"Introduction a la théorie des structures infinitésimales et des pseudo-groupes de Lie," Colloque Internationale de Géometrie Différentielle, *C. N. R. S.* (1953), 97-110.

Introduction to the theory of infinitesimal structures and Lie pseudogroups.

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This presentation constitutes only an introduction to the study of infinitesimal structures that I would like to develop, along with its applications. Since 1951, I have presented these notions and their applications in my course at Strasbourg and Rio de Janeiro, as well as in a series of conferences that were given, for example, in Bologna, Pisa, Southampton, Leeds, Manchester, and Louvain, but I have published only very brief summaries on this subject. (*)

At the International Colloquium on Differential Geometry in Strasbourg, I briefly indicated the following applications (which will be developed in another article):

The notion of a system of partial differential equations is generalized by the given of a subset ϕ in $J^r(V_n, V_m)$ or a manifold that is embedded in or extracted from $J^r(V_n, V_m)$. One may define the prolongation of such a system. One is then led to the notion of a complete pseudogroup of transformations of order r, which, in the analytic case, gives the notion of Lie pseudogroup: Its associated groupoid is then an analytic submanifold of $\Pi^r(V_n)$. One has the definition of a pure infinitesimal structure (a notion that generalizes that of geometric object) and a regular infinitesimal structure. The pseudogroup of local automorphisms of such a structure is complete of order r. Its determination leads to the problem of generalized Cartan equivalence. Any complete Lie pseudogroup of order r has a prolongation that forms a complete Lie pseudogroup of first order. One can define Lie pseudogroups of finite type k: The associated groupoid of order k is isomorphic to the associated groupoid of order $k - 1$. The LIE GROUPS are of finite type. One can define infinitesimal transformations. A pseudogroup of transformations of finite type on a simply-connected, compact manifold is deduced, by localization, from a Lie GROUP of transformations that is defined on V_n.

1. The notion of local jet. – Let one be given a map f from a set U onto a set $f(U)$, and call U the *source* of f, while $f(U)$ is the *target* of f. Let one be given three sets E, E',

(*) References:

1. "Les prolongements d'une variété différentiable" (Atti del IV Congresso I. M. I., Taormina, 1951).
2. "Les prolongements d'une variété différentiable" (Comtpes rendus Acad. Sciences **233** (1951), 598, 777, 1081; *ibid.* **234** (1952), 1028, 1424.
3. "Structures locales et structures infinitésimales" (Comptes rendus **234** (1952), pp. 587).
4. "Structures locales" (Conference polycopies from Rome, 1952, to appear in the Annali di Matematica, 1953).
5. Polycopied notes from a course in Rio de Janeiro, 1952).

E'', and let f be a map with source $A \subset E$ and target $B \subset E'$, while f' is a map with source $B' \subset E'$ and target $C \subset E''$. We let $f'f$ denote the map $x \to f'((x))$ whose source is the set A' of elements x such that $f'(f(x))$ is defined.

Let E and E' be two topological spaces, and consider the continuous maps that have a subspace of E for their source and a subspace of E' for their target. Let $C_x(E, E')$ be the set of pointed continuous maps (f, x) where f is a continuous map that has an arbitrary neighborhood of x for its source and a subspace of E' for its target. Two elements (f, x) and (f', x) of $C_x(E, E')$ are said to have *the same local class at* x when the restrictions of f and f' to a neighborhood if x are identical. The relation thus defined is an equivalence relation in $C_x(E, E')$. An equivalence class for that relation will be called a *local jet* of E into E' with source x and target $f(x)$, where (f, y) is an arbitrary element of the class. The local jet of (f, x) will be denoted by $j_x^\lambda f$. Let $J^\lambda(E, E')$ be the set of local jets of E into E'. The source of $X \in J^\lambda(E, E')$ will be denoted by $\alpha(X)$ and its target, by $\beta(X)$. The symbol j^λ will also denote the map $(f, x) \to j_x^\lambda f$ of $C(E, E')$ onto $J^\lambda(E, E')$, where $C(E, E') = \bigcup_{X \in E} C_x(E, E')$.

If f is a continuous map that has an open set U of E for its source and a subspace of E for its target then the map $x \to j_x^\lambda f$ of U into $J^\lambda(E, E')$ may be denoted by $j^\lambda f$. One may consider $j^\lambda f$ as a local chart of E in $J^\lambda(E, E')$. The set of these local charts is an atlas $\mathcal{A}$ for E in $J^\lambda(E, E')$ that is compatible with the pseudogroup of transformations that is composed of the set of identity maps of the open sets of E. Indeed, $j_x^\lambda f = j_{x'}^\lambda f'$ is equivalent to $x = x'$ for x that belongs to an open set of E. This atlas $\mathcal{A}$ induces a topology on $J^\lambda(E, E')$ that is generated by the targets of the charts $j^\lambda f$, which are called the ELEMENTARY OPEN SETS of $J^\lambda(E, E')$, and THE SPACE $J^\lambda(E, E')$ IS AN ÉTALÉ SPACE OVER E UNDER THE MAP α, whose restriction to the target of $j^\lambda f$ is the inverse homeomorphism of $j^\lambda f$. The map β is a continuous map of $J^\lambda(E, E')$ into E. The map f admits the CANONICAL COMPOSITION $f = \beta(j^\lambda f)$. The space $J^\lambda(E, E')$ is not separable. A separable open set of $J^\lambda(E, E')$, or furthermore, the restriction of β to such a set, is the notion that one may call a MULTIFORM CONTINUOUS MAP of E into E'.

Let E'' be a third topological space. If one is given $X \in J^\lambda(E, E')$ and $X' \in J^\lambda(E', E'')$ then one may compose $X'X$ when $\beta(X) = \alpha(X')$. If $(f, x) \in X$ and $(f', x') \in X'$, where $x' = f(x)$ then the pointed space $(f'f, x)$ belongs to $C_x(E, E'')$. The jet $j_x^\lambda(f f')$ depends only upon X and X' and will be, by definition, the composition $X'X$. One may also consider it to be the composition of f' with $j_x^\lambda f$:

$$j_x^\lambda(f' f) = \left(j_{x'}^\lambda f\right)\left(j_x^\lambda f\right) = f'\left(j_x^\lambda f\right), \qquad \text{where } x' = f(x).$$

In particular, if one denotes the local jet of the identity map on E whose source is x by $j_x^\lambda f$ then one has: $j_x^\lambda f = f\, j_x^\lambda$. The law of composition $(X', X) \to X'\, X$ is continuous with respect to the topology considered.

