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FOURTH MEMOIR

ON THE PATHS THAT ARE FOLLOWED BY LIGHT AND ELASTIC BODIES IN GENERAL UNDER THE PHENOMENA OF REFLECTION AND REFRACTION.

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§ I.

Geometric properties of light under the phenomena of refraction.

The theory of the paths that are traced, when applied to the work that was done on cutting and filling [†], shows us in a general manner that for a sheaf of rectilinear paths that normal to the same given surface, we can always:

1. Consider that sheaf to be the set of paths of advance or return of a complete system of paths, the most proper of which are either ones of cutting or ones of filling.

2. Stop the same paths at a sequence of points that form an absolutely arbitrary limiting surface.

3. Find a new sheaf of paths for the returns or advances – namely, the ones that correspond to the original advances or return – that are capable of all being normal to a certain surface, like the first sheaf, and consequently are also capable of forming two groups of developable surfaces that always cross at a right angle.

We have proved that the laws that the corresponding advances and returns in this general system of paths must obey are:

1. The paths of advance and return that agree at the same point of the limiting surface are both in a plane that is normal to that surface at that point.

[†] Translator: From the French *déblais et remblais*. That theory is discussed earlier in the book, but, as Dupin himself says later in this memoir, the present theory will established geometrically, with no further reference to cutting and filling beyond some basic axioms that derive from that theory.

2. At that point, the two paths form angles with the limiting surface whose *cosines* have a constant ratio, and consequently form angles with the normal to that limiting surface whose *sines* have a constant ratio.

3. In the particular case where that same force is sufficient to make one traverse the same space at the same cost, the ratio of those cosines and sines will be unity along the paths of advance or return. The path of advance and the path of return, which agree at the same point of the limiting surface, must then subtend the same angle with that surface at that point.

For the cases in which the paths meet, either in the cutting or the filling, we have shown that how they then present two sheaves that are separated completely by a surface and how that surface enjoys the same properties with respect to the two sheaves as a surface that is limited by both paths of advance and return. Therefore, when two paths agree at a point of the *separating* surface:

1. They will both be in a plane that is normal to that surface at that point.

2. Those two paths will form angles with the normal to the separating surface at that same point whose sines have a constant ratio. It will be the ratio of the spaces that are traversed on those same paths for the same cost.

3. In the particular case where the costs are the same for the paths that belong to the two sheaves, the two paths that agree at a point of the separating surface will form that same angle with that surface.

We have further proved that the various sheaves of paths of advance or return, whether they do or do not agree under cutting and filling, each have the constant, general character that they represent the system of normals to a certain surface. That is why all of the paths of the same sheaf will always form two systems of developable surfaces such that the developables of the one system will cross all of the developables of the other system at a right angle.

We have seen that for the most advantageous paths (such as the ones that can be normal to the same surface), when they are considered to be the set of paths of advance that stop on a *limiting* surface of arbitrary form, one can always find a sheaf of the most advantageous paths of return, which will, consequently, enjoy all of the properties that we just enumerated with respect to the limiting surface that the first sheaf does.

Finally, we have seen, analogously, that if a sheaf of the most advantageous paths terminates at a *separating* surface of arbitrary form during the cutting and filling then one can always find a second sheaf of the most advantageous paths on the other side of the separatrix that enjoys all of the properties that we have just recapitulated with respect to the separating surface as the first one.

These properties, by their nature, apply immediately to the phenomena of reflection and refraction of light.

When a light ray passes from one medium into another one whose density differs from that of the first medium, that ray will experience a deviation that is known by the name of *refraction* and is subject to the following law: If one draws the normal to the surface that separates the two media through the point on that surface where the refraction takes places then the first ray, which one calls *incident*, and the second one, which one calls *refracted*, will be in one and the same plane through that normal. Moreover, if one measures the angle that is formed, first of all, by the normal and the incident ray – i.e., the *angle of incidence* – and secondly by the normal and the refracted ray – i.e., the *angle of refraction* – then one will find that the sine of the angle of incidence and the sine of the angle of refraction have a constant ratio for all of the rays that traverse the two media that one considers in any direction.

Now suppose that a sheaf of incident rays that can be normal to the same surface passes from one medium to the other, and regard the separating surface of the two media as having the most general form. The theory of cutting and filling, whose principal results we just summarized, will then lead us immediately to the following theorems:

First consider the separation of the media to be a *limiting surface* and the sheaf of incident rays are paths of advance. The system of paths of return for which the cost of transport is *m* times more dear than the cost of advance will form a new sheaf of the most advantageous paths on the same side of the boundary. Moreover, they will be capable of being:

1. Normal to the same surface.
2. Decomposing into two series of developables that cross at a right angle everywhere, etc.

Regard the separation between the two media, in turn, as a *separating surface* between the two systems of paths (under cutting or filling). On the other side of the separating surface, a third sheaf whose rays always form the same angle with the separating surface as the paths of the second sheaf will correspond to the second sheaf that we just found.

Now, the third sheaf of paths, thus-determined, is precisely the sheaf of rays that are refracted upon passing from the first medium to the second one. Indeed, each refracted ray is in a plane that is normal to the separating surface of the media, along with the incident ray, and the sine of the angle of incidence will not cease to have a constant ratio *m* with the sine of the angle of refraction, and that ratio is given by experiment.

Therefore:

When a sheaf of light rays is decomposable into two series of developable surfaces that cross at a right angle everywhere, one can make it traverse an arbitrary number of homogeneous media that are separated by arbitrary surfaces (with simple or double curvature) without that sheaf ceasing to enjoy the following properties:

1. *It is composed of normals to a surface.*
2. *It is decomposable into two series of developable surfaces that cross at a right angle everywhere.*

The proofs upon which we have based the general properties of paths are purely geometric. We could apply them immediately to the search for laws that would correlate the rays of one sheaf of light that is refracted as it passes from one medium to another upon traversing a surface of the most general form. However, we think that instead of giving the same proofs twice, while changing only some terminology, it would be more interesting to science to present an agreement that seems remarkable to us between the laws of most advantageous transport and the general laws of refraction. It is in that same spirit that we shall consider the phenomena of reflection.

§ II.

Fundamental properties of the paths of light under the phenomena of reflection.

As we have learned from experience, if a sheaf of light rays falls upon a surface that does not let it pass through and does not absorb it – when the surface is, in a word, in a limited space from which the light cannot leave – then each reflected ray will obey the following relations with respect to the incident ray that produced it:

1. The plane that contains those two rays is normal to the reflecting surface at their point of agreement.
2. That surface will form the same angle with one and the other ray at that point.

We can then consider the paths of advance to be the incident rays and the paths of return to be the reflected rays in a system of paths that are most advantageous for cutting and filling (upon assuming that it requires the same force to traverse the space under advance or return). The general laws that we have proved for similar systems of paths apply immediately to the sheaves of light rays that represent them.

Moreover, we first conclude the following principle:

When a sheaf of light rays is decomposable into two systems of developable surfaces that cross at a right angle (which will be the case whenever those rays can be considered to be the normals to a unique surface), if one receives that sheaf on a mirror of an arbitrary form then the reflected rays must form a new sheaf that is, like the first one, decomposable into two systems of developables that cross at a right angle.

Consequently, once the character of being normal to a unique surface belongs to a sheaf of light, it will be ineffaceable despite all of the reflections that the rays of that sheaf can possibly experience by a sequence of mirrors of arbitrary form; similarly, it will persist despite all of the refractions that the rays of the sheaf can possible experience when it is led to traverse media that are separated by surface of arbitrary form.

If all of the rays of the original sheaf emanate from just one luminous point then they will obviously all be normal to each sphere whose center is at that point. Therefore, a sheaf of rays that emanates from a luminous point and reflects or refracts as many times as one likes from mirrors or separating surfaces of an arbitrary form will always present a sheaf of rays that can be normal to the same surface, and like them, can form two systems of developable surfaces that cross at a right angle.

A sheaf of parallel paths can be considered to be produced by the normals to a plane. Therefore, when a sheaf of parallel rays is reflected by a sequence of mirrors of arbitrary form, it will always present a sheaf of reflected or refracted rays that can be normal to the same surface.

The Sun, by reason of the immensity of its distance in comparison to the magnitude of the objects around us that it illuminates, projects rays whose parallelism seems perfect, even for the observer that is endowed with the most delicate instruments. It follows from this that when a sheaf of solar rays is refracted or reflected by a sequence of mirrors or arbitrary separating surfaces, it will always be transformed into sheaves whose rays can both be normal to the same surface for the same sheaf.

Malus made these latter consequences of the general principle that we just presented known in his paper on optics. However, he thought that the principle was true only when the light rays either emanated from a unique point or were parallel and only for a single reflection and a single refraction. He asserted that the reflection from a second mirror (and, *a fortiori*, the refractions upon traversing a second separating surface) would no longer define a sheaf of rays that were decomposable into two systems of developable surfaces. In that sense, our results differ from those of that geometer, and the sciences will deplore the discrepancy (*).

Since Malus made use of an extremely complicated analytical procedure, just one error in calculation led him to believe that it would no longer be possible to satisfy the equations of condition upon which the orthogonality of the developable surfaces that are defined by the reflected rays would depend for reflections on a second mirror (**). However, the false conclusion that resulted from that error (to some extent, mechanically) should not mean that he does not deserve to be credited with having discovered one of the most beautiful theorems of the application of geometry to optics.

In order to further leave no doubt about the extension that we claim to have given to Malus's theorem, we shall prove that principle directly for catoptrics, with the aid of only geometry, and no considerations that are deduced from the theory of cutting and filling. At the same time, we have preserved that advantage of having made known some extremely striking analogies with questions of a very different nature, and several identical laws. (*See* Note I at the end of the memoir.)

Consider a sheaf of incident rays that fall upon a mirror whose form is arbitrary. Suppose only that these rays are all normal to the same surface (Σ) and seek to determine the reflected sheaf of rays.

For that, suppose that a sphere of variable radius has its center constantly located on the mirror, and its surface constantly tangent to the surface (Σ) that has all of the incident rays for normals. That surface (Σ) will be the envelope of the space that is traversed by the sphere in front of the mirror.

The space that is occupied by the part of the sphere that is found behind the mirror is analogously bounded by an enveloping surface whose normal at each point agrees with the radius of the sphere that touches that envelope at the same point. We shall now prove that this new normal is the prolongation of a reflected ray behind the mirror.

When an arbitrary surface, which does or does not have a variable parameter, moves in such a manner that one of its points traverses an arbitrary director line, the space that the entire surface traverses will be bounded by another surface that one calls the

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(*) *See* the papers that were inserted into the Journal de l'École Polytechnique, 14th letter, and the special article that was published much later under the title "Théorie de la double réfraction de la lumière," art. 11.

(**) Cauchy corrected the calculations. Analogously, he proved that the equations of condition for the orthogonality of the developables that are formed by light rays after a second refraction will be satisfied.

envelope, while one calls the moving generator the *enveloper*. (See the *Géometrie analytique* of G. Monge.)

Each enveloper cuts the one that follows it immediately along a curve that is found completely on the envelope: viz., the curve of contact of these two surfaces.

