“Bemerkung über die angenäherte Gültigkeit der klassischen Mechanik innerhalb der Quantenmechanik,” Zeit. Phys. **45** (1927), 455-457.

Remark on the approximate validity of classical mechanics within quantum mechanics.

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(Received on 5 September 1927)

The relation:

$$m\,\frac{d^2}{dt^2}\iiint d\tau\cdot\Psi\Psi^*\cdot \boldsymbol{x} = \iiint d\tau\cdot\Psi\Psi^*\left(-\frac{\partial V}{\partial x}\right)$$

can be derived from the **Schrödinger** equation by a brief elementary calculation *without leaving anything out*, and for a small wave packet that stays small (with m of order 1 g), that says that the acceleration of its position coordinates will come about in the sense of **Newton**’s equations of motion with a position-dependent force $-\partial V/\partial x$.

It is desirable to be able to answer the following question in the most elementary way that is possible: *How does one see Newton’s equations of classical mechanics from the standpoint of quantum mechanics?* In a whole series of recent publications, it has been, in essence, clarified completely that classical mechanics remains valid for macroscopic processes to a high degree of approximation, and in what way that would be true ([1]). However, please permit me to prove briefly a special elementary relation that follows *exactly* from **Schrödinger**’s equation without leaving anything out, because it will perhaps make the connection between wave mechanics and classical mechanics even easier to survey.

The formulas will be presented for the case of a single degree of freedom, and thus for the following form of the **Schrödinger** equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\,\Psi = i\hbar\frac{\partial\Psi}{\partial t}, \tag{1}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V(x)\,\Psi^* = -\,i\hbar\frac{\partial\Psi^*}{\partial t}. \tag{1*}$$

([1]) **Louis de Broglie**, Thèse, 1924; Journ. de phys. et le Rad. (6) **7** (1926), 1, 32; C. R. **180** (1925), 498; *ibid.* **183** (1926), 272. – **L. Brillouin**, Journ. de phys. et le Rad. **7** (1926), 353. – **E. Schrödinger**, Naturwiss. **14** (1926), 664. – **P. Debye**, Phys. Zeit. **28** (1927), 170. – **W. Heisenberg**, Zeit. Phys. **43** (1927), 172. – **E. H. Kennard**, Zeit. Phys. **44** (1927), 326.

Then, define ([1]):

$$\int_{-\infty}^{+\infty} dx \;\, x\, \Psi\Psi^* \equiv Q(t), \tag{2}$$

$$i\hbar \int_{-\infty}^{+\infty} dx\, \Psi \frac{\partial \Psi^*}{\partial x} \equiv P(t), \tag{3}$$

and calculate dQ / dt and dP / dt with the use of (1) and (2). By substitution and partial integration, one will get immediately (and *with nothing left out*):

$$\frac{dQ}{dt} = \frac{1}{m} P, \tag{4}$$

$$m \frac{d^2 Q}{dt^2} = \frac{dQ}{dt} = \int dx\, \Psi\Psi^* \left(-\frac{\partial V}{\partial x} \right). \tag{5}$$

However, equation (5) obviously says: Any time that the width of the (probability) wave packet $\Psi\Psi^*$ is relatively small (in comparison to macroscopic distances), *the acceleration (of the center of mass Q) of the wave packet will behave as it would in Newton's equations under the force* $\left(-\frac{\partial V}{\partial x} \right)$ *"that acts upon the position of the wave packet."*

Remarks: The gradual dissipation of a wave packet was discussed thoroughly by Heisenberg, *loc. cit.*. His calculation for the example of a force-free motion of a material point in a one-dimensional space can perhaps be made more familiar with the help of its close relationship to known calculations in heat conduction. For $V(x) = 0$, **Schrödinger**'s equation has the structure of the heat equation:

$$\frac{\partial \Psi}{\partial t} = a^2 \frac{\partial^2 \Psi}{\partial x^2}, \tag{6}$$

with

$$a^2 = i\, \frac{\hbar}{2m}. \tag{7}$$

If one substitutes the general solution (see, e.g., **B. Riemann-Weber**, Bd. II):

$$\Psi(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} d\xi\, e^{-\frac{(x-\xi)^2}{4a^2 t}}\, \Psi(0, x) \tag{8}$$

in it for the starting state:

([1]) If one develops Ψ in eigenfunctions: $\Psi = \sum c_n e^{iE_n t/\hbar} \varphi_n(x)$ then that will yield its relationship to the matrices $q_{nm} = e^{i(E_n - E_m)t/\hbar} \int dx \cdot x\, \varphi_n\, \varphi_m$ and p_{nm}.

$$\Psi(0,\ \xi) = C\, e^{-\frac{t^2}{4a^2 t} + i\mu\xi}, \tag{9}$$

so

$$(\Psi\Psi^*)_{t=0} = C^2 \cdot e^{-\frac{t^2}{\omega^2}} \tag{10}$$

(μ is an arbitrary *real* constant), then one will find, just like **Heisenberg**, that the later position and distribution of the "wave packet" will be:

$$\Psi\Psi^* = c(t) \cdot e^{-\frac{(x - \frac{\hbar\mu}{m} t)^2}{\Omega^2}}, \tag{11}$$

where:

$$\Omega^2 = \omega^2 + \frac{\hbar^2 t^2}{m^2 \omega^2}, \tag{12}$$

and therefore a displacement of the wave packet with the velocity $\hbar\mu / m$ and an increasing dissipation in time. A doubling of the initial width (i.e., $\Omega^2 = 4\omega^2$) will then come about after a time:

$$T = \sqrt{3}\, \frac{m\omega^2}{\hbar} \qquad \left(\hbar = \frac{6.6\times 10^{-27}}{2\pi}\right). \tag{13}$$

For $m = 1$ g, $\omega = 10^{-3}$ cm, one will have $T = 10^{21}$ sec; by contrast, for $m = 1.7 \times 10^{-24}$ g and $\omega = 10^{-8}$ cm, one will have $T = 10^{-13}$ sec!