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On the theory of magnetic and electric phenomena

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1. – Introduction.

The theoretical electrodynamics of superfluous components ("rudimentary concepts") was made more orderly by the work of **H. Hertz** and **E. Cohn**, and the entire system of experimental facts in the realm of magnetic, electromagnetic, and electric phenomena was summarized in a system of six first-order partial differential equations for media at rest and just as many equations for the media in motion ([1]).

One must then ask whether the development of the theory of the aforementioned realm of phenomena must be considered to be concluded as a result of that. **H. Hertz** was inclined to believe that it would suffice to place the aforementioned system of twelve partial differential equations at the forefront of the theory. The theory would then have to solve only the problem of deriving the detailed phenomena from those equations and verifying them by experiments. The fact that this standpoint is not generally satisfactory is shown by the numerous attempts that have been made to base those equations on mechanics. Exactly as one considers the phenomena in other domains to be purely processes of motion with advancing knowledge, so must one regard the goal of an interpretation of electric phenomena as being likewise the reduction of them to phoronomic processes. That must go hand in hand with the elimination of special "imponderable fluids" that are still being maintained up to now.

The concepts of "electricity" and "amount of electricity" and the amount of "free magnetism" that were adopted from the older pictures still enter at the focus of the systematics of electric and magnetic phenomena of **Hertz** and **Cohn**, since they are necessary in it if one is to define those vector quantities that are introduced into it as "electric" ("magnetic," resp.) forces ([2]). A further question to ask is whether that system of twelve differential equations also actually defines the physically-"simplest" expression for those processes that play out where magnetic and electric lines of force are found throughout. Finally, methodological considerations require an "intuitive" foundation for the theory of electric phenomena.

([1]) **H. Hertz**, Wied. Ann. **40** (1890), pp. 577 or "Untersuchungen," pp. 208, 1892, which will be cited as **Hertz I** in what follows. Wied Ann. **41** (1890), pp. 369, or "Untersuchungen," pp. 256, 1892, which will be cited as **Hertz II** in what follows. **E Cohn**, Wied. Ann. **40** (1890), pp. 625.

([2]) Cf., also **L. Boltzmann**, *Vorles. über Maxwell's Theorie der Electricität und des Lichtes*, Part II, J. A. Barth, 1893, pp. 13, *et seq.*

In the attempts to go further in that direction, one must reduce either the **Maxwell** equations or their clarified form, namely, **Hertz**'s equations, to simple processes of motion and stress in the ether. What is initially quite striking is the fact that so many mechanical interpretations are possible that often starts from the opposite assumptions. The problem is then a multi-valued one. **H. Poincaré** suggested ([1]) that this was based upon the multiplicity of solutions that the partial differential equation:

$$f' \equiv \sum_i \frac{\partial f}{\partial p_i} q_i$$

admits, when we use **Helmholtz**'s notation and represent the general parameter of the problem by p_i, while $q_i = p_i'$ is the rate of change in its velocity, and the prime represents the derivative with respect to time. If we look for the infinitesimal transformations (with the terminology of **Sophus Lie**) that this differential equation admits under the more detailed circumstances that are present here then the group-theoretic treatment of the equation will also prove itself in the present case to be an important reference point for the peculiarities of the system of solutions that correspond to electric phenomena in particular ([2]).

Among the possible solutions, the simplest one will offer an advantage from the outset when it likewise explains all phenomena in a manner that is free of any objections. That is still not entirely the case for the present theories since each of them encounters certain difficulties that the authors themselves mostly acknowledge. However, it seems to me that the facts of observation themselves point unambiguously to a path that leads to an extremely-simple solution:

If we would like to go deeper into the processes that lie at the basis for the equations of electrodynamics then we must establish that the *magnetic* phenomena are reducible to vorticial motions that repeat themselves or more generally cyclic motions in the **Helmholtz** sense. Once **Maxwell** had brought **Faraday**'s picture of autonomous states that exist in space and advance within it into a mathematical form, he sought to find the mechanical foundation for those states. Starting from the assumption that vortex motions exist in the neighborhood of the magnetic field lines, he explained all phenomena of magnetism and electromagnetism with no further hypotheses ([3]). In that way, to him, the galvanic current was initially just a special form of a system of magnetic force lines.

It was only in the fundamental work of **H. von Helmholtz** in the statics of monocyclic systems that the properties of those systems have been understood more precisely as repeating motions that subsume vorticial motions as a special case, and thus it will not be possible to give a more definite foundation to those investigations. I have attempted to give a theory of electric and magnetic

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([1]) **H. Poincaré**, *Electricité et Optique*, Paris, 1890, pp. IX, *et seq.*

([2]) I believe that I will give such a systematic treatment in which numerous special problems shall be treated in detail, as well as certain mathematical developments that cannot find their place here, in a special article later. It shall also go deeper into the work of **Vito Volterra** [Atti R. Accad. dei Lincei, Rendiconti (4) **7** (1891), pp. 177 and Il nuovo Cimento (3) **29** (1891), pp. 147] and **O. Heaviside**, as well as discussing the relationships between the individual theories and the present one more thoroughly.

([3]) **Cl. Maxwell**, Phil. Mag. (4) **21** (1861), pp. 161, 281, 338, and **23** (1862), pp. 12, 85. The logical continuation of that attempt at explaining things failed upon reaching the "rudimentary" concepts, which **Maxwell** still did not succeed in eliminating completely. The friction rollers that he embedded between the vortices, and which he identified with "electricity," were an unnecessary complication.

phenomena that starts from **Helmholtz**'s investigations. It assumes only the possibility that repeating "cyclic motions" are the carrier for those phenomena. It differs from the first part of **L. Boltzmann**'s book ([1]) on **Maxwell**'s theory essentially. We shall derive the basic equations of electrodynamics from the theory for bodies that are isotropic, as well as anisotropic, non-conducting, as well as conducting, homogeneous, as well as inhomogeneous, and at rest, as well as in motion.

The phenomena of dispersion and absorption can also be classified within that system with no further assumptions.

That theory can be made accessible to the imagination in a simple way by models.

2. – Foundations of the theory.

Ampère had already assumed that magnetic phenomena are basically something that can be regarded kinematically as a system of repeating motions. Later, Sir **W. Thomson** showed that the electromagnetic rotation of the polarization plane of light can only be interpreted as the fact that along with the undulatory changes of state in the ether along the polarized light ray, rotational recurrent motions along the magnetic lines of force that use the light ray as an axis will enter into the magnetic field. Those considerations, which are independent of the sense in which the rotation actually results (i.e., whether with or against the current) can be implemented ([2]). In fact, that assumption is at the basis for all theories that seek to explain that phenomenon, as well as the closely-related **Kerr** phenomenon.

It seems to me that by far the most important argument for the presence of such recurrent elementary motion in magnetic fields, which has still not been sufficiently emphasized, was given by **Helmholtz** in his treatises on the general theory of kinematical systems that include recurring motions – viz., "cyclic motions" – (static monocyclic systems) ([3]) and in his treatises on the general meaning of the principle of least action ([4]); some very general properties of electromagnetic phenomena were derived in them. We shall abstract the following result from those treatises, which shall be fundamental for us: In the cases where the forces that act upon certain coordinates are continually equal to zero, we can eliminate the corresponding coordinates from the problem completely, so the equations of motion can indeed be brought into the usual form, but the kinetic energy is no longer a homogeneous function of degree two in the (cyclic) velocities, as usual, but will also include other (e.g., first) powers, in general. If we actually preserve all coordinates then **Helmholtz** called the system a "complete" one. For him, kinetic energy had the form of an expression of degree two in the velocities. However, if we carry out their elimination then we will get an "incomplete system of equations."

([1]) **L. Boltzmann**, *Vorlesungen über **Maxwell**'s Theorie der Electricität und des Lichtes*, J. A. Barth, Leipzig, Part I, 1891.

([2]) Cf., **Maxwell**, *Treatise* **2**, Chaps. 21 and 22, §§ 806-845.

([3]) **H. von Helmholtz**, Sitzungsber. d. Preuss. Akad. d. Wiss. Math. naturwiss. Mittheil. (1884), 67-86, 153-160, 375-380, 693-698. (**Crelle-Borchardt**'s) Journal für reine und angew. Math. **97** (1884), 111-140, 317-336.

([4]) **H. von Helmholtz**, Journal für reine und angew. Math. **100** (1887), 137-166, 213-222.

An example of that type is a rotating body that hangs from a Cardan suspension and rotates around its axis ([1]). If the body rotates very rapidly then we will see nothing from its motion. However, if we move the axis from its position then a certain opposing force will act upon each coordinate that we are trying to change (viz., rotation of the rings in the **Cardan** suspension).

