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On an inverse problem in the calculus of variations.

By PAUL DEDECKER (*),
Assistant at the royal meteorological institute.

Translated by D. H. Delphenich

Abstract. – A necessary and sufficient condition for a system of n differential equations of any order to be identical to the Euler-Lagrange equations of a variational problem is established with the aid of some properties of exterior differential forms that are defined in certain spaces $E_{(p)}$.

1. Introduction. – Let:

(1) $$F_i\left(t, y^i, \frac{dy^i}{dt}, \ldots, \frac{d^s y^i}{dt^s}\right) = 0 \qquad (i, j = 1, 2, \ldots, n)$$

be a system of n differential equations of order s in the n unknown functions of the independent variable t. I shall address the following proposition here:

Theorem:

A necessary and sufficient condition for there to exist a Lagrangian function of a certain order r:

(2) $$L = L\left(t, y^i, \frac{dy^i}{dt}, \ldots, \frac{d^r y^i}{dt^r}\right)$$

whose variational derivatives with respect to the y^i are identical to the F_i :

(3) $$\frac{\delta L}{\delta y^i} \equiv F_i$$

is that the system of variations of the F_i should be self-adjoint ([1]).

(*) Presented by **Lepage** and **Van den Dungen**.
([1]) For the notion of self-adjointness, see no. **4**.

An equivalent condition was given by L. Koenigsberger [**7**], along with a proof of the *necessity* for arbitrary n and s. As far as *sufficiency* is concerned, it led to some long and complicated calculations that were limited to some particular cases ($n = 1, s = 2, 4$; $n = s = 2$), all while asserting that his method would extend to the general case. **D. R. Davis** [**3**] has taken up the case of $n = s = 2$ once more by using the concept of self-adjointness, but his calculations were still quite long and complicated. In his treatise [**4**] (pp. 204), **Th. De Donder** indicated that *necessity* would be an application of his methods. Finally, **I. M. Rapoport** [**9**] gave a proof of Koenigsberger's condition for arbitrary n and $s = 2$ (necessity and sufficiency).

I propose to show that the use of *exterior differential forms* ([1]) that are defined in certain spaces $E_{(p)}$ (spaces of contact elements) will simplify the calculations considerably in the case of arbitrary n and s. My method is based on a fundamental lemma that is coupled with the structure of the spaces $E_{(p)}$, and when one starts from it, the assertion will be proved in just a few sentences (no. **5**).

In addition, I will show that one can arrange that the order r of the function L should be $s / 2$ or $(s + 1) / 2$ according to whether s is even or odd, resp., which is a fact that is already contained in Koenigsberger's book.

2. The spaces $E_{(\rho)}$ and their differential forms. – Let ρ denote a non-negative integer and let $(t, y_0^i, y_1^i, \ldots, y_\rho^i)$ denote a system of $n(\rho + 1) + 1$ *independent* variables. We consider them to be the coordinates of points in a space $E_{(\rho)}$ that we shall call the *space of contact elements of order* ρ.

The system (1) is obtained for $\rho = s$ by starting from n functions F_i that are defined in $E_{(\rho)}$ that one annuls after replacing y_0^i with $y^i(t)$, y_1^i with dy^i / dt, etc.

We consider exterior differential forms of degree $\nu = 0, 1, \ldots$ on $E_{(\rho)}$. They amount to homogeneous polynomials of a certain degree n in the *indeterminates* dt, dy_0^i, dy_1^i, …, dy_ρ^i with coefficients that are functions ([2]) of t, y_0^i, y_1^i, …, y_ρ^i, and in which the multiplication of the indeterminates is associative and alternating.

One knows that the operation of *exterior differentiation* transforms any form Ω_ν of degree ν into a form of degree $\nu + 1$ that is denoted $d\,\Omega_\nu$. The condition that $d\,\Omega_\nu = 0$ is valid in a domain of $E_{(\rho)}$ that is homeomorphic to a hypersphere is necessary and sufficient for the existence of a $\Omega_{\nu-1}$ in that domain such that $\Omega_\nu = d\,\Omega_{\nu-1}$.