The associativity of the composition of maps implies the associativity of the composition of local jets: If $X'X$ and $X''X'$ exist then the compositions $(X''\,X')\,X$ and $X''(X'\,X)$ exist and are equal.

$j_x^\lambda f$ is a left unity and a right unity; i.e.:

$$j_{x'}^\lambda(j_x^\lambda f) = j_x^\lambda f, \qquad \text{where } x' = f(x),$$
$$(j_{x'}^\lambda f)\, j_x^\lambda = j_{x'}^\lambda f,$$
$$j_x^\lambda f\ \ j_x^\lambda f = j_x^\lambda f.$$

The jet $X \in J^\lambda(E, E')$ is *invertible* when there exists $X' \in J^\lambda(E', E)$ such that $X'\,X = j_x^\lambda$ and $XX' = j_{x'}^\lambda$, where $x = \alpha(X)$, $x' = \beta(X)$; this condition is equivalent to the existence of a pointed map (f, x) that belongs to X, such that f is a homeomorphism of a neighborhood of x onto a neighborhood of $f(x)$.

The set of the invertible elements of $J^\lambda(E', E)$ is a groupoid $\Pi^\lambda(E)$. The subset of invertible jets with source and target x is a group $\Pi_x^\lambda(E)$ that one may call the LOCAL ISOTROPY GROUP of E at x, relative to the topological structure $\mathcal{T}$ of E.

Let Γ be a pseudogroup of transformations that is defined in E and is contained in the pseudogroup Γ_* of local automorphisms of E; the subjacent topology to Γ may be coarser than $\mathcal{T}$. The set of local jets $j_x^\lambda\varphi$, where $\varphi \in \Gamma$, is a SUBGROUPOID $J^\lambda(\Gamma)$ of $\Pi^\lambda(E) = J^\lambda(\Gamma_*)$. The intersection of $J^\lambda(\Gamma)$ and $\Pi_x^\lambda(E)$ is a group $J_x^\lambda(\Gamma)$ that is called the LOCAL ISOTROPY GROUP at x of the local structure that is defined by Γ. We remark that that $J^\lambda(\Gamma)$ is an open set of $\Pi^\lambda(E)$, which is itself open in $J^\lambda(E, E')$.

If we are given a subset Φ of $J^\lambda(E, E')$ then a SOLUTION or INTEGRAL of Φ is a continuous map of f with source U, which is an open set of E, such that $j_x^\lambda f \in \Phi$ for any $x \in U$. The solutions of Φ correspond to elementary open sets that are contained in Φ in a bijective fashion.

Let Φ be an open subgroupoid of $\Pi^\lambda(E)$ such that the projection of Φ by α is E; i.e., it contains the set of units of $\Pi^\lambda(E)$. The set of solutions of Φ is then a pseudogroup of transformations Γ that is defined in E and admits $\mathcal{T}$ as its subjacent topology; one has $f = J^\lambda(\Gamma)$. However, Γ may contain a sub-pseudogroup Γ' whose subjacent topology is coarser that $\mathcal{T}$ and is such that $J^\lambda(\Gamma') = J^\lambda(\Gamma) = \Phi$. We say that Γ is deduced from Γ' by *localization*. In particular, Γ' may be a group of automorphisms of E.

Let Γ be a pseudogroup of local automorphisms of E let Γ' be a pseudogroup of local automorphisms of E'. THE GROUPOID $J^\lambda(\Gamma) \times J^\lambda(\Gamma')$ IS A GROUPOID OF OPERATORS ON $J^\lambda(E, E')$ ACCORDING TO THE LAW OF COMPOSITION:

$$(s, s', X) \to s'\,X\,s^{-1}, \quad \text{where } X \in J^\lambda(E, E'),\ s \in J^\lambda(\Gamma), s' \in J^\lambda(\Gamma).$$

LIKEWISE, $J^\lambda(\Gamma)$ AND $J^\lambda(\Gamma')$ ARE GROUPOIDS OF OPERATORS ON $J^\lambda(E, E')$.

If Φ is an open set of $J^{\lambda}(E, E')$ that is invariant under $J^{\lambda}(\Gamma) \times J^{\lambda}(\Gamma')$ then the set of solutions of Φ is invariant $\Gamma \times \Gamma$.

Suppose, to simplify, that $\mathcal{T}$ is the subjacent topology to Γ. Let $\tilde{E}$ be a topological space and let $\mathcal{A}$ be a complete atlas for E on $\tilde{E}$ that is compatible with Γ and is such that any $g \in \mathcal{A}$ is a homeomorphism of an open set of E onto an open set of $\tilde{E}$. Let $J^{\lambda}(\mathcal{A})$ be the set of jets j_x^{λ} where $g \in \mathcal{A}$. An element of h in $J^{\lambda}(\mathcal{A})$ will be called a LOCAL FRAME at the point $\beta(h)$ of $\tilde{E}$ relative to the structure that is defined by $\mathcal{A}$.

The set $J^{\lambda}(\mathcal{A})$ is an open set of $J^{\lambda}(E, \tilde{E})$ that has the following characteristic properties:

1. Any element of $J^{\lambda}(\mathcal{A})$ is invertible.
2. If $h \in J^{\lambda}(\mathcal{A})$, $h' \in J^{\lambda}(\mathcal{A})$, and $\beta(h) = \beta(h')$ then one has $h^{-1}h' \in J^{\lambda}(\Gamma)$.
3. $J^{\lambda}(\mathcal{A})$ is invariant under $J^{\lambda}(\Gamma)$.

The set of elements $h^{-1}h'$, where $h \in J^{\lambda}(\mathcal{A})$, $h' \in J^{\lambda}(\mathcal{A})$, and $\alpha(h) = \alpha(h')$ is the open subgroupoid $J^{\lambda}(\tilde{\Gamma})$ of $\Pi^{\lambda}(\tilde{E})$ whose solutions for the pseudogroup of local automorphisms of $\tilde{E}$ relative to the structure that is defined by $\mathcal{A}$.

Conversely, if one is given an open set of $J^{\lambda}(E, \tilde{E})$ that verifies the preceding three properties then the set of its solutions is a complete atlas $\mathcal{A}$ of E on $\tilde{E}$ that is compatible with Γ.

If $\mathcal{A}$ is an incomplete atlas then $J^{\lambda}(\mathcal{A})$ is an open set of $J^{\lambda}(E, \tilde{E})$ that is characterized by properties 1) and 2).

