"Les prolongements d'une variété differentiable. I. Calcul de jets, prolongement principal." Comptes rendus, **233** (1951), 598-600.

The prolongations of a differentiable manifold
I. Calculus of jets, principal prolongation

By **CHARLES EHRESMANN**

Translated by D. H. Delphenich

This Note continues a previous Note (1) and summarizes a conference talk that was given at Oberwolfach on 19 August 1951. The jet as fundamental element of differential geometry. Prolongations of order r of a differentiable manifold. Study of the fiber structure of prolongations. The ones that depend only upon the fiber structure of the principal prolongation of first order.

We call any r times continuously differentiable homeomorphism of an open subset of $\mathbb{R}^n$ into $\mathbb{R}^n$ that is everywhere of rank n a *local automorphism of order r* of the numerical space $\mathbb{R}^n$; let Λ_n^r be the pseudogroup composed of these automorphisms. An *r-manifold structure* on V_n is defined by an atlas A on V_n with values in $\mathbb{R}^n$ that is compatible (1) with Λ_n^r. If V_n and V_m are two r-manifolds then a map f from a neighborhood of $x \in V_n$ into V_m is called an *r-map* at the point x if, with the aid of admissible local coordinates in neighborhoods of x and $f(x)$, it is expressed by functions f_i that admit continuous partial derivatives of each type up to order r. Let $C_x^k(V_n, V_m)$ be the set of pointed functions (f, x), where f is an r-map at the point $x \in V_n$; let $C^r(V_n, V_m)$ be the union $\bigcup_{x\in V_n} C_x^k(V_n, V_m)$. Two elements (f, x) and (g, x) of $C_x^k(V_n, V_m)$ are said to have the same *r-class* when $f(x) = g(x)$ and when the partial derivatives of the same type and order for the functions f and g take the same value at x.

DEFINITIONS (2). – We call an r-class X in $C_x^k(V_n, V_m)$ an *r-jet with source x*, and the image of x under one of the elements of X the *target* of X.

---

(1) *Comptes Rendus*, **216** (1943), pp. 268. *See* also: C. EHRESMANN, *Sur la théorie des espaces fibrés* (*Colloque de Topologie algébrique, C. N. R. S.*, Paris, 1947).

(2) The definitions also apply to the case of $r = \infty$ and to the real or complex analytic case, as well as the algebraic case (Λ_n^r must then be replaced by the pseudogroup of birational transformations of the real or complex projective space to itself, f being an algebraic map of one algebraic variety without singularities V_n to another one.)

Let $J_x^r(V_n, V_m)$ be the set of r-jets with source x and let $J^r(V_n, V_m)$ be the union $\bigcup_{x \in V_n} J_x^r(V_n, V_m)$. The r-jet that is determined by $(f, x) \in C^r(V_n, V_m)$ will be denoted by $j_x^r f$.

The function $x \to j_x^r f$, which is defined in a neighborhood of x and which will be denoted $j^r f$, is the *r-flow* of f.

The elements $(f, x) \in C^r(V_n, V_m)$ and $(g, f(x)) \in C^r(V_m, V_p)$ admit the composition $(gf, x) \in C^r(V_n, V_p)$. *Upon passing to the quotients, this composition implies one law of composition for r-jets, a second one for r-maps and r-jets, and a third one between r-jets and pointed maps:*

$$j_x^r(gf) = (j_{f(x)g}^r g)(j_x^r f) = g(j_x^r f) = (j_{f(x)g}^r)(f, x).$$

The r-jet of the pointed identity map on V_n at $x \in V_n$ is the *neutral r-jet* at x; one may identify it with x. A *stable r-jet* at x is an r-jet of V_n into V_m with source and target x. An *isotropy r-jet* at x is a stable, invertible r-jet at x – i.e., one of rank n, which is the usual rank of an element of the r-jet at x. (We also define a rank of order $k \leq r$.) The isotropy r-jets at x define a group $L_n^r(V_n, x)$, namely, the *infinitesimal isotropy group at x*, which is isomorphic to the group $L_n^r(\mathbb{R}^n, 0)$ that we denote by L_n^r. The group L_n^1 is canonically identified with the homogeneous linear group L_n of $\mathbb{R}^n$. The group L_n^r is an inessential extension of L_n by a solvable group that is homeomorphic to a numerical space. L_n^r is an extension of L_n^{r-1} by a group that is isomorphic to the additive group $\mathbb{R}^r$.

We call an r-jet of $\mathbb{R}^p$ into V_n with source 0 and target x a p^r-*velocity* in V_n; let $T_p^r(V_n)$ be the set of p^r-velocities in V_n. We call an r-jet of V_n into $\mathbb{R}^p$ with source x and target 0 a p^r-*covelocity* in V_n with origin x; let $T_p^{r*}(V_n)$ be the set of p^r-covelocities in V_n. For $p = r = 1$, one thus defines *velocities* and *covelocities* in V_n, which are also called *vectors* and *covectors*. Let $L_{n,p}^r$ be the space of p^r-velocities in $\mathbb{R}^n$ at 0 or n^r-covelocities in $\mathbb{R}^p$ at 0. L_n^r is a group of operators that acts on $L_{n,p}^r$ on the left and acts on $L_{p,n}^r$ on the right. Let t_x be the translation of $\mathbb{R}^n$ that takes $x \in \mathbb{R}^n$ to 0. The set $T_p^r(\mathbb{R}^n)$ is canonically identified $\mathbb{R}^n \times L_{n,p}^r$ by means of $X \to (x, t_x X)$, where X is a p^r-velocity with origin x, and $T_p^{r*}(\mathbb{R}^n)$ is canonically identified with $\mathbb{R}^n \times L_{p,n}^r$. We call an n^r-velocity in V_n of rank n an *r-frame*. The set $H^r(V_n)$ of these r-frames is the *principal prolongation of order r* in V_n; there are analogous definitions for r-coframes and $H^{r*}(V_n)$.