Helmholtz appealed to the analogy with ponderable bodies in mechanics and called such cases *cases with hidden motions*, and in the cited treatise (cf. 147) he showed that the case of the interaction between magnetic and electrical currents belongs to them. The fundamental recurring motions here, so the "cyclic" motions, are less accessible to our intuition than they are in the case of the rapidly-rotating top, but when we perform, e.g., a change of position for a conductor in the magnetic field, we will note the appearance of a so-called "electromotive" force that generates a current. Along the lines of magnetic force, there also exist hidden recurring motions.

In fact, **Helmholtz** had derived the electrodynamical law of induction from the equations of motion by using precisely the same general argument that gave the relationship between changes of angle and the corresponding forces on the top. Indeed, **Helmholtz** indicated (pp. 142, *loc. cit.*) that it was virtually the investigations into the form of the kinetic potential that **Maxwell**'s theory of electrodynamics requires that would come under consideration that gave him the impetus to carry out later incisive investigations into the nature the phenomena here that were very general and deep.

In conjunction with those results of **H. von Helmholtz**, we can consider it to have been proved that everywhere that magnetic lines of force are found in space, a motion will be carried out that is, of course, "hidden" to us, but about which we know only that it is a cyclic motion, i.e., a circular or vorticial recurring motion.

If we now denote the components of the vector that gives the magnetic force at each point in space by L, M, N and the components of the cyclic velocity of the monocycle that exists there and is equivalent to the latter by ξ', η', ζ' (in what follows, primes shall always mean derivatives with respect to time) then we must set:

$$L = k \cdot \xi', \quad M = k \cdot \eta', \quad N = k \cdot \zeta'. \tag{1}$$

One will have:

$$\xi' = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \eta' = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \zeta' = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

here, if u, v, w are the rectangular components of the velocity. k is an "*internal* constant of the ether" in the **Hertzian** sense ([2]).

It might seem as though we have to introduce two special assumptions that would compromise the generality of our considerations. First of all, we might get the impression that the introduction of the constant k would necessitate a more detailed insight into the fundamental processes. However, we will show later that it plays no explicit role in electric phenomena but must be combined with another similar auxiliary constant l that will be introduced later into the **Hertz**

([1]) **H. von Helmholtz**, Journal für reine und angew. Math. **100** (1887), pp. 153.
([2]) Cf. **H. Hertz**, I, pp. 583 (Untersuch., pp. 214).

constant A (reciprocal speed of light), which will then take on a well-defined physical sense. Conversely, certain conclusions about the special nature of the fundamental processes can be derived from that. Secondly, it might seem preposterous that we have made L, M, N proportional to ξ', η', ζ', resp. However, that is justified by experiments. It tells us that we have to set the energy per unit volume of the magnetically-stressed medium equal to:

$$T = \frac{\mu}{8\pi}(L^2 + M^2 + N^2).$$

The magnetic energy has the kinetic form, from what was said above. According to **Helmholtz**, it must be homogeneous of degree two in the cyclic velocities (since we have not eliminated any of them), and as a result:

$$T = \frac{\mu k^2}{8\pi}(\xi'^2 + \eta'^2 + \zeta'^2). \tag{2}$$

We then have a "complete" **Helmholtz** system (cf., pp. 3).

We have thus established the expressions for the magnetic forces and energy. They must give the electric force and energy when we consider the experimental facts.

Experiments teach us that electric phenomena are always coupled with magnetic ones in such a way that the temporal variation of a vector quantity that varies with position X, Y, Z, and which we shall call the electric force, is equal to the momentary spatial distribution of the magnetic forces L, M, N in the neighborhood of that position. That fact finds its expression in the **Hertzian** system of equations (we would like to call it briefly "the first one") ([1]):

$$\text{(I)} \qquad \begin{cases} A\varepsilon \dfrac{dX}{dt} = \dfrac{\partial M}{\partial z} - \dfrac{\partial N}{\partial y}, \\ A\varepsilon \dfrac{dY}{dt} = \dfrac{\partial N}{\partial x} - \dfrac{\partial L}{\partial z}, \\ A\varepsilon \dfrac{dZ}{dt} = \dfrac{\partial L}{\partial y} - \dfrac{\partial M}{\partial x}. \end{cases}$$

If we substitute the values in (1) for L, M, N and integrate over time then that will imply that we have to make X, Y, Z proportional to expressions like $\partial\eta / \partial z - \partial\zeta / \partial y$, etc. (One can always assign the special value of zero to the time-independent integration constants.)

We then write:

$$X = \frac{l}{\varepsilon}\left(\frac{\partial\eta}{\partial z} - \frac{\partial\zeta}{\partial y}\right), \quad Y = \frac{l}{\varepsilon}\left(\frac{\partial\zeta}{\partial x} - \frac{\partial\xi}{\partial z}\right), \quad Z = \frac{l}{\varepsilon}\left(\frac{\partial\xi}{\partial y} - \frac{\partial\eta}{\partial x}\right), \tag{3}$$

([1]) Cf. **H. Hertz**, I, pp. 584 (Untersuch., pp. 215).

in which l is an auxiliary constant for which the same thing is true that was true for k above. As we see, those values for X, Y, Z follow uniquely from our basic assumption. They depend upon the cyclic coordinates themselves. Unfortunately, there is no simple expression for the form of that dependency. (In the language of quaternions, one would say that the vector of the electric force $i\,X + j\,Y + k\,Z$ is equal to the vector component of the operator:

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

when one lets it act upon the vector of the cyclic (angular) motion $\int (i\,\xi' + j\,\eta' + k\,\zeta')\,dt$. That statement is equivalent to the statement that the line integral along a closed curve of one vector is equal to the surface integral of the other [$\iint$ vector projection onto the normal times surface element], which has the curve for its complete (and only) boundary, or equal to the vector flux through the surface). **O. Heaviside** has introduced the term "curl" for that ([1]), and we would like to preserve it, as well. We will then say that the vector of the electric force is equal to the "curl" of the cyclic motion. Mechanically, we would be able to represent the electric force as stresses that appear in the ether as a consequence of the positional variations in the rotations that the individual neighboring elements have experienced with respect to each other, as one assumes in **Mac Cullagh**'s theory of light and in **Sir William Thomson**'s quasi-elastic ether ([2]).

As is known, based upon experiments, we have to write:

$$\Phi = \frac{\varepsilon}{8\pi}(X^2 + Y^2 + Z^2)$$

for those "electric stresses." It would have the character of a potential in the aforementioned mechanical interpretation. If we substitute the values from (3) in it then we will get:

([1]) **O. Heaviside**, *Electrical Papers*, **1**, 1892, pp. 199.

([2]) **Remark:** Rotations of the ether also play a role in other electromagnetic theories. However, either they would explain the magnetic forces [**Lord Kelvin** [1]), **E. Padova** [2]), **L. Boltzmann** [3])], and assign some other meaning to the electric forces, as is done here, or it would, in fact, be the electric forces that are attached to the rotations, but other expressions would enter into the magnetic forces than the angular velocities, such as the linear velocities found in **A. Sommerfeld** [4]) or **R. Reiff** [5]). Moreover, the rotations of the ether themselves play a role there, but here it is only their differences from one location to a neighboring one that play the role. That is how those theories will differ from the one developed here, more or less.

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[1]) **Sir William Thomson**, *Math. and Phys. Papers*, **3**, art. 99, namely, § 39, pp. 436, 1890.

[2]) **E. Padova**, Il nuovo Cimento (3) **29** (1891), pp. 225; cf., also Atti della Accad. dei Lincei, Rendiconti (4) **7** (1891), pp. 204.

[3]) **L. Boltzmann**, Wied. Ann. **48** (1893), pp. 78 and *Vorlesungen über **Maxwell**'s Theorie*, Part II, Leipzig 1893.

[4]) **A. Sommerfeld**, Wied. Ann. **46** (1892), pp. 139.

[5]) **R. Reiff**, *Ueber Wirbelbew. reibeinder Flüssigkeiten*, Mittheil. math. naturw. Vereins in Württemberg, 1893. – *Elasticität und Electricität*, Freiburg i. B. u. Leipzig, 1893.

$$\Phi = \frac{l^2}{8\pi}\left[\frac{l}{\varepsilon}\left(\frac{\partial\eta}{\partial z}-\frac{\partial\zeta}{\partial y}\right)^2+\frac{l}{\varepsilon}\left(\frac{\partial\zeta}{\partial x}-\frac{\partial\xi}{\partial z}\right)^2+\frac{l}{\varepsilon}\left(\frac{\partial\xi}{\partial y}-\frac{\partial\eta}{\partial x}\right)^2\right]. \tag{4}$$

Φ itself would depend upon the cyclic variables then ([1]).

