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Super-energy in general relativity (*).

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INTRODUCTION.

If the problems of the definition of the energy of gravitation and the nature of gravitational waves are indeed important in the relativistic theory of gravitation then one must admit that the various solutions that have been proposed up until now present various drawbacks.

The object of this study is essentially that of indicating a path that one might follow in defining these problems, not a solution to them.

One may, upon schematizing things, say that the oldest assumptions rest upon an analogy between classical mechanics and relativity. The conserved quantities of classical mechanics: momentum, moment of momentum, and energy are linked with the transformations of the fundamental group, namely, spatial translations, spatial rotations, and time translations, respectively. Having said this, one sees the difficulties that one might encounter when one transposes such a mode of thought into the context of general relativity on a Riemannian manifold, which possesses no such group, but only a pseudo-group of isometries. It is precisely this criticism that A. Trautmann has brought to light at the recent colloquium on the relativistic theories of gravitation that was held at Royaumont in June of 1959, under the auspices of the C.N.R.S. [**1**].

A new path was introduced by Lichnerowicz [**2**], Pirani [**3**], and Trautmann [**4**]. It emphasizes the analogy between the electromagnetic field and the tensor field that is defined by the Riemann curvature tensor.

It is Pirani who, in several recent articles, has insisted on the role and physical significance of the Riemann tensor, a role and significance that has been obscured by the fact that only the Ricci tensor appears in the field equations.

The contribution of Trautmann is based on the manner of posing the boundary-value problem in special relativity, as well as in general relativity. Here, we retain the idea of the analogy that is thus described, and we propose to see how and up to what point one may pursue it. This analogy has also inspired an article of R. Penrose that will appear, in which use is made of the spinorial representation in the study of the electromagnetic field, as well as the Riemann tensor field.

1. *The energy-momentum tensor of the electromagnetic field* [**5**]. The electromagnetic field is defined by an anti-symmetric tensor field, or 2-form:

$$F_{\alpha\beta} = - F_{\beta\alpha}, \qquad \alpha, \beta = 0, 1, 2, 3, \tag{1.1}$$

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† Translated by D.H. Delphenich.

which is defined, in all that follows, in a Riemannian manifold of normal hyperbolic signature +, –, –, – ; in the sequel, $g_{\alpha\beta}$ will denote the metric tensor.

The energy-momentum tensor of the field is given by:

$$-T_{\alpha\beta} = F_{\alpha\gamma}F_{\beta}^{\gamma} - \tfrac{1}{4}g_{\alpha\beta}(F_{\rho\sigma}F^{\rho\sigma}) . \tag{1.2}$$

$\eta_{\alpha\beta\lambda\mu}$ will denote the volume element tensor, and we agree that $\eta_{0123} = \sqrt{-g}$.

$$*F_{\alpha\beta} = \tfrac{1}{2}\,\eta_{\alpha\beta\lambda\mu}\,F^{\lambda\mu} \tag{1.3}$$

will denote the tensor that is adjoint to $F_{\alpha\beta}$ for the given metric. We shall recall eight remarkable properties of the tensor $T_{\alpha\beta}$:

a. Symmetry:

$$T_{\alpha\beta} = T_{\beta\alpha} . \tag{1.4}$$

b. Null trace:

$$T_{\alpha\beta}\,g^{\alpha\beta} = 0. \tag{1.5}$$

c. Conservation. In the case of an electromagnetic field in the absence of charges:

$$\nabla_{\alpha}T^{\alpha\beta} = 0, \tag{1.6}$$

in which ∇_{α} denotes the covariant derivative.

d. Its involutive character, and the two types of fields [**6**]:

$$T_{\alpha}^{\beta}\,T_{\beta\gamma} = \tfrac{1}{4}g_{\alpha\gamma}I^{2} \tag{1.7}$$

in which:

$$4I^{2} = (F_{\rho\sigma}F^{\rho\sigma})^{2} + (F_{\rho\sigma}*F^{\rho\sigma})^{2}. \tag{1.8}$$

The field is non-singular if $I \neq 0$ and singular if $I = 0$.

e. Characteristic isotropic vectors:

1. Non-singular field: there exist two real isotropic vectors $l_{\alpha}^{(1)}$ and $l_{\alpha}^{(2)}$ such that:

$$T_{\alpha\beta} = -\,\mathrm{I}\left(g_{\alpha\beta} - 2\,\frac{l_{\alpha}^{(1)}l_{\beta}^{(2)} + l_{\alpha}^{(2)}l_{\beta}^{(1)}}{l_{12}}\right), \tag{1.9}$$

in which:

$$l_{12} = l_{\alpha}^{(1)}\;l_{(2)}^{\alpha} . \tag{1.10}$$

2. Singular field: there exists a real isotropic vector l^{α} such that:

$$T_{\alpha\beta} = \mathcal{T} l_\alpha \, l_\beta \,, \tag{1.11}$$

in which $\mathcal{T}$ is a scalar; one may also say that l_α is a double isotropic vector and that the vectors $l_\alpha^{(1)}$ and $l_\alpha^{(2)}$ in the non-singular case coincide in the singular case.

The isotropic vectors $l_{(i)}^\alpha$ are also defined by the fact that they are the real isotropic proper vectors of $T_{\alpha\beta}$:

$$T_{\alpha\beta} \, l_{(i)}^\beta = + \, I \; l_\alpha^{(i)} \,, \qquad\qquad i = 1, 2. \tag{1.12}$$

f. Geometric interpretation [**7**]: the linear space that is tangent to a point of the Riemannian manifold is a centered Minkowski space. If one studies the restriction of that space to one of its hyperplane sections ("space at infinity") then one obtains a Cayley space C_3 whose absolute space is the hyperplane section of the hyper-cone of lightlight directions, and is an oval quadric.

Two real null or isotropic directions $l_{(i)}^\alpha$ are associated with the tensor $T_{\alpha\beta}$, namely, two points $l_{(i)}$ of the absolute space. A quadric is associated with $T_{\alpha\beta}$, and in the non-singular case, it is characterized by the following fact: the quadric $T_{\alpha\beta}$ cuts the absolute space along the left quadrilateral of the generatrices of the absolute space that pass through the points $l^{(1)}$ and $l^{(2)}$, and is apolar to the absolute space. ((1.5) expresses the property of apolarity) (cf. Fig. 1):

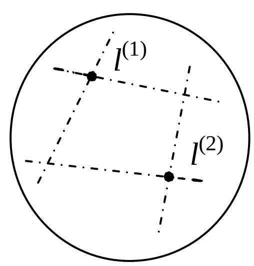

Fig. 1.

In the singular case, the quadric $T_{\alpha\beta}$ is formed from the plane tangent to the point l, counted twice.

g. Energy for an observer: The introduction of an observer may be accomplished by introducing a timelike direction that is defined by a unitary vector u^α ($u^\alpha u_\alpha = 1$). Under these conditions:

$$E_\alpha = F_{\alpha\beta} \, u^\beta \,, \qquad H_\alpha = - \, {}^*F_{\alpha\beta} \, u^\beta \tag{1.13}$$

are the electric field and magnetic field vectors for the observer. If:

$$T(u) = T_{\alpha\beta} \, u^\alpha \, u^\beta \tag{1.14}$$

then $T(u)$ is the energy function of the observer. One has:

$$T(u) = \tfrac{1}{2}(E_a E^a + H_a H^a)\ . \tag{1.15}$$

$T(u)$ is a positive-definite function.

h. Mariot's theorem [**8**]. In the case of a singular field the trajectories of the vector l^α are isotropic geodesics, and when the conservation condition is applied to (1.11), one obtains:

$$l^\alpha \nabla_\alpha l_\beta = k\, l_\beta\ . \tag{1.16}$$

2. *Energy-momentum tensor and the electromagnetic field.* In the preceding section, the emphasis was on the tensor $T_{\alpha\beta}$, but the analysis that we carried out may also be applied to the field $F_{\alpha\beta}$. In the non-singular case, $F_{\alpha\beta}$ may always be put into the form:

$$F_{\alpha\beta} = E\, a_{\alpha\beta} + H\, {}^*a_{\alpha\beta} \tag{2.1}$$

in which:

$$a_{\alpha\beta} = \frac{l_\alpha^{(1)} l_\beta^{(2)} - l_\alpha^{(2)} l_\beta^{(1)}}{l_{12}}\ , \tag{2.2}$$

and E and H are two scalar field invariants, and $I^2 = E^2 + H^2$. A well-defined field $T_{\alpha\beta}$ corresponds to the field $F_{\alpha\beta}$, and conversely, a family of fields $F_{\alpha\beta}$ corresponds to the tensor $T_{\alpha\beta}$, since only I itself is well-defined. The field $F_{\alpha\beta}$ is then determined up to a "duality rotation," in the sense that if:

$$\mathrm{F}'_{\alpha\beta} = F_{\alpha\beta} \cos\theta + {}^*F_{\alpha\beta} \sin\theta\ , \tag{2.3}$$

$\mathrm{F}'_{\alpha\beta}$ and $F_{\alpha\beta}$ have the same energy-momentum tensor [**9**].

The vectors $l_\alpha^{(i)}$ are also the real isotropic proper vectors of $F_{\alpha\beta}$:

$$F_{\alpha\beta}\, l_{(i)}^\beta = k\ l_\alpha^{(i)}\ . \tag{2.4}$$

If the field $F_{\alpha\beta}$ is singular then, for the direction l^α, one has both:

$$F_{\alpha\beta}\ l^\beta = {}^*F_{\alpha\beta}\ l^\beta = 0\ . \tag{2.5}$$

The tensor $F_{\alpha\beta}$ and ${}^*F_{\alpha\beta}$ may then be put into the form:

$$\begin{aligned} F_{\alpha\beta} &= l_\alpha u_\beta - l_\beta u_\alpha \\ {}^*F_{\alpha\beta} &= l_\alpha v_\beta - l_\beta v_\alpha\ , \end{aligned} \tag{2.6}$$

with:

$$l^\alpha l_\alpha = l^\alpha u_\alpha = l^\alpha v_\alpha = v^\alpha u_\alpha = 0\ , \qquad u^\alpha u_\alpha = v^\alpha v_\alpha\ .$$

The tensors $F_{\alpha\beta}$ and $*F_{\alpha\beta}$ are then the coordinates in C^3 of the two lines that are conjugate and tangent to the absolute space at the point l .

3. *The tensor of L. Bel.* [**10**] *Main properties.* In the sections that follow, we shall first consider the vacuum Einstein spaces, i.e., ones such that:

$$R_{\alpha\mu} = R_{\alpha\mu}{}^{\beta}{}_{\mu} = 0 \tag{3.1}$$

in which $R_{\alpha\beta\lambda\mu}$ is the Riemann tensor.

L. Bel has introduced the tensor:

$$T_{\alpha\beta\lambda\mu} = 2(R_{\alpha\rho\lambda\sigma}R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + R_{\alpha\rho\mu\sigma}R_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - A g_{\alpha\beta}g_{\lambda\mu})\ , \tag{3.2}$$

in which:

$$A = \tfrac{1}{8} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}\ . \tag{3.3}$$

This tensor possesses the following symmetry properties:

$$T_{\alpha\beta\lambda\mu} = T_{\beta\alpha\lambda\mu} = T_{\alpha\beta\mu\lambda} = T_{\lambda\mu\alpha\beta}\ . \tag{3.4}$$

I. Robinson has drawn attention to the fact that this tensor is completely symmetric and L. Bel has communicated a proof to me, and we recall the essentials here.

If:

$$(R^*)_{\alpha\beta\lambda\mu} = \tfrac{1}{2} R_{\alpha\beta}{}^{\rho\sigma}\,\eta_{\rho\sigma\lambda\mu}\ , \qquad (^*R)_{\alpha\beta\lambda\mu} = \tfrac{1}{2}\,\eta_{\rho\sigma\lambda\mu}\,R_{\alpha\beta}{}^{\rho\sigma},$$
$$(^*R^*)_{\alpha\beta\lambda\mu} = \tfrac{1}{4}\,\eta_{\alpha\beta\rho\sigma}\,R^{\rho\sigma\gamma\delta}\,\eta_{\gamma\delta\lambda\mu} \tag{3.5}$$

then one has, thanks to (3.1):

$$^*R^* = -R \qquad \text{and} \qquad R^* = {}^*R = \overset{*}{\mathrm{R}}\ . \tag{3.6}$$

This being the case, one has the following identity:

$$R_{\alpha\rho\mu\sigma}R_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \overset{*}{\mathrm{R}}{}_{\alpha\rho\lambda\sigma}\overset{*}{\mathrm{R}}{}_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} = g_{\lambda\mu}\,g_{\alpha\beta}\,A\ . \tag{3.7}$$

Indeed:

$$\begin{aligned} -\overset{*}{\mathrm{R}}{}_{\alpha\sigma\lambda\sigma}\overset{*}{\mathrm{R}}{}_{\beta}{}^{\rho\mu\sigma} &= \tfrac{1}{4}\,\mathrm{R}_{\alpha\rho}{}^{\lambda'\mu'}\eta_{\lambda'\sigma'\lambda\sigma}\eta^{\mu\sigma\mu'\sigma''}\mathrm{R}_{\beta}{}^{\rho}{}_{\mu'\sigma''} \\ &= \tfrac{1}{4}\,\delta^{\mu\mu'\sigma''}_{\lambda'\sigma'\lambda}\,\mathrm{R}_{\alpha\rho}{}^{\lambda'\mu'}\mathrm{R}_{\beta}{}^{\rho}{}_{\mu'\sigma''} \\ &= \tfrac{1}{2}(\mathrm{R}_{\alpha\rho}{}^{\mu\sigma'}\mathrm{R}_{\beta}{}^{\rho}{}_{\sigma'\lambda} + \mathrm{R}_{\alpha\rho}{}^{\lambda'\mu}\mathrm{R}_{\beta}{}^{\rho}{}_{\lambda\lambda'} + \delta^{\mu}_{\lambda}\mathrm{R}_{\alpha\rho}{}^{\lambda\sigma}\mathrm{R}_{\beta}{}^{\rho}{}_{\lambda\sigma})\ . \end{aligned}$$

(3.7) then results, thanks to the identity:

$$R_{\alpha\rho\lambda\sigma}R_{\beta}{}^{\rho\lambda\sigma} = 2A\,g_{\alpha\beta}\ . \tag{3.8}$$

One thus has:

$$T_{\alpha\beta\lambda\mu} = 2(R_{\alpha\rho\lambda\sigma} R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + \overset{*}{R}_{\alpha\rho\lambda\sigma} \overset{*}{R}_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma}) \,. \tag{3.9}$$

Now, one has both:

$$R_{\alpha\rho\lambda\sigma} + R_{\rho\lambda\alpha\sigma} + R_{\lambda\alpha\rho\sigma} = 0 \tag{3.10}$$

$$\overset{*}{R}_{\alpha\rho\lambda\sigma} + \overset{*}{R}_{\rho\lambda\alpha\sigma} + \overset{*}{R}_{\lambda\alpha\rho\sigma} = 0 \,; \tag{3.11}$$

the first identity is true for any tensor $R_{\alpha\beta\lambda\mu}$, and the second one results from (3.1).