For a map f of V_n into $\mathbb{R}^p$, one calls the p^r-covelocity $d_x^r f = j_x^r(t_{f(x)} f)$ its *differential of order r* at $x \in V_n$, t_x always denoting the translation of $\mathbb{R}^p$ that takes u to 0. In particular, if f is the identity map of $\mathbb{R}^n$ then $d_x^r f$ is denoted by $d^r x$. The function $x \to d_x^r f$ is denoted by $d^r f$. It corresponds to a map of $T_q^r(V_n)$ into $L_{p,q}^r$ that is defined by $X \to (d_x^r f)X$, where X is a q^r-velocity of origin x.

Let f be an r-map of V_n into V_n. By composition, f defines a map of $T_p^r(V_n)$ to $T_p^r(V_m)$ that is called the *prolongation* of f and is again denoted by f; the pseudogroup Λ_n^r thus prolongs to both $T_p^r(\mathbb{R}^n)$ and $T_p^{r*}(\mathbb{R}^n)$. The prolongation of $\varphi \in \Lambda_n^r$ is written:

$$(x, y) \to (\varphi(x), \varphi_x^r y),$$

where $x \in \mathbb{R}^n$, $y \in L_{n,p}^r$, and:

$$\varphi_x^r = j_0^r(t_{\varphi(x)} \varphi t_x^{-1}) = d_x^r(\varphi t_x^{-1}).$$

The element φ_x^r of L_n^r is the *derivative* of order r of φ at x. The atlas A admits a prolongation that defines an atlas on $T_p^r(V_n)$ with values in $T_p^r(\mathbb{R}^n) = \mathbb{R}^n \times L_{n,p}^r$ that is compatible with the prolongation of Λ_n^r. It determines a fiber structure on $T_p^r(V_n)$ with symbol $T_p^r(V_n,\ M_{n,p}^r,\ L_n^r,\ H^r(V_n))$. For $h \in H^r(V_n)$, the map $y \to hy$ is an isomorphism of $L_{n,p}^r$ onto a fiber of that fiber structure. The associated principal fiber bundle is $H^r(V_n)$, and its structure group is L_n^r. It is an extension ([1]) of $H^1(V_n)$ that is associated with the canonical homomorphism of L_n^r onto L_n. Since its kernel is homeomorphic to $\mathbb{R}^k$, its fiber structure is determined up to an isomorphism by that of $H^1(V_n)$.

DEFINITION. – A *prolongation of order r of* V_n is a fiber bundle that is associated to the principal prolongation $H^r(V_n)$.

The preceding shows:

THEOREM. – *The fiber structures that are defined on the prolongations of order r of* V_n *are determined, up to isomorphism, by the fiber structure of the principal prolongation of first order.*

([1]) C. EHRESMANN, *Les connexions infinitésimales* (*Colloque de Topologique, C. B. R. M.,* Brussels, 1950).

"Les prolongements d'une variété differentiable. II. L'espace des jets d'ordre r de V_n dans V_m." Comptes rendus **233** (1951), 777-779.

The prolongations of a differentiable manifold
II. The space of jets of order r of V_n into V_m

By **CHARLES EHRESMANN**

Translated by D. H. Delphenich

This Note continues a previous Note ([1]). Being given two r-manifolds V_n and V_m, we shall define three fiber structures on the space $J^r(V_n, V_m)$ of r-jets of V_n into V_m: The first one has base $V_n \times V_m$, the second one has base V_n, and the third one has base V_m. The fibers are isomorphic to $L^r_{m,n}$, $T^r_n(V_m)$, $T^{r*}_n(V_m)$, respectively, endowed with the structure groups $L^r_n \times L^r_m$, L^r_n, L^r_m.

For $X \in J^r(V_n, V_m)$, let $\alpha(X)$ be the source of X and let $\beta(X)$ be the target of X; α is the canonical map of $J^r(V_n, V_m)$ onto V_n, β is the canonical map onto V_m, and $\gamma = (\alpha, \beta)$ is the canonical map onto $V_n \times V_m$.

For $X \in J^r(\mathbb{R}^p, V_m)$, set $\partial^r X = X\, \partial^r x$, where $x = \alpha(X)$, $\partial^r x = j_0^r(t_x^{-1})$. For $(f, x) \in C^r(\mathbb{R}^p, V_m)$, set $\partial_x^r f = f\, \partial^r x$. The p^r-velocity of V_n with origin $x' = \beta(X)$ [$f(x)$, resp.] that is defined by $\partial^r X$ ($\partial_x^r f$, resp.) is called the *velocity of order r* of X (f, resp.) at x, or at the instant x.

For $X \in J^r(V_m, \mathbb{R}^p)$, the *differential of order r* of X is:

$$d^r X = (d^r x')X,$$

where $x' = \beta(X)$, $d^r x' = j_{x'}^r(t_{x'})$. If X is invertible (of rank $n = p$) then $d^r X \in H^{r*}(V_n)$ is the inverse of $\partial^r(X^{-1})$; $d^r x$ is the inverse of $\partial^r x$, where $x \in \mathbb{R}^p$. Recall the notation:

$$d_x^r f = (d^r x')(f, x),$$

where $(f, x) \in C^r(V_n, \mathbb{R}^p)$, $x' = f(x)$.