One can, in turn, suppose that the envelope varies in form and position in such a manner that one of its points will describe an arbitrary line. The space that is then traversed by the enveloper will be bounded by an enveloping surface of the envelopes which will have no other relationships with the original enveloper than that they should touch at a point or at several isolated points.

If one considers three infinitely-close envelopers that do not belong to the same first envelope then the points that are common to these three envelopers will be their points of contact with the envelope of the envelopes.

For example, take the particular case in which the enveloping surfaces are spheres whose center moves on an arbitrary mirror. If one considers three infinitely-close spheres whose centers do not lie along the same line then there will be two common points to those three spheres, which will produce two disjoint sheets on the enveloping surface. The first sheet will be in front of the mirror, and the second one will be behind it. The two points that are thus found – one on each sheet – will be placed symmetrically with respect to the plane that is drawn through the centers of the three spheres. Consequently, the rays that are drawn through the center of one of these three spheres to one or the other of these two points will be in a plane that is perpendicular to the plane of the three centers. Moreover, these two rays define the same angle with that plane. Finally, since each of them is normal to the sphere that they belong to, they will, analogously, be normal to the general envelope that touches that sphere at the end of that ray.

Since the three centers are taken infinitely close to each other on the mirror, and not on a straight line, the plane that contains them will be tangent to the mirror. However, of the two rays that we considered, the one that is found in front of the mirror is, by hypothesis, an incident light ray. Therefore, the one that is found on the other side of the mirror is the prolongation of the reflected ray. Therefore, it finally suffices that the incident rays should be normal to an arbitrary surface (Σ) for which the reflected rays will be analogously normal to a surface for whatever form the mirror should take.

Elastic bodies have the property that when they strike an immovable body (elastic or rigid and bounded by an arbitrary surface) they are reflected in such a manner that the angle of reflection is equal to the angle of incidence and that the two paths will be in a plane that is normal to that of the rigid body at the point where the reflection takes place. Therefore, if one supposes that an infinitude of elastic molecules start from an arbitrary point and follow arbitrary rectilinear directions then when those molecules strike the surface of the immovable body they will reflect in such a manner that their paths will define a double sheaf of developable surfaces that cross at a right angle. Finally, the property that these paths will be normals to a unique surface will be preserved no matter how many times the elastic molecules experience successive reflections.

If one makes a sonorous point vibrate then the disturbance that it experiences will be communicated step-by-step in the atmosphere with a velocity and intensity that depends upon the density of the atmosphere and the distance from the sonorous point to the point where the observer is placed. Now, suppose that the atmosphere has the same density in the extent where sound rays that are not too extensive can propagate. Imagine that a sequence of spheres has the original sonorous point for their common center. The propagated sounds will diminish as they grow distant from their origin. However, they will have the same intensity at all points of the same sphere. Finally, the radii of those spheres will obviously be the shortest-possible lines that one can follow in order to pass from the original sonorous point to the points where the sound no longer has a given intensity. These rays are what one calls *sound rays.*

If the propagation of sound is presented with a limiting surface that does not permit the sound to extend beyond the space that is it defined in then it will be reflected, not in such a manner that it extends in all possible directions indifferently, but in such a manner that:

1. The reflected sound rays are in a plane that is normal to the reflecting surface with the incident sound ray.

2. The two rays define the same angle with the plane that is drawn tangentially to that surface through the point where the reflection takes place.

From this, it is obvious that the reflected sound rays define a sheaf of lines that can all be normals to the same surface. If one determines the two surfaces that are the locus of centers of the largest and smallest curvature of that surface, respectively, then they will be the locus of successive intersection of the reflected sound rays. They will then be the locus of echoes of the sonorous point with respect to the reflecting surface that produces the echo.

In general, when a sound emanates from a unique point and is reverberated by a sequence of arbitrary surfaces, the locus of echoes will be, for each repercussion, the system of two surfaces that are the loci of centers of curvature of a third surface that is perpendicular to the reflected sound rays.

Sometimes, the two surfaces that are loci of centers of curvatures can be situated at the same time in the part of space that is filled with the atmosphere; there will then be two series of real echoes. Sometimes, just one of those surfaces is located in that way, while the other one is found in the space that is intercepted by the reflecting surface. Only the first surface will then be the locus of real echoes, and the other echoes will be imaginary. Finally, the two surfaces can be in the space that is intercepted by the reflecting surface; all of the echoes will be imaginary then.

When one of the surfaces that is a locus of echoes reduces to a line, the echoes will acquire incomparably more intensity. They will then acquire even more when that surface reduces to a point and above all, when the two surfaces reduce to the same point.

We now return to the principal object of this memoir. The comparison of spheres that helped us to arrive at the proof of the general theorem that was stated above on pp. 4 can also make some remarkable properties of sheaves of reflected light rays known to us.

Suppose that the radius of the moving sphere becomes equal to zero. The curve that is traversed by the center of that sphere will then coincide with the enveloping surface of the space that is traversed by the sphere itself. Therefore, not only is that curve situated on the mirror where it must constantly remain the center of the sphere, but the space itself that is traversed by one and the other envelope of the moving sphere will reduce to a point.

Consequently, if one determines, on the one hand, all of the surfaces that have the incident rays for their normals and on the other hand, all of the surfaces that have the reflected rays for normals then the surfaces of the different systems will intersect pair-wise along a curve that will be located completely on the mirror.

Since each of these curves is, at the same time, on one surface whose incident rays are just as many normals and on another surface whose reflected rays are, analogously, just as many normals, we first conclude that the curve itself has all of the incident or reflected rays that end on it for its normals.

If one draws a plane through the incident ray and the reflected ray that cross at a point of that curve then, as we know, it will be normal to that point on the mirror; moreover, it will necessarily be normal to that curve.

Therefore, if one traces a new system of curves on the mirror that are the orthogonal trajectories to the first curves at each point of them then the two incident and reflected rays will project onto the mirror tangential to the trajectory that passes through that same point.

That tells us that if the incident rays, for example, that end on each of the first curves form just as many developable surfaces then the first curves will be the lines of curvature of those developables. However, the sheaf of incident rays is decomposable into two series of developable surfaces that cross at a right angle. Therefore, the surfaces of the other series will obviously pass through the orthogonal trajectories of the first curves. Meanwhile, those trajectories are not necessarily lines of curvature of the second developable surfaces; in order for that to be true, it would be necessary that they should be the lines of curvature of the mirror itself.

§ III.

Properties of cyclide surfaces, as well as second-degree curves and surfaces.

Before we can examine the properties that are enjoyed by light or sound rays that are reflected from a surface later on, it is necessary to discuss some general principles of geometry that were developed for the first time in our "Mémoire sur le contact des cercles et des spheres" (*).

We shall first prove that there exists a family of surfaces whose characteristic property is that they have only circles for the lines of greatest and least curvature; that is why we call those surfaces *cyclides*.

We first remark that the sphere is included in that family of surfaces, since two systems of circles that are traced on it at right angles to each other can be considered to be its lines of curvature.

Surfaces of revolution, whether conical or cylindrical, are also part of the family that we would like to study. Indeed, their lines of curvature are, on the one hand, parallel circles and on the other hand, straight-line meridians, which one can regard as circles that have an infinite radius. It is, moreover, obvious that no developable surface other than the cylinder or the cone will have circles for its lines of curvature, because each point of

(*) The analysis of the results of this paper is found in the *Correspondence polytechnique*; the paper itself was not printed.

the edge of regression of that developable must be a point of regression for one of the lines of curvature, and the circle has no point of regression.

In order for the circle to be a line of curvature on an arbitrary surface, it is first of all necessary that the lines that are drawn normal to that surface and each point of such a circle must form a developable surface. It will then, in turn, be necessary for that developable surface to have that circle for a line of curvature. Consequently, that developable surface must be a right circular cone that has the circle for its base.

Therefore, the surfaces whose lines of curvature are all circles will have the characteristic property of being cut normally by a right circular cone along the entire extent of each circle.

Take the summit of each of those cones to be the center of a sphere on which that circle is placed. Since the sphere has the same normals as the desired surface along the entire extent of that circle, it will have the same tangent planes.

Consequently, the general surface whose lines of curvature are all circles can be generated in two different ways by the motion of a sphere whose radius varies conveniently. The first manner of generation will give the lines of one curvature of the cyclide surface, while the second one will give the lines of the other curvature.

That is why a cone of revolution, for example, can be generated: *first of all*, by a sphere whose center moves along a straight line, while it radius increases or decreases in proportion to the distance that is traversed by its center; the lines of curvature that are produced by that manner of generation will be circles. *Secondly,* the same cone of revolution can be generated by a sphere of infinite radius; i.e., by a plane that constantly defines the same angle with the axis of the cone. The lines of curvature that are produced by this second manner of generation will be the meridian lines that serve as the edge of the cone.

We return to the general case. It is obvious that each sphere of the first manner of generation must be tangent to all of those of the second, since each line of one curvature of the cyclide must be cut by all of those of the other curvature, and each line of that second curvature will belong to a sphere of the second manner of generation.

However, if one is content to take three spheres of the first manner of generation (*) then that will suffice to determine all spheres of the second one. It is then necessary that if one takes the first spheres three at a time and determines all of the second ones from that simple given then the second spheres will constantly be the same ones. Consequently, it is the possibility or impossibility of that identity alone that can show whether there do or do not exist other surfaces than the sphere, the cone, and the cylinder of revolution that have only circles for their lines of curvature.

We have seen that two spheres of a different manner of generation will touch at a point that is located on the cyclide surface. When a sphere of the second manner of generation is determined by the condition that it must be tangent to three of the first spheres, one will have three points of contact. Those three points will belong to the cyclide; they will be located on the same circle – namely, a line of curvature. Consequently, they will determine that circle completely.

(*) In general, there are only a finite number of spheres (16) that can be tangent to four given spheres at the same time. One needs only to take three fixed spheres in order to get the infinitude of moving spheres whose envelope is the desired surface.

Draw a plane through those three points of contact. It will cut the generating sphere along the circle that is the line of curvature of which we speak. It will cut each of the three fixed spheres along a particular circle.

We regard each of the four circles thus-determined as the base of a right cone that is tangent to the sphere on which that circle is located. In order to get the axes of those four cones, one must draw a perpendicular through the center of each sphere to the common plane of the four bases. The four axes will then be mutually parallel.

At the point of contact of the first sphere with each of the other three, two of the four cones will be, analogously, in contact. Therefore, the cone that is circumscribed by the first sphere will be touched by each of the other three cones at a different point.

Now, two cones of revolution that have parallel axes and touch at a point will be necessarily similar. Indeed, the same plane that is tangent to these two cones will make the same angle with the two axes.

Therefore, the four cones that we consider are mutually similar. Consequently, the spherical segments that they circumscribe will have volumes that are proportional to those of the spheres that they belong to. Therefore, the bases of those segments, which are also the bases of the circumscribed cones, will be separated from the center of the sphere that they belong to by quantities that are proportional to the radii of those same sphere, respectively.