One may then verify that:

$$T_{\alpha\beta\lambda\mu} = T_{\lambda\beta\alpha\mu} \tag{3.12}$$

and the complete symmetry results from the properties (3.4).

Indeed, thanks to (4.10), one has, in turn:

$$\begin{aligned} R_{\alpha\rho\lambda\sigma} R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} - R_{\lambda\rho\alpha\sigma} R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} &= R_{\alpha\lambda\rho\sigma} R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma}, \\ = \tfrac{1}{2} R_{\alpha\lambda\rho\sigma} (R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} - R_{\beta}{}^{\sigma}{}_{\mu}{}^{\rho}) &= \tfrac{1}{2} R_{\alpha\lambda\rho\sigma} R_{\beta\mu}{}^{\rho\sigma}. \end{aligned}$$

Similarly, thanks to (3.11), one has:

$$\overset{*}{R}_{\alpha\rho\lambda\sigma} \overset{*}{R}_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} - \overset{*}{R}_{\lambda\rho\alpha\sigma} \overset{*}{R}_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} = \tfrac{1}{2} \overset{*}{R}_{\alpha\lambda\rho\sigma} \overset{*}{R}_{\beta\mu}{}^{\rho\sigma} ,$$

and, thanks to the identity:

$$R_{\alpha\lambda\rho\sigma} R_{\beta\mu}{}^{\rho\sigma} + \overset{*}{R}_{\alpha\lambda\rho\sigma} \overset{*}{R}_{\beta\mu}{}^{\rho\sigma} = 0 \tag{3.13}$$

the relation (3.12) results.

The tensor $T_{\alpha\beta\lambda\mu}$ enjoys the following properties:

a) Complete symmetry.

b) Null trace. One has:

$$g^{\beta\mu} \, T_{\alpha\beta\lambda\mu} = 0 \,. \tag{3.14}$$

This results immediately from $R_{\alpha\mu} = \overset{*}{R}_{\alpha\mu} = 0$.

c) Conservation. One has:

$$\nabla_{\alpha} T^{\alpha}{}_{\beta\lambda\mu} = 0 \,. \tag{3.15}$$

For example, start with (3.9). Thanks to the Bianchi identities, and by virtue of (3.6), one has both:

$$\nabla_{\alpha} R^{\alpha}{}_{\beta\lambda\mu} = \nabla_{\alpha} \overset{*}{R}{}^{\alpha}{}_{\beta\lambda\mu} = 0 \,. \tag{3.16}$$

If one then takes into account (3.10) and (3.11) then one obtains:

$$\nabla_\alpha T^\alpha{}_{\beta\lambda\mu} = R^{\alpha\rho}{}_\lambda{}^\sigma \nabla_\beta R_{\alpha\beta\lambda\mu} + \overset{*}{R}{}^{\alpha\rho}{}_\lambda{}^\sigma \nabla_\alpha \overset{*}{R}{}_{\alpha\beta\lambda\mu} ,$$

which is null, thanks to the identity (3.13).

4. *Two completely isotropic planes and characteristic isotropic vectors. Petrov classification* [**11**] . A 6×6 matrix is associated with the tensor $R_{\alpha\beta\lambda\mu}$ that, from the hypothesis (3.1), may always be put into the form:

$$C_{AB} = \left(\begin{array}{c|c} \alpha & \beta \\ \hline \beta & -\alpha \end{array}\right), \qquad A, B, \ldots = 1, 2, \ldots, 6, \tag{4.1}$$

in an orthonormal frame, in which α and β are symmetric 3×3 matrices with null trace (in the sequel, we agree to set: 1 = 23, 2 = 31, 3 = 12, 4 = 10, 5 = 20, 6 = 30). In C_3, where the absolute equation is:

$$g_{\alpha\beta} X^\alpha X^\beta = 0 , \tag{4.2}$$

the coordinates of the 2-planes or bivectors:

$$p^{\alpha\beta} = X^\alpha Y^\beta - X^\beta Y^\alpha, \quad \text{or } p^A, \tag{4.3}$$

are the Plückerian coordinates of the lines in C_3 .

C_{AB} is associated with a quadratic complex of lines in C_3 such that two quadruples of complex conjugate lines belong to the absolute. To these two quadruples of lines there correspond two quadruples of completely isotropic 2-planes, which we qualify with the term *characteristic*. The two quadruples of characteristic lines have four real points of the absolute in common, which are also four characteristic isotropic vectors [**12**].

Indeed, one may map the space C_5 of lines in C_3 onto a pair of complex conjugate Cayley planes by the map:

$$Z^a = p^A + i\, p^{A+3}, \quad Z^{a+3} = p^{A+3} + i\, p^A = i\, Z^a, \qquad (a, b, \ldots = 1, 2, 3). \tag{4.4}$$

The image of the complex C_{AB} is then the conic:

$$C_{ab} = (\alpha - i\beta) \tag{4.5}$$

in a plane whose absolute is:

$$g_{ab} Z^a Z^b = Z_1^2 + Z_2^2 + Z_3^2 \tag{4.6}$$

and the complex conjugate conic in the complex conjugate Cayley plane. We fix our attention on the conic (4.6). That conic is the image of a ruling of the quadric (4.2). The four points that are common to the conics C_{ab} and g_{ab} are the images of four lines in the complex C_{AB} that belong to the absolute are the *generatrices* or *completely isotropic characteristic 2-planes,* which we denote by $g_{(i)}$ and $p^{\alpha\beta}_{(i)}$ in the sequel; there also exist conjugate generatrices $g_{(\bar{i})}$ and 2-planes $p^{\alpha\beta}_{(\bar{i})}$.

$l_{(i)}$ will denote the points of the absolute that are common to $g_{(i)}$ and $g_{(\bar{i})}$.

Analytically, one may associate the aforementioned notions in some detail in the following manner: First of all, suppose that the four points that are common to C_{ab} and g_{ab} are distinct, so the matrix C_{ab} is then diagonalizable:

$$C_{ab} = \begin{pmatrix} \mathrm{A}_{11} & 0 & 0 \\ 0 & \mathrm{A}_{22} & 0 \\ 0 & 0 & \mathrm{A}_{33} \end{pmatrix}, \quad A_{aa} = \alpha_a - i\beta_a . \tag{4.7}$$

If we represent the conic (4.6) parametrically by:

$$Z^1 = 2\xi\eta, \qquad Z^2 = \xi^2 - \eta^2, \qquad Z^3 = i\,(\xi^2 + \eta^2) \tag{4.8}$$

then the generatrices g_i are defined by the equation:

$$A_{11}(Z^1)^2 + A_{22}(Z^2)^2 + A_{33}(Z^3)^2 = 0 , \tag{4.9}$$

or also:

$$r(\xi^4 + \eta^4) + 2s\xi^2\eta^2 = 0 \tag{4.10}$$

or:

$$r = A_{22} - A_{33} \qquad \text{and} \qquad s = 3A_{11} . \tag{4.11}$$

We also set:

$$A_1 = A_{22} - A_{33} , \qquad A_2 = A_{33} - A_{11} , \qquad A_3 = A_{11} - A_{22} , \tag{4.12}$$

$$r = A_1 \qquad \text{and} \qquad s = A_3 - A_2 . \tag{4.13}$$

Since the three complex numbers A_{aa} and A_a have a null sum, one may construct the following figure: an oriented triangle with vertices A_{11}, A_{22}, A_{33}, and the three numbers A_a are equipollent to the sides of the triangle, such that the origin of the complex plane coincides with the center of mass of the triangle (Fig. 2).

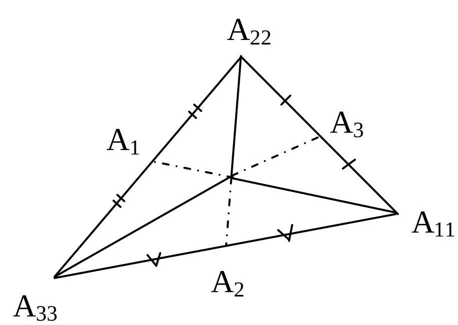

Fig. 2.

The Petrov classification amounts to the classes of conics C_{ab} in the Cayley plane of the absolute g_{ab} [**13**]. We denote the general case that we just considered by 1*a*.

1*b* will denote the case in which the conic C_{ab} is bitangent to the absolute; there are thus two doubly characteristic 2-planes (and their conjugates) and two doubly isotropic vectors $l^{\alpha}_{(i)}$.

In the case where the two characteristic values A_{aa} are equal, one of the numbers A_a is null, so $r = 0$, as a result; the characteristic equation (4.10) then reduces to:

$$\xi^2 \eta^2 = 0 \tag{4.14}$$

II*a*: the conic C_{ab} is tangent to the absolute and intersects the absolute in two more distinct points. There is thus a doubly characteristic 2-plane and two simply characteristic 2-planes (and their conjugates), and therefore also a doubly isotropic vector l^{α} and two simple isotropic vectors $l^{\alpha}_{(i)}$. The matrix (4.5) possesses the following reduced form:

$$C_{ab} = \begin{pmatrix} -2\mathrm{A} & 0 & 0 \\ 0 & \mathrm{A} - \sigma & i\sigma \\ 0 & i\sigma & \mathrm{A} + \sigma \end{pmatrix} \quad \text{and } A \neq 0 . \tag{4.15}$$

Equation (4.10) takes the form:

$$\xi^2 (3A\eta^2 + s\xi^2) = 0 . \tag{4.16}$$

II*b*: The conic C_{ab} is sur-osculating to the absolute, so there is a quadruply characteristic 2-plane (and its conjugate) and quadruply isotropic vector l^{α}. One then has the reduced form for the matrix C_{ab} that was given in (4.15) with $A = 0$. Equation (4.16) reduces to $\xi^4 = 0$.

III. The conic C_{ab} is osculating to the absolute and intersects the absolute at a single point. There is a triply characteristic 2-plane and a simple 2-plane (and their conjugates), and there is a triply isotropic vector $l^{\alpha}_{(1)}$ and a simple vector $l^{\alpha}_{(2)}$.

$$C_{ab} = \begin{pmatrix} 0 & -\sigma & i\sigma \\ -\sigma & 0 & 0 \\ i\sigma & 0 & 0 \end{pmatrix}, \tag{4.17}$$

and equation (4.10) takes the form:

$$\xi^3 \eta = 0 \tag{4.18}$$

Finally, observe that the conic C_{ab} may not coincide with the absolute, due to the condition of apolarity; however, it may, of course, be identically null.

Remarks on the subject of case 1*a*:

Three remarkable cases are worthy of attention:

a-I. The triangle $A_{11}A_{22}A_{33}$ has null area: the three complex numbers A_{aa} and A_a are proportional, and one has:

$$\alpha_a = \lambda\beta_a \,. \tag{4.19}$$

a-II. The four points $Z^a_{(i)}$ that are common to C_{ab} and g_{ab} form a harmonic group, and the same is true for the four isotropic vectors $l^a_{(i)}$. This is a particular case of *a*-I, in which one of the numbers A_{aa} is null, one of the pairs (α_a, β_a) is null, and equation (4.10) takes the form:

$$\xi^4 + \eta^4 = 0 \,; \tag{4.20}$$

indeed, $s = 0$.

a-III. The four points $Z^a_{(i)}$ form an equianharmonic group, so the four isotropic vectors $l^a_{(i)}$ do, as well. One has:

$$\alpha_{k+1} = A\cos\left(\theta + k\frac{2\pi}{3}\right), \qquad \beta_{k+1} = A\sin\left(\theta + k\frac{2\pi}{3}\right), \qquad k = 0, 1, 2. \tag{4.21}$$

5. *Isotropic 2-planes and characteristic isotropic vectors. Analytical expressions.* Let:

$$(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 = 0 \tag{5.1}$$

be the equation of the absolute in C_3 in an orthonormal frame. One may give it the parametric representation:

$$\begin{aligned} X^0 &= \xi_1\xi_2 + \eta_1\eta_2 \,, \qquad X^1 = \xi_1\xi_2 - \eta_1\eta_2 \,, \\ X^2 &= \xi_1\eta_2 + \xi_1\eta_2 \,, \qquad X^3 = i(\xi_2\eta_1 - \xi_1\eta_2), \end{aligned} \tag{5.2}$$

in which (ξ_1, η_1) and (ξ_2, η_2) are two complex homogeneous parameters.