Let $X \in J^r(\mathbb{R}^n, \mathbb{R}^p)$, $x = \alpha(X)$, $x' = \beta(X)$. Call the element:

$$(d^r x')\, X\, \partial^r x = (d^r X)\, \partial^r x = (d^r x')\, \partial^r X$$

([1]) *Comptes rendus* **233** (1951), pp. 598.

the *derivative of order r* of X; it is an element of $L^r_{m,n}$ that one may also denote by d^rX / d^rx. For $(f, x) \in C^r(\mathbb{R}^n, \mathbb{R}^p)$, the derivative of order r of f at x is the element f^r_x of $L^r_{m,n}$ that is defined by:

$$f^r_x = (d^r x')\, f\, \partial^r x,$$

where $x' = f(x)$. One has:

$$d^r_x f = f^r_x d^r x, \qquad \partial^r_x f = (\partial^r x')\, f^r_x,$$

which justifies the notation:

$$f^r_x = \frac{d^r_x f}{d^r x},$$

If $X \in J^r(\mathbb{R}^n, \mathbb{R}^m)$ and $X' \in J^r(\mathbb{R}^n, \mathbb{R}^p)$ admit the composition $X'X \in J^r(\mathbb{R}^n, \mathbb{R}^p)$ then the associativity of the composition of the jets implies that:

$$\frac{d^r(X'X)}{d^r x} = \left(\frac{d^r X'}{d^r x'}\right)\left(\frac{d^r X}{d^r x}\right), \qquad x = \alpha(X), \qquad x' = \beta(X) = \alpha(X').$$

For $(f, x) \in C^r(\mathbb{R}^n, \mathbb{R}^m)$ and $(g, x') \in C^r(\mathbb{R}^m, \mathbb{R}^p)$, where $x' = f(x)$, one has:

$$(gf)^r_x = g^r_{x'} f^r_x, \qquad d^r_x(gf) = g^r_{x'} d^r_x f, \qquad \partial^r_x(gf) = g\partial^r_x f = (\partial^r_{x'} g)\, f^r_x.$$

$J^r(\mathbb{R}^n, \mathbb{R}^m)$ is canonically identified with $\mathbb{R}^n \times \mathbb{R}^m \times L^r_{m,n}$ by the map $X \to (x, x', y)$, where $X \in J^r(\mathbb{R}^n, \mathbb{R}^m)$, $x = \alpha(X)$, $x' = \beta(X)$, $y = d^rX$ / $d^rx \in L^r_{m,n}$. Let $\Lambda^r_n \times \Lambda^r_m$ be the pseudogroup of transformations in $\mathbb{R}^n \times \mathbb{R}^m$ defined by the set of transformations (φ, ψ): $(x, x') \to (\varphi(x), \psi(x'))$, where $\varphi \in \Lambda^r_n$, $\psi \in \Lambda^r_m$, $(x, x') \in \mathbb{R}^n \times \mathbb{R}^m$, such that $\varphi(x)$ and $\psi(x')$ are defined. The pseudogroup $\Lambda^r_n \times \Lambda^r_m$ prolongs to $J^r(\mathbb{R}^n, \mathbb{R}^m)$:

$$(\varphi, \psi)\, X = (\, j^r_{x'},\, \psi)\, X\, (j^r_{\varphi(x)} \varphi^{-1})\, \varphi^{-1}).$$

The transformation $X \to (\varphi, \psi)X$ corresponds in $\mathbb{R}^n \times \mathbb{R}^m \times L^r_{m,n}$ to the transformation:

$$(x, x', y) \to (\varphi(x), \psi(x'), \psi^r_{x'} y (\varphi^r_x)^{-1}), \qquad \text{or} \qquad \varphi^r_x \in L^r_n, \qquad \psi^r_{x'} \in L^r_m.$$

Let V_n and V_m be two r-manifolds, A, an atlas for V_n with values in $\mathbb{R}^n$, and let A' be an atlas for V_m with values in $\mathbb{R}^m$; $g \in A$ is an r-isomorphism of an open set of $\mathbb{R}^n$ onto a open set of V_n . The set of pairs (g, g'), where $g \in A$, $g' \in A'$, defines an atlas $A \times A'$ for $V_n \times V_m$ with values in $\mathbb{R}^n \times \mathbb{R}^m$. The chart (g, g') admits the following prolongation, which defines a local chart of $J^r(V_n , V_m)$ with values in $J^r(\mathbb{R}^n, \mathbb{R}^m)$:

$$X \to (j^r_{x'} g') X (j^r_{g(x)} g^{-1}) = h_{x'} y h_x^{-1} \in J^r(V_n , V_m),$$

where $X \in J^r(\mathbb{R}^n, \mathbb{R}^m)$, $x = \alpha(X)$, $x' = \beta(X)$, $y = d^r X / d^r x = (d^r x')\, X\, \partial^r x$,

$$h_x = \partial^r_x g = g\partial^r x \in H^r(V_n), \qquad h_{x'} = g'\partial^r x \in H^r(V_m).$$

Upon identifying $J^r(\mathbb{R}^n, \mathbb{R}^m)$ with $\mathbb{R}^n \times \mathbb{R}^m \times L^r_{m,n}$, this prolongation of (g, g') may also be written: $(x, x', y) \to h_{x'} y h_x^{-1}$. The chart $(g\varphi^{-1}, g'\psi^{-1})$, where $(\varphi, \psi) \in \Lambda^r_n \times \Lambda^r_m$, admits the prolongation $(\varphi(x), \psi(x'), \psi) \to h_{x'} (\psi^r_{x'})^{-1} y \varphi^r_x h_x^{-1}$, upon taking into account the equality:

$$\partial^r_{\varphi(x)} (g\varphi^{-1}) = h_x (\varphi^r_n)^{-1}.$$

These two charts of $J^r(V_n , V_m)$ correspond to the following change of chart:

$$(x, x', y) \to (\varphi(x), \psi(x'), \psi^r_x y (\varphi^r_{x'})^{-1}).$$

The following theorem results from this:

THEOREM 1. – *The prolongation of the atlas* $A \times A'$ *determines a fiber structure on the space* $J^r(V_n , V_m)$ *with base* $V_n \times V_m$, *projection* γ, *fibers isomorphic to* $L^r_{m,n}$, *structure group* $L^r_n \times L^r_m$, *and admitting* $H^r(V_n) \times H^r(V_m)$ *as associated principal fiber bundle.*

The element (h, h') *of* $H^r(V_n) \times H^r(V_m)$ *determines the isomorphism* $y \to h' y h^{-1}$ *of* $L^r_{m,n}$ *onto a fiber of that fiber structure.*

One must also remark that (g, g') admits the following prolongation, which defines a local chart for $H^r(V_n) \times H^r(V_m)$ in $V_n \times L^r_n \times V_m \times L^r_m$: $(x, s, x', s') \to (h_x s, h_{x'} s')$, where $s \in L^r_n$, $s' \in L^r_m$.