An *intransitivity class* of an invertible element X of $J^{\lambda}(E, \tilde{E})$ relative to $J^{\lambda}(\Gamma)$ (i.e., the set of elements $X\, s^{-1}$, where $s \in J^{\lambda}(\Gamma)$) may be called the STRUCTURE GERM ON $\tilde{E}$ THAT IS ASSOCIATED WITH Γ. By passing to the quotient, one deduces the space of structure germs that are associated with Γ from $J^{\lambda}(E, \tilde{E})$. It is an étalé space $G(\tilde{E}, \Gamma)$ over $\tilde{E}$. A structure on $\tilde{E}$ that is associated with Γ corresponds to a lift of $\tilde{E}$ in $G(\tilde{E}, \Gamma)$.

Suppose that the topology $\mathcal{T}'$ of E' is also the topology subjacent to the pseudogroup Γ'. Let $\tilde{E}'$ be a topological space and let $\mathcal{A}'$ be a complete atlas for E' on $\tilde{E}'$ that is compatible with Γ' and is such that any $g' \in \mathcal{A}'$ is a homeomorphism of an open set of E' onto an open set of $\tilde{E}'$. ANY OPEN SET Φ OF $J^{\lambda}(E, E')$ THAT IS INVARIANT UNDER $J^{\lambda}(\Gamma) \times J^{\lambda}(\Gamma)$ THEN CORRESPONDS TO AN OPEN SET Φ' OF $J^{\lambda}(E, \tilde{E})$ THAT IS INVARIANT UNDER $J^{\lambda}(\tilde{\Gamma}) \times J^{\lambda}(\tilde{\Gamma}')$. The set Φ' is the set of elements $h'\, X\, h^{-1}$, where $X, h \in J^{\lambda}(\mathcal{A})$, $h' \in J^{\lambda}(\mathcal{A}')$.

For example, let Λ_n^r be the pseudogroup of r-times continuously differentiable local automorphisms of the numerical space $\mathbb{R}^n$ that are everywhere of rank n. Let f be a continuous map that a neighborhood of $x \in \mathbb{R}^n$ for its source and a subspace of $\mathbb{R}^m$ for its target. One will say that f is an *r-times continuously differentiable map* (or r-MAP) at the point x when f has continuous partial derivatives of each type up to order r with respect to the canonical coordinates that are defined in $\mathbb{R}^n$ in a neighborhood of x. If f is an r-map at the point x then the local jet $j_x^\lambda f$ will be called *r-times differentiable.* Let $J^{\lambda,r}(\mathbb{R}^n, \mathbb{R}^m)$ be the set of r-times differentiable local jets of $\mathbb{R}^n$ to $\mathbb{R}^m$. It is an open set of $J^\lambda(\mathbb{R}^n, \mathbb{R}^m)$ that invariant under $J^\lambda(\Lambda_n^r) \times J^\lambda(\Lambda_m^r)$.

Let V_n be an r-times differentiable manifold (or *r-manifold*) whose structure is defined by a complete atlas $\mathcal{A}$ in $\mathbb{R}^n$ on V_n that is compatible with Λ_n^r. Likewise, let V_m be an r-manifold whose structure is defined by a complete atlas $\mathcal{A}'$ that is compatible with Λ_m^r. Let $J^{\lambda,r}(V_n, V_m)$ be the set of local jets $h' X h^{-1}$, where $h \in J^\lambda(\mathcal{A})$, $h' \in J^\lambda(\mathcal{A}')$, $X \in J^{\lambda,r}(\mathbb{R}^n, \mathbb{R}^m)$; i.e., h is a local frame of V_n at the source of X and h' is a local frame of V_m at the target of X. A local jet that belongs to $J^{\lambda,r}(V_n, V_m)$ will be called *r-times differentiable.* In order for $Y \in J^\lambda(V_n, V_m)$ to be r-times differentiable, it is necessary and sufficient that $h'^{-1}\, Y\, h$ be an r-times differentiable element of $J^\lambda(\mathbb{R}^n, \mathbb{R}^m)$. If $j_x^\lambda f$ is r-times differentiable then the map f will be called an r-MAP AT THE POINT x.

The set $J^{\lambda,r}(V_n, V_m)$ is an open set in $J^\lambda(V_n, V_m)$ that is invariant under $\Pi^{\lambda,r}(V_n) \times \Pi^{\lambda,r}(V_m)$, where $\Pi^{\lambda,r}(V_n)$ is the groupoid that is associated with $J^\lambda(\Lambda_n^r)$ by $\mathcal{A}$; i.e., $\Pi^{\lambda,r}(V_n)$ is the set of elements $h_1\, h^{-1}$, where $h \in J^\lambda(\mathcal{A})$ and $h_1 \in J^\lambda(\mathcal{A})$. A solution to $J^{\lambda,r}(V_n, V_m)$ wil be called an r-MAP of V_n into V_m. The solutions of $\Pi^{\lambda,r}(V_n)$ are r-maps and form the pseudogroup $\Lambda_n^r(V_n)$ of local automorphisms of V_n.

One easily proves that THE COMPOSITION OF TWO LOCAL r-TIMES DIFFERENTIABLE JETS IS r-TIMES DIFFERENTIABLE: The composition of $X \in J^{\lambda,r}(V_n, V_m)$ and $X' \in J^{\lambda,r}(V_m, V_p)$ is an element of $\in J^{\lambda,r}(V_n, V_p)$, where V_p is also an r-manifold.

Suppose that V_n is endowed simply with the structure of a topological manifold of n dimensions; i.e., it is associated with a pseudogroup Λ_n of all the local automorphisms of $\mathbb{R}^n$. One then obtains the notions that relate to indefinitely differentiable manifolds (the case of $r = \infty$), real or complex analytical manifolds, real or complex locally algebraic manifolds (1), and real or complex locally rational manifolds.

It is important to remark that in the analytic case the space $J^\omega(V_n, V_m)$ of analytic jets of V_n into V_m is an open, SEPARABLE subspace of $J^\lambda(V_n, V_m)$. The connected component of $X \in J^\lambda(V_n, V_m)$ is the COMPLETE ANALYTIC PROLONGATION of the analytic jet X.