If (ξ_1, η_1) and (ξ_2, η_2) are two complex conjugate parameters then one will have the real points of (5.1).

If ξ_1, η_1 are fixed and ξ_2, η_2 are variable in (5.2) then one obtains the parametric equations of a generatrix of the absolute, and the calculation of its Plückerian coordinates shows, upon passing to the coordinates Z^a, that (ξ_1, η_1) are the coordinates of this generatrix in the parameterization (4.8). Therefore, if $\xi^{(i)}$, $\eta^{(i)}$ denotes one of the roots of the equation (4.10) then the corresponding isotropic vector will have coordinates that are equal to the numbers:

$$l^0_{(i)} = \xi^{(i)}\bar{\xi}^{(i)} + \eta^{(i)}\bar{\eta}^{(i)} \,, \ldots, \tag{5.3}$$

that are obtained by replacing (ξ_1, η_1) and (ξ_2, η_2) in (5.2) by $\xi^{(i)}$, $\eta^{(i)}$ and $\bar{\xi}^{(i)}, \bar{\eta}^{(i)}$, respectively.

If *a*, *b* is a root of (4.10), which are therefore the coordinates of a point Z^a on C_{ab}, then the four points $Z^a_{(i)}$ will have the ξ, η coordinates:

$$(a, b), \qquad (b, a), \qquad (a, -b), \qquad (b, -a), \tag{5.4}$$

equation (4.10) may indeed be factorized into the form:

$$r(\xi^4 + \eta^4) + 2s\xi^2\eta^2 = (a\xi + b\eta)(b\xi + a\eta)(a\xi - b\eta)(b\xi - a\eta)\,, \tag{5.5}$$

and the four corresponding isotropic vectors will have the coordinates:

$$\begin{aligned} &a\bar{a} + b\bar{b}\,, a\bar{a} - b\bar{b}\,, a\bar{b} + \bar{a}b\,, i(\bar{a}b - a\bar{b}) \\ &b\bar{b} + a\bar{a}\,, b\bar{b} - a\bar{a}\,, b\bar{a} + \bar{b}a\,, i(\bar{b}a - b\bar{a}) \\ &a\bar{a} + b\bar{b}\,, a\bar{a} - b\bar{b}\,, -a\bar{b} - \bar{a}b\,, -i(a\bar{b} - \bar{a}b) \\ &b\bar{b} + a\bar{a}\,, b\bar{b} - a\bar{a}\,, -b\bar{a} - \bar{b}a\,, -i(\bar{b}a - b\bar{a})\;. \end{aligned} \tag{5.6}$$

Let:

$$a_0 = a\bar{a} + b\bar{b}\,,\; a_1 = a\bar{a} - b\bar{b}\,,\; a_2 = a\bar{b} + \bar{a}b\,,\; a_3 = i(\bar{a}b - a\bar{b}) \tag{5.7}$$

be the coordinates of one of the isotropic vectors. One easily verifies that one has, as result:

$$\begin{gathered} a_0^2 = a_1^2 + a_2^2 + a_3^2 \\ (a_1^2 + a_2^2)^2 = 16\mathrm{A}_3\bar{\mathrm{A}}_3, \quad (a_2^2 + a_3^2)^2 = 16\mathrm{A}_1\bar{\mathrm{A}}_1, \quad (a_3^2 + a_1^2)^2 = 16\mathrm{A}_2\bar{\mathrm{A}}_2\,. \end{gathered} \tag{5.8}$$

If $r_a^2 = \mathrm{A}_a\bar{\mathrm{A}}_a$ are the squares of the moduli of the A_a then one finds:

$$\begin{aligned} a_0^2 &= 2(r_1 + r_2 + r_3) = 4p \\ a_1^2 &= 2(r_2 + r_3 - r_1) = 4(p - r_1) \\ a_2^2 &= 2(r_3 + r_1 - r_2) = 4(p - r_2) \\ a_3^2 &= 2(r_1 + r_2 - r_3) = 4(p - r_3)\,, \end{aligned} \tag{5.9}$$

and the matrix of components of the characteristic isotropic vectors may be written:

$$l_{(i)}^a = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_0 & -a_1 & a_2 & -a_3 \\ a_0 & a_1 & -a_2 & -a_3 \\ a_0 & -a_1 & -a_2 & a_3 \end{pmatrix}, \qquad i = 1, 2, 3, 4, \tag{5.10}$$

and one finally has:

$$\det(l_{(i)}^a) = 16 a_0 a_1 a_2 a_3 = 16 \times 16\ \mathrm{S}, \tag{5.11}$$

in which S is the area of the triangle $\mathrm{A}_{11}\mathrm{A}_{22}\mathrm{A}_{33}$.

Particular cases:

I*a*. I. The four vectors $l^a_{(i)}$ are coplanar: one has $S = 0$.

I*a*. II. If $A_{11} = 0$ then one has $A_{22} = A$, $A_{33} = -A$, $A_1 = 2A$, $A_2 = -A$, $A_3 = -A$, $r = 2A$, $s = 0$, from which it follows that:

$$l^a_{(i)} = a\begin{pmatrix} \sqrt{2} & 0 & 1 & 1 \\ \sqrt{2} & 0 & 1 & -1 \\ \sqrt{2} & 0 & -1 & -1 \\ \sqrt{2} & 0 & -1 & 1 \end{pmatrix} \tag{5.12}$$

if $a^2 = A\bar{A}$. The four vectors $l^a_{(i)}$ are in the plane $X^2 = 0$.

I*a*. III. $A_{11} = A$, $A_{22} = A\,e^{2\pi i/3}$, $A_{33} = A\,e^{4\pi i/3}$, and one has:

$$l^a_{(i)} = a\begin{pmatrix} \sqrt{3} & 1 & 1 & 1 \\ \sqrt{3} & -1 & 1 & -1 \\ \sqrt{3} & 1 & -1 & -1 \\ \sqrt{3} & -1 & -1 & -1 \end{pmatrix}. \tag{5.13}$$

I*b*. $A_{11} = -2A$, $A_{22} = A$, $A_{33} = -A$, $A_1 = 0$, $A_2 = 3A$, $A_3 = -3A$, $s = 0$. There are two distinct isotropic vectors, and:

$$l^a_{(i)} = a\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}. \tag{5.14}$$

II*a*. Starting with equation (4.16), one finds:

$$l^a_{(i)} = a\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 & -a_3 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \tag{5.15}$$

in which:

$$a_0 = \sigma^2 + 3\sqrt{A\bar{A}}, \qquad a_1 = \sigma^2 - 3\sqrt{A\bar{A}}, \tag{5.17}$$
$$a_2 = i\sigma(\sqrt{3A} - \sqrt{3\bar{A}}), \qquad a_3 = -\sigma(\sqrt{3A} + \sqrt{3\bar{A}}) .$$

II*b*. There are four isotropic vectors whose components (1, 1, 0, 0) all coincide.

III. (1, 1, 0, 0) is the triply isotropic vector and (1, −1, 0, 0) is the simply isotropic vector.

6. *Study of the surface* T(X). We set:

$$T(X) = T_{\alpha\beta\lambda\mu}\, X^{\alpha} X^{\beta} X^{\lambda} X^{\mu} . \tag{6.1}$$

We place ourselves in case I*a* by starting with the reduced form (4.1). One has:

$$T_{0000} = T_{1111} = T_{2222} = T_{3333} \tag{6.2}$$

$$T_{0000} = 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \beta_1^2 + \beta_2^2 + \beta_3^2) \tag{6.3}$$

$$= 2(A_{11}\bar{A}_{11} + A_{22}\bar{A}_{22} + A_{33}\bar{A}_{33}) = \tfrac{2}{3}(A_1\bar{A}_1 + A_2\bar{A}_2 + A_3\bar{A}_3)$$

$$T_{1122} = -T_{0033} = 4(\alpha_1\alpha_2 + \beta_1\beta_2) = -\tfrac{4}{3}A_3\bar{A}_3 + \tfrac{1}{3}T_{0000} \tag{6.4}$$

$$T_{2233} = -T_{0011} = 4(\alpha_2\alpha_3 + \beta_2\beta_3) = -\tfrac{4}{3}A_1\bar{A}_1 + \tfrac{1}{3}T_{0000} \tag{6.5}$$

$$T_{3311} = -T_{0022} = 4(\alpha_3\alpha_1 + \beta_3\beta_1) = -\tfrac{4}{3}A_2\bar{A}_2 + \tfrac{1}{3}T_{0000} \tag{6.6}$$

$$T_{0123} = \tfrac{4}{3}(\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_1\beta_3 - \alpha_3\beta_1)$$

$$= \tfrac{2i}{3}(A_2\bar{A}_3 - \bar{A}_2 A_3) = \tfrac{8}{3} S .$$

These are the only non-null components of the tensor $T_{\alpha\beta\lambda\mu}$.

THEOREM. – The surface T(X) intersects the absolute of C_3 in two quadruples of complex conjugate generatrices, which are the characteristic generatrices g_i and $g_{\bar{i}}$.

One has:

$$\begin{aligned} T(X) = {} & T_{0000}[(X^0)^4 + (X^1)^4 - (X^2)^4 + (X^3)^4] \\ & + 6T_{1122}[(X^1)^2(X^3)^2 - (X^0)^2(X^3)^2] + 6T_{2233}[(X^2)^2(X^3)^2 - (X^0)^2(X^1)^2] \\ & + 6T_{3311}[(X^3)^2(X^1)^2 - (X^0)^2(X^2)^2] + 24T_{0123}\, X^0 X^1 X^2 X^3 , \end{aligned} \tag{6.8}$$

or further:

$$\begin{aligned} T(X) = {} & T_{0000}[(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2]^2 \\ & -8A_3\bar{A}_3 [(X^1)^2(X^2)^2 - (X^0)^2(X^3)^2] + -8A_2\bar{A}_2 [(X^3)^2(X^1)^2 - (X^0)^2(X^2)^2] \\ & -8A_3\bar{A}_3 [(X^1)^2(X^2)^2 - (X^0)^2(X^1)^2] + 64S\, X^0 X^1 X^2 X^3 , \end{aligned} \tag{6.9}$$

On the absolute, taking (5.2) into account, one has:

$$\begin{aligned} T(X) = {} & 8A_1\bar{A}_1 (\xi_1^4 + \eta_1^4)(\xi_2^4 + \eta_2^4) \\ & + 2(A_2\bar{A}_2 - A_3\bar{A}_3 - A_2\bar{A}_3 + \bar{A}_2 A_3)(\xi_1^4 + \eta_1^4)\, \xi_2^2\eta_2^2 \\ & + 2(A_2\bar{A}_2 - A_3\bar{A}_3 + A_2\bar{A}_3 - \bar{A}_2 A_3)(\xi_1^4 + \eta_1^4)\, \xi_1^2\eta_1^2 \\ & + 4(2A_2\bar{A}_2 + 2A_3\bar{A}_3 - A_1\bar{A}_1)\, \xi_1^2\eta_1^2\, \xi_2^2\eta_2^2 \end{aligned} \tag{6.10}$$

$$= 8[\mathrm{A}_1(\xi_1^4 + \eta_1^4) + 2(\mathrm{A}_3 - \mathrm{A}_2)\,\xi_1^2\eta_1^2\,]\,[\,\bar{\mathrm{A}}_1\,(\xi_2^4 + \eta_2^4) + 2\,(\bar{\mathrm{A}}_3 - \bar{\mathrm{A}}_2)\,\xi_2^2\eta_2^2\,]\,.$$

On the absolute, the condition T(X) = 0 is therefore equivalent to the two systems:

$$r(\xi_1^4 + \eta_1^4) + 2s\xi_1^2\eta_1^2 = 0$$
$$\bar{r}(\xi_1^4 + \eta_1^4) + 2\bar{s}\,\bar{\xi}_1^2\bar{\eta}_1^2 = 0\,,$$

which define the generatrices g_i and $g_{\bar{i}}$ precisely.

An analogous proof is possible in the other cases. The givens of two quadruples of generatrices g_i and $g_{\bar{i}}$ projectively determine the surface T(X) thanks to the conditions of "apolarity" (3.14). This will be the subject of the following paragraph.

