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The Dirac equations for semi-vectors

By A. EINSTEIN and W. MAYER

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Translated by D. H. Delphenich

CONTENTS:

Introduction [1]): Brief report on semi-vectors and semi-tensors. [2])

In this paper, the most necessary facts from the theory of semi-vectors shall be introduced, to the extent that is required for the understanding of the reader. Corresponding to the scope of the previous paper, we restrict ourselves to the space of special relativity. By the introduction of rectangular Cartesian coordinates ($g_{11} = g_{22} = g_{33} = - g_{44} = 1$, the rest of the $g_{ik} = 0$), we define certain structural tensors $c_{\sigma\tau}$, σ, $\tau = 1, \ldots, 4$:

$$c_{\sigma\tau} = c\, g_{\sigma\tau} + v_{\sigma\tau}\,, \tag{1}$$

where the skew-symmetric $v_{\sigma\tau}$ satisfies the defining v-relation [3]):

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[1]) "Semivektoren und Spinoren," Sitzber. der Preuss. Akad. 1932.

[2]) We should also be thankful here for the fact that we were prompted to carry out this investigation by the urgent request of EHRENFEST that we look for a logically simple and transparent analysis of spinors.

[3]) $\eta_{\sigma\tau\mu\nu}$ is anti-symmetric in the indices and $\eta_{1234} = 1$.

$T_{\sigma\tau\mu\nu} = \sqrt{g}\ \eta_{\sigma\tau\mu\nu}$ has a tensor character and one has $T^{\sigma\tau\mu\nu} = \dfrac{1}{\sqrt{g}}\ \eta^{\sigma\tau\mu\nu}$.

$$v_{\sigma\tau} = -\tfrac{1}{2}\sqrt{g}\,\eta_{\sigma\tau\mu\nu}\,v^{\mu\nu}. \qquad g = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}. \tag{1′}$$

Since $\sqrt{g} = i$, from (1′), the $v_{\sigma\tau}$ are complex tensors, and, as (1′) further shows, completely determined by three of their components (e.g., v_{12}, v_{13}, v_{14}). The most general $c_{\sigma\tau}$ that are defined by (1) thus include four complex constants.

Since the $c_{\sigma\tau}$ are complex, along with them, and with an equal significance to them, the conjugate quantities $\bar{c}_{\sigma\tau}$ also appear, which are, however, different from the quantities $c_{\sigma\tau}$, because, from (1′), $\bar{v}_{\sigma\tau}$ satisfies the relation:

$$\bar{v}_{\sigma\tau} = \tfrac{1}{2}\sqrt{g}\,\eta_{\sigma\tau\mu\nu}\,\bar{v}^{\mu\nu} \qquad (\overline{\sqrt{g}} = -\sqrt{g}\ !). \tag{1″}$$

Now, if two c's, $c_{\sigma\tau}$ and $c'_{\sigma\tau}$, are given then there exists the *fundamental* commutation law:

$$c_{\sigma\tau}\bar{c}'^{\tau}{}_{\rho} = \bar{c}'_{\sigma\tau}c^{\tau}{}_{\rho}\ . \tag{2}$$

However, the content of the relation (2) goes further. If $c'_{\sigma\tau} = c'\,g_{\sigma\tau} + v'_{\sigma\tau}$ is the most general c tensor then the totality of all tensors $c_{\sigma\tau}$ for which (2) is true – so they all "commute" with the most general c-tensor – coincides with the totality of just these tensors.

However, it follows from this that the composition $c_{\sigma\tau}c'^{\tau}{}_{\rho}$ of two c's ($c_{\sigma\tau}$, $c'_{\sigma\tau}$) is again a c-tensor. From (1), one recognizes that, along with $c_{\sigma\tau}$, $c'_{\sigma\tau} = c_{\tau\sigma}$ is also a c-tensor. Thus, along with $c_{\sigma\tau}$, $c_{\sigma\tau}c^{\sigma}{}_{\rho}$ is also a c-tensor, and due to the symmetry of these quantities in τ and σ, from (1), one must have:

$$c_{\sigma\tau}c^{\sigma}{}_{\rho} = \Phi\, g_{\tau\rho}\,.$$

By summation ($\tau = \rho$), one calculates Φ and obtains the important formula:

$$c_{\sigma\tau}c^{\sigma}{}_{\rho} = \left(\tfrac{1}{4}c_{\alpha\beta}c^{\alpha\beta}\right)g_{\tau\rho}\,. \tag{3}$$

If $|\,c_{\sigma\tau}| = \Delta$ is the determinant of the $c_{\sigma\tau}$ then (3) gives:

$$\Delta^2 = \left(\tfrac{1}{4}c_{\alpha\beta}c^{\alpha\beta}\right)^4. \tag{4}$$

The vanishing of the determinant $| c_{\alpha\beta} |$ – i.e., the "degeneracy of the $c_{\alpha\beta}$" – will then also be characterized by $c_{\alpha\beta}\, c^{\alpha\beta} = 0$ [1]).

If one considers the $c_{\alpha\beta}$ to be a complex transformation matrix then, from (3), it represents a rotation with a simultaneous dilatation [2]). This suggests that we examine the following "product construction":

$$a_{\sigma\rho} = c_{\sigma\tau}\bar{c}^{\tau}{}_{\rho} . \tag{5}$$

Due to (2), $a_{\sigma\rho}$ is real ($\bar{a}_{\sigma\rho} = a_{\sigma\rho}$); if one further sets $c_{\alpha\beta}\, c^{\alpha\beta} = 4\Phi$ then it follows immediately from (2), (3), and (5) that:

$$a_{\sigma\mu}\, a^{\sigma}{}_{\nu} = 4\Phi\,\bar{\Phi}\, g_{\mu\nu} . \tag{5′}$$

When regarded as a transformation matrix, $a_{\sigma\rho}$ means a real LORENTZ rotation with a simultaneous dilatation. If $\Phi = 1$ then $c_{\sigma\rho}$ is a complex pure LORENTZ rotation and $a_{\sigma\rho}$ is a real one. Conversely, one also shows that any real LORENTZ rotation $a_{\sigma\rho}$, when it can be generated from the identity $g_{\sigma\rho}$ by real infinitesimals (we then call it a *proper* rotation) has a representation (5). This representation is unique, up to the case $a_{\sigma\rho} = (-c_{\sigma\tau})(-\bar{c}^{\tau}{}_{\rho})$, which, from (5), is trivial when $c_{\sigma\tau}$ is also a LORENTZ rotation.

The association $a_{\sigma\rho} \rightleftarrows c_{\sigma\rho}$ exists between the elements $a_{\sigma\rho}$ of the group of proper LORENTZ rotations and the elements $c_{\sigma\rho}$ of the LORENTZ rotations ($c_{\alpha\beta}\, c^{\alpha\beta} = 4$), which is mediated by (5).

However, this association is, as one again sees from (2), an isomorphism, and this represents the mathematical basis for the introduction of semi-vectors and tensors.

If one performs the LORENTZ transformation:

$$x'_i = a_i^k\, x_k \tag{6}$$

of Cartesian coordinates then, by definition, a *semi-vector "of the first kind"* $\rho_{\bar{\alpha}}$ transforms according to [3]):

$$\rho'_{\bar{\alpha}} = \bar{c}_{\alpha}{}^{\beta} \rho_{\bar{\beta}} , \tag{6′}$$

and a *semi-vector "of the second kind"* $\sigma_{\bar{\bar{\alpha}}}$ transforms according to:

$$\sigma'_{\bar{\bar{\alpha}}} = c_{\alpha}{}^{\beta} \sigma_{\bar{\bar{\beta}}} . \tag{6″}$$

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[1]) The exact relation between Δ and $c_{\alpha\beta}\, c^{\alpha\beta}$ is: (4′) $\Delta = -\left(\frac{1}{4} c_{\alpha\beta} c^{\alpha\beta}\right)^2$.

[2]) We speak of it briefly as a "generalized" LORENTZ transformation.

[3]) A singly overbarred (Greek) index characterizes a semi-quantity of the first kind, while the doubly overbarred index characterizes a semi-quantity of the second kind.

Later on, whenever the character of an index has been established (as a spatial index or semi-index of the first or second kind), we shall omit the inconvenient overbars.

Therefore, the relation (5) exists between the a_i^k and the LORENTZ transformations $c_\alpha^{\ \beta}$.

The conjugate quantity to a semi-vector of one kind is, by definition, a semi-vector of the other kind.

