta}{\partial F}+\frac{T_\zeta}{G\,\Omega}\frac{\partial T_\zeta}{\partial F}+\frac{M_\xi}{G\,I_\xi}\frac{\partial M_\xi}{\partial F}+\frac{M_\eta}{E\,I_\eta}\frac{\partial M_\eta}{\partial F}+\frac{M_\zeta}{E\,I_\zeta}\frac{\partial M_\zeta}{\partial F}\right)ds\,, \tag{19}$$

$$\varphi = \int\left(\frac{N}{E\,\Omega}\frac{\partial N}{\partial C}+\frac{T_\eta}{G\,\Omega}\frac{\partial T_\eta}{\partial C}+\frac{T_\zeta}{G\,\Omega}\frac{\partial T_\zeta}{\partial C}+\frac{M_\xi}{G\,I_\xi}\frac{\partial M_\xi}{\partial C}+\frac{M_\eta}{E\,I_\eta}\frac{\partial M_\eta}{\partial C}+\frac{M_\zeta}{E\,I_\zeta}\frac{\partial M_\zeta}{\partial C}\right)ds\,. \tag{20}$$

Furthermore, in order to calculate $\lambda$ and $\varphi$, it will suffice to perform the reduction of the external forces that are applied to the left of any section to the center of gravity of that section (including constraint forces), which will yield $N, \ldots, M_\zeta$ as functions of *F* and *C*, and then form the partial derivatives $\frac{\partial N}{\partial F}, \ldots, \frac{\partial M_\zeta}{\partial F}, \frac{\partial N}{\partial C}, \ldots, \frac{\partial M_\zeta}{\partial C}$, and substitute those results in formulas (19) and (20).

If the body (or system of bodies) is subject to a *planar flexure* (no. **9**) then the expression for the internal potential will simplify considerably, and formulas (19) and (20) will reduce to:

(19′) $$\lambda = \int\left(\frac{N}{E\,\Omega}\frac{\partial N}{\partial F}+\frac{T}{G\,\Omega}\frac{\partial T}{\partial F}+\frac{M}{E\,I}\frac{\partial M}{\partial F}\right)ds\,,$$

(20′) $$\varphi = \int\left(\frac{N}{E\,\Omega}\frac{\partial N}{\partial C}+\frac{T}{G\,\Omega}\frac{\partial T}{\partial C}+\frac{M}{E\,I_\xi}\frac{\partial M}{\partial C}\right)ds\,.$$

The preceding formulas yield the projected displacements and projected rotations only for the points of application of the external forces and the sections to which the external couples are applied. However, it is easy to extend them to an arbitrary point and an arbitrary section by a very simple trick.

Hence, one must determine the elastic displacement of an arbitrary point $A$ that is projected onto an arbitrarily-chosen direction $\Delta$. Apply *an auxiliary force* $\mathcal{F}$ of arbitrary magnitude to that point along that direction, in addition to the given external forces and couples $F$ and $C$. Moreover, the elements $N$, $T_\eta$, …, $M_\zeta$ of the reduction of the elastic forces in an arbitrary section (or external forces to the left) at the center of gravity of that section will become $N + \mathcal{N}$, $T_\eta + \mathcal{T}_\eta$, …, $M_\zeta + \mathcal{M}_\zeta$, if one lets $\mathcal{N}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\zeta$ denote what the elements of that reduction would become if only the force $\mathcal{F}$ were applied to the exclusion of the forces and couples $F$ and $C$. When the elastic displacement of the point $A$ under the influence of the system $F$, $C$, $\mathcal{F}$ is projected onto the direction $\Delta$, it will have the expression:

$$\lambda = \int\left[\frac{N+\mathcal{N}}{E\,\Omega}\frac{\partial(N+\mathcal{N})}{\partial\mathcal{F}}+\frac{T_\eta+\mathcal{T}_\eta}{G\,\Omega}\frac{\partial(T_\eta+\mathcal{T}_\eta)}{\partial\mathcal{F}}+\cdots+\frac{M_\zeta+\mathcal{M}_\zeta}{E\,I_\zeta}\frac{\partial(M_\zeta+\mathcal{M}_\zeta)}{\partial\mathcal{F}}\right]ds\,,$$

by virtue of formula (19).

In order to get the desired projected displacement, it will obviously suffice to make $\mathcal{F} = 0$, which will imply that $\mathcal{N} = 0$, $\mathcal{T}_\eta = 0$, …, $\mathcal{M}_\zeta = 0$. If one observes that $N$, $T_\eta$, …, $M_\zeta$ are independent of $\mathcal{F}$ and that, as a result, $\frac{\partial N}{\partial\mathcal{F}}$, $\frac{\partial T_\eta}{\partial\mathcal{F}}$, …, $\frac{\partial M_\zeta}{\partial\mathcal{F}}$ are zero then one will get:

(21) $$\lambda = \int\left(\frac{N}{E\,\Omega}\frac{\partial\mathcal{N}}{\partial\mathcal{F}}+\frac{T_\eta}{G\,\Omega}\frac{\partial\mathcal{T}_\eta}{\partial\mathcal{F}}+\cdots+\frac{M_\zeta}{E\,I_\zeta}\frac{\partial\mathcal{M}_\zeta}{\partial\mathcal{F}}\right)ds\,.$$

Similarly, the projection of the rotation of an arbitrary section ($S$) onto an arbitrarily-chosen direction $\Gamma$ is expressed by the formula:

(22) $$\varphi = \int\left(\frac{N}{E\,\Omega}\frac{\partial\mathcal{N}}{\partial\mathcal{C}}+\frac{T_\eta}{G\,\Omega}\frac{\partial\mathcal{T}_\eta}{\partial\mathcal{C}}+\cdots+\frac{M_\zeta}{E\,I_\zeta}\frac{\partial\mathcal{M}_\zeta}{\partial\mathcal{C}}\right)ds\,,$$

in which $\mathcal{M}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\zeta$ are the elements of the reduction of the elastic forces in an arbitrary section at the center of gravity of that section when the body (or system of bodies) is supposed to be subjected to exclusively an *auxiliary couple* of arbitrary intensity $\mathcal{C}$ that is applied to the section (*S*) and the direction of the axis Γ.

In the case of planar flexure, the direction Δ must be situated in the plane of flexure and the direction Γ must normal to that plane, and formulas (21) and (22) will reduce to:

$$\lambda = \int\left(\frac{N}{E\,\Omega}\frac{\partial\mathcal{N}}{\partial\mathcal{F}} + \frac{T}{G\,\Omega}\frac{\partial\mathcal{T}}{\partial\mathcal{F}} + \frac{M}{E\,I}\frac{\partial\mathcal{M}}{\partial\mathcal{F}}\right)ds, \tag{21′}$$

$$\varphi = \int\left(\frac{N}{E\,\Omega}\frac{\partial\mathcal{N}}{\partial\mathcal{C}} + \frac{T}{G\,\Omega}\frac{\partial\mathcal{T}}{\partial\mathcal{C}} + \frac{M}{E\,I}\frac{\partial\mathcal{M}}{\partial\mathcal{C}}\right)ds. \tag{22′}$$

$\varphi$ is then the rotation of the section (*S*).

**17.** – $N$, $T_\eta$, …, $M_\zeta$ are subject to given forces and couples *F* and *C* in the system of bodies considered, and the elements of the reduction at the center of gravity of an arbitrary section of any of the bodies are subject to external forces that act to the left of that section, including constraint forces.

$\mathcal{M}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\zeta$ are elements of the same nature in that system that are subject to either the auxiliary force $\mathcal{F}$ or the auxiliary couple $\mathcal{C}$.

The calculation of those various elements of reduction demands the prior determination of the constraint forces.

If the system considered is *isostatic* then that determination will involve only pure statics and will offer no difficulty.

If the system is *hyperstatic* then that determination will demand the intervention of the theory of elasticity; we shall treat that question later. Nonetheless, one can avoid that intervention as far as the calculation of the elements $\mathcal{M}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\zeta$ is concerned; here is how:

In an arbitrary hyperstatic system that is acted upon by given forces and couples *F* and *C*, let $F_s$ be the redundant external and internal constraint forces, which are unknown forces. Consider the isostatic system that is obtained by suppressing all of the redundant constraints on the hyperstatic system. Subject it to forces and couples *F*, *C*, and unknown forces $F_s$. It will take on a state of deformation that is identical to that of the hyperstatic system if it were subject to only the forces and couples *F* and *C*. Consequently, instead of calculating the projected elastic displacements $\lambda$ and projected elastic rotations $\varphi$ in the hyperstatic system, it would amount to the same thing to calculate them in the isostatic system. Furthermore, in formulas (21) and (22) (no. **16**):

1. The elements $\mathcal{M}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\zeta$ of the reduction become ones that relate to the isostatic system when it is subject to either the auxiliary force $\mathcal{F}$ or the auxiliary couple $\mathcal{C}$.

2. The elements $N$, $T_\eta$ , …, $M_\zeta$ of the reduction will likewise become ones that relate to the same isostatic system when it is subject to the forces and couples $F$, $C$, and the unknown forces $F_s$ , but the latter elements will have the same values that they would have in the hyperstatic system if it were subject to only the given forces and couples $F$ and $C$, since the state of deformation of those two systems will be the same.

Hence, when one applies formulas (21) and (22) [or (21′) and (22′)] to a hyperstatic system, one can consider the elements of reduction $\mathcal{N}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\zeta$ (or $\mathcal{N}$, $\mathcal{T}$, $\mathcal{M}$) to be attached to the isostatic system that is obtained by suppressing the redundant constraints from that hyperstatic system, which will render it calculable by means of pure statics. As for the elements of the reduction $N$, $T_\eta$, …, $M_\zeta$, they will be, by contrast, attached to the hyperstatic system, and the determination will necessitate the intervention of the theory of elasticity.

## 18. – Theorem of General Menabrea. Determining the redundant constraint forces

Consider an isotropic body or one with a mean fiber, or more generally, a system of such bodies. That system is hyperstatic. It is in elastic equilibrium under the action of forces and couples $F$ and $C$ that are applied directly, and we suppose (and this is the most complex case) that its redundant constraints are external in one case and internal in the other. Make them isostatic (no. **11**) by suppressing the aforementioned *redundant constraints* by applying forces to them that correspond to the constraints that are thus suppressed, moreover, in order that its state of elastic equilibrium should not be perturbed. Let $F_{es}$ and $C_{es}$ be an arbitrary force and couple, resp., of redundant external constraints, while $F_{is}$ and $C_{is}$ are an arbitrary force and couple, resp., of redundant internal constraints. In relation to a system that has been rendered isostatic, those forces must be considered to be forces that are applied directly, as well as the forces and couples $F$ and $C$.

Now, from the very fact of the existence of external constraints itself, the projection $\lambda$ of the elastic displacement of the point of application of the force $F_{es}$ onto the direction of that force is zero, and the rotation of the section of application of the couple $C_{es}$ is likewise zero. From the general formulas (18) (no. **14**), one will then have:

$$\frac{\partial \Pi}{\partial F_{es}} = 0, \qquad \frac{\partial \Pi}{\partial C_{es}} = 0.$$

On the other hand, an arbitrary internal constraint, and in particular, a redundant internal constraint, ordinarily consists of saying that a point $A$ of one piece $P$ of the system is restricted to remain invariably coupled to a point $B$ and another piece $Q$. The corresponding constraint forces are a force $F_{is}$ that is applied to $A$ on the piece $P$ and an equal and opposite force that is applied to $B$ on the piece $Q$. The distance $AB$ between those two points, which is zero before deformation, will then once more be zero afterward. As a result, by virtue of the corollary I (no. **15**) to Castigliano's theorem, one will have:

$$\frac{\partial \Pi}{\partial F_{is}} = 0.$$

Finally, if the redundant constraints of one piece $P$ with respect to a piece $Q$ consist of saying that two points $a$ and $a'$ of the former are restricted to remain invariably linked with two points $b$ and $b'$ of the latter then the corresponding constraint forces will be composed of two equal and opposite forces that are applied at $a'$ and $b'$. If the two forces that are applied at $a$ and $a'$ constitute a couple $C_{is}$ then the two forces that are applied at $b$ and $b'$ will constitute an equal and opposite couple. In that case, since the elastic rotation of the line $aa'$ relative to the line $bb'$ is zero, one will have:

$$\frac{\partial \Pi}{\partial C_{is}} = 0,$$

by virtue of corollary II (no. **15**).

In summary, the partial derivatives of the internal potential of the deformed system with respect to the forces and couples of the redundant constraints, whether external or internal, are zero. Hence, the values of those forces and couples will render that potential a maximum or minimum. It remains for us to decide between those two alternatives.

Let $A$ be the value of any of the forces $F_{es}$, which is a value that will consequently satisfy the equation:

$$\frac{\partial \Pi}{\partial F_{es}} = 0.$$

For any other value that is attributed arbitrarily to the force $F_{es}$, the projection of the displacement the point of application of the force onto the direction of that force will not be zero. It will have the expression:

$$\lambda = \frac{\partial \Pi}{\partial F_{es}}.$$

Now, it is obvious that if one starts from that arbitrary value of the force $F_{es}$ and it takes on an increment $dF_{es}$ then the corresponding variation $d\lambda$ of the projected displacement $\lambda$ will have the same sense as $F_{es}$ (i.e., positive), in such a way that one will have:

$$\frac{\partial \lambda}{\partial F_{es}} > 0,$$

and as a result:

$$\frac{\partial^2 \Pi}{\partial F_{es}^2} > 0.$$

That inequality, which was established for any value of $F_{es}$ that is not equal to $A$, will be likewise true for $F_{es} = A$, whereas, since the potential Π has degree two in $F_{es}$ ([1]), its second derivative with respect to that variable will be independent of the value that is attributed to it.

One shows that the second derivatives of Π with respect to $C_{es}$ , $F_{is}$ , and $C_{is}$ are positive in the same fashion. Certain authors then conclude that Π is a minimum and state the theorem of General Menabrea in the following manner (or something that approaches it):

*In a hyperstatic system that is composed of isotropic bodies or ones with mean fibers, the values that the forces and couples of the redundant external and internal constraints take will, in fact, will render the internal potential of the system a minimum when it is considered to be a function of those forces and couples.*

That statement is very seductive, but it goes beyond the one that was actually proved. Indeed, the condition for Π to be a minimum is that the total differential of that function should be positive for the values of $F_{es}$ , …, $C_{is}$ that annul the first partial derivatives $\frac{\partial \Pi}{\partial F_{es}}$, …, $\frac{\partial \Pi}{\partial C_{is}}$. Now, one can only establish that the second partial derivatives $\frac{\partial^2 \Pi}{\partial F_{es}^2}$, …, $\frac{\partial^2 \Pi}{\partial C_{is}^2}$ are positive for those values. We also believe that one must take the following statement into account:

**Theorem. –** *In a hyperstatic system that is composed of isotropic bodies or ones with mean fibers, the values the forces and couples of redundant external and internal constraints actually take will annul the first partial derivatives of the internal potential, when it is considered to be a function of those forces and couples. In addition, if one replaces those forces and couples with their effective values in that function, except for one of those forces or one of those couples, then the function of one variable that is thus obtained will be a minimum for the effective value of that force or couple.*

In addition, as far as applications are concerned, the question of knowing whether an internal potential is a minimum or not is devoid of interest. The only point that matters is that the effective values of the forces and couples of the redundant constraints annul the first partial derivatives of that potential.

As one knows, that theorem is a corollary to Castigliano's theorem. However, before the work of that engineer, it was stated for articulated systems by General Menabrea ([2]) under the name of the "principle of minimum elastic work," and one can likewise find that it was given for an arbitrary system of bodies with mean fibers in that era, since those

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([1]) It is a function of degree two in the normal and tangential fatigues. Now, those fatigues are linear functions of the external forces that produce them. Hence, Π is a function of degree two in the external forces, and in particular, of the force $F_{es}$ .

([2]) MENABREA, "Principe général pour déterminer les pressions et les tensions dans un système élastique," Turin (1868). *See* also a note that General Menabrea read at the session of the Académie des Sciences on 31 May 1858, Comptes rendus, t. XLVI, pp. 1056.

bodies can be considered to be a particular case of articulated systems ([1]), according to a remark of Mohr and Winkler.

We shall indicate the way that one employs Menabrea's theorem in order to calculate the forces and couples of redundant constraints in bodies or systems of bodies with mean fibers after we have extended that theorem to the case in which the deformation is both elastic and caloric.

### 19. – Extension of Castigliano's theorem to the case in which the deformation is both elastic and caloric

That extension was made by Ernest Flamard in his previously-cited thesis in the context of systems of bodies with mean fibers, by means of the theorem of virtual work. We shall do that for the systems of isotropic bodies, as well as the ones with mean fibers, by means of the *vis viva* theorem.

Consider a system of *isotropic* bodies (whether isostatic or hyperstatic) that is deformed by some external forces and couples $F$ and $C$ and a rise in temperature $\tau$, which is measured by starting from the temperature at which the system constraints have been realized. (In the case of a drop in temperature, it will suffice to endow $\tau$ with the negative sign. Apply the generalized Clapeyron equation (15) (no. **13**) to that deformation:

$$\sum F\,\lambda + \sum C\,\varphi = 2\Pi - \iiint \alpha\tau\,(n_x + n_y + n_z)\,dx\,dy\,dz\,,$$

in which:

- $\lambda$ is the projection of the elastic and caloric displacement of the point of application of any of the external forces $F$ onto the direction of that force

- $\varphi$ is the projection of the elastic rotation of the line that joins the points of application of the two forces that form an external couple $C$ onto the axis of that couple

- $\Pi$ is the internal potential of the deformed system of bodies

- $n_x$, $n_y$, $n_z$ are the normal fatigues along the three elements that are drawn normally to the coordinate axes at an arbitrary point $(x, y, z)$ of the system

That equation is true no matter what system of values is attributed to $F$, $C$, and $\tau$. As a result, the equation that is obtained by differentiating it with respect to those independent variables and with respect to the quantities $\lambda$, $\varphi$, $\Pi$, $n_x$, $n_y$, $n_z$ that it depends upon will likewise be true. One can then write:

$$(a)\qquad \sum F\,d\lambda + \sum \lambda\,dF + \sum C\,d\varphi + \sum \varphi\,dC$$
$$= 2\,d\Pi - \iiint \alpha[(n_x + n_y + n_z)\,d\tau + \tau\,d(n_x + n_y + n_z)\,dx\,dy\,dz\,,$$

([1]) Maurice LEVY, *La Statique graphique et ses applications aux constructions*, Part 4, 1888, pp. 141.

and the increments $dF$, $dC$, and $d\tau$ in that equation are arbitrary.

Imagine the infinitely-small deformation that takes the system of bodies between two states of deformation, the first of which corresponds to the values $F$, $C$, and $\tau$ of the external forces and couples and the rise in temperature, resp., while the second one corresponds to the values $F + dF$, $C + dC$, and $\tau + d\tau$ of those quantities. During that deformation, the work done by external forces and couples is equal to the work done by elastic forces; the equation that expresses that fact was established before: It was (14) (no. **13**), which was:

$$(b) \qquad \sum F\,d\lambda + \sum C\,d\varphi = d\,\Pi - \iiint \alpha\,(n_x + n_y + n_z)\,d\tau\,dx\,dy\,dz\,,$$

when one suppresses the primes (which is only a question of notation).

Subtract corresponding sides of equations (*a*) and (*b*). That will give:

$$(c) \qquad \sum \lambda\,dF + \sum \varphi\,dC = d\,\Pi - \iiint \alpha\,\tau\,(n_x + n_y + n_z)\;dx\,dy\,dz\,.$$

The fatigues $n_x$, …, $\gamma_{xy}$, and as a result, the internal potential $\Pi$, depend upon external forces and couples $F$ and $C$ that are applied directly and the constraint forces. If the system is hyperstatic then they will depend upon not only $F$ and $C$, but also on the temperature, so such a system cannot be freely dilatable, in such a way that $n_x$, $n_y$, $n_z$, and $\Pi$ will be functions of $F$, $C$, and $\tau$ in equation (*c*). If the system were isostatic then those quantities would be, on the contrary, independent of $\tau$, so such a system would be freely dilatable.