7. *Expression for the tensor* $\mathrm{T}_{\alpha\beta\gamma\mu}$ *with the aid of the characteristic isotropic vectors* [**14**]. If $l_a^{(i)}$ are the characteristic isotropic vectors then one has, thanks to (3.8) and (5.10):

$$\begin{aligned}\prod_{i=1}^{4}(l_a^{(i)}\mathrm{X}^a) &= \\ &= [(\mathrm{X}^0)^2 - (\mathrm{X}^1)^2 - (\mathrm{X}^2)^2 - (\mathrm{X}^3)^2]\,[\,a_0^4(\mathrm{X}^0)^2 - a_1^4(\mathrm{X}^1)^2 - a_2^4(\mathrm{X}^2)^2 - a_3^4(\mathrm{X}^3)^2\,] \\ &+ (a_2^2 + a_3^2)[(\mathrm{X}^0)^2(\mathrm{X}^1)^2 - (\mathrm{X}^2)^2(\mathrm{X}^3)^2] + (a_3^2 + a_1^2)[(\mathrm{X}^0)^2(\mathrm{X}^2)^2 - (\mathrm{X}^3)^2(\mathrm{X}^1)^2] \\ &+ (a_1^2 + a_2^2)[(\mathrm{X}^0)^2(\mathrm{X}^3)^2 - (\mathrm{X}^1)^2(\mathrm{X}^2)^2] - 8a_0a_1a_2a_3\,\mathrm{X}^0\,\mathrm{X}^1\,\mathrm{X}^2\,\mathrm{X}^3\,,\end{aligned} \tag{7.1}$$

and, modulo the absolute, one further has:

$$\begin{aligned}\prod_{i=1}^{4}(l_a^{(i)}\mathrm{X}^a) &= 16\mathrm{A}_1\bar{\mathrm{A}}_1[(\mathrm{X}^0)^2(\mathrm{X}^1)^2 - (\mathrm{X}^2)^2(\mathrm{X}^3)^2] + 16\mathrm{A}_2\bar{\mathrm{A}}_2\,[(\mathrm{X}^0)^2(\mathrm{X}^2)^2 - (\mathrm{X}^1)^2(\mathrm{X}^3)^2] \\ &+ 16\mathrm{A}_3\bar{\mathrm{A}}_3\,[(\mathrm{X}^0)^2(\mathrm{X}^3)^2 - (\mathrm{X}^1)^2(\mathrm{X}^2)^2] - 8\times 16\mathrm{S}\,\mathrm{X}^0\,\mathrm{X}^1\,\mathrm{X}^2\,\mathrm{X}^3 = \mathrm{T}(\mathrm{X})\,.\end{aligned} \tag{7.2}$$

The same is true in all the other cases.

Having said this, we set:

$$\mathrm{L}_{\alpha\beta\lambda\mu} = \tfrac{1}{24}\prod_{1,2,3,4} l_\alpha^{(1)} l_\beta^{(2)} l_\lambda^{(3)} l_\mu^{(4)}\,, \tag{7.3}$$

in which the sum is taken over all 24 permutation of the sequence 1, 2, 3, 4. One has:

$$\mathrm{L(X)} = \mathrm{L}_{\alpha\beta\lambda\mu}\;\mathrm{X}^\alpha\,\mathrm{X}^\beta\,\mathrm{X}^\lambda\,\mathrm{X}^\mu = \prod_{i=1}^{4}(l_a^{(i)}\mathrm{X}^a)\,. \tag{7.4}$$

Furthermore, let:

$$\mathrm{L}_{\lambda\mu} = \mathrm{L}_{\alpha\lambda\mu}^{\alpha}\,, \qquad \mathrm{L} = \mathrm{L}_\lambda^\lambda\,. \tag{7.5}$$

Thanks to (7.2):

$$L_{\alpha\beta\lambda\mu} = T_{\alpha\beta\lambda\mu} \text{ modulo } g_{\alpha\beta} . \tag{7.6}$$

From this, it results that one may write, if L ≠ 0:

$$T_{\alpha\beta\lambda\mu} = \frac{c}{8}(g_{\alpha\beta}\, b_{\lambda\mu} + g_{\lambda\mu}\, b_{\alpha\beta} + g_{\alpha\lambda}\, b_{\beta\mu} + g_{\beta\mu}\, b_{\alpha\lambda} + g_{\alpha\mu}\, b_{\beta\lambda} + g_{\beta\lambda}\, b_{\alpha\mu} - \frac{96}{L}\, L_{\alpha\beta\lambda\mu}), \tag{7.7}$$

in which $b_{\beta\mu}$ is another undetermined tensor and c is a scalar that is also indeterminate.

The conditions (3.14):

$$g^{\alpha\beta} T_{\alpha\beta\lambda\mu} = 0$$

give:

$$b_{\lambda\mu} = \frac{12}{L} L_{\lambda\mu} - g_{\lambda\mu} . \tag{7.8}$$

Therefore, by starting with the scalar c, the tensor $T_{\alpha\beta\lambda\mu}$ is completely determined by the given of the characteristic isotropic vectors $l_a^{(i)}$.

One will observe that the tensors that figure in (7.7) are independent of the choice of the isotropic vectors $l_a^{(i)}$, up to a choice of factor.

If:

$$l_{ij} = g_{\alpha\beta}\, l_{(i)}^{\alpha} l_{(j)}^{\beta} \tag{7.9}$$

then one verifies, by starting with the definitions (7.3) and (7.5), that:

$$L = \frac{1}{3}(l_{12}l_{34} + l_{13}l_{24} + l_{14}l_{23}) , \tag{7.10}$$

so one has:

$$L \neq 0 \text{ in the cases I}a, \text{ I}b, \text{ II}a . \tag{7.11}$$

If L = 0 then one will have:

$$T_{\alpha\beta\lambda\mu} = \chi [-\frac{1}{8}(g_{\alpha\beta} L_{\lambda\mu} + g_{\lambda\mu} L_{\alpha\beta} + g_{\alpha\lambda} L_{\beta\mu} + g_{\beta\mu} L_{\alpha\lambda} + g_{\alpha\mu} L_{\beta\lambda} + g_{\beta\lambda} L_{\alpha\mu}) + L_{\alpha\beta\lambda\mu}] , \tag{7.12}$$

in which χ is a scalar that depends upon the choice of the isotropic vector $l_{(i)}^{a}$ for the tensor $T_{\alpha\beta\lambda\mu}$.

In the case IIb, one has, moreover:

$$L_{\lambda\mu} = 0 , \tag{7.13}$$

and:

$$T_{\alpha\beta\lambda\mu} = \chi\, l_\alpha l_\beta l_\lambda l_\mu . \tag{7.14}$$

In conclusion, generalizing the situation that was described in paragraph I, Fig. 1, the surface T(X) intersects the absolute in two quadruples of complex conjugate generatrices that pass through the points $l_{(i)}$, and the surface T(X) is projectively determined by this situation of one adds the conditions of "apolarity" (3.14), Fig. 3.

The various cases that were prevdicted by the Petrov classification are the limiting cases of the general case.

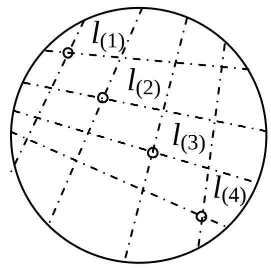

Fig. 3.

8. *Involutive character of the tensor* $\mathrm{T}_{\alpha\beta\lambda\mu}$. We propose to establish that:

$$\mathrm{T}_{\alpha\beta\lambda\mu}\mathrm{T}_{\gamma}^{\ \beta\lambda\mu} = \tfrac{1}{4}\mathrm{K}g_{\alpha\gamma}, \tag{8.1}$$

in which K is a scalar.

We have not found a direct tensorial proof of that property from starting with the tensorial definition of $\mathrm{T}_{\alpha\beta\lambda\mu}$.

Meanwhile, if L ≠ 0 then a very long calculation that starts with the expression (7.7) gives us that:

$$\mathrm{T}_{\alpha\beta\lambda\mu}\mathrm{T}_{\gamma}^{\ \beta\lambda\mu} = \tfrac{9}{4}c^2(1-3k)\mathrm{g}_{\alpha\gamma}, \tag{8.2}$$

in which:

$$k = 1 - 2\,\frac{(l_{23})^2(l_{14})^2 + (l_{13})^2(l_{24})^2 + (l_{12})^2(l_{34})^2}{(l_{23}l_{14} + l_{13}l_{24} + l_{12}l_{34})^2}. \tag{8.3}$$

If L = 0 then one painlessly verifies, by starting with (7.12) and (7.14), that:

$$\mathrm{T}_{\alpha\beta\lambda\mu}\mathrm{T}_{\gamma}^{\ \beta\lambda\mu} = 0\,. \tag{8.4}$$

In the case I_a one may, moreover, easily verify, on starting with expressions (6.2) to (6.7) for the components of $\mathrm{T}_{\alpha\beta\lambda\mu}$ that one indeed has (8.1).

Furthermore, one has:

$$\mathrm{T}_{0\alpha\beta\gamma}\mathrm{T}_0^{\ \alpha\beta\gamma} = (\mathrm{T}_{0000})^2 + 3(\mathrm{T}_{0011})^2 + 3(\mathrm{T}_{0022})^2 + 3(\mathrm{T}_{0033})^2 - 6(\mathrm{T}_{0123})^2\,. \tag{8.5}$$

Thanks to (5.8) and (5.10), one verifies that one has:

$$l_{24} = l_{13} = 32\,A_1\bar{A}_1, \qquad l_{12} = l_{34} = 32\,A_2\bar{A}_2, \quad l_{14} = l_{23} = 32\,A_3\bar{A}_3, \tag{8.6}$$

which permits us to put expression (8.3) for k into the form:

$$k = \frac{16S^2}{(A_1\bar{A}_1 + A_2\bar{A}_2 + A_3\bar{A}_3)^2}\,. \tag{8.7}$$

One thus obtains:

$$T_{0\alpha\beta\gamma}T_0^{\ \alpha\beta\gamma} = \frac{16}{9}(A_1\bar{A}_1 + A_2\bar{A}_2 + A_3\bar{A}_3)\,(1 - 3k)\,. \tag{8.8}$$

If we compare (8.2) and (8.8) then:

$$c = \pm\frac{8}{9}(A_1\bar{A}_1 + A_2\bar{A}_2 + A_3\bar{A}_3) = \pm\frac{8}{3}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \beta_1^2 + \beta_2^2 + \beta_3^2)\,. \tag{8.9}$$

If one refers back to the definitions, to (7.7), and, for example, to the expression (6.3) for T_{0000}, then one sees that there is good reason to adopt the – sign in (8.9). One thus has:

$$24c = -\,L\,. \tag{8.10}$$

If we refer to the definition (3.9) then we have:

$$\begin{aligned} T_{0000} &= 2(R_{0\rho0\sigma}R_0^{\ \rho}{}_0^{\ \sigma} + \overset{*}{R}_{0\rho0\sigma}\overset{*}{R}_0^{\ \rho}{}_0^{\ \sigma}) \\ &= 2(R_{0a0b}\,R_0^{\ b}{}_0^{\ a} + \overset{*}{R}_{0a0b}\overset{*}{R}_0^{\ a}{}_0^{\ b}), \end{aligned} \tag{8.11}$$

due to the fact that:

$$R_{0a0b} = -\,a_{ab}\,, \qquad \text{and} \qquad \overset{*}{R}_{0a0b} = -\,\beta_{ab}\,. \tag{8.12}$$

One thus obtains:

$$T_{0000} = 2(\Sigma(\alpha_{ij})^2 + \Sigma(\beta_{ij})^2) = 2(\Sigma\,A_{ij}\bar{A}_{ij}). \tag{8.13}$$

One will observe that T_{0000} is always positive and it is null only if $R_{\alpha\beta\lambda\mu}$ is identically null.

The expression (8.7) for k permits us to geometrically verify, for example, that:

$$0 \le k \le \frac{1}{3}\,. \tag{8.14}$$

$k \neq 0$ if the vectors $l_{(i)}^a$ are four in number, distinct, and linearly independent.

k = 1/3 corresponds to the case in which the triangle $A_{11}A_{22}A_{33}$ is equilateral, i.e., in which the vectors $l_{(i)}^a$ form an equi-anharmonic group (case Ia-III).

Finally:

$$K = 64\sum_a\sum_b\ [(\alpha_{ab}\alpha_{ab} - \beta_{ab}\beta_{ab})^2 + 4(\alpha_{ab}\beta_{ab})^2] \tag{8.15}$$

and:

$$K = (R_{\alpha\beta\lambda\mu} R^{\alpha\beta\lambda\mu})^2 + (\overset{*}{R}_{\alpha\beta\lambda\mu} \overset{*}{R}{}^{\alpha\beta\lambda\mu})^2. \tag{8.16}$$

9. *The tensor* $T_{\alpha\beta\lambda\mu}$ *and the Petrov classification.* In this paragraph, we propose to assemble all of the formulas and geometric interpretations that relate to the various cases of the Petrov classification.

I*a*. One has:

$$T_{\alpha\beta\lambda\mu}\, l^{\beta}_{(i)} l^{\lambda}_{(i)} l^{\mu}_{(i)} = -\frac{3}{4} c f_i l^{(i)}_{\alpha}, \tag{9.1}$$

in which:

$$f_i = \frac{l_{ij} l_{ik} l_{il}}{L}, \tag{9.2}$$

in which i, j, k, l forms the sequence 1, 2, 3, 4.

The surface T(X) passes through the points $l^{(i)}_{\alpha}$, and its tangent planes are isotropic at these points.