Draw a line in the plane of three fixed spheres whose distances to their centers are proportional to the radii of those spheres. Any plane that is drawn through that line will cut out segments on it that are proportional to their volumes, and no other line can enjoy that property.

Thus, the three points of contact of the generating sphere with the three fixed spheres will determine a particular plane that always passes through the unique line that we speak of, and which we call the *director line.*

All of the tangents to the circles of curvature that are traced on the generating spheres that are cut by those planes will obviously pass through the same line as all of those planes in which they are found to be located, respectively. Among those various tangents, the ones that belong to the various points of a line of second curvature will form a developable surface by that fact already.

However, a developable surface whose rectilinear edges must all pass through a *director line* can be only a plane or a cone. If it is a plane then that plane that passes through the director line will be one of the ones that contain the lines of first curvature. The lines of second curvature, far from cutting the first ones at a right angle, will coincide with them everywhere. It follows from this that the tangents to the lines of first curvature that are drawn through each of the points of a line of second curvature will define a cone.

Now, consider the sequence of points of contact of the generating spheres with one of the fixed spheres. These points will necessarily form a second line of curvature for the envelope of the generating sphere, since the common normals to the envelope and each fixed sphere will form a conic developable surface.

Hence, the line of second curvature that is traced on each of the three fixed spheres will have lines that form a cone for its normals. We prove that this line must be planar and circular.

Since each of those two circles of second curvature are determined on a fixed sphere, those two circles will suffice to determine the position of each generating sphere, and as a result, the enveloping surface of all those generating spheres.

It is obvious that the three fixed spheres are symmetric with respect to the plane that is drawn through their three centers. Consequently:

1. The generating spheres are arranged symmetrically above and below that plane.

2. Any sphere that has its center on that plane and touches two of the generating spheres above the same plane will, analogously, touch two other ones below it.

We shall determine a new fixed sphere that touches four of the generating spheres that were determined previously. We can replace one of the three old spheres with that new one. The other two, which preserve four points of contact with four generating spheres (which are assumed to stay the same) will each have the same circle of contact with the envelope as well as with all the generating spheres. Consequently, that enveloping surface will not change under the substitution of the new fixed sphere for one of the old three. Therefore, we finally have that the same enveloping surface can be produced by an infinitude of new spheres, each of which are tangent to all of the generating spheres, and each of which will produce a circle for the line of second curvature of that envelope.

Therefore, it is first of all possible to find other surfaces than the sphere, the cylinder, and the cone of revolution that have only circles for lines of curvature. Secondly, in order to define those cyclide surfaces, it will suffice to demand that a sphere of variable radius should constantly touch three fixed, but arbitrary, spheres. Thirdly, there is no other possible manner of generation that will produce other surfaces of the same family.

The lines that are loci of the centers of curvature of the cyclide surfaces have some remarkable properties: We shall discuss the ones that can be applied to the subject that we are treating.

We have seen that the centers of the fixed spheres are all situated in the same plane. We could have taken the generators to be the spheres that we have called fixed, and conversely, take the fixed spheres to be the ones that we have regarded as generators. Therefore, the spheres of each manner of generation will have their centers located in a unique particular plane. It is easy to see that these two planes must intersect at a right angle.

The cyclide surface is, as we have seen, symmetric with respect to the first of those planes; the same thing is true with respect to the second one. Now, when a surface is symmetric with respect to two planes, it will be necessary that these planes must intersect at a right angle (*).

In order to distinguish the two manner of generating the cyclide surface by successive intersections of enveloping spheres, we call the spheres of one manner of generation *first spheres* and those of the other manner of generation, *second spheres.*

Draw all possible normals to the cyclide surface from the center of one of the first spheres. Those normals, which start from the same point, will form a cone. They will end on the cyclide at various points of a circle of curvature that will serve as the basis for that cone, which will be right and circular. Finally, the axis of that cone will be tangent

(*) At least, when the surface is one of revolution and has the intersection of the two given planes for its axis.

(**) to the curve that is the locus of centers of the first spheres. However, each normal edge of that cone will pass through the center of one of the second spheres. Therefore, the entire system of edges of the cone will pass through the curve that is the locus of the centers of the second spheres. Now, we have proved that this curve is planar. Consequently, the curve that is the locus of centers of second curvature of the cyclide surface will be a plane section of the cone – i.e., a second-degree curve. It is obvious that the planar curve that is the locus of centers of first curvature will, analogously, have degree two.

If we compare all of the cones that one can define by lines that are drawn from each center of the first spheres to the various centers of the second spheres, while supposing, moreover, that each line that is terminated by the two centers that it connects, then they will consist of:

1. A radius of the first sphere.
2. A radius of the second sphere.

Upon terminating all the edges of the same cone in that way, one will see that they have in common the radius of the first sphere whose center serves as the summit of the cone. Consequently, the difference between these edges, when taken pair-wise, will be equal to the difference between the radii of the second spheres that correspond to those edges.

Consider two of those cones. Trace two edges on each of them that begin at the centers of those two spheres of the second manner of generation, respectively. The difference between the two edges of the first cone and the difference between the two edges of the second cone will be equal to the difference of the radii of the two spheres of the second manner of generation, respectively.

The following theorem results from this:

For all of the cones whose summits are located on the first curve of centers and whose common base is the second curve of centers, I say that the difference between the two edges that begin at two given points of the base, respectively, will be the same.

One can then regard the edges of each of them as being defined by the edges of only one of them by lengthening or shortening all of the edges of an initial cone by the same quantity.

The property that is enjoyed by the second curve of centers gives us a means of describing that curve by a very simple continuous motion.

One attaches a string to each point of the first curve. One lets all of those strings become united at any point of the second curve. One then lengthens or shortens all of the strings by the same quantity. While always keeping three of those strings tense, which will suffice to determine the position of the point where they are united:

1. All of the other strings will be equally tense.
2. All of the strings will form a right circular cone whose center describes the second curve of centers.
3. Finally, the axis of that cone will always be tangent to that curve.

(**) The latter property is general for the surfaces that are generated by the successive intersections of spheres of variable radius in an arbitrary manner.

One can then regard those strings as radius vectors and the points of the first curve of centers as foci with respect to the second curve.

Since all of the properties of these two curves are reciprocal, one must also conclude that all points of the second curve can be considered to be foci of the first one.

Each of these curves is symmetric with respect to the plane of the other one. It follows from this that both of them must have an axis that is located on the common intersection of those two planes. The summits of the first curve that are located on that axis will be the ordinary foci of the second curve, and conversely. That general property of second-degree curves results from this.

Imagine a first curve of degree two in a horizontal plane and then a new line of the same degree that is drawn in a vertical plane through the major axis of that curve and has the foci of the first curve for its summits and the summits of that same curve for its foci.

> "All points of the first one will be foci of the second one, and all points of the second one will be foci of the first one. All of the radius vectors that are directed from the same point of one of those curves to all points of the other one will form a right circular cone whose axis is tangent to the first curve at a point that serves as the origin of all the radii. Hence, each curve will define the same angle with all the radius vectors that intersect at one of its points."

One sees how these properties generalize the ones that are known about the radius vectors that emanate from just two foci.

Up to now, we have supposed that all of the radius vectors are located on the same sheet of each cone, and we have proved that their difference would then be constant. That must be true for the generation of the hyperbola. Indeed, for the hyperbola, the foci are outside of the summits. However, in order to generate the second-degree curve that is the locus of all foci of that hyperbola, one must change the foci into summits, and conversely, the summits into foci. Therefore, the second curve will have its summits outside of its foci. It will consequently be an ellipse, and that ellipse will be on the sheet of the right circular cone that has it for its oblique base.

On the contrary, we would like to generate the ellipse by means of the hyperbola. We will see that all of the radius vectors that start at two points of the same branch of the hyperbola will have a constant *difference*, while all of the radii that start at two points that are taken on different branches will have a constant *sum*, taken pair-wise. That is obviously what is true for the two summits of the ellipse that serve as the foci of the hyperbola.

In the case where one of the curves becomes parabolic, the other one will be parabolic, analogously. One of the parabolas has its branches turned towards the right and the other one, towards the left. It is then the difference between the radius vectors that must be constant.

§ IV.

Application of the properties that were discussed in § III *to the search for new properties of the path of light under the phenomena of reflection.*

When a second-degree surface is formed by revolving a curve of the same order around the axis on which its foci are located, those points will be, at the same time, the foci of the entire surface; we call them *general foci*.

If we make an arbitrary plane section of that surface then it will be obvious that the section will include the two general foci of the surface among its particular foci. Therefore, the cones that have that section for their base and one or the other general focus for their summit will be right and circular.

The following general property of second-degree surfaces that have either ellipsoids, paraboloids, or hyperboloids for general foci results from that: All of the planar curves that one can possibly trace on those surfaces will appear to be circles when they are regarded from one of the foci. Therefore, when one takes ones perspective table to be the surface of a sphere that has its center at one of the foci of the second-degree surface, all of the plane sections that are traced on it, in perspective on the spherical surface, will form circles.

Now consider the infinitely-small plane section that serves as the indicatrix of the curvature of the surface at a point. We first see that for the surfaces whose properties we are studying, when all of the indicatrices are viewed from one of the general foci, they will seem to be circles that have the visual ray that goes from a general focus to the point P of the surface whose indicatrix gives the curvature for their axis.

Suppose that all of the elements of that surface grow proportionally and with an infinite ratio, while starting from the point P. The indicatrix will then take on a finite extent, and the summit of the cone of revolution whose oblique base it is will be stretched to infinity. The cone will become a cylinder of revolution.

When the indicatrix (which is the oblique base of the cylinder) is projected onto the circle (which is the direct base of that cylinder), the diameters of the indicatrix will project onto the diameters of the circle. Any parallelogram will circumscribe the indicatrix (which is a second-degree curve), and the two conjugate diameters that are parallel to the opposite sides of that parallelogram, respectively, will project onto the circle with ceasing to be parallel. The parallelogram will not cease to circumscribe the circle and will become a square (*quarré*). Finally, the two conjugate diameters that are parallel to the sides of that square, respectively, will cross at a right angle in that projection.

Two arbitrary conjugate diameters of an indicatrix of a second-degree surface with general foci are then in two planes that pass through the same general focus and cut at right angles.

While discussing the theory of indicatrices and conjugate tangents of surfaces (*Développements de géométrie*, *First memoir*), we proved that those conjugate tangents are always located on two conjugate diameters of the indicatrix. Therefore, the second-degree surfaces that have general foci enjoy the remarkable characteristic property: *Two arbitrary conjugate tangents are on two planes that pass through the same general focus, respectively, and they cut at a right angle.*

One sees from this that when one views a second-degree surface from one of its general foci, its conjugate tangents will seem to cut at a right angle everywhere.

Now, let us extend those properties of second-degree surfaces of revolution to the surfaces of an arbitrary form. For that, we can appeal to the theory of *Indicatrices* and *Conjugate tangents.* As one sees, that theory casts a very remarkable light on the question; it makes it easy to find some solutions that were inaccessible to simple geometry up to now.

Let P be the point that one considers on an arbitrary surface (S). Take the plane of horizontal projection to be the tangent plane to that surface at P. Moreover, let PO and $P\Omega$ be the projections of two incident and reflected rays [that are reflected by (S)].