THEOREM 2. – *The atlas A admits a prolongation that defines an atlas on* $J^r(V_n, V_m)$ *in* $J^r(\mathbb{R}^n, V_m)$ *and determines a fiber structure on* $J^r(V_n , V_m)$ *with base* V_n , *projection* α, *fibers isomorphic to* $T_n^r(V_m)$, *structure group* L_n^r, *and admitting* $H^r(V_n)$ *as associated principal fiber bundle.*

The element h of $H^r(V_n)$ *determines the isomorphism* $z \to zh^{-1}$ *of* $T_n^r(V_m)$ *onto a fiber of that fiber structure when* $z \in T_n^r(V_m)$.

THEOREM 2′. – *The atlas* A' *admits a prolongation that defines an atlas on* $J^r(V_n, V_m)$ *in* $J^r(V_n, \mathbb{R}^m)$ *and determines a fiber structure on* $J^r(V_n, V_m)$ *with base* V_m, *projection* β, *fibers isomorphic to* $T_m^{r*}(V_n)$, *structure group* L_m^r, *and admitting* $H^r(V_m)$ *as its associated principal fiber bundle.*

The element h' *of* $H^r(V_m)$ *determines the isomorphism* $z \to h'z'$ *of* $T_m^{r*}(V_n)$ *onto a fiber of that fiber structure when* $z' \in T_m^{r*}(V_n)$.

To prove theorem 2, one canonically identifies the space $J^r(\mathbb{R}^n, V_m)$ with $\mathbb{R}^n \times T_n^r(V_m)$ by the map $X \to (x, \partial^r X)$, where $X \in J^r(\mathbb{R}^n, V_m)$, $x = \alpha(X)$. The chart $g \in A$ admits the prolongation $X \to (\partial^r X)h_x^{-1}$, which is equivalent to $(x, z) \to zh_x^{-1}$, $z = \partial^r X \in T_n^r(V_m)$, $zh_x^{-1} \in J^r(V_n, V_m)$, which defines a local chart on $J^r(V_n, V_m)$ with values in $\mathbb{R}^n \times T_n^r(V_m)$. The change of chart $\varphi \in \Lambda_n^r$ admits the prolongation $(x, z) \to (\varphi(x), z(\varphi_x^r)^{-1})$.

Definitions.

A section of the fiber structure on $J^r(V_n, V_m)$ with base V_n will be called an *r-flow* of V_n in V_m.

A section of the fiber structure on $J^r(V_n, V_m)$ with base V_m will be called an *r-field* of V_n in V_m.

A flow whose elements are p^r-covelocities will be called a *differential form* ([1]) *of order r* on V_n.

In another Note, we will study the singularities of a flow or a field.

([1]) One must not confuse this notion with that of *exterior form of degree p*, which is derived from it in the case of $r = 1$.

"Les prolongements d'une variété differentiable. III. Transitivité des prolongements." Comptes rendus **233** (1951), 1081-1083.

The prolongation of a differentiable manifold
III. Transitivity of prolongations

By **CHARLES EHRESMANN**

Translated by D. H. Delphenich

This Note continues two previous Notes ([1]). The transitivity of prolongations is expressed by Theorem 1. A regular prolongation admits a certain infinitesimal structure ([2]) with a property that is given in Theorem 2.

Let V_n and V_m be two r-manifolds whose structures are defined by two atlases A and A' with values in $\mathbb{R}^n$ and $\mathbb{R}^m$. Any r-jet $X \in J^r(V_n, V_m)$ determines a k-jet $\gamma_k(X) \in J^k(V_n, V_m)$, where $0 \leq k \leq r$. The map γ_k of $\in J^r(V_n, V_m)$ onto $\in J^k(V_n, V_m)$ is continuous. In particular, γ_0 is the projection γ of $J^r(V_n, V_m)$, if one identifies $J^0(V_n, V_m)$ with $V_n \times V_m$; i.e., $\gamma_0(X)$ is identified with the pair $(\alpha(X), \beta(X))$.

The prolongation to $J^k(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m \times L^r_{m,n}$ of the pseudogroup $\Lambda^r_n \times \Lambda^r_m$ is a pseudogroup of l-times differentiable transformations upon setting $r = k + l$. The prolongation to $J^r(V_n, V_m)$ of the atlas $A \times A'$ is compatible with the pseudogroup and *thus defines an l-times differentiable fiber structure on* $J^r(V_n, V_m)$ *with base* $V_n \times V_m$.