I*b*. Let $l^{\alpha}_{(1)}$ and $l^{\alpha}_{(2)}$ be two doubly isotropic vectors. One has:

$$T_{\alpha\beta\lambda\mu}\, l^{\beta}_{(i)} l^{\lambda}_{(i)} l^{\mu}_{(i)} = 0, \qquad i = 1, 2 \tag{9.3}$$

$$T_{\alpha\beta\lambda\mu}\, l^{\lambda}_{(i)} l^{\mu}_{(i)} = -\frac{c}{2} l^{(i)}_{\alpha} l^{(i)}_{\beta}\ . \tag{9.4}$$

The points $l^{(1)}$ and $l^{(2)}$ of T(X) are biplanar singular points: the surface of tangent directions at these points is composed of isotropic planes that are tangent to the absolute at these points, counted twice. If one sets:

$$t_{\alpha\beta} = g_{\alpha\beta} - 2\frac{l^{(1)}_{\alpha} l^{(2)}_{\beta} + l^{(2)}_{\alpha} l^{(1)}_{\beta}}{l_{12}} \tag{9.5}$$

then one obtains:

$$T_{\alpha\beta\lambda\mu} = \frac{c}{8}(g_{\alpha\beta}\, g_{\lambda\mu} + g_{\alpha\lambda}\, g_{\beta\mu} + g_{\alpha\mu}\, g_{\beta\lambda} - 3t_{\alpha\beta}\, t_{\lambda\mu} - 3t_{\alpha\lambda}\, t_{\beta\mu} - 3t_{\alpha\mu}\, t_{\lambda\beta}) \tag{9.6}$$

and:

$$T(X) = \frac{3}{8} c((g(X)^2 - 3(t(X))^2)\ . \tag{9.7}$$

T(X) is a combination of the absolute, counted twice, and the quadric t(X), counted twice.

II*a*. Let $l_\alpha^{(1)}$ be a doubly isotropic vector and let $l_\alpha^{(2)}$ and $l_\alpha^{(3)}$ be simply isotropic vectors.

One has:

$$\mathrm{T}_{\alpha\beta\lambda\mu}\, l_{(i)}^\beta l_{(i)}^\lambda l_{(i)}^\mu = -\frac{9}{8} c \frac{l_{1i} l_{ii'}}{l_{1i'}} l_\alpha^{(i)}, \qquad \begin{cases} i = 2 \text{ or } 3 \\ i' = 3 \text{ or } 2 \end{cases} \tag{9.10}$$

and:

$$\mathrm{T}_{\alpha\beta\lambda\mu}\, l_{(i)}^\beta l_{(i)}^\lambda l_{(i)}^\mu = 0 \tag{9.11}$$

$$\mathrm{T}_{\alpha\beta\lambda\mu}\, l_{(1)}^\lambda l_{(1)}^\mu = -\frac{c}{2} l_\alpha^{(1)} l_\beta^{(1)} . \tag{9.12}$$

The tangent planes are isotropic at the points $l^{(2)}$ and $l^{(3)}$, and the point $l^{(1)}$ is biplanar singular.

II*b*. Let l_α be the quadruply isotropic vector:

$$\mathrm{T}_{\alpha\beta\lambda\mu} = c\, l_\alpha\, l_\beta\, l_\lambda\, l_\mu \tag{9.13}$$

$$\mathrm{T}_{\alpha\beta\lambda\mu}\, l^\mu = 0 . \tag{9.14}$$

The surface T(X) is composed of an isotropic plane that is tangent to the absolute, counted four times.

III. Let $l_\alpha^{(1)}$ be a triply isotropic vector, and let $l_\alpha^{(2)}$ be a simply isotropic vector. One has:

$$\mathrm{T}_{\alpha\beta\lambda\mu}\, l_{(1)}^\lambda l_{(1)}^\mu = 0 \tag{9.15}$$

$$\mathrm{T}_{\alpha\beta\lambda\mu}\, l_{(1)}^\mu = \frac{\chi}{16} l_{12} l_\alpha^{(1)} l_\beta^{(1)} l_\lambda^{(1)} \tag{9.16}$$

$$\mathrm{T}_{\alpha\beta\lambda\mu}\, l_{(2)}^\beta l_{(2)}^\lambda l_{(2)}^\mu = \frac{\chi}{16} (l_{12})^3 l_\alpha^{(2)} . \tag{9.17}$$

One has:

$$\mathrm{L}_{\lambda\mu} = \frac{1}{2} l_{12} l_\lambda^{(i)} l_\mu^{(i)} \tag{9.18}$$

$$\mathrm{T(X)} = \chi (l^1(\mathrm{X}))^2\, [l^1(\mathrm{X})\, l^2(\mathrm{X}) - \frac{3}{8} g(\mathrm{X})] . \tag{9.19}$$

The surface T(X) degenerates into a product of two quadrics, one of which is composed of a plane tangent to the absolute at the point $l^{(1)}$, counted twice.

10. *Trajectories of multiple isotropic vectors.* First of all, consider case I*b*. Starting with the relations (9.4), and taking (3.15) into account, one has:

$$\mathrm{T}^\alpha{}_{\beta\lambda\mu} l_{(1)}^\lambda \nabla_\alpha l_{(1)}^\mu = -\frac{1}{4} l_\beta^{(1)} \nabla_\alpha c\, l_{(1)}^\alpha - \frac{c}{4} l_{(1)}^\alpha \nabla_\alpha l_\beta^{(1)} . \tag{10.1}$$

Thanks to (9.6):

$$\mathrm{T}^{\alpha}{}_{\beta\lambda\mu} l^{\lambda}_{(1)} = \frac{c}{2}[\delta^{\alpha}_{\beta} l^{(1)}_{\beta} + g_{\beta\mu} l^{\alpha}_{(1)} + \delta^{\alpha}_{\mu} l^{(1)}_{\beta} - \frac{3}{l_{12}}(l^{\alpha}_{(2)} l^{(1)}_{\beta} l^{(1)}_{\mu} + l^{(2)}_{\beta} l^{\alpha}_{(1)} l^{(1)}_{\mu} + l^{(2)}_{\mu} l^{\alpha}_{(1)} l^{(1)}_{\beta})]. \quad (10.2)$$

Upon substituting this expression into (10.1), and taking into account the fact that $l^{(1)}_{\alpha}$ and $l^{(2)}_{\alpha}$ are isotropic, one finds that:

$$l^{\alpha}_{(1)} \nabla_{\alpha} l^{(1)}_{\mu} = l^{(1)}_{\beta}[-\frac{1}{c}\nabla_{\alpha} c l^{\alpha}_{(1)} + \frac{2}{l_{12}} l^{\alpha} \nabla_{\alpha} l^{\mu}_{(1)} l^{(2)}_{\mu}], \quad (10.3)$$

which proves that the trajectories of the vector $l^{(1)}_{\alpha}$ are isotropic geodesics; an analogous calculation shows that the same is true for $l^{(2)}_{\alpha}$. L. Bel [**15**] has shown that this proposition is also valid in the cases II-*a*, II-*b*, and III. One thus has:

THEOREM. – The trajectories of multiply isotropic vectors are isotropic geodesics. [**16**], [**27**].

11. *The tensors* $\mathrm{E}_{\alpha\beta}$ *and* $\mathrm{H}_{\alpha\beta}$ [**17**]. *Characterization of the characteristic isotropic 2-planes.* Set:

$$\mathrm{E}_{\alpha\beta} = \mathrm{R}_{\alpha\lambda\beta\mu} u^{\lambda} u^{\mu} \quad (11.1)$$

$$\mathrm{H}_{\alpha\beta} = -\overset{*}{\mathrm{R}}_{\alpha\lambda\beta\mu} u^{\lambda} u^{\mu},$$

in which u^{λ} is a vector of length 1.

It results from (3.9) that one has:

$$\mathrm{T}(u) = \frac{1}{2}(\mathrm{E}_{\alpha\beta}\mathrm{E}^{\alpha\beta} + \mathrm{H}_{\alpha\beta}\mathrm{H}^{\alpha\beta}); \quad (11.2)$$

T(u) defines the *super-energy* for the observer u^{λ}.

It is a positive-definite function. Indeed, we have observed in (8.13) that T_{0000} is always positive, so T(u) is positive in the time direction. One sees, moreover, thanks to (6.2), that the same is again true in the spatial directions, as well. Meanwhile, T(X) is annulled in the real isotropic directions $l^{(i)}_{\alpha}$.

Geometrically, $\mathrm{E}_{\alpha\beta}$ and $\mathrm{H}_{\alpha\beta}$ define two cones with vertex (u), which are the quadratic cones that are formed from lines that belong to the quadratic complexes of lines that are associated with $\mathrm{R}_{\alpha\beta\lambda\mu}$ and $\overset{*}{\mathrm{R}}_{\alpha\lambda\beta\mu}$, respectively, and pass through (u).

If:

$$\mathrm{R}^{+} = \mathrm{R} + i\overset{*}{\mathrm{R}} \quad (11.3)$$

$$\mathrm{R}^{-} = \mathrm{R} - i\overset{*}{\mathrm{R}} \quad (11.4)$$

and:

$$\mathrm{K}^{+}_{\alpha\beta} = \mathrm{E}_{\alpha\beta} + i\mathrm{H}_{\alpha\beta}, \qquad \mathrm{K}^{-}_{\alpha\beta} = \mathrm{E}_{\alpha\beta} - i\mathrm{H}_{\alpha\beta}, \tag{11.5}$$

then one has:

$$\mathrm{T}(u) = \frac{1}{2}\mathrm{K}^{+}_{\alpha\beta}\ \mathrm{K}^{-\alpha\beta}. \tag{11.6}$$

The cones K^+ and K^- may be constructed in the following manner: the polar plane of the point (u) intersects the absolute in a conic g, and the four generatrices g_i have points P_i in common with that quadric. The cone of vertex u whose four generatrices are $u\mathrm{P}_i$ and which is apolar to the absolute is one of the cones K, and the other is obtained by starting with the generatrices $g_{\bar{i}}$.

On the absolute, the cones $\mathrm{K}^{+}_{\alpha\beta}$ and $\mathrm{K}^{-}_{\alpha\beta}$ degenerate into a pair of planes.

If $l_{i\bar{j}}$ denotes a point of the absolute that is the intersection of g_i and $g_{\bar{i}}$ then at the point the cones $\mathrm{E}_{\alpha\beta}$ and $\mathrm{H}_{\alpha\beta}$ degenerate into the isotropic plane that is tangent to $l_{i\bar{j}}$ and a second plane $p_{i\bar{j}}$ that is conjugate to the first one with respect to the absolute.

For example, one has:

$$\mathrm{E}_{\alpha\beta}\,(l_{ij}) = l^{i\bar{j}}_{\alpha} l^{i\bar{j}}_{\beta} + l^{i\bar{j}}_{\beta} l^{i\bar{j}}_{\alpha}\,. \tag{11.7}$$

This property permits us to give a characterization of the generatrices g_i and $g_{\bar{i}}$, as well as the associated completely isotropic 2-planes, and, in turn, the characteristic vectors $l^{(i)}$.

If $l_{i\bar{j}}$ and $l_{i\bar{k}}$ are two points of the same generatrix g_i , and if:

$$p^{\alpha\beta}_{i} = l^{\alpha}_{i\bar{j}} l^{\beta}_{i\bar{k}} - l^{\beta}_{i\bar{j}} l^{\alpha}_{i\bar{k}} \tag{11.8}$$

$$\Omega^{\alpha}{}_{\beta} = \mathrm{R}_{\alpha\beta\lambda\mu}\ l^{\lambda}_{i\bar{j}} l^{\mu}_{i\bar{k}} \tag{11.9}$$

$$\overset{*}{\Omega}{}^{\alpha}{}_{\beta} = \overset{*}{\mathrm{R}}{}^{\alpha}{}_{\beta\lambda\mu} l^{\lambda}_{i\bar{j}} l^{\mu}_{i\bar{k}}$$

are the components of the infinitesimal rotations of the curvature and the mixed curvature in the direction of the plane $p^{\alpha\beta}_{i}$ then one has:

THEOREM. – If n^{λ} is a vector that is situated in the 2-plane $p^{\alpha\beta}_{i}$ then the vectors:

$$n'^{\alpha} = \Omega^{\alpha}{}_{\beta}\, n^{\beta}\,, \qquad\qquad n''^{\alpha} = \overset{*}{\Omega}{}^{\alpha}{}_{\beta}\, n^{\beta}\,, \tag{11.10}$$

are also situated in the 2-planes $p^{\alpha\beta}_{i}$.

The completely isotropic characteristic 2-planes are such that the infinitesimal rotations of the curvature and mixed curvature preserve these 2-planes.