The two rays are in a plane that is perpendicular to the tangent plane P that is taken to be the plane of horizontal projection; consequently, they will project horizontally along the same line $OP\Omega_h$.

Now, regard the points O and Ω as the foci of a second-degree surface of revolution that passes through P. That surface will be tangent to the general surface (S) at that point, since, by hypothesis, the tangent plane to (S) at P makes the same angle with the two radius vectors that project onto OP_v and $P\Omega_v$, and it will be perpendicular to the plane of those two rays.

One determines, here and now, the indicatrix of the second-degree surface for the point P by the condition that the indicatrix must be the trace of a right circular cylinder whose axis is PO or $P\Omega$ on the horizontal plane of projection.

Indeed, if we take a vertical plane of projection that is parallel to the rays PO, $P\Omega$ then those rays will be equal to their vertical projections PO_v, $P\Omega_v$, respectively. Since the major axis of the indicatrix is equal to $\alpha P\gamma_h$, which has an arbitrary magnitude, its projection $\alpha P\gamma_v$ will be equal to $\alpha P\gamma_h$, and the minor axis $\beta P\delta_h$ will be twice the distance from the point α_v to the ray PO_v .

Now, if the focus O remains the same, as well as the point P on the surface (S), then one can arbitrarily vary the magnitude of the indicatrix $\alpha\beta\gamma\delta$ of the second-degree surface whose focus Ω will become increasingly distant or approach the point P conveniently. We demand that among these similar curves, $\alpha\beta\gamma\delta$, $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$, …, they are the ones that are tangent to the indicatrix $ABCD$ of the surface (S).

In order to do that, regard $\alpha\beta\gamma\delta$ and $ABCD$ as the horizontal traces of the two cylinders that have PO for their axis. By hypothesis, the first cylinder will one of revolution, so all of its diametral planes will be normal to its surface. Therefore, in order for that cylinder to be tangent to the one whose trace is $ABCD$, it is necessary that the diametral plane that is drawn through the edge that is the locus of contact should also be normal to the cylinder whose base is $ABCD$ along the entire extent of that edge.

When a second-degree cylinder is not one of revolution, there are only two planes that pass through its axis that enjoy that property: They are the two principal planes, and those two planes will necessarily intersect at a right angle.

However, we know that the conjugate diameters of the various plane sections of a cylinder are placed on the conjugate diametral planes of that cylinder, respectively. Therefore, the two principal planes of the cylinder that has $ABCD$ for its oblique base will trace out two conjugate diameters $M'N'$, $M''N''$ on $ABCD$. The two curves $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$, which are similar to $\alpha\beta\gamma\delta$ and tangent to $ABCD$ at MN and MN, respectively, will pass through the extremities of those diameters. (*See* note II at the end of the memoir.)

It is easy to see that the second-degree surface whose indicatrix is $\alpha'\beta'\gamma'\delta'$ has the three points M', P, N' in common with the surface (S), so the tangents to the arc $M'PN'$ will be the same at M', N' for the two surfaces. However, the indicatrices $ABCD$, $\alpha'\beta'\gamma'\delta'$ at the points M and N, resp., will be mutually tangent. Therefore, the same tangents $M'm'$, $N'n'$ are common to the two indicatrices, and consequently, to the two surfaces. Therefore, the tangent planes to those surfaces at either M' or N' will be identical.

Consequently, if we draw an incident ray from the point O to the point M' or N' then the reflected ray will be the same for the original surface and for the second-degree surface of revolution that has $\alpha'\beta'\gamma'\delta'$ for its indicatrix. However, since O is one of the general foci of the latter surface, all of the rays that emanate from the point O will be reflected to the other focus Ω'. Therefore, the rays that are reflected by the arbitrary surface (S) at the points M' and N' will combine at the point Ω', which is the second general focus of the second-degree surfaces whose indicatrix is $\alpha'\beta'\gamma'\delta'$.

Similarly, the rays that are reflected by the arbitrary surface (S) at the points M'' and N'' will meet at the focus Ω' of the second second-degree surface whose indicatrix is $\alpha''\beta''\gamma''\delta''$.

Since there are no other points on $ABCD$ besides M', N' and M'', N'' for which the tangent plane to (S) coincides with the tangent plane to one of the second-degree surfaces of revolution that have O for a focus, there will be no other incident rays than OM', ON', OM'', ON'' that will meet $P\Omega$ when they are reflected.

We can then assert the general principles:

1. When one starts from a point P of an arbitrary surface, there will be two directions $M'N'$, $M''N''$ such that the rays that emanate from the arbitrary point O and are reflected by the surface at M', N', M'', or N'', which are infinitely-close to P, meet the ray $P\Omega$ that is reflected by the point P.

2. The two directions $M'N'$, $M''N''$ are those of the two *conjugate diameters* of the indicatrix $ABCD$, and consequently, they are also those of the two *conjugate tangents* to the surface (S) at the point P.

3. Those two directions are situated on two planes that intersect at a right angle along the incident ray, since they are the directions of the two principal planes of the cylinder that has $ABCD$ for its base and OP for its axis.

4. The planes that are drawn through those two directions and the reflected ray $P\Omega$ define the same angle between them as the preceding two did, and analogously intersect at a right angle.

In order to prove the last property, we imagine three new cylinders that have the reflected ray $P\Omega$ for their common axis and $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$, $ABCD$, resp. for their bases. The first two cylinders are once more cylinders of revolution. They touch the third one along four edges that pass through the points M', N', M'', N'', respectively, and which are located pair-wise on the principal planes of the last cylinder. Therefore, the two planes that are drawn through the reflected rays $\Omega'M'$, $\Omega'P'$, $\Omega'N'$, on the one hand, and $\Omega''M''$, $\Omega''P''$, $\Omega''N''$, on the other, will intersect at a right angle along ΩP.

The direction of the incident ray OP will remain the same, no matter what the position of the focus O is along that ray, so the form of the indicatrix $\alpha\beta\gamma\delta$ will not vary. That is why the two conjugate directions $M'N'$, $M''N''$ will not vary. Suppose that as many incident rays as one desires start from the various points of OP. The reflected rays can meet $P\Omega$ only by passing through the points M', N' or M'', N'' that are located on two conjugate diameters of $ABCD$, which is the indicatrix of the original surface (S), and consequently, they will be located on two conjugate tangents to that original surface at the point P.

We now shall show what modifications the general principles that we just presented can experience according to the various forms that the indicatrices of the mirror and those of the auxiliary second-degree surfaces can present.

The second-degree auxiliary surfaces – simply because they have two general foci – will have their two curvatures directed in the *same* sense everywhere. Therefore, the indicatrix of their curvature will be everywhere *elliptic*.

However, since the mirror is supposed to have an arbitrary form, its indicatrix can be elliptic, hyperbolic, or parabolic. We examine these three cases, in turn, which comprise all of the general forms that the curvature of surfaces can be affected with.

In the first case, where the indicatrix is an ellipse, as is represented in Fig. 1 [†], the cylinder that has the incident ray for its axis and that ellipse for its horizontal trace will be an elliptic cylinder. There thus exist two principal planes that pass through that axis and four principal edges that are located on those places, respectively. Those edges have the points M', N' and M'', N'' for their horizontal traces on the plane of projection. As we have seen already, we can always find two indicatrices $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$ that are tangent to the indicatrix $ABCD$ of the mirror at M', N' and M'', N'', respectively.

We demand to know how to determine the position of the points Ω', Ω'' where the reflected rays $M'\Omega'$, $N'\Omega'$ and $M''\Omega''$, $N''\Omega''$, resp., meet the ray $P\Omega$ that is reflected at the point P. Since the indicatrices $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$ are determined by the means that we have presented, their major axes $\alpha'P\gamma'$, $\alpha''P\gamma''$ will be determined analogously. Let R be the radius of curvature of the normal section of the mirror that is made in the direction of that axis. Let r' and r'' be the radii of curvature of the sections that are made by that plane in the two auxiliary second-degree surfaces that have $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$ for their indicatrices, respectively. We have:

$$r' = R \cdot \frac{\alpha'P^2}{AP^2}; \qquad r'' = R \cdot \frac{\alpha''P^2}{AP^2}.$$

Now, with r' as its radius, trace the osculating circle of the normal section that is made along $\alpha P\gamma$ in the auxiliary surface that has $\alpha'\beta'\gamma'\delta'$ for its indicatrix, as in Fig. 2. We can replace that section with that circle without changing the point Ω that is the locus of points where the rays that are reflected from the infinitely-close points P, M' meet.

[†] Translator: The cited figures were not available at the time of translation.

However, the chords *Pp* and *Pr*, *Mm* and *Ms* in the osculating circle are equal, respectively, since they define the same angle with the circumference of that circle pairwise. Therefore, one has the following relations between the arcs that are subtended by the chords:

$$\overline{Pp} = \overline{Pr}, \qquad \overline{Mm} = \overline{Ms}.$$

Hence:

$$\overline{Mm} - \overline{Pp} = \overline{Ms} - \overline{Pr},$$

and upon suppressing the part *Pp* that is common to the first two arcs, as well as the part *Mr* that is common to the second two:

$$\overline{PM} + \overline{pm} = \overline{rs} - \overline{PM};$$

therefore:

$$\overline{rs} = 2\cdot\overline{PM} + \overline{pm}.$$

However, the triangles $\Omega'PM$, $\Omega''rs$ are similar. The angle $P\Omega'M$ is infinitely small, and therefore the arcs $\overline{PM}$, $\overline{pm}$, $\overline{rs}$ are equal to their chords *PM*, *pm*, *rs*, respectively. Therefore, one has:

$$P\Omega'Pr :: PM : PM + rs = PM + 2PM + pm = 3PM + pm,$$

so

$$P\Omega' = \frac{PM\cdot Pr}{3PM + pm}.$$

The similar triangles *OPM*, *Opm* likewise give:

$$PO : Pp :: PM : PM - pm;$$

hence:

$$PM - pm = \frac{PM\cdot Pp}{PO}.$$

However:

$$P\Omega' = \frac{PM\cdot Pr}{3PM + pm} = \frac{PM\cdot Pp}{4PM - (PM - pm)};$$

therefore:

$$P\Omega' = \frac{PM\cdot Pp}{4PM - \dfrac{PM\cdot Pp}{PO}} = \frac{PO\cdot Pp}{4PO - Pp}.$$

Let α denote the angle that the incident ray *PO* makes with the plane tangent to the mirror at *P*. Twice the radius r', multiplied by sine of that angle will be the chord *Pp* of the osculating circle. Let Δ denote the distance from the point *O* to the point *P*, so we will have:

$$P\Omega' = \frac{\Delta\cdot 2r'\cdot\sin\alpha}{4\Delta - 2r'\sin\alpha} = \frac{\Delta\cdot r'\cdot\sin\alpha}{2\Delta - r'\sin\alpha}$$

for the value of $P\Omega'$. (*See* Note III.)