Equations (1) to (4) suffice completely for one to be able to derive all equations of electrodynamics for non-conductors.

In order to exhibit the equation for a conductor, we must observe the following: Up to now, we have assumed that the cyclic motions in the ether suffered no loss of energy. However, we know that there are bodies in which such losses occur. For elastically-coupled monocycles, which would correspond to the present case, that phenomenon will find its mechanical expression in a partial differentiation of the mutually-rotated elements and the product of heat that is coupled with it. In **Hertz**'s system of equations, that situation is introduced into the calculations by the addition of terms $-4\pi\lambda A X$, $-4\pi\lambda A Y$, $-4\pi\lambda A Z$ on the right-hand sides of equations (I) (λ: "conductivity")([2]). **Hertz** arrived at that conclusion by assuming that the dissipation over time that entered into the values of the electric vector as a result of conduction was proportional to the values of the vector themselves. In our case, the "curl" corresponds to the cyclic coordinates of that vector, so we have to assume equality for them and must therefore also extend the equations that we obtained for non-conductors to conductors in such a way that we will add analogous terms.

In that way, we will also arrive at the equations for the conductor without having to introduce any special assumption about the process of energy conversion that exists in it. Later on (pp. 22), some special representations of the mechanical foundations of electrical conduction processes shall be given.

3. – Deriving the fundamental equations for isotropic media at rest.

If we set:

$$X = \frac{l}{\varepsilon}\left(\frac{\partial\eta}{\partial z}-\frac{\partial\zeta}{\partial y}\right), \qquad \text{etc.,}$$

as in equations (3), then upon differentiating with respect to time, we will get:

$$\varepsilon\frac{dX}{dt} = l\left(\frac{\partial\eta'}{\partial z}-\frac{\partial\zeta'}{\partial y}\right),$$

and when we replace η', ζ' with the values in (1), we will have:

([1]) The cyclic variables themselves are missing from the expression for energy, and therefore from the kinetic potential $H = \Phi - T$ (free energy), as is true for **Helmholtz**'s monocycles. By contrast, derivatives of the cyclic variables with respect to position appear here, which correspond to "elastically-coupled monocycles" in terms of the cycles of mechanics, as is easy to verify more generally.

([2]) **H. Hertz**, I, pp. 586 (Untersuch., pp. 217).

$$\frac{k}{l}\frac{dX}{dt} = \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y}.$$

If we write the value A for k / l then we will get the first **Hertz** system:

(I) $$A\varepsilon\frac{dX}{dt} = \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y}, \text{ etc.}$$

for isotropic non-conductors ([1]). For conductors, according to pp. 7, we must add terms $-4\pi\lambda\cdot AX$, etc., such that we will get:

$$A\varepsilon\frac{dX}{dt} = \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y} - 4\pi\lambda\cdot AX, \quad \text{etc.},$$

which is the first **Hertz** system for isotropic conductors ([2]).

We will get the second **Hertzian** system, which represents the time variation of the magnetic state in terms of the spatial distribution of the electric states, by applying **Hamilton**'s principle to the expressions for the energy. If we substitute the values for T and Φ that are given in (2) and (4), resp., in:

(H) $$\delta\int_{t_0}^{t_1}\int\int_{-\infty}^{+\infty}\int\{(T-\Phi)\,d\tau\}\,dt = 0$$

then when we take the variation relative to the cyclic coordinates ξ, η, ζ, ξ', η', ζ' (while regarded the factors μ and ε as constant, although they depend upon x, y, z in inhomogeneous media), we will get:

$$\begin{aligned}0 = \int_{t_0}^{t_1} dt \int\int_{-\infty}^{+\infty}\int \Bigg\{ & \frac{\mu k^2}{4\pi}[\xi'\cdot\delta\xi' + \eta'\cdot\delta\eta' + \zeta'\cdot\delta\zeta'] \\ & - \frac{l^2}{4\pi}\left[\frac{l}{\varepsilon}\left(\frac{\partial\eta}{\partial z} - \frac{\partial\zeta}{\partial y}\right)\left(\frac{\partial\delta\eta}{\partial z} - \frac{\partial\delta\zeta}{\partial y}\right) + \frac{l}{\varepsilon}\left(\frac{\partial\zeta}{\partial x} - \frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\delta\zeta}{\partial x} - \frac{\partial\delta\xi}{\partial z}\right)\right. \\ & \left. + \frac{l}{\varepsilon}\left(\frac{\partial\xi}{\partial y} - \frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\delta\xi}{\partial y} - \frac{\partial\delta\eta}{\partial x}\right)\right]\Bigg\}\,d\tau,\end{aligned}$$

in which we imagine that the spatial integral extends far enough that the velocities and rotations will be equal to zero at the limits. For any discontinuity surfaces that lie between them, we shall apply the "**Helmholtz** trick," i.e., we shall assume that our functions ξ, η, ζ, ξ', η', ζ' indeed

([1]) **H. Hertz**, I, pp. 584, eq. (4.b) (Untersuch., pp. 215).
([2]) **H. Hertz**, I, pp. 587, eq. (6.b) (Untersuch., pp. 218).

vary quite rapidly from location to location, but still continuously (viz., the "principle of the continuity of the transition") ([1]). We can then set the surface integral equal to zero that comes about by partially integrating the variations by which we converted the previous equation that are differentiated with respect to the coordinates. The simultaneous partial completion of the time integral in the first part:

$$\left(\delta\xi' = \delta\frac{d\xi}{dt} = \frac{d\,\delta\xi}{dt}\right)$$

and the passage to the time element will lead to:

$$\frac{\mu k^2}{4\pi}\cdot\xi'' = \frac{l^2}{4\pi}\left[\frac{\partial}{\partial y}\frac{l}{\varepsilon}\left(\frac{\partial\xi}{\partial y}-\frac{\partial\eta}{\partial x}\right)-\frac{\partial}{\partial z}\frac{l}{\varepsilon}\left(\frac{\partial\zeta}{\partial x}-\frac{\partial\xi}{\partial z}\right)\right], \qquad \text{etc.,}$$

when we set the terms that are multiplied by those variations equal to zero. However, it will follow from (1) and (3) that:

$$\frac{\mu k}{l}\cdot\frac{dL}{dt} = \frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z},$$

or since we have set $k / l = A$ above, that is:

$$\text{(II)} \qquad A\mu\frac{dL}{dt} = \frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}, \text{ etc.,}$$

which is the second **Hertzian** system, which is valid for non-conductors and conductors in the same way ([2]).

The constant A is once more an "internal constant of the ether" (cf., *supra*, pp. 4). If one differentiates one of the two **Hertzian** systems with respect to time and substitutes the values of the time derivatives on the right-hand side that one gets from the other system then one will see that A is the reciprocal velocity with which periodic changes of state propagate through the free ether ($\varepsilon = 1$, $\mu = 1$) since one will be led to the well-known wave equation.

It is very simple to get the extension that the **Hertzian** equations must experience if they are to also encompass dispersion and absorption phenomena:

In our way of looking at things, the changes of state at one point are determined by the states in its neighborhood. However, that transition cannot result with infinite velocity. Therefore, if the changes of state in an electromagnetic field are to happen very rapidly, as they do with light, then the time that is necessary for a state of motion at one point to carry over to the other ones must come under consideration. If ξ, η, ζ are the values of the cyclic variables (rotations) in the neighborhood of a point then the quantities:

([1]) **L. Boltzmann**, *Vorlesungen*, Part II, pp. 9.
([2]) **H. Hertz**, I, pp. 584, eq. (4.a) and pp. 587, eq. (6.a) (Untersuch., pp. 215 and 219).

$$l\left(\frac{\partial\eta}{\partial z}-\frac{\partial\zeta}{\partial y}\right), \qquad \text{etc.,}$$

will no longer determine a value of X, as it pertains to time t, but as it pertains to a somewhat-later time $t + dt$. When we likewise employ the **Taylor** development, we must then set:

$$X + c_0\frac{dX}{dt} + c_1\frac{d^2X}{dt^2} + \cdots = \frac{l}{\varepsilon}\left(\frac{\partial\eta}{\partial z}-\frac{\partial\zeta}{\partial y}\right),$$

or when only harmonic changes of state of the form $A\cdot e^{i2\pi m t}$ ($m = 1 / T =$ oscillation number) come under consideration ([1]):

$$X + c_1\frac{d^2X}{dt^2} + c_3\frac{d^4X}{dt^4} + \cdots = \frac{l}{\varepsilon}\left(\frac{\partial\eta}{\partial z}-\frac{\partial\zeta}{\partial y}\right).$$

Differentiation with respect to time will now give:

$$A\left(\varepsilon\frac{dX}{dt} + \varepsilon'\frac{d^3X}{dt^3} + \varepsilon''\frac{d^5X}{dt^5} + \cdots\right) = \frac{\partial M}{\partial z}-\frac{\partial N}{\partial y}, \quad \text{etc.}$$

P. Drude, who had inserted the **Hertzian** system of equations into those equations by hypothesis, had already showed that, in conjunction with the second **Hertzian** system (pp. 9), they could represent all dispersion phenomena completely, and that the associated boundary conditions are compatible with the energy principle ([2]).