Having said that, first suppose that the system considered is *isostatic*, and under that hypothesis, develop the total differentials $d\,\Pi$ and $d\,(n_x + n_y + n_z)$ in equation (*c*). If one assumes that the variation of temperature is the same at all points of the system then it will become:

$$\begin{aligned}
\sum \lambda\,dF &+ \sum \varphi\,dC \\
&= \sum \frac{\partial \Pi}{\partial F}\,dF + \sum \frac{\partial \Pi}{\partial C}\,dC - \alpha\tau \iiint \left( \sum \frac{\partial (n_x + n_y + n_z)}{\partial F}\,dF \right) dx\,dy\,dz \\
&\qquad - \alpha\tau \iiint \left( \sum \frac{\partial (n_x + n_y + n_z)}{\partial C}\,dC \right) dx\,dy\,dz \\
&= \sum \frac{\partial \Pi}{\partial F}\,dF + \sum \frac{\partial \Pi}{\partial C}\,dC - \alpha\tau \sum \frac{\partial}{\partial F}\left[ \iiint (n_x + n_y + n_z)\,dx\,dy\,dz \right] dF \\
&\qquad - \alpha\tau \sum \frac{\partial}{\partial C}\left[ \iiint (n_x + n_y + n_z)\,dx\,dy\,dz \right] dC\,,
\end{aligned}$$

or, upon setting:

$$(23) \qquad H = \Pi - \alpha\tau \iiint (n_x + n_y + n_z)\,dx\,dy\,dz\,,$$

$$\sum \lambda\,dF + \sum \varphi\,dC = \sum \frac{\partial H}{\partial F}\,dF + \sum \frac{\partial H}{\partial C}\,dC\,.$$

In order for the latter equation to be satisfied, it is necessary that one must have:

$$\lambda = \frac{\partial H}{\partial F}, \qquad \varphi = \frac{\partial H}{\partial C}, \tag{24}$$

separately, since the increments $dF$ and $dC$ are arbitrary.

Formulas (23) and (24), which were established for *isostatic systems*, remain valid for *hyperstatic* systems. Indeed, let $F_s$ be the redundant constraint forces, whether internal or external, for a given arbitrary hyperstatic system. Consider the isostatic system that is obtained by suppressing all of the redundant constraints from the hyperstatic system. Subject it to the forces and couples $F$, $C$, $F_s$ , and the variation of temperature $\tau$; so formulas (23) and (24) will apply to it. Now, its state of deformation is identical to that of the given hyperstatic system when it is subject to the forces and couples $F$ and $C$ and the variation of temperature $\tau$, and as a result, the various quantities $\lambda$, $\varphi$, $\Pi$, $n_x$ , $n_y$ , $n_z$ will have the same values in that isostatic system as they do in the hyperstatic system. Hence, formulas (23) and (24) will be likewise applicable to the latter system.

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The two formulas (24) differ from (18) (no. **14**) only by the replacement of the function $\Pi$ with the function $H$. *Except for that replacement, they will then translate into a proposition that is identical to Castigliano's theorem.*

When the preceding proof is applied to the case of a system of bodies with *mean fibers*, that will once more lead to the formulas (24), but with:

$$H = \Pi - \alpha\tau \int N\, ds . \tag{25}$$

One can, moreover, pass directly from the expression (23) for $H$ to the expression (25), upon remarking that from formula (17) (no. **13**), one will have:

$$\iiint (n_x + n_y + n_z)\, dx\, dy\, dz = \int N\, ds$$

for any system of bodies with mean fibers.

We note, in passing, that if one replaces $\Pi$ in formula (25) with its resultant expression from formulas (11) and (9) (no. **9**) then one will get:

$$H = \frac{1}{2}\int \left[ \left( \frac{N^2}{E\,\Omega} - 2\alpha\tau N \right) + \frac{T_\eta^2}{G\,\Omega} + \cdots + \frac{M_\zeta^2}{E\,I_\zeta} \right] ds . \tag{26}$$

In the case of planar flexure, the last formula will reduce to:

$$H = \frac{1}{2}\int \left[ \left( \frac{N^2}{E\,\Omega} - 2\alpha\tau N \right) + \frac{T^2}{G\,\Omega} + \frac{M^2}{E\,I} \right] ds . \tag{26$'$}$$

**20.** – The quantity $H$ can be interpreted as follows:

Let $F_s$ be the redundant constraint forces, whether internal or external, for a hyperstatic system of bodies with mean fibers that are subject to forces and couples $F$ and $C$, as well as a temperature variation $\tau$. Let the forces and couples $F$, $C$, $F_s$ , and the temperature variation $\tau$ act upon the isostatic system that one obtains by suppressing the redundant constraints. It will take on a state of deformation that is identical to that of the hyperstatic system that is subject to the given forces and couples $F$ and $C$ and the temperature variation. Imagine that this deformation is achieved over two time intervals as follows:

During the first time interval, the isostatic system is subjected to forces and couples $F$, $C$, $F_s$ , in such a way that its deformation is purely elastic and that as a result the work that is done by elastic forces will be $\Pi$.

During the second time interval, it is subject to the temperature variation. The elastic forces remain constant, since the isostatic system is freely dilatable, and it will accomplish an amount of work whose value is $-N\,\alpha\tau\,ds$ for a slice of thickness $ds$ and $-\alpha\tau\int N\,ds$ for the entire system.

The total work done by the elastic forces during the complete deformation, both elastic and caloric, will then be:

$$\Pi - \alpha\tau\int N\,ds = H,$$

and if the system is composed of isotropic bodies then one will likewise find that the total work done is:

$$\Pi - \alpha\tau\iiint (n_x + n_y + n_z)\,dx\,dy\,dz = H.$$

Ernest Flamard called the quantity $H$ *the total work done by deformation of an elastic system that is subject to a temperature variation.*

It should be pointed out that this terminology is quite conventional. It is exact only with the reservation that the deformation is performed during two time intervals and in the order that was indicated above. Indeed, if, on the contrary, the temperature variation precedes the application of the force and couples $F$, $C$, $F_s$ then the work done by elastic forces will reduce to $\Pi$. If the two actions are simultaneous then the work done by elastic forces will be somewhere between $\Pi$ and $H$.

Formulas (24) nonetheless constitute an extension of Castigliano's theorem to the case in which the deformation is both elastic and caloric.

### 21. – Application of the generalized Castigliano theorem to the calculations of elastic and caloric displacements in bodies and systems of bodies with mean fibers

The general expressions (24) (no. **19**) for the elastic and caloric displacements and rotations are developed in exactly the same manner as the expressions (18) (no. **14**) of the purely elastic displacements and rotations in no. **16**. One will then find that:

(27) $$\lambda = \int \left[ \left( \frac{N}{E\,\Omega} - \alpha\tau \right) \frac{d\mathcal{N}}{d\mathcal{F}} + \frac{T_\eta}{G\,\Omega} \frac{d\mathcal{T}}{d\mathcal{F}} + \cdots + \frac{M_\eta}{E\,I_\eta} \frac{d\mathcal{M}_\eta}{d\mathcal{F}} \right] ds$$

(28) $$\varphi = \int \left[ \left( \frac{N}{E\,\Omega} - \alpha\tau \right) \frac{d\mathcal{N}}{d\mathcal{C}} + \frac{T_\eta}{G\,\Omega} \frac{d\mathcal{T}}{d\mathcal{C}} + \cdots + \frac{M_\eta}{E\,I_\eta} \frac{d\mathcal{M}_\eta}{d\mathcal{C}} \right] ds .$$

Those formulas differ from the corresponding ones (21) and (22) (no. **16**) only by the replacement of $\frac{N}{E\,\Omega}$ with $\left( \frac{N}{E\,\Omega} - \alpha\tau \right)$. When one applies them to a hyperstatic system, from the caveats that were made before (no. **17**, *in fine*), one must take care to calculate the reduced elements $\mathcal{N}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\zeta$, not in the hyperstatic system considered, but in the isostatic system that is obtained suppressing the redundant constraints of that hyperstatic system.

In the case where the flexure is planar, one will have simply:

(27′) $$\lambda = \int \left[ \left( \frac{N}{E\,\Omega} - \alpha\tau \right) \frac{d\mathcal{N}}{d\mathcal{F}} + \frac{T}{G\,\Omega} \frac{d\mathcal{T}}{d\mathcal{F}} + \frac{M}{E\,I} \frac{d\mathcal{M}}{d\mathcal{F}} \right] ds$$

(28′) $$\varphi = \int \left[ \left( \frac{N}{E\,\Omega} - \alpha\tau \right) \frac{d\mathcal{N}}{d\mathcal{C}} + \frac{T}{G\,\Omega} \frac{d\mathcal{T}}{d\mathcal{C}} + \frac{M}{E\,I} \frac{d\mathcal{M}}{d\mathcal{C}} \right] ds .$$

### 22. – Extension of General Menabrea's theorem to the case in which the deformations are both elastic and caloric. Determining the redundant constraint forces

The extension of Castigliano's theorem implies a corresponding extension of its corollary, namely, General Menabrea's theorem. That extension is obtained by replacing the internal potential $\Pi$ with the function $H$ in that theorem.

In order to apply the theorem, thus-generalized, to the determination of the constraint forces and couples of a hyperstatic system, one introduces redundant constraint forces and couples into the function $H$ and forms its partial derivatives with respect to those forces and couples. The equations that are obtained by equating those derivatives to zero will yield the forces and couples of the *redundant* constraints. Pure statics will then provide the equations that are necessary to calculate the forces and couples of the *static* constraints.

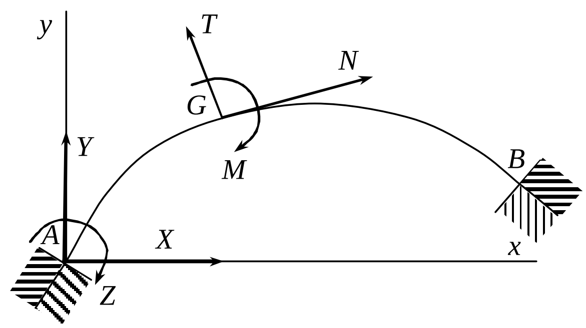


Figure 4.

### 23. – Example of the determination of redundant *external* constraint forces

Let (Fig. 4) *AB* be an *arch that is anchored at its two extremities,* which signifies that the two transverse sections whose centers of gravity are *A* and *B* cannot be displaced elastically.

That arch has a plane curve for its mean fiber and it is structurally symmetric with respect to the plane of that curve. It is subject to arbitrary forces that are situated in that plane, as well as a temperature variation $\tau$, which is measured from the temperature that is realized at its anchors. Its flexure is therefore planar.

The elementary reaction of either of the two anchors (the one on the left, for example) is reducible to a resultant translation at the point *A* and a resultant couple whose axis is normal to the plane of the mean fiber. Let *X* and *Y* be the components of that resultant translation along two arbitrary rectangular axes *Ax* and *Ay*, and let *Z* be the resultant couple. Suppressing the anchor on the left will obviously have the effect of rendering the arch isostatic, since that anchor constitutes a redundant constraint of that arch (no. **11**) and as a result, *X*, *Y*, and *Z* will be the two forces and the couple, resp., of the redundant external constraint. By virtue of the generalized theorem of General Menabrea (no. **22**):

$$\frac{\partial H}{\partial X} = 0, \qquad \frac{\partial H}{\partial Y} = 0, \qquad \frac{\partial H}{\partial Z} = 0.$$

The function *H* is expressed by the formula (26) (no. **19**), which is valid in the case of planar flexure:

$$H = \frac{1}{2}\int\left[\left(\frac{N^2}{E\,\Omega} - 2\alpha\tau N\right) + \frac{T^2}{G\,\Omega} + \frac{M^2}{E\,I}\right] ds\,,$$

and in which the integral extends along the total length of the mean fiber of the arch. As a result, the three equations above can be written:

$$(a)\qquad \begin{cases} \displaystyle\int\left[\left(\frac{N}{E\,\Omega}-\alpha\tau\right)\frac{\partial N}{\partial X}+\frac{T}{G\,\Omega}\frac{\partial T}{\partial X}+\frac{M}{E\,I}\frac{\partial M}{\partial X}\right]ds=0, \\ \displaystyle\int\left[\left(\frac{N}{E\,\Omega}-\alpha\tau\right)\frac{\partial N}{\partial Y}+\frac{T}{G\,\Omega}\frac{\partial T}{\partial Y}+\frac{M}{E\,I}\frac{\partial M}{\partial Y}\right]ds=0, \\ \displaystyle\int\left[\left(\frac{N}{E\,\Omega}-\alpha\tau\right)\frac{\partial N}{\partial Z}+\frac{T}{G\,\Omega}\frac{\partial T}{\partial Z}+\frac{M}{E\,I}\frac{\partial M}{\partial Z}\right]ds=0. \end{cases}$$

Having said that, let $\nu$, $\theta$, and $\mu$ be the values that are taken by the elements of the reduction $N$, $T$, and $M$, respectively, at the center of gravity $G$ $(x, y)$ of an arbitrary section of the external forces that act from the left of the section if the arch is rendered isostatic by suppressing its left anchor. $\nu$, $\theta$, and $\mu$ are immediately calculable by pure statics and can consequently be considered to be known in what follows.

One obviously has:

$$(b)\qquad \begin{cases} \displaystyle N=\nu+X\,\frac{dx}{ds}+Y\,\frac{dy}{ds}, \\ \displaystyle T=\theta-X\,\frac{dx}{ds}+Y\,\frac{dy}{ds}, \\ M=\mu-Xy+Yx+Zm, \end{cases}$$

and as a result:

$$\frac{\partial N}{\partial X}=\frac{dx}{ds},\qquad \frac{\partial T}{\partial X}=-\frac{dy}{ds},\qquad \frac{\partial M}{\partial X}=-y,$$

$$\frac{\partial N}{\partial Y}=\frac{dy}{ds},\qquad \frac{\partial T}{\partial Y}=\frac{dx}{ds},\qquad \frac{\partial M}{\partial Y}=x,$$

$$\frac{\partial N}{\partial Z}=0,\qquad \frac{\partial T}{\partial Z}=0,\qquad \frac{\partial M}{\partial Z}=1.$$

Upon substituting these twelve expressions in equations ($a$) and letting $a$ and $b$ denote the coordinates of the point $B$, one will find that:

(*c*)
$$\left\{\begin{aligned}
&X\int\left[\frac{\left(\dfrac{dx}{ds}\right)^2}{E\,\Omega}+\frac{\left(\dfrac{dy}{ds}\right)^2}{G\,\Omega}+\frac{y^2}{E\,I}\right]ds+Y\int\left[\left(\frac{1}{E\,\Omega}-\frac{1}{G\,\Omega}\right)\frac{dx}{ds}\frac{dy}{ds}\right]ds-Z\int\frac{y}{E\,I}\,ds\\
&\qquad=\tau a+\int\left[-\frac{\nu}{E\,\Omega}\frac{dx}{ds}+\frac{\theta}{G\,\Omega}\frac{dy}{ds}+\frac{\mu\,y}{E\,I}\right]ds,\\
&X\int\left[\left(\frac{1}{E\,\Omega}-\frac{1}{G\,\Omega}\right)\frac{dx}{ds}\frac{dy}{ds}-\frac{xy}{E\,I}\right]ds+Y\int\left[\frac{\left(\dfrac{dx}{ds}\right)^2}{E\,\Omega}+\frac{\left(\dfrac{dy}{ds}\right)^2}{G\,\Omega}+\frac{x^2}{E\,I}\right]ds+Z\int\frac{x}{E\,I}\,ds\\
&\qquad=\tau a-\int\left[\frac{\nu}{E\,\Omega}\frac{dx}{ds}+\frac{\theta}{G\,\Omega}\frac{dy}{ds}+\frac{\mu\,y}{E\,I}\right]ds,\\
&X\int\frac{y}{E\,I}\,ds-Y\int\frac{x}{E\,I}\,ds-Z\int\frac{1}{E\,I}\,ds=\int\frac{\mu}{E\,I}\,ds.
\end{aligned}\right.$$

These three equations yield the two forces and couple of the redundant constraints. One can then calculate the elements of the reduction *N*, *T*, and *M* that are attached to an arbitrary section by means of formulas (*b*).

### 24. – Example of the determination of redundant *internal* constraint forces

Consider an *arch that is anchored at its two extremities and includes a ball O* (Fig. 5).

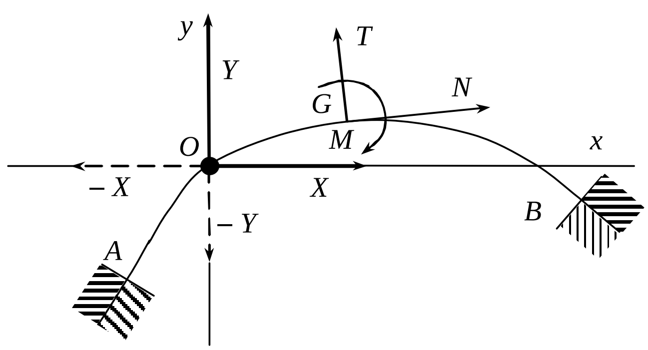


Figure 5.

That arch has a plane curve for its mean fiber and it is structurally symmetric with respect to the plane of that curve. It is subjected to arbitrary forces that are situated in that plane, as well as a temperature variation $\tau$, which is measured by starting from the temperature that is realized at its anchors. Its flexure will then be planar.

In reality, the arch considered is a system of two anchored arches *AO* and *OB*, each of which starts from one of the extremities of the arch and is joined to the other at the ball *O*. The action of the arch *AO* on the arch *OB* is that of a force that is applied to its extremity $O$, and the components of that force along the two rectangular axes $Ox$ and $Oy$, will be denoted by $X$ and $Y$, resp. The reaction of the arch $OB$ on the arch $AO$ is equal and opposite to that action.

It is clear that the system will become isostatic if one suppresses the constraint on the two arches at $O$. Consequently, $X$ and $Y$ will be two redundant internal constraint forces. By virtue of the generalized theorem of General Menabrea, one will then have:

$$\frac{\partial H}{\partial X} = 0, \qquad \frac{\partial H}{\partial Y} = 0,$$

with

$$H = \frac{1}{2}\int\left[\left(\frac{N^2}{E\,\Omega} - \alpha\tau\right) + \frac{T^2}{G\,\Omega} + \frac{M^2}{E\,I}\right] ds\,,$$

or rather:

$$(a) \qquad \begin{cases} \displaystyle\int\left[\left(\frac{N}{E\,\Omega} - \alpha\tau\right)\frac{\partial N}{\partial X} + \frac{T}{G\,\Omega}\frac{\partial T}{\partial X} + \frac{M}{E\,I}\frac{\partial M}{\partial X}\right] ds, \\ \displaystyle\int\left[\left(\frac{N}{E\,\Omega} - \alpha\tau\right)\frac{\partial N}{\partial Y} + \frac{T}{G\,\Omega}\frac{\partial T}{\partial Y} + \frac{M}{E\,I}\frac{\partial M}{\partial Y}\right] ds, \end{cases}$$

in which the integrals are extended along the total length $AOB$ of the mean fibers of the two arches.

Let $\nu$, $\theta$, and $\mu$ denote the values that the elements $N$, $T$, $M$, resp., of the reduction of the external forces that act from the left on an arbitrary section of either of the two arches take at the center of gravity $G$ ($x$, $y$) of that section if the system is made isostatic by suppressing the constraint on the two arches at $O$. $\nu$, $\theta$, and $\mu$ are immediately calculable by means of pure statics and can consequently be considered to be known in what follows.

It is easy to see that, regardless of whether the section belongs to one or the other of the two arches, one will have:

$$(b) \qquad \begin{cases} N = \nu + X\dfrac{dx}{ds} + Y\dfrac{dy}{ds}, \\ T = \theta - X\dfrac{dy}{ds} + Y\dfrac{dx}{ds}, \\ M = \mu - Xy + Yx, \end{cases}$$

and as a result:

$$\frac{\partial N}{\partial X} = \frac{dx}{ds}, \qquad \frac{\partial T}{\partial X} = -\frac{dy}{ds}, \qquad \frac{\partial M}{\partial X} = -y,$$

$$\frac{\partial N}{\partial Y} = \frac{dy}{ds}, \qquad \frac{\partial T}{\partial Y} = \frac{dx}{ds}, \qquad \frac{\partial M}{\partial Y} = x.$$

Upon substituting these nine expressions into equations (*a*) and letting *a* and *b* denote the projections of the entire mean fiber *AOB* onto *Ox* and *Oy*, resp., one will find that:

$$
(c)\quad \left\{
\begin{aligned}
&X\int\left[\frac{\left(\frac{dx}{ds}\right)^2}{E\,\Omega}+\frac{\left(\frac{dy}{ds}\right)^2}{G\,\Omega}+\frac{y^2}{E\,I}\right]ds+Y\int\left[\left(\frac{1}{E\,\Omega}-\frac{1}{G\,\Omega}\right)\frac{dx}{ds}\frac{dy}{ds}-\frac{xy}{E\,I}\right]ds\\
&\qquad=\alpha\tau a+\int\left[-\frac{\nu}{E\,\Omega}\frac{dx}{ds}+\frac{\theta}{G\,\Omega}\frac{dy}{ds}+\frac{\mu y}{E\,I}\right]ds,\\
&X\int\left[\left(\frac{1}{E\,\Omega}-\frac{1}{G\,\Omega}\right)\frac{dx}{ds}\frac{dy}{ds}-\frac{xy}{E\,I}\right]ds+Y\int\left[\frac{\left(\frac{dx}{ds}\right)^2}{E\,\Omega}+\frac{\left(\frac{dy}{ds}\right)^2}{G\,\Omega}+\frac{x^2}{E\,I}\right]ds\\
&\qquad=\alpha\tau a+\int\left[-\frac{\nu}{E\,\Omega}\frac{dx}{ds}+\frac{\theta}{G\,\Omega}\frac{dy}{ds}+\frac{\mu y}{E\,I}\right]ds.
\end{aligned}\right.
$$

These two equations yield the two redundant internal constraint forces. One can then calculate the elements of the reduction *N*, *T*, and *M* that are attached to an arbitrary section by formulas (*b*).