2. The notion of infinitesimal jet. – Let $C_x^r(\mathbb{R}^n, \mathbb{R}^m)$ be the set of pointed r-maps (f, x), where f is an r-map at the point $x \in \mathbb{R}^n$ into $\mathbb{R}^m$. Two elements (f, x) and (f', x) of $C_x^r(\mathbb{R}^n, \mathbb{R}^m)$ are said to have *the same r-class* when the partial derivatives of the same type of f and f' admit the same values at the point x for all the partial derivatives of order $\leq r$. One thus defines an equivalence relation in $C_x^r(\mathbb{R}^n, \mathbb{R}^m)$. An equivalence class for that relation will called an INFINITESIMAL JET OF ORDER r – or r-JET – from $\mathbb{R}^n$ to $\mathbb{R}^m$. The r-jet of (f, x) will be denoted by $j_x^r f$. The point x will be called the *source* of the jet and the point $f(x)$, the *target* of the jet. Let $J^r(\mathbb{R}^n, \mathbb{R}^m)$ be the set of infinitesimal jets of order r from $\mathbb{R}^n$ to $\mathbb{R}^m$. The source of $X \in J^r(\mathbb{R}^n, \mathbb{R}^m)$ will be denoted by $\alpha(X)$ and it target, by $\beta(X)$. This defines two canonical maps α and β of $J^r(\mathbb{R}^n, \mathbb{R}^m)$ onto $\mathbb{R}^n$ and $\mathbb{R}^m$.

Set $C^r(\mathbb{R}^n, \mathbb{R}^m) = \bigcup_{x\in\mathbb{R}^n} C_x^r(\mathbb{R}^n, \mathbb{R}^m)$, which is the set of pointed r-maps of $\mathbb{R}^n$ into $\mathbb{R}^m$. In $C^r(\mathbb{R}^n, \mathbb{R}^m)$, consider the two equivalence relations whose equivalence classes are: 1. The local jets. 2. The r-jets. The first of these relations implies the second one. On thus has a canonical map j^r of $J^{\lambda,r}(\mathbb{R}^n, \mathbb{R}^m)$ onto $J^r(\mathbb{R}^n, \mathbb{R}^m)$: $j^r(j_x^\lambda f) = j_x^r f$. This permits us to canonically identify an r-jet with a class of local jets. One may set $j^r = j^r j^\lambda$, which amounts to considering j^r as an operator that operates on $J^r(\mathbb{R}^n, \mathbb{R}^m)$ and on $C^r(\mathbb{R}^n, \mathbb{R}^m)$: $j^r(f, x) = j_x^r f$.

If one is given $(f, x) \in C_x^r(\mathbb{R}^n, \mathbb{R}^m)$ and $(f', x') \in C_x^r(\mathbb{R}^n, \mathbb{R}^m)$ then if $x' = f(x)$, set $(f', x')\,(f, x) = (f'f, x)$. This law of composition is compatible with not only the first one, but also with the second of the preceding equivalence relations. Indeed, the partial derivatives of $f'f$ of order $\leq r$ at the point x can be expressed in the form of a polynomial as a function of the partial derivatives of order $\leq r$ of the canonical components of f and f' at the points x and x'. By passing to the quotient, one defines the law of composition that is expressed by the formulas:

$$(j_{x'}^\lambda f')(j_x^\lambda f) = j_x^\lambda(f'\,f),$$

$$(j_{x'}^r f')(j_x^r f) = j_x^r(f'\,f),$$

$$(j_{x'}^\lambda f')(j_x^r f) = (j_{x'}^r f')(j_x^\lambda f) = f'(j_x^r f) = (j_{x'}^r f')(f, x) = j_x^r(f'\,f).$$

In order for the composition of $X \in J^r(\mathbb{R}^n, \mathbb{R}^m)$ and $X' \in J^r(\mathbb{R}^m, \mathbb{R}^p)$ to be defined, it is necessary and sufficient that $\beta(X) = \alpha(X')$. These laws of composition are associative, and the operator j^r is compatible with them:

$$j^r(\xi', \xi) = j^r(\xi')\, j^r(\xi) = \xi'\, j^r(\xi) = j^r(\xi')\,\xi,$$

where:

$$\xi \in J^{\lambda,r}(\mathbb{R}^n, \mathbb{R}^m) \text{ and } \xi' \in J^{\lambda,r}(\mathbb{R}^m, \mathbb{R}^p).$$

The groupoid $\Pi^{\lambda,\, r}(\mathbb{R}^n) \times \Pi^{\lambda,\, r}(\mathbb{R}^m)$ is a groupoid of operators on $J^{\lambda,r}(\mathbb{R}^n, \mathbb{R}^m)$ and $J^{\lambda,r}(\mathbb{R}^n, \mathbb{R}^m)$, and the map j^r is invariant with respect to this groupoid:

$$(s, s', \xi) \to s' \,\xi\, s^{-1},$$
$$(s, s', X) \to s' \,X\, s^{-1},$$
$$j^r(s' \,\xi\, s^{-1}) = s' \,j^r(\xi)\, s^{-1},$$

where:

$$s \in \Pi^{\lambda,\, r}(\mathbb{R}^n),\ s' \in \Pi^{\lambda,\, r}(\mathbb{R}^m),\ \xi \in J^{\lambda,r}(\mathbb{R}^n, \mathbb{R}^m),\ X \in J^{\lambda}(\mathbb{R}^n, \mathbb{R}^m).$$

This amounts to saying that the groupoid of operators $\Pi^{\lambda,\, r}(\mathbb{R}^n) \times \Pi^{\lambda,\, r}(\mathbb{R}^m)$ leaves invariant the equivalence relation that is associated with j^r in $J^{\lambda,r}(\mathbb{R}^n, \mathbb{R}^m)$, and whose equivalence classes are canonically identified with elements $J^{\lambda}(\mathbb{R}^n, \mathbb{R}^m)$.

This equivalence relation is associated with an equivalence relation in $J^{\lambda,r}(V_n, V_m)$ that is invariant under the groupoid $\Pi^{\lambda,\, r}(V_n) \times \Pi^{\lambda,\, r}(V_m)$, where V_n and V_m are two r-manifolds. This equivalence relation is defined in the following manner: Two elements ξ and ξ_1 in $J^{\lambda,r}(V_n, V_m)$ will be said to have the same *r-class* when they have the same source and the same target, and when:

$$j^r(h'^{-1} f\, h) = j^r(h'^{-1} f'\, h),$$

where h is a local frame on V_n at the point x, and h' is a local frame on V_m at the point $f(x)$. AN EQUIVALENCE CLASS OF $C^r(V_n, V_m)$ FOR THIS RELATIONS WILL BE CALLED AN INFINITESIMAL JET OF ORDER r – OR r-JET – OF V_n TO V_m.

The r-jet of (f, x) is denoted $j^r_x f$. The set of r-jets of V_n to V_m is denoted by $J^r(V_n, V_m)$. If $X = j^r_x f$ then the point x is called the *source* of X and the point $f(x)$ is the *target* of X. The source of X is denoted by $\alpha(X)$, the target, by $\beta(X)$; this defines two canonical maps α and β of $J^r(V_n, V_m)$ onto V_n and V_m.