4. – Deriving the fundamental equations for anisotropic media at rest.

If we now move on to anisotropic media then we must once more make the magnetic forces proportional to the velocities ξ, η, ζ of any cyclic motions that might be present:

$$L = r\xi', \qquad M = r\eta', \qquad N = r\zeta' \tag{5}$$

(r is an auxiliary constant). However, the effect of those cyclic motions is not expressed immediately in the phenomena, it will be modified by the anisotropy in the medium. What we observe are the magnetic polarizations whose components are linear components of the three components of the magnetic forces that emanate from the cyclic motions. We find that the relationship between the two vector quantities of force and polarization is determined completely

([1]) That consideration is entirely-analogous to the ones that I had previously posed (Wied. Ann. **48** (1893), pp. 1), in which I addressed the problem of extending **Maxwell**'s system of equations in such a way that dispersion phenomena would also find their explanation in that system.

([2]) **P. Drude**, Nachr. Gött. Ges. d. Wissensch. (1892), pp. 366.

by six quantities that are functions of position for inhomogeneous media in such a way that the components of the magnetic polarization $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ are coupled with the force by the equations ([1]):

$$(6)\qquad \begin{cases} \mathfrak{L} = \mu_{11}L + \mu_{12}M + \mu_{13}N = \mu_{11}\, r\,\xi' + \mu_{12}\, r\,\eta' + \mu_{13}\, r\,\zeta', \\ \mathfrak{M} = \mu_{21}L + \mu_{22}M + \mu_{23}N = \mu_{21}\, r\,\xi' + \mu_{22}\, r\,\eta' + \mu_{23}\, r\,\zeta', \\ \mathfrak{N} = \mu_{31}L + \mu_{32}M + \mu_{13}N = \mu_{31}\, r\,\xi' + \mu_{32}\, r\,\eta' + \mu_{33}\, r\,\zeta', \end{cases}$$

in which $\mu_{23} = \mu_{32}$, $\mu_{31} = \mu_{13}$, $\mu_{12} = \mu_{21}$.

The kinetic energy has the value:

$$T = \frac{1}{8\pi}(\mathfrak{L}\, L + \mathfrak{M}\, M + \mathfrak{N}\, M)$$

here, or:

$$(7)\qquad T = \frac{r^2}{8\pi}(\mu_{11}\,\xi'^2 + \mu_{22}\,\eta'^2 + \mu_{33}\,\zeta'^2 + 2\mu_{23}\,\eta'\zeta' + 2\mu_{31}\,\zeta'\xi' + 2\mu_{12}\,\xi'\eta') \;\ldots$$

Spatial differences in the "rotations" ξ, η, ζ will also produce "stresses" in the medium here. The anisotropy in the medium will have a certain influence on the stress distribution itself since the medium can be stressed differently in different directions. The stresses that are produced are those of the dielectric polarization, which are determined uniquely by the three direction quantities $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, which are referred to the three axis directions for each point. As before, we must then set, analogously:

$$(8)\qquad \begin{cases} \mathfrak{X} = \mathfrak{s}\left(\dfrac{\partial\eta}{\partial z} - \dfrac{\partial\zeta}{\partial y}\right), \\ \mathfrak{Y} = \mathfrak{s}\left(\dfrac{\partial\zeta}{\partial x} - \dfrac{\partial\xi}{\partial z}\right), \\ \mathfrak{Z} = \mathfrak{s}\left(\dfrac{\partial\xi}{\partial y} - \dfrac{\partial\eta}{\partial z}\right) \end{cases}$$

($\mathfrak{s}$ is an auxiliary constant). However, in addition to representing the stress state of the medium by those three direction quantities $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, there is yet another way that involves three quantities X, Y, Z that are linearly coupled with the first ones. If we set:

$$(9.a)\qquad \begin{cases} \mathfrak{X} = \varepsilon_{11}\, X + \varepsilon_{12}\, Y + \varepsilon_{13}\, Z, \\ \mathfrak{Y} = \varepsilon_{21}\, X + \varepsilon_{22}\, Y + \varepsilon_{23}\, Z, \\ \mathfrak{Z} = \varepsilon_{31}\, X + \varepsilon_{32}\, Y + \varepsilon_{33}\, Z, \end{cases}$$

([1]) **H. Hertz**, I, pp. 592, formula 9.c (Untersuch., pp. 224).

or

$$(9.b)\qquad \begin{cases} X = \alpha_{11}\,\mathfrak{X} + \alpha_{12}\,\mathfrak{Y} + \alpha_{13}\,\mathfrak{Z}, \\ Y = \alpha_{21}\,\mathfrak{X} + \alpha_{22}\,\mathfrak{Y} + \alpha_{23}\,\mathfrak{Z}, \\ Z = \alpha_{31}\,\mathfrak{X} + \alpha_{32}\,\mathfrak{Y} + \alpha_{33}\,\mathfrak{Z}, \end{cases}$$

in which the α are proportional to the corresponding subdeterminants of the ε, so one calls X, Y, Z the components of the electric force. For anisotropic bodies, it is not directly accessible to observation. It is only in isotropic bodies that the X, Y, Z go to the $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$. For the ε, one will again have:

$$\varepsilon_{23} = \varepsilon_{32}, \qquad \varepsilon_{31} = \varepsilon_{13}, \qquad \varepsilon_{12} = \varepsilon_{21}$$

and as a result, one will also have:

$$\alpha_{23} = \alpha_{32}, \qquad \alpha_{31} = \alpha_{13}, \qquad \alpha_{12} = \alpha_{21}\ .$$

If the anisotropic medium is inhomogeneous then the ε and α will be functions of position.

The potential energy Φ is now:

$$\Phi = \frac{1}{8\pi}(\mathfrak{X}\,X + \mathfrak{Y}\,Y + \mathfrak{Z}\,Z)$$

or

$$(10)\quad \begin{cases} \Phi = \dfrac{s^2}{8\pi}\Bigg[\alpha_{11}\left(\dfrac{\partial\eta}{\partial z} - \dfrac{\partial\zeta}{\partial y}\right)^2 + \alpha_{22}\left(\dfrac{\partial\zeta}{\partial x} - \dfrac{\partial\xi}{\partial z}\right)^2 + \alpha_{33}\left(\dfrac{\partial\xi}{\partial y} - \dfrac{\partial\eta}{\partial x}\right)^2 \\ \quad + 2\alpha_{23}\left(\dfrac{\partial\zeta}{\partial x} - \dfrac{\partial\xi}{\partial z}\right)\left(\dfrac{\partial\xi}{\partial y} - \dfrac{\partial\eta}{\partial x}\right) + 2\alpha_{31}\left(\dfrac{\partial\xi}{\partial y} - \dfrac{\partial\eta}{\partial x}\right)\left(\dfrac{\partial\eta}{\partial z} - \dfrac{\partial\zeta}{\partial y}\right) + 2\alpha_{12}\left(\dfrac{\partial\eta}{\partial z} - \dfrac{\partial\zeta}{\partial y}\right)\left(\dfrac{\partial\zeta}{\partial x} - \dfrac{\partial\xi}{\partial z}\right)\Bigg]. \end{cases}$$

We would like to assume that the μ, ε, and α are independent of time, so the medium will experience no variations.

The derivation of the first **Hertzian** system of equations is also quite immediate here. If one differentiates equations (8) with respect to time, substitutes the values of ξ', η', ζ' in (5) and writes the quantity A for r / s then one will get:

$$A\frac{d\mathfrak{X}}{dt} = A\left(\varepsilon_{11}\frac{dX}{dt} + \varepsilon_{12}\frac{dY}{dt} + \varepsilon_{13}\frac{dZ}{dt}\right) = \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y},$$

and similarly for the other components. However, that is the **Hertzian** system of equations for the time variation of the electric polarization when it depends upon the instantaneous spatial distribution of the magnetic forces for crystalline non-conductors.

For conductors, time variations of the electric polarization state will also appear as a consequence of conduction. Due to the anisotropy, that dissipation in electric stress will generally be a linear function of the three polarization components again, and therefore of the force components. With **Hertz**, if we then set:

$$u = \lambda_{11}(X - X') + \lambda_{12}(Y - Y') + \lambda_{12}(Z - Z'),$$

in which the X', Y', Z' are components of electromotive forces of hydroelectric or thermoelectric origin that possibly act at certain locations in the medium, then we will get the equations:

$$A\frac{d\mathfrak{X}}{dt} = \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y} - 4\pi A\, u\,, \qquad \text{etc.,}$$

i.e., equations (9.b) in **Hertz** for the anisotropic conductor.