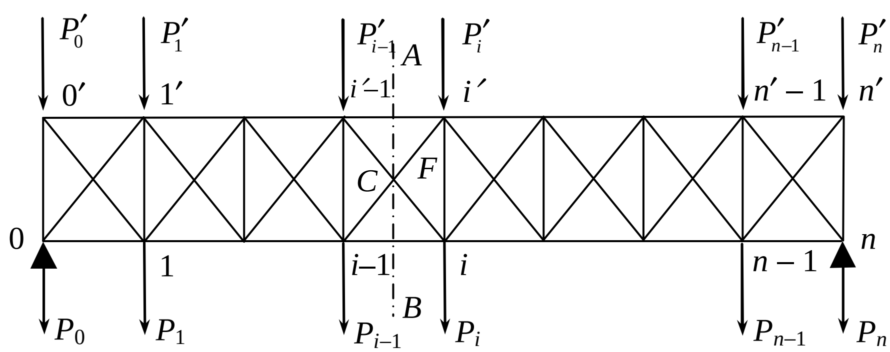


Figure 6.

## 25. – Another example of the determination of redundant *internal* constraint forces

As an example, take an *articulated straight beam with double lattice* that rests upon two simple supports (Fig. 6).

This beam consists of rectilinear bars whose mean fibers are situated in a vertical plane and are concurrent at points that are called *nodes*. At each node, the bars are assembled together by an articulation axis that is normal to the plane that contains their mean fibers.

One calls:

The horizontal bars *frame elements,*
The vertical bars *uprights,*
The inclined bars *diagonals*.

The part of the beam that consists of two consecutive uprights, along with the two frame elements and the two diagonals that are located between those two uprights bears the name of a *panel*. We shall denote the panels by the ordinal numbers 1, 2, …, $i$, …, $n - 1$, $n$, the lower nodes by the numbers 0, 1, …, $i$, …, $n - 1$, $n$, and the upper nodes by the same numbers with primes.

The beam is subject to vertical loads $P_0$, $P_1$, …, $P_i$, …, $P_{n-1}$, $P_n$, and loads of $P_0'$, $P_1'$, …, $P_i'$, …, $P_{n-1}'$, $P_n'$ are applied to the upper nodes.

The elastic forces in any section of any bar are reducible to a single force that is directed along the mean fiber of the bar. It is the normal effort, or if one uses the term that is currently adopted, the *effort in the bar* (the shearing effort and the flexure couple are zero).

We propose to calculate the efforts that are produced by the loads in all of the bars of the beam.

Let:

$X_i$ and $X_i'$ be the efforts in the two frame elements $(i - 1, i)$ and $(i' - 1, i')$. $\Omega_i$ and $\Omega_i'$ are the transverse sections of those two bars, and $a$ is their common length.

$Y_{i-1}$ and $Y_i$ be the efforts in the two uprights $(i - 1, i' - 1)$ and $(i, i')$. $\omega_{i-1}$ and $\omega_i$ are the transverse sections of those two uprights, and $b$ is their common length.

$Z_i$ and $Z_i'$, $S_i$ and $S_i'$, $c$ be the analogous quantities that relate to the two diagonals $(i' - 1, i)$ and $(i - 1, i')$.

Each of those efforts will be considered to be positive or negative according to whether the corresponding bar is compressed or tensed, resp.

The beam is externally isostatic, but internally hyperstatic. There exist a large number of ways of rendering it completely isostatic. We shall adopt the one that consists of suppressing the diagonal constraints $(0, 1')$, … $(i - 1, i')$, …, $(n - 1, n')$ whose lower nodes of assembly are 0, …, $i - 1$, …, $n - 1$, resp. Furthermore, those diagonals will no longer play a role in the beam. They will behave as if they were suppressed and will, in turn, become a *reticulated* system. Now, one knows that such a system is internally isostatic.

The constraints that are thus suppressed are the redundant internal constraints (no. **11**). The corresponding constraint forces are – for example, for the diagonal $(i - 1, i')$ – two equal and opposite forces that have the same direction as the effort $Z_i'$ in that diagonal, and one of them will apply a force to the extremity of the aforementioned that has been rendered free and the other one will apply a force to the node $i - 1$, and those forces will be repulsive or attractive according to whether $Z_i'$ is an effort of compression or an effort of tension, resp. One can then say that the redundant internal constraint forces are the efforts $Z'_1, \ldots, Z'_i, \ldots, Z'_n$ in the diagonals (0, 1′), …, ($i$ – 1, $i'$), …, ($n$ – 1, $n'$). Those diagonals bear the name of *redundant bars.*

If $Z'_i$ is a redundant constraint force then one will have:

$$(a) \qquad \frac{\partial H}{\partial Z'} = 0,$$

by virtue of General Menabrea’s theorem, where $H$ denotes the internal potential of the beam that is deformed by the loads.

That potential is equal to the sum of the potentials of all the bars that constitute the beam. Now, for an arbitrary bar, if one generally denotes its section by $\Omega$, its length by $s$, and the effort in that bar by $N$ then, from formulas (9′) and (11) (no. **9**), the internal potential will have the expression:

$$\frac{1}{2}\int_0^s \frac{N^2}{E\,\Omega}\,ds = \frac{1}{2}\frac{N^2 s}{E\,\Omega},$$

since the shearing effort and flexure couple will be zero in any section. Consequently:

$$\Pi = \sum \frac{1}{2}\frac{N^2 s}{E\,\Omega},$$

in which the sum is taken over all bars of the beam, or rather, if one lets $A$ denote the sum of the potentials in the bars that do not belong to the panel ($i$) and recalls the notations that were introduced above in relation to the bars of the panel:

$$\Pi = A + \frac{1}{2}\frac{X_i^2\,a}{E\,\Omega_i} + \frac{1}{2}\frac{X_i'^2\,a}{E\,\Omega'_i} + \frac{1}{2}\frac{X_{i-1}^2\,b}{E\,\omega_{i-1}} + \frac{1}{2}\frac{Y_i^2\,b}{E\,\omega_i} + \frac{1}{2}\frac{Z_i^2\,c}{E\,S_i} + \frac{1}{2}\frac{Z_i'^2\,c}{E\,S'_i}.$$

Substitute that expression for the potential in equation ($a$) and cancel the denominator $E$; one will get:

$$(b) \qquad E\frac{\partial A}{\partial Z'_i} + a\left(\frac{X_i}{\Omega_i}\frac{\partial X_i}{\partial Z'_i} + \frac{X'_i}{\Omega'_i}\frac{\partial X'_i}{\partial Z'_i}\right) + b\left(\frac{Y_{i-1}}{\omega_{i-1}}\frac{\partial Y_i}{\partial Z'_i} + \frac{Y_i}{\omega_i}\frac{\partial Y_i}{\partial Z'_i}\right) + c\left(\frac{Z_i}{S_i}\frac{\partial Z_i}{\partial Z'_i} + \frac{S'_i}{S'_i}\right) = 0.$$

Now calculate the expressions for $X_i$, $X'_i$, $Y_{i-1}$, $Y_i$, and $Z_i$ as functions of $Z'_i$ and substitute them in the last equation.

To that effect, cut the panel by a vertical plane $AB$ that passes through the point of intersection $C$ of the mean fibers of the diagonals of the panel ($i$). The external forces that act from the left of that section (loads and support reactions to the left of the panel) are reducible at the point $C$ to a vertical shearing effort $T_i$ that we measure as positive in the ascending sense and a flexure couple $M_i$. Those two quantities can be calculated immediately by means of pure statics, in such a way that we can consider them to be known in what follows. The four efforts $X_i$, $X'_i$, $Z_i$, and $Z'_i$ in the bar that is cut by the plane $AB$ define a system that is equivalent to the force $T_i$ and the couple $M_i$. One can then write the two equations for the projections onto a vertical axis and a horizontal axis, and the equations for the moments with respect to the point $C$ as these:

$$
\begin{gathered}
- Z_i \cos\alpha + Z'_i \cos\alpha = T_i\,, \\
X_i + X'_i + Z_i \sin\alpha + Z'_i \sin\alpha = 0, \\
- X_i \frac{b}{2} + X'_i \frac{b}{2} = M_i\,,
\end{gathered}
$$

in which $\alpha$ denotes the acute angle that is defined by the mean fibers of the diagonal and the vertical.

On the other hand, since the node $i' - 1$ is in equilibrium under the influence of the load $P'_{i-1}$ and the actions that are exerted upon it by the bars that are assembled at it, one will have the following equation for the projections onto a vertical axis:

$$Z'_{i-1} \cos\alpha + Y_{i-1} + Z_i \cos\alpha - P'_{i-1} = 0.$$

One infers from these four equations:

$$(c) \qquad Z_i = Z'_i - \frac{T_i}{\cos\alpha}\,,$$

$$(d) \qquad X_i = - Z'_i \sin\alpha + \frac{T_i \tan\alpha}{2} - \frac{M_i}{b}\,,$$

$$(e) \qquad X'_i = - Z'_i \sin\alpha + \frac{T_i \tan\alpha}{2} + \frac{M_i}{b}\,,$$

$$(f) \qquad Y_{i-1} = -(Z'_{i-1} + Z'_i)\cos\alpha + T_i + P'_{i-1}\,,$$

and when one changes $i$ into $i + 1$ in the last of those formulas:

$$(g) \qquad Y_i = -(Z'_i + Z'_{i+1})\cos\alpha + T_{i+1} + P'_i\,.$$

Upon differentiating the five formulas above with respect to $Z'_i$, one will have:

$$
(h) \qquad \left\{
\begin{array}{c}
\dfrac{\partial Z_i}{\partial Z'_i} = 1, \quad \dfrac{\partial X_i}{\partial Z'_i} = -\sin\alpha, \\[2ex]
\dfrac{\partial X'_i}{\partial Z'_i} = -\sin\alpha, \quad \dfrac{\partial Y_{i-1}}{\partial Z'_i} = -\cos\alpha, \quad \dfrac{\partial Y_i}{\partial Z'_i} = -\cos\alpha.
\end{array}
\right.
$$

If one sets $i = 1, 2, \ldots, i - 1$, and then $i = i + 1, i + 2, \ldots, n$, in succession, in formulas ($c$) to ($g$) then one will see immediately that the efforts in all of the bars of the beam besides those of the panel ($i$) are independent of $Z_i'$. As a result, the same thing will be true for the internal potential of those bars, and one will have, consequently:

$$\frac{\partial A}{\partial Z_i'} = 0. \tag{$i$}$$

Upon performing the substitutions in equation ($b$) that permit one to deduce formulas ($c$) and ($i$), and taking into account that:

$$\frac{a}{c} = \sin\alpha, \qquad \frac{b}{c} = \cos\alpha,$$

one will finally find that:

$$\frac{\cos^3\alpha}{\omega_{i-1}} Z_{i-1}' + \left[\sin^3\alpha\left(\frac{1}{\Omega_i} + \frac{1}{\Omega_i'}\right) + \cos^3\alpha\left(\frac{1}{\omega_{i-1}} + \frac{1}{\omega_i}\right) + \frac{1}{S_i} + \frac{1}{S_i'}\right] Z_i' + \frac{\cos^3\alpha}{\omega_i} Z_{i+1}' \tag{$k$}$$

$$= \left[\frac{\sin^3\alpha}{2\cos\alpha}\left(\frac{1}{\Omega_i} + \frac{1}{\Omega_i'}\right) + \frac{1}{S_i\cos\alpha}\right] T_i + \cos^3\alpha\left(\frac{T_i + P_{i-1}'}{\omega_{i-1}} + \frac{1}{\omega_i}\right) + \frac{\sin^2\alpha}{b}\left(\frac{1}{\Omega_i} - \frac{1}{\Omega_i'}\right) M_i \ .$$

That equation couples the efforts $Z_{i-1}'$, $Z_i'$, $Z_{i+1}'$ in the three consecutive redundant bars ($i - 2$, $i' - 1$), ($i - 1$, $i'$), and ($i$, $i' + 1$), resp. Upon successively setting $i = 1, 2, 3, \ldots, n - 1, n$ and taking into account the fact that $Z_0' = 0$, $Z_{n+1}' = 0$, $T_{n+1} =$ ([1]), one will obtain a system of $n$ equations, the first of which will contain $Z_1'$ and $Z_2'$, the second of which will contain $Z_1'$, $Z_2'$, and $Z_3'$, the third of which will contain $Z_2'$, $Z_3'$, and $Z_4'$, …, the ($n$ – 1)th of which will contain $Z_{n-2}'$, $Z_{n-1}'$, and $Z_n'$, and the $n$th of which will contain $Z_{n-1}'$ and $Z_n'$.

Solving that system will yield the efforts in the $n$ redundant bars (0, 1′), (1, 2′), …, ($n$ – 2, $n'$ – 1), ($n$ – 1, $n'$). The application of the general formulas ($c$) to ($g$) will then give the efforts in all of the other bars of the beam.

([1]) Which one will see immediately upon supposing that the beam is fictitiously prolonged to the left of its left supports by a panel that is indexed by 0 and to the right of its right support by a panel that is indexed by $n + 1$. The shearing efforts in those two addition panels will obviously be zero, and the same will be true for the efforts in their diagonals.

## 26. – Reciprocity principles

**Theorem:**

1. *If a force $F_\lambda$ that is equal to unity is applied to a point A of a body that is isotropic or has a mean fiber (or a system of such bodies that is isostatic or hyperstatic) along an arbitrarily-chosen direction $\Delta_A$ and produces an elastic displacement to a point B whose projection onto a likewise-arbitrary direction $\Delta_B$ is $\lambda_B^A$ then conversely, if a force $F_B$ that is equal to unity is applied to the point B along the direction $\Delta_B$ then it will produce an elastic displacement to the point A whose projection $\lambda_A^B$ onto the direction $\Delta_A$ will be equal to $\lambda_B^A$.*

2. *If a couple $C_{aa'}$ that is equal to unity and its two forces are applied at the two points a and a′ and its axis has a direction $\Gamma_{aa'}$ that is chosen arbitrarily from the perpendiculars to the line aa′ produces an elastic rotation of a line bb′ whose projection* ([1]) *onto a direction $\Gamma_{bb'}$ that is chosen arbitrarily from the perpendicular to the line bb′ is $\varphi_{bb'}^{aa'}$ then conversely a couple $C_{bb'}$ that is equal to unity whose two forces are applied at b and b′ and whose axis has a direction $\Gamma_{bb'}$ will produce an elastic rotation of the line aa′ whose projection $\varphi_{aa'}^{bb'}$ onto the direction $\Gamma_{aa'}$ is equal to $\varphi_{bb'}^{aa'}$.*

3. *If a couple $C_{aa'}$ that is equal to unity and its two forces are applied at the two points a and a′ and its axis has a direction $\Gamma_{aa'}$ that is chosen arbitrarily from the perpendiculars to the line aa′ produces an elastic displacement of the point B whose projection onto an arbitrary direction $\Delta_B$ is $\lambda_B^{aa'}$ then conversely a force $F_B$ that is equal to unity and is applied to B along the direction $\Delta_B$ will produce an elastic rotation of the line aa′ whose projection $\varphi_{aa'}^B$ onto the direction $\Gamma_{aa'}$ is equal to $\lambda_B^{aa'}$.*

(One should understand that from the viewpoint of homogeneity, that rotation is not measured by its angle, but by the lengths of the arch that is swept out by that angle on a circumference with a radius of unit length.)

Here is the proof of the first part of that theorem:

Let:

| | |
|---|---|
| $\lambda_A^A$ and $\lambda_B^A$ | be the projections onto the directions $\Delta_A$ and $\Delta_B$, respectively, of the elastic displacements that are produced at the points *A* and *B* when the unit force $F_A$ is applied to *A* along the direction $\Delta_B$ |

([1]) One should recall that one intends the term "projection of a rotation" to mean the *projection of the vector that represents that rotation* (no. **12**).

$\lambda_A^B$ and $\lambda_B^B$ be the projections onto those same directions of the elastic displacements that are produced at those same points when the unit force $F_B$ is applied to $B$ along the direction $\Delta_A$

(In these notations, the lower index denotes the point that submits to the displacement considered, while the upper index refers to the force that produced that displacement.)

Suppose that two forces $F'_A$ and $F'_B$, which differ from unity, are applied *simultaneously*, the former to $A$ along the direction $\Delta_A$, while the latter is applied to $B$ along the direction $\Delta_B$. Let $\lambda'_A$ and $\lambda'_B$ be the projections onto those two directions of the elastic displacements that they produce at the points $A$ and $B$, respectively. Now, from the principle of superposition of the elastic effects of the forces:

$$(a) \qquad \lambda'_A = \lambda_A^A F'_A + \lambda_A^B F'_B, \qquad \lambda'_B = \lambda_B^A F'_A + \lambda_B^B F'_B,$$

and by virtue of Castigliano's theorem (no. **14**):

$$(b) \qquad \lambda'_A = \frac{\partial \Pi}{\partial F'_A}, \qquad \lambda'_B = \frac{\partial \Pi}{\partial F'_B},$$

in which $\Pi$ denotes the internal potential of the body (or system of bodies) when it is deformed by the forces $F'_A$ and $F'_B$, acting simultaneously.

Now, analytically, one must have:

$$\frac{\partial^2 \Pi}{\partial F'_A \partial F'_B} = \frac{\partial^2 \Pi}{\partial F'_B \partial F'_A},$$

and as a result, because of relations ($b$):

$$\frac{\partial \lambda'_A}{\partial F'_B} = \frac{\partial \lambda'_A}{\partial F'_B},$$

or, upon replacing those two partial derivatives with their value that one would deduce from formulas ($a$):

$$\lambda_A^B = \lambda_B^A. \qquad \text{Q. E. D.}$$

The same method of proof will apply to the other two parts of the theorem.

The three reciprocity principles that were just stated are called *Maxwell's principles* abroad, although that scholar established only the first one and only for articulated systems ([1]).

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([1]) Clerk MAXWELL, "On the calculation of the equilibrium and stiffness of frames," Phil. Mag. **27** (1864), pp. 294.

## 27. – Lines of influence

The reciprocity principles lead to a general method for determining the *lines of influence* in bodies and systems of bodies with mean fibers that are subject to redundant constraints ([1]). Those lines, which were introduced into the resistance of materials by Fränkel ([2]) and were studied in a remarkable way by Winkler ([3]) and Maurice Levy ([4]), play an important role in the calculations for metal bridges or reinforced concrete. Indeed, they provide the means for determining the maximum efforts that are produced in the various elements of those bridges by the passage of moving overloads. However, the limited scope of the present note does not permit us to expand upon that subject.

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([1]) BERTRAND DE FONTVIOLANT, "Sur la détermination des forces élastiques et de leurs lignes d'influence dans les poutres assujetties à des liaisons surabondant," Comptes rendus de l'Académie des Sciences **108** (1889), pp. 45.

"Methode générale de determination des lignes d'influence dans les poutres pleines ou réticulaires, assujetties à des conditions surabondant," Bulletin de la Société des Ingenieurs civils de France, November 1890, pp. 742.

"Ponts métalliques à travées continues. Methode de calcul satisfaisant aux prescriptions du Réglement ministériel du 29 août 1891," Comptes rendus de l'Académie des Sciences **115** (1892), pp. 996 and Bulletin de la Société des Ingenieurs civils de France, December 1892, pp. 1105.

([2]) W. FRÄNKEL, "Ueber die ungünstige Einstellung eines Systems von Einzellasten auf Fachwerkungen mit Hilfe von Influenzkurven," Der Civilingenieur **22** (1876), 218, 441.

([3]) E. WINKLER, "Beitrag zur Theorie der Bogenträger," Zeitschrift des Architekten- und Ingenieur-Vereins zu Hannover **15** (1879), 199.

([4]) Maurice LEVY, *La Statique graphique et ses applications aux constructions*, Part 2, 1886, Part 3, 1887.

## CHAPTER III

# SECOND METHOD BASED UPON THE *VIS VIVA* THEOREM. GENERAL EQUATION OF ELASTICITY

**28.** – The presentation that follows will be noticeably different in form, but not in principle, from the one that was presented in our "Mémoire sur les déformations élastiques des pièces et des systèmes des pièces à fibres moyennes planes ou gauches" ([1]). It is more general, because it is concerned with not just pieces, but also isotropic bodies. Finally, it is simpler and quicker and necessitates no integration.