For the same values of r' and sin α, that quantity can be infinite, positive, or negative, according to the value of Δ; i.e., according to the position of the point *O* on the incident ray, whose direction is assumed to be constant.

1. When $2\Delta - r' \sin\alpha = 0$, the reflected ray $M\Omega'$ will become parallel to the ray $P\Omega'$, since Ω', as the point of intersection of two rays, will be transported to infinity.

Therefore, when the radius r' projects onto the incident ray, with a projection $r' \sin\alpha = 2\Delta$ = twice *PO*, the auxiliary surface will have $\alpha'\beta'\gamma'\delta'$ for its indicatrix with its reflected rays mutually parallel. It will be a paraboloid, and along the direction $M'PN'$, the light rays that emanate from the point *O* and reflect from the mirror at the point *P* will be parallel.

When Pp', which is one-half of the chord *Pp*, is smaller or larger than $2PO = 2\Delta$, the rays that are reflected by the auxiliary surface that has $\alpha'\beta'\gamma'\delta'$ for its indicatrix will no longer be parallel. In the former case, 2Δ will be larger than r' sin α. Therefore, the value $P\Omega' = \dfrac{\Delta r'\sin\alpha}{2\Delta - r'\sin\alpha}$ will have the same sign as Δ (since r' is assumed to be positive, as well as sin α). The point Ω will then be on the same side of the mirror as the point *P*. Consequently, the second-degree auxiliary surface that has two points *O*, Ω' that are located on the same side of the tangent plane for its general foci will be elliptic. In that case, the image Ω' of the focus *O* will be on the same side of the mirror as that point *O*.

When r' is positive, if 2Δ is found to be less than r' sin α then the quantity $P\Omega' = \dfrac{\Delta r'\sin\alpha}{2\Delta - r'\sin\alpha}$ will become negative, and the second focus Ω will pass to the other side of the tangent plane at *P*. In that case, the second-degree auxiliary surface will be hyperbolic.

When Δ becomes negative, $P\Omega' = \dfrac{-\Delta r'\sin\alpha}{-2\Delta - r'\sin\alpha} = \dfrac{\Delta r'\sin\alpha}{2\Delta + r'\sin\alpha}$ will necessarily become positive. Hence, when the point *O* passes along *PO*, when prolonged to the other side of the mirror, the point of intersection Ω' will stay on the same side. Therefore, the second-degree auxiliary surface will again be hyperbolic.

Among all of the cases that one can study, the one for which the indicatrix *ABCD* is elliptic is remarkable: That is the case in which that indicatrix is similar to the curve $\alpha\beta\gamma\delta$. That indicatrix will then coincide with one of the two curves $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$, instead of being touched by them, as in Fig. 1. In that case, all of the incident rays that fall on the curve $\alpha\beta\gamma\delta$ will reflect and pass through the focus Ω of the second-degree auxiliary surface whose indicatrix at *P* is $\alpha\beta\gamma\delta$. The infinitely-close rays to *PO* that emanate from the point *O* will all meet at the same point Ω of the reflected ray $P\Omega$ after reflection.

We pass on to the case in which the indicatrix of the mirror at *P* is a hyperbola $AM'B$, $CN'D$, as in Fig. 3, instead of an ellipse *ABCD*, as in Fig. 1. No matter what direction the incident ray *PO* has, that hyperbola can only be the trace of a hyperbolic cylinder that has *PO* for its axis. Of the two principal planes that pass through that axis, only one of them will cut that cylinder along two edges that have the point M', N' for their traces on the horizontal plane of projection. From this, it will seem that there is only one real direction $M'PN'$ along which the reflected ray $P\Omega$ is met by the infinitely-close reflected rays when the mirror has a hyperbola for its indicatrix.

However, the indicatrix at the point *P* is nothing but a section of the mirror that is infinitely close to the tangent plane at *P* and parallel to that plane. That section can be made above or below the tangent plane to *P* when the two curvatures are directed in contrary senses. The first indicatrix will then indicate the curvature that is convex downward, while the second one will indicate the curvature that is convex upward.

Since indicatrices are determined by only the ratio of their axis, and not their absolute magnitude, one of those axes will pass from real to imaginary, and the other one will simultaneously pass from imaginary to real without their ratio changing in the process. That is why their asymptotes will be the same. For elliptic indicatrices, the two axes will pass from real to imaginary together, while no longer exhibiting a real curve. Consequently, they must exhibit only curves of one and the same form.

The trace of the cylinder is the second hyperbolic indicatrix with two real principal edges. It distinguishes two points M'', N'' on the plane that is tangent to the mirror at *P*. Those points indicate the new direction $M''PN''$ along which the ray that emanates from the point *O* and reflects from the point *P* is cut (along Ω'') by the infinitely-close reflected rays. Since the two hyperbolas $AM'B$, $CN'D$, and $aM''b$, $cN''d$ are similar, the parallelograms that have their summits on the asymptotes that are common to those hyperbolas will be the same for one or the other of them. Consequently, those two curves will have systems of conjugate diameters that are superimposed exactly. However, each of the hyperbolas can have only one system of conjugate diameters in common with the similar ellipses $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$. That system will then be the same for both hyperbolas. Consequently, the diameters $M'PN'$, $M''PN''$ will be mutually conjugate.

It follows from this that the two directions $M'PN'$, $M''PN''$ represent a system of conjugate diameters with respect to the oblique base $\alpha'\beta'\gamma'\delta'$ of a right circular cylinder that has *PO* or $P\Omega$ for its axis. Consequently, those two directions are also seen to cut at a right angle at an arbitrary point of the incident ray *PO* or the reflected ray $P\Omega$.

Suppose that the surface of the mirror has one of its curvatures zero at *P*, instead of having both of them pointing in opposite directions; i.e., it is developable at that point. Its indicatrix at the same point will be the system of two lines $M'm$, $N'n$ that are parallel to the edge of the developable that passes through *P*. The problem will again have two solutions.

The first of them will be given by the curve $\alpha'\beta'\gamma'\delta'$ that is tangent to those parallel at M' and N'. The second of them will be given by the conjugate diameter to $M'PN'$; i.e., by $M''PN''$, which is the true diameter of the system of two parallel lines $M'm$, $N'n$ when they are considered to be the branches of the same second-degree line.

Up to now, we have studied the laws of reflection only by starting at a point *P* of the mirror and ones that are infinitely close to that point. We shall now pass from the reflected ray at that point *P* to one of the infinitely-close reflected rays that meet it. We then pass on (and always in the same direction) from the second reflected ray to a third one that is infinitely close to the second and also cuts it, and then from the third to a fourth, etc. We shall then define a surface that is obviously developable, since it will be the system of infinitely-close straight lines that each meet the one that precedes it and the one that follows it.

We define a second surface that is analogously developable by starting with the reflected ray at the point *P* and passing to a new infinitely-close ray that cuts the first one along the second direction (which is conjugate to the first one). From that new ray, one passes on to a third that analogously cuts it, and which will also be in the second direction, and then from the third ray to a fourth, etc.

We then conclude, first of all, that when an arbitrary surface is illuminated by a sheaf of light rays that emanate from the same point, they will reflect along a new sheaf whose rays form two systems of developable surfaces, in such a way that each reflected ray will be found, at the same time, on two developable surface that belong to each of those systems, respectively.

If one determines the trace of the developable surfaces of reflected rays on the reflecting surface or mirror then a first system of developables will produce a first series of curves that succeed them on the mirror while varying infinitely little in form and position. The second system of developables will, analogously, trace a second series of curves that succeed it on the mirror while varying only infinitely little in form and position.

The general law that links the developables of the first system to those of the second is that the first ones will be cut at a right angle by the second ones along the entire extent of various rays that are the loci of their common intersections.

The general law that links the curves that are traced on the mirror by these two systems of developable surfaces is that the first curves constantly cross the second one along directions that are given by the *conjugate directions* to the surface of the mirror. In a word: the second curves will be everywhere *conjugate* to the first ones.

If one regards the first curves as the bases for just as many cones that have their common summit at the point where the light emanates, and similarly regards the second curves as based upon a new series of conical surfaces that have the same summit as the preceding ones then all of the cones of the first series will cross the cones of the second series at a right angle.

A general property of surfaces that are illuminated by light rays will result from the theorems that we just presented:

If two reflecting surfaces touch along line that is the trace of a developable surface of reflected rays for one of them (while the incident rays are assumed to emanate from a unique point) then that same developable will be analogously a system of reflected ray for the second reflecting surface when it is substituted for the first one. All of the developable surfaces of reflected rays of a system that is different from the first developable will traverse it on one and the other mirror along curves that will be tangent to it at each point and at the conjugate tangents to the common curve to the two surfaces.

Instead of supposing that the rays emanate from one and the same fixed point, suppose that the point advances or retreats along the incident ray that passes through the point *P* of the mirror. We know that the ray that is reflected by the point *P* will not change. The conjugate tangents of the surface are in two planes at a right angle that both pass through that incident ray or the reflected ray, respectively, so I say that these those tangents will not change either. Therefore, the same planes will be tangent, on the one hand, to all cones of incident rays that have their summit on the incident rays that falls on *P*, and on the other hand, to all developable surfaces of rays that are reflected by the mirror where one finds the point *P*.

The following general property of light rays then results from this:

If other light rays emanate from the various points of a line that is regarded as an incident ray in arbitrary directions then:

1. *When those rays are reflected by an arbitrary mirror, they will form a series of developable surfaces.*

2. *All of the developables that one can possibly form in that way will be tangent to one or the other of the conjugate tangents that are seen to cut at a right angle to the ray that is common to all those developables.*

3. *All of the cones that are formed from the incident rays that pass through the traces of those developables on the mirror will analogously cross at a right angle and will be tangent to one or the other of the conjugate tangents to the system that we just determined.*

At the beginning of this memoir, we proved (and along two absolutely independent paths) that a sheaf of light rays that is decomposable into two series of developable surfaces that cross at a right angle will further preserve the property that it is decomposable into two series of developable surfaces such that all of the surfaces of one series will cut those of the other series at a right angle after it is reflected by an arbitrary mirror.

Suppose, moreover, that the developable surfaces of a first series of incident rays correspond to the developable surfaces of a first series of reflected rays in such a manner that those curves are each the common trace on the mirror of a developable surface of reflected rays and a developable surface of incident rays. I say that all of the developables of the one series enjoy the same property. Therefore, in the second series, as in the first, each developable surface of incident rays and a developable surface of reflected rays will have the same trace on the mirror. Moreover, two of those traces that belong to two different series of developable surfaces will always be touched by two conjugate tangents to the surface of the mirror at their common point of intersection.

In order to make the proof of that principle clearer, let (D_i') and (D_i'') represent the two developable surfaces of incident rays that pass through a point *P* of the mirror, and let (D_r') and (D_r'') represent the two developable surfaces of reflected rays that pass through the same point *P*. Finally, suppose that (D_i') and (D_r') are the two developables (from different systems) that have the same trace on the mirror.

Take two infinitely-close points *P*, M' on that trace. Since the two incident rays – one of which falls on *P*, while the other one falls on M' – are located on the developable (D_i'), they will meet at a certain point *O*. Similarly, since the two reflected rays at *P* and M' are located on the developable (D_r'), they will meet at a certain point Ω'.