All of the forms that we must consider are *truncated*, i.e., the coefficients of the terms in dt in them will always be zero. (Meanwhile, the other coefficients can depend upon t.)

In what follows, the word "form" will always signify "truncated form."

One sees that one is dealing with the inverse problem in the formal sense. One can also look for the conditions for there to exist a Lagrangian such that the corresponding equations in variational derivatives are *equivalent* to (1). See **J. Douglas [5]**, [**6**].

([1]) **E. Cartan** [**1**], (chaps. 6, 7); [**2**].

([2]) We shall suppose, once and for all, that the functions that are being used are differentiable a sufficient number of times.

In place of the differential $d\,\Omega_\nu$, one will apply the *truncated differential* $\delta\,\Omega_\nu$ to the (truncated) forms, which are obtained by suppressing the terms in dt from $d\,\Omega_\nu$. For example, upon starting from the linear form (Pfaff form):

(4) $$\Omega_1 = A_i^k\, dy_k^i\,,$$

one will have:

$$\delta\,\Omega_\nu = \frac{1}{2}\left(\frac{\partial A_j^l}{\partial y_k^i} - \frac{\partial A_i^k}{\partial y_l^j}\right) dy_k^i\, dy_l^j \qquad \begin{pmatrix} i, j = 1, 2, \ldots, n \\ k, l = 0, 1, \ldots, \rho \end{pmatrix}.$$

One easily verifies that for any form Ω_ν, the condition that $\delta\,\Omega_\nu = 0$ is valid in a neighborhood of a point of $E_{(\rho)}$ is necessary and sufficient for the existence of a $\Omega_{\nu-1}$ such that $\Omega_\nu = \delta\,\Omega_{\nu-1}$ in the neighborhood of that point.

3. The operator d / dt. – It is obvious that any form Ω_ν in $E_{(\rho)}$ can be regarded as a form on $E_{(\rho+1)}$ for which the coefficients of the terms in $dy_{\rho+1}^i$ will be zero. Conversely, a form on $E_{(\rho+1)}$ will belong to $E_{(\rho)}$ on the conditions that:

1. Its coefficients are independent of the $y_{\rho+1}^i$, and

2. The coefficients of the terms in $dy_{\rho+1}^i$ are zero.

We say that a form Ω_ν has *order* ρ when it belongs to $E_{(\rho)}$ without belonging to $E_{(\rho-1)}$. In order to specify the order ρ of Ω_ν, we shall use the notation Ω_ν^ρ.

An infinitesimal transformation (1) in $E_{(\rho-1)}$ is written:

$$X \equiv T\,\frac{\partial}{\partial t} + X_k^i\,\frac{\partial}{\partial y_k^i} \qquad (k = 0, 1, \ldots, \rho + 1),$$

in which the T, X_k^i are functions of a variable point in $E_{(\rho-1)}$. We shall consider the particular transformation for which $T = 1$, $X_k^i = y_{k+1}^i$ ($k < \rho$), $X_{\rho+1}^i = 0$, and we shall let d / dt denote the operator:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + y_1^i\,\frac{\partial}{\partial y_0^i} + \cdots + y_{\rho+1}^i\,\frac{\partial}{\partial y_\rho^i}\,.$$

(1) For that notion, see, e.g., **E. Cartan** [**1**], chap. 9.

We shall make that transformation operate on the differential forms *of order at most* ρ. One sees that *any (truncated) form transforms into another (truncated) form.*

For example, consider the function $A = A(t, y_0^i, \ldots, y_\rho^i)$ (form of degree zero). One has:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + y_1^i \frac{\partial A}{\partial y_0^i} + \cdots + y_{\rho+1}^i \frac{\partial A}{\partial y_\rho^i}.$$

As for the form (4), upon setting $\rho = 1$, to simplify, it will become:

$$\frac{d}{dt}\Omega_1 = \frac{dA_i^0}{dt} dy_0^i + \left(\frac{dA_i^1}{dt} + A_i^0\right) dy_1^i + A_i^1\, dy_2^i. \tag{5}$$