Indeed: if n^{λ} is contained in the 2-plane $p^{\alpha\beta}_{i}$ then one has:

$$n^{\beta} = a l^{\beta}_{\bar{i}\bar{j}} + b l^{\beta}_{\bar{i}\bar{k}} . \tag{11.11}$$

Thanks to (11.7), this yields:

$$\begin{aligned} n'^{\alpha} &= \Omega^{\alpha}{}_{\beta}\, n^{\beta} = \mathrm{R}^{\alpha}{}_{\beta\lambda\mu}\,(a l^{\beta}_{\bar{i}\bar{j}} + b l^{\beta}_{\bar{i}\bar{k}})\, l^{\lambda}_{\bar{i}\bar{j}} l^{\mu}_{\bar{i}\bar{k}} \\ &= -\, a(l^{\alpha}_{\bar{i}\bar{j}}\, p^{\bar{i}\bar{k}}_{\mu} + l^{\bar{i}\bar{j}}_{\mu}\, p^{\alpha}_{\bar{i}\bar{k}})\, l^{\mu}_{\bar{i}\bar{k}} + b(l^{\alpha}_{\bar{i}\bar{k}}\, p^{\bar{i}\bar{k}}_{\lambda} + q^{\alpha}_{\bar{i}\bar{k}} l^{\bar{i}\bar{k}}_{\lambda} l^{\lambda}_{\bar{i}\bar{j}}) \\ &= -\, a(l^{\alpha}_{\bar{i}\bar{j}}\, (l^{\mu}_{\bar{i}\bar{k}}\, p^{\bar{i}\bar{j}}_{\mu}) + b\, l^{\alpha}_{\bar{i}\bar{k}} (l^{\lambda}_{\bar{i}\bar{j}}\, p^{\bar{i}\bar{k}}_{\lambda})) . \end{aligned} \tag{11.12}$$

Thus, n'^{α} also belongs to the 2-plane $p_i^{\alpha\beta}$. An analogous calculation is valid upon starting with $\overset{*}{\mathrm{R}}{}_{\alpha\lambda\beta\mu}$.

THEOREM. – The characteristic vectors $l_{\alpha}^{(i)}$ are real isotropic vectors such that:

$$\begin{aligned} \mathrm{R}_{\alpha\beta\lambda\mu}\, l^{\beta}_{(i)} l^{\mu}_{(i)} n^{\lambda} &= k\, l^{(i)}_{\alpha} \\ \overset{*}{\mathrm{R}}{}_{\alpha\lambda\beta\mu}\, l^{\beta}_{(i)} l^{\mu}_{(i)} n^{\lambda} &= \overset{*}{k}\, l^{(i)}_{\alpha} \end{aligned} \tag{11.13}$$

for any isotropic vector n^{λ} such that $n^{\lambda} l^{(i)}_{\lambda} = 0$.

Since this statement is a direct consequence of the preceding theorem, the bivector (l^{α}, n^{β}) actually determines a generatrix of the absolute.

In conclusion, we have recovered the equivalent for $\mathrm{T}_{\alpha\beta\lambda\mu}$ of all of the properties of the tensor $\mathrm{T}_{\alpha\beta}$ that were stated in paragraph 1.

12. *Irreducible components of the Riemann tensor* [**18**]. Suppose, for the moment, that the Riemann tensor is not subject to any algebraic condition. In this case, $\mathrm{R}_{\alpha\beta\lambda\mu}$ admits a decomposition into irreducible components:

$$\mathrm{R}_{\alpha\beta\lambda\mu} = \mathrm{C}_{\alpha\beta\lambda\mu} + \mathrm{E}_{\alpha\beta\lambda\mu} + \mathrm{G}_{\alpha\beta\lambda\mu} . \tag{12.1}$$

$\mathrm{C}_{\alpha\beta\lambda\mu}$ is the conformal curvature tensor of H. Weyl:

$$\begin{aligned} \mathrm{C}_{\alpha\beta\lambda\mu} = \mathrm{R}_{\alpha\beta\lambda\mu} &+ \tfrac{1}{2}(g_{\alpha\lambda}\, \mathrm{R}_{\beta\mu} + g_{\beta\mu}\, \mathrm{R}_{\alpha\lambda} - g_{\alpha\mu}\, \mathrm{R}_{\beta\lambda} - g_{\beta\lambda}\, \mathrm{R}_{\alpha\mu}) \\ &+ \tfrac{1}{6}\mathrm{R}(g_{\alpha\mu}\, g_{\beta\lambda} - g_{\beta\lambda}\, g_{\alpha\mu}) , \end{aligned} \tag{12.2}$$

in which $\mathrm{R} = g^{\alpha\beta}\mathrm{R}_{\alpha\beta}$ is the scalar curvature.

$\mathrm{C}_{\alpha\beta\lambda\mu} = 0$ is the necessary and sufficient condition for the Riemannian manifold to be conformally Euclidian.

$\mathrm{E}_{\alpha\beta\lambda\mu}$ is a tensor that was introduced by Einstein. If:

$$\mathrm{S}_{\alpha\beta} = \mathrm{R}_{\alpha\beta} - \tfrac{1}{4}\mathrm{R}\, g_{\alpha\beta} , \tag{12.3}$$

then one has:

$$E_{\alpha\beta\lambda\mu} = -\tfrac{1}{2}(g_{\alpha\lambda} S_{\beta\mu} + g_{\beta\mu} S_{\alpha\lambda} - g_{\alpha\mu} S_{\beta\lambda} - g_{\beta\lambda} S_{\alpha\mu}) . \tag{12.4}$$

The conditions $E_{\alpha\beta\lambda\mu} = 0$ and $S_{\alpha\beta} = 0$ are equivalent. Finally:

$$G_{\alpha\beta\lambda\mu} = -\tfrac{1}{12} R(g_{\alpha\lambda} g_{\beta\mu} - g_{\alpha\mu} g_{\beta\lambda}) . \tag{12.5}$$

In the case of the gravitational vacuum, one has:

$$R_{\lambda\mu} = 0 , \quad E_{\alpha\beta\lambda\mu} = 0 , \quad G_{\alpha\beta\lambda\mu} = 0 , \quad R_{\alpha\beta\lambda\mu} = C_{\alpha\beta\lambda\mu} . \tag{12.6}$$

All of the algebraic considerations that were developed in the preceding paragraphs are applicable if one replaces the tensor $R_{\alpha\beta\lambda\mu}$ with the tensor $C_{\alpha\beta\lambda\mu}$.

Therefore, in any Riemannian manifold of hyperbolic normal signature there exist four real isotropic characteristic vectors $l_\alpha^{(i)}$, which are defined by:

$$\begin{aligned} C_{\alpha\beta\lambda\mu}\, l_{(i)}^{\beta} l_{(i)}^{\mu} n^{\lambda} &= k\, l_\alpha^{(i)} \\ \overset{*}{C}_{\alpha\beta\lambda\mu}\, l_{(i)}^{\beta} l_{(i)}^{\mu} n^{\lambda} &= \overset{*}{k}\, l_\alpha^{(i)} , \end{aligned} \tag{12.7}$$

for any vector n^λ such that $n^\lambda l_\lambda^{(i)} = 0$.

One may, with L. Bel, introduce the tensor:

$$\begin{aligned} T_{\alpha\beta\lambda\mu} = {} & R_{\alpha\rho\lambda\sigma} R_{\beta\ \mu}^{\ \rho\ \sigma} + R_{\alpha\rho\mu\sigma} R_{\beta\ \mu}^{\ \rho\ \sigma} \\ & + (^*R^*)_{\alpha\rho\lambda\sigma} (^*R^*)_{\beta\ \mu}^{\ \rho\ \sigma} + (^*R^*)_{\alpha\rho\mu\sigma} (^*R^*)_{\beta\ \mu}^{\ \rho\ \sigma} - 2A g_{\alpha\beta}\, g_{\lambda\mu} , \end{aligned} \tag{12.8}$$

a tensor that no longer has complete symmetry, as in the particular case of the gravitational vacuum, but enjoys the symmetry properties:

$$T_{\alpha\beta\lambda\mu} = T_{\alpha\beta\mu\lambda} = T_{\beta\alpha\lambda\mu} = T_{\lambda\mu\alpha\beta} . \tag{12.9}$$

We therefore set:

$$V_{\alpha\beta\lambda\mu} = \tfrac{1}{3}(T_{\alpha\beta\lambda\mu} + T_{\alpha\lambda\beta\mu} + T_{\alpha\mu\beta\lambda}) , \tag{12.10}$$

which we call the *super-energy tensor*.

If one takes (12.1) into account, along with the definitions and properties (3.5) and (3.6), then one finds [**19**]:

$$V_{\alpha\beta\lambda\mu} = V(C)_{\alpha\beta\lambda\mu} + V(E)_{\alpha\beta\lambda\mu} + V(G)_{\alpha\beta\lambda\mu} , \tag{12.11}$$

in which each term of the right-hand side is calculated by starting with formulas (12.8) and (12.10) and replacing R with C, E, and G, respectively.

By virtue of the properties that were established in paragraph 3, which also apply to the tensor C, $T(C)_{\alpha\beta\lambda\mu}$ possesses complete symmetry, and one has:

$$V(C)_{\alpha\beta\lambda\mu} = T(C)_{\alpha\beta\lambda\mu} . \tag{12.12}$$

The same is not also true for V(E) and V(G).

One has:

$$V(E)_{\alpha\beta\lambda\mu} = \tfrac{2}{3}(S_{\alpha\beta}\,S_{\lambda\mu} + S_{\alpha\lambda}\,S_{\beta\mu} + S_{\alpha\mu}\,S_{\beta\lambda}) - \tfrac{1}{2}(g_{\alpha\beta}\,W_{\lambda\mu} + g_{\alpha\lambda}\,W_{\beta\mu} + g_{\alpha\mu}\,W_{\beta\lambda} + g_{\beta\mu}\,W_{\alpha\lambda} + g_{\alpha\mu}\,W_{\beta\lambda} + g_{\beta\lambda}\,W_{\alpha\mu}), \quad (12.13)$$

in which:

$$W^{\lambda\mu} = S_{\lambda}^{\sigma}\,S_{\mu\sigma} + \tfrac{1}{4}g_{\lambda\mu}\,S_{\rho\sigma}S^{\rho\sigma}. \quad (12.14)$$

One must remark that $W_{\lambda\mu} = 0$ in the case where $R_{\alpha\beta}$ represents the energy-momentum tensor of an electromagnetic field.

Finally:

$$V(G)_{\alpha\beta\lambda\mu} = \tfrac{1}{72}R^2(g_{\alpha\beta}\,g_{\lambda\mu} + g_{\alpha\lambda}\,g_{\beta\mu} + g_{\alpha\mu}\,g_{\beta\lambda}). \quad (12.15)$$

13. *Weyl tensor and the Petrov classification.* From the foregoing, the tensor $T(C)_{\alpha\beta\lambda\mu}$ that is associated with the Weyl tensor plays a role in the conformal transformations of the Riemannian metric. This results, in particular, from the properties that were described in paragraphs 2 through 11.

In particular, we emphasize the following proposition: it results from the remarkable fact that was expressed by (11.2) that:

$$T(C)_{\alpha\beta\lambda\mu} = 0 \quad (13.1)$$

is a condition for a space to be conformally Euclidian. In this paragraph, we propose to carry out an analysis for Weyl tensor that is similar to the one that was carried out in paragraph 2 for an electromagnetic field. We shall nevertheless limit ourselves to the singular case in the Petrov classification.

I*b*. In this case, if one refers to (4.7) then one may write:

$$C_{\alpha\beta}\,Z^{\alpha}\,Z^{\beta} = -3(\alpha - i\beta)(Z^1)^2 + (\alpha - i\beta)((Z^1)^2 + (Z^2)^2 + (Z^3)^2). \quad (13.2)$$

If $l_{\alpha}^{(1)}$ and $l_{\alpha}^{(2)}$ are doubly isotropic vectors that are associated with $C_{\alpha\beta\lambda\mu}$, and:

$$b_{\alpha\beta} = \frac{l_{\alpha}^{(1)}l_{\beta}^{(2)} - l_{\alpha}^{(2)}l_{\beta}^{(1)}}{l_{12}} \quad (13.3)$$

and:

$$b_{\alpha\beta}^{+} = b_{\alpha\beta} + i\,{}^{*}b_{\alpha\beta} \quad (13.4)$$

$$C_{\alpha\beta\lambda\mu}^{-} = C_{\alpha\beta\lambda\mu} - i\overset{*}{C}_{\alpha\beta\lambda\mu} \quad (13.5)$$

then one will have:

$$C_{\alpha\beta\lambda\mu}^{-} = -3(\alpha - i\beta)\,b_{\alpha\beta}^{+}b_{\lambda\mu}^{+} + (a - ib)(g_{\alpha\beta\lambda\mu} - i\,\eta_{\alpha\beta\lambda\mu}) \quad (13.6)$$

$$(g_{\alpha\beta\lambda\mu} = g_{\alpha\beta}\, g_{\lambda\mu} - g_{\alpha\mu}\, g_{\beta\lambda}) \,.$$

From this, one deduces that:

$$\begin{aligned} \mathrm{C}_{\alpha\beta\lambda\mu} = {} & 3\alpha\,(b_{\alpha\beta}\, b_{\lambda\mu} - {}^*b_{\alpha\beta}\, {}^*b_{\lambda\mu}) + \alpha\, g_{\alpha\beta\lambda\mu} \\ & - 3\beta\,(b_{\alpha\beta}\, {}^*b_{\lambda\mu} + {}^*b_{\alpha\beta}\, b_{\lambda\mu}) - \beta\, \eta_{\alpha\beta\lambda\mu} \;. \end{aligned} \tag{13.7}$$

One then has the following characteristic properties in this case:

$$\mathrm{C}_{\alpha\beta\lambda\mu}\; l^{\beta}_{(i)} l^{\mu}_{(i)} = 2\alpha\; l^{(i)}_{\alpha} l^{(i)}_{\lambda} \tag{13.8}$$

$$\overset{*}{\mathrm{C}}_{\alpha\beta\lambda\mu} l^{\beta}_{(i)} l^{\mu}_{(i)} = 2\beta\; l^{(i)}_{\alpha} l^{(i)}_{\lambda} \,.$$

The cones $\mathrm{E}_{\alpha\beta}$ and $\mathrm{H}_{\alpha\beta}$ that are associated with the points $l^{\alpha}_{(i)}$ are composed of the tangent planes to the absolute at these points, counted twice.