It consists of giving a new proof to the theorem of Betti ([2]), Boussinesq ([3]), and Maurice Levy ([4]) that is very elementary and appends a complement to that theorem that will imply a general relation between the elastic displacements and the external forces that produce them.

Upon introducing the caloric displacements into that relation (which was not done in our aforementioned paper), we will then obtain the *general equation of elasticity*, which synthesizes the entire theory of deformations and permits us to determine the elastic and caloric displacements of an arbitrary construction and to form the equations that are necessary for the calculation of the constraint forces in all cases, and without special analyses, in bodies and hyperstatic systems of bodies that are subject to arbitrary external forces, as well as caloric actions.

One will then find that a new proof of the *general equation of elasticity* has been presented that is based upon the *vis viva* theorem, and in our paper that relates to it ([5]), it was established by means of the virtual work theorem.

### 29. – Completion of the theorem of Betti, Boussinesq, and Maurice Levy

Consider an isostatic or hyperstatic (no. **11**) system of isotropic bodies or ones with mean fibers. (The case of a single body will be regarded as a special case.) Subject it to the action of a system of *m* arbitrary external forces that we shall call *system* (*A*), which are forces that increase slowly from zero up to certain final values. Let:

| | |
|---|---|
| $F_{A_i}$ | the final value of any of those forces |
| $A_i$ | its point of application |
| $\Delta_{A_i}$ | its direction |
| $\lambda_{A_i}^{A}$ | the projection onto the direction $\Delta_{A_i}$ of the elastic displacement of the point $A_i$ |

([1]) Comptes rendus de l'Académie des Sciences **107** (1888), pp. 383 and Bulletin de la Société des Ingénieurs civil de France, August 1888, pp. 291 and March 1889, pp. 416.

([2]) Betti, *Teoria del Elasticità*, 1872.

([3]) BOUSSINESQ, *Cours d'Analyse infinitésimale*, t. I, fasc. 2, 1887, pp. 127 and 128.

([4]) Maurice LEVY, Comptes rendus de l'Académie des Sciences **107** (1888), pp. 414.

([5]) Bulletin de la Société des Ingénieurs civil de France, October 1907, pp. 365. – *See* also LECORNU, *Cours de Mécanique professé à l'École Polytechnique*, t. III, 1918, pp. 45, 63, 76.

By virtue of Clapeyron's equation (13) (no. **12**), one will have:

$$(a) \qquad \tfrac{1}{2}\sum_{i=1}^{m} F_{A_i}\lambda_{A_i}^{A} = \Pi_{F_A},$$

when one denotes the internal potential of the deformed system of bodies by $\Pi_{F_A}$.

Replace the system of forces (*A*) with a second system (*B*) that is composed of *n* arbitrary external forces that increase like the first one. Let:

- $F_{B_i}$ be the final value of any of those forces
- $B_i$ be its point of application
- $\Delta_{B_i}$ be its direction
- $\lambda_{B_i}^{B}$ be the projection onto the direction $\Delta_{B_i}$ of the elastic displacement of the point $B_i$

As before, one will have:

$$(b) \qquad \tfrac{1}{2}\sum_{i=1}^{n} F_{B_i}\lambda_{B_i}^{B} = \Pi_{F_B},$$

when one denotes the internal potential of the deformed system of bodies by $\Pi_{F_B}$.

Now let:

- $\lambda_{A_i}^{B}$ be the projection onto the direction $\Delta_{A_i}$ of the elastic displacement of the point $A_i$ under the action of the system of forces (*B*)
- $\lambda_{B_i}^{A}$ be the projection onto the direction $\Delta_{B_i}$ of the elastic displacement of the point $B_i$ under the action of the system of forces (*A*)

Consider the deformation of the system of bodies, no longer under the action of only one of those two systems (*A*) and (*B*), but under the simultaneous action of both of them. By virtue of the superposition principle, the projection of the elastic displacement of the point $A_i$ onto the direction $\Delta_{A_i}$ will be $\lambda_{A_i}^{A} + \lambda_{A_i}^{B}$, and the projection of the point $B_i$ onto the direction $\Delta_{B_i}$ will be $\lambda_{B_i}^{A} + \lambda_{B_i}^{B}$. That deformation can be realized in two different ways, as follows:

1. First apply the system of forces (*A*), which increase from zero to their final values, and then apply the system (*B*), which increase in the same manner as the latter ones do. The deformation is thus accomplished over two time intervals.

During the first interval, the points of application $A_i$ of the forces in the system (*A*) experience the projections of the displacements $\lambda_{A_i}^{A}$ onto the direction $\Delta_{A_i}$, and the work done by those forces, which increase from zero to $F_{A_i}$, will be (no. **12**):

$$\tfrac{1}{2}\sum_{i=1}^{m} F_{A_i} \lambda_{A_i}^{A} ,$$

whereas the points $B_i$ will experience the projections of the displacements $\lambda_{B_i}^{A}$ onto the direction $\Delta_{B_i}$ .

During the second time interval, the points $A_i$ will experience the projections of the displacements $\lambda_{A_i}^{B}$ onto the directions $\Delta_{A_i}$ of the forces $F_{A_i}$ , and the work done by those forces, which remain constant, will be:

$$\tfrac{1}{2}\sum_{i=1}^{m} F_{A_i} \lambda_{A_i}^{B} ,$$

whereas the points of application $B_i$ of the forces in the system ($B$) will experience the projections $\lambda_{B_i}^{B}$ of the displacements onto the directions $\Delta_{B_i}$ of the $F_{B_i}$ , and the work done by those forces, which increase from zero to $F_{B_i}$ , will be:

$$\tfrac{1}{2}\sum_{i=1}^{n} F_{B_i} \lambda_{B_i}^{B} .$$

The total work done by the two systems of forces ($A$) and ($B$) is the sum of the three partial works above, and from Clapeyron's equation (13) (no. **12**), it will be equal to the internal potential of the deformed system of bodies under the simultaneous action of those two systems of forces. If one denotes that potential by $\Pi_{F_{A'},F_B}$ then one will have:

$$(c) \qquad \tfrac{1}{2}\sum_{i=1}^{m} F_{A_i} \lambda_{A_i}^{A} + \tfrac{1}{2}\sum_{i=1}^{m} F_{A_i} \lambda_{A_i}^{B} + \tfrac{1}{2}\sum_{i=1}^{n} F_{B_i} \lambda_{B_i}^{B} = \Pi_{F_{A'},F_B} .$$

Conversely, when one first applies the system of forces ($B$) and then the system of forces ($A$), that will lead to the equation:

$$(d) \qquad \tfrac{1}{2}\sum_{i=1}^{n} F_{B_i} \lambda_{B_i}^{B} + \tfrac{1}{2}\sum_{i=1}^{n} F_{B_i} \lambda_{B_i}^{A} + \tfrac{1}{2}\sum_{i=1}^{m} F_{A_i} \lambda_{A_i}^{A} = \Pi_{F_{A'},F_B} ,$$

in the same way as before, and that equation differs from equation ($c$) only by the permutation of the two systems of forces ($A$) and ($B$) and the corresponding elastic displacements.

Upon subtracting equation ($a$) and ($b$) from equation ($c$), and then from equation ($d$), one will obtain the new equations:

$$(e) \qquad \tfrac{1}{2}\sum_{i=1}^{m} F_{A_i} \lambda_{A_i}^{B} = \Pi_{F_{A'},F_B} - (\Pi_{F_{A'}} + \Pi_{F_B}) ,$$

(*f*) $$\tfrac{1}{2}\sum_{i=1}^{n} F_{B_i}\lambda_{B_i}^{A} = \Pi_{F_{A'},F_B} - (\Pi_{F_{A'}} + \Pi_{F_B}),$$

which translate into:

**Theorem –** *If one applies the two systems of forces* (*A*) *and* (*B*) *in succession to an isostatic or hyperstatic system that is isotropic or has a mean fiber then:*

1. *The sum of the works done by the forces in the system* (*A*) *under the elastic displacements of the system* (*B*) *will be equal to the sum of the works done by the forces of the system under the elastic displacements of the system* (*A*).

2. *The two sums of the works will be equal to the difference between the internal potential of the body (or system of bodies) that has been deformed by the two systems of forces* (*A*) *and* (*B*) *when they are applied simultaneously and the sum of the internal potentials of the body (or system of bodies) when they are deformed by each of those two force systems being applied to the exclusion of the other one.*

The first part of that proposition is the theorem of Betti, Boussinesq, and Maurice Levy. The second part is the complement that was announced above (no. **28**).

**Remark. –** Among the forces of the two systems (*A*) and (*B*), or only one of those systems, there can be ones that form couples. Hence, suppose that the system (*A*) is composed of *p* forces that do not form couples and 2*q* forces that form *q* couples. Let:

$a_i$ and $a_i'$ be the points of application of the two forces that form a couple

$C_{a_i a_i'}$ be the value of that couple

$\varphi^{B}_{a_i a_i'}$ be the projection of the elastic rotation that is given to the line $a_i\, a_i'$ by the system of forces (*B*) onto the axis of the couple $C_{a_i a_i'}$.

The work done by that couple for the rotational displacement is $C_{a_i a_i'}\,\varphi^{B}_{a_i a_i'}$, and as a result, one will have:

$$\tfrac{1}{2}\sum_{i=1}^{m} F_{A_i}\lambda_{A_i}^{B} = \tfrac{1}{2}\sum_{i=1}^{p} F_{A_i}\lambda_{A_i}^{B} + \sum_{i=1}^{q} C_{a_i a_i'}\,\varphi^{B}_{a_i a_i'},$$

which will permit one to write equation (*e*) in the new form:

(29) $$\tfrac{1}{2}\sum_{i=1}^{p} F_{A_i}\lambda_{A_i}^{B} + \sum_{i=1}^{q} C_{a_i a_i'}\,\varphi^{B}_{a_i a_i'} = \Pi_{F_{A'},F_B} - (\Pi_{F_{A'}} + \Pi_{F_B}),$$

if one would like to exhibit the projected rotations $\varphi^{B}_{a_i a_i'}$.

## 30. – Reciprocity principles

The three reciprocity principles that were established before (no. **26**) are particular cases of the theorem of Betti, Boussinesq, and Maurice Levy. The first one corresponds to the case in which the systems ($A$) and ($B$) are each composed of a single unit force, the second one, to the case in which they are each composed of a unit couple, and the third, to the case in which system ($A$) is composed of a unit couple and the system ($B$) is composed of a unit force.

## 31. – General relation between the elastic displacements and the external forces that produce them

Let a system consist of bodies that are isotropic or have mean fibers. Suppose that it is externally and internally hyperstatic (no. **11**). (The case of an isostatic system will be regarded as a special case, and likewise that of a single body.) That system of bodies is deformed by external forces $F$ that are applied directly.

Let:

1.
$$\lambda_1, \lambda_2, \lambda_3, \ldots$$

denote the projections of the elastic displacements at a certain number of points:

$$A_1, A_2, A_3, \ldots$$

of the system of bodies onto the arbitrarily-chosen directions:

$$\Delta_1, \Delta_2, \Delta_3, \ldots,$$

and let:

2.
$$\varphi_1, \varphi_2, \varphi_3, \ldots$$

denote the projections of the elastic rotations of a certain number of lines:

$$a_1 a_1', \qquad a_2 a_2', \qquad a_3 a_3'$$

that are contained in the system of bodies onto the directions:

$$\Gamma_1, \Gamma_2, \Gamma_3, \ldots,$$

which are chosen arbitrarily from the directions normal to those lines.

Now consider the isostatic system that is obtained by suppressing the redundant constraints in the given hyperstatic system ([1]).

Call the redundant external constraint forces $F_{es}$ and call the redundant internal constraint forces of the hyperstatic system $F_{is}$ . Among those forces, one can have ones that form couples, which are of little importance, so it would be pointless to exhibit those couples here.

If one subjects the aforementioned isostatic system to the forces $F$, $F_{es}$ , and $F_{is}$ then it will take on a state of elastic equilibrium that is identical to that of the hyperstatic system when it is subject only the forces $F$. In particular, the elastic displacements of the points $A_1$, $A_2$, $A_3$, … and the elastic rotations of the lines $a_1\,a_1'$, $a_2\,a_2'$, $a_3\,a_3'$, … in that isostatic system are the same as the ones in the hyperstatic system.

Suppress the forces $F$, $F_{0s}$ , $F_{1s}$ from the isostatic system, which will no longer be subject to any forces then, and apply:

1. Some forces of arbitrary magnitude:

$$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$$

that one calls *auxiliary forces* to the points:

$$A_1 , A_2 , A_3 , \ldots$$

of that system along the directions:

$$\Delta_1 , \Delta_2 , \Delta_3 , \ldots$$

2. Some forces that form couples of arbitrary magnitude:

$$\mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_3 , \ldots,$$

which are called *auxiliary couples*, along axes in the directions:

$$\Gamma_1 , \Gamma_2 , \Gamma_3 , \ldots$$

at the points:

$$a_1 , a_1' , a_2 , a_2' , a_3 , a_3' , \ldots$$

Having said that, apply equation (29) (no. **29**, the complete theorem of Betti, Boussinesq, and Maurice Levy) to the isostatic system, while one considers the system of forces ($A$) in that equation to be composed of auxiliary forces and couples $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, …,

([1]) It was pointed out before (no. **11**) that depending upon the number and nature of those constraints, a hyperstatic system can be made isostatic in just one way or several of them. Here, we shall consider any one of the isostatic systems that are obtained in some way.

$\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, …, and the system of forces (*B*) to be composed of the forces *F*, $F_{es}$, and $F_{is}$. One will get:

$$\mathcal{F}_1\,\lambda_1 + \mathcal{F}_2\,\lambda_2 + \mathcal{F}_3\,\lambda_3 + \ldots,\ \mathcal{C}_1\,\varphi_1 + \mathcal{C}_2\,\varphi_2 + \mathcal{C}_3\,\varphi_3 + \ldots$$

$$= \bar{\Pi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} - (\bar{\Pi}_{F,F_{es},F_{is}} + \bar{\Pi}_{\mathcal{F},\mathcal{C}}),$$

in which $\bar{\Pi}_{F,F_{es},F_{is}}$, $\bar{\Pi}_{\mathcal{F},\mathcal{C}}$, $\bar{\Pi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}}$ denote the total internal potential of the isostatic system when it is deformed by:

1. The given forces *F* and the redundant constraint forces $F_{es}$ and $F_{is}$ of the hyperstatic system,

2. The auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$,

3. The totality of all forces and couples,

respectively. (The overbar on the symbol Π is intended to indicate that those potentials relate to the isostatic system.)

We write:

(30) $$\sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = \bar{\Pi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} - (\bar{\Pi}_{F,F_{es},F_{is}} + \bar{\Pi}_{\mathcal{F},\mathcal{C}}),$$

to abbreviate.

### 32. – General equation of elasticity

Suppose that the hyperstatic system of bodies that was considered before (no. **31**) is subject to not only some external forces *F* that are applied directly, but also to a temperature variation $\tau$ that is measured by starting from the temperature that is realized at its constraints and will be positive for an increase and negative for a decrease. The deformation of that system will then be both elastic and caloric.

Consider a certain number of points and lines in the system, and let:

1. $\lambda$ be the projection of the elastic and caloric displacement of any one of those points *A* onto an arbitrarily-chosen direction.

2. $\varphi$ be the projection of the elastic and caloric rotation of any one of those lines $aa'$ onto a direction that is chosen arbitrarily from the normals to $aa'$.

Imagine the isostatic system that is obtained by suppressing the redundant constraints from the given hyperstatic system. Subject it to the external forces *F*, redundant constraint forces $F_{es}$ and $F_{is}$ (no. **31**) of the hyperstatic system, and the temperature variation $\tau$. It will take on a state of elastic and caloric deformation that is identical to that of the hyperstatic system when it is subject to only the forces *F* and the temperature variation $\tau$. In particular, the values of $\lambda$ and $\varphi$ are the same in the hyperstatic system and in the isostatic one.

Let $\lambda'$ and $\varphi'$ denote the values that $\lambda$ and $\varphi$ would take if the isostatic system were subject to only the forces $F$, $F_{es}$, and $F_{is}$ ; i.e., if the deformation were purely elastic. Let $\lambda_\tau$ and $\varphi_\tau$ be the values that those same quantities would take if the system were subject to only the variation of temperature $\tau$; i.e., if its deformation were purely caloric.

By virtue of the principle of superposition of elastic and caloric effects, one will have:

$$(a) \qquad \lambda = \lambda' + \lambda_\tau , \qquad \varphi = \varphi' + \varphi_\tau .$$

Having said that, we shall examine the aforementioned elastic deformation and caloric deformation separately.

As far as the first one is concerned, from the relation (30) (no. **31**), one will have immediately:

$$(b) \qquad \sum \mathcal{F}\,\lambda' + \sum \mathcal{C}\,\varphi' = \bar{\Pi}_{F, F_{es}, F_{is}, \mathcal{F}, \mathcal{C}} - (\bar{\Pi}_{F, F_{es}, F_{is}} + \bar{\Pi}_{\mathcal{F}, \mathcal{C}}) ,$$

in which $\mathcal{F}$ and $\mathcal{C}$ denote the auxiliary forces that are applied to the points $A$ along the directions $\Delta$ and the auxiliary couples whose two forces are applied at the extremities of the lines $aa'$ and whose axes have the directions $\Gamma$, respectively.

In order to study the second deformation, imagine that one has suppressed the forces $F$, $F_{es}$, $F_{is}$ and applied the same auxiliary forces and couples as above to the isostatic system. The system will take on a certain state of deformation. Let:

$\lambda''$ be the projection of the elastic displacement of any one of the points $A$ onto the direction $\Delta$.

$\varphi''$ be the projection of the elastic rotation of any one of the lines $aa'$ onto the direction $\Gamma$.

One has:

$$\frac{1}{2}\left(\sum \mathcal{F}\,\lambda'' + \sum \mathcal{C}\,\varphi''\right) = \Pi_{\mathcal{F}, \mathcal{C}} ,$$

by virtue of Clapeyron's equation (13) (no. **12**).

Now, subject the system to the temperature variation $\tau$. It will take on a caloric deformation that is superimposed with its elastic deformation, in such a way that the projection of the displacement of any of the points $A$ and the projection of the rotation of any of the lines $aa'$ will become:

$$\lambda'' + \lambda_\tau , \qquad \varphi'' + \varphi_\tau ,$$

respectively, and by virtue of the generalized Clapeyron equation (15) (no. **13**), one will have:

$$\frac{1}{2}\Big[\sum \mathcal{F}(\lambda'' + \lambda_\tau) + \sum \mathcal{C}(\varphi'' + \varphi_\tau)\Big] = \bar{\Pi}_{\mathcal{F},\mathcal{C}} - \tfrac{1}{2}\alpha\tau \iiint (n'_x + n'_y + n'_z)\, dx\, dy\, dz\,,$$

in which $n'_x$, $n'_y$, and $n'_z$ denote the normal fatigues that are produced by the auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$ on three elements that are normal to the coordinate axes, when taken at an arbitrary point $(x, y, z)$ of the isostatic system.

Upon subtracting corresponding sides of the two equations above, one will obtain:

$$(c) \qquad \sum \mathcal{F}\,\lambda_\tau + \sum \mathcal{C}\,\varphi_\tau = -\alpha\tau \iiint (n'_x + n'_y + n'_z)\, dx\, dy\, dz\,.$$

Add the corresponding sides of the two equations (*b*) and (*c*), while recalling the relations (*a*). One will finally have:

$$(31) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = \bar{\Pi}_{F, F_{es}, F_{is}, \mathcal{F}, \mathcal{C}} - (\bar{\Pi}_{F, F_{es}, F_{is}} + \bar{\Pi}_{\mathcal{F},\mathcal{C}}) - \alpha\tau \iiint (n'_x + n'_y + n'_z)\, dx\, dy\, dz\,.$$

That is the *general equation of elasticity for isotropic systems of bodies.*

In the mathematical theory of elasticity, if one knows how to form the expressions for the normal and tangential fatigues, and as a result, from formulas (3) and (5) (no. **4**), that of the internal potential, as functions of the external forces that produce those fatigues then the equation above will permit one to solve all of the problems that relate to the elastic and caloric deformation of constructions without it being necessary to appeal to the hypotheses of the resistance of materials. An important advance would then have been achieved if one managed to overcome the difficulties that are presented by the integration of the partial differential equations of the mathematical theory of elasticity.

**33.** – Upon repeating the preceding proof in the case of a system with mean fibers, one will find that:

$$(32) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = \bar{\Pi}_{F, F_{es}, F_{is}, \mathcal{F}, \mathcal{C}} - (\bar{\Pi}_{F, F_{es}, F_{is}} + \bar{\Pi}_{\mathcal{F},\mathcal{C}}) - \alpha\tau \iiint \mathcal{N}\, ds\,,$$

in which $\mathcal{N}$ denotes the normal effort that is produced on an arbitrary section of any one of the bodies of the isostatic system that is obtained by suppressing the redundant constraints of the given hyperstatic system under the action of the auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$.