More generally, any form Ω_ν^ρ will correspond to a form $\Omega_\nu^{\rho+1}$ such that:

$$\frac{d}{dt}\Omega_\nu^\rho = \Omega_\nu^{\rho+1}.$$

Suppose that $\frac{d}{dt}\Omega_1 = 0$. Formula (5) shows that this implies that $A_i^1 = A_i^0 = 0$, i.e., $\Omega_1 = 0$. That result extends immediately to any form of non-zero degree and constitutes the:

Fundamental lemma:

For $\nu \geq 1$, *the relation* $d / dt\, \Omega_\nu = 0$ *implies that* $\Omega_\nu = 0$.

In the foregoing, the integer ρ was indeterminate. When the operation d / dt is applied to a form of order $\rho + 1$, $\rho + 2$, …, it will then be defined by replacing ρ with $\rho + 1$, $\rho + 2$, … everywhere. One defines the operators $\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right)$, … by recurrence, which will give rise to the general formula:

$$\frac{d^\sigma}{dt^\sigma}\Omega_\nu^\rho = \Omega_\nu^{\rho+\sigma}.$$

We further make the following remark: We know that the operator of an arbitrary infinitesimal transformation X commutes with that of exterior differentiation. The same thing will be true in the context of the operator δ for any operator X that:

(*a*) transforms any truncated form into a truncated form, and (*b*) verifies the relation $X(dt) = 0$ ([1]).

Therefore, the same thing will be true for the operator d/dt, and we will have the law:

$$\frac{d}{dt}\delta \equiv \delta\frac{d}{dt}.$$

4. Self-adjoint systems. – The notion of self-adjoint system is usually defined for a system of linear, homogeneous expressions in certain variables that depend upon *t* and their derivatives with coefficients that are functions of *t* [**4**] (pp. 196).

We extend that notion to a system of linear differential forms. Let *n* Pfaff forms ϖ_i have order ρ. We say that the system of ϖ_i is *self-adjoint* if there exists a quadratic exterior form Ω_2 of order ρ' such that one will have the identity from the exterior differential calculus ([2]):

$$\varpi_i\, dy_0^i + \frac{d}{dt}\Omega_2 = 0. \tag{6}$$

5. Proof of the theorem. – In the statement of the theorem, there is good reason to interpret the phrase *system of variations* of the F_i to mean the system of *n* Pfaff forms δF_i. One must then show that the existence of a Ω_2 such that:

$$\delta F_i\, dy^i + \frac{d}{dt}\Omega_2 = 0 \tag{7}$$

is necessary and sufficient for the existence of a function *L* that gives rise to (3).

a) *Necessary condition.* – Suppose that *L* exists.

([1]) One has for any Ω (truncated or not): $dX\,\Omega = X\,d\,\Omega$. If Ω is truncated then we will have the decomposition $d\,\Omega = \delta\,\Omega + \omega'\,dt$, so from (*b*): $X\,d\,\Omega = X\,\delta\,\Omega + (X\omega')dt$. On the other hand, $dX\,\Omega = \delta X\,\Omega + \omega''\,dt$, so $X\,\delta\,\Omega - \delta X\,\Omega = dt(X\omega' - \omega'')$. From (*a*), the left-hand side does not contain *dt*. The two sides can only be zero then.

([2]) One recovers the classical definition by supposing that the coefficients of the ϖ_i depend upon only *t*. Upon letting $\delta_1 y_0$, $\delta_2 y_0^i$; $\delta_1 y_k^i$, $\delta_2 y_k^i$ denote arbitrary functions of *t* and their derivatives, the relation (6) can be written, with the usual algebraic notations:

$$\varpi_i(\delta_1)\,\delta_2 y^i - \varpi_i(\delta_2)\,\delta_1 y^i = \frac{d}{dt}\Omega_2(\delta_1, \delta_2),$$

in which Ω_2 denotes an alternating bilinear form in $\delta_1 y_k^i$ and $\delta_2 y_k^i$, this time.

