Graphical illustration of De Broglie’s phase waves in O. Klein’s five-dimensional universe (1)

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With one figure

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De Broglie’s conception of electrons as groups of phase waves is interpreted in Klein’s theory in the case of force-free motion and the connection is illustrated graphically.

§ 1. – According to de Broglie’s ideas (2), the motion of an electron “in reality” amounts to the propagation of groups of waves in a dispersive ether that lies in the ordinary four-dimensional universe. Schrödinger (3) then extended that wave theory of matter quite appreciably. O. Klein also arrived at similar thoughts independently. The most important difference is that for him the waves propagate in a dispersion-less medium that lies in a five-dimensional universe (4). The de Broglie waves are then the “traces” of those five-dimensional waves in ordinary space.

Here, we will consider that connection more closely in the case of force-free motion and illustrate it graphically.

§ 2. – According to Klein, in the case of the force-free motion of an electron, the five-dimensional phase waves are plane waves that first of all satisfy the dispersion-less wave equation:

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + m^2 c^2 \frac{\partial^2}{\partial x_0^2} \right] U = 0 ,
\]

(1)

(1) O. Klein, Zeit. Phys. 37 (1926), 895.
(2) L. de Broglie, Ann. de Phys. (10) 3 (1925), 22.
(4) Which, as Klein assumed, is periodic in its fifth dimension with a period that is connected with Planck’s constant.
in which $x_0$ means the fifth dimension. Secondly, the $x_0$-direction has the prescribed period $h$.

We can then represent those phase or $U$-waves by:

$$U = u e^{\frac{2\pi i (h\nu x - py - qz - rz - x_0)}{h}},$$

(2)

in which the periods $1/\nu, h/p, h/q, h/r$ in the $t, x, y, z$ directions, resp., are coupled with each other by:

$$p^2 + q^2 + r^2 + m^2 c^2 = \left(\frac{h \nu}{c}\right)^2,$$

(3)

as a result of (1). Their “traces” in the ordinary universe (1) are also once more plane waves with the superluminal velocity:

$$\nu_{ph} = \frac{h \nu}{c \sqrt{p^2 + q^2 + r^2}},$$

(4)

and therefore, from (3), with the de Broglie dispersion law:

$$\nu_{ph} = \frac{h \nu}{\sqrt{h^2 \nu^2 - m^2 c^4}}.$$ 

(5)

§ 3. – When we take just one of the three ordinary spatial dimensions $x, y, z$ – say $x$ – we can then illustrate the relationship graphically.

If we write (2):

$$\tau = c t, \quad \xi_0 = \frac{x_0}{mc}$$

then one will see, first of all, that the planes of equal phase in $(x, \tau, \xi_0)$-space always define an angle of $45^\circ$ with the $\tau$-axis, and will thus be tangent to the cone $x^2 + \xi_0^2 - \tau^2 = 0$ at the origin and secondly, it repeats in the $\xi_0$-direction with a period of $h/mc$.

The periods in the $x$ and $\tau$-directions are then coupled to each other by (2). Both of them will also be established when one gives the tangent $OB$.

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(1) I. e., its intersection with an $R_4$ that is established by $x_0 = \text{const}$.

(2) For protons, one must replace $m$ with $M$ everywhere. The yardstick in the $\xi_0$-direction is different for protons and electrons.
If we now assert that the electron moves like a group that is modelled on U-waves, in the sense of the ideas of de Broglie, Schrödinger, and Klein, then it will be clear that $OB$ represents the five (three, here) dimensional world-line of the electron ($^1$), because the phases of two neighboring $U$-waves will reinforce each other only along the line $OB$.

In $(x, \tau, \xi_0)$-space, all electrons have the same velocity then ($^2$). In the ordinary four (two, here) dimensional universe, the world-lines of the electron is represented by the projection $OD$ of $OB$ onto the $x_0\tau$-plane. All velocities can then appear to be smaller than the speed of light. Now, one can further easily show that:

a) The quantities $p, q, r$, which have determined only the periods of the phase waves in the $x, y, z$ direction, up to now, also means the impulse components of the electron now.

Proof:

The equation of $OD$ is $(h\nu/c) x - pt = 0$, so the velocity of the electron in $x_0\tau$-space is:

$$v_e = \frac{cp}{h\nu},$$

and (3) will become:

$$p^2 + m^2c^2 = \left(\frac{h\nu}{c}\right)^2.$$  

If $\beta = v_e/c$ then one will find from this that:

$$p = \frac{mv_e}{\sqrt{1 - \beta^2}}.$$  

b) In the ordinary four (two, here) dimensional space, the velocity of the electron is also once more the group velocity of the “traces” of the phase waves ($^3$).

Proof:

The equation of $OA$ is: $(h\nu/c) \tau - px = 0$, so the velocity of the wave trace will be:

$$v_{ph} = \frac{h\nu}{cp} = \frac{h\nu}{\sqrt{h^2\nu^2 - m^2c^4}},$$

($^1$) Naturally, just as one does with phase waves, one must also think of $OB$ are repeating in the $\xi_0$-direction with a period of $h/mc$.

($^2$) In $(x, t, x_0)$-space, that will be a certain velocity region that is different for protons and electrons.

($^3$) As one sees from the figure and (4), they will always have superluminal velocity.
in agreement with (4) and (5). Now, one must show that the known connection between group and phase velocity exists between $v_e$ and $v_{ph}$:

$$\frac{1}{v_e} = \frac{d}{dv} \left( \frac{v}{v_{ph}} \right).$$

§ 4. – When the electron moves with a certain period (e.g., in a ring or between two walls), the phase wave must also have that period then. Now, one sees from the figure and (3) that when a certain period $L$ in the $x$-direction is given, in addition to the period $h$ in the $x_0$-direction, only a discrete number of tangents $OB$, and therefore only certain velocities, can appear in four (two, here) dimensional space. In fact, if $n$ is a whole number then one must have:

$$L = \frac{n h}{p}$$

or

$$p L = \int p dq = n h.$$  \hspace{1cm} (8)

Since $p$ also means the electron impulse, that determines the possible velocities. That is the ordinary quantum condition for this case. As one sees from the figure and (3), the period in time will also be determined then.

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