"Lies Invariantentheorie der Berührungstransformationen und ihre Erweiterung," from the Jahresbericht der deutschen Mathematiker-Vereinigung, v. 5, H. Liebmann and F. Engel, *Die Berührungstransformationen: Geschichte und Invariantentheorie*, B. G. Teubner, Leipzig, 1914.

Lie's invariant theory of contact transformations and its extensions.

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In the year 1872, the Proceedings of the Scientific Society of Christiania received a brief – barely three pages – communication of Sophus Lie: "Zur Invarianten-Theorie der Berührungstranfromationen." This is noteworthy in more than one way. First, due to the particularly important applications that Lie made of his new theory to the integration of partial differential equations of first order. Second, especially for the fact that it treated the invariants of a special infinite group, the group of all contact transformations. Before then, there was, at first, only one example of such an invariant theory that began with Gauss, was further developed by Codazzi, Minardi, and Beltrami, and became the invariant theory of quadratic differential forms in several variables by Riemann, Christoffel, and Lipschitz. Finally, one observes that thirdly, Lie's invariant theory existed in precisely the same time period in which F. Klein developed the general ideas that he laid down in his Erlanger Programm, so Lie had already worked out an important example of what Klein had proposed as a program for the future. Indeed, Lie knew of these ideas, for the most part, from speaking with Klein, and had himself contributed to its development, but the novelty in these ideas was that for him many realms of existing mathematics could be described as invariant theories of groups, while, on the other hand,

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[1]) Talk submitted to the German Society of Mathematicians and excerpts presented at its meeting in Vienna, September 1913.

the question of the invariant properties of the transformed picture under a given group was completely natural to him.

Lie's first presentation of his invariant theory of contact transformations suffered from the fact that he still was in no position to give a simple derivation of the formulas for contact transformations, but based it on the general theory of Pfaffian problems. He did that in 1874 in the great treatise in volume VIII of the Mathematischen Annalen. Soon after, Adolph Mayer gave a relatively simple, direct presentation of the theory of contact transformations (Gött. Nachr. 1874), but Lie could still not make up his mind whether to accept the Mayer approach outright. This is a peculiarity of Lie himself: He made it his ambition to found his new theory only in such a way that he himself had thought of, and he went out of his way to avoid use of any simplifications that originated with anyone else. By the same inducement, one is obviously also led to the remarkable fact that Lie made no mention whatsoever of the bilinear covariants of a Pfaffian expression that Lipschitz [1]) had already given in 1869 and had then been utilized to great effect by Frobenius [2]) in 1877, nor did he use them anywhere. As S. Kantor justifiably suggested in a 1901 paper [3]), it is precisely with the help of these covariants of the theory of Pfaffian problems, and especially the Mayer formulation, as well, by which the theory of contact transformations can be greatly simplified, namely, when one notices in addition that the Poisson bracket symbol can be presented as a form in plane coordinates that is covariant to these bilinear covariants.

Since the presentation of the analytical theory of contact transformations that was given in the second volume of *Transformationsgruppen* was also strongly affected by Lie's aforementioned peculiarity, it does not seem superfluous to me to present the theory of contact transformations and the associated invariant theory as it is now possible to do by using the modern tools. In general, many things that are known will also have to be reiterated. However, it will yield that the generalization of the theory that Lie himself already had in mind to the case where one does not base everything on the contact transformations for a specific normal form for the Pfaffian expression is not in slightest more difficult. I have the aforementioned paper of S. Kantor and another one that was published in the meantime [4]) to thank for many essential inspirations. Furthermore, I cannot, by any means, claim that the insights of S. Kantor were my own. Similarly, I cannot help but stress that some of the very elevated claims that Kantor made seemed completely unjustified to me. Along with the many good ideas that one finds in the papers on how to approach matters, one also finds a large number of flawed or outright false ones, and the lack of organization in the presentation has the effect that overall, it suffers from a lack of clarity that the reader of the paper does not find edifying.

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[1]) In the paper: "Untersuchungen in Betreff der ganzen homogenen Funktionen von *n* Differentialen." Crelles Journal, v. 70, pp. 72, *et seq*.

[2]) "Über das Pfaffsche Problem," Crelles Journal, v. 82, pp. 230-315.