Let $f \in C^r_x(V_n, V_m)$, $X = j^r_x f$. The k-flow $j^k f$ is an l-map of a neighborhood of $x \in V_n$ in $\mathcal{J}^r(V_n, V_m)$, where the l-jet at x depends only upon X; set $\gamma^r_k(X) = j^l_x(j^k f)$. The map γ^l_k is a canonical isomorphism of $\mathcal{J}^r(V_n, V_m)$ onto a subspace of $\mathcal{J}^l(V_n, \mathcal{J}^k(V_n, V_m))$. In particular, one thus obtains an isomorphism of $T^r_n(V_m)$ onto a subspace of $T^l_n(T^k_n(V_m))$ and an isomorphism of $T^{r*}_n(V_m)$ onto a subspace of $\mathcal{J}^l(V_n, T^{k*}_m(V_n))$; this permits us identify $\partial^r_x f$ with $\partial^l_x(\partial^k f)$ when $f \in C^r_x(\mathbb{R}^n, V_m)$ and $d^r_x f$ with $j^l_x(d^k f)$ when $f \in C^r_x(V_n, \mathbb{R}^m)$. One likewise defines a canonical isomorphism of $L^r_{m,n}$ onto a subspace of $T^l_n(L^k_{m,n})$, which permits us identify f^r_x with $\partial^l_x(f^k)$, where $f \in C^r(\mathbb{R}^n, \mathbb{R}^m)$ and f^k is the function $x \to f^r_x$.

Let F be a space that admits L^k_n as a group of operators, the law of composition $(s, y) \to sy$ being continuous when $s \in L^k_n$, $y \in F$. The canonical homomorphism of L^r_n onto L^k_n,

([1]) *Comptes rendus* **233** (1951), pp. 598 and 773. Correction: pp. 599 ([2]), the algebraic case corresponds to a pseudogroup of local algebraic automorphisms of the real or complex numerical space.

([2]) A notion that will be defined in a general fashion in a later Note.

when composed with $(s, y) \to sy$, also defines L_k^r as a group of operators on F. Let $E(V_n, F)$ be the prolongation of order k of V_n whose fibers are isomorphic to F. The prolongation to $\mathbb{R}^n \times F$ of the pseudogroup Λ_n^r is the set of transformations:

$$(1) \qquad (x, y) \to (\varphi(x),\ \varphi_x^k y), \qquad x \in \mathbb{R}^n, y \in F, \varphi \in \Lambda_n^r.$$

The atlas A of V_n with values in $\mathbb{R}^n$ admits a prolongation $A(E)$ that define an atlas on E with values in $\mathbb{R}^n \times F$, which is compatible with the prolongation of Λ_k^r. We say that $E(V_n, F)$ is a *regular prolongation* of order k of V_n if the map $(s, y) \to sy$ is l-times differentiable. Since $x \to \varphi_x^r$ is l-times differentiable, the transformations (1) are then l-times differentiable and *the atlas $A(E)$ defines an l-times differentiable fiber structure on E with base V_n*.

THEOREM 1. – *If $E(V_n, F)$ is a regular prolongation of order k of V_n then any prolongation of order l of E is a prolongation of order r of V_n, upon setting $r = k + l$.*

This theorem is first proved for the prolongation $T_p^l(E)$. The atlas A admits a prolongation that defines an atlas of $T_p^l(E)$ with values in $T_p^l(\mathbb{R}^n \times F)$ that is compatible with the prolongation of Λ_n^r to $T_p^l(\mathbb{R}^n \times F)$. Upon identifying $T_p^l(\mathbb{R}^n \times F)$ with $T_p^l(\mathbb{R}^n) \times T_p^l(F) = \mathbb{R}^n \times L_{n,p}^l \times T_p^l(F) = \mathbb{R}^n \times F'$, where $F' = L_{n,p}^l \times T_p^l(F)$, the prolongation of $\varphi \in \Lambda_n^r$ – or of the transformation in (1) – is written:

$$(2) \qquad (x, u, Y) \to (\varphi(x),\ \varphi_x^l u\ , Y'),$$

where $x \in \mathbb{R}^n$, $u \in L_{n,p}^l$, $Y \in T_p^l(F)$, $Y' = g(j_x^l(\varphi^k)(\partial_x^l)u\ , Y)$, upon letting g denote the map $(s, y) \to sy$, as well as its prolongation to $T_p^l(L_n^k \times F)$. Set $z = (u, Y) \in F'$. The element $z' = (\varphi_x^l u\ , Y')$ depends uniquely upon (φ_x^r, z), and one may denote it by $\varphi_x^r \cdot z$. For $\psi \in \Lambda_n^r$, one has $\psi_{x'}^r \cdot (\varphi_x^r \cdot z) = (\varphi(x), \varphi_x^r \cdot z)$. The prolongation of A thus defines the fiber structure on $T_p^l(E)$ that is associated with the principal prolongation $H^r(V_n)$ with fibers isomorphic to F'.

For a fiber space associated with $T_p(E)$ the theorem gives the following results:

1. A fiber space that is associated with $T_p(E)$ is of the form $\mathbb{R}^n \times F''$.

2. Being given two fiber spaces $\mathcal{E}$ and $\mathcal{E}'$ with fibers that are isomorphic to F and two associated fiber space $\mathcal{E}_1$ and $\mathcal{E}_1'$ with fibers that are isomorphic to F_1, any representation

(isomorphism, resp.) of $\mathcal{E}$ in $\mathcal{E}'$ is associated with a representation (isomorphism, resp.) of $\mathcal{E}_1$ in $\mathcal{E}'_1$.

The prolongation of Λ_n^r to $\mathbb{R}^n \times F$ defines a *local structure* on $\mathbb{R}^n \times F$ whose infinitesimal isotropy group of order l at the point (x, y) is isomorphic to the subgroup of L_n^r that leaves y invariant. The atlas $A(E)$ determines a local structure on E that is subordinate to its l-times differentiable fiber structure. At $\xi \in E$ the *infinitesimal isotropy group of order l* of that local structure is isomorphic to that of $\mathbb{R}^n \times F$ at the point (x, y) that corresponds to x in a local chart of $A(E)$. If L_n^k operates transitively on F then these isotropy groups are isomorphic to the subgroup of L_n^k that leaves $y_0 \in F$ invariant. From this, one deduces:

THEOREM 2. – *If $E(V_n, F)$ is a regular prolongation of order k of V_n such that L_n^k operates transitively on F then any prolongation of order l of E admits a subordinate fiber structure with base E whose structure group is the subgroup of L_n^r that leaves $y_0 \in F$ invariant; this group may likewise be reduced to its projection on L_n (which is identified with a subgroup of L_n^r).*

It results from this, for example, that the principal prolongation $H^k(V_n)$ is parallelizable.