If we regard the points O, Ω' as the two general foci of a second-degree auxiliary surface that passes through P then that surface will analogously pass through M'. It will have the same tangent plane as the mirror at those two points, since otherwise the two surfaces could not have the two incident rays OP, OM' and the two reflected rays $P\Omega'$ and $M'\Omega'$ in common. Therefore, the intersection of those two consecutive tangent planes will be common to the two surfaces. Consequently, that intersection and the line that is drawn through the two point P, M' will form a system of conjugate tangents for one and the other surface. (See *Dévelop. de Géom.*, First *Mémoire.*)

Now, two conjugate tangents to a second-degree surface with the general foci O, Ω' will be on two planes that both pass through the same focus – either O or O' – and they will cut and a right angle. Therefore, the two conjugate tangents to the mirror, one of which passes through the element PM', will also be on two planes that both pass through the incident rays OP or through the reflected ray $P\Omega'$ and cut at a right angle.

One of these planes that passes through the infinitely-close edges OP, OM' of the developable (D_i') is tangent to that developable. Therefore, the plane that passes through OP and it perpendicular to it will be tangent to the second developable (D_i''), since, by hypothesis, that second developable cuts the first one at a right angle all along the extent of the ray OP.

Therefore, the trace of the second developable (D_i'') in the mirror is touched by the conjugate tangent to the direction PM' of the first developable.

One likewise proves that the two planes at a right angle that pass through the reflected ray $P\Omega'$ and two conjugate tangents to the mirror at P, respectively, are tangents to the developables (D_r'), (D_r'), respectively, that pass through the reflected ray $P\Omega'$.

Therefore, the two developables (D_i') and (D_r') are touched by the same conjugate tangent to PM' on the mirror. Consequently, their traces on the mirror are also touched by the same conjugate tangent. Therefore, they themselves will mutually touch.

If the (D_i') have the same trace as the (D_r') on the entire extent of the mirror then since the (D_i'') and (D_r'') must have their traces mutually tangent everywhere, it is necessary that the latter traces must be identical. That consequence will complete the general theorem that we would like to prove.

The converse of that principle is likewise true. When a sheaf of incident rays decomposes into two series of developable surfaces that cross at a right angle and trace out two series of curves on the mirror whose directions are conjugate everywhere, the same curves will be the traces of developable surfaces that are composed of reflected rays, and those developable surfaces, like those of the incident rays, will again present two series such that the surfaces of one series are cut at a right angle by all of those of the other series.

Here is what will happen in the particular case where the incident rays that are infinitely close to the point P meet the incident ray that falls on P at the same point O, while those rays meet at another point Ω' that is analogously the same for all reflected rays after reflection:

I say that the indicatrix of the mirror at the umbilic P will be a circle when it is seen from an arbitrary point of the incident or reflected ray that that passes through P.

All of the developable surfaces of incident rays of the first series (D_i') pass through the incident ray that falls on the umbilical point *P*.

All of the developable surfaces of the reflected rays of a first series (D_r') pass through the ray that is reflected by the umbilical point.

Each of those two rays (viz., incident or reflected) represent just one surface of the second series $(D_i''), (D_r'')$. Upon starting from each ray, those developable surfaces will present the form of an infinitely-acute right circular cone that has one or the other rays for its axis, respectively, and the luminous point or its image for its summit. As a result, the developable surfaces $(D_i''), (D_r'')$ will get gradually larger and take on a form that depends upon the curvature of the mirror.

One then sees that on the mirror, the traces of the first series will all pass through the point *P* while those of the second series will be:

1. The point *P*.
2. (At an infinitely-small distance) the indicatrix of the curvature of the mirror at the point *P*.
3. Curves that are more or less different from that indicatrix, and whose figure depends upon the form of the mirror itself.

Therefore, the points of the mirror that provide only one point of intersection for the reflected rays after they emanate from just one focus will be the same thing with respect to a system of conjugate traces that are common to the incident or reflected developable surface that the *umbilics* are for the lines of curvature. Furthermore, the latter umbilics are only a particular case of the systems of conjugated traces. They are the ones for which the rays are perpendicular to the mirror, and for which it is not only necessary that the traces should be conjugate, but that they should again cut at a right angle (*).

One can subsume all of the umbilical points that are provided by reflection of light under the general term of *catoptric umbilics.*

If an arbitrary reflecting surface is illuminated by a luminous point *O* then we propose to determine the points *P* of that mirror around which the rays that are infinitely close to the ray that are reflected by each of those points *P* meet the same reflected ray at a unique point Ω.

Since the indicatrix of the mirror must appear to be a circle when one regards it from the focus *O* or the point Ω' for the umbilical point *P*, it is, first and foremost, necessary that the major axis of that indicatrix must be in a plane that is normal to the mirror and passes through the point *O* from which the incident rays begin.

However, the major axis of the indicatrix of the mirror at *P* is tangent to the line of least curvature at that point. Therefore, the plane that is normal to the mirror at *P* and which passes through the major axis of the indicatrix, moreover, is tangent to the developable surface that is defined by the rays of least curvature of the mirror that pass through the point *P*.

Draw a plane through the point *O* that is required to touch the surface that is the locus of the centers of greatest curvature of the mirror. Analogously, that plane touches one of

(*) In the third and fifth memoirs that comprise the *Développements de géométrie*, we have presented the theory of the various kinds of umbilics of surfaces in considerable detail. That theory can find its application here.

the developable surfaces that are defined by the rays of least curvature of the mirror. That developable surface traces a line of least curvature on the mirror. Finally, the edge that is the locus of contact points of that developable with its tangent plane that passes through the point *O* distinguishes a point on that line of curvature where it is obviously touched by the plane that is both normal to the mirror and tangent to the that developable.

The sequence of points that is thus determined on each line of least curvature defines a line on which one necessarily finds the desired umbilic points.

At every umbilical point, the cylinder that has the indicatrix for its base and the incident ray for its axis will be a cylinder of revolution. When the major axis of the indicatrix is projected onto a plane that is perpendicular to the incident ray, it will be equal to the minor axis as a result. Therefore, the ratio of these two axes will be equal to the sine of the angle that is defined by the major axis with the incident ray: It is the cosine of the angle of incidence.

Therefore, the ratio of the squares of the axes of the indicatrix is equal to the square of that cosine. However, that ratio is the ratio of the radii of curvature of the mirror for the point to which the indicatrix belongs.

We conclude: Any catoptric umbilic of a mirror of arbitrary form enjoys the following properties:

1. *The normal plane that contains the major axis of the indicatrix belongs to the umbilic point that passes through the focus from which the incident rays emanate.*

2. *The ratio of the two radii of curvature of the mirror at that point is equal to the square of the cosine of the angle of incidence.*

Wit the help of the various methods that were presented in the course of this memoir, one can see that it becomes easy to reduce the most complicated questions of catoptrics to simple graphical operations of descriptive geometry. Moreover, that path has the advantage that it makes several general laws that the sheaves of light rays are subject to under the phenomena of reflection that pertain to mirrors of arbitrary form.

MAIN NOTES
ON THE FOURTH MEMOIR

FIRST NOTE

On the orthogonality of the intersections of developable surfaces that are defined by incident rays and reflected rays on a mirror of arbitrary form.

In this note, we shall translate the proof that was given in our memoir into analysis. In order to do that, let a, b, c represent the rectangular coordinates of the surface of the mirror, and let $c = \varphi(a, b)$ be the equation of that surface.

A sphere that has radius r and its center at a, b, c has the equation:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 = \overline{\pi(a,b)}^2 .$$

In this equation, π is an arbitrary function of a and b.

In order to get the equation of the enveloping surface of the space that is traversed by that sphere when its center passes through the various points of the mirror, one must first suppose that r and a, c, and then r and b, c are the only ones that vary in the equation of the sphere. It is then necessary to differentiate with respect to those quantities, which will give:

$$x - a + (z-c)\frac{dc}{da} = -r \cdot \frac{dr}{da},$$
$$y - b + (z-c)\frac{dc}{db} = -r \cdot \frac{dr}{db}.$$

These equations in x, y, z are those of two planes that are perpendicular to the tangent plane to the surface of the mirror at the point a¸ b, c.

The line of intersection of these two planes is then itself perpendicular to that tangent plane. Finally, it is on that line that one finds the point x, y, z of the envelope, which corresponds to the point a, b, c.

In order to calculate more rapidly, let:

$$\frac{dc}{da} = p, \qquad \frac{dc}{db} = q,$$
$$\frac{dr}{da} = m, \qquad \frac{dr}{db} = n,$$

with the notations that one receives.

If we now combine the three equations:

$$\begin{aligned}(x-a)^2 + (y-b)^2 + (z-c)^2 &= r^2,\\ (x-a) + \qquad\qquad + p\,(z-c) &= -r\cdot m,\\ (y-b) + q\,(z-c) &= -r\cdot n\end{aligned}$$

then it will be easy to deduce $x - a$, $y - b$, $z - c$ from them.

In order to do that, we first substitute the values of $x - a$ and $y - b$ that are deduced from the other two; that will give:

$$(rm + p\cdot\overline{z-c})^2 + (rm + q\cdot\overline{z-c})^2 + (z-c)^2 = r^2,$$

or rather, upon expanding that:

$$(z-c)^2\,(1+p^2+q^2) + 2r\,(z-c)\,(pm+qn) = r^2\,(1-m^2-n^2).$$

When that equation is solved for $(z - c)$, one will have:

$$z-c = -r\left[\frac{pm+qn}{1+p^2+q^2} \pm \sqrt{\frac{1-m^2-n^2}{1+p^2+q^2} + \left(\frac{pm+qn}{1+p^2+q^2}\right)^2}\,\right],$$

and upon reducing all of the terms to the same denominator:

$$z-c = -r\,\frac{pm+qn \pm \sqrt{(1+p^2+q^2)(1-m^2-n^2) + (pm+qn)^2}}{1+p^2+q^2}.$$

Let:

$$pm + qn = A, \quad (1+p^2+q^2)\,(1-m^2-n^2) + (pm+qn)^2 = B,$$

to abbreviate, so:

$$z-c = -r\cdot\frac{A\pm\sqrt{B}}{1+p^2+q^2}.$$

By means of the preceding equations, that will give:

$$\begin{aligned} x-a &= -rm - p\,(z-c) = -r\left[m - p\cdot\frac{A\pm\sqrt{B}}{1+p^2+q^2}\right],\\ y-b &= -rn - q\,(z-c) = -r\left[n - q\cdot\frac{A\pm\sqrt{B}}{1+p^2+q^2}\right].\end{aligned}$$

Since there are two values of $x - a$, $y - b$, $z - c$, they will give two points on each sphere for the enveloping surface. Consequently, that envelope must have two disjoint sheets.