If $C_{\sigma\tau}$ is any c-tensor and $c_{\sigma\tau}$ is a LORENTZ-c ($c_{\alpha\beta}\, c^{\alpha\beta} = 4$) then, from (3), it follows from:

$$C_{\sigma\tau}\bar{c}^{\tau}_{\ \rho} = \bar{c}_{\sigma\tau}C^{\tau}_{\ \rho}, \tag{2'}$$

after multiplying with $\bar{c}_\nu^{\ \rho}$, that:

$$C_{\sigma\nu} = \bar{c}_\sigma^{\ \tau}\bar{c}_\nu^{\ \rho}C_{\tau\rho}. \tag{7}$$

A comparison with (6′) shows that every $C_{\sigma\tau}$ quantity, as a semi-tensor of the first kind, is numerically invariant. With our semi-index notation, we can also say that briefly as: $C_{\bar{\sigma}\bar{\tau}}$ is numerically invariant.

In particular, this is true for $g_{\bar{\sigma}\bar{\tau}}$ ($c_{\sigma\tau}$, the transformation matrix of the semi-vector, is indeed a LORENTZ transformation).

One likewise shows that all $C_{\bar{\bar{\sigma}}\bar{\bar{\tau}}}$ – in particular, $g_{\bar{\bar{\sigma}}\bar{\bar{\tau}}}$ – are numerically invariant.

(The "raising" and "lowering" of semi-indices is done with the metric tensor $g_{\sigma\tau} = g_{\bar{\sigma}\bar{\tau}} = g_{\bar{\bar{\sigma}}\bar{\bar{\tau}}}$ of R_4.)

Along with the *numerical invariance* of these second-rank tensors, one has the fundamental third-rank E tensor $E^{r\bar{\sigma}\bar{\bar{\tau}}}$, which depends upon four constants $a_{(t)}$, for an example of the simplest such tensor.

Its form is:

$$E_{r\bar{s}\bar{\bar{t}}} = g_{rs}\, a_{(t)} + g_{rt}\, a_{(s)} - g_{st}\, a_{(r)} - \sqrt{g}\ \eta_{rstw}\, a^{(w)}, \qquad a^{(w)} = g^{wt}\, a_{(t)}\,. \tag{8}$$

If the $a_{(t)}$ are real then (8) gives:

$$\bar{E}_{r\bar{s}\bar{\bar{t}}} = E_{r\bar{s}\bar{\bar{t}}} \qquad (\text{real } a_{(t)}). \tag{9}$$

In the present paper, we will examine the most general linear first order system of equations for two semi-vectors $\psi_{\bar{\sigma}}$ and $\chi_{\bar{\bar{\sigma}}}$:

$$\left.\begin{aligned} E^{r\bar{\sigma}\bar{\bar{\tau}}}(\chi_{\bar{\bar{\rho}},r} - i\varepsilon\chi_{\bar{\bar{\rho}}}\varphi_r) &= c^{\bar{\nu}\bar{\sigma}}\psi_{\bar{\nu}} \\ E^{*r\bar{\sigma}\bar{\bar{\tau}}}(\psi_{\bar{\sigma},r} - i\varepsilon\psi_{\bar{\sigma}}\varphi_r) &= -\bar{c}^{\bar{\bar{\rho}}\bar{\bar{\nu}}}\chi_{\bar{\bar{\nu}}} \end{aligned}\right\} \tag{10}$$

(E, with the constants $a_{(t)}$, E^*, with the constants $a^*_{(t)}$) that is produced by variation of the most general HAMILTON function that comes into consideration (§ 1). In (10), φ_ν is the electromagnetic potential vector.

§ 1. Hamilton function and field equations.

The Hamilton scalar of the total field is of the form:

$$H = H_1 + H_2 + H_3 ,$$

in which H_1 means the metric curvature scalar, H_2, the electromagnetic field scalar ($\varphi_{\alpha\beta}\varphi^{\alpha\beta}$), and H_3 means a scalar that we must look for that depends upon the electromagnetic potential vector φ_r and two semi-vectors $\psi_{\bar{\sigma}}$ and $\chi_{\bar{\bar{\sigma}}}$. Thus, H_3 proves to be determined, by and large, by two conditions:

a. It shall be real.

b. It shall include the aforementioned quantities in such a way that the resulting system of equations becomes linear and of first order relative to the semi-quantities and essentially depends upon only the anti-symmetric derivatives of the φ_r (φ-condition).

These two conditions then lead to the following form for H_3, in which we temporarily restrict ourselves to the special theory of relativity. (From the results of our previous paper, this restriction is not essential, since it was shown there how one had to construct the general-relativistic expressions.) If A, B are real constants then it is:

$$\left.\begin{aligned} H_3 = \; & E^i_{\sigma\tau}(\psi^\sigma_{,i} - i\varepsilon\psi^\sigma\varphi_i)\bar{\psi}^\tau + \bar{E}^i_{\sigma\tau}(\bar{\psi}^\sigma_{,i} + i\varepsilon\bar{\psi}^\sigma\varphi_i)\psi^\tau \\ & + iA[E^i_{\sigma\tau}(\psi^\sigma_{,i} - i\varepsilon\psi^\sigma\varphi_i)\bar{\psi}^\tau - \bar{E}^i_{\sigma\tau}(\bar{\psi}^\sigma_{,i} + i\varepsilon\bar{\psi}^\sigma\varphi_i)\psi^\tau] \\ & + E^{i*}_{\sigma\tau}(\bar{\chi}^\sigma_{,i} + i\varepsilon\bar{\chi}^\sigma\varphi_i)\chi^\tau + \bar{E}^{i*}_{\sigma\tau}(\chi^\sigma_{,i} - i\varepsilon\chi^\sigma\varphi_i)\bar{\chi}^\tau \\ & + iB[E^{i*}_{\sigma\tau}(\bar{\chi}^\sigma_{,i} + i\varepsilon\bar{\chi}^\sigma\varphi_i)\chi^\tau - \bar{E}^{i*}_{\sigma\tau}(\chi^\sigma_{,i} - i\varepsilon\chi^\sigma\varphi_i)\bar{\chi}^\tau] \\ & + C_{\sigma\tau}\psi^\sigma\bar{\chi}^\tau + \bar{C}_{\sigma\tau}\bar{\psi}^\sigma\chi^\tau . \end{aligned}\right\} \tag{1}$$

It is therefore important to remark that the "φ-condition" (i.e., the invariance of H_3 whenever φ_i, ψ^σ, χ^σ are replaced with $\varphi_i + \frac{1}{\varepsilon}\frac{\partial\alpha}{\partial x_i}$, $\psi^\sigma e^{i\alpha}$, $\chi^\sigma e^{i\alpha}$ in sequence) involves the introduction of only a *single* constant ε.

If $a_{(t)} = \alpha_t + i\beta_t$ ($a^*_{(t)} = \alpha^*_t + i\beta^*_t$, resp.) is the system of constants in E (E^*, resp.) then one has:

$$E^i_{\sigma\tau} = E^i_{\sigma\tau}(\alpha) + iE^i_{\sigma\tau}(\beta) \qquad \text{and} \qquad \bar{E}^i_{\sigma\tau} = E^i_{\tau\sigma}(\alpha) - iE^i_{\tau\sigma}(\beta) .$$

The first two rows of (1) then read:

$$\begin{aligned} & E^i_{\sigma\tau}(\alpha)(\psi^\sigma_{,i} - i\varepsilon\psi^\sigma\varphi_i)\bar{\psi}^\tau + E^i_{\sigma\tau}(\alpha)(\bar{\psi}^\sigma_{,i} + i\varepsilon\bar{\psi}^\sigma\varphi_i)\psi^\tau \\ + \; & [E^i_{\sigma\tau}(\beta)(\psi^\sigma_{,i} - i\varepsilon\psi^\sigma\varphi_i)\bar{\psi}^\tau - E^i_{\sigma\tau}(\beta)(\bar{\psi}^\sigma_{,i} + i\varepsilon\bar{\psi}^\sigma\varphi_i)\psi^\tau]\; i \\ + \; & iA[E^i_{\sigma\tau}(\alpha)(\psi^\sigma_{,i} - i\varepsilon\psi^\sigma\varphi_i)\bar{\psi}^\tau - E^i_{\sigma\tau}(\alpha)(\bar{\psi}^\sigma_{,i} + i\varepsilon\bar{\psi}^\sigma\varphi_i)\psi^\tau] \\ - \; & A[E^i_{\sigma\tau}(\beta)(\psi^\sigma_{,i} - i\varepsilon\psi^\sigma\varphi_i)\bar{\psi}^\tau + E^i_{\sigma\tau}(\beta)(\bar{\psi}^\sigma_{,i} + i\varepsilon\bar{\psi}^\sigma\varphi_i)\psi^\tau] \end{aligned}$$