The present condition formally extends to forms ϖ_i of arbitrary degree *n* on the condition that one must replace Ω_2 with $\Omega_{\nu+1}$ in (6).

The fundamental formula of the calculus of variations [**4**] (pp. 8) consists of replacing the linear form of order r:

$$\delta L = \frac{\partial L}{\partial y_0^i} dy_0^i + \cdots + \frac{\partial L}{\partial y_r^i} dy_r^i$$

with a sum of two forms that generally have order $2r$:

$$\delta L = \frac{\delta L}{\delta y_0^i} dy_0^i + \frac{d}{dt}\omega , \qquad \omega = \frac{\delta L}{\delta y_k^i} dy_{k-1}^i \qquad (k = 1, 2, \ldots, r),$$

which is realized with the aid of a series of integrations by parts ([1]).

By virtue of (3), that will become:

$$\delta L = F_i \, dy_0^i + \frac{d}{dt}\omega , \tag{8}$$

which will give rise to (7) when one applies the operation δ to both sides, but on the condition that one must set $\Omega_2 = \delta\omega$.

b) *Sufficient condition*. – One starts from formula (7).

Upon applying the operation δ to both sides of that relation, one will get $\frac{d}{dt}\delta\Omega_2 = 0$, so by virtue of the fundamental lemma: $\delta\Omega_2 = 0$. There will then exist a linear form ω such that $\delta\omega = \Omega_2$. Moreover, the formula (7) can be written:

$$\delta\left(F_i \, dy_0^i + \frac{d}{dt}\omega\right) = 0 ,$$

and that expresses the existence of a function L that gives rise to (8), and therefore, to (3).

6. Remarks:

1. The same method will permit one to prove that *the condition* $\frac{\delta L}{\delta y_0^i} = 0$ *is necessary and sufficient for* L *to have the form* $L = dN / dt$, $N = N(t, y_0^i, y_1^i, \ldots)$ in a few sentences.

([1]) The variational derivatives of L are defined by recurrence:

$$\frac{\delta L}{\delta y_r^i} = \frac{\partial L}{\partial y_r^i}, \qquad \frac{\delta L}{\delta y_k^i} = \frac{\partial L}{\partial y_k^i} - \frac{d}{dt}\frac{\delta L}{\delta y_{k+1}^i} \qquad (k = 0, 1, 2, \ldots, r-1).$$

2. The variational derivatives with respect to y_0^i of the function L generally have order $s = 2r$. What can one say about the order r of the function L that was constructed in 5.*b*) by starting with F_i of order s ?

We shall show that one can always arrange to have $r = s / 2$ or $r = (s + 1) / 2$ according to whether s is even or odd, resp. It will suffice to show that the order of L can be reduced for $2r > s + 1$.

If the variational derivatives of L with respect to y_0^i are independent of the y_{2r}^i then one will have:

$$\frac{\partial^2 L}{\partial y_r^i \, \partial y_r^j} = 0\,, \quad \text{hence} \quad L = M_i \, y_r^i + M\,,$$

in which the functions M_i, M depend upon only t, y_0^i, …, y_{r-1}^i. If the variational derivatives are independent of the y_{2r-1}^i, in addition, then one will have:

$$\frac{\partial M_i}{\partial y_{r-1}^j} = \frac{\partial M_j}{\partial y_{r-1}^i}\,,$$

so a function $N = N(t, y_0^i, \ldots, y_{r-1}^i)$ will exist such that:

$$\frac{\partial N}{\partial y_{r-1}^i} = M_i\,.$$

The function $L' = L - dN / dt$ has order $r - 1$ and possesses the same variational derivatives with respect to the y_0^i as L.

3. There is good reason to extend the method that was used here to system of partial differential equations. I shall confine myself to merely pointing to the fact that this extension presents some difficulties that are similar to the ones that numerous authors encountered while seeking to extend the theories in the calculus of variations that are valid for simple integrals to multiple integrals, and in particular, the theory of Weierstrass fields ([1]).

([1]) For the bibliography of this question, see [**8**] and [**10**].
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