The map $(f, x) \to j^r_x f$ admits a canonical decomposition $(f, x) \to j^{\lambda}_x f \to j^r_x f$. We consider j^r to be an operator that operates on $C^r(V_n, V_m)$ and $J^{\lambda}(V_n, V_m)$:

$$(f, x) \to j^{\lambda}_x f \to j^r_x f \to j^k_x f\,,$$

$$C^r(V_n, V_m) \to J^{\lambda,r}(V_n, V_m) \to J^r(V_n, V_m) \to J^k(V_n, V_m)\,.$$

The k-jet that is canonically associated to the r-jet X will be further denoted by $j^k X$; i.e., we consider j^k to operate on $C^r(V_n, V_m)$, $J^{\lambda,r}(V_n, V_m)$ and $J^r(V_n, V_m)$, where $k \leq r$. One has:

$$j^k j^r = j^r, \qquad j^k j^\lambda = j^k, \qquad j^r j^r = j^r.$$

In particular, $j^0 X$ is identified with the pair $(\alpha(X), \beta(X))$ and j^0 defines the canonical map of $J^r(V_n, V_m)$ onto $V_n \times V_m$.

Let V_p be a third r-manifold. Let:

$$(f, x) \in C^r(V_n, V_m), \qquad (f', x') \in C^r(V_n, V_m).$$

The law of composition:

$$(f', x')\,(f, x) = (f'f, x), \qquad \text{if } x' = f(x)$$

is compatible with the equivalence relations considered. The laws of composition that one deduces from them by passing to the quotients are thus compatible with the preceding canonical projections. One has:

$$\left(j^r_{x'} f'\right)\left(j^r_x f\right) = \left(j^\lambda_{x'} f'\right)\left(j^r_x f\right) = \left(j^r_{x'} f'\right)\left(j^\lambda_x f\right) = f'\left(j^r_x f\right) = \left(j^r_x f'\right)(f, x) = j^r_x(f' f),$$

$$j^k(X'X) = (j^k X')\,(j^k X) = (j^k X')\,X = X'(j^k X),$$

where:

$$X \in J^r(V_n, V_m), \qquad X' \in J^r(V_m, V_p).$$

Let j^r_x denote the r-jet with source x of the identity map on V_n. One then has $j^r_x f = f\, j^r_x$, and j^r_x is a unity on the left and right for the composition of r-jets. The jet $X \in J^r(V_n, V_m)$ is invertible when there exists $X' \in J^r(V_m, V_n)$ such that $X'X = j^r_x$ and $XX' = j^r_{x'}$, where $x = \alpha(X)$, $x' = \beta(X)$. In order for X to be invertible, it is necessary and sufficient that $n = m$ and that the rank of X be equal to n. The rank of the jet $j^r_x f$ is equal to the rank of the jet $h'^{-1}(j^r_{x'} f)h$, where h is a local frame on V_n at the point x and h' is a local frame on V_m at the point $f(x)$. Now, $h'^{-1}(j^r_{x'} f)h$ is the jet of the first order of a linear map whose rank is the desired invariant.

We call any invertible r-jet of $\mathbb{R}^n$ into V_n whose target is x a *frame of order r on V_n at the point x.* Unless indicated to the contrary, it will be assumed that the frames considered will all have source 0. The set $H^r(V_n)$ of frames of order r on V_n with source 0 will be called the *principal prolongation of order r of the manifold V_n*. The inverse of a frame of order r at the point x will be called a *coframe of order r at the point x.* The set of coframes of order r and target 0 on V_n will be denoted by $H^{r*}(V_n)$.

The set of invertible elements of $J^r(V_n, V_m)$ is a groupoid $\Pi^r(V_n)$. The subset of elements of $\Pi^r(V_n)$ whose source and target is the given x is a group $L^r_n(V_n, x)$ that is called the infinitesimal isotropy group of order r on V_n at the point x. In particular, the group $L_n^r(\mathbb{R}^n, 0)$ will be denoted by L_n^r. If h is a frame of order r on V_n at the point x (with source 0) then the map $y \to hyh^{-1}$, where $y \in L_n^r$ is an isomorphism of L_n^r onto $L_n^r(V_n, x)$; the set of isomorphisms that one thus obtains is a class modulo the group of interior automorphisms of L_n^r.

Let $L_{m,n}^r$ be the set of elements of $J^r(\mathbb{R}^n, \mathbb{R}^m)$ that have their source and target at the common origin 0 of $\mathbb{R}^n$ and $\mathbb{R}^m$. Any $y \in L_{m,n}^r$ is the r-jet with source 0 of a well-defined map of the form:

$$u_i = \sum a_{ij} x_j + \sum a_{ij_1, j_2} x_{j_1} x_{j_2} + \cdots + \sum a_{ij_1, j_2 \cdots j_r} x_{j_1} x_{j_2} \cdots x_{j_r} ,$$

where the x_j are the canonical coordinates in $\mathbb{R}^n$ and the u_i are the canonical coordinates in $\mathbb{R}^m$, while the coefficients $a_{ij_1, j_2 \cdots j_k}$ are symmetric with respect to the indices $j_1, j_2, \ldots, j_k$; this map will be called the *representative polynomial* of y. Upon considering $a_{ij_1, j_2 \cdots j_k}$, where $j_1 \leq j_2 \leq \ldots \leq j_k$, to be the canonical coordinates of y, we endow $L_{m,n}^r$ with the structure of a real analytic manifold that is isomorphic to a numerical space. In particular, $L_{m,n}^1$ may be canonically identified with the space $L_{m,n}$ of homogeneous linear maps of $\mathbb{R}^n$ to $\mathbb{R}^m$, and in turn, also to the space of sequences of n vectors with origin 0 in $\mathbb{R}^m$. Therefore, $L_{m,1}^1$ is canonically identified with $\mathbb{R}^m$: The element y of $L_{m,1}^1$ first corresponds to its linear representative and then to the transformed vector of the unit vector in $\mathbb{R}$ by that linear transformation. The rank of $y \in L_{m,n}^r$ is the rank of the matrix (a_{ij}). The group L_n^r is an analytic submanifold of $L_{m,n}^r$ and L_n^1 that is canonically identified with the homogeneous linear group L_n. In the case $r = \infty$, the representative polynomial becomes a formal series with vectorial values (or a sequence of m ordinary formal series). $L_{m,n}^\infty$ may be identified with the set (1) of these formal series.

The operator j^k defines a homomorphism of L_n^r onto $L_{n'}^r$ whose kernel is a solvable group that is homeomorphic to a numerical space; if $k = r - 1$, this kernel is isomorphic to an additive group $\mathbb{R}^d$. The group L_n^r is an inessential extension of the group L_n, since it may be identified with a subgroup of $L_{n'}^r$. This subgroup is the set of elements L_n^r whose representative polynomial is linear; it is not invariant under $L_{n'}^r$.