One can, moreover, pass directly from equation (31) to equation (32) by applying the relation (17) (no. **13**), which will give:

$$\iiint (n'_x + n'_y + n'_z)\, dx\, dy\, dz = \int \mathcal{N}\, ds$$

here.

The right-hand side of equation (32) is transformed as follows:

From formula (11) (no. **9**), one can write:

$$(a) \qquad \bar{\Pi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} - (\bar{\Pi}_{F,F_{es},F_{is}} + \bar{\Pi}_{\mathcal{F},\mathcal{C}}) = \int \left[ \varpi_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} - (\varpi_{F,F_{es},F_{is}} + \varpi_{\mathcal{F},\mathcal{C}}) \right] ds ,$$

in which one lets $\varpi_{F,F_{es},F_{is}}$, $\varpi_{\mathcal{F},\mathcal{C}}$, $\varpi_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}}$ denote the internal potentials per unit length of the mean fiber of any one of the bodies of the previously-defined *isostatic system* when one deforms by means of:

1. Forces $F$ that are applied directly and redundant constraint forces $F_{es}$ and $F_{is}$ of the given hyperstatic system,

2. Auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$,

3. The totality of those forces and couples,

respectively. Let:

1. $\qquad N, \quad T_\eta, \quad T_\zeta, \quad M_\xi, \quad M_\eta, \quad M_\zeta$

denote the elements of the reduction (at the center of gravity of an arbitrary section of any of the bodies) of the elastic forces that are developed in that section by applying the force $F$, $F_{es}$, and $F_{is}$ to the *isostatic* system (those elements are identically the same as in the hyperstatic system when it is subject to only the forces $F$).

Let:

2. $\qquad \mathcal{N}, \quad \mathcal{T}_\eta, \quad \mathcal{T}_\zeta, \quad \mathcal{M}_\xi, \quad \mathcal{M}_\eta, \quad \mathcal{M}_\zeta$

be the analogous reduction elements when the isostatic system is assumed to be subject to the forces and couples $\mathcal{F}$ and $\mathcal{C}$.

By virtue of the superposition principle, if that system were subject to the simultaneous action of forces and couples $F$, $F_{es}$, $F_{is}$, $\mathcal{F}$, and $\mathcal{C}$ then the reduction elements would become:

$$N + \mathcal{N}, \qquad T_\eta + \mathcal{T}_\eta, \qquad T_\zeta + \mathcal{T}_\zeta, \qquad M_\xi + \mathcal{M}_\xi, \qquad M_\eta + \mathcal{M}_\eta, \qquad M_\zeta + \mathcal{M}_\zeta .$$

Having said that, from formula (9) (no. **9**), one will have:

$$\varpi_{F,F_{es},F_{is}} = \frac{1}{2}\left( \frac{N^2}{E\Omega} + \frac{T_\eta^2}{G\Omega} + \cdots + \frac{M_\zeta^2}{E I_\eta} \right).$$

Now, that expression for $\bar{\varpi}_{F,F_{es},F_{is}}$ is a homogeneous function of degree two in the six quantities $N$, $T_\eta$, …, $M_\zeta$, and in order to obtain the expressions for $\bar{\varpi}_{\mathcal{F},\mathcal{C}}$ and $\bar{\varpi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}}$, it will suffice to replace those six quantities with, first of all $\mathcal{N}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\zeta$, and then $N + \mathcal{N}$, $T_\eta + \mathcal{T}_\eta$, …, $M_\zeta + \mathcal{M}_\zeta$. From a property of homogeneous functions of degree two ([1]), one will then have:

$$\bar{\varpi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} - (\bar{\varpi}_{F,F_{es},F_{is}} + \bar{\varpi}_{\mathcal{F},\mathcal{C}}) = \mathcal{N}\,\frac{\partial \bar{\varpi}_{F,F_{es},F_{is}}}{\partial N} + \mathcal{T}_\eta\,\frac{\partial \bar{\varpi}_{F,F_{es},F_{is}}}{\partial T_\eta} + \cdots + \mathcal{M}_\eta\,\frac{\partial \bar{\varpi}_{F,F_{es},F_{is}}}{\partial M_\zeta},$$

or, upon replacing the partial derivatives with their values that one deduces from the expression for $\bar{\varpi}_{F,F_{es},F_{is}}$:

$$\bar{\varpi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} - (\bar{\varpi}_{F,F_{es},F_{is}} + \bar{\varpi}_{\mathcal{F},\mathcal{C}}) = \mathcal{N}\,\frac{\mathcal{N}}{E\,\Omega} + \mathcal{T}_\eta\,\frac{T_\eta}{G\,\Omega} + \cdots + \mathcal{M}_\zeta\,\frac{M_\zeta}{E\,I_\zeta}.$$

As a result, upon substituting this into formula (*a*), one will have:

$$(b) \qquad \bar{\Pi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} - (\bar{\Pi}_{F,F_{es},F_{is}} + \bar{\Pi}_{\mathcal{F},\mathcal{C}}) = \int \left[\mathcal{N}\,\frac{\mathcal{N}}{E\,\Omega} + \mathcal{T}_\eta\,\frac{T_\eta}{G\,\Omega} + \cdots + \mathcal{M}_\zeta\,\frac{M_\zeta}{E\,I_\zeta}\right] ds,$$

and equation (32) will become:

$$(33) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi$$
$$= \int \left[\mathcal{N}\left(\frac{N}{E\,\Omega} - \alpha\tau\right) + \mathcal{T}_\eta\,\frac{T_\eta}{G\,\Omega} + \mathcal{T}_\zeta\,\frac{T_\zeta}{G\,\Omega} + \mathcal{M}_\xi\,\frac{M_\xi}{G\,I_\xi} + \mathcal{M}_\eta\,\frac{M_\eta}{E\,I_\eta} + \mathcal{M}_\zeta\,\frac{M_\zeta}{E\,I_\zeta}\right] ds,$$

moreover.

In the applications, it is important to not lose sight of the facts that:

1. The projections of the displacements $\lambda$ and $\gamma$, as well as the elements $N$, $T_\eta$, …, $M_\zeta$ of the reduction at the center of gravity of an arbitrary section of any of the bodies of the elastic forces that act in that section (or, what amounts to the same thing, of the

([1]) Here is the statement of that property, which was established by Euler. Let:
$f(x, y, …)$ be a function that is homogeneous of degree two in an arbitrary number of variables $x, y, …$
$x', y', …$ be a system of arbitrary values that one attributes to those variables.
One will have identically:

$$f[(x + x'), (y + y', …)] - [f(x, y, …) + f(x', y', …)]$$
$$= x'\,\frac{\partial f(x, y, \cdots)}{\partial x} + y'\,\frac{\partial f(x, y, \cdots)}{\partial y} + \ldots = x\,\frac{\partial f(x', y', \cdots)}{\partial x'} + y\,\frac{\partial f(x', y', \cdots)}{\partial y'} + \ldots$$

external forces that are applied to the left of that section), relate to the given *hyperstatic* system when it is subjected to the given forces $F$.

2. The similar reduction elements $\mathcal{N}$, $\mathcal{T}_\eta$ , …, $\mathcal{M}_\zeta$ relate to the *isostatic* system that is obtained by suppressing the redundant external and internal constraints from the given hyperstatic system, which is an isostatic system that is subjected to auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$. The latter elements can be calculated very easily, since they are statically determinate.

It is obvious that equation (33) will remain valid in the case where the given system is isostatic. All of the quantities that enter into it then relate to that isostatic system.

**34. – Elastic and caloric rotation of an arbitrary section.** – From the standpoint of applications, it is useful to introduce into equation (33) (no. **33**) the projection onto an arbitrarily-chosen direction of the elastic and caloric rotation of an arbitrary transverse section ($S$) of any of the bodies. Here is how one does that:

Draw a line $aa'$ in the section considered ($S$) in a direction that is normal to $\Gamma$. [That is always possible; in order to see that, it will suffice that $aa'$ should be directed along the intersection of the section ($S$) and a plane normal to $\Gamma$]. Apply two forces at $a$ and $a'$ that form an auxiliary couple $\mathcal{C}$ whose axis is directed along $\Gamma$. A term $\mathcal{C}\ \varphi$ in equation (33) will correspond to that that couple $\mathcal{C}$, where $\varphi$ denotes the projection of the rotation of the line $aa'$ onto the direction $\Gamma$ of rotation. Now, one can use the kinematical considerations that used already (no. **15**) in order to prove Corollary III of Castigliano's theorem to easily establish that this projection is equal to that of the rotation of the section ($S$) around the direction $\Gamma$. One can then say that in each of the terms $\mathcal{C}\ \varphi$ of equation (33), $\varphi$ represents the projection of the rotation of an arbitrary section ($S$) onto an arbitrarily-chosen direction $\Gamma$ under the conditions that the auxiliary couple $\mathcal{C}$ must be applied to that section and that its axis must have the direction $\Gamma$.

**35. Case of planar flexure.** – If the body or (isostatic or hyperstatic) system of bodies considered is subject to planar flexure in every section then the elements of the reduction of the elastic forces will be exclusively a normal effort $N$, a shearing effort $T$ that is situated in the plane of reduction or of the mean fibers (viz., the plane of flexure), and a flexure couple $M$ with an axis that is normal to that plane. As a result, equation (33) (no. **33**) will reduce to:

$$\sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = \int \left[ \mathcal{N}\left(\frac{N}{E\,\Omega} - \alpha\tau\right) + \mathcal{T}\,\frac{T}{G\,\Omega} + \mathcal{M}\,\frac{M}{E\,I}\right] ds\,. \tag{33′}$$

In order to use the latter formula, one applies:

1. Auxiliary forces $\mathcal{F}$ to points that are situated in the plane of flexure and along the direction $\Delta$ that are drawn in that plane.

2. Couples $\mathcal{C}$ to the transverse section whose axes are directed normally to that plane.

It will then follow that the quantities $\lambda$ will be the projections of the displacements of the points considered onto the directions $\Delta$, and that the quantities $\varphi$ will be the rotations of the sections considered.

### 36. – General expression for elastic and caloric displacements in bodies and systems of bodies with mean fibers

We propose to form the general expression for:

1. The projection $\lambda$ of the elastic and caloric displacement of an arbitrary point $A$ onto an arbitrarily-chosen direction $\Delta$.

2. The projection $\varphi$ of the elastic and caloric rotation of an arbitrary section ($S$) onto an arbitrarily-chosen axis $\Gamma$.

Those two expressions follow immediately from the general equation of elasticity, when it is taken in the form (33) (no. **33**).

In order to obtain the first one, it will suffice to introduce into that equation just one auxiliary force $\mathcal{F}$ of arbitrary magnitude that is applied at $A$ in the direction $\Delta$. One will then obtain:

$$\lambda = \int \left[ \frac{\mathcal{N}}{\mathcal{I}} \left( \frac{N}{E\,\Omega} - \alpha\tau \right) + \frac{\mathcal{T}_\eta}{\mathcal{I}} \frac{T_\eta}{G\,\Omega} + \cdots + \frac{\mathcal{M}_\zeta}{\mathcal{I}} \frac{M_\zeta}{E\,I_\zeta} \right] ds \,. \tag{34}$$

In order to obtain the second one, it will suffice to introduce into that same equation just one auxiliary couple $\mathcal{C}$ of arbitrary magnitude about an axis in the direction $\Gamma$, which will give:

$$\varphi = \int \left[ \frac{\mathcal{N}}{\mathcal{I}} \left( \frac{N}{E\,\Omega} - \alpha\tau \right) + \frac{\mathcal{T}_\eta}{\mathcal{C}} \frac{T_\eta}{G\,\Omega} + \cdots + \frac{\mathcal{M}_\zeta}{\mathcal{C}} \frac{M_\zeta}{E\,I_\zeta} \right] ds \,. \tag{35}$$

One should not lose sight of the fact that if the system considered is hyperstatic then the reduction elements $\mathcal{N}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\zeta$ must be calculated in the isostatic system that is obtained by suppressing the redundant constraints from that hyperstatic system (no. **33**, *in fine*).

**Case of a system subject to planar flexure.** – In that case, the preceding formulas will reduce to:

$$\lambda = \int \left[ \frac{\mathcal{N}}{\mathcal{I}} \left( \frac{N}{E\Omega} - \alpha\tau \right) + \frac{\mathcal{T}}{\mathcal{I}} \frac{T}{G\Omega} + \frac{\mathcal{M}}{\mathcal{I}} \frac{M}{EI} \right] ds , \tag{34′}$$

$$\varphi = \int \left[ \frac{\mathcal{N}}{\mathcal{I}} \left( \frac{N}{E\Omega} - \alpha\tau \right) + \frac{\mathcal{T}}{\mathcal{C}} \frac{T}{G\Omega} + \frac{\mathcal{M}}{\mathcal{C}} \frac{M}{EI} \right] ds . \tag{35′}$$

Recall that in the latter case, the axis of the auxiliary couple $\mathcal{C}$ that is applied to the section considered ($S$) must be directed normally to the plane of flexure, and that $\varphi$ is the rotation of that section.

## 37. – Ernest Flamard's formulas

In the general equation of elasticity (32) (no. **33**), which relates to systems of bodies with mean fibers, one can first introduce just one auxiliary force $\mathcal{F}$ and then just one auxiliary couple $\mathcal{C}$ and obtain the two formulas:

$$(a) \qquad \left\{ \begin{aligned} \lambda &= \frac{1}{\mathcal{F}} \left[ \bar{\Pi}_{F, F_{es}, F_{is}, \mathcal{F}} - (\bar{\Pi}_{F, F_{es}, F_{is}} + \bar{\Pi}_{\mathcal{F}}) - \alpha\tau \int \mathcal{N}\, ds \right], \\ \varphi &= \frac{1}{\mathcal{C}} \left[ \bar{\Pi}_{F, F_{es}, F_{is}, \mathcal{C}} - (\bar{\Pi}_{F, F_{es}, F_{is}} + \bar{\Pi}_{\mathcal{C}}) - \alpha\tau \int \mathcal{N}\, ds \right], \end{aligned} \right.$$

and if the system of bodies is not subject to any temperature variation then they will reduce to:

$$(b) \qquad \left\{ \begin{aligned} \lambda &= \frac{1}{\mathcal{F}} \left[ \bar{\Pi}_{F, F_{es}, F_{is}, \mathcal{F}} - (\bar{\Pi}_{F, F_{es}, F_{is}} + \bar{\Pi}_{\mathcal{F}}) \right], \\ \varphi &= \frac{1}{\mathcal{C}} \left[ \bar{\Pi}_{F, F_{es}, F_{is}, \mathcal{C}} - (\bar{\Pi}_{F, F_{es}, F_{is}} + \bar{\Pi}_{\mathcal{C}}) \right], \end{aligned} \right.$$

just as one can easily verify that if, on the contrary, the system is subject to a temperature variation then one can write the two formulas ($a$) in the form:

$$(c) \qquad \left\{ \begin{aligned} \lambda &= \frac{1}{\mathcal{F}} \left[ \bar{H}_{F, F_{es}, F_{is}, \mathcal{F}} - (\bar{H}_{F, F_{es}, F_{is}} + \bar{H}_{\mathcal{F}}) \right], \\ \varphi &= \frac{1}{\mathcal{C}} \left[ \bar{H}_{F, F_{es}, F_{is}, \mathcal{C}} - (\bar{H}_{F, F_{es}, F_{is}} + \bar{H}_{\mathcal{C}}) \right], \end{aligned} \right.$$

when one introduces the function $H$ that is expressed by formula (25) (no. **19**) and agrees that $\bar{H}_{F, F_{es}, F_{is}}$ represents the value of the function $H$ that is attached to the isostatic system that is subjected to the forces $F$, $F_{es}$, $F_{is}$, and the temperature variation $\tau$, and that $\bar{H}_{F,F_{es},F_{is},\mathcal{F}}$ and $\bar{H}_{F,F_{es},F_{is},\mathcal{C}}$ represent the values that the same function takes when the isostatic system is subjected to the auxiliary force $\mathcal{F}$ or the auxiliary couple $\mathcal{C}$, in addition.

Formulas ($b$) and ($c$) were established by Ernest Flamard in his previously-cited doctoral thesis, although in a different form from what we just indicated. They remain valid in the case of isotropic systems.

## 38. – Agreement between the results of the two methods based upon the *vis viva* theorem

Let us compare the general expressions (27) (no. **21**) and (34) (no. **36**) for the projection of the elastic and caloric displacement of an arbitrary point $A$ onto an arbitrarily-chosen direction Δ:

$$\lambda = \int\left[\left(\frac{N}{E\,\Omega}-\alpha\tau\right)\frac{d\mathcal{N}}{d\mathcal{I}}+\frac{T_\eta}{G\,\Omega}\frac{d\mathcal{T}_\eta}{d\mathcal{I}}+\cdots+\frac{M_\zeta}{E\,I_\zeta}\frac{d\mathcal{M}_\zeta}{d\mathcal{I}}\right]ds\,,$$

$$\lambda = \int\left[\frac{\mathcal{N}}{\mathcal{I}}\left(\frac{N}{E\,\Omega}-\alpha\tau\right)+\frac{\mathcal{T}_\eta}{\mathcal{I}}\frac{T_\eta}{G\,\Omega}+\cdots+\frac{\mathcal{M}_\zeta}{\mathcal{I}}\frac{M_\zeta}{E\,I_\zeta}\right]ds\,,$$

which are expressions that are obtained by the first and second method, respectively, based on the *vis viva* theorem.

In those two formulas, $\mathcal{N}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\zeta$ are the elements of the reduction at the center of gravity of an arbitrary section of any of the bodies of the elastic forces that are developed in that section by applying the auxiliary force $\mathcal{F}$ to the point $A$ along the direction Δ, and if the system is hyperstatic then those elements that one uses in one or the other of the two formulas must be calculated in the isostatic system that is obtained suppressing the redundant constraints of that hyperstatic. Now, by virtue of the principle of superposition of the effects of the forces, the elastic forces in an arbitrary section, and as a result, the elements $\mathcal{N}$, $\mathcal{T}$, …, $\mathcal{M}$ of their reduction, will be proportional to the external force $\mathcal{F}$ that produces them. One can then write:

$$\mathcal{N} = a\,\mathcal{F}, \qquad \mathcal{T}_\eta = b\,\mathcal{F}, \;\ldots, \quad \mathcal{M}_\zeta = f\,\mathcal{F},$$

in which $a$, $b$, …, $f$ denote six constants that are independent of $\mathcal{F}$. Hence:

$$\frac{d\mathcal{N}}{d\mathcal{I}} = a, \qquad \frac{d\mathcal{T}_\eta}{d\mathcal{I}} = b, \qquad \ldots, \qquad \frac{d\mathcal{M}_\zeta}{d\mathcal{I}} = f,$$

and as a result:

$$\frac{d\mathcal{N}}{d\mathcal{I}} = \frac{\mathcal{N}}{\mathcal{I}}, \quad \frac{d\mathcal{T}_\eta}{d\mathcal{I}} = \frac{\mathcal{T}_\eta}{\mathcal{I}}, \quad \ldots, \quad \frac{d\mathcal{M}_\zeta}{d\mathcal{I}} = \frac{\mathcal{M}_\zeta}{\mathcal{I}},$$

which proves the agreement between the two expressions (27) and (34) for the projected displacement $\lambda$. The agreement between the two expressions (28) (no. **21**) and (35) (no. **36**) of the projected rotation $\varphi$ is proved in the same manner.

### 39. – Determination of the redundant constraint forces in systems of bodies with mean fibers. – Equation of redundant constraints

The method of calculating the redundant constraint forces follows from the general equation of elasticity. In order to simplify the presentation, we shall suppose that the hyperstatic system considered, which is deformed by forces $F$ that are applied directly and a temperature variation $\tau$, is subject to planar flexure. In that case (no. **35**), the general equation of elasticity will reduce to:

$$(33') \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = \int \left[ \mathcal{N}\left( \frac{N}{E\,\Omega} - \alpha\tau \right) + \mathcal{T}\,\frac{T}{G\,\Omega} + \mathcal{M}\,\frac{M}{E\,I} \right] ds\,.$$

The auxiliary forces and couples to be applied to the isostatic system that is obtained by suppressing the redundant constraints of the hyperstatic system considered that we shall adopt are:

1. Forces $\mathcal{F}$ of arbitrary magnitude that have same points of application and the same direction as the redundant *external* constraint forces of the hyperstatic system.

2. Couples $\mathcal{C}$ of arbitrary magnitude that have the same sections of application as the couples of the redundant *external* constraint forces of the hyperstatic system, and like them, they have their axes normal axis of the plane of flexure.

3. Forces $\mathcal{F}'$ of arbitrary magnitude that have the same points of applications and the same direction as the redundant *internal* constraint forces of the hyperstatic system, and like them, they are pair-wise equal and opposite.