II*b*. One has, by virtue of (4.15), when A = 0:

$$\mathrm{C}_{\alpha\beta}\, \mathrm{Z}^{\alpha}\, \mathrm{Z}^{\beta} = -\, \sigma\,(\mathrm{Z}^1 - i\, \mathrm{Z}^2)^2 \,. \tag{13.9}$$

Suppose l^{α} is a quadruply isotropic vector, and u and v are two vectors such that:

$$u^{\alpha}\, l_{\alpha} = v^{\alpha}\, l_{\alpha} = u^{\alpha}\, v_{\alpha} = 0 \,, \qquad u^{\alpha}\, u_{\alpha} = v^{\alpha}\, v_{\alpha}, \tag{13.10}$$

and that:

$$a_{\alpha\beta} = l_a\, u_{\beta} - l_{\beta}\, u_{\alpha} \tag{13.11}$$

$${}^*\, a_{\alpha\beta} = l_a\, v_{\beta} - l_{\beta}\, v_{\alpha} \tag{13.12}$$

and:

$$\bar{a}_{\alpha\beta} = a_{\alpha\beta} - i\; {}^*a_{\alpha\beta} - l_{\alpha}\, w_{\beta} - l_{\beta}\, w_{\alpha} \tag{13.13}$$

$$w^{\alpha}\, w_{\alpha} = 0 \,. \tag{13.14}$$

The $\bar{a}_{\alpha\beta}$ are therefore the coordinates of one generatrix of the absolute or a completely isotropic 2-plane. (13.9) thus permits us to write:

$$\mathrm{C}_{\alpha\beta\lambda\mu} = a_{\alpha\beta}\, a_{\lambda\mu} - {}^*a_{\alpha\beta}\, {}^*a_{\lambda\mu} \,, \tag{13.15}$$

and one has:

$$\mathrm{C}_{\alpha\beta\lambda\mu}\, l^{\mu} = \overset{*}{\mathrm{C}}_{\alpha\beta\lambda\mu}\, l^{\mu} = 0 \,. \tag{13.16}$$

One says that $a_{\lambda\mu}$ is a 2-form that is singular for l^{α} and $\mathrm{C}_{\alpha\beta\lambda\mu}$ is a double 2-form that is singular for l^{α} [**20**].

III. One has, thanks to (4.17):

$$\mathrm{C}_{\alpha\beta}\, \mathrm{Z}^{\alpha}\, \mathrm{Z}^{\beta} = 2\sigma\, \mathrm{Z}^1(\mathrm{Z}^3 - i\mathrm{Z}^2) \,. \tag{13.17}$$

$C_{\alpha\beta\lambda\mu}$ may then be put into the form:

$$C_{\alpha\beta\lambda\mu} = a_{\alpha\beta}\, b_{\lambda\mu} + a_{\lambda\mu}\, b_{\alpha\beta} - {}^*a_{\alpha\beta}{}^*b_{\lambda\mu} - {}^*a_{\lambda\mu}{}^*b_{\alpha\beta} \qquad (13.17^{\dagger})$$

and if $l^{\alpha}_{(1)}$ is a triply isotropic vector and $l^{\alpha}_{(2)}$ is a simply isotropic vector then:

$$a_{\alpha\beta} = l^{(1)}_{\alpha}\, u_{\beta} - l^{(1)}_{\beta}\, u_{\alpha}\,, \qquad b_{\alpha\beta} = \frac{l^{(1)}_{\alpha} l^{(2)}_{\beta} - l^{(2)}_{\beta} l^{(1)}_{\alpha}}{l_{12}}\,. \qquad (13.18)$$

One has:

$$C_{\alpha\beta\lambda\mu}\, l^{\beta}_{(1)} l^{\mu}_{(1)} = \overset{*}{C}_{\alpha\beta\lambda\mu} l^{\beta}_{(1)} l^{\mu}_{(1)} = 0 \qquad (13.19)$$

$$C_{\alpha\beta\lambda\mu}\, l^{\alpha}_{(1)} = l_{12}\, a_{\alpha\beta}\, l^{(1)}_{\lambda}\,, \qquad \overset{*}{C}_{\alpha\beta\lambda\mu}\, l^{\mu}_{(1)} = l_{12}\, {}^*a_{\alpha\beta}\, l^{(1)}_{\mu}\,. \qquad (13.20)$$

II*a*. The expression for $C_{\alpha\beta\lambda\mu}$ is easily obtained by adding the corresponding expressions that were given in II*b* and III. Indeed, let $l^{(1)}_{\alpha}$ be a doubly isotropic vector, and let $l^{(2)}_{\alpha}$ and $l^{(3)}_{\alpha}$ be simply isotropic vectors. If:

$$a_{\alpha\beta} = l^{(1)}_{\alpha}\, u_{\beta} - l^{(1)}_{\beta}\, u_{\alpha}\,, \qquad b_{\alpha\beta} = \frac{l^{(2)}_{\alpha} l^{(3)}_{\beta} - l^{(3)}_{\beta} l^{(2)}_{\alpha}}{l_{23}}\,. \qquad (13.21)$$

then one has:

$$\begin{aligned} C_{\alpha\beta\lambda\mu} &= 3\alpha\,(c_{\alpha\beta}\, c_{\lambda\mu} - {}^*c_{\alpha\beta}{}^*c_{\lambda\mu}) + \alpha\, g_{\alpha\beta\lambda\mu} - 3\beta\,(c_{\alpha\beta}{}^*c_{\lambda\mu} - {}^*c_{\alpha\beta}\, c_{\lambda\mu}) \qquad (13.22)\\ &\quad - 3\beta\,(c_{\alpha\beta}{}^*c_{\lambda\mu} + {}^*c_{\alpha\beta}\, c_{\lambda\mu}) - \beta\, \eta_{\alpha\beta\lambda\mu} + a_{\alpha\beta}\, a_{\lambda\mu} - {}^*a_{\alpha\beta}{}^*a_{\lambda\mu}\,. \end{aligned}$$

One has, moreover:

$$C_{\alpha\beta\lambda\mu}\, l^{\beta}_{(1)} l^{\mu}_{(1)} = 2\alpha\, l^{(1)}_{\alpha} l^{(1)}_{\lambda}\,, \qquad \overset{*}{C}_{\alpha\beta\lambda\mu} l^{\beta}_{(1)} l^{\mu}_{(1)} = 2\rho\, l^{(1)}_{\alpha} l^{(1)}_{\lambda}\,. \qquad (13.23)$$

We finally remark that if l^{λ} is a simply isotropic characteristic vector of (11.7) then:

$$C_{\alpha\beta\lambda\mu}\, l^{\beta}\, l^{\mu} = l_{\alpha}\, p_{\lambda} + l_{\lambda}\, p_{\alpha}\,; \qquad \overset{*}{C}_{\alpha\beta\lambda\mu}\, l^{\beta}\, l^{\mu} = l_{\alpha}\, q_{\lambda} + l_{\lambda}\, q_{\alpha}\,, \qquad (13.28)$$

in which $l^{\alpha}\, p_{\alpha} = l^{\alpha}\, q_{\alpha} = 0$.

14. *Weyl tensor and the tensor* $T(C)_{\alpha\beta\lambda\mu}$. *Duality rotations.* As in the case of the energy-momentum tensor of the electromagnetic field, the given of the tensor $C_{\alpha\beta\lambda\mu}$ entails that of a well-defined tensor $T(C)_{\alpha\beta\lambda\mu}$. Conversely, to a given tensor $T(C)_{\alpha\beta\lambda\mu}$, there corresponds a family of tensors $C_{\alpha\beta\lambda\mu}$ that are determined up to a "duality rotation." If:

† Trans. note: (sic).

$$C'_{\alpha\beta\lambda\mu} = C_{\alpha\beta\lambda\mu} \cos\theta + \overset{*}{C}_{\alpha\beta\lambda\mu} \sin\theta \tag{14.1}$$

then all of the tensors $C'_{\alpha\beta\lambda\mu}$ have the same tensor T(C), and:

$$T(C')_{\alpha\beta\lambda\mu} = T(C)_{\alpha\beta\lambda\mu} . \tag{14.2}$$

Indeed, it suffices to recall that:

$$T(C)_{\alpha\beta\lambda\mu} = 2(C_{\alpha\rho\lambda\sigma} C_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + \overset{*}{C}_{\alpha\rho\lambda\sigma} \overset{*}{C}_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma}), \tag{14.3}$$

If one refers to the analysis that was carried out in paragraph 4 then one sees that the effect of a duality rotation of the triangle $A_{11}A_{22}A_{33}$ is to turn it around its center of mass in the complex plane without changing geometric elements that enter into the expression for the tensor $T(C)_{\alpha\beta\lambda\mu}$ [cf. 6.2 through 6.7].

15. *Conservation conditions.* Returning, for the moment, to the case of the electromagnetic field, the Maxwell equations, in the absence of charges, may be written:

$$\nabla_\alpha F_\beta{}^\alpha = 0 \tag{15.1}$$
$$\nabla_\alpha {}^*F_\beta{}^\alpha = 0 . \tag{15.2}$$

In the presence of charged matter, the first equations, namely, (15.1), are replaced with:

$$\nabla_\alpha F_\beta{}^\alpha = J_\beta , \tag{15.3}$$

in which J_β is the electric current vector. The second equations (15.2) remain unchanged.

In the case of vacuum, i.e., equations (15.1) and (15.2), the energy-momentum tensor satisfies the conservation relations:

$$\nabla_\alpha T_\beta{}^\alpha = 0 , \tag{15.4}$$

and in the presence of charged matter, one has:

$$\nabla_\alpha T_\beta{}^\alpha = F_{\rho\beta} J^\rho . \tag{15.5}$$

There exists a completely similar situation in the case of Riemannian manifolds. The Riemann tensor satisfies the Bianchi identities, which one put into the form:

$$\nabla_\alpha ({}^*R^*)^\alpha{}_{\beta\lambda\mu} = 0 , \tag{15.6}$$

for example. In the case of the gravitational vacuum:

$$R_{\lambda\mu} = 0 \qquad \text{and} \qquad R_{\alpha\beta\lambda\mu} = -\,({}^*R^*)_{\alpha\beta\lambda\mu}\,. \tag{15.7}$$

Therefore, one also has:

$$\nabla_\alpha R^\alpha{}_{\beta\lambda\mu} = 0\,, \tag{15.8}$$

By contrast, in general, the relations (15.8) become:

$$\nabla_\alpha R^\alpha{}_{\beta\lambda\mu} = \nabla_\mu R_{\beta\lambda} - \nabla_\lambda R_{\beta\mu} = J_{\beta\lambda\mu}\;. \tag{15.9}$$

In the theory of Riemannian manifolds, $J_{\beta\lambda\mu}$ plays the role of a super-current.

Let us then see what happens for the super-energy tensor. In the case of the gravitational vacuum $V_{\alpha\beta\lambda\mu} = T(C)_{\alpha\beta\lambda\mu} = T_{\alpha\beta\lambda\mu}$ and:

$$\nabla_\alpha V^\alpha{}_{\beta\lambda\mu} = \nabla_\alpha T^\alpha{}_{\beta\lambda\mu}\;. \tag{15.10}$$

We shall now establish the equivalent of (15.5) in the general case. Set:

$$G({}^*R^*)_{\alpha\beta\lambda\mu\nu} = \nabla_\nu ({}^*R^*)_{\alpha\beta\lambda\mu} + \nabla_\lambda ({}^*R^*)_{\alpha\beta\mu\nu} + \nabla_\mu ({}^*R^*)_{\alpha\beta\nu\lambda}\;. \tag{15.11}$$

One easily verifies that one has:

$$G({}^*R^*)_{\alpha\beta\lambda\mu\nu} = \tfrac{1}{2}\,\eta_{\alpha\beta\rho\sigma}\,\eta_{\lambda\mu\nu\gamma}\,J^{\gamma\rho\sigma}\,. \tag{15.12}$$

If we start with the expression (12.8) for the tensor $T_{\alpha\beta\lambda\mu}$ then it results from (15.6) that:

$$\begin{aligned}\nabla_\alpha T^\alpha{}_{\beta\lambda\mu} = {} & J_{\gamma\rho\sigma} R_\beta{}^\rho{}_\mu{}^\sigma + \text{id.} + R^{\alpha\rho}{}_\lambda{}^\sigma\,\nabla_\alpha R_{\beta\rho\mu\sigma} + \text{id.}\\ & + ({}^*R^*)^{\alpha\rho}{}_\lambda{}^\sigma\,\nabla_\alpha ({}^*R^*)_{\beta\rho\mu\sigma} + \text{id.} - 2\partial_\beta A\, g^{\lambda\mu}\,,\end{aligned} \tag{15.13}$$

in which "id." denotes terms that are derived from the preceding term by a permutation of λ and μ.