Denote the representative polynomial of $y \in L_{m,n}^r$ by $\bar{y}$. The representative polynomial of $j^k y$ is obtained upon suppressing the terms of degree $> k$ in $\bar{y}$. If $y' \in L_{p,n}^r$ then the representative polynomial of $y'y$ is obtained by suppressing the terms of degree $> k$ in $\bar{y}'\bar{y}$ if $r > k$. In certain cases, one may define a composition of y and y' that belongs to $L_{p,n}^l$, where $l > k$. Let r and k be the largest integers such that $j^{n'-1}y = 0$ and $j^{k'-1}y' = 0$

(i.e., the canonical coordinates of these jets are zero). If l is the smallest of the numbers $k'r$ and $r'k$ then the jet $j_0^l(\bar{y}'\bar{y})$ depends only upon y' and y and defines a composition that we again denote by $y'y$. Its representative polynomial is obtained by suppressing the terms of degree $> l$ in $\bar{y}'\bar{y}$.

The product $L_m^r \times L_n^r$ is a group of analytic operators on $L_{m,n}^r$:

$$(s', s)\, y = s'\, y\, s^{-1}, \qquad \text{where } s \in L_n^r,\ s' \in L_m^r,\ y \in L_{m,n}^r .$$

Likewise, L_n^r and L_m^r are groups of operators on $L_{m,n}^r$:

$$(s, y) \to y\, s^{-1}, \qquad (s', y) \to s'\, y.$$

Let $h \in H^r(V_n)$ and $h' \in H^r(V_m)$.

The map $y \to h'\, y\, h^{-1}$ is a bijective map of $L_{m,n}^r$ onto the set $J_{x,x'}^r(V_n, V_m)$, which consists of the elements of $J^r(V_n, V_m)$ with source $\beta(h)$ and target $x' = \beta(h')$. Upon supposing that (x, x') is given, one then obtains a class of bijective maps of $L_{m,n}^r$ onto $J_{x,x'}^r(V_n, V_m)$; this will be an equivalence class with respect to the group of operators $L_m^r \times L_n^r$ (If G is a group operators on the set F then it is also a group of operators on the set of maps of F into F' and an intransitivity class in that set will be called an *equivalence class with respect to G*.) Any structure on $L_{m,n}^r$ that is invariant under $L_m^r \times L_n^r$ is then transported canonically onto $J_{x,x'}^r(V_n, V_m)$. For example, if $r = 1$ then one obtains a vector space structure.

We call any element of $J^r(\mathbb{R}^p, V_n)$ with source 0 and target x' a p^r-VELOCITY (or *velocity* of dimension p and order r) on V_n with origin x; let $T_p^r(V_n)$ be the set of these p^r-vectors in V_n. We call any element of $J^r(V_n, \mathbb{R}^p)$ with source x and target 0 a p^r-COVELOCITY on V_n with origin x; let $T_p^{r*}(V_n)$ be the set of p^r-covelocities of V_n. For $p = r = 1$, one then calls the VELOCITIES and COVELOCITIES on V_n, VECTORS and COVECTORS, resp.

Let $T_{p,x}^r(V_n)$ be the set of p^r-velocities with origin x, let $T_{p,x}^{r*}(V_n)$ be the set of p^r-covelocities with origin x, and let $H_p^r(V_n)$ be the set of frames with origin x. The frame $h \in H_p^r(V_n)$ corresponds to the bijective map $y \to hy$ of $L_{n,p}^r$ onto $T_{p,x}^r(V_n)$, where $y \in L_{n,p}^r$. For a given x, one then obtains a set of bijective maps of $L_{n,p}^r$ onto $T_{p,x}^r(V_n)$, which forms an equivalence class with respect to the group L_n^r that operates (on the left) on $L_{n,p}^r$. Any structure on $L_{n,p}^r$ that is invariant under L_n^r is then transported to $T_{p,x}^r(V_n)$ in a canonical way. Likewise, the set of bijective maps $y \to yh^{-1}$ of $L_{p,n}^r$ onto $T_{p,x}^{r*}(V_n)$, where $y \in L_{p,n}^r$ and $h \in H_x^r(V_n)$, forms an equivalence class with respect to the group L_n^r that operates (on the left) on $L^r_{p,n}$; thus, structures on $L^r_{p,n}$ that are invariant under L^r_n are canonically transported to $T^{r*}_{p,x}(V_n)$. In particular, $T^1_{p,x}(V_n)$ and $T^{1*}_{p,x}(V_m)$ are endowed with vector space structures.

Any jet $Z \in J^r(V_n, V_m)$ defines the following map of $T^r_{p,x}(V_n)$ into $T^r_{p,x}(V_m)$:

$$X \to ZX, \qquad \text{where } X \in T^r_{p,x}(V_n).$$

It likewise defines the following map of $T^{r*}_{p,x}(V_m)$ into $T^{r*}_{p,x}(V_n)$:

$$Y \to YZ, \qquad \text{where } Y \in T^{r*}_{p,x}(V_m).$$

Upon choosing two frames h and h' at the source and target of Z, resp., the former map comes down to a representation $y \to zy$ of $L^r_{n,p}$ in $L^r_{m,p}$, where $y \in L^r_{n,p}$ and $z \in L^r_{m,p}$; the latter map likewise comes down to a representation $y' \to y'z$ of $L^r_{p,m}$ in $L^r_{p,n}$, where $y' \in L^r_{p,m}$. In the case $r = 1$, the maps considered are all homogeneous linear ones.

Any r-map f of V_n into V_m has a PROLONGATION $X \to fX$ that maps $T^r_p(V_n)$ into $T^r_p(V_m)$. In the case $r = 1$, the restriction of that map to $T^1_{p,x}(V_n)$ is linear.

The group L^r_p operates on $T^r_p(V_m)$:

$$(s, X) \to Xs^{-1}, \qquad \text{where } s \in L^r_p,\ X \in T^r_p(V_m).$$

The intransitivity class XL^r_p will be called a p^r-*contact element* of V_n with origin x.