Let $E(B, F, G, H)$ be a fiber space, ρ an equivalence relation in F that is invariant under G, and let K be the subgroup of G that leaves each class mod ρ invariant. By the isomorphisms of F onto the fibers of E, ρ determines an equivalence relation $\bar{\rho}$ in E. *If ρ is an open equivalence relation then $E / \bar{\rho}$ is endowed with a fiber structure that is associated with $E(B, F, G, H)$ with fibers that are isomorphic F/ρ and structure group G/K.* We say that the fiber space E is an *extension* of the fiber space $E / \bar{\rho}$ that is associated with the canonical homomorphism of G onto G/K.

A prolongation of order r of V_n is not always a prolongation of order l of a prolongation of order k of V_n, but *it is always an extension of order k that is associated the canonical homomorphism of L_n^r onto L_n^k.* To each prolongation of V_n there corresponds an extension of the same type of an arbitrary fiber space with structure group L_n, a remark that permits one to extend to these extensions the study that is made in these Notes.

"Les prolongements d'une variété differentiable. IV. Éléments de contact et éléments d'enveloppe." Comptes rendus **234** (1952), 1028-1030.

The prolongation of a differentiable manifold
IV. Contact elements and enveloping elements

By **CHARLES EHRESMANN**

Translated by D. H. Delphenich

This Note continues four previous Notes ([1]). Groupoid associated with a fiber space, intransitivity classes, covariant maps. These notions, which we put at the basis for the theory of covariant differentials of an infinitesimal structure, are applied to define the contact elements and enveloping elements of order *r* and to indicate the fiber structure that is defined by these elements.

1. Let $E(B, F, G, H)$ be a fiber space with topological structure group G. We call the groupoid $\Pi = HH^{-1}$ of isomorphisms of a fiber onto a fiber the *associated groupoid*. It is endowed with a fiber structure with base $B\times B$, fiber G, and structure group $G\times G$ that operates on G in the following manner:

$$(s', s)t = s'ts^{-1},$$

where $s, s', t \in G$. Let p be the projection of E onto B, $\hat{p}$, the projection of H onto B, α and β, the projections of Π onto B that are defined by:

$$\alpha(h'h^{-1}) = \hat{p}(h), \qquad \beta(h'h^{-1}) = \hat{p}(h'),$$

where $h, h' \in H$. Π is a *groupoid of operators* for *E:* The composition θz of $\theta \in \Pi$ and $z \in E$ is defined when $p(z) = \alpha(\theta)$; one then has $p(\theta z) = \beta(\theta)$. One has $\theta'(\theta z) = (\theta'\theta)z$ when one of these compositions is defined. The left and right neutral elements of Π are the identity automorphisms of the fibers.

We call the set of compositions θz, where $\theta \in \Pi$ the *intransitivity class* of $z \in E$. Two points $z \in E$ and $y \in F$ will be called equivalent when there exists an $h \in H$ such that $z = hy$. The intransitivity class of z is the set of points that are equivalent to a point $y \in F$. The set of points of F that are equivalent to z is an intransitivity class of F relative to G.

A subspace E' of E will be called *invariant* (under Π) when it is the union of intransitivity classes. E' is then the set of points that are equivalent to an arbitrary point of a subspace F' of F that are invariant under G. *It is endowed with a fiber structure* $E'[B, F', G/N, H/N]$ *where N is the subgroup of G which leaves each point of F′ invariant.*

Let φ be a representation of G on a group of automorphism $\bar{G}$ of a space $\bar{F}$ and let $\bar{E}(B, \bar{F}, \bar{G}, \bar{H})$ be the fiber space associated with $E(B, F, G, H)$ by φ. We also let φ

([1]) Comptes rendus **233** (1951), pp. 598, 777, and 1081, **234** (1952), pp. 587.

denote the associated representation of H on $\bar{H}$, as well as that of Π on $\bar{\Pi}$, which is defined by:

$$\varphi(h'h^{-1}) = \varphi(h')\varphi(h)^{-1},$$

where h, $h' \in H$. A map ψ of E in $\bar{E}$ will be called the *covariant map* (*invariant*, if $\bar{G}$ reduces to the neutral element, resp.) when $\psi\theta = \varphi(\theta)\psi$ for any $\theta \in \Pi$. The covariant map ψ projects to the identity map on B; it corresponds to a covariant map ψ_0 of F in $\bar{F}$; i.e., such that $\psi_0 s = \varphi(s)\psi_0$, where $s \in G$. An intransitivity class is mapped under ψ to an intransitivity class. Let ρ be the equivalence relation in F that is associated with ψ_0 and let $\bar{\rho}$ be the equivalence relation in E that is associated with ψ, and suppose that ρ is an open equivalence relation. ψ then admits the *canonical decomposition:*

$$\psi = \psi'' \psi',$$

where ψ' is the canonical covariant map of E onto $E/\bar{\rho}$, which is endowed ([1]) with an associated fiber structure $E/\bar{\rho}$ [B, F/ρ, G/K, H/K], K being the kernel of φ, and ψ'' being an isomorphism of $E/\bar{\rho}$ onto an invariant subspace of $\bar{E}$. *In order for the classes* mod $\bar{\rho}$ *on E to be the fibers of a fiber structure whose pseudogroup of local automorphisms includes the local automorphisms of the structure* E(B, F, G, H), *it is necessary and sufficient that the classes* mod ρ *in F are the fibers of a fiber structure that is invariant under G* (a condition that is verified, in particular, when G is a transitive Lie group in F and a class mod ρ is closed).