In order to know the ratios of the position of those sheets with the surface of the mirror, we look for the distance from the two points *x*, *y*, *z* to the tangent plane to the mirror at *a*, *b*, *c*, which is a plane whose equation is given by:

$$dc = p\,da + q\,db,$$

namely:

$$Z - c = p\,(X - a) + q\,(Y - b).$$

That equation, when combined with these two:

$$X - a = -\,rm - p\,(Z - c),$$
$$Y - b = -\,rn - q\,(Z - c),$$

will give:

$$Z - c = -\,rmp - p^2\,(Z - c) - rnq - q^2\,(Z - c),$$

or

$$(Z - c)\,(1 + p^2 + q^2) = -\,r(mp + nq).$$

Therefore:

$$Z - c = -\,r \cdot \frac{mp+nq}{1+p^2+q^2} = \frac{-r \cdot A}{1+p^2+q^2}.$$

Hence:

$$X - a = -\,r\left[m - p \cdot \frac{A}{1+p^2+q^2}\right],$$
$$Y - b = -\,r\left[n - q \cdot \frac{A}{1+p^2+q^2}\right].$$

We now remark that these values are one-half the sum of the two values of $x - a$, $y - b$, $z - c$, respectively, that are found on the two sheets of the enveloping surface. We conclude that the two points of the envelope that are indicated by those double values are not only equidistant from the point *a*, *b*, *c*, but they are on a perpendicular to the tangent plane to the mirror and located at the same distance on that plane – viz., one in front of it and one behind it.

From that, it is obvious that the two radii of the spheres that is drawn through the center *a*, *b*, *c* to the double points *x*, *y*, *z* define the same angle with the tangent plane to the mirror, and are both in a plane that is normal to that tangent plane.

One can, moreover, verify this *a posteriori* by calculating the cosine of that angle. Indeed, that cosine is equal to:

$$\sqrt{\frac{(X-a)^2+(Y-b)^2+(Z-c)^2}{(x-a)^2+(y-b)^2+(z-c)^2}} = \frac{\sqrt{(X-a)^2+(Y-b)^2+(Z-c)^2}}{r}.$$

Consequently, it is equal to:

$$\sqrt{\left(m-\frac{pA}{1+p^2+q^2}\right)^2+\left(n-\frac{qA}{1+p^2+q^2}\right)^2+\frac{A^2}{1+p^2+q^2}}\;;$$

i.e:

$$\sqrt{m^2-n^2-2\cdot\frac{(mp+nq)A}{1+p^2+q^2}+\frac{(1+p^2+q^2)A^2}{(1+p^2+q^2)^2}}$$

or

$$\sqrt{\frac{(m^2+n^2)(1+p^2+q^2)-(mp+nq)^2}{1+p^2+q^2}}\;.$$

When that expression is simplified, it will finally give:

$$\sqrt{\frac{m^2+n^2+(mp-nq)^2}{1+p^2+q^2}}\;.$$

That quantity presents two values. One of them belongs to the angle that is defined by the radius of the sphere that is drawn through the point *a*, *b*, *c* to the first point *x*, *y*, *z* where that sphere touches the envelope. The other value belongs to the second point where the same sphere touches that envelope. Consequently, the two radii that are drawn from *a*, *b*, *c* to those two points will define the same angle with the tangent plane. Moreover, they are both in a plane that is perpendicular to the tangent plane, as we shall see in just a moment. Therefore:

If one of the two rays is considered to be incident then the other one will be the ray that is reflected by the point a, b, c on the surface of the mirror.

However, all of the incident rays are normal to the envelope of spheres. That envelope will have the greatest generality possible, moreover, since we have supposed that the function $r = \pi\,(a,\,b)$ is perfectly arbitrary; finally, the reflected rays are, analogously, normal to the second sheet of that envelope.

Therefore:

In general, if a sheaf of light rays that is normal to an arbitrary initial surface falls upon a mirror of an analogously arbitrary form then those rays will be reflected in such a manner that they form a new sheaf that will generally be normal to the same surface as the incident rays.

Often, the double values that we have found for the coordinates of the envelope of the spheres will lose their radical form, since the quantities under the square root sign can be perfect squares. The two sheets of the enveloping surface will have the incident and reflected rays for normals, respectively, so I say that that those two sheets belong to two essentially distinct surfaces.

In order to give an example of the general case, we can suppose that a luminous semi-ellipsoid is illuminated by rays that are normal to its surface, and that there is a mirror located in the principal plane that serves as the base of that semi-ellipsoid. Each reflected ray must then be normal to the other half of the ellipsoid that one considers.

We shall pursue the analytic study that was begun in that note. In order to obtain the equations that relate to the rays themselves – both incident and reflected – recall the equations:

$$\begin{aligned} (x-a)^2 + (y-b)^2 + (z-c)^2 &= r^2, \\ (x-a) + \qquad\qquad + p\,(z-c) &= -r\cdot m, \\ (y-b) + q\,(z-c) &= -r\cdot n\,. \end{aligned}$$

If we set:

$$\frac{x-a}{z-c} = F, \qquad \frac{y-b}{z-c} = f$$

in those equations, in which F and f are convenient functions of a, b, c then we will have:

$$\frac{x-a}{r} = \frac{F}{\sqrt{F^2+f^1+1}},$$

$$\frac{y-b}{r} = \frac{f}{\sqrt{F^2+f^1+1}},$$

$$\frac{z-c}{r} = \frac{1}{\sqrt{F^2+f^1+1}}.$$

Once we have determined F and f, their double values will give the direction of the incident and reflected rays, respectively.

Now, upon combining the three equations that we just obtained with the preceding ones, we will have:

$$\frac{F}{\sqrt{F^2+f^1+1}} + \frac{p}{\sqrt{F^2+f^1+1}} = -m \ldots F + p = -m\sqrt{F^2+f^1+1},$$

$$\frac{f}{\sqrt{F^2+f^1+1}} + \frac{q}{\sqrt{F^2+f^1+1}} = -n \ldots f + q = -n\sqrt{F^2+f^1+1}.$$

These equations will give us F and f in terms of p, q, m, n.

In order to do that, divide the corresponding sides of the two equations; one will get:

$$\frac{F+p}{f+q} = \frac{m}{n} \ldots F = \frac{mf + mq - np}{n}, \qquad f = \frac{nF + np - mq}{m},$$

so

$$F + p = \sqrt{(nF + np - mq)^2 + m^2F^2 + m^2},$$

$$f + q = \sqrt{(mf + mq - np)^2 + n^2 f^2 + n^2} \; .$$

Upon solving these two second-degree equations for *F* and *f*, one will obtain the values of the functions *F* and *f* at *p*, *q*, *m*, *n*. Those values will necessarily become explicit as soon as one knows that equation of the mirror $c = \varphi\,(a, b)$ and the equation $r = \pi\,(a, b)$ that belongs to the spheres that cut the *incident* rays.

NOTE II

Necessary analytical conditions for two surfaces that touch at a point to have conjugate tangents that are common to them; application to catoptrics.

We have proved (*Développements de géométrie*, pp. 147) that if a surface has second-order differential elements that are represented by:

$$dz^2 = r\,dx^2 + 2s\,dx\,dy + t\,dy^2$$

then the equation of its indicatrix at the point x, y, z will be presented in the form:

$$r\,(X - x)^2 + 2s\,(X - x)\,(Y - y) + t\,(Y - y)^2 = C,$$

in which C is an arbitrary constant. Now, if one represents the differential equations of the projection of the conjugate tangents onto the xy-plane by $dy / dx = \varphi$, $dy / dx = \psi$ then the equations between those tangents will be:

$$r + s\,(\varphi + \psi) + t\,\varphi\psi = 0. \tag{a}$$

Analogously, let:

$$r' + s'\,(\varphi' + \psi') + t'\varphi'\,\psi' = 0 \tag{a'}$$

be the equation between the conjugate tangents of a second surface that touches the first one at x', y', z'. (*Développements de géométrie*, pp. 95)

The conjugate tangents at the point x', y', z' are found to be located on the same plane that is tangent to both surfaces, so it will obviously suffice to express the idea that the projections of the conjugate tangents onto the xy-plane are identical in order to express the identity of those tangents in their proper plane.

Upon supposing that equations (a), (a') are both true at the same time, let Φ and Ψ be the values that satisfy both of those equations simultaneously; we will have:

$$\left.\begin{aligned} r + s(\Phi + \Psi) + t\Phi\Psi = 0, \\ r' + s'(\Phi + \Psi) + t'\Phi\Psi = 0, \end{aligned}\right\} \tag{A}$$

so we infer immediately that:

$$-\Psi = \frac{r + s\Phi}{s + t\Phi} = \frac{r' + s'\Phi}{s' + t'\Phi}.$$

Upon clearing the denominators, we will have:

$$rs' - r's + (rt' - r't)\,\Phi + (st' - s't)\,\Phi^2 = 0. \tag{1}$$

Since that equation will give two values for Φ, one might believe that there are two systems of conjugate tangents. However, if one observes that the two equations (*A*) are symmetric in Φ and Ψ then one will see that the values of Ψ at *r*, *s*, *t* and *r*′, *s*′, *t*′ must be identical with those of Φ. Therefore, the two values of Φ that are deduced from the preceding equation are precisely the ones that belong to the two conjugate tangents.

We first conclude from this that in general two surfaces that touch at a point *x*′, *y*′, *z*′ can have only one system of conjugate tangents in common at that point.

If we desire that the two surfaces should have more than one system of conjugate tangents at the same point *x*′, *y*′, *z*′ then we must satisfy the equations of condition:

$$\frac{rs' - r's}{st' - s't} = \frac{0}{0}, \quad \frac{rt' - r't}{st' - s't} = \frac{0}{0}, \qquad \text{which give } \frac{r}{r'} = \frac{s}{s'} = \frac{t}{t'}.$$

It is then obvious that the indicatrices of the two surfaces whose equations are:

$$r \ (X-x)^2 + 2s \ (X-x)(Y-y) + t \ (Y-y)^2 = C,$$
$$r' (X-x')^2 + 2s' (X-x')(Y-y') + t' (Y-y')^2 = C',$$

respectively, will become similar, concentric curves and will have parallel homologous lines, moreover. Consequently, all of their corresponding conjugate diameters will be superposed, and the conjugate tangents on which they are found to be located will be the same for one and the other surface. Therefore, in this case – and only in this case – all systems of conjugate tangents will be common to the two surfaces. If we recall the equation:

$$rs' - r's + (rt' - r't)\,\Phi + (st' - s't)\,\Phi^2 = 0 \tag{1}$$

then we will see that the two roots for Φ will both be real or imaginary. In order to make the distinction between those two cases as simple as possible, observe that we can always make *s* disappear from the equation:

$$r' (X-x')^2 + 2s' (X-x')(Y-y') + t' (Y-y')^2 = C$$

by a simple transformation of coordinates. Therefore, we can suppose that $s' = 0$, with no loss of generality in the results. We will then have:

$$- r's + (rt' - r't)\,\Phi + st'\,\Phi^2 = 0, \tag{2}$$

instead of equation (1), which is an equation whose two roots will be real when the quantity:

$$\frac{r'}{t'} + \tfrac{1}{4}\left(\frac{rt' - r't}{st'}\right)^2$$

is positive, or when the quantity:

$$4s^2 r' t' + r^2 t'^2 + r'^2 t^2 - 2rr'tt'$$

is positive.