The group L^r_p also operates on $T^{r*}_p(V_n)$:

$$(s, Y) \to sY, \qquad \text{where } s \in L^r_p,\ Y \in T^{r*}_p(V_n).$$

The class L^r_pY will be called an *enveloping* p^r*-element.*

Let Z be an element of $J^r(V_n, V_m)$, let h be a frame at the source of Z, and let h' a frame at the target of Z. The element Z is canonically associated with an n^r-contact element with origin $\beta(Z)$ and an m^r-enveloping element with origin $\alpha(Z)$: They are the classes $Z\,h\,L^r_n$ and $L^r_m\,h'^{-1}\,Z$. The set of frames of order r at the point $x \in V_n$ is the *fundamental contact element of order r* – or the *infinitesimal structure element of order r* – to V_n at the point x. If f is an r-map of V_n to V_m then the pair (f, V_p) may be called an *embedded manifold* in V_n. The map f also prolongs to the set of contact elements to V_n; the images of these elements are the contact elements of the embedded manifolds. The theory of contact for embedded manifolds is the study of the incidence relation between contact elements that is defined in the following manner: If one is given $X \in T^r_p(V_n)$ and

$X' \in T_q^r(V_n)$ then one will say that X *is contained in* X' when $X = X'y$, or $y \in L_{q,p}^r$; in this case, the contact element XL will be said to be *contained in* $X'L$. One likewise defines an incidence relation between covelocities or enveloping elements.

One is also led to consider classes of local jets that are analogous to contact elements and enveloping elements. A class $XJ_0^\lambda(\Lambda_n^r)$, where $X \in J^{\lambda,r}(\mathbb{R}^p, V_n)$ and $a(X) = 0$, may be called a GERM OF THE EMBEDDED r-MANIFOLD. In the set of these germs, one may introduce a topology that is analogous to the one on the space of structure germs considered in paragraph 1. An embedded r-manifold corresponds to a separable open set in that space of germs of the embedded r-manifold. The class $J_0^\lambda(\Lambda_m^r)Y$, where $Y \in J^{\lambda,r}(V_n, \mathbb{R}^m)$ and $\beta(Y) = 0$, may be called a GERM OF THE FOLIATION in V_n. The space of these germs, which is again defined as above, will be an étalé space over V_n. A lifting of V_n into that space is an r-TIMES DIFFERENTIABLE FOLIATION with singularities admitted into it. One easily deduces a stronger equivalence relation: $Y \sim Y'$ when the intersection of $\bar{f}^1(0)$ and $\bar{f}'^1(0)$ is a neighborhood of x relative to the subspaces $\bar{f}^1(0)$ and $\bar{f}'^1(0)$, where Y and Y' are elements of $J^{\lambda,r}(V_n, \mathbb{R}^m)$ with the same source x and target 0, $j_x^\lambda f = Y$, $j_x^\lambda f' = Y'$. the equivalence classes thus defined may be called GERMS OF THE EXTRACTED MANIFOLD. One further defines the topological space of these germs: the separable open sets of that space are the extracted r-manifolds of V_n. An r-times differentiable foliation of V_n defines a set of extracted r-manifolds on V_n that are called the *leaves* of the foliation.

The intransitivity class of $y \in L_{m,n}^r$ with respect to $L_m^r \times L_n^r$ will be called the *equivalence class of y*. A fundamental problem of local differential geometry consists in finding a canonical representation in each equivalence class, and more generally, finding the invariants and covariants of y with respect to $L_m^r \times L_n^r$ or certain subgroups.

The element y of $L_{m,n}^r$ is called REGULAR when the rank of the matrix (a_{ij}) is equal to the smaller of the numbers m and n. The set of regular elements forms just one equivalence class that has a canonical representative in the map $x_i' = x_i$ for $i \leq m$ if $m \leq n$, or the map $x_i' = x_i$ for $i \leq n$ and $x_j' = 0$ for $j > n$ if $n \leq m$.

If one is given $y \in L_{m,n}^r$ then let p and q be the smallest integers such that y admits the decompositions: $y = y_1 z = x y_2$, where z is a REGULAR element of $L_{p,n}^r$, and z' is a REGULAR element of $L_{m,q}^r$, with $y_1 \in L_{m,p}^r$, $y_2 \in L_{q,n}^r$. One then has $p \leq n$, $q \leq m$, and y admits the decomposition $y = z'y'z$, where $y' \in L_{q,p}^r$. If z_0 denotes the r-jet of the canonical map of $\mathbb{R}^n$ onto $\mathbb{R}^p$ and z_0' denotes the r-jet of the canonical map of $\mathbb{R}^q$ into $\mathbb{R}^m$ then y is equivalent to $z_0' y' z_0$; one may say that y is equivalent to $y' \in L_{q,p}^r$ *in the larger sense*. Call p the rank of order r at the source and q, the rank of order r at the target of y. For $k < r$, the ranks of order k at the source and target of y are those of $j^k y$. These numbers are increasing functions of k.

Consider k elements $y_1, y_2, \ldots, y_k$ of $L^r_{m,n}$. One may identify the sequence of $y_1, y_2, \ldots, y_k$ with an element of $L^r_{km,n}$. Let z be an element of $L^r_{m,kn}$. The composition $z(y_1, y_2, \ldots, y_k)$ will be an element of $L^r_{m,n}$, and may be called the *composition* of $y_1, y_2, \ldots, y_k$ along z; this notion generalizes the notion of linear combination of vectors. In particular, the element y of $L^r_{m,n}$ is identified with a sequence of m elements $(y_1, y_2, \ldots, y_k)$ of $L^r_{1,n}$. (They are the components of y, but that notion is not invariant with respect to L^r_m.) Any element z of $L^r_{1,n}$ determines a composition $z(y_1, y_2, \ldots, y_k)$. One is then led to the notion of a subspace of $L^r_{1,n}$ that is generated by a given set of elements of $L^r_{1,n}$. The rank of order r at the source of y is the minimum number of regular elements of $L^r_{1,n}$ that generate a subspace that contains $y_1, y_2, \ldots, y_m$.

The same considerations lead to the notion of the prolongation of a law of composition. An r-times differentiable law of composition $(x, x') \to xx'$ that is defined in V_m prolongs to $J^r(V_n, V_m)$. If one is given two elements X and X' such that $\alpha(X) = \alpha(X') = u$ then one sets:

$$XX' = j^r_u(ff'), \qquad \text{where } X = j^r_u f, \qquad X' = j^r_u f',$$

upon denoting the map $x \to f(u)\, f'(u)$ by ff' here. In particular, if G is an r-times differentiable group then $T^r_p(G)$ will be a group. If G operates on an r-manifold V_n then since the law of exterior multiplication is r-times differentiable, the group $T^r_p(G)$ operates on $T^r_p(V_n)$.