2. Consider two r-manifolds V_n and V_m. Let $\Pi^r(V_n)$ be the groupoid associated with $H^r(V_n)$; it is the set of invertible elements of $J^r(V_n, V_n)$. Consider $J^r(V_n, V_m)$, endowed with its fiber structure ([1]) that has base $V_n \times V_m$, fibers isomorphic to $L^r_{m,n}$, structure group $L^r_m \times L^r_n$, and associated with the principal fiber space $H^r(V_n) \times H^r(V_m)$; its associated groupoid is $\Pi^r(V_n) \times \Pi^r(V_m)$. The intransitivity class of $z \in J^r(V_n, V_m)$ is the set of elements $h'yh^{-1}$, where $h \in H^r(V_n)$, $h' \in H^r(V_m)$, y being an element of $L^r_{m,n}$ that is equivalent to z. We call the intransitivity classes of y and z the *equivalence elements* of y and z.

Also, consider the fiber structure on $J^r(V_n, V_m)$ that has base V_n, fibers isomorphic to $T^r_n(V_m)$, structure group L^r_n, and associated with $H^r(V_n)$. The composition of $s \in L^r_n$ and $Y \in T^r_n(V_m)$ is Ys^{-1}. The associated groupoid is $\Pi^r(V_n)$, the composition of $z \in J^r(V_n, V_m)$ and $\theta \in \Pi^r(V_n)$ being $z\theta^{-1}$. The intransitivity class of z relative to $\Pi^r(V_n)$ corresponds to the class YL^r_n in $T^r_n(V_m)$, where $Y = zh$. *This class* YL^r_n *is called the contact element of Y or of z; we also say that it is an contact* n^r*-element in* V_m.

Upon considering the fiber structure on $T^r_n(V_m)$ that has base V_m, fibers isomorphic to $L^r_{m,n}$, and structure group L^r_n, the equivalence relation $Y \sim Ys^{-1}$ corresponds in $L^r_{m,n}$ to the equivalence relation $y \sim ys^{-1}$, which is invariant under L^r_n.

Let $P^r_{m,n}$ be the quotient space of $L^r_{m,n}$ under this equivalence relation; it is the space of contact n^r-elements with origin O in $\mathbb{R}^m$. This space, upon which L^r_n operates, is not separable, but each of its intransitivity classes is a Lie homogeneous space. *The space* $P^r_n(V_m)$ *of contact n^r-elements on V_m is the prolongation of order r of V_m with fibers that are isomorphic to* $P^r_{m,n}$. The map $Y \to YL^r_n$ is a covariant map ψ^r of $T^r_n(V_m)$ onto $P^r_n(V_m)$. The reciprocal image under ψ^r of an intransitivity class of $P^r_n(V_m)$ is endowed with a fiber structure that is associated with that projection. For $k \leq r$, the intransitivity classes of $P(V_m)$ are the *regular prolongations* ([1]) of V_n and canonical map ψ^k of $T^r_n(V_m)$ onto $P^r_n(V_m)$ reduces for each of them to an l-map, where $k + l = r$. If f is an r-map of V_n into V_m then the pair (f, V_n) is called an *embedded r-manifold in V_m*, $f(V_n)$, its support, $\psi^k(j^k_x f)$ is its contact element of order k at x. The map $\psi^k(j^k f)$ of V_n into $P^r_n(V_m)$ defines *the prolongation of order k of an embedded manifold. If the equivalence element of $j^k_x f$ is fixed then this prolongation will be an embedded l-manifold in an intransitivity class of* $P^r_n(V_m)$; *the contact element* $\psi^r(j^r_x f)$ *is canonically identified with the contact element of order l at x of the prolongation* $\psi^k(j^k f)$.

Upon likewise considering the fiber structure on $J^r(V_n, V_m)$ with base V_m, one defines the *enveloping element of* $z \in J^r(V_n, V_m)$ or an equivalence element Z of $T^r_n(V_m)$; it is the class $T^{r*}_m(V_n)$ that we also call the *enveloping m^r-element in V_n*. The set $P^{r*}_m(V_n)$ of these elements is the prolongation of order r of V_n with fibers isomorphic to $P^{r*}_{m,n}$, the set of classes $L^r_m y$ (viz., the enveloping element of $y \in L^r_{m,n}$).

"Les prolongements d'une variété différentielle. V. Covariants différentiels et prolongements d'une structure infinitésimale," Comptes rendus (1952), 1424-1425.

The prolongations of a differentiable manifold V. Differential covariant and prolongations of an infinitesimal structure

By CHARLES EHRESMANN

Translated by D. H. Delphenich

This Note continues five previous Notes ([1]). Definition of the notion of differential covariant with respect to a pure infinitesimal structure. Prolongations of an infinitesimal structure. The successive prolongations of a general affine connection of order *r*.

1. Let V_n be an *r*-manifold, $H^r(V_n)$, its principal prolongation of order *r*, and $\Pi^r(V_n)$, its associated groupoid. Consider two prolongation *E* and $\bar{E}$ of V_n of order *k* and $\bar{k}$, where $k \leq r$, $\bar{k} \leq r$. $\Pi^r(V_n)$ is a groupoid of operators on *E* and $\bar{E}$. Let ψ be a covariant map of *E* into $\bar{E}$; i.e., $\psi\theta = \theta\psi$, where $\theta \in \Pi^r(V_n)$. We say that $\psi(z)$ is a *differential covariant* of $z \in E$. Such a ψ corresponds to a covariant map ψ_0 of *F* into $\bar{F}$, which are the fiber types of *E* and $\bar{E}$: $\psi_0 s = s\psi_0$, where $s \in L_n^r$.

Let $\mathfrak{s}$ be a pure infinitesimal structure that is defined by a section σ of *E*. If $z = s(x)$, where $x \in V_n$, then the element $y(z)$ is a differential covariant of $\mathfrak{s}$ at the point *x*; the section $\psi\sigma$ of $\bar{E}$ is a differential covariant of $\mathfrak{s}$.