Under variation, the first and last row produce nothing, and can therefore be omitted, while the other two rows, when one sets:

$$\gamma_t = \beta_t + A\alpha_t,$$

yield:

$$i[E^i_{\sigma\tau}(\gamma)(\psi^\sigma{}_{,i} - i\varepsilon\psi^\sigma\varphi_i)\bar{\psi}^\tau - E^i_{\sigma\tau}(\bar{\psi}^\sigma{}_{,i} + i\varepsilon\bar{\psi}^\sigma\varphi_i)\bar{\psi}^\tau].$$

Therefore, with no loss of generality, one can choose the additional Hamilton term to be (up to the factor *i*):

$$\left.\begin{array}{l} E^i_{\sigma\tau}(\psi^\sigma{}_{,i} - i\varepsilon\psi^\sigma\varphi_i)\bar{\psi}^\tau - E^i_{\sigma\tau}\psi^\sigma(\bar{\psi}^\tau{}_{,i} + i\varepsilon\bar{\psi}^\tau\varphi_i) \\ -E^{i*}_{\sigma\tau}(\bar{\chi}^\sigma{}_{,i} + i\varepsilon\bar{\chi}^\sigma\varphi_i)\chi^\tau + E^{i*}_{\sigma\tau}\bar{\chi}^\sigma(\bar{\chi}^\tau{}_{,i} - i\varepsilon\bar{\chi}^\tau\varphi_i) \\ +C_{\sigma\tau}\psi^\sigma\bar{\chi}^\tau - \bar{C}_{\sigma\tau}\bar{\psi}^\sigma\chi^\tau, \end{array}\right\} \tag{2}$$

instead of (1), in which the constants in E (E^*, resp.) are now real; in the sequel, they will be again denoted by a and a^*. By varying the ψ and χ in (2), we obtain the "Dirac equations" [1]):

$$\left.\begin{array}{l} E^i_{\sigma\tau}(\psi^\sigma_{,i} - i\varepsilon\psi^\sigma\varphi_i) = +\bar{C}_{\tau\sigma}\chi^\sigma, \\ E^{i*}_{\sigma\tau}(\chi^\tau_{,i} - i\varepsilon\chi^\tau\varphi_i) = -C_{\tau\sigma}\psi^\tau. \end{array}\right\} \tag{3}$$

One obtains the current vector from (2) by varying φ_i in the form:

$$J^i = E^i_{\sigma\tau}\psi^\sigma\bar{\psi}^\tau + E^{i*}_{\sigma\tau}\bar{\chi}^\sigma\chi^\tau. \tag{4}$$

As it must be, one has:

$$J^i{}_{,i} = 0 \tag{5}$$

as a consequence of the system (3).

§ 2. The Dirac equations.

The system § 1, (3):

$$\left.\begin{array}{l} E^{i\sigma\tau}(\psi_{\sigma,i} - i\varepsilon\psi_\sigma\varphi_i) = +\bar{C}^{\tau\rho}\chi_\rho, \\ E^{i\sigma\tau*}(\chi_{\tau,i} - i\varepsilon\chi_\tau\varphi_i) = -C^{\rho\sigma}\psi_\rho, \end{array}\right\} \tag{1}$$

resp., has, along with the constant ε, the eight real constants a_i, a_i^* in the E, and four more complex constants (C_{11}, C_{12}, C_{13}, C_{14}) that fix the $C^{\rho\sigma}$ on the right-hand sides. Along with ε, there are 16 real constants, in all.

However, we can reduce this number of constants substantially, if we make use of the possibility that, instead of the quantities χ_τ, we can introduce quantities $\underline{\chi}_\nu$ by a non-singular transformation $\bar{c}_\tau^{\ \nu}$:

[1]) In these equations, it is essential that the transformation invariant factor C enter on the right-hand side.

$$\chi_\tau = \bar{c}_\tau{}^\nu \underline{\chi}_\nu , \tag{2}$$

and, instead of the quantities ψ_σ, we can introduce the quantities $\underline{\psi}_\mu$ by a non-singular transformation $c'^{\ \mu}_\sigma$:

$$\psi_\sigma = c'^{\ \mu}_\sigma \underline{\psi}_\mu . \tag{3}$$

A glance at the Hamilton function § 1, (2) shows that E therefore goes to $\underset{\sim}{E}$:

$$\underset{\sim}{E}{}^i_{\mu\nu} = E^i_{\sigma\tau} c'^\sigma{}_\mu \bar{c}^\tau{}_\nu , \tag{4}$$

and E* goes to $\underset{\sim}{E}{}^*$:

$$\underset{\sim}{E}{}^{i*}_{\mu\nu} = E^{i*}_{\sigma\tau} c^\sigma{}_\mu \bar{c}^\tau{}_\nu , \qquad ^1). \tag{4′}$$

Which transformations of the quantities a into $\underset{\sim}{a}$ (a^* into $\underset{\sim}{a}{}^*$, resp.) will be realized by such an E transformation (4) [(4′), resp.)]?

If one sets i, μ, τ equal to the values 1, 1, 1; 2, 2, 2,; 3, 3, 3, and 4, 4, 4 in succession in the relation:

$$E^{i\ \tau'}_{\mu} = E^{i\sigma\tau} c_{\mu\sigma} \tag{5}$$

then one obtains immediately:

$$a'_i = c_{ik}\, a^k, \tag{5′}$$

where a, a' are the values of the constants in E, E'. Likewise, it follows from the relation:

$$E^{i\sigma}{}_\nu{}'' = E^{i\sigma\tau'} \bar{c}_{\nu\tau'} \tag{6}$$

that there is a relation:

$$a''_i = \bar{c}_{ik} a^{k'} . \tag{6′}$$

Thus, one has:

$$E^{i\ ''}_{\sigma\nu} = E^{i\mu\tau} c_{\sigma\mu}\ \bar{c}_{\nu\tau} , \tag{7}$$

along with:

$$a''_i = \bar{c}_{ik} c^k{}_j a^j . \tag{7′}$$

A transformation (7) thus induces a “generalized” real LORENTZ transformation relative to the constants a in the E.

We can therefore speak of a “pseudo-vector” (a_i), instead of a system of four constants (a_i), so we think about this situation.

Likewise, it seems preferable to speak of space-like and time-like $E^{i\sigma\tau}$, and indeed, E might be called space-like (time-like, resp.) when (a_i) is space-like (time-like, resp.). We also speak of a singular E when the corresponding pseudo-vector is a null vector ($a_i\, a^i$ = 0). Since we are dealing with a proper (generalized) LORENTZ transformation in (7′),

[1]) Naturally, $\underset{\sim}{E}$ is again numerically invariant, so it is an E quantity of the same structure.

we can bring a space-like E into the form $E^{i\sigma\tau}(1, 0, 0, 0)$, $(a^i = \delta^i{}_1)$ [$E^{i\sigma\tau}(0, 0, 0, -1) = -E^{i\sigma\tau}(0, 0, 0, 1)$, $(a^i = \delta^i{}_4)$, resp.] by a $c_{\sigma\mu}$ transformation (7) [1]). In this paper, we shall omit the singular E.

Thus, one must distinguish the main cases:

I. E and E^* are space-like in (2), § 1.
II. " " time-like " ".
III. E is space-like, E^* is time-like.

In this paper, we shall treat the case (I) exclusively, but at various points we shall discuss the analogous examinations of cases II and III, and why their physical meaning does not come into question for us.

§ 3. The main case: *E* and *E** are space-like. First reduction.

The system (1), § 2, now reads:

$$\left.\begin{aligned} E^{r\sigma\tau}(\psi_{\sigma,r} - i\varepsilon\psi_\sigma\varphi_r) &= +\bar{C}^{\tau\rho}\chi_\rho, \\ E^{r\sigma\tau *}(\chi_{\tau,r} - i\varepsilon\chi_\tau\varphi_r) &= -C^{\rho\sigma}\psi_\rho, \end{aligned}\right\} \tag{1}$$

where the two E are equal, and in fact, equal to $E(1, 0, 0, 0)$, so the system includes, along with the constant ε, only the four mutually independent complex constants (C_{11}, C_{22}, C_{33}, C_{44}).