We can also obtain the corresponding system of equations here that represents time variations of the magnetic polarization as something that is required by the distribution of electric forces in the neighborhood when we apply **Hamilton**'s principle to our system of cyclic velocities and curls of cyclic coordinates. When we likewise perform the variations that relate to the cyclic coordinates (and in that way, the μ and α, which are indeed functions of the position coordinates x, y, z, will remain unvaried), the expression for (H) (pp. 8) will assume the form:

$$\begin{aligned}
0 = \int_{t_0} dt \iiint_{-\infty} \Bigg\{ \frac{r^2}{4\pi} [&\mu_{11}\,\xi'\cdot\delta\xi' + \mu_{22}\,\eta'\cdot\delta\eta' + \mu_{33}\,\zeta'\cdot\delta\zeta' \\
&+ \mu_{23}\,\eta'\cdot\delta\zeta' + \mu_{23}\,\zeta'\cdot\delta\eta' + \mu_{31}\,\zeta'\cdot\delta\xi' + \mu_{31}\,\xi'\cdot\delta\zeta' + \mu_{12}\,\xi'\cdot\delta\eta' + \mu_{12}\,\eta'\cdot\delta\xi'] \\
&- \frac{s^2}{4\pi}\Bigg[\alpha_{11}\left(\frac{\partial\eta}{\partial z} - \frac{\partial\zeta}{\partial y}\right)\left(\frac{\partial\delta\eta}{\partial z} - \frac{\partial\delta\zeta}{\partial y}\right) + \alpha_{22}\left(\frac{\partial\zeta}{\partial x} - \frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\delta\zeta}{\partial x} - \frac{\partial\delta\xi}{\partial z}\right) \\
&+ \alpha_{33}\left(\frac{\partial\xi}{\partial y} - \frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\delta\xi}{\partial y} - \frac{\partial\delta\eta}{\partial y}\right) + \alpha_{23}\left(\frac{\partial\zeta}{\partial x} - \frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\delta\xi}{\partial y} - \frac{\partial\delta\eta}{\partial y}\right) \\
&+ \alpha_{23}\left(\frac{\partial\xi}{\partial y} - \frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\delta\zeta}{\partial x} - \frac{\partial\delta\xi}{\partial z}\right) + \alpha_{31}\left(\frac{\partial\xi}{\partial y} - \frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\delta\eta}{\partial z} - \frac{\partial\delta\zeta}{\partial x}\right) \\
&+ \alpha_{12}\left(\frac{\partial\eta}{\partial z} - \frac{\partial\zeta}{\partial y}\right)\left(\frac{\partial\delta\zeta}{\partial x} - \frac{\partial\delta\xi}{\partial z}\right) + \alpha_{12}\left(\frac{\partial\zeta}{\partial x} - \frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\delta\eta}{\partial z} - \frac{\partial\delta\zeta}{\partial y}\right)\Bigg]\Bigg\} d\tau\,.
\end{aligned}$$

Upon performing the time integral in the first part of that expression and the partial integrations over the spatial coordinates in the second part, applying the limiting values as in pp. 8, and setting expressions that multiply the variations in them equal to zero, one will get:

$$r^2[\mu_{11}\cdot\xi''+\mu_{12}\cdot\eta''+\mu_{13}\cdot\zeta''] = s^2\left[\frac{\partial}{\partial y}\left(\alpha_{31}\left(\frac{\partial\eta}{\partial z}-\frac{\partial\zeta}{\partial y}\right)+\alpha_{32}\left(\frac{\partial\zeta}{\partial x}-\frac{\partial\xi}{\partial z}\right)+\alpha_{33}\left(\frac{\partial\xi}{\partial y}-\frac{\partial\eta}{\partial x}\right)\right)\right.$$

$$\left.-\frac{\partial}{\partial z}\left(\alpha_{21}\left(\frac{\partial\eta}{\partial z}-\frac{\partial\zeta}{\partial y}\right)+\alpha_{22}\left(\frac{\partial\zeta}{\partial x}-\frac{\partial\xi}{\partial z}\right)+\alpha_{23}\left(\frac{\partial\xi}{\partial y}-\frac{\partial\eta}{\partial x}\right)\right)\right],$$

or when one recalls (5), (8), and (9.b) and sets $r / s = A$:

$$A\left(\mu_{11}\frac{dL}{dt}+\mu_{12}\frac{dM}{dt}+\mu_{13}\frac{dN}{dt}\right) = A\frac{d\mathfrak{L}}{dt} = \frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z},$$

and analogously for the remaining coordinates. Those are equations (9.a) in **Hertz**, which collectively make up the second **Hertzian** system for the general case of anisotropic homogeneous or inhomogeneous bodies.

By considerations that are completely analogous to the ones above on pp. 14, one can also extend that system in such a way that it will be valid for rapidly-alternating electromagnetic states. Higher derivatives of the polarization vectors with respect to time will then appear, and the equations will also encompass the phenomena of normal and anomalous dispersion in anisotropic media ([1]).

5. – Deriving the fundamental equations for moving media.

We have the reduced the magnetic and electric phenomena to processes of motion that are bound with the positions of the magnetic lines of force and to the "curls" that result from inequalities in those motions at the neighboring locations. From the basic hypothesis upon which **Hertz** constructed the electrodynamics of moving media, the individual particles of the medium will advance along the lines of force under the motion. That will also be the case when the fundamental processes are located completely in the ether since one can likewise assume that it moves completely with the material substrate medium without contradicting the facts ([2]). We must then be able to obtain the fundamental equations for moving bodies with no further assumptions about the nature of the electromagnetic processes. According to **Hertz**, we only have to refer the magnetic and electric polarizations (forces, resp., i.e., cyclic motions and curls) to the system of polarization lines themselves and introduce parameters *a*, *b*, *c* whose various systems of values characterize the various particles of the medium as coordinates ([3]).

([1]) Cf., **P. Drude**, Nachr. Gött. Ges. d. Wissensch. 1893.

([2]) **H. Hertz**, II, pp. 371. (Untersuch., pp. 257).

([3]) In direct connection with **Hertz**'s equations, **Vito Volterra** has shown [Il nuovo Cimento (3) **29** (1891), pp. 53] that by introducing such a reference system, it will be possible to put the fundamental equations of electrodynamics into such a form that they will be simultaneously valid for bodies in motion, as well as at rest, which is even based upon the admissibility of **Hertz**'s fundamental hypothesis.

We use the term "polarization lines" to refer to lines that give the direction of polarization at each point in the same sense as when we speak of lines of force in isotropic media.

Let a certain particle of the medium be determined by the parameters a, b, c of its position. The a, b, c are mutually independent variables, but they will depend upon time t. We consider the rectangular coordinates x, y, z that a particle will have at time t to be functions of a, b, c, and time:

$$x = f(a, b, c, t)\,, \qquad y = g(a, b, c, t)\,, \qquad z = h(a, b, c, t)\,,$$

and denote the functional determinant of the x, y, z with respect to the a, b, c by D :

$$D = \begin{pmatrix} x & y & z \\ a & b & c \end{pmatrix} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}. \tag{11}$$

We can then consider the a, b, c to be the position coordinates that the individual particles of the medium had at time $t = 0$, just as we do in hydrodynamics when we write the equations of motion in the "second **Euler** form" ([1]).