Thanks to the Bianchi identities, which may be written $G(R)_{\alpha\beta\lambda\mu\nu} = 0$, one has:

$$\begin{aligned}R^{\alpha\rho}{}_\lambda{}^\sigma\,\nabla_\alpha R_{\beta\rho\mu\sigma} &= \tfrac{1}{2} R^{\alpha\rho}{}_\lambda{}^\sigma\,(\nabla_\alpha R_{\beta\rho\mu\sigma} - \nabla_\rho R_{\beta\alpha\mu\sigma})\\ &= \tfrac{1}{2} R^{\alpha\rho}{}_\lambda{}^\sigma\,\nabla_\beta R_{\alpha\rho\mu\sigma}\;,\end{aligned} \tag{15.14}$$

and thanks to (15.11):

$$\begin{aligned}&({}^*R^*)^{\alpha\rho}{}_\lambda{}^\sigma\,\nabla_\alpha ({}^*R^*)_{\beta\rho\mu\sigma}\\ &\quad = \tfrac{1}{2}({}^*R^*)^{\alpha\rho}{}_\lambda{}^\sigma\,\nabla_\beta ({}^*R^*)_{\alpha\rho\mu\sigma} + \tfrac{1}{2}({}^*R^*)^{\alpha\rho}{}_\lambda{}^\sigma\,G({}^*R^*)_{\mu\sigma\alpha\beta\rho}\,.\end{aligned} \tag{15.15}$$

Upon collecting the various terms of (15.13), and taking into account the identity:

$$R^{\alpha\beta}{}_{\lambda\mu}\,R_{\alpha\beta}{}^{\lambda\mu} + ({}^*R^*)^{\alpha\beta}{}_{\lambda\mu}\,({}^*R^*)_{\alpha\beta}{}^{\lambda\mu} = 4\,A\,g^\nu_\mu\,, \tag{15.16}$$

one obtains:

$$\nabla_\alpha T^\alpha{}_{\beta\lambda\mu} = J_{\gamma\rho\sigma} R_\beta{}^\rho{}_\mu{}^\sigma + \text{id.} + \tfrac{1}{2}({}^*R^*)^{\alpha\rho}{}_\lambda{}^\sigma\,G({}^*R^*)_{\mu\sigma\alpha\beta\rho} + \text{id.} \tag{15.17}$$

Thanks to the definition (3.5), as well as (15.12):

$$(^*R^*)^{\alpha\rho}{}_{\lambda}{}^{\sigma}\, G(^*R^*)_{\mu\sigma\alpha\beta\rho} = - J^{\gamma ab}\,(R_{\beta\lambda ab} + 2g_{\lambda\alpha}\, R_{\beta\lambda b\mu}) \,. \qquad (15.18)$$

One thus obtains:

$$\nabla_{\alpha}\, T^{\alpha}{}_{\beta\lambda\mu} = 2\, J_{\rho\lambda\sigma} R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + \text{id.} - J^{\gamma ab}\, R_{\beta\gamma ab}\, g_{\lambda\mu} \,. \qquad (15.19)$$

If one sets:

$$M_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} = R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + R_{\mu}{}^{\rho}{}_{\beta}{}^{\sigma} \,, \qquad (15.20)$$

one then finds that:

$$\nabla_{\alpha}\, V^{\alpha}{}_{\beta\lambda\mu} = \tfrac{1}{3}\, J^{\gamma\rho\sigma}(R_{\beta\gamma ab}\, g_{\lambda\mu} + R_{\gamma\lambda ab}\, g_{\mu\beta} + R_{\lambda\mu ab}\, g_{\beta\lambda} + 2M_{\beta\gamma\mu b}\, g_{a\lambda} + 2M_{\lambda\gamma\beta b}\, g_{a\mu} + 2M_{\mu\gamma ab}\, g_{\alpha\beta}) \,, \qquad (15.21)$$

namely, a relation of the form:

$$\nabla_{\alpha}\, V^{\alpha}{}_{\beta\lambda\mu} = J^{\gamma ab}\, V_{\gamma ab\,\beta\lambda\mu} \,, \qquad (15.22)$$

which generalizes (15.5) [**21**].

16. *Case in which there exists an electromagnetic field.* We now address the case in which there exists a pure electromagnetic field (without any motion of charged matter).

First of all, consider the case in which the field is singular, and let l^{α} be the associated isotropic direction, and let:

$$T_{\alpha\beta} = \mathcal{T}\, l_{\alpha}\, l_{\beta} \,, \qquad (16.1)$$

be the expression for the energy-momentum tensor of electromagnetic field. One thus has:

$$R_{\alpha\beta} = T_{\alpha\beta} \,. \qquad (16.2)$$

It is natural to assume that the tensor $C_{\alpha\beta\lambda\mu}$ possesses the vector l^{α} as a quadruply isotropic vector, wherever it is identically null. One thus has:

$$R_{\alpha\beta\lambda\mu}\, l^{\mu} = (R^*)_{\alpha\beta\lambda\mu}\, l^{\mu} = 0 \,. \qquad (16.3)$$

A. Lichnerowicz [**22**] has called a space that satisfies conditions (16.3) a *state of pure total radiation*.

It is then possible to put the tensor $R_{\alpha\beta\lambda\mu}$ into the form:

$$R_{\alpha\beta\lambda\mu} = a_{ij}\, a^{(i)}_{\alpha\beta}\, a^{(j)}_{\lambda\mu} \,, \qquad i, j = 1, 2 \qquad (16.4)$$

in which:

$$a_{ij} = a_{ji} \,, \qquad a^{(2)}_{\alpha\beta} = {}^*a^{(1)}_{\alpha\beta} \,, \qquad a_{\alpha\beta}\, l^{\beta} = {}^*a_{\alpha\beta}\, l^{\beta} = 0 \,. \qquad (16.5)$$

One has:

$$\mathcal{T} = -(a_{11} + a_{22})\ . \tag{16.5†}$$

If $\mathcal{T} = 0$ then one is concerned with a state of pure gravitational radiation. One then has:

$$V_{\alpha\beta\lambda\mu} = 4\mu\, l_\alpha\, l_\beta\, l_\lambda\, l_\mu\ , \qquad V(C)_{\alpha\beta\lambda\mu} = 2(\mu - \det a_{ij})\, l_\alpha\, l_\beta\, l_\lambda\, l_\mu\ , \tag{16.6}$$
$$V(E)_{\alpha\beta\lambda\mu} = 2\mathcal{T}^2\, l_\alpha\, l_\beta\, l_\lambda\, l_\mu\ ,$$

and $\mu = \sum_i \sum_j (a_{ij})^2$.

One has:

$$\nabla_\alpha\,[V(E)^\alpha{}_{\beta\lambda\mu} - V(C)^\alpha{}_{\beta\lambda\mu}] \neq 0\ . \tag{16.7}$$

Now:

$$V(E)^\alpha{}_{\beta\lambda\mu} - V(C)^\alpha{}_{\beta\lambda\mu} = 4(\mu - \mathcal{T}^2)\, l^\alpha\, l_\beta\, l_\lambda\, l_\mu\ . \tag{16.8}$$

If one assumes, moreover, that the electromagnetic field is of integrable type $l_\alpha = \partial_\alpha f$, then it results from (16.8) that:

$$\nabla_\alpha\,[(\mu - \mathcal{T}^2)\, l^\alpha] = 0\ , \tag{16.9}$$

and under the same conditions, by virtue of the conservation of $T_{\alpha\beta}$, one has:

$$\nabla_\alpha\,(\mathcal{T} l^\alpha) = 0\ . \tag{16.10}$$

In the conformally Euclidian case:

$$\mu = \mathcal{T}^2 \qquad \text{and} \qquad \det a_{ij} = 0\ . \tag{16.11}$$

Some states of pure total radiation of integrable type are:

1) The Bondi solution [**23**]:

$$ds^2 = e^{2\varphi}\,(dt^2 - dx^2) - (\xi^2\, dy^2 + \eta^2\, dz^2)\ , \tag{16.12}$$

in which φ, $\xi > 0$, $\eta > 0$ are three functions of $(t - x)$.

2) The solutions of J. Hely [**24**]:

$$g_{\alpha\beta} = \delta_{\alpha\beta} + l_\alpha\, u_\beta + u_\beta\, l_\alpha\ , \tag{16.13}$$

in which:

$$u_\alpha = a l_\alpha + \partial_\alpha b, \tag{16.14}$$

† Translators note: [sic].

and if $q = \dfrac{l^\alpha \partial_\alpha a}{1 + l^\alpha \partial_\alpha b}$ then q is a function of $(l_\gamma x^\gamma)$.

3) There also exist conformally Euclidian solutions:

$$ds^2 = u(dt^2 - dx^2 - dy^2 - dz^2), \tag{16.15}$$

in which u is a function of $(t - x)$, for example. In this latter case, the space may not represent a pure gravitational state without being Euclidian.

We now address the case of a non-singular field. In this case, there exists a pair of isotropic vectors, namely, $l_\alpha^{(1)}$ and $l_\alpha^{(2)}$. One has the field equations:

$$\mathrm{R}_{\alpha\beta} = \mathrm{I}\left(2\frac{l_\alpha^{(1)} l_\beta^{(2)} + l_\alpha^{(2)} l_\beta^{(1)}}{l_{12}} - g_{\alpha\beta}\right). \tag{16.16}$$

It is then natural to assume that the tensor $\mathrm{C}_{\alpha\beta\lambda\mu}$ possesses two characteristic isotropic vectors, so it is of type Ia, or of type III, or identically null. Since type III makes the two fields $l_\alpha^{(1)}$ and $l_\alpha^{(2)}$ play different roles, we assume that $\mathrm{C}_{\alpha\beta\lambda\mu}$ is of type Ia. $\mathrm{C}_{\alpha\beta\lambda\mu}$ is then given by (13.3), $\mathrm{E}_{\alpha\beta\lambda\mu}$ is known, thanks to (16.16), and $\mathrm{G}_{\alpha\beta\lambda\mu} = 0$. One may then put $\mathrm{R}_{\alpha\beta\lambda\mu}$ into the form:

$$\mathrm{R}_{\alpha\beta\lambda\mu} = a_{ij} b_{\alpha\beta}^{(j)} b_{\lambda\mu}^{(i)} + \mathrm{A} g_{\alpha\beta} g_{\lambda\mu} + \mathrm{B}\ \eta_{\alpha\beta\lambda\mu}, \tag{16.17}$$

with:

$$b_{\alpha\beta} = \frac{l_\alpha^{(1)} l_\beta^{(2)} - l_\alpha^{(2)} l_\beta^{(1)}}{l_{12}}, \qquad b_{\alpha\beta}^{(1)} = {}^*b_{\alpha\beta}^{(2)}, \tag{16.18}$$

$$\mathrm{I} = \tfrac{1}{2}(a_{11} + a_{22}), \qquad \mathrm{A} = \tfrac{1}{6}(a_{11} - a_{22}), \qquad \mathrm{B} = \tfrac{1}{3} a_{12}. \tag{16.19}$$

One has:

$$\begin{aligned}\mathrm{V}_{\alpha\beta\lambda\mu} &= \mathrm{C}(\mathrm{R}_{\alpha\beta}\mathrm{R}_{\lambda\mu} + \mathrm{R}_{\sigma\lambda}\mathrm{R}_{\beta\mu} + \mathrm{R}_{\alpha\mu}\mathrm{R}_{\beta\lambda}) \\ &\quad - \mathrm{D}(g_{\alpha\beta} g_{\lambda\mu} + g_{\sigma\lambda} g_{\beta\mu} + g_{\alpha\mu} g_{\beta\lambda}),\end{aligned} \tag{16.20}$$

and:

$$\mathrm{C} = \frac{1}{3\mathrm{I}^2}\left(\sum_i \sum_j (a_{ij})^2\right), \qquad \mathrm{D} = \frac{1}{18}\left(\sum_i \sum_j (a_{ij})^2\right) - 2 \det a_{ij}). \tag{16.21}$$

One must remark that there is good reason to take into account the Bianchi identities and the Maxwell-Rainich equations of the electromagnetic field in order to further couple the functions a_{ij} to the electromagnetic field.

An example of such a space is provided by the Nordstrom solution; L. Witten has given another example [**26**].

The Riemann tensor that was given in (16.17), along with (16.18), is entirely characterized by the conditions:

$$\mathrm{R} = 0\,, \qquad \mathrm{R}_{\alpha\beta\lambda\mu}\; l^{\beta}_{(i)} l^{\mu}_{(i)} = \frac{5a_{11} + a_{22}}{6}\, l^{(i)}_{\alpha} l^{(i)}_{\lambda}\,,$$

$$({}^*\mathrm{R}^*)_{\alpha\beta\lambda\mu}\; l^{\beta}_{(i)} l^{\mu}_{(i)} = \frac{a_{11} + 5a_{22}}{6}\, l^{(i)}_{\alpha} l^{(i)}_{\lambda}\,, \qquad (16.22)$$

$$(\mathrm{R}^*)_{\alpha\beta\lambda\mu}\; l^{\beta}_{(i)} l^{\mu}_{(i)} = ({}^*\mathrm{R})_{\alpha\beta\lambda\mu}\; l^{\beta}_{(i)} l^{\mu}_{(i)} = -\frac{2}{3}\, a_{12}\, l^{(i)}_{\alpha} l^{(i)}_{\lambda}\,.$$

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[**26**] L. WITTEN, *Colloque C.B.R.M.*, Brussels, June 1959; La Relativité générale; to appear.

[**27**] After the editing of this text, BEL made it known to me that he presented the theorem that we proved in §10 at a conference on 16 May 1959 at the Faculté des Sciences de Paris.