A transformation of the ψ_σ into $\underline{\psi}_\sigma$:

$$\underline{\psi}_\sigma = c_\sigma{}^\mu\,\psi_\mu \tag{2}$$

gives, as one sees immediately from (1), or from the Hamilton function (2), § 1, new values for the first $E^{r\sigma\tau}$ in (1) (for $C^{\rho\sigma}$, resp.):

$$\underset{\sim}{E}{}^r_{\mu\nu} = E^r_{\sigma\tau}\,c^\sigma_\mu\,\bar{c}^\tau_\nu\,, \tag{3}$$

$$\underset{\sim}{C}_{\mu\tau} = C_{\sigma\tau}\,c^\sigma{}_\mu\,, \tag{3$'$}$$

while the $E^{r\sigma\tau}$ in the second system in (1) remains unchanged.

The new $\underset{\sim}{a}$ in $\underset{\sim}{E}$ are calculated from (3) as in § 2, (7), (7′):

$$\underline{a}^r = c_j{}^k\,\bar{c}_k{}^r\,a^j\,. \tag{4}$$

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[1]) $E^{i\sigma\tau}(a^1, a^2, a^3, a^4)$ means: The values in brackets give the values of the "contravector" a^i.

Without giving the general form (1) for $E(1, 0, 0, 0)$, we can now subject any generalized LORENTZ transformation c_j^k that transforms the $a^j(1, 0, 0, 0)$ into $\underline{a}^j(\pm 1, 0, 0, 0)$ to further constant reductions.

Our demand on the c_j^k leads to the equation $\underset{\sim}{a}^r = c_{jk}\bar{c}^{kr}a^j$:

$$\left.\begin{aligned} \pm 1 &= c_{11}\bar{c}^{11} + c_{12}\bar{c}^{21} + c_{13}\bar{c}^{31} + c_{14}\bar{c}^{41}, \\ 0 &= c_{11}\bar{c}^{12} + c_{12}\bar{c}^{22} + c_{13}\bar{c}^{32} + c_{14}\bar{c}^{42}, \\ 0 &= c_{11}\bar{c}^{13} + c_{12}\bar{c}^{23} + c_{13}\bar{c}^{33} + c_{14}\bar{c}^{43}, \\ 0 &= c_{11}\bar{c}^{14} + c_{12}\bar{c}^{24} + c_{13}\bar{c}^{34} + c_{14}\bar{c}^{44}. \end{aligned}\right\} \tag{4′}$$

We would like to express the c_{ik} in this system by the following four constants:

$$c_{11} = a, \qquad c_{23} = b, \qquad c_{34} = c, \qquad c_{42} = d, \tag{5}$$

so, from the symmetry properties of the c^{ik}, one has:

$$c_{12} = i\,c, \qquad c_{13} = i\,d, \qquad c_{14} = -\,i\,b, \qquad c_{22} = c_{33} = -\,c_{44} = a. \tag{5′}$$

If we introduce (5) and (5′) into (4′) then we obtain:

$$\pm 1 = a\bar{a} + b\bar{b} - c\bar{c} - d\bar{d}\,, \tag{6}$$

$$\left.\begin{aligned} 0 &= a\bar{c} - c\bar{a} - b\bar{d} + d\bar{b}, \\ 0 &= a\bar{d} - d\bar{a} - c\bar{b} + b\bar{c}, \\ 0 &= a\bar{b} - b\bar{a} + d\bar{c} - c\bar{d}. \end{aligned}\right\} \tag{6′}$$

We next discuss the system (6′). If:

$$a : b \neq \bar{a} : \bar{b} \tag{7}$$

then the complex numbers c and d can be represented in terms of a and b:

$$c = pa + qb, \quad d = ra + sb; \quad p, q, r, s \text{ real.} \tag{8}$$

The system (6′) then gives:

$$q + r = 0, \qquad s - p = 0, \qquad 1 + rq - sp = 0, \tag{9}$$

so:

$$1 - r^2 - s^2 = 0. \tag{9′}$$

One can then set: $r = \cos\alpha$, $s = \sin\alpha$, $p = \sin\alpha$, $q = -\cos\alpha$, which makes:

$$c = a\sin\alpha - b\cos\alpha, \qquad d = a\cos\alpha + b\sin\alpha. \tag{10}$$

It follows from (10) that:

$$a\bar{a} + b\bar{b} - c\bar{c} - d\bar{d} = 0, \tag{11}$$

in contrast to (6). Therefore, it is necessary that:

$$\frac{a}{b} = \frac{\bar{a}}{\bar{b}} = \lambda,$$

where λ is naturally real. It then follows from the last equation of (6′) that:

$$\frac{c}{d} = \frac{\bar{c}}{\bar{d}} = \mu,$$

with μ real. We have then reduced the first two equations (6′) to $(\mu\lambda - 1)(b\bar{d} - \bar{b}d) = 0$ and $(\lambda + \mu)(b\bar{d} - \bar{b}d) = 0$; since $\mu\lambda = 1$ and $\mu = -\lambda$ real is not fulfilled, one must have:

$$\frac{b}{d} = \frac{\bar{b}}{\bar{d}}.$$

As the single solution of (6′), we then have:

$$\frac{a}{\bar{a}} = \frac{b}{\bar{b}} = \frac{c}{\bar{c}} = \frac{d}{\bar{d}}. \tag{12}$$

From (12), the complex numbers a, b, c, and d lie in one direction of the Gaussian number plane. From (6), the direction also remains arbitrary.

In the sequel, we shall fix our attention on the $c_{\tau\sigma}$ for which the a, b, c, d are real:

$$c_{11},\ c_{23},\ c_{34},\ c_{42} \qquad \text{are real.} \tag{13}$$

However, for any $c_{\tau\sigma}$ that satisfies (13), so (6′) is fulfilled, (6) reads:

$$\pm 1 = c_{11}\, c^{11} + c_{21}\, c^{21} + c_{34}\, c^{34} + c_{42}\, c^{42} = \tfrac{1}{4}\, c_{\alpha\beta}\, c^{\alpha\beta}. \tag{13′}$$

Such a $c_{\tau\sigma}$ is therefore either a complex LORENTZ transformation, or i-times it is one.

One now has the following two theorems:

I. If $c_{\tau k}$ and $c'_{\tau k}$ are two c's that satisfy the condition (13) then their composition $c_{\tau k}\, c'^{k}{}_{\sigma}$ is such a c.

II. Any $C_{\tau\sigma}$ has the unique decomposition:

$$C_{\tau\sigma} = \underset{1}{C}_{\tau\sigma} + i\,\underset{2}{C}_{\tau\sigma}, \tag{14}$$

where $\underset{1}{C}_{\tau\sigma}$ and $\underset{2}{C}_{\tau\sigma}$ verify the condition (13).

If we set $c^{\mu}_{\sigma} = \delta^{\mu}_{\sigma} e^{i\alpha}$ in (2) then E remains unchanged in (3) and only $C^{\rho\sigma}$ is multiplied by $e^{i\alpha}$ in (3′). This means that there is the possibility of replacing $C^{\rho\sigma}$ with $C^{\rho\sigma}e^{i\alpha}$ in the system (1). If (14) is the decomposition of the $C_{\tau\rho}$ on the right-hand side of (1) into $\underset{1}{C}$ and $\underset{2}{C}$ with the property (13) then:

$$\underset{\sim}{C}_{\tau\rho} = C_{\tau\rho}\, e^{i\alpha} = (\underset{1}{C}_{\tau\sigma}\cos\alpha - \underset{2}{C}_{\tau\sigma}\sin\alpha) + i(\underset{1}{C}_{\tau\sigma}\sin\alpha + \underset{2}{C}_{\tau\sigma}\cos\alpha) \tag{14′}$$

is the analogous decomposition for $\underset{\sim}{C}_{\tau\rho}$. One then has:

$$\underset{\underset{1}{\sim}}{C}_{\tau\rho} = \underset{\underset{1}{\sim}}{C}_{\tau\sigma}\cos\alpha - \underset{\underset{2}{\sim}}{C}_{\tau\sigma}\sin\alpha\,. \tag{15}$$

Now, if $\underset{\underset{1}{\sim}}{C}$ is singular for any choice of α then one always has:

$$\underset{\underset{1}{\sim}}{C}_{\tau\rho}\underset{\underset{1}{\sim}}{C}^{\tau\rho} = 0, \tag{16}$$

so one must also necessarily have:

$$\underset{1}{C}_{\tau\rho}\underset{1}{C}^{\tau\rho} = \underset{2}{C}_{\tau\rho}\underset{2}{C}^{\tau\rho} = \underset{1}{C}_{\tau\rho}\underset{2}{C}^{\tau\rho} = 0. \tag{17}$$

However, from (14), $C_{\tau\rho}$ is also singular then.