The square of the line element that is placed arbitrarily in space is then a homogeneous function of degree two in the differentials da, db, dc. We would like to denote the coefficients of that quadratic function by H_{ik} in such a way that:

$$\left\{ \begin{aligned} ds^2 = \quad & H_{11} \cdot da^2 + H_{22} \cdot db^2 + H_{33} \cdot dc^2 \\ & + 2H_{32} \cdot db\, dc + 2H_{31} \cdot dc\, da + 2H_{12} \cdot da\, db. \end{aligned} \right. \tag{12}$$

Since $H_{ik} = H_{ki}$, the determinant of the H_{ik} is equal to D^2, so:

$$D^2 = \begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix}.$$

We then form the subdeterminants of the determinant and set:

([1]) The method for representing fluid motions analytically in such a way that the coordinates x, y, z of any particle of the fluid at time t are considered to be functions of time t and its position a, b, c at time $t = 0$ is ordinarily ascribed to **Lagrange**, but it was already found by **Euler** in full generality twenty-nine years earlier. [**Euler**, "De principiis motus fluidorum," Novi comm. acad. sc. imp. Petropolitanae **14** (1769), pp. 358]. We shall then call them the second **Euler** form of the differential equations of hydrodynamics.

$$\frac{1}{D}\begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix} = \mathfrak{u}_a\,, \qquad \frac{1}{D}\begin{vmatrix} H_{21} & H_{23} \\ H_{31} & H_{33} \end{vmatrix} = \mathfrak{u}_b\,, \qquad \frac{1}{D}\begin{vmatrix} H_{21} & H_{22} \\ H_{31} & H_{32} \end{vmatrix} = \mathfrak{u}_c\,,$$

$$\frac{1}{D}\begin{vmatrix} H_{12} & H_{13} \\ H_{32} & H_{33} \end{vmatrix} = \mathfrak{v}_a\,, \qquad \frac{1}{D}\begin{vmatrix} H_{11} & H_{13} \\ H_{31} & H_{33} \end{vmatrix} = \mathfrak{v}_b\,, \qquad \frac{1}{D}\begin{vmatrix} H_{11} & H_{12} \\ H_{31} & H_{32} \end{vmatrix} = \mathfrak{v}_c\,,$$

$$\frac{1}{D}\begin{vmatrix} H_{12} & H_{13} \\ H_{22} & H_{23} \end{vmatrix} = \mathfrak{w}_a\,, \qquad \frac{1}{D}\begin{vmatrix} H_{11} & H_{13} \\ H_{21} & H_{23} \end{vmatrix} = \mathfrak{w}_b\,, \qquad \frac{1}{D}\begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix} = \mathfrak{w}_c\,.$$

We imagine that we have constructed electric and magnetic polarization lines that would coincide with the lines of force in isotropic bodies. Analytically, they are represented by the lines of intersection of the two families of surfaces: $\varphi_1\,(a, b, c) = E_1$ and $\varphi_2\,(a, b, c) = E_2$ for the electric polarizations and $\psi_1\,(a, b, c) = M_1$ and $\psi_2\,(a, b, c) = M_2$ for the magnetic ones. The products of the three two-rowed functional determinants will then be:

$$\text{(19)} \qquad \begin{cases} A\dfrac{d\,(D\mathfrak{B}_a)}{dt} = \dfrac{\partial U_c}{\partial b} - \dfrac{\partial U_b}{\partial c}, \\ A\dfrac{d\,(D\mathfrak{B}_b)}{dt} = \dfrac{\partial U_a}{\partial c} - \dfrac{\partial U_c}{\partial a}, \\ A\dfrac{d\,(D\mathfrak{B}_c)}{dt} = \dfrac{\partial U_b}{\partial a} - \dfrac{\partial U_a}{\partial b}, \end{cases}$$

$$\text{(20)} \qquad \begin{cases} A\dfrac{d\,(D\mathfrak{U}_a)}{dt} = \dfrac{\partial V_c}{\partial c} - \dfrac{\partial V_b}{\partial b} - 4\pi A\cdot W_a, \\ A\dfrac{d\,(D\mathfrak{U}_b)}{dt} = \dfrac{\partial V_a}{\partial a} - \dfrac{\partial V_c}{\partial c} - 4\pi A\cdot W_b, \\ A\dfrac{d\,(D\mathfrak{U}_c)}{dt} = \dfrac{\partial V_b}{\partial b} - \dfrac{\partial V_a}{\partial a} - 4\pi A\cdot W_c. \end{cases}$$

Those are the electrodynamical equations in their most-general form. They are valid in the same form for *moving* media, as well as ones at rest ([1]).

In order to also derive them from our basic representation, we would like to consider the rectangular coordinates x, y, z as functions of the parameters a, b, c whose various systems of values characterize the individual particle of the medium. The angular velocities at one location are known to have the following values then:

$$\xi' = A\,\frac{\partial x}{\partial a} + B\,\frac{\partial x}{\partial b} + C\,\frac{\partial x}{\partial c},$$

([1]) Cf., **Vito Volterra**, *loc. cit.*, pp. 60.

$$\eta' = A\frac{\partial y}{\partial a} + B\frac{\partial y}{\partial b} + C\frac{\partial y}{\partial c},$$

$$\zeta' = A\frac{\partial z}{\partial a} + B\frac{\partial z}{\partial b} + C\frac{\partial z}{\partial c},$$

$$A = -\frac{1}{2D}\left[\begin{pmatrix} x & u \\ b & c \end{pmatrix} + \begin{pmatrix} y & v \\ b & c \end{pmatrix} + \begin{pmatrix} z & w \\ b & c \end{pmatrix}\right], \qquad \text{etc.}$$

The ξ, η, ζ are functions of the a, b, c now.

If we again make the magnetic forces proportional to the cyclic velocities then we will have:

$$V_a = mA\frac{\partial x}{\partial a} + mB\frac{\partial x}{\partial b} + C\frac{\partial x}{\partial c}, \qquad \text{etc.}$$

From (17), the components of the magnetic polarization are:

$$\mathfrak{B}_a = M_{11}\, V_a + M_{12}\, V_b + M_{13}\, V_c\,, \quad \text{etc.}$$

The kinetic energy now has the value:

$$T = \frac{m^2}{8\pi}(M_{11}\,\xi'^2 + M_{22}\,\eta'^2 + M_{33}\,\eta'^2 + 2M_{23}\,\eta'\zeta' + 2M_{31}\,\zeta'\xi' + 2M_{12}\,\xi'\eta')\,. \tag{21}$$

The rotations are now:

$$\xi = \int_0^1 dt\left(A\frac{\partial x}{\partial a} + B\frac{\partial x}{\partial b} + C\frac{\partial x}{\partial c}\right),$$

$$\eta = \int_0^1 dt\left(A\frac{\partial y}{\partial a} + B\frac{\partial y}{\partial b} + C\frac{\partial y}{\partial c}\right),$$

$$\zeta = \int_0^1 dt\left(A\frac{\partial z}{\partial a} + B\frac{\partial z}{\partial b} + C\frac{\partial z}{\partial c}\right).$$

For the components of the dielectric polarization, one now has:

$$(22)\qquad \begin{cases} D\,\mathfrak{U}_a = n\left(\dfrac{\partial\eta}{\partial c} - \dfrac{\partial\zeta}{\partial b}\right), \\ D\,\mathfrak{U}_b = n\left(\dfrac{\partial\zeta}{\partial a} - \dfrac{\partial\xi}{\partial c}\right), \\ D\,\mathfrak{U}_c = n\left(\dfrac{\partial\xi}{\partial b} - \dfrac{\partial\eta}{\partial a}\right). \end{cases}$$

One gets the electric forces by solving that system of equations while recalling equations (15):

$$U_a = A_{11}\frac{n}{D}\left(\frac{\partial\eta}{\partial c} - \frac{\partial\zeta}{\partial b}\right) + A_{12}\frac{n}{D}\left(\frac{\partial\zeta}{\partial a} - \frac{\partial\xi}{\partial c}\right) + A_{13}\frac{n}{D}\left(\frac{\partial\xi}{\partial b} - \frac{\partial\eta}{\partial a}\right), \quad \text{etc.}$$

The potential energy Φ now has the value:

$$\Phi = \frac{1}{8\pi}\frac{n^2}{D^2}\left[A_{11}\left(\frac{\partial\eta}{\partial c} - \frac{\partial\zeta}{\partial b}\right)^2 + A_{22}\left(\frac{\partial\zeta}{\partial a} - \frac{\partial\xi}{\partial c}\right)^2 + A_{33}\left(\frac{\partial\xi}{\partial b} - \frac{\partial\eta}{\partial a}\right)^2\right.$$

$$\left. + 2A_{23}\left(\frac{\partial\zeta}{\partial a} - \frac{\partial\xi}{\partial c}\right)\left(\frac{\partial\xi}{\partial b} - \frac{\partial\eta}{\partial a}\right) + 2A_{31}\left(\frac{\partial\xi}{\partial b} - \frac{\partial\eta}{\partial a}\right)\left(\frac{\partial\eta}{\partial c} - \frac{\partial\zeta}{\partial b}\right) + 2A_{12}\left(\frac{\partial\eta}{\partial c} - \frac{\partial\zeta}{\partial b}\right)\left(\frac{\partial\zeta}{\partial a} - \frac{\partial\xi}{\partial c}\right)\right].$$

Upon differentiating that with respect to time, one will get the first generalized **Hertzian** system of equations from the system of equations (22) immediately:

$$A\frac{d\,(D\,\mathfrak{U}_a)}{dt} = \frac{\partial V_b}{\partial c} - \frac{\partial V_c}{\partial b}, \qquad \text{etc.,}$$

when one sets $m / n = A$. If currents are present then terms like $-4\pi A\, W_a$ must be added to the right-hand side, and one will get:

$$A\frac{d\,(D\,\mathfrak{U}_a)}{dt} = \frac{\partial V_b}{\partial c} - \frac{\partial V_c}{\partial b} - 4\pi A \cdot W_a, \qquad \text{etc.,}$$

as above in (20) on pp. 16.