Consequently, if one manages to make that quantity negative then equation (2) will have only imaginary roots. For this to be true, it will suffice to make:

$$4s^2 r't' \text{ negative and } > (rt' - r't)^2;$$

i.e.:

$$r't' \text{ negative and } > \left(\frac{rt' - r't}{2s}\right)^2.$$

It is therefore necessary, first of all, that r' and t' must have different signs, and consequently, that the indicatrix:

$$r'(X - x')^2 + 2s'(X - x')(Y - y') + t'(Y - y')^2 = C$$

must be a hyperbola.

If one takes the other indicatrix, instead of this one, in order to make s disappear from its equations then one will likewise arrive at this consequence:

The values of Φ *cannot be imaginary, at least when the two indicatrices are not both* HYPERBOLAS.

Now suppose that a mirror of arbitrary form receives rays that emanate from a luminous point. Consider the incident ray that goes from that focus to the arbitrary point P of the mirror to be the axis of a circular cylinder. The trace of that cylinder on the tangent plane to the mirror at P will be the indicatrix of the auxiliary second-degree surface that has the luminous focus for one of its general foci, and for the second general focus, the point where the reflected ray at P is met by a reflected ray at P', which is a point that is infinitely close to P. One will find two auxiliary surfaces, which will give two points P', P'' on the mirror that are infinitely close to P. Those points P', P'' are located on two conjugate diameters that are common to the indicatrix of the mirror and to the auxiliary surface that has the point P. Now, since the latter indicatrix is the trace of an elliptic cylinder on a plane, it is an ELLIPSE. Therefore, not matter what indicatrix the mirror might have, it will *always* have a system of *real* conjugate tangents in common with that ellipse. Therefore, there are always two conjugate directions for which a ray that is reflected by the mirror infinitely close to P will meet the ray that is reflected by the point P.

Let X, Y, Z be the coordinates of the luminous focus. Let x', y', z' be those of the mirror whose indicatrix is defined by the equation:

$$r(x - x')^2 + 2s(x - x')(y - y') + t(y - y')^2 = C.$$

We demand to know the directions of the two conjugate tangents at the point x', y', z' of that surface that touch the two developable surfaces, respectively, that are composed of reflected rays and both pass through the point x', y', z'.

We already know the general equation of the conjugate tangents:

$$r + s\,(\varphi + \psi) + t\,\varphi\psi = 0.$$

It then suffices to express the idea that the two planes that are drawn through the first and second tangent, respectively, cut at a right angle. For greater ease, suppose that the *xy*-plane is parallel to the tangent plane to the mirror at x', y', z'. The equations of the two planes will be:

$$\varphi\,(x - x') - (y - y') + m\,(z - z') = 0,$$
$$\psi\,(x - x') - (y - y') + n\,(z - z') = 0,$$

in which m, n are two arbitrary constants that are determined by the condition that the two planes must pass through the focus X¸Y, Z, which gives:

$$\varphi\,(X - x') - (Y - y') + m\,(Z - z') = 0, \qquad \ldots, \qquad m = -\,\frac{\varphi(X - x') - (Y - y')}{Z - z'},$$

$$\psi\,(X - x') - (Y - y') + n\,(Z - z') = 0, \qquad \ldots, \qquad n = -\,\frac{\psi(X - x') - (Y - y')}{Z - z'}.$$

Upon substituting these values for m and n into the preceding two equations, they will become:

$$\varphi\,(X - x') - (Y - y') - \frac{\varphi(X - x') - (Y - y')}{Z - z'}\,(Z - z') = 0,$$
$$\psi\,(X - x') - (Y - y') - \frac{\psi(X - x') - (Y - y')}{Z - z'}\,(Z - z') = 0.$$

In order for those two planes to cut at a right angle, it is necessary that one must have:

$$1 + \varphi\psi + \frac{\varphi(X - x') - (Y - y')}{Z - z'} \cdot \frac{\psi(X - x') - (Y - y')}{Z - z'} = 0.$$

For more simplicity, represent $X - x'$ by ξ, $Y - y'$ by ν, and $Z - z'$ by ζ; the preceding equation will become:

$$\zeta^2 + \varphi\psi \cdot \zeta^2 + (\varphi\xi - \nu)\,(\psi\xi - \nu) = 0,$$

or

$$\zeta^2 + \nu^2 + \xi\nu\,(\varphi - \psi) + (\zeta^2 + \xi^2)\,\varphi\psi = 0.$$

That equation will have the form:

$$r' + s'\,(\varphi + \psi) + t'\,\varphi\psi = 0$$

if one sets:

$$\zeta^2 + \nu^2 = r', \qquad -\,\xi\nu = s', \qquad \zeta^2 + \xi^2 = t'.$$

It then belongs to the conjugate tangents of a certain surface, and those conjugate tangents will all be seen to cut at a right angle when one regards them from the focus X,

Y, Z. They then belong to second-degree surfaces of revolution that have X, Y, Z for their focus and touch the mirror at x', y', z'.

When one replaces r', s', t', with their values in equation (1) [viz., $rs' - r's + (rt' - r't)\,\Phi + (st' - s't)\,\Phi^2 = 0$] that will give the two desired directions. In order to do that, it will suffice to solve the second-degree equation:

$$- r \cdot \xi v - s \cdot (\zeta^2 + v^2) + [r\,(\zeta^2 + \xi^2) - t\,(\zeta^2 + v^2)]\,\Phi + [s\,(\zeta^2 + \xi^2) + t \cdot \xi v]\,\Phi^2 = 0,$$

which will always have two real roots, in the usual manner.

If one regards r, s, t as functions of x', y', z', which is the point on the mirror where the reflection takes place, then if one sets Φ equal to its value dy' / dx', the preceding equation will become one of ordinary differentials. It will then represent the two systems of curves that are traced on the mirror by the reflected rays.

That is why the two systems of lines of curvature of the surface are given to us by a second-degree equation in dy / dx that is expressed in terms of the coefficients of the partial differentials p, q, r, s, r of first and second order.

NOTE III

Graphical method of determining the points of intersection of infinitely-close reflected rays.

Nothing is easier than to determine points of intersection of the rays that are reflected by a mirror around a point P of the reflecting surface when the incident rays start from a unique focus O by means the results that were obtained in the course of this memoir.

If the directions of greatest and least curvature for the mirror are given at P then those directions will be those of the axes of the indicatrix $ABCD$ of the mirror. One can then immediately trace out that indicatrix by employing the simple means that we have presented in our *Développements de géométrie*.

One must then trace an ellipse $\alpha\beta\gamma\delta$ that is big enough to cut $ABCD$. In the course of this memoir, we have indicated how one can determine the axes of that curve.

The ellipse $\alpha\beta\gamma\delta$ cuts $ABCD$, which is the indicatrix of the mirror, at four points, and when they are joined pair-wise by lines, they will form a parallelogram whose opposite sides are parallel to the conjugate diameters $M'PN'$, $M''PN''$, respectively. Those two diameters each cut the ellipse $\alpha\beta\gamma\delta$ at two points M, N and m, n, resp. The ellipses $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$, are similar to it, and their homologous lines are, moreover, parallel or superposed, so one will have:

$$Pm : PM' :: P\alpha : P\alpha' :: P\beta : P\beta',$$
$$Pm : PM'' :: P\alpha : P\alpha'' :: P\beta : P\beta''.$$

These proportions make the semi-axes $P\alpha'$, $P\beta'$, $P\alpha''$, $P\beta''$ of the two auxiliary indicatrices $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$, resp., which touch the indicatrix $ABCD$ of the mirror at M', N' and M'', N'', resp., known immediately.

We shall now determine the points Ω', Ω'' that are the loci of the intersections of the reflected rays $M'\Omega'$, $N'\Omega'$; $M''\Omega''$, $N''\Omega''$ that are infinitely close to $P\Omega$.

In order to do that, take Figure 2 to be the new vertical projection that corresponds to Figure 1, so as to not make the first vertical projection too confused.

By means of the quarter-circle $\Gamma'\gamma'$, carry $P\gamma'_v = P\gamma'_h$ to $P\Gamma'_v$. Take PC_v to be equal to PC_h, which is the semi-diameter of $ABCD$ that is located on the major axis of the auxiliary indicatrix.

Let PR be the magnitude of the radius of curvature at the point P of the section of the mirror at P by a normal plane that is directed along the line APC. We determine immediately the radius at P of the section that is made by the same plane in the second-degree surface whose indicatrix is $\alpha'\beta'\gamma'\delta'$ by means of this proportion:

$$PC^2 : P\Gamma'^2 = P\gamma'^2 :: R : \text{desired radius } r' = \frac{P\gamma'^2}{PC^2} \cdot R.$$

In order to construct that value, upon referring PC_h (Fig. 1) to PC_v (Fig. 2), draw the line $C\Gamma'$ and the perpendicular PF to that line. We will then have $PC^2 : P\Gamma'^2 = P\gamma'^2 :: CF : \Gamma'F$.

Prolong FC in such a manner that one can take $FK = R$ on that line. Then draw KK' through the point K parallel to PF. Draw GIL parallel to KF through the point G where KK' meets the horizontal PCG. The similar triangles PFC, PIG; $PF\Gamma'$, PIL give:

$$PC^2 : P\Gamma'^2 = P\gamma'^2 :: FC : F\Gamma' :: GI = FK = R : LI = \frac{P\gamma'^2}{PC^2} \cdot R = r'.$$

LI will then be the desired radius.

Presently, with PL' as radius and L' as center, we shall trace the circle MPQ that is the osculator of the normal section that is made along $\alpha' P\, \gamma'$ in the auxiliary surface that has $\alpha'\beta'\gamma'\delta'$ for its indicatrix.

Upon determining the point p where that circle cuts the incident ray OP, we can immediately construct the value:

$$P\Omega' = \frac{PO \cdot Pp}{4PO - Pp}$$

in the memoir on pp. 18.

In order to do that, put $\frac{PO \cdot Pp}{4PO - Pp}$ into the form $\frac{\frac{1}{2}PO \cdot \frac{1}{2}Pp}{PO - \frac{1}{4}Pp}$, and we will have:

$$PO - \tfrac{1}{4}Pp : \tfrac{1}{2}PO :: \tfrac{1}{2}Pp : P\Omega'.$$

Take the point O' at the middle of PO, the point p' at the middle of Pp, and p'' at the middle of Pp. Carry $Op'' = PO - \frac{1}{4}Pp$ from P to Z and Pp' to Pp on the vertical PZ. Finally, draw the line ps parallel to the line that passes through the points O' and Z. We will have:

$$Ps = \frac{\frac{1}{2}PO \cdot \frac{1}{2}Pp}{PO - \frac{1}{4}Pp} = P\Omega'.$$

If we carry Ps along the reflected ray from P to Ω' then we will immediately determine the point Ω' where the reflected rays $M\Omega'$, $P\Omega'$, $N'\Omega'$ meet.

Upon employing the same method, one will determine the second point Ω'' where the reflected rays $M''\Omega''$, $P\Omega''$, $N''\Omega''$ meet with equal facility.