When we thus assume that $C_{\tau\rho}$ is non-singular:

$$C_{\tau\rho}\, C^{\tau\rho} \neq 0, \tag{18}$$

we know that $\underset{1}{C}_{\tau\rho}$ is also non-singular (possibly under an allowed variation). We may then set:

$$c^{\sigma}{}_{\mu} = \rho\, \underset{1}{C}^{\sigma}{}_{\mu} \tag{19}$$

in (2), (3), and (3′), where the real number ρ is determined from (13′) in the form:

$$\pm 4 = \rho^2\, \underset{1}{C}_{\alpha\beta}\underset{1}{C}^{\alpha\beta}. \tag{20}$$

From (3′) and (14), one has:

$$\underset{\sim}{C}_{\mu\tau} = \rho(\underset{1}{C}_{\sigma\tau} + i\,\underset{2}{C}_{\sigma\tau})\underset{1}{C}^{\sigma}{}_{\mu} = \rho\left[\pm\frac{1}{\rho^2} g_{\tau\mu} + i\,\underset{2}{C}_{\sigma\tau}\underset{1}{C}^{\sigma}{}_{\mu}\right] = \pm\frac{1}{\rho} g_{\tau\mu} + i\,\underset{3}{C}_{\tau\mu}\,. \tag{21}$$

According to the ± sign in (20) (thus, also in (13′)), the E is reproduced in (3) ($\underset{\sim}{E} = -E$, resp.).

With (21), *we have thus arrived at the fact that the* C_{23}, C_{34}, C_{42} *in the right-hand side of the system* (1) *are pure imaginary*. Therefore, the system (1) either preserves its form, or what appears in its place is:

$$\left.\begin{aligned} E^{r\sigma\tau}(\psi_{\sigma,r} - i\varepsilon\psi_\sigma\varphi_r) &= \bar{C}^{\tau\rho}\chi_\rho, \\ E^{r\sigma\tau}(\chi_{\tau,r} - i\varepsilon\chi_\tau\varphi_r) &= C^{\rho\sigma}\psi_\rho, \end{aligned}\right\} \tag{1′}$$

with the current vector:

$$I^k = E^{k\sigma\tau}\bar{\chi}_\sigma\chi_\tau - E^{k\sigma\tau}\bar{\psi}_\sigma\psi_\tau . \tag{1″}$$

With this, the main case splits into the sub-cases (1) and (1′), so along with ε, there are now five remaining constants in the system.

In order to clarify the difference between the systems (1) and (1′), in the next paragraph we would like to consider those solutions that correspond to the DE BROGLIE waves of a particle at rest.

§ 4. The DE BROGLIE waves for the main case of § 3.

We consider the system:

$$\left.\begin{aligned} E^{r\sigma\tau}(\chi_{\tau,r} - i\varepsilon\chi_\tau\varphi_r) &= +C^{\rho\sigma}\psi_\rho, \\ E^{r\sigma\tau}(\psi_{\sigma,r} - i\varepsilon\psi_\sigma\varphi_r) &= -\bar{C}^{\tau\rho}\chi_\rho, \end{aligned}\right\} \tag{1}$$

in which

$$E = E(1, 0, 0, 0),$$

and therefore the:

$$b = C^{23}, \qquad c = C^{34}, \qquad \text{and } d = C^{42}$$

are real.

This case then corresponds completely to the one in which the C^{23}, C^{34}, C^{42} are pure imaginary, since the C in (1) indeed admit the factor $e^{i\alpha}$. We choose the simpler calculation, due to the aforementioned reality assumption.

We then set the DE BROGLIE wave in (1), for a vanishing electromagnetic potential, equal to:

$$\chi_\tau = \alpha_\tau e^{i\nu x_4}, \qquad \psi_\sigma = \beta_\sigma e^{i\nu x_4}, \tag{2}$$

and obtain

$$\left.\begin{aligned} i\nu\alpha_\tau E^{4\sigma\tau} &= C^{\rho\sigma}\beta_\rho, \\ -i\nu\beta_\sigma E^{4\sigma\tau} &= \bar{C}^{\rho\sigma}\alpha_\rho, \end{aligned}\right\} \tag{3}$$

and, more thoroughly ($C^{12} = -ic$, $C^{13} = -i\alpha$, $C^{14} = -ib$, $C^{11} = C^{22} = C^{33} = -C^{44} = a$):

$$\left.\begin{aligned} -i\nu\alpha_4 &= a\beta_1 + ic\beta_2 + id\beta_3 - ib\beta_4, \\ \nu\alpha_3 &= -ic\beta_1 + a\beta_2 - b\beta_3 + d\beta_4, \\ -\nu\alpha_2 &= id\beta_1 + b\beta_2 + a\beta_3 - c\beta_4, \\ -i\nu\alpha_1 &= ib\beta_1 - d\beta_2 + c\beta_3 - a\beta_4, \end{aligned}\right\} \quad (4), \qquad \left.\begin{aligned} i\nu\beta_4 &= \bar{a}\alpha_1 + ic\alpha_2 + id\alpha_3 - ib\alpha_4, \\ \nu\beta_3 &= -ic\alpha_1 + \bar{a}\alpha_2 + b\alpha_3 - d\alpha_4, \\ -\nu\beta_2 &= -id\alpha_1 - b\alpha_2 + \bar{a}\alpha_3 + c\alpha_4, \\ i\nu\beta_1 &= ib\alpha_1 + d\alpha_2 - c\alpha_3 - \bar{a}\alpha_4, \end{aligned}\right\} \quad (4')$$

and by eliminating the α_i, one finally obtains:

$$\left.\begin{aligned} (-\nu^2 - a\bar{a} - b^2 + c^2 + d^2)\beta^1 - i(\bar{a}-a)c\beta^2 - i(\bar{a}-a)d\beta^3 - i(\bar{a}-a)b\beta^4 &= 0, \\ i(\bar{a}-a)c\beta^1 + (-\nu^2 - a\bar{a} - b^2 + c^2 + d^2)\beta^2 + (\bar{a}-a)b\beta^3 + (\bar{a}-a)d\beta^4 &= 0, \\ i(\bar{a}-a)d\beta^1 - (\bar{a}-a)d\beta^2 + (-\nu^2 - a\bar{a} - b^2 + c^2 + d^2)\beta^3 - (\bar{a}-a)c\beta^4 &= 0, \\ i(\bar{a}-a)b\beta^1 - (\bar{a}-a)d\beta^2 + (\bar{a}-a)c\beta^3 - (-\nu^2 - a\bar{a} - b^2 + c^2 + d^2)\beta^4 &= 0. \end{aligned}\right\} \quad (5)$$

The matrix of the system (5) is obviously a $c_{\sigma\tau}$ matrix; its determinant is therefore equal to the square of $\frac{1}{4}\ c_{\sigma\tau}c^{\sigma\tau}$, up to sign, or to:

$$(\nu^2 + a\bar{a} + B^2)^2 + (\bar{a}-a)^2 B^2, \qquad B^2 = b^2 - c^2 - d^2. \tag{6}$$

Therefore, (5) has a system of solutions $\beta^\sigma \neq 0$ that is a singular semi-vector [1]) when and only when:

$$0 = (\nu^2 + a\bar{a} + B^2)^2 + (\bar{a}-a)^2 B^2 \tag{7}$$

is fulfilled. Since $(\bar{a}-a) < 0$, one must have $B^2 > 0$. We thus obtain the condition:

$$b^2 > c^2 + d^2. \tag{8}$$

If we further set:

$$a = \alpha + i\beta, \qquad \text{then (7) becomes} \qquad \nu^2 + \alpha^2 + \beta^2 + B^2 = \pm\, 2B\beta, \tag{9}$$

and therefore:

$$\nu^2 + \alpha^2 + (\beta \pm B)^2 = 0. \tag{10}$$

Therefore, there are no DE BROGLIE waves in this case.