One will get the second **Hertzian** system in a way that is completely analogous to the one above by setting the variation of the following integral equal to zero:

$$\int_0^{t_1} dt\left(\int\int_{-\infty}^{+\infty}\int D\,(T - \Phi)\,d\tau\right),$$

in which triple integral is, however, extended over all systems of values for the a, b, c, and the definite integrals that are obtained by the partial integrations that must be performs are always defined for those variables. One can apply the same reasoning as on pp. 9 above to them, as well as the time integral. From a known theorem of **Jacobi**, the functional determinant D of the x, y, z with respect to the a, b, c will appear under the integral when one no longer expresses the volume element $d\tau$ in terms of dx, dy, dz now, but in terms of da, db, dc. The calculations take a form that is completely analogous to the one on pp. 13, and passing to the time element and setting the terms in it that are multiplied by the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$ equal to zero, one will ultimately get:

$$\begin{aligned}
& m\frac{d}{dt}D(M_{11}\,m\xi' + M_{12}\,m\eta' + M_{13}\,m\zeta') \\
& - n^2\left[\frac{\partial}{\partial b}\left(\frac{A_{31}}{D}\left(\frac{\partial\eta}{\partial c}-\frac{\partial\zeta}{\partial b}\right)+\frac{A_{32}}{D}\left(\frac{\partial\zeta}{\partial a}-\frac{\partial\xi}{\partial c}\right)+\frac{A_{33}}{D}\left(\frac{\partial\xi}{\partial b}-\frac{\partial\eta}{\partial a}\right)\right)\right. \\
& \left. - \frac{\partial}{\partial c}\left(\frac{A_{21}}{D}\left(\frac{\partial\eta}{\partial c}-\frac{\partial\zeta}{\partial b}\right)+\frac{A_{22}}{D}\left(\frac{\partial\zeta}{\partial a}-\frac{\partial\xi}{\partial c}\right)+\frac{A_{23}}{D}\left(\frac{\partial\xi}{\partial b}-\frac{\partial\eta}{\partial a}\right)\right)\right] = 0\,, \text{ etc.}
\end{aligned}$$

However, when one recalls (17), (22), and (15), that is:

$$A\frac{d\,(D\mathfrak{B}_a)}{dt} = \frac{\partial U_b}{\partial b} - \frac{\partial U_b}{\partial c}\,, \qquad \text{etc.,}$$

i.e., the second generalized **Hertzian** system.

In regard to the phenomena of dispersion and absorption, the two systems of equations admit precisely the same extension as the equations on pp. 9.

6. – Illustrating the theory.

Our assumptions about the nature of the magnetic and electric phenomena can be explained with the following ether model of **George Fr. Fitzgerald** ([1]). A vertical board carries several horizontal rows of axles that are arranged vertically above each other and are fixed perpendicular to the board. Brass rollers are fixed along those axles that rotate as easily as possible, and four V-shaped, circular indentations are cut into them. Each roller is coupled with its four neighboring ones by rubber bands. From our basic picture, that will explain electromagnetic phenomena since it represents an elastically-coupled monocycle. The energy of the brass roller rotating around its axle represents the energy of the magnetic forces, and the tensions that appear in the rubber bands when neighboring rollers rotate with different speeds (viz., curl) represent the electric forces. In that way, one side of the band will be tensed, while the other one will be more or less loosened.

([1]) **George Fr. Fitzgerald**, Proc. R. Dublin Soc. (New series) **4** (1885), pp. 407. Cf., also Transactions of the Royal Society of London **171**, Part 2 (1880), pp. 691, where the author derived some formulas from **Maxwell**'s equations that are closely related to the ones that were developed here.

The difference in the tensions can serve as a measure of **Maxwell**'s dielectric change of state (i.e., displacement), which is itself proportional to the electric force.

That is because, for **Maxwell**, the dielectric polarization (i.e., displacement) is not produced by opposing displacements of something substantial (so polar differentials), but generally refers to the state of a medium in which polar opposite states are found on opposite sides of an element, which is precisely like the tensing and loosening of the elastic couplings between rollers in **Fitzgerald**'s model. The direction of the "displacement" has the same place in the model then that the perpendiculars to direction of the magnetic forces (i.e., roller axles) has in the theory.

Fitzgerald (*loc. cit.*) showed in detail how one can use the model to illustrate electromagnetic processes in the free ether, and above all, the advancement of electromagnetic waves, as well.

The dissipation of stress in or on conductors corresponds to the partial slippage of the rubber bands, which is coupled with friction and therefore the production of heat. **O. Lodge** called that process "slip" in the elementary motions on conductors, in contrast to the "spin" of the rotations in free ether that result with no loss of energy (*Modern Views of Electricity*).

However, it seems to me that the most important property of this model, which has still not been emphasized since then, is that it makes the basic ideas of **Hertz**'s theory immediately intuitive and also allows one to easily find **Hertz**'s fundamental equation for the case of the isotropic media at rest without having to employ **Hamilton**'s principle. That is because it shows how the time variation of one of the quantities that determines the electromagnetic state is caused by the positional variation (viz., the "curl") of the others. If neighboring rollers rotate equally fast, so no difference in the magnetic force will then exist, then the tensions that might exist in the elastic bands will not change. That will first occur when some rollers are rotating faster than the ones in their neighborhood. Conversely, the state of motion of one roller can be changed only in such a way that immediately beforehand, the stress relationships between the rubber bands that are attached to the roller will be different in different directions. If we formulate those two statements then we will get the two **Hertzian** systems of equations.

The first system of equations (I) on pp. 5 is obtained immediately from the consideration of the time variation that the tensions experience as a result of positional differences in the rotations that was just described. If we recall the notations that were introduced into equations (1) and (3) on pps. 4 and 5 then we will get:

$$\text{(I)} \qquad A\varepsilon \frac{dX}{dt} = \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y}, \quad \text{etc.,}$$

In order to obtain the second system of equations (II) on pp. 9, we imagine that the large, widely-separated rollers of the model are replaced with a large number of small and densely-packed rollers that are coupled to each other with rubber bands. We consider the processes on a surface patch $dy \cdot dz$ that includes *one* roller. (The plane of the board is thought of as being the *yz*-plane, where the *y*-direction shall run parallel to a series of rollers and the *z*-direction shall run parallel to the series that is perpendicular to it.) Y is then the difference in tensions on one side of the roller, $Y + (\partial Y / \partial z) \cdot dz$ is the tension on the opposite side, Z and $Z + (\partial Z / \partial y) \cdot dy$ are the tensions in the pair of bands that are perpendicular to it, and:

$$\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right) dy\, dz$$

is the algebraic sum of all tension differences all around it [the line integral of the vectors Y and Z around the surface $dy \cdot dz$, the "version," as **Heaviside** called it ([1])]. However, since every impulse that changes the motion of a roller is present immediately beforehand as a difference in tension in one of the rubber bands that supplies it, the total state of tension in all bands that exists at an instant all around a roller will determine the time variation $\mu\, dL\, /\, dt$ of the state of motion of the roller that results immediately afterwards. The same consideration will be valid when the surface patch includes not just one, but several rollers. The state of tension on the periphery that is caused by the motion of the rollers that lie outside of it will then determine the mean variation of the state of motion, which must exist on the surface immediately afterwards: $A\ \mu\ (dL\ /\ dt)\ dy\ dz$. (We shall denote the mean value by an overbar.) If we imagine that the rollers are very small and very densely packed, so the surface $dy\ dz$ will be likewise very small, then $\overline{\mu\, dL\, /\, dt}$, viz., the mean of $\mu\, dL\, /\, dt$ over a surface, will go to the value at a certain location. It is then equal to:

$$\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right) dy\, dz\,,$$

so we will get:

$$\mu\, A\frac{dL}{dt}\, dy\, dz\ = \left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right) dy\, dz\,,$$

and if we divide both sides by the surface elements then the second **Hertzian** system of equations will follow:

(II) $$\mu\, A\frac{dL}{dt}\ = \frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\,, \qquad \text{etc.}$$

Analogous equations are valid for the other coordinate plane.

We can similarly explain the relationships and derive the equations on pp. 10 in the more-complicated case when we include the finite speed of propagation of the changes of state in our calculations, and in that way we will obtain the equations that imply the dispersion phenomena. If a roller rotates rapidly through a certain angle then initially only the neighboring bands will be tensed. It is only when the tension wave reaches the next rollers that they will also be set in motion, etc.