[3]) "Über einen neuen Gesichtspunkt in der Theorie des Pfaffschen Problemes, der Funktionengruppen und der Berührungstransformationen." Wiener Berichte, Math.-naturw. Klasse, Bd. CX, Abt. IIa, December 1901, pp. 1147, *et seq*.

[4]) "Neue Grundlagen für die Theorie und Weiterentwicklungen der Lieschen Funktionengruppen," *ibid*., Bd. CXII, Abt. IIa, July 1903, pp. 755, *et seq*. Obviously, the paper that follows this one immediately on pp. 678-754 ("Über eine neue Klasse gemischter Gruppen") belongs with this one, as well.

§ 1. The bilinear covariant.

First, I must briefly discuss the bilinear covariant of a Pfaffian expression.

Let:

(1) $$D = \sum_i^n \alpha_i(x_1, \cdots, x_n)\, dx_i$$

be an arbitrary Pfaffian expression, and the same expression, when formed in another system of differentials δx_i would be denoted by Δ if:

(2) $$x_i = \varphi_i(y_1, \ldots, y_r) \qquad (i = 1, \ldots, n),$$

where the φ_i are completely arbitrary functions, and thus D is converted onto a new Pfaffian expression in the r variables $y_1, \ldots, y_r$:

(3) $$D = \sum_{i=1}^n \alpha_i(x_1, \cdots, x_n)\, dx_i = \sum_{k=1}^r \beta_i(y_1, \cdots, y_n)\, dy_k$$

and one thus obtains, by means of (2):

(4) $$\begin{cases} d\Delta - \delta D = \sum_{i=1}^n (d\alpha_i \delta x_i - \delta\alpha_i dx_i) + \sum_{i=1}^n \alpha_i (d\delta x_i - \delta dx_i) \\ \qquad = \sum_{k=1}^r (d\beta_k \delta y_k - \delta\beta_k dy_k) + \sum_{k=1}^r \beta_k (d\delta y_k - \delta dy_k). \end{cases}$$

On the other hand, however, from (2), one gets:

$$dx_i = \sum_{k=1}^r \frac{\partial \varphi_i}{\partial y_k} dy_k ,$$

so, as one easily sees:

(5) $$d\,\delta x_i - \delta\, dx_i = \sum_{k=1}^r \frac{\partial \varphi_i}{\partial y_k} (d\delta y_k - \delta dy_k) ;$$

from (3), it then follows that:

$$\sum_{i=1}^n \alpha_i (d\delta x_1 - \delta dx_n) = \sum_{k=1}^r \beta_k (d\delta y_k - \delta dy_k) ,$$

such that one can deduce from equation (4) that:

(6) $$\sum_{i=1}^n (d\alpha_i \delta x_1 - \delta\alpha_i dx_i) = \sum_{k=1}^r (d\beta_k \delta y_k - \delta\beta_k dy_k) .$$

With this, one shows that the expression:

$$\sum_{i=1}^{n}(d\alpha_i \delta x_i - \delta\alpha_i dx_i) = \sum_{i,\nu}^{1\cdots n}\left(\frac{\partial \alpha_i}{\partial x_\nu} - \frac{\partial \alpha_\nu}{\partial x_i}\right) dx_\nu \delta x_i \tag{7}$$

is covariant to the Pfaffian expression, and indeed is not merely an introduction of new variables, since under any substitution (2) the functions φ_i may or may not be independent of each other.

In all of the representations that are known to me, one constructs the bilinear covariant of (1) by regarding the expressions $d\delta x_i$ and δdx_i in the expression $d\Delta - \delta D$ as equal, and one must then verify the covariance property by computation. Here, this property emerges as an immediate consequence of the simple fact that from (5) the expressions $d\delta x_i - \delta dx_i$ are cogredient to the dx_i and δx_i . Also, it is important that the covariance property of (7) is not assured merely by the introduction of new independent variables, but for any arbitrary substitution of the form (2).