In the case (1′), § 3, instead of (1), the system emerges in which the second equation of (1) has the sign changed on its right-hand side. Likewise, the current vector has the form § 3, (1″).

In the second system (3) and in (4′), calculation yields a sign change on the left-hand side, which then again leads to the system (5), but with $-\nu^2$, instead of $+\nu^2$.

The condition (8) still remains valid, while instead of (10), one now has:

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[1]) And indeed the most general spin vector of the type that is given by (5) (for a given ν). A spin vector is then defined to be a semi-vector $\beta^{\bar{\sigma}}$ for which a relation $c_{\tau\sigma}\beta^\sigma = 0$ is true (naturally, with a singular $c_{\tau\sigma}$).

$$v^2 = \alpha^2 + (\beta \pm B)^2. \tag{11}$$

Therefore, DE BROGLIE waves appear with two numerically vanishing v, as long as only β and B are non-vanishing.

One surmises from these results that a further reduction of the system (1′) is still possible, since apparently only two of the five remaining constants seem to have a physical significance. We must then see the validity of this conjecture.

We will encounter a "canonical representation" of the system (1′), for which:

$$C_{34} = C_{42} = 0,$$

and C_{11} is pure imaginary and C_{23} is real, moreover.

If we set (in anticipation) c and $d = 0$ in (4) then (4) splits into:

$$\left.\begin{aligned} -iv\alpha_4 &= a\beta_1 - ib\beta_4, \\ -iv\alpha_1 &= ib\beta_1 - a\beta_4, \\ v\alpha_3 &= a\beta_2 - b\beta_3, \\ -v\alpha_2 &= b\beta_2 + a\beta_3, \end{aligned}\right\} \quad (12) \qquad \left.\begin{aligned} -iv\beta_4 &= \bar{a}\alpha_1 - ib\alpha_4, \\ -iv\beta_1 &= ib\alpha_1 - \bar{a}\alpha_4, \\ -v\beta_3 &= \bar{a}\alpha_2 + b\alpha_3, \\ v\beta_2 &= -b\alpha_2 + \bar{a}\alpha_3, \end{aligned}\right\} \quad (12')$$

where the system that corresponds to (1′), § 3 has the sign changed from the corresponding system (12′). By eliminating α, we then obtain:

$$\left.\begin{aligned} (v^2 - a\bar{a} - b^2)\beta_1 + i(\bar{a} - a)b\beta_4 &= 0, \\ i(\bar{a} - a)b\beta_1 + (v^2 - a\bar{a} - b^2)\beta_4 &= 0, \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} (v^2 - a\bar{a} - b^2)\beta_2 + i(\bar{a} - a)b\beta_3 &= 0, \\ -(\bar{a} - a)b\beta_2 + (v^2 - a\bar{a} - b^2)\beta_3 &= 0. \end{aligned}\right\} \tag{13'}$$

There are two roots v^2 for which one has [1]):

$$v^2 - a\bar{a} - b^2 = \pm i(\bar{a} - a)b\,. \tag{14}$$

This gives, together with (13), $\beta_1 \pm \beta_4 = 0$, $i\beta_2 \pm \beta_3 = 0$, so:

$$\beta_4 = \mp\beta_1, \qquad \beta_3 = \mp\, i\beta_2\,. \tag{15}$$

From (12), we then obtain:

[1]) In the sequel, the two cases will be treated together in which two signs enter in one above the other, such that the upper one relates to the first of the roots and the lower one refers to the second one (v_1 or v_2, resp.).

$$\left.\begin{aligned} -iv\alpha_1 &= (ib \pm a)\beta_1, \\ -iv\alpha_4 &= (a \pm ib)\beta_1, \\ v\alpha_3 &= (a \pm ib)\beta_2, \\ -v\alpha_2 &= (b \mp ia)\beta_2, \end{aligned}\right\} \tag{16}$$

so

$$\alpha_4 = \pm\,\alpha_1, \qquad \alpha_3 = \mp\, i\alpha_2\,. \tag{16′}$$

For the current density:

$$I^4 = E^{4\sigma\tau}\,\alpha_\sigma\,\alpha_\tau - E^{4\sigma\tau}\,\beta_\sigma\bar{\beta}_\tau\,,$$

we obtain, after some calculation:

$$I^4 = \mp 4(\beta_2\bar{\beta}_2 + \beta_1\bar{\beta}_1)\,. \tag{17}$$

This proves to be an important result, namely, that the electricity densities that belong to the two v possess opposite signs. The two ponderable masses then seem (by the usual interpretation) to be assigned electric charges of opposite signs. This is then the place to discuss the other two main cases II and III of § 2. The main case (II), in which two DE BROGLIE wave do indeed appear, is invalid from the physical standpoint because the current densities always have a single sign then: That leads to a theory of electromagnetism in which only electrical measure densities that have a definite sign appear.

In the main case III, only *one* DE BROGLIE wave exists.

§ 5. The further reduction of the Dirac equations.

Up to now, we could arrange that the $C_{\sigma\tau}$ in the right-hand side of the Dirac equations already had the form:

C_{23}, C_{34}, C_{42} are pure imaginary.

The E on the left-hand side would then have the values $a^i = \delta^i{}_1$ as constants.

In this paragraph, it shall be shown that one should have, additionally:

C_{11} real.

Following the reasoning of § 3, we thus consider *one* $\underline{c}^{\sigma\tau}$, whose (1, 1), (2, 3), (3, 4), and (4, 2) components are real:

$$\left.\begin{aligned} &\underline{c}^{\sigma\tau} = c^{\sigma\tau} e^{i\alpha}, \\ &c_{11}, c_{23}, c_{34}, c_{42} \text{ are real.} \end{aligned}\right\} \tag{1}$$

From (3′), § 3, one has:

$$\underset{\sim}{C}_{\mu\tau} = C_{\sigma\tau}\underline{c}^{\sigma}{}_{\mu} = C_{\sigma\tau}c^{\sigma}{}_{\mu}e^{i\alpha}\,. \tag{2}$$

For $C_{\sigma\tau}$, we would like to assume that it has the form that was arrived at up to now:

$$C_{\sigma\tau} = A\, g_{\sigma\tau} + i \underset{1}{C}_{\sigma\tau}, \qquad A \text{ real}, \tag{3}$$

where the $\underset{1}{C}_{\sigma\tau}$ already fulfill the reality conditions:

$$\underset{1}{C}_{11},\ \underset{1}{C}_{23},\ \underset{1}{C}_{34},\ \underset{1}{C}_{42} \text{ real.} \tag{4}$$

For $\underset{\sim}{C}_{\sigma\mu}$, we make a similar assumption:

$$\underset{\sim}{C}_{\sigma\mu} = \underset{\sim}{A} g_{\sigma\mu} + i \underset{2}{C}_{\sigma\mu}. \tag{5}$$

where now, in addition to $\underset{2}{C}_{11},\ \underset{2}{C}_{23},\ \underset{2}{C}_{34},\ \underset{2}{C}_{42}$ being real, one also has:

$$\underset{2}{C}_{11} = 0. \tag{6}$$

This shows that under the assumption:

$$\underset{1}{C}_{11} \neq 0, \qquad A \neq 0, \quad \underset{1}{C}{}^2_{23} - \underset{1}{C}{}^2_{34} - \underset{1}{C}{}^2_{42} > 0$$

one can find the $c^{\sigma}{}_{\mu}\, e^{i\alpha}$ that correspond to (2) and (1).