4. Couples $\mathcal{C}'$ of arbitrary magnitude that have the same sections of application as the couples of the redundant *internal* constraints of the hyperstatic system, and like them, they are pair-wise equal and in the opposite senses,

Moreover, equation (33′) is written:

$$(a) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi + \sum \mathcal{F}'\lambda + \sum \mathcal{C}'\varphi = \int \left[ \mathcal{N}\left(\frac{N}{E\,\Omega} - \alpha\tau\right) + \mathcal{T}\,\frac{T}{G\,\Omega} + \mathcal{M}\,\frac{M}{E\,I} \right] ds\,.$$

It is easy to see that its left-hand side is zero.

Indeed, the projections $\lambda$ of the elastic displacements of the points of application of the auxiliary forces $\mathcal{F}$ and $\mathcal{F}'$ onto the directions of those forces, as well as the rotations $\varphi$ of the sections of application of the auxiliary couples $\mathcal{C}$ and $\mathcal{C}'$, are attached to the hyperstatic system that is deformed by the forces $F$ and by the temperature variation $\tau$. Now:

1. The projections $\lambda$ in the directions of the forces $\mathcal{F}$ (which are, by hypothesis, those of the redundant external constraint forces) are zero, by reason of those constraints themselves, and as a result, the sum $\sum \mathcal{F}\,\lambda$ will likewise be zero.

2. The rotations $\varphi$ of the sections of application of the couples of the redundant external constraints are zero by reason of those constraints themselves, so as a result, the sum $\sum \mathcal{C}\,\varphi$ will likewise be zero.

3. By hypothesis, the auxiliary forces $\mathcal{F}'$ are pair-wise equal and opposite, in such a way that any force + $\mathcal{F}'$ corresponds to a force – $\mathcal{F}'$. Now, those two forces + $\mathcal{F}'$ and $-\mathcal{F}'$ are applied at two points, which are constrained to remain in contact in the hyperstatic system, and consequently, the projected displacement $\lambda$ will be the same. Hence, each term + $\mathcal{F}'\lambda$ in the sum $\sum \mathcal{F}'\lambda$ will correspond to a term – $\mathcal{F}'\lambda$, and as a result, that sum will be zero.

4. The sum $\sum \mathcal{C}\,\varphi$ is zero for the same reason.

Equation ($a$) then reduces to:

$$(36) \qquad \int \left[ \mathcal{N}\left(\frac{N}{E\,\Omega} - \alpha\tau\right) + \mathcal{T}\,\frac{T}{G\,\Omega} + \mathcal{M}\,\frac{M}{E\,I} \right] ds = 0.$$

That is the *equation of the redundant constraints* in which the elements of reduction $N$, $T$, $M$ are attached to the hyperstatic system considered, which is subject to forces $F$ and the temperature variation $\tau$, and those $\mathcal{N}$, $\mathcal{T}$, $\mathcal{M}$ of the isostatic system are subject to the auxiliary forces and couples $\mathcal{F}$, $\mathcal{C}$, $\mathcal{F}'$, and $\mathcal{C}'$ that were defined above. That equation is used in the following manner:

Suppose that one has:

*m* forces and *n* couples of the redundant external constraints

2*p* forces and 2*q* couples of the redundant internal constraints.

The total number of forces and couples of the redundant constraints is therefore $m + n + 2\,(p + q)$. However, the forces and couples of the redundant internal constraints are pair-wise equal and opposite, so the number of unknown forces and couples will reduce to $m + n + p + q$.

The use of the equation of the redundant constraints demands the application (to the system that is made isostatic by suppressing the redundant constraints and subtracting the action of the forces *F* and the temperature variation *t*) of:

*m* auxiliary forces $\mathcal{F}_1, \ldots, \mathcal{F}_m$ ,
*n* auxiliary couples $\mathcal{C}_1, \ldots, \mathcal{C}_m$ ,
2*p* auxiliary forces $\mathcal{F}_1', -\mathcal{F}_1', \ldots, \mathcal{F}_p', -\mathcal{F}_p'$,
2*q* auxiliary couples $\mathcal{C}_1', -\mathcal{C}_1', \ldots, \mathcal{C}_q', -\mathcal{C}_q'$,

which are forces and couples of arbitrary magnitude, while the forces have the same points of application and the same directions as the redundant constraint forces and the couples have the same sections of application as the redundant constraint couples.

The corresponding elements of reduction $\mathcal{N}$, $\mathcal{C}$, and $\mathcal{M}$ for an arbitrary section of any body of the system are linear, homogeneous functions of those auxiliary forces and couples. One can then write:

$$(a)\qquad \begin{cases} \mathcal{N} = \mathcal{I}_1\alpha_1 + \cdots + \mathcal{I}_m\alpha_m + \mathcal{C}_1\beta_1 + \cdots + \mathcal{C}_n\beta_n \\ \qquad + \mathcal{I}_1'\alpha_1' + \cdots + \mathcal{I}_p'\alpha_p' + \mathcal{C}_1'\beta_1' + \cdots + \mathcal{C}_q'\beta_q', \\ \mathcal{T} = \mathcal{I}_1\gamma_1 + \cdots + \mathcal{I}_m\gamma_m + \mathcal{C}_1\delta_1 + \cdots + \mathcal{C}_n\delta_n \\ \qquad + \mathcal{I}_1'\gamma_1' + \cdots + \mathcal{I}_p'\gamma_p' + \mathcal{C}_1'\delta_1' + \cdots + \mathcal{C}_q'\delta_q', \\ \mathcal{M} = \mathcal{I}_1\varepsilon_1 + \cdots + \mathcal{I}_m\varepsilon_m + \mathcal{C}_1\theta_1 + \cdots + \mathcal{C}_n\theta_n \\ \qquad + \mathcal{I}_1'\varepsilon_1' + \cdots + \mathcal{I}_p'\varepsilon_p' + \mathcal{C}_1'\theta_1' + \cdots + \mathcal{C}_q'\theta_q', \end{cases}$$

in which

$$\alpha, \beta, \gamma, \varepsilon, \theta, \ \alpha', \beta', \gamma', \varepsilon', \theta'$$

are functions of the coordinates *x* and *y* of the center of gravity of the section considered.

Substitute those expressions into equation (36) for the redundant constraints and group the terms that contain the auxiliary forces and couples:

$$\mathcal{F}_1, \ldots, \mathcal{F}_m, \mathcal{C}_1, \ldots, \mathcal{C}_m, \mathcal{F}_1', \ldots, \mathcal{F}_p', \mathcal{C}_1', \ldots, \mathcal{C}_q',$$

respectively, as factors. The equation thus-transformed includes $m + n + p + q$ groups, and in order for it to be satisfied, it is necessary that each of those groups must be separately zero, insofar as the auxiliary forces and couples are arbitrary magnitudes. Moreover, one will have the following $m + n + p + q$ equations:

$$(b)\qquad \begin{cases} \int\left[\alpha_1\left(\frac{N}{E\Omega}-\alpha\tau\right)+\gamma_1\frac{T}{G\Omega}+\varepsilon_1\frac{M}{EI}\right]ds=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \int\left[\alpha_m\left(\frac{N}{E\Omega}-\alpha\tau\right)+\gamma_m\frac{T}{G\Omega}+\varepsilon_m\frac{M}{EI}\right]ds=0, \\ \int\left[\beta_1\left(\frac{N}{E\Omega}-\alpha\tau\right)+\delta_1\frac{T}{G\Omega}+\theta_1\frac{M}{EI}\right]ds=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \int\left[\beta_n\left(\frac{N}{E\Omega}-\alpha\tau\right)+\delta_n\frac{T}{G\Omega}+\theta_n\frac{M}{EI}\right]ds=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

$$(b')\qquad \begin{cases} \int\left[\alpha_1'\left(\frac{N}{E\Omega}-\alpha\tau\right)+\gamma_1'\frac{T}{G\Omega}+\varepsilon_1'\frac{M}{EI}\right]ds=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \int\left[\alpha_p'\left(\frac{N}{E\Omega}-\alpha\tau\right)+\gamma_p'\frac{T}{G\Omega}+\varepsilon_p'\frac{M}{EI}\right]ds=0, \\ \int\left[\beta_1'\left(\frac{N}{E\Omega}-\alpha\tau\right)+\delta_1'\frac{T}{G\Omega}+\theta_1'\frac{M}{EI}\right]ds=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \int\left[\beta_q'\left(\frac{N}{E\Omega}-\alpha\tau\right)+\delta_q'\frac{T}{G\Omega}+\theta_q'\frac{M}{EI}\right]ds=0. \end{cases}$$

If one replaces $N$, $T$, and $M$ in those equations as functions of the $m + n + p + q$ unknown redundant force and couple constraints then they will provide the values of those forces and couples.

**Remark.** – Equation ($b$) expresses the redundant external constraints and equations ($b'$) provide the redundant internal constraints.

One can form those equations in a slightly-different manner that will be more convenient in certain cases. Hence, in order to form the first of equations ($b$), rather than applying all of the auxiliary forces and couples that were indicated above to the system that has been rendered isostatic, it will suffice to apply only the single auxiliary force $\mathcal{F}_1$. Formulas ($a$) will then reduce to:

$$\mathcal{N} = \mathcal{F}_1\, \alpha_1\,, \qquad \mathcal{T} = \mathcal{F}_1\, \gamma_1\,, \qquad \mathcal{M} = \mathcal{F}_1\, \varepsilon_1\,,$$

and the substitution of those expressions for $\mathcal{N}$, $\mathcal{T}$, and $\mathcal{M}$ into equation (36) will give the first of equations (*b*). The same procedure is applicable to the successive formation of the other equations (*b*).

Similarly, in order to form the first of equations (*b*), it will suffice to apply the two auxiliary forces $\mathcal{F}_1'$ and $-\mathcal{F}_1'$ to the system that has been rendered isostatic, to the exclusion of the other auxiliary forces and couples. Formulas (*a*) will then reduce to:

$$\mathcal{N} = \mathcal{F}_1'\alpha_1', \qquad \mathcal{T} = \mathcal{F}_1'\gamma_1', \qquad \mathcal{M} = \mathcal{F}_1'\varepsilon_1',$$

and the substitution of those expressions for $\mathcal{N}$, $\mathcal{T}$, and $\mathcal{M}$ into equation (36) will give the first of equations (*b*′). The same procedure is applicable to the successive formation of the other equations (*b*′).

## 40. – Example of the determination of redundant external constraint forces

Consider the *arch that is anchored at its two extremities* that was taken in no. **23** (Fig. 4) to be an example of the generalized theorem of General Menabrea for the determination of the redundant external constraint forces. That arch is subject to forces that are located in the plane of its mean fiber, as well as a temperature variation $\tau$, which is measured by starting from the temperature that is realized by those anchors.

We have seen that the reactions of the left anchor are reducible at the center of gravity *A* of the anchored section to two mutually-rectangular forces *X* and *Y* and a couple *Z* with an axis that is normal to the plane of the mean fiber of the arch. They are the two forces and the couple of the redundant constraints. The problem is now to determine them.

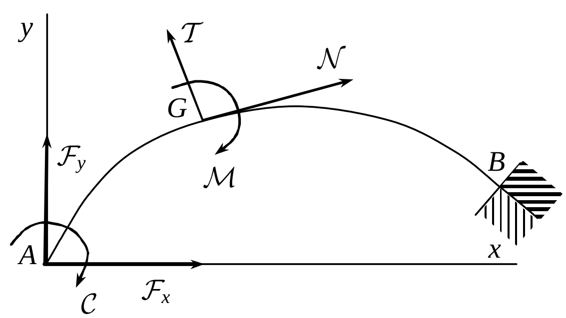


Figure 7.

To that effect, following the method that was presented in no. **39**, make the arch isostatic by suppressing the left anchor (Fig. 7), and then apply two auxiliary forces $\mathcal{F}_x$ and $\mathcal{F}_y$ to the point *A* with the same directions as *X* and *Y*, resp., to the section whose point is the center of gravity, and apply a couple $\mathcal{C}$ whose axis is normal to the plane of the mean fiber (forces and couple arbitrary magnitudes).

The elements of the corresponding reduction $\mathcal{N}$, $\mathcal{T}$, and $\mathcal{M}$ to the center of gravity $G(x, y)$ of an arbitrary section of the arch have the following expressions:

$$\mathcal{N} = \mathcal{F}_x \frac{dx}{ds} + \mathcal{F}_y \frac{dy}{ds},$$

$$\mathcal{T} = -\mathcal{F}_x \frac{dy}{ds} + \mathcal{F}_y \frac{dx}{ds},$$

$$\mathcal{M} = -\mathcal{F}_x\, y + \mathcal{F}_y\, x + \mathcal{C}.$$

Substitute those expressions into equation (36) (no. **39**) for the redundant constraints and group the terms that contain $\mathcal{F}_x$, $\mathcal{F}_y$, and $\mathcal{C}$, respectively, as factors. That will give:

$$\mathcal{F}_x \int \left[\left(\frac{N}{E\,\Omega} - \alpha\tau\right)\frac{dx}{ds} - \frac{T}{G\,\Omega}\frac{dy}{ds} - \frac{M\,y}{E\,I}\right] ds$$

$$+ \mathcal{F}_y \int \left[\left(\frac{N}{E\,\Omega} - \alpha\tau\right)\frac{dy}{ds} + \frac{T}{G\,\Omega}\frac{dx}{ds} + \frac{M\,x}{E\,I}\right] ds + \mathcal{C}\int \frac{M}{E\,I}\, ds = 0,$$

in which $\mathcal{F}_x$, $\mathcal{F}_y$, and $\mathcal{C}$ are arbitrary quantities, so in order for that equation to be satisfied, it is necessary that one must have:

$$\int \left[\left(\frac{N}{E\,\Omega} - \alpha\tau\right)\frac{dx}{ds} - \frac{T}{G\,\Omega}\frac{dy}{ds} - \frac{M\,y}{E\,I}\right] ds = 0,$$

$$\int \left[\left(\frac{N}{E\,\Omega} - \alpha\tau\right)\frac{dy}{ds} + \frac{T}{G\,\Omega}\frac{dx}{ds} + \frac{M\,x}{E\,I}\right] ds = 0,$$

$$\int \frac{M}{E\,I}\, ds = 0,$$

separately.

If one replaces $N$, $T$, and $M$ with their expressions ($b$) (no. **23**) as functions of the unknowns $X$, $Y$, and $Z$ then one will obtain the three equations ($c$) (same number) that determine those unknowns.

### 41. – Example of the determination of redundant internal constraint forces

Recall the example that was considered before in no. **24** (Fig. 5): viz., *an arch that is anchored at its two extremities and includes a ball O* and is subjected to forces that are located in the plane of the mean fiber, as well as a temperature variation $\tau$, which is measured by starting from the temperature that is realized at the anchors.

We have seen that the arch is, in reality, a system of two anchored arches *AO* and *OB*, each of which has one of its extremities joined to an extremity of the other by the ball *O*, that the redundant constraint forces are two forces *X* and *Y* that are normal to each other, which are applied to the extremity *O* of the arch *OB* and two forces – *X* and – *Y* that are applied to the extremity *O* of the arch *AO*. Those are the two forces that one must determine.

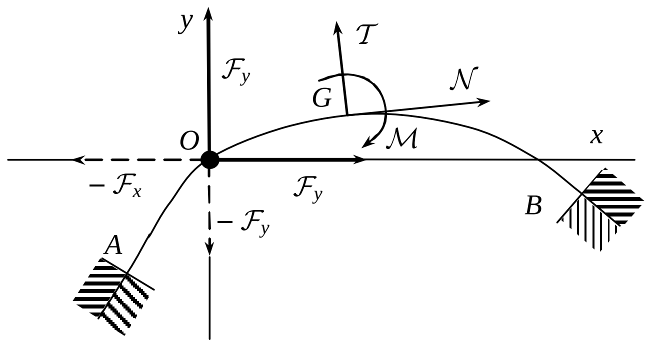


Figure 8.

To that effect, following the general method of no. **39**, the system is rendered isostatic by suppressing the ball *O*, which creates the redundant internal constraints. One then applies (Fig. 8) auxiliary forces $\mathcal{F}_x$, $\mathcal{F}_y$, $-\mathcal{F}_x$, and $-\mathcal{F}_y$ of arbitrary magnitude to it, which have the same point of application and the same direction as the redundant constraint forces *X*, *Y*, – *X*, and – *Y*. The corresponding reduction elements $\mathcal{N}$, $\mathcal{T}$, and $\mathcal{M}$ at the center of gravity *G* (*x*, *y*) of any section of either of the arches *AO* and *OB* will have the expressions:

$$\mathcal{N} = \mathcal{F}_x \frac{dx}{ds} + \mathcal{F}_y \frac{dy}{ds},$$

$$\mathcal{T} = -\mathcal{F}_x \frac{dy}{ds} + \mathcal{F}_y \frac{dx}{ds},$$

$$\mathcal{M} = -\mathcal{F}_x\, y + \mathcal{F}_y\, x .$$

Substitute those expressions into equation (36) (no. **39**) for the redundant constraints and separate the terms into two groups that contain $\mathcal{F}_x$ and $\mathcal{F}_y$ as factors, respectively. In order for the equation thus-transformed to be satisfied, it is necessary that each of those two groups must be zero separately, since the auxiliary forces $\mathcal{F}_x$ and $\mathcal{F}_y$ are arbitrary quantities. In that way, we will obtain the two equations:

$$\int\left[\left(\frac{N}{E\,\Omega}-\alpha\tau\right)\frac{dx}{ds}-\frac{T}{G\,\Omega}\frac{dy}{ds}-\frac{M\,y}{E\,I}\right]ds = 0,$$

$$\int\left[\left(\frac{N}{E\,\Omega}-\alpha\tau\right)\frac{dy}{ds}+\frac{T}{G\,\Omega}\frac{dx}{ds}+\frac{M\,x}{E\,I}\right]ds = 0.$$

If one replaces *N*, *T*, and *M* in those two equations with their expressions (*b*) (no. **24**) as functions of *X* and *Y* then one will obtain two equations (*c*) (same number) that determine those unknowns.

## 42. – Another example of the determination of redundant external constraint forces

Recall the example that was considered before in no. **25** (Fig. 6): viz., *a straight beam with a double articulated lattice* that rests upon two simple supports and is subject to loads that are applied at the nodes.

Before making use of equation (36) (no. **39**) for the redundant constraints, we remark that in the present case, things can be simplified as follows:

The elastic forces that are produced by the loads in any section of an arbitrary bar of the beam are reducible to exclusively the normal effort *N* (the shearing effort and the flexure couple are zero). Moreover, that normal effort is the same in all sections of the bar, and as was said in no. **23**, it bears the name of the *effort* in the bar. We shall see later on that, similarly, the elastic forces that are produced by the auxiliary forces $\mathcal{F}$ in any section of any bar are reducible to exclusively the normal effort $\mathcal{N}$ and that this effort is the same in all sections of the bar. When one suppresses the term in equation (36) that relates to the temperature variation, which is not included here, it will then reduce to:

$$\int \mathcal{N}\,\frac{N}{E\,\Omega}\,ds = 0,$$

in which the integral is taken along the total length of the mean fiber of any bar.

Now, for a bar of length *s*, since *N* and $\mathcal{N}$ are constant, as well as **Ω**, one will have:

$$\int_0^s \frac{N}{E\,\Omega}\,ds = \mathcal{N}\,\frac{N\,s}{E\,\Omega}.$$

Consequently, upon letting *m* denote the number of bars that constitute the beam, one will have:

$$(a)\qquad \sum_1^m \mathcal{N}\,\frac{N\,s}{E\,\Omega} = 0.$$

Having said that, we saw in no. **25** that the redundant constraints on the beam are constraints on the diagonals $(0, 1')$, …, $(i - 1, i')$, …, $(n - 1, n')$, with the lower attaching nodes 0, …, $i - 1$, …, $n - 1$, and that for the diagonal $(i - 1, i')$, for example, the redundant constraint forces are two forces of the same magnitude and the same direction as the effort $Z_i'$ in that diagonal, one of which is applied to the extremity of the aforementioned diagonal that was made free, and the other of which is applied to the node $i - 1$; those forces are repulsive or attractive according to whether $Z_i'$ is an effort of compression or tension, respectively.

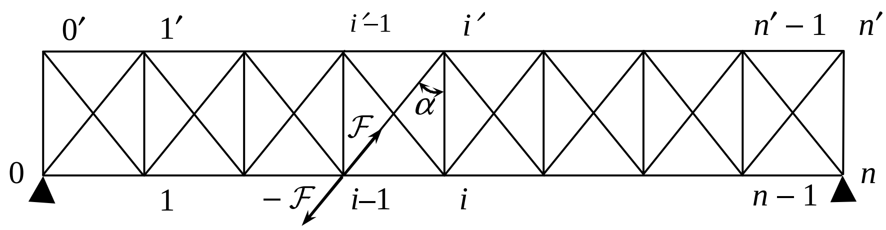


Figure 9.