In that fashion, one defines algebraic structures on $L^r_{m,n}$ that are invariant under L^r_n, but not L^r_m. An algebraic structure over $\mathbb{R}^n$ thus prolongs to an algebraic structure on $L^r_{m,n}$ that is invariant under L^r_n. We remark that the product yy' of two elements of $L^2_{m,n}$ is zero in $L^1_{m,n}$; however, one may define yy' as an element of $L^2_{m,n}$. One likewise defines the product of k elements of $L^1_{m,n}$ as an element of $L^k_{m,n}$. More generally, one may substitute elements of $L^1_{m,n}$ for the variables in a homogeneous polynomial of degree k, with coefficients that are taken from the base field of the algebra considered over $\mathbb{R}^n$, and one thus obtains an element of $L^k_{m,n}$. Its polynomial representative is obtained by composing the linear representatives of the given elements.

Local differential geometry essentially amounts to the study of the space $L^r_{m,n}$, when it is endowed with the group of operators $L^r_m \times L^r_n$ or groups of operators that are subgroups of it. The subgroups of L^r_n and L^r_m intervene when one replaces Λ^r_n and Λ^r_m by the sub-pseudogroups Γ and Γ'. Let $J^r(\Gamma)$ denote the groupoid that is formed by the set of jets $j^r_x\varphi$, where $\varphi \in \Gamma$. The group $J^r_0(\Gamma)$, which is the intersection of $J^r(\Gamma)$ and L^r_n, will be called the *infinitesimal isotropy group of order r* at the point 0 relative to Γ, which we assume to be transitive, in order to simply matters. One will then have to study the structures on $L^r_{m,n}$ that are invariant under $J^r_0(\Gamma)\times J^r_0(\Gamma)$; these structures transport canonically onto $J^r_{x,x'}(V_n, V_m)$ if V_n and V_m are endowed with structures that are associated with Γ and Γ′, respectively. In particular, $L^1_{m,n}$ may have an invariant subspace under $J^r_0(\Gamma)\times J^r_0(\Gamma')$. This will determine a class of distinguished r-jets in $J^r(V_n, V_m)$. More generally, this class of distinguished r-jets will be defined in $J^r(V_n, V_m)$ when V_n and V_m are endowed with structures that call *regular infinitesimal structures* (4) with structure groups $J^r_0(\Gamma)$ and $J^r_0(\Gamma')$, respectively. For example, in the case of complex analytic manifolds ($\Gamma = \Lambda^c_n$, $\Gamma' = \Lambda^c_m$) one obtains the class of complex analytic r-jets that are contained in $J^r(V_n, V_m)$. This notion also applies to almost-complex manifolds of order r (manifolds that are endowed with a regular infinitesimal structure with structure group $J^r_0(\Lambda^c_n)$; they are equivalent to complex analytic manifolds at each point up to order r). Having distinguished a subspace that is invariant under $L^r_{m,n}$, one is led to also consider the equivalent relations that are invariant under this subspace, as well as the corresponding quotient spaces. The general notion of differential covariants of infinitesimal structure comes down to that of covariants with respect the representations of $J^r_0(\Gamma)\times J^r_0(\Gamma')$ are groups of operators. In the geometries that one call *non-holonomic*, the subgroups of L^r_n and L^r_m that intervene are not associated with pseudogroups of transformations Γ and Γ′.

We remark that $L^k_{m,n}$ also admits $L^r_m \times L^{r'}_n$ as a group of operators when $k \leq r$ and $k \leq r'$. In particular, $L^\infty_m \times L^\infty_n$ is a group of operators on $L^r_{m,n}$; the same is true for $J^\lambda_0(\Lambda^\infty_n)\times J^\lambda_0(\Lambda^\infty_m)$. The operator j^r is invariant under the group of operators $L^\infty_m \times L^\infty_n$ and $L^r_{m,n} = j^r(L^\infty_{m,n})$.

In a general fashion, one is led to consider the subspaces of $L^\infty_{m,n}$ that are invariant under $L^\infty_m \times L^\infty_n$ or one of its subgroups. On such a distinguished subspace, one will have to consider the invariant structures – in particular, the invariant equivalence relations and the corresponding quotient spaces. The latter spaces may be considered as generalizations of the spaces of jets. We confirm this by examples in the study of foliated structures, product manifold structures, or structures that are prolongation of an r-manifold.

For example, in $\mathbb{R}^{p+q}$, which is identified with $\mathbb{R}^p \times \mathbb{R}^q$, the product $\Lambda^k_p \times \Lambda^l_q$ generates a pseudogroup that may likewise denote by $\Lambda^k_p \times \Lambda^l_q$. Consider the pointed maps (f, x) of $\mathbb{R}^p \times \mathbb{R}^q$ into $\mathbb{R}^m$, where $x = (u, v) \in \mathbb{R}^p \times \mathbb{R}^q$, such that f admits continuous partial derivatives of every type and that satisfy the following conditions: They are of order $\leq k$ with respect to the coordinates u_i of u, of order $\leq l$ with respect to the coordinates v_j of v, of order $\leq r$ with respect to the set of these coordinates. In the set of these pointed maps, one has the equivalence relation whose classes each correspond to a given system of values of the derivatives considered at the point x. This relation is invariant under

$\Lambda_p^k \times \Lambda_q^l$. The equivalence classes are generalized jets that one may denote by $j_x^{k,l,r} f$. As in the preceding, one defines jets of this type on V_{p+q} in V_m, where V_m is an r-manifold and V_{p+q} is endowed with a local product structure that is locally isomorphic to $\mathbb{R}^p \times \mathbb{R}^q$ (i.e., it is defined by an atlas of $\mathbb{R}^p \times \mathbb{R}^q$ on V_{p+q} that is compatible with $\Lambda_p^k \times \Lambda_q^l$). The set of jets of this type on $\mathbb{R}^p \times \mathbb{R}^q$ in $\mathbb{R}^m$ with source and target 0 is a quotient space of $L_{1,p+q}^{\infty}$. For $m = 1$, it is the quotient algebra of $L_{1,p+q}^{\infty}$ by the ideal that is generated by the products of the coordinates that are of degree > k with respect to the u_i, of degree > l with respect to the v_j, and of degree r with respect to the set of u_i *and* v_j. This rejoins the viewpoint that was presented by A. Weil. However, the spaces of jets with a given source are not always endowed with algebraic structures. (*)

(*) For the notions of pseudogroup of transformations and structure associated with a pseudogroup, see (4). The notion of the germ of a structure is defined for all types of local structures. The notion of the germ of a subspace also extends to arbitrary local structures (See P. Dedecker, Comptes rendus, Paris **233** (1953) pp. 771). One will find the general definition of prolongation structures and infinitesimal structures in (1), (2), and (3).