If we introduce the quantities (3) and (5) into (2) then we obtain:

$$\left.\begin{aligned} \underset{\sim}{A} g_{\tau\mu} + i \underset{2}{C}_{\mu\tau} = (A g_{\sigma\tau} + i \underset{1}{C}_{\sigma\tau})\, c^{\sigma}{}_{\mu} (\cos\alpha + i \sin\alpha) = \\ (A g_{\sigma\tau}\cos\alpha - \underset{1}{C}_{\sigma\tau}\sin\alpha)\, c^{\sigma}{}_{\mu} + i (A g_{\sigma\tau}\sin\alpha + \underset{1}{C}_{\sigma\tau}\cos\alpha)\, c^{\sigma}{}_{\mu}. \end{aligned}\right\} \tag{7}$$

Since the decomposition into c quantities with the aforementioned reality behavior is unique, one has:

$$\underset{\sim}{A} g_{\tau\mu} = (A g_{\sigma\tau}\cos\alpha - \underset{1}{C}_{\sigma\tau}\sin\alpha)\, c^{\sigma}{}_{\mu}, \tag{8}$$

$$\underset{2}{C}_{\mu\tau} = (A g_{\sigma\tau}\sin\alpha + \underset{1}{C}_{\sigma\tau}\cos\alpha)\, c^{\sigma}{}_{\mu}. \tag{8′}$$

We set:

$$c^{\sigma}{}_{\mu}\, c_{\rho}{}^{\mu} = \Phi\ \delta_{\rho}^{\sigma}. \tag{9}$$

Since we first look for c, $\Phi \neq 0$ is an assumption that will be justified later on. If we multiply (8) by $c_{\rho}{}^{\mu}$ then we obtain:

$$\frac{1}{\Phi} \underset{\sim}{A} c_{\rho\tau} = A g_{\rho\tau}\cos\alpha - \underset{1}{C}_{\rho\tau}\sin\alpha. \tag{10}$$

For (8′), that gives:

$$\left.\begin{aligned}\frac{1}{\Phi}\underset{\sim}{A}\underset{2}{C}_{\rho\tau} &= (Ag_{\sigma\tau}\sin\alpha + \underset{1}{C}_{\sigma\tau}\cos\alpha)(Ag_{\mu}^{\ \sigma}\cos\alpha - \underset{1}{C}^{\sigma}{}_{\mu}\sin\alpha)\\ &= A^2 g_{\tau\mu}\sin\alpha\cos\alpha - \underset{1}{C}_{\sigma\tau}\underset{1}{C}^{\sigma}{}_{\mu}\sin\alpha\cos\alpha + A\underset{1}{C}_{\mu\tau}\cos^2\alpha - A\underset{1}{C}_{\tau\mu}\sin^2\alpha\end{aligned}\right\}\tag{11}$$

Due to (6), one has ($\underset{2}{C}_{11} = 0$ and $\underset{1}{C}_{\sigma\tau}\underset{1}{C}^{\sigma}{}_{\mu} = \frac{1}{4}\underset{1}{C}_{\alpha\beta}\underset{1}{C}^{\alpha\beta}g_{\tau\mu}$):

$$0 = (A^2 - \tfrac{1}{4}\underset{1}{C}_{\alpha\beta}\underset{1}{C}^{\alpha\beta})\sin\alpha\cos\alpha + A\,\underset{1}{C}_{11}(\cos^2\alpha - \sin^2\alpha),\tag{12}$$

or

$$0 = (A^2 - \tfrac{1}{4}\underset{1}{C}_{\alpha\beta}\underset{1}{C}^{\alpha\beta})\sin 2\alpha + 2A\,\underset{1}{C}_{11}\cos 2\alpha.\tag{12′}$$

This yields the desired angle α, with which the right-hand side of (10) – and thus $c_{\rho\tau}$ – is given, up to a factor.

Under the assumptions that were made for A and $\underset{1}{C}_{11}$, this shows that $c_{\rho\tau}$ is non-singular.

The quadratic condition § 3, (13′) then gives:

$$\tfrac{1}{4}c_{\alpha\beta}\,c^{\alpha\beta} = +\,1, \qquad \text{so} \qquad \Phi = I \text{ in (9)},\tag{13}$$

since indeed the two sub-cases of our main case I are not transformable into each other due to the vanishing behavior relative to the DE BROGLIE wave. (A DE BROGLIE wave naturally still exists under our c transformations of the ψ.)

Equation (13) ultimately serves to determine the factor $\underset{\sim}{A}$ in (10).

In the next paragraph, we will arrive at the *reduction to normal form:* All that will remain in the Dirac equation are two constants: *the real* C_{11} *and the pure imaginary* C_{23} .

§ 6. The normal form for the Dirac equations.

The last form that we arrived at for the $C^{\sigma\tau}$ on the right-hand side of the Dirac equations was:

$$C^{11} \text{ pure imaginary, } C^{23}, C^{34}, C^{42} \text{ real.}\tag{1}$$

We now seek to determine two c – thus c and $\bar{c}$ – that correspond to the conditions (13), (13′) of § 3 and satisfy the relations:

$$c_{\alpha\beta}\,C^{\alpha\beta} = \tilde{C}_{\alpha\beta}\tilde{c}^{\beta\tau},\tag{2}$$

where the reality behavior (1) is true for C and $\tilde{C}$, and one has $\tilde{C}_{34} = \tilde{C}_{42} = 0$, moreover.

Since the quadratic relations (13′) of § 3 are true for c and $\tilde{c}$:

$$\tfrac{1}{4}c_{\alpha\beta}\bar{c}^{\alpha\beta} = \pm\tfrac{1}{4}\tilde{c}_{\alpha\beta}\tilde{c}^{\alpha\beta} = \pm 1, \tag{3}$$

so it follows from (2), after multiplying by $\tilde{c}_{\gamma\tau}$, that:

$$\tilde{C}_{\alpha\gamma} = \pm c_{\alpha\beta}\tilde{c}_{\gamma\tau}C^{\beta\tau}. \tag{4}$$

As long as (2) is fulfilled, we can then go from the Dirac equations with the $C_{\rho\sigma}$ to the ones with the $\tilde{C}_{\rho\tau}$ by simultaneously transforming the two semi-quantities.

The system (2) is again true for all combinations of the values (τ, α) when it exists for (1, 1), (1, 2), (1, 3), (1, 4). This gives four equations:

$$\left.\begin{aligned}
c_{11}C^{11}+c_{12}C^{21}+c_{13}C^{31}+c_{14}C^{41} &= \tilde{C}_{11}\tilde{c}^{11}+\tilde{C}_{12}\tilde{c}^{21}+\tilde{C}_{13}\tilde{c}^{31}+\tilde{C}_{14}\tilde{c}^{41},\\
c_{21}C^{11}+c_{22}C^{21}+c_{23}C^{31}+c_{24}C^{41} &= \tilde{C}_{21}\tilde{c}^{11}+\tilde{C}_{22}\tilde{c}^{21}+\tilde{C}_{23}\tilde{c}^{31}+\tilde{C}_{24}\tilde{c}^{41},\\
c_{31}C^{11}+c_{32}C^{21}+c_{33}C^{31}+c_{34}C^{41} &= \tilde{C}_{31}\tilde{c}^{11}+\tilde{C}_{32}\tilde{c}^{21}+\tilde{C}_{33}\tilde{c}^{31}+\tilde{C}_{34}\tilde{c}^{41},\\
c_{41}C^{11}+c_{42}C^{21}+c_{43}C^{31}+c_{44}C^{41} &= \tilde{C}_{41}\tilde{c}^{11}+\tilde{C}_{42}\tilde{c}^{21}+\tilde{C}_{43}\tilde{c}^{31}+\tilde{C}_{44}\tilde{c}^{41}.
\end{aligned}\right\} \tag{5}$$

We convert (5), using the c-relations:

$$\left.\begin{aligned}
c^{12}=-ic^{34},\ c^{13}=-ic^{42},\ c^{14}=-ic^{23},\\
c^{11}=c^{22}=c^{33}=-c^{44},
\end{aligned}\right\} \tag{6}$$

and get

$$\left.\begin{aligned}
c_{11}C^{11}-c_{34}C^{34}-c_{42}C^{42}-c_{23}C^{23} &= \tilde{C}_{11}\tilde{c}^{11}-\tilde{C}_{34}\tilde{c}^{34}-\tilde{C}_{42}\tilde{c}^{42}-\tilde{C}_{23}\tilde{c}^{23},\\
ic_{34}C^{11}+ic_{11}C^{34}+ic_{23}C^{42}+ic_{24}C^{23} &= -i\tilde{C}_{34}\tilde{c}^{11}+i\tilde{C}_{11}\tilde{c}^{34}+i\tilde{C}_{23}\tilde{c}^{42}+i\tilde{C}_{42}\tilde{c}^{23},\\
-ic_{42}C^{11}-ic_{23}C^{34}+ic_{11}C^{42}-ic_{34}C^{23} &= -i\tilde{C}_{42}\tilde{c}^{11}-i\tilde{C}_{23}\tilde{c}^{34}+i\tilde{C}_{11}\tilde{c}^{42}-i\tilde{C}_{34}\tilde{c}^{23},\\
ic_{23}C^{11}+ic_{42}C^{34}-ic_{34}C^{42}+ic_{11}C^{23} &= i\tilde{C}_{23}\tilde{c}^{11}+i\tilde{C}_{42}\tilde{c}^{34}-i\tilde{C}_{34}\tilde{c}^{42}+i\tilde{C}_{11}\tilde{c}^{23}.
\end{aligned}\right\} \tag{7}$$