It must be further remarked that equation (5), which follows from (2), yields:

$$\sum_{i=1}^{n}(d\delta x_i - \delta dx_i)\,\frac{\partial F}{\partial x_i} = \sum_{k=1}^{r}(d\delta y_k - \delta dy_k)\,\frac{\partial F}{\partial y_k},$$

when F denotes an arbitrary function of $x_1, \ldots, x_n$. If one understands dy_k and δy_k here to mean the increases that the y_k experience under two arbitrary infinitesimal transformations $Y_1 f$ and $Y_2 f$ in $y_1, \ldots, y_k$ then one immediately recognizes that $d\delta y_k - \delta dy_k$ is the increase that y_k experiences under the infinitesimal transformations, which will be represented by the bracket expression:

$$(Y_1 Y_2) f = Y_1 Y_2 f - Y_2 Y_1 f .$$

Equations (5) thus express the known fact that this bracket expression represents an infinitesimal transformation that is covariant to both infinitesimal transformations.

§ 2. The integration problem of a Pfaffian equation and a Pfaffian expression.

Lie's first paper on the theory of partial differential equations (*loc. cit.*) was the one in which he originally posed the question of Pfaff once more, which consisted in the question of whether any manifold in the space of $x_1, \ldots, x_n$ on which the Pfaff equation is fulfilled could be regarded as an integral manifold of a Pfaffian equation:

$$\sum_{i=1}^{n}\alpha_i(x_1, \cdots, x_n)\,dx_i = 0.$$

If one thinks of an m-fold extended manifold as represented by $n - m$ independent equations:

$$\Phi_k(x_1, \dots, x_n) = 0 \qquad (k = 1, \dots, n - m)$$

then it is an integral manifold when and only when any system of values $x_1, \dots, x_n, dx_1, \dots, dx_n$ that satisfies the equations $\Phi_k = 0$, $d\Phi_k = 0$ also satisfies the equation $\sum \alpha_i \, dx_i = 0$. On the other hand, if one thinks of the manifold as represented with the help of m independent variables $u_1, \dots, u_m$ in the form:

$$x_i = \varphi_i(u_1, \dots, u_m) \qquad (i = 1, \dots, n)$$

then it is an integral manifold when and only when the equation $\sum \alpha_i \, dx_i = 0$ becomes an identity under the substitution $x_i = \varphi_i$.

It is convenient to introduce, in a corresponding way, the notion of an "integral manifold of a Pfaffian expression $\sum \alpha_i \, dx_i$," and understand this to mean a manifold on which the expression $\sum \alpha_i \, dx_i$ becomes a complete differential. [1])

If one recalls the known theorem that a Pfaffian expression is a complete differential when and only when its bilinear covariant vanishes identically then one immediately recognizes that a manifold:

$$x_i = \varphi_i(u_1, \dots, u_m) \qquad (i = 1, \dots, n)$$

represents an integral manifold of the Pfaffian expression $\sum \alpha_i \, dx_i$ when and only when its bilinear covariant vanishes identically under the substitution $x_i = \varphi_i$. However, this, in turn, yields that a system of equations:

$$\Phi_k(x_1, \dots, x_n) = 0 \qquad (k = 1, \dots, n - m),$$

represents an integral manifold when and only when this bilinear covariant vanishes for all systems of values x_i, dx_i, δx_i that satisfy the equations $\Phi_k = 0$, $d\Phi_k = 0$, $\delta\Phi_k = 0$.

Had Lie introduced the notion of an integral manifold of a Pfaffian expression then he would have been able to represent a non-trivial part of his investigations much more conveniently.

§ 3. Unions in the space of elements $x_1, \dots, x_n, p_1, \dots, p_n$.

The integration of those partial differential equations in $z, x_1, \dots, x_n$ in which the unknown function z itself does not appear subsumes the problem of finding the n-fold extended integral manifolds of the Pfaffian expression:

[1]) I have already been using this formulation in my lectures for many years. One also finds, moreover, as I have remarked – after the fact – that it was already used by S. Kantor in: "Über eine Klasse gemischter Gruppen," *loc. cit.*, Bd. CXII, Abt. IIa, July 1903, pp. 721 in no. 5.