Furthermore, according to the final remark in no. **39**, if the beam is made isostatic by suppressing the redundant constraints and subtracting the action of the loads then apply an auxiliary force $\mathcal{F}$ to the lower extremity of the diagonal $(i - 1, i')$ (Fig. 9) that is directed along the mean fiber to that diagonal and a force $-\mathcal{F}$ to the node $i - 1$. It is clear that:

1. Those two auxiliary forces produce no elastic forces in the bars of the beam other than the ones of the panel $(i)$, in such a way that $\mathcal{N} = 0$ in those bars.

2. In each of the bars of the panel $(i)$, the elastic forces that the auxiliary forces generate in any section are reducible to exclusively the normal effort $\mathcal{N}$, and that effort is constant all along that bar.

The values of that normal effort $\mathcal{N}$ in the six bars of the panel $(i)$ are calculated immediately by pure statics. They are indicated in the second column of the Table below. The special notations that were adopted in no. **25** in order to represent the efforts $N$ that are produced in the bars by the loads, as well as the lengths $s$ and sections $\Omega$ of those bars, are reproduced in the last three columns of that Table.

Each of the six bars in the panel $(i)$ corresponds to a term in the sum $\sum_1^m$ in equation $(a)$. The corresponding terms in the other bars of the beam are zero, since $\mathcal{N} = 0$ for each of those bars. As a result, equation $(a)$ can be written in the following form, when one suppresses the common factor $\mathcal{F}$ and the common denominator $E$:

$$-a\sin\alpha\left(\frac{X_i}{\Omega_i}+\frac{X_i'}{\Omega_i'}\right)-b\cos\alpha\left(\frac{Y_{i-1}}{\omega_i}+\frac{Y_i}{\omega_i}\right)+c\left(\frac{Z_i}{S_i}+\frac{Z_i'}{S_i'}\right)=0.$$

If one replaces $Z_i$, $X_i$, $X_i'$, $Y_{i-1}$, and $Y_i$ in the latter equation with their expressions (*c*) and (*g*) (no. **25**) that are obtained by pure statics, and if one takes into account the facts that:

$$\frac{a}{c}=\sin\alpha,\qquad \frac{b}{c}=\cos\alpha$$

then one will recover the final equation (*k*) in no. **25**.

| | Effort produced by | | | |
|---|---|---|---|---|
| Type of bar | the auxiliary forces $\mathcal{F}$ and $-\mathcal{F}$, $\mathcal{N}$ | the load $N$ | Length of the bars $s$ | Section of the bars $\Omega$ |
| Elements of the members $(i-1, i')$ … | $-\mathcal{F}\sin\alpha$ | $X_i$ | $a$ | $\Omega_i$ |
| Elements of the members $(i'-1, i')$ … | $-\mathcal{F}\sin\alpha$ | $X_i'$ | $a$ | $\Omega_i'$ |
| Uprights $(i-1, i'-1)$ … | $-\mathcal{F}\cos\alpha$ | $Y_{i-1}$ | $b$ | $\omega_{i-1}$ |
| Uprights $(i, i')$ … | $-\mathcal{F}\cos\alpha$ | $Y_i$ | $b$ | $\omega_i$ |
| Diagonals $(i-1, i)$ … | $\mathcal{F}$ | $Z_i$ | $c$ | $S_i$ |
| Diagonals $(i-1, i')$ … | $\mathcal{F}$ | $Z_i'$ | $c$ | $S_i'$ |

$\alpha$ = acute angle between the mean fiber of the diagonal and the vertical

**CHAPTER IV**

# METHOD BASED UPON THE VIRTUAL WORK THEOREM

**43.** – The theorem of virtual work was used for the first time in the resistance of materials by Mohr ([1]) for the determination of the efforts in articulated systems with redundant bars.

Since then, the applications of that theorem to the other systems that are employed in construction have been largely developed by various authors, and notably Müller-Breslau ([2]).

Finally, the same theorem permits us to establish the *general equation of elasticity* ([3]), to which we just gave a new proof that was deduced from the *vis viva* theorem much earlier (nos. **32** and **33**).

## 44. – General equation of elasticity

Let a system of bodies be isotropic. Suppose that it is externally and internally hyperstatic (no. **11**). (The case of an isostatic system will be regarded as a particular case, and likewise that of a single body.) That system of bodies is deformed under the action of external forces that are applied directly and a temperature variation of $\tau$ degrees, which is measured from the temperature that is realized by the various constraints of the system.

Let:

1. $$\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots$$

denote the projections of the elastic and caloric displacements of a certain number of points:

$$A_1,\ A_2,\ A_3,\ \ldots$$

of the system of bodies onto the arbitrarily-chosen directions:

$$\Delta_1\,,\ \Delta_2\,,\ \Delta_3\,,\ \ldots$$

Let:

2. $$\varphi_1,\ \varphi_2,\ \varphi_3,\ \ldots$$

be the projections of the elastic and caloric rotations of a certain number of lines:

$$a_1\,a_1',\quad a_2\,a_2',\quad a_3\,a_3',\quad \ldots$$

that are contained in the system of bodies onto the directions:

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([1]) Zeitschrift der Achitekten und Ingenieur Vereins zu Hannover (1874), pp. 223.

([2]) *Die Methoden der Festigkeitlehre und der Statik der Baukonstructionen*, 1886.

([3]) "L'équation générale de l'élasticité des constructions et ses applications," Bulletin de la Société des Ingenieurs civils de France, October 1907. pp. 365.

$$\Gamma_1, \Gamma_2, \Gamma_3, \ldots,$$

which are chosen arbitrarily from the directions that are normal to those lines.

Let:

3. $$\varepsilon_x - \alpha\tau, \ \varepsilon_2 - \alpha\tau, \ \varepsilon_3 - \alpha\tau, \ \ldots$$

be the six parameters of the elastic and caloric deformation (no. **5**) of an elementary parallelepiped that is taken at an arbitrary point $(x, y, z)$ of the system.

4. Let $\varpi_F$ be the internal potential per unit volume at the same point $(x, y, z)$.

5. Let $F_{es}$ and $F_{is}$ be the redundant external and internal constraint forces on the system.

Now consider the isostatic system that is obtained by suppressing the redundant constraints from the given hyperstatic system. If one subjects it to the forces $F$, $F_{es}$, and $F_{is}$, as well as a temperature variation of $\tau$ degrees then it will take on a state of elastic equilibrium that is identical to that of the hyperstatic system that is subject to only forces $F$ and the variation of temperature. In particular, in that isostatic system, the elastic and caloric displacements of the points $A_1$, $A_2$, $A_3$, …, the elastic and caloric rotations of the lines $a_1 a_1'$, $a_2 a_2'$, $a_3 a_3'$, …, and the deformation parameters of an arbitrary parallelepiped will be the same as in the hyperstatic system.

The internal potential is likewise the same, in such a way that, from formula (4) (no. **4**) and the final remark in no. **5**, one can write:

$$(a) \qquad \bar{\varpi}_{F,F_{es},F_{is}} = \varpi_F = \frac{\lambda}{2}(\varepsilon_x + \varepsilon_y + \varepsilon_z) + \mu(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + \frac{\mu}{2}(\gamma_{yz}^2 + \gamma_{zx}^2 + \gamma_{xy}^2),$$

upon letting $\bar{\varpi}_{F,F_{es},F_{is}}$ denote the internal potential per unit volume of $(x, y, z)$ of the isostatic system that is deformed by the forces $F$, $F_{es}$, $F_{is}$ . (The overbar above the letter $\varpi$ is intended to indicate that this potential relates to the isostatic system.)

Suppose that the action of the forces $F$, $F_{es}$, $F_{is}$, and the temperature variation has been subtracted from the isostatic system and apply:

1. *Auxiliary forces* to the points:

$$A_1, A_2, A_3, \ldots$$

whose magnitudes:

$$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$$

are arbitrary, and which point in the directions:

$$\Delta_1, \Delta_2, \Delta_3, \ldots$$

2. Forces that form *auxiliary couples* to the points:

$$a_1, a_1', a_2, a_2', a_3, a_3', \ldots,$$

whose magnitudes:

$$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots$$

are arbitrary and which point in the directions:

$$\Gamma_1, \Gamma_2, \Gamma_3, \ldots$$

The isostatic system will take on a new state of deformation under the action of the auxiliary forces and couples. Let:

$\lambda_1'$, $\lambda_2'$, $\lambda_3'$, …, $\varphi_1'$, $\varphi_2'$, $\varphi_3'$, … be the new values of the projected displacements and projected rotations $\lambda_1, \lambda_2, \lambda_3, \ldots, \varphi_1, \varphi_2, \varphi_3, \ldots$, resp.

$\varepsilon_x'$, $\varepsilon_y'$, $\varepsilon_z'$, $\gamma_{yz}'$, $\gamma_{zx}'$, $\gamma_{xy}'$ be the new values of the deformation parameters of an elementary parallelepiped that is taken at an arbitrary point (*x*, *y*, *z*).

$\bar{\varpi}_{\mathcal{F},\mathcal{C}}$ be internal potential per unit volume at the point (*x*, *y*, *z*), which is a potential whose expression, from formula (4) (no. **4**), has the expression:

$$(a')\qquad \bar{\varpi}_{\mathcal{F},\mathcal{C}} = \frac{\lambda}{2}(\varepsilon_x' + \varepsilon_y' + \varepsilon_z') + \mu\,(\varepsilon_x'^2 + \varepsilon_y'^2 + \varepsilon_z'^2) + \frac{\mu}{2}(\gamma_{yz}'^2 + \gamma_{zx}'^2 + \gamma_{xy}'^2)\,.$$

$\bar{\Pi}_{\mathcal{F},\mathcal{C}}$ be the total internal potential of the system, which, from formula (5) (no. 4), has the value:

$$\bar{\Pi}_{\mathcal{F},\mathcal{C}} = \iiint \bar{\varpi}_{\mathcal{F},\mathcal{C}}\, dx\, dy\, dz\,.$$

Now, subject the isostatic system, thus-deformed, to a virtual deformation that is compatible with the constraints on that system, and let $\delta\lambda_1'$, $\delta\lambda_2'$, $\delta\lambda_3'$, …, $\delta\varphi_1'$, $\delta\varphi_2'$, $\delta\varphi_3'$, …, $\delta\varepsilon_x'$, $\delta\varepsilon_y'$, $\delta\varepsilon_z'$, $\delta\gamma_{yz}'$, $\delta\gamma_{zx}'$, $\delta\gamma_{xy}'$ be the corresponding virtual variations of the projected displacements, projected rotations, and deformation parameters, resp. The virtual work theorem immediately gives the equation:

$$\mathcal{F}_1\,\delta\lambda_1' + \mathcal{F}_2\,\delta\lambda_2' + \mathcal{F}_3\,\delta\lambda_3' + \cdots + \mathcal{C}_1\,\delta\varphi_1' + \mathcal{C}_2\,\delta\varphi_2' + \mathcal{C}_3\,\delta\varphi_3'$$

$$= \iiint \left( \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \varepsilon_x'}\,\delta\varepsilon_x' + \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \varepsilon_y'}\,\delta\varepsilon_y' + \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \varepsilon_z'}\,\delta\varepsilon_z' + \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \gamma_{yz}'}\,\delta\gamma_{yz}' + \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \gamma_{zx}'}\,\delta\gamma_{zx}' + \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \gamma_{xy}'}\,\delta\gamma_{xy}' \right) dx\, dy\, dz\,.$$

Indeed, the left-hand side of that equation represents the virtual work done by auxiliary forces and couples ([1]), and the right-hand side, which is the virtual variation of the internal potential $\overline{\Pi}_{\mathcal{F},\mathcal{C}}$, represents the virtual work, with the opposite sign, done by internal forces.

Since the virtual deformation is subject to only being compatible with the constraints of the isostatic system considered, one can take that deformation to be the actual deformation that the system experiences under the action of the forces $F$, $F_{es}$, $F_{is}$, and the temperature variation of $\tau$ degrees, which implies that:

$$\begin{array}{llll}
\delta\lambda_1' = \lambda_1, & \delta\lambda_2' = \lambda_2, & \delta\lambda_3' = \lambda_3, & \dots, \\
\delta\varphi_1' = \varphi_1, & \delta\varphi_2' = \varphi_2, & \delta\varphi_3' = \varphi_3, & \dots, \\
\delta\varepsilon_x' = \varepsilon_x - \alpha\tau, & \delta\varepsilon_y' = \varepsilon_y - \alpha\tau, & \delta\varepsilon_z' = \varepsilon_z - \alpha\tau, & \\
\delta\gamma_{yz}' = \gamma_{yz}, & \delta\gamma_{zx}' = \gamma_{zx}, & \delta\gamma_{xy}' = \gamma_{xy}, &
\end{array}$$

and as a result:

$$(b) \qquad \mathcal{F}_1\,\lambda_1 + \mathcal{F}_2\,\lambda_2 + \mathcal{F}_3\,\lambda_3 + \dots + \mathcal{C}_1\,\varphi_1 + \mathcal{C}_2\,\varphi_2 + \mathcal{C}_3\,\varphi_3 + \dots$$

$$= \iiint\left(\frac{\partial\overline{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial\varepsilon_x'}\varepsilon_x' + \dots + \frac{\partial\overline{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial\gamma_{xy}'}\gamma_{xy}'\right)dx\,dy\,dz - \alpha\tau\iiint\left(\frac{\partial\overline{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial\varepsilon_x'} + \frac{\partial\overline{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial\varepsilon_y'} + \frac{\partial\overline{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial\varepsilon_z'}\right)dx\,dy\,dz\,,$$

in which one assumes that the temperature variation $\tau$ is the same at all points of the system.

Transform the two integrals in the right-hand side of the equation ($b$). As far as the left-hand side is concerned, imagine that the isostatic system of simultaneously subjected to:

1. Forces $F$, $F_{es}$, $F_{is}$,

2. Auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$,

without that system being subjected to any temperature variation, moreover. From the superposition principle, the elastic deformation parameters that are produced by the set of all those forces and couples have the values:

$$\varepsilon_x + \varepsilon_x', \quad \varepsilon_y + \varepsilon_y', \quad \varepsilon_z + \varepsilon_z', \quad \gamma_{yz} + \gamma_{yz}', \quad \gamma_{zx} + \gamma_{zx}', \quad \gamma_{xy} + \gamma_{xy}'$$

at an arbitrary point ($x$, $y$, $z$). As a result, upon letting $\overline{\varpi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}}$ denote the internal potential per unit volume at that point of the system that has been deformed by the aforementioned forces and couples, one will have:

([1]) The constraint forces of the isostatic system that is assumed to be subjected to the forces $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, …, and the couples $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, … do not enter into the expression for work, insofar as the virtual works done by the constraint forces are zero by reason of those constraints.

$$(a'') \qquad \bar{\varpi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} = \frac{\lambda}{2}[(\varepsilon_x + \varepsilon'_x) + (\varepsilon_y + \varepsilon'_y) + (\varepsilon_z + \varepsilon'_z)]^2 + \mu\,[(\varepsilon_x + \varepsilon'_x)^2 + (\varepsilon_y + \varepsilon'_y)^2 + (\varepsilon_z + \varepsilon'_z)^2] + \frac{\mu}{2}[(\gamma_{yz} + \gamma'_{yz})^2 + (\gamma_{zx} + \gamma'_{zx})^2 + (\gamma_{xy} + \gamma'_{xy})^2] .$$

The expression ($a$) for $\bar{\varpi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}}$ is a homogeneous function of degree two of the six quantities $\varepsilon_x$, …, $\gamma_{xy}$, and in order to pass from that expression to the ones ($a'$) and ($a''$) for $\bar{\varpi}_{\mathcal{F},\mathcal{C}}$ and $\bar{\varpi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}}$, resp., it will suffice to replace those six quantities, first, with $\varepsilon'_x$, …, $\gamma'_{xy}$, and then with $(\varepsilon_x + \varepsilon'_x)$, …, $(\gamma_{xy} + \gamma'_{xy})$. Consequently, by virtue of the property of homogeneous functions of degree two that utilized before in no. **33**, one will have:

$$\frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \varepsilon'_x}\varepsilon_x + \cdots + \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \gamma'_{xy}}\gamma_{xy} = \bar{\varpi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} - (\bar{\varpi}_{F,F_{es},F_{is}} + \bar{\varpi}_{\mathcal{F},\mathcal{C}}) .$$

Upon multiplying the two sides of that identity by $dx\,dy\,dz$ and integrating over the total volume that is occupied by the system of bodies, one will get:

$$(c) \qquad \iiint \left( \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \varepsilon'_x}\varepsilon_x + \cdots + \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \gamma'_{xy}}\gamma_{xy} \right) dx\,dy\,dz = \bar{\Pi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} - (\bar{\Pi}_{F,F_{es},F_{is}} + \bar{\Pi}_{\mathcal{F},\mathcal{C}}) .$$

That is the transformed expression for the first integral in the right-hand side of equation ($b$), which is an expression in which $\bar{\Pi}_{F,F_{es},F_{is}}$, $\bar{\Pi}_{\mathcal{F},\mathcal{C}}$, and $\Pi_{F,F_{es},F_{is}}$ represent the values of the total internal potential of the isostatic system, when it is deformed by:

1. Forces $F$, $F_{es}$, $F_{is}$ ,

2. Auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$,

3. The totality of all those forces and couples,

respectively.

As for the second integral in equation ($b$), by virtue of formulas (6) (no. **4**), one can write:

$$(d) \qquad \iiint \left( \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \varepsilon'_x} + \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \varepsilon'_y} + \frac{\partial \bar{\varpi}_{\mathcal{F},\mathcal{C}}}{\partial \varepsilon'_z} \right) dx\,dy\,dz = \iiint (n'_x + n'_y + n'_z)\,dx\,dy\,dz ,$$

in which $n'_x, n'_y, n'_z$ denote the normal fatigues on three mutually-rectangular elements that are drawn through an arbitrary point ($x$, $y$, $z$) of the isostatic system when it is subjected to the auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$.

Upon replacing the two integrals on the right-hand side of equation (*b*) with their expressions (*c*) and (*d*) and writing the left-hand side in the abbreviated form, one will finally get:

$$(37) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = \bar{\Pi}_{F,F_{es},F_{is},\mathcal{F},\mathcal{C}} - (\bar{\Pi}_{F,F_{es},F_{is}} + \bar{\Pi}_{\mathcal{F},\mathcal{C}}) - \alpha\tau \iiint (n'_x + n'_y + n'_z)\,dx\,dy\,dz\,.$$

We thus recover the general equation (31) (no. **32**) of elasticity for systems of isotropic bodies.

Upon repeating the preceding proof in the case of a system of bodies with mean fibers, one will likewise recover equation (32) (no. **33**), whose development will lead to the general equation (33) (same number) of elasticity for the systems in question:

$$(38) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = \int \left[ \mathcal{N}\left( \frac{N}{E\,\Omega} - \alpha\tau \right) + \mathcal{T}_\eta \frac{T_\eta}{G\,\Omega} + \cdots + \mathcal{M}_\zeta \frac{M_\zeta}{E\,I} \right] ds\,.$$

The application of the latter equation to the calculation of displacements and elastic and caloric rotations, as well as the determination of the constraint forces in hyperstatic system, was presented before (nos. **36** and **39** to **42**); it shall not be repeated.

### 45. – Variants of the general equation of elasticity

I. *Case of systems of isotropic bodies.* – In order to establish the general equation (37) of elasticity for systems of isotropic bodies in no. **44**, we considered the *isostatic* system that was obtained suppressing the redundant constraints in the given hyperstatic system, and we applied the virtual work theorem to that system, which is assumed to be subjected to auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$ whose magnitude is arbitrary, by taking the virtual displacements to be the actual displacements that will result from the deformation of the hyperstatic system when it is subjected to the given forces *F* and the given temperature variation $\tau$.

If one repeats that proof exactly, while considering, not the isostatic system, but the given *hyperstatic* system, then one will get the following variant of equation (37):

$$(39) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = \Pi_{F,\mathcal{F},\mathcal{C}} - (\Pi_F + \Pi_{\mathcal{F},\mathcal{C}}) - \alpha\tau \iiint (n''_x + n''_y + n''_z)\,dx\,dy\,dz\,,$$

in which $n''_x$, $n''_y$, $n''_z$ denote the fatigues that are normal to the three rectangular elements that are drawn through an arbitrary point (*x*, *y*, *z*) of the hyperstatic system when it is subjected to the auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$, resp.