([1]) **Heaviside**, *Electrical Papers*, **1**, 1892, pp. 211.

7. – Mechanical meaning of electrical conductivity.

A certain difficulty in the theories up to now is raised by the question of how the intrinsic nature of a conductor differs from that of a non-conductor. One says that the conductor holds the field energy, but obviously that expresses only a fact of experience. According to **Poynting**, we can define a conductor in full generality as a body, whether solid, liquid, or gaseous, that is in a position to continually give rise to a partial conversion of energy of the field into the energy of disordered motions, so into heat. Whereas in the free field, the individual particles of the ether will be pulled into cyclic (vortex) motion, on the boundary surface of the free ether, something will oppose such things that is connected with the molecules of a solid, liquid, or gaseous body, which corresponds mechanically to a sliding motion, and with how it is coupled with a transformation of energy in a high state of order into energy of a lower state of order, so into thermal motion. However, it in just that way that the conductor will take on the property that it can carry the energy of organized motion (e.g., the electromagnetic field) over long distances with minor dissipation, such as would be possible by, e.g., free radiation, so the conductor would become a "leader" of the energy.

If we believe in the basic picture that is developed here then the field energy of a current-carrying conductor will have the form of the kinetic energy of the ether particles that might orbit around its concentrically-enclosed magnetic field lines in a vorticial motion. In fact, the work that is necessary to construct the field by way of a current that is ultimately brought up to the current strength J is equal $\frac{1}{2} L J^2$, in which L denotes the coefficient of self-induction of the current path, is completely analogous to the way that the (kinetic) energy $\frac{1}{2} M q^2$ will be present in a monocycle whose cyclic velocity has achieved the value q, when M denotes the total moment of inertia. **Maxwell** already showed that the centrifugal forces of that vortex would suffice completely ([1]) to explain the lateral pressure that the ether masses exerted upon each other while in their vortical motions that **Faraday** required, as well as the tensions along the force lines themselves ([2]). The force lines then seek to contract together around the conductor like extended rubber bands. If the vortices that lie on the outer surface were initially devoid of any energy then they would have to contract since, according to **H. von Helmholtz**, the product $Q\, k \cdot l$ of the vortex intensity (cross-section Q times the vortex velocity k) with the length l is a constant. At the locations that become free, some of them will push together, while other ones push apart, in the way that the theory of vortices would indicate. Thus, purely-mechanical sources will imply the migration of energy through the field that takes place perpendicular to the surface of the conductor that Poynting showed would be included as an essential component of **Maxwell**'s theory, as well as the phenomenon of the advance of field energy. The dissipation of energy that takes place in the molecules of the surface of the conductor will be the guiding principle for the energy. A cable of purest copper at a temperature of – 273° will lose its capacity to transform energy of ordered motions into heat motion, since its molecules will no longer displace with respect to each other, so it will no longer serve as the guiding principle for electrical field energy; we say that "it no longer

([1]) **Cl. Maxwell**, Phil. Mag. (4) **21**, pp. 165, *et seq.*

([2]) Cf., also **L. Grätz**, who showed that one could arrive at entirely-plausible values for the diameters of those molecular vortices, as well as for the density of the ether that is connected with them. Wied. Ann. **25** (1885), pp. 165.

conducts." (Cf., the attempt of **Cailletet** and **Bounty**, **Wroblewski**, and **Dewar**, which came more or less close to showing that.) For ordinary solid or liquid conductors of galvanic currents, we shall point out only the final result that a transformation into heat motion will take place without giving any insight into the mechanism of that transformation, just as we cannot track the conversion of visible kinetic energy into heat energy in a collision. Above all, we will not see whether (luminescent) motions in the molecules will be excited, which are initially only intramolecular ones, essentially, and which will then be first converted into thermal motions, or whether the latter are the primary ones. The behavior of gases will give us information about that, and it shows that the former is the case. For dilute gases, under favorable conditions, that transformation of energy will produce light, and we can then observe those processes that make a body into a conductor, i.e., the guiding principle for field energy, directly. Here, I shall refer to the experiment that **E. Wiedemann** and myself have described ([1]). There, one will see, e.g., how a process of scattered radiation whose energy has a sufficient magnitude will draw the field lines over to locations whether they can never intersect themselves, according to the laws of electrostatics or those of the spreading of rapid electrical oscillations.

The fact that certain substances (above all, the conductors of the first class and electrolytes, to some degree) are capable of converting electromagnetic energy and attracting it can be connected with the property that ionization will take place in them under the influence of electric forces. For metals, the ease with which that can exist in the form of isolated atoms might affect that. That is the case when we already have free ions present from the outset. The capacity of the molecules to dissociate also seems to play a major role for gases ([2]).

8. – Application to gas discharges.

It seems to me that the theory of electrical current that was developed here also tells us that we are forced to ascribe a certain direction to the electric current, which finds its expression, above all, in the phenomena of gas discharges between anodes and cathodes.

H. von Helmholtz has shown that a vortex ring will always advance in the sense in which its particles advanced through its opening and that a vortex ring that impinges upon a wall will not be reflected by it but will remain attached to it and spread across it ([3]). If we then create electric stresses at the ends of a discharge tube that is equipped with metallic electrodes then, from our basic assumption, and complete agreement with **Maxwell**, those stresses cannot be constructed at the individual points of the *field* at all without rotational motions, i.e., magnetic motions or vortices. (In **Fitzgerald**'s model, the previously-unstressed rubber bands in the dielectric can be put into a stressed state only when the individual rollers are rotated.) The time variation that produces polarization (i.e., displacement) in dielectrics is equal to the distribution of the rotation

([1]) **H. Ebert** and **E. Wiedemann**, Wied. Ann. **50** (1893), pp. 1, and in particular, pp. 18, *et seq.*

([2]) **H. Ebert** and **E. Wiedemann**, Wied. Ann. **50** (1893), pp. 22. Cf., also **J. J. Thomson**, *Recent Researches in Electricity and Magnetism*, Oxford, 1893, in particular, pps. 54 and 56.

([3]) In that way, those vortex rings will differ essentially from elastic rings, with which they otherwise have many similarities. Moreover, it seems to me that this forms the basis for an essential difference between the purely-kinematical theory that was presented here and those theories that consider the ether to be a more or less elastic medium, if not quasi-elastic.

around it (pp. 6). Therefore, a vortex ring will be formed that encircles each particle of the medium that is put into a stressed state by the source of electricity, and its plane will be perpendicular to the direction of the electric force.

In dilute gases, vortex rings can drift unperturbed over considerable distances from the anode to the cathode, accumulate on the cathode, and produce continuing motions, and even before the actual discharge begins. If the latter is triggered then different phenomena must take place at the anode and cathode when the gas pressure is sufficiently low that those continuing motions will remain for a long time and can draw together sufficiently-extended parts in their neighborhoods in during the process of their motion. If we rapidly apply electric forces whose magnitudes and directions alternate, e.g., **Hertz**ian oscillations, then we will get two cathodes ([1]). The cathode does not need to consist of metal. Characteristic cathode phenomena will be produced at each wall (e.g., glass, sealing wax) that is exposed to the impacts of molecular vortices ([2]). That will explain a whole series of gas discharge phenomena on the basis of the same assumption that led to all other phenomena of electrodynamics [e.g., the appearance of continuing motions and accumulation phenomena that might lead to the heating or sputtering of the cathode material ([3])], the influence of the supply of electricity, and a series of phenomena in a volume of space that is filled with dilute gas and exposed to a high-frequency electric field that were described by **E. Wiedemann** and myself. We will soon communicate further details regarding other phenomena that are connected with that and explain them in terms of the picture that was developed here.

We have occasionally called our derivation of the fundamental equations of electrodynamics a "mechanical" one. However, what we have mainly employed are the expressions for energy (2) and (4) on pps. 5 and 6. We were just as well able to introduce the so-called forces as mere factors from which the energy is composed. Our derivation would then be a purely "energetic" one, in the **Ostwald** sense, and as long as it could remain purely energetic, we would also not have to deal with the special nature of the processes and the intuitive interpretation of the energy factors.

In any event, that would seem to show that one could arrive at a complete, and therefore simple, interpretation of magnetic and electric phenomena by a rational implementation of the cyclic theory of them.

Erlangen, October 1893.

([1]) **H. Ebert** and **E. Wiedemann**, Wied. Ann. **50** (1893), pp. 33.

([2]) *Loc. cit.*, pp. 41.

([3]) I shall go no further into the meaning of the electrical oscillations for the cathode phenomena at this point (cf., **E. Wiedemann** and **H. Ebert**, Sitzungsber. der Physikal.-med. Soc. zu Erlangen, Session on 14 Dec. 1891.)