(8) $$\sum_{i=1}^{n} p_i dx_i$$

in the space $x_1, \ldots, x_n, p_1, \ldots, p_n$ that satisfy one or more given equations between the $x_1, \ldots, x_n, p_1, \ldots, p_n$. We must therefore first say a few things on the integral manifolds of (8). We then briefly call a system of values x_i, p_i an *element.*

It is clear that any family of ∞^1 elements is an integral manifold of the Pfaffian expressions (8). Thus, integral manifolds of (8) always go through any two neighboring elements x_i, p_i and $x_i + dx_i$, $p_i + dp_i$. On the contrary, if we demand that the integral manifold still includes a second infinitely close element $x_i + \delta x_i$, $p_i + \delta p_i$ then, from § 2, the condition:

(9) $$\sum_{i=1}^{n} (dx_i \delta p_i - \delta x_i dp_i) = 0$$

must be fulfilled. When two of the elements x_i, p_i are infinitely close and fulfill (9), we would like to say that they are *united*, and accordingly, we would like to briefly say that the integral manifolds of (8) are *unions.*

A union now includes the element x_i^0, p_i^0 and m infinitely close elements $x_i^0 + dx_i$, $p_i^0 + dp_i$ ($k = 1, \ldots, m$) that belong to no manifold of dimension less than m, for which not all m-rowed determinants of the matrix:

(10) $$\left| d_k x_1 \cdots d_k x_n d_k p_1 \cdots d_k p_n \right| \qquad (k = 1, \ldots, m)$$

necessarily vanish. Therefore, one must first fulfill the equations:

(11) $$\sum_{i=1}^{m} (d_k x_i d_j p_i - d_k p_i d_j x_i) = 0 \qquad (k, j = 1, \ldots, m),$$

and secondly, all of the elements x_i, p_i that are infinitely close elements of the union must satisfy the m mutually independent equations:

(12) $$\sum_{i=1}^{m} (d_k x_i dp_i - d_k p_i dx_i) = 0 \qquad (k = 1, \ldots, m).$$

On the other hand, since (12) possesses the m linearly independent solutions:

$$dx_i = d_k x_i, \qquad dp_i = d_k p_i, \qquad (k = 1, \ldots, m),$$

one then demands that one must have $m \le n$, such that there are no unions of more than ∞^n elements. However, if $m \le n$ then there is always a union of ∞^m elements that includes the element x_i^0, p_i^0 and the m given infinitely close elements. Namely, if one sets:

$$d_k x_i = a_k \,\delta t, \qquad d_k p_i = b_k \,\delta t$$

then one has:

$$x_i = x_i^0 + \sum_{k=1}^{m} a_k u_k , \qquad p_i = p_i^0 + \sum_{k=1}^{m} b_k u_k \qquad (i = 1, \ldots, n),$$

when one considers $u_1, .., u_m$ to be independent variables of such a union. [1])

From the previous statements, it emerges that any union will be represented by equation of the form:

$$x_i = \Phi_i(v_1, \ldots, v_m), \qquad p_i = X_i(v_1, \ldots, v_m) \qquad (i = 1, \ldots, n),$$

where $m \leq n$, and among the $2n$ functions Φ_i, X_i , m of them are mutually independent. Among the n functions $\Phi_1, \ldots, \Phi_n$, let exactly $l \leq m$ of them be mutually independent, so one can represent the equations $x_i = \Phi_i$ in the form:

$$x_i = \varphi_i(u_1, \ldots, u_l) \qquad (i = 1, \ldots, n), \tag{13}$$

of which, l of them can be solved for $u_1, \ldots, u_l$. Now, should the expression:

$$\sum_{i=1}^{n} p_i dx_i = \sum_{i=1}^{n} p_i d\varphi_i$$

be a complete differential, then it could obviously include no other independent variables than just $u_1, \ldots, u_l$, so one must have:

$$\sum_{i=1}^{n} p_i d\varphi_i = d\Omega(u_1, \ldots, u_l),$$

an equation that can be subdivided into the following ones:

$$\sum_{i=1}^{n} p_i \frac{\partial \varphi_i}{\partial u_k} = \frac{\partial \Omega}{\partial u_k} \qquad (k = 1, \ldots, l). \tag{14}$$