Due to the reality behavior, the system splits into:

$$\left.\begin{aligned}
c_{11}C_{11} &= \tilde{C}_{11}\tilde{c}_{11},\\
c_{34}C_{11} &= \tilde{C}_{11}\tilde{c}_{34},\\
c_{42}C_{11} &= \tilde{C}_{11}\tilde{c}_{42},\\
c_{23}C_{11} &= \tilde{C}_{11}\tilde{c}_{23},
\end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned}
c_{34}C_{34}+c_{42}C_{42}-c_{23}C_{23} &= \tilde{C}_{34}\tilde{c}_{34}+\tilde{C}_{42}\tilde{c}_{42}-\tilde{C}_{23}\tilde{c}_{23},\\
c_{11}C_{34}+c_{23}C_{42}-c_{42}C_{23} &= \tilde{C}_{34}\tilde{c}_{11}+\tilde{C}_{23}\tilde{c}_{42}-\tilde{C}_{42}\tilde{c}_{23},\\
c_{23}C_{34}-c_{11}C_{42}-c_{34}C_{23} &= -\tilde{C}_{42}\tilde{c}_{11}+\tilde{C}_{23}\tilde{c}_{34}-\tilde{C}_{34}\tilde{c}_{23},\\
c_{42}C_{34}-c_{34}C_{42}-c_{11}C_{23} &= -\tilde{C}_{23}\tilde{c}_{11}+\tilde{C}_{42}\tilde{c}_{34}-\tilde{C}_{34}\tilde{c}_{42}.
\end{aligned}\right\} \tag{8′}$$

When one arrives at:

$$C_{11} = \bar{C}_{11}, \tag{9}$$

it then follows from (8) that:

$$c_{\alpha\beta} = \tilde{c}_{\alpha\beta}. \tag{10}$$

(Due to (3), this can read $c_{\alpha\beta} = \pm\ \tilde{c}_{\alpha\beta}$, but if one fixes (9) then (10) is true.)

Since one has $c_{\alpha\beta}\, c^{\alpha\beta} = \tilde{c}_{\alpha\beta}\tilde{c}^{\alpha\beta}$, the transformation of the equations in the C into the ones in the $\tilde{C}$ is linked with either no change of sign in the E (i.e., in the equations) or a simultaneous change of sign; naturally, this changes nothing at all regarding the type of the Dirac system. With consideration to (10), (8′) becomes:

$$\left.\begin{aligned} -c_{23}(C_{23}-\tilde{C}_{23})+c_{34}(C_{34}-\tilde{C}_{34})+c_{42}(C_{42}-\tilde{C}_{42}) &= 0,\\ c_{11}(C_{34}-\tilde{C}_{34})+c_{23}(C_{42}+\tilde{C}_{42}) \qquad -c_{42}(C_{23}-\tilde{C}_{23}) &= 0,\\ c_{11}(C_{42}-\tilde{C}_{42})-c_{23}(C_{34}+\tilde{C}_{34})+c_{34}(C_{23}+\tilde{C}_{23}) &= 0,\\ c_{11}(C_{23}-\tilde{C}_{23}) \qquad +c_{34}(C_{42}+\tilde{C}_{42})-c_{42}(C_{34}-\tilde{C}_{34}) &= 0. \end{aligned}\right\} \tag{11}$$

Should this system be fulfilled with $c_{\sigma\tau} \neq 0$, the determinant of (11) must then vanish. This (Δ) is:

$$\Delta = -\ [(C_{23}^2-\tilde{C}_{23}^2)-(C_{34}^2-\tilde{C}_{34}^2)-(C_{42}^2-\tilde{C}_{42}^2)]^2 . \tag{12}$$

We fulfill $\Delta = 0$ by way of:

$$\tilde{C}_{23}^2 = C_{23}^2 - C_{34}^2 - C_{42}^2, \qquad \tilde{C}_{34} = \tilde{C}_{42} = 0. \tag{13}$$

For the $c_{\sigma\tau}$ in (11), one then easily calculates that $c_{\sigma\tau}\, c^{\sigma\tau} \neq 0$. We have thus fulfilled the system (4) with the calculated $c_{\sigma\tau}$, $\tilde{c}_{\sigma\tau}$, and $\tilde{C}_{\sigma\tau}$, and exhibited the normal form [1]).

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[1]) We have also arrived at the reduction of the Dirac equations for semi-vectors in a second way when we converted the Hamilton function H_3 by a c-transformation of the $C_{\sigma\tau}$ into $g_{\sigma\tau}$ at the first step. We then had only the eight real constants a_i (a_i^*, resp.) (i = 1, …, 4) left in the E (E^*, resp.).

Any further transformation $\psi_\sigma = c_\sigma^\alpha \underline{\psi}_\alpha$, $\bar{\chi}^\sigma = c'^{\sigma}_{\beta} \underline{\chi}^\beta$ of the ψ_σ, $\bar{\chi}^\sigma$ into $\underline{\psi}_\sigma$, $\underline{\chi}_\sigma$ must then leave $\psi_\sigma \bar{\chi}^\sigma$ invariant. This gives the condition $c'^{\sigma}_{\beta} c^{\alpha}_{\sigma} = \delta^\alpha_\beta$; i.e., the transformations of the ψ and χ must involve mutually inverse matrices.

The second step then brings the one E in H_3 into normal form; e.g., E(1, 0, 0, 0), when it is "space-like."

However, that still allows all transformations of this E into itself – i.e., all c-transformations for which $\chi'_i = c_i^{\ j} \bar{c}_j^{\ k} \chi_k$ is a rotation around the χ_1-axis.

One can then arrive at the final form in which – e.g. – only the first and fourth "components" a^{*k} are non-zero.

Thus, according to whether E^* is space-like or time-like, one will have:

$$(a_1^*)^2 - (a_4^*)^2 \gtrless 0 .$$

The normal form that one arrives at in this way is completely equivalent to ours, on the basis of symmetry, but in this paper we preferred the final form that we developed.

§ 7. Summary and physical interpretation.

In the foregoing, we have presented a field theory in which, along with the metric and electromagnetic field quantities, two semi-vectors ψ and χ appeared as new field quantities. In the Hamilton function, in addition to the curvature scalar and the electromagnetic field scalar, a scalar entered in that is quadratic in the ψ and χ, as well as their first derivatives. Along with the gravitational field equations and Maxwell equations, completed with the electrical current density, a system of generalized Dirac equations for the semi-vectors appeared.

It was shown that the original 17 arbitrary constants that entered into this system of equations could be reduced to three real ones, namely, ε, a, and b. One thus obtains the normal equations of § 3, (I′), (I″). In these normal equations, the four E-constants a_1 to a_4 have the value (1, 0, 0, 0), and the (covariant) matrix of the $C_{\sigma\tau}$ in the right-hand side has the form:

$$\begin{Vmatrix} ia & 0 & 0 & -ib \\ 0 & ia & b & 0 \\ 0 & -b & ia & 0 \\ ib & 0 & 0 & -ia \end{Vmatrix}.$$

By neglecting the electromagnetic field (i.e., the limiting case), the equations admit two solutions that correspond to a special species of elementary particles at rest whose DE BROGLIE frequencies (particle masses, resp.) are determined by the equations:

$$\nu_1^2 = (a + b)^2, \qquad \nu_2^2 = (a - b)^2.$$

The electrical charge densities that correspond to these two DE BROGLIE waves are of opposite sign. This therefore seems to be the first time that an explanation was given for the fact that there are two electric elementary particles of different masses whose electrical charges possess opposite signs. It is further essential that only one constant with the dimension of an electrical charge enters into the equations, which is given by ε, up to a universal factor. This is obviously connected with the fact that there is only one elementary electrical charge (in absolute value).

The fact that negative, along with positive, values entered into ν as measurement constants is perhaps connected with the apparent appearance of “positive electrons,” which were generally to be regarded as electronegative particles of negative ponderable mass. Correspondingly, one would expect that the theory also applies to protons.

It is clear that such a field theory cannot admit BORN’s probability interpretation of the ψ field, and in the meantime there thus remains an open question of whether such a theory admits an interpretation of the atomistic structure of matter that is free from contradictions to begin with.

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