Finally, if one recalls that same proof once more, while considering the *internally-isostatic and externally-completely-free* system that is obtained by suppressing the redundant internal constraints and *all* of the external constraints on the given hyperstatic system and supposing that the system is not subjected to auxiliary forces and couples of absolutely-arbitrary magnitudes, but to auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$, resp., that are restricted by the condition of equilibrium on that system then one will get the second variant:

$$(40) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = \bar{\bar{\bar{\Pi}}}_{F,F_e,F_{is},\mathcal{F},\mathcal{C}} - (\bar{\bar{\bar{\Pi}}}_{F,F_e,F_{is}} + \bar{\bar{\bar{\Pi}}}_{\mathcal{F},\mathcal{C}}) - \alpha\tau \iiint (n_x''' + n_y''' + n_z''')\,dx\,dy\,dz\,,$$

in which $\bar{\bar{\bar{\Pi}}}_{F,F_e,F_{is}}$, $\bar{\bar{\bar{\Pi}}}_{\mathcal{F},\mathcal{C}}$, and $\bar{\bar{\bar{\Pi}}}_{F,F_e,F_{is},\mathcal{F},\mathcal{C}}$ denote the total internal potential of the internally-isostatic and externally-completely-free system when it is deformed by:

1. The given forces *F*, all of the external constraint forces $F_e$ of the given hyperstatic system, and the redundant internal constraints $F_{is}$ of the latter system,

2. Auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$,

3. The totality of those forces and couples,

respectively.

$n_x'''$, $n_y'''$, $n_z'''$ are the normal fatigues on three rectangular elements that are drawn through an arbitrary point (*x*, *y*, *z*) of the internally-isostatic and externally-completely-free system that is subject to the auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$.

II. *Case of systems of bodies with mean fibers.* – The two variants of the general equation (37) (no. **44**) of elasticity for system of isotropic bodies that were just pointed out correspond to two variants of the general equation (38) (same number) of elasticity for systems of bodies with mean fibers.

The last two variants show that in equation (38), one can, if desired, consider the elements of reduction $\mathcal{N}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\xi$ to be the resultant of either the application of auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$, resp., of arbitrary magnitude to the *isostatic* system that was defined before or the application of auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$, resp., that are restricted by the condition of equilibrium on that system to the *internally-isostatic and externally-completely-free* system that was defined before.

It was the last of those three viewpoints that we adopted in our previously-cited paper on the *general equation of elasticity and its applications*. Here, we have adopted the first one, because it attaches the second method, which was based upon the *vis viva* theorem, to the method that is based upon the virtual work theorem in a more direct manner. As far as ease of application is concerned, the two viewpoints are absolutely equivalent, moreover. Essentially, no matter whether one chooses one or the other, the reduction elements $\mathcal{N}$, $\mathcal{T}_\eta$, …, $\mathcal{M}_\xi$ will always be statically-determinate, and as a result, they can be easily calculated.

The second viewpoint has only a purely-theoretical interest.

### 46. – Completion of the theorem of Betti, Boussinesq, and Maurice Levy

That theorem follows immediately from the general equation of elasticity. Consider a system of bodies that are isotropic or have mean fibers. Suppose that it is hyperstatic. (The case of an isostatic system will be regarded as a particular case, and likewise, that of a single body.)

Apply a first system of forces (*A*) to it. Let: *m* be the number of forces, $F_{A_i}$, any one of them, $A_i$ , its point of application, $\Delta_i$ , its direction, and let $\Pi_{F_A}$ be the total internal potential of the system of bodies that are deformed by those forces.

Replace the system of forces (*A*) with a second system of forces (*B*). Let *n* be the number of forces of that second system, $F_{B_i}$, any one of them, $B_i$, its point of application, $\Delta_{B_i}$, its direction, and $\Pi_{F_B}$, the total internal potential of the system of bodies that is deformed by those forces.

Let:

$\lambda_{A_i}^B$ denote the projection of the elastic displacement of the point $A_i$ under the action of the system of forces (*B*) onto the direction $\Delta_{A_i}$.

$\lambda_{B_i}^A$ denote the projection of the elastic displacement of the point $B_i$ under the action of the system of forces (*A*) onto the direction $\Delta_{B_i}$.

$\Pi_{F_A,F_B}$ or $\Pi_{F_B,F_A}$ denote the value that the total internal potential of the system of bodies will taken when it is deformed by the two systems of forces (*A*) and (*B*) acting simultaneously.

Having said that, if one annuls the variation of temperature $\tau$, which does not relate to the present question, and one likewise annuls the auxiliary couples $\mathcal{C}$ then the variant (39) (no. **45**) of the general equation of elasticity will reduce to:

$$\sum \mathcal{F}\,\lambda = \Pi_{F,\mathcal{F}} - (\Pi_F + \Pi_{\mathcal{F}})\,.$$

When that equation is applied to the deformation of the system of bodies by the system of forces (*B*), that will immediately give:

$$(a) \qquad \sum_{i=1}^{m} F_{A_i}\,\lambda_{A_i}^B = \Pi_{F,\mathcal{F}} - (\Pi_F + \Pi_{\mathcal{F}})\,,$$

if one takes the auxiliary forces $\mathcal{F}$ to be the forces of the system *A*.

When it is applied to the deformation of the system of bodies under the system of forces (*A*), that same equation will give:

$$(b) \qquad \sum_{i=1}^{n} F_{B_i}\,\lambda_{B_i}^{A} = \Pi_{F_A,F_B} - (\Pi_{F_A} + \Pi_{F_B}),$$

if one takes the auxiliary forces to be the forces of the system (*B*).

The two relations (*a*) and (*b*) are precisely the ones (*e*) and (*f*) (no. **29**) that translated into the complete theorem of Betti, Boussinesq, and Maurice Levy.

## 47. – Application of the general equation of elasticity to the graphical statics of elastic arches

In his masterful treatise on graphical statics, Maurice Levy presented some very remarkable graphical methods for the calculation of the elastic arches that are restricted by redundant constraints and subject to external forces that act in the planes of their mean fibers, as well as a temperature variation.

Those methods take into account only the elastic deformations that correspond to a flexural couple. They neglect the ones that correspond to the normal and shearing efforts.

They are based upon the property that is possessed by *fictitious forces* that are mutually-parallel and are applied to each element $ds$ of the mean fiber of the arch and are equal to $(M / EI)\,ds$, where $M$ denotes the moment of flexure in the section whose curvilinear abscissa is $s$, $I$ is the moment of inertia of that section, and $E$ is the longitudinal elastic modulus of the matter that comprises the arch. Those properties, which are particular to each type of arch, result from the conditions to which they are restricted during its elastic deformation by the fact of its redundant constraints. In the treatise of Maurice Levy, they were established by the geometrical and kinematical methods of the calculus of deformations. Furthermore, they are deduced more simply from the equation of redundant constraints (no. **39**) or (which basically amounts to the same thing) the general equation of elasticity for systems of bodies with mean fibers:

$$(a) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = \int \left[ \mathcal{N}\left(\frac{N}{E\,\Omega} - \alpha\tau\right) + \mathcal{T}\,\frac{T}{G\,\Omega} + \mathcal{M}\,\frac{M}{E\,I} \right] ds,$$

which will reduce to:

$$(b) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = -\alpha\tau \int \mathcal{N}\,ds + \int \mathcal{M}\,\frac{M}{E\,I}\,ds,$$

when one suppresses the terms that correspond to the deformations that are due to the normal effort $N$ and the shearing effort $T$.

The degree of approximation that one obtains by neglecting those deformations can often be considered to be sufficient. However, in certain cases – notably when one is dealing with lowered arches (*arcs surbaissés*) – that will no longer be true, and it will become necessary to take into account at least the deformations that are due to the normal effort, if not the ones that are due to the shearing effort, which are always the weakest. One can arrive at that fact by means of some very simple corrections that alter nothing in the structure of the graphical methods, but permit one to involve either just the latter deformations or likewise just the ones that are due to the shearing effort, if desired ([1]). Here is how:

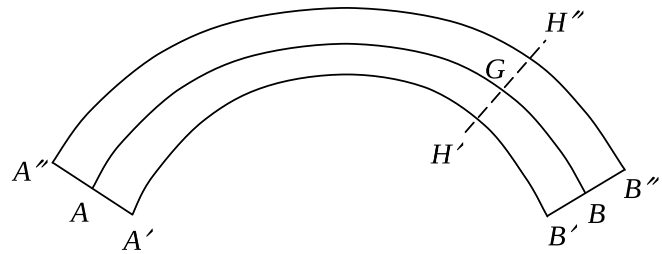


Figure 10.

**48. Introduction of deformations due to normal effort. –** If $r$ denotes the radius of gyration of an arbitrary section ($S$) of the arch around the axis that is drawn through the center of gravity $G$ of that section (Fig. 10) normal to the plane of the mean fiber, which lies along the intersection of that plane with the plane of the section ($S$), then:

$$GH' = GH'' = r.$$

Call:

The two points $H'$ and $H''$, which are nothing but the two summits of the central ellipse of inertia of the section, *conjugate points* relative to the section ($S$).

The two loci $A'B'$ and $A''B''$ of points $H'$ and $H''$ that relate to all sections of the arch *conjugate lines*. Each element $ds$ of the mean fiber $AB$ of the arch corresponds to two elements $ds'$ and $ds''$ of conjugate lines.

The two sums of moments with respect to $H'$ and $H''$ of the elastic forces that are developed in a section ($S$) (or external forces that act to the left of that section) *conjugate moments* that relate to the section.

The values of those moments in the hyperstatic arch considered, when it is subjected to given external forces $F$ and a temperature variation $\tau$, $M'$ and $M''$.

The values of those same moments $\mathcal{M}'$ and $\mathcal{M}''$ in the isostatic arch that is obtained by suppressing the redundant constraints from the hyperstatic arch and subjecting it to auxiliary forces and couples $\mathcal{F}$ and $\mathcal{G}$.

One will have immediately:

([1]) BERTRAND DE FONTVIOLANT, "Mémoire sur la Statique graphique des arcs élastiques," Comptes rendus de l'Académie des Sciences **110** (1890), pp. 697 and Bulletin de la Société des Ingenieurs civils de France, April 1890, pp. 403.

$$M' = M + Nr, \qquad M'' = M - Nr\,;$$

hence:

$$N = \frac{M' - M''}{2r}, \qquad M = \frac{M' + M''}{2r}.$$

Similarly:

$$\mathcal{N} = \frac{\mathcal{M}' - \mathcal{M}''}{2r}, \qquad \mathcal{M} = \frac{\mathcal{M}' + \mathcal{M}''}{2r}.$$

Now, when one suppresses the terms that correspond to the deformations that are due to the shearing effort, the general equation (*a*) (no. **47**) can be written:

$$\sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = -\,\alpha\tau \int \mathcal{N}\,ds + \int \left( \mathcal{N}\,\frac{N}{E\,\Omega} + \mathcal{M}\,\frac{M}{E\,I} \right) ds\,.$$

Upon replacing the normal efforts and moments of flexure in the second integral with their expressions above and keeping in mind the relation $I = \Omega\, r^2$, after some reductions, one will get:

$$\sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = -\,\alpha\tau \int \mathcal{N}\,ds + \int \left( \mathcal{M}'\,\frac{M'}{2E\,I} + \mathcal{M}''\,\frac{M''}{2E\,I} \right) ds\,,$$

in which the integrals extend the whole length of the mean fiber *AB* of the arch.

Set:

$$I' = 2I$$

and agree to represent (indistinctly):

One or the other of the two conjugate moments $M'$ and $M''$ by $M'$,

One or the other of the two conjugate moments $\mathcal{M}'$ and $\mathcal{M}''$ by $\mathcal{M}'$,

One or the other of the elements $ds'$ and $ds''$ of the two conjugate lines that correspond to an element $ds$ of the mean fiber by $ds'$.

In that way, one can write:

$$(c) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = -\,\alpha\tau \int \mathcal{N}\,ds + \int \mathcal{M}'\,\frac{M'}{E\,I'}\,\frac{ds}{ds'}\,ds'\,,$$

in which the second integral extends along the whole length of the two conjugate lines $A'B'$ and $A''B''$.

Upon comparing equation (*b*) (no. **47**), which neglects the deformations that are due to the normal effort, with equation (*c*), which takes them into account, one will see that in order to pass from the former to the latter, it will suffice to replace the moments of flexure $M$ and $\mathcal{M}$ with the conjugate moments $M'$ and $\mathcal{M}'$, the moment of inertia $I$ with $I'$

= $2I$, and to regard the second integral as extending, no longer along the total length of the mean fiber, but indeed the total length of the two conjugate lines.

One can, moreover, imagine (without entering into the proof, which is given in our paper on "Statique graphique des arcs élastiques") that:

*In order to introduce the deformations that are due to the normal effort into the methods of graphical statics that neglect them, it will suffice to replace the fictitious parallel forces* ($M$ / $E\,I$) *ds that are applied to the various elements ds of the mean fiber of the arch considered in those methods with the fictitious forces:*

$$\frac{M'}{E\,I'}\frac{ds}{ds'}ds' = \frac{M'}{E\,I'}ds\,,$$

*which are parallel to the first ones and applied to each element ds′ of the two conjugate lines.*

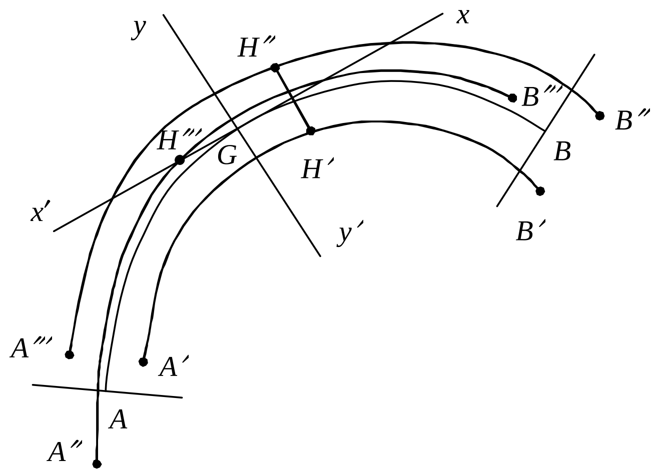


Figure 11.

**49. Introduction of deformations due to normal and shearing efforts.** – Call the three points $H'$, $H''$, $H'''$, which are situated in plane of the mean fiber and have coordinates referred to the tangent $Gx$ and normal $Gy$ to that fiber:

$$x' = \frac{r\sqrt{2a}}{2}, \qquad y' = -\frac{r\sqrt{6}}{2},$$
$$x'' = \frac{r\sqrt{2a}}{2}, \qquad y'' = \frac{r\sqrt{6}}{2},$$
$$x''' = -r\sqrt{2a}\,, \qquad y''' = 0,$$

in which $a$ denotes the ratio of the longitudinal modulus of elasticity $E$ to the transverse modulus of elasticity $G$, *conjugate points* relative to an arbitrary section ($S$) whose center of gravity is $G$ (Fig. 11).

Furthermore, by analogy with what was said before, the arch admits *three conjugate lines* $A'B'$, $A''B''$, $A'''B'''$, and each element $ds$ of its mean fiber corresponds to three elements $ds'$, $ds''$, and $ds'''$ of the conjugate lines. Moreover, any section ($S$) corresponds to *three conjugate moments.*

Let:

$M'$, $M''$, $M'''$ be the values of those moments in the hyperstatic arch considered when it is subject to external forces $F$ and a temperature variation $\tau$.

$\mathcal{M}'$, $\mathcal{M}''$, $\mathcal{M}'''$ be the values of those same moments in the isostatic arch that is obtained by suppressing the redundant constraints from the hyperstatic arch when one subjects it to auxiliary forces and couples $\mathcal{F}$ and $\mathcal{C}$.

One will have immediately:

$$M' = M + N\frac{r\sqrt{6}}{2} + T\frac{r\sqrt{2a}}{2},$$
$$M'' = M - N\frac{r\sqrt{6}}{2} + T\frac{r\sqrt{2a}}{2},$$
$$M''' = M - Tr\sqrt{2a},$$

from which, one will infer that:

$$N = \frac{M' - M''}{r\sqrt{6}},$$
$$T = \frac{M' + M'' - 2M'''}{3r\sqrt{2a}},$$
$$M = \frac{M' + M'' + M'''}{3}.$$

Likewise:

$$\mathcal{N} = \frac{\mathcal{M}' - \mathcal{M}''}{r\sqrt{6}},$$
$$\mathcal{T} = \frac{\mathcal{M}' + \mathcal{M}'' - 2\mathcal{M}'''}{3r\sqrt{2a}},$$
$$\mathcal{M} = \frac{\mathcal{M}' + \mathcal{M}'' + \mathcal{M}'''}{3}.$$

Upon substituting these values for the normal effort, shearing effort, and moment of flexure, resp., in the second integral of the general equation ($a$) (no. **47**), when it is written in the form:

$$\sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = -\alpha\tau \int \mathcal{N}\,ds + \int \left( \mathcal{N}\frac{N}{E\,\Omega} + \mathcal{T}\frac{T}{G\,\Omega} + \mathcal{M}\frac{M}{E\,I} \right) ds\,,$$

and upon taking into account the facts that $a = E / G$ and $I = \Omega\,r^2$, one will get:

$$\sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = -\alpha\tau \int \mathcal{N}\,ds + \int \left( \mathcal{M}'\frac{M'}{3E\,I} + \mathcal{M}''\frac{M''}{3E\,I} + \mathcal{M}'''\frac{M'''}{3E\,I} \right) ds\,,$$

in which the integrals extend along the whole length of the mean fiber *AB* of the arch.

Set:

$$I' = 3I$$

and agree to represent (indistinctly):

Any one of the three conjugate moments $M'$, $M''$, and $M'''$ by $M'$,

Any one of the three conjugate moments $\mathcal{M}'$, $\mathcal{M}''$, and $\mathcal{M}'''$ by $\mathcal{M}'$,

Any one of the three elements $ds'$, $ds''$, $ds'''$ of the three conjugate lines that correspond to an element $ds$ of the mean fiber by $ds'$ .

Furthermore, one can write:

$$(d) \qquad \sum \mathcal{F}\,\lambda + \sum \mathcal{C}\,\varphi = -\alpha\tau \int \mathcal{N}\,ds + \int \mathcal{M}'\frac{M'}{E\,I}\frac{ds}{ds'}\,ds'\,,$$

in which the second integral extends along the whole length of the three conjugate lines $A'B'$, $A''B''$, and $A'''B'''$.

That equation has exactly the same form as equation (*c*) (no. **48**), and one concludes, as before, that:

*In order to introduce deformations that are due to the normal and shearing efforts into the methods of graphical statics that neglect them, it will suffice to replace the parallel fictitious forces* ($M$ / $EI$) *ds that are applied to each element ds of the mean fiber of the arch in those methods with the fictitious forces:*

$$\frac{M'}{EI'}\frac{ds}{ds'}\,ds' = \frac{M'}{EI'}\,ds\,,$$

*which are parallel to the first ones and applied to each element $ds'$ of the three conjugate lines.*

We add that, as we established in our previously-cited paper, a section of an arch does not correspond to just one system of three conjugate points, but indeed, to an infinitude of them, and as a result, an infinitude of systems with three conjugate moments, and that an arch will consequently admit an infinitude of systems of three conjugate lines. The proposition above applies to any one of those systems, but the most convenient of three, for the sake of applications, is the one that was defined above.

# COMPARING THE THREE METHODS

**50. –** The following remarks emerge from the preceding expose:

The methods that were presented are in complete agreement, and all three of them take into account, not just the effects of external forces, but also the caloric effects.

Although there are close relationships between them, they have neither the same significance nor the same theoretical character:

The first method, like the other two, leads to reciprocity principles that are particular cases of the theorem of Betti, Boussinesq, and Maurice Levy. However, in its current state, it will not permit one to either establish that beautiful theorem directly or the general equation of elasticity, which is a synthesis of the entire theory of deformations. In that regard, it is less satisfactory than the last two. From the standpoint of the presentation, the first method based upon the *vis viva* theorem is very delicate, at least the part of it that relates to caloric deformations, which necessitates the extension of Clapeyron's equation that was made here.

On the contrary, the second method, which was likewise deduced from the *vis viva* theorem, is simple and elementary.

The third method, which is inferred from the virtual work theorem, appeals to the most advanced notions from general mechanics, but it permits one to take into account the caloric and elastic deformation from the outset and simultaneously.

The third method has been criticized for the fact that it rests upon a basis that is hardly solid, because, as we said, there exists no rigorous proof of the theorem of virtual work. Without entering into a discussion of that subject that would find no place here, we believe that we must recall that in one of the notes in Lagrange's celebrated *Mécanique analytique* ([1]), one of the most sophisticated mathematicians of the last century, Joseph Bertrand, expressed it thus: "The first *rigorous* proof of the principle of virtual work was due to Fourier (J. de l'École Polytechnique, t. II, year VII)." Nonetheless, in the numerous applications that have been made of it, to my knowledge, the virtual work theorem has never been found to be wrong, and as is clear from the present Note, the results to which it leads, as far as the calculation of the deformation is concerned, in particular, are in complete agreement with the ones that are deduced from *vis viva* theorem.

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([1]) *Œuvres de Lagrange* (published with the attention of J.-A, Serret and Gaston Darboux), t. XI, 1888, pp 263.