Suppose that σ is *l*-times differentiable, where $k + l = r$. Let $\sigma'(x)$ be the contact element of $j_x^l\sigma$. We say that σ', the prolongation of order *l* of the section σ, defines the prolongation $\mathfrak{s}'$ of order *l* of $\mathfrak{s}$. $\mathfrak{s}'$ is a pure infinitesimal structure of order $k + l$, and its differential covariants will again be called differential covariants of $\mathfrak{s}$.

If (f, V_p) is an embedded *r*-manifold in V_n then its differential covariants are those of its contact elements.

2. A groupoid $\Pi(\mathfrak{s})$ is associated with the pure infinitesimal structure $\mathfrak{s}$: viz., the groupoid of infinitesimal automorphisms of $\mathfrak{s}$. It is a subgroup of $\Pi^k(V_n)$, and its solutions are the local automorphisms of $\mathfrak{s}$. A *covariant map with respect to* $\mathfrak{s}$ is defined by the condition $\psi\theta = \theta\psi$, where $\theta \in \Pi(s)$, ψ being a map of a prolongation of order *k* of V_n to a prolongation of order k_2 ; $k_1 \leq k$, $k_2 \leq k$. If $\mathfrak{s}$ is *l*-times differentiable then consider the prolongation $\mathfrak{s}'$ of order *l* of $\mathfrak{s}$ and its associated groupoid $\Pi(\mathfrak{s}')$. The covariant maps with respect to $\mathfrak{s}'$ are again called covariant with respect to $\mathfrak{s}$ and one thus has the notion of differential covariant of order $\leq k + l$ with respect to $\mathfrak{s}$.

([1]) Comptes rendus **233** (1951), pp. 598, 777, and 1081; **234** (1952), pp. 587 and 1028.

One must, above all, consider the covariant maps in a *tensorial prolongation* of V_n (a prolongation of first order that is associated with a linear representation of L_n^1).

Let $\mathfrak{s}$ be a *regular* infinitesimal structure of order k on V_n. It corresponds to a fiber subspace $\bar{H}(V_n)$ of $H^k(V_n)$, with structure group G that is a subgroup of L_n^k. A fiber subspace E that is associated with $\bar{H}(V_n)$ that corresponds to a representation φ of G on a group of automorphisms of F will be called a *prolongation of* V_n *relative to* $\mathfrak{s}$. In order for E to also be a prolongation relative to the r-manifold structure on V_n, it is necessary and sufficient that φ prolongs to L_n^k. The notion of covariant map relative to $\mathfrak{s}$ is defined for the prolongations relative to $\mathfrak{s}$.

3. The groupoid $\Pi^k(V_n)$ is endowed with an l-manifold structure, where $r = k + l$. The three fiber structures on $\Pi^k(V_n)$ that correspond to the projection γ into $V_n \times V_n$ and the two projections α and β onto V_n are l-times differentiable. Let E be a regular prolongation of order k of V_n and let p be the projection of E of V_n. The law of composition $(\theta, z) \to \theta z$, where $\theta \in \Pi^k(V_n)$ and $z \in E$ prolongs to the set of pairs (Θ, Z), where $\Theta \in T_p^l[\Pi^k(V_n)]$ and $Z \in T_p^l(E)$ such that $pZ = \alpha\Theta$.

If G is a Lie group that operates in an r-differentiable manner on F then $T_p^k(G)$ *is a group that operates in an l-fold differentiable manner on* $T_p^k(F)$. *The group* $T_p^k(G)$ *is l-times differentiable, and in general it is a non-trivial extension of G.*

In particular, set $L_n^{[r]} = T_n^1(L_n^{[r-1]})$, $L_n^{[1]} = L_n^1 = L_n$. The group L_n^r is canonically isomorphic to a subgroup of $L_n^{[r]}$. Also, set $L_n^{k,l} = T_n^l(L_n^k)$.

Upon setting $l \leq k$ let $\Pi^{k,l}(V_n)$ be the set of $X \in \mathcal{J}^l[V_n, \Pi^k(V_n)]$, such that αX is the neutral l-jet and βX be the canonical l-jet that is deduced from the target k-jet of X, $\Pi^{k,l}(V_n)$ is a groupoid that operates on $T_p^l(E)$.

Let $\Pi^{k,l}(V_n)$ be the set of $Y \in T_p^l[\Pi^k(V_n)]$ such that the projection of Y onto V_n is the canonical n^l-velocity that is deduced from the n^k-velocity at the origin of Y. $H^r(V_n)$ is canonically identified with a subspace of $H^{k,l}(V_n)$, which is the associated principal fiber space to $H^r(V_n)$ by enlarging L_n^r to $L_n^{k,l}$. The groupoid associated to $H^{k,l}(V_n)$ is $\Pi^{k,l}(V_n)$.

We call an infinitesimal connection in $H^k(V_n)$ a *special affine connection of order k.* It is a regular infinitesimal structure of order $k + 1$ that is defined ([1]) by a certain field $\tilde{C}$ of contact n^1-elements in $\Pi^k(V_n)$. The prolongation of order 1 of that structure is an infinitesimal connection in $H^{k,1}(V_n)$. We call an infinitesimal connection in $H^{[r]}(V_n)$, which is the principal fiber space associated with $H^r(V_n)$ by enlarging L_n^r to $L_n^{[r]}$, a *general affine connection of order r. By successive prolongations of a special affine connection one obtains a general affine connection of order r;* however, one does not obtain all connections of this type in this manner.

([1]) See the precise definition in: C. EHRESMANN, *Les connexions infinitésimales (Colloque Topologie,* Bruxelles, 5-8 June 1950).

A general study of the infinitesimal connections of order r will be made in another article.