However, it is clear that equations (13) and (14) collectively always represent a union of ∞^n elements when one chooses the functions $\varphi_1, \ldots, \varphi_n$ and Ω completely arbitrarily and cares only whether the l functions $\varphi_1, \ldots, \varphi_n$ are mutually independent. Likewise, one demands that all unions of ∞^n elements can be found in this manner. On the other hand, each union of ∞^m elements ($l \leq m \leq n$), among whose equations n of them of the form (13) can be found, must belong to a union that is determined by equations of the form (13) and (14), and any family of elements in it that is included in a union must obviously define a union in its own right, and with this, the determination of all possible unions is achieved. One obtains all of them when one chooses the functions $\varphi_1, \ldots, \varphi_n, \Omega$ in the

[1]) Cf., G. Kowalewski, Leipz. Ber. 1900, pp. 96, *et seq.*

most general manner for all possible values of l ($0 \leq l \leq n$), and thus adds to equations (13), in the most general manner, the equations:

(15) $$p_i = \chi_i(u_1, \ldots, u_l, u_{l+1}, \ldots, u_m) \qquad (i = 1, \ldots, n)$$

($l \leq m \leq n$), such that equations (14) are fulfilled identically.

The union of ∞^n elements, on which, as we have seen, all unions must lie, must then be considered in particular.

Let the equations:

(16) $$\Phi_\nu(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0 \qquad (\nu = 1, \ldots, n)$$

be mutually independent, and let x_i^0, p_i^0 be a system of values that satisfies (16) and that does not make all of the n-rowed determinants of the matrix:

(17) $$\left| \frac{\partial \Phi_\nu}{\partial x_1} \cdots \frac{\partial \Phi_\nu}{\partial x_n} \frac{\partial \Phi_\nu}{\partial p_1} \cdots \frac{\partial \Phi_\nu}{\partial p_n} \right| \qquad (\nu = 1, \ldots, n).$$

vanish. When does the totality of all elements that lie in a certain neighborhood of the element x_i^0, p_i^0 and satisfy equations (16) represent a union of ∞^n elements?

From § 2, it is necessary and sufficient that for all systems of values x_i, p_i, dx_i, dp_i, δx_i, δp_i that satisfy the equations $\Phi_\nu = 0$, $d\Phi_\nu = 0$, $\delta\Phi_\nu = 0$ the equation:

(9) $$\sum_{i=1}^n (dx_i \delta p_i - dp_i \delta x_i) = 0$$

is fulfilled. Since we restrict ourselves to such elements that lie in a certain neighborhood of the elements x_i^0, p_i^0, we can, and would prefer to moreover, consider only such elements x_i, p_i that fulfill (16) and also do not make all of the n-rowed determinants of (17) vanish. For each element x_i, p_i, the equations $d\Phi_\nu = 0$ represent n independent equations for the differentials dx_i, dp_i. If the n systems of values $d_k x_i$, $d_k p_i$ ($k = 1, \ldots, n$) are linearly independent systems of solutions to the equations $d\Phi_\nu = 0$ then the n equations:

(18) $$\sum_{i=1}^n (d_k x_i \delta p_i - d_k p_i \delta x_i) = 0 \qquad (k = 1, \ldots, n)$$

are linearly independent of each other, and since these equations must be satisfied for all systems of values dx_i, dp_i that satisfy the equations $\delta\Phi_\nu = 0$, this demands that the system of equations (18) must be equivalent to the system $\delta\Phi_\nu = 0$. In this fact, one finds that the expressions:

(19) $$dx_i = \sum_{\mu=1}^n \lambda_\mu \frac{\partial \Phi_\mu}{\partial p_i} dt, \qquad dp_i = -\sum_{\mu=1}^n \lambda_\mu \frac{\partial \Phi_\mu}{\partial x_i} dt$$

with the n arbitrary parameters $\lambda_1, \ldots, \lambda_n$, represent the most general system of values that satisfy the equations $d\Phi_\nu = 0$. If we therefore set, with the employment of the Poisson bracket symbol:

(20)
$$\sum_{i=1}^{n}\left(\frac{\partial\varphi}{\partial p_i}\frac{\partial\chi}{\partial x_i}-\frac{\partial\varphi}{\partial x_i}\frac{\partial\chi}{\partial p_i}\right) = (\varphi\,\chi),$$

then the expressions:

$$d\Phi_\nu = \sum_{\mu=1}^{n}\lambda_\mu(\Phi_\mu\Phi_\nu)\,dt$$

must vanish for arbitrary λ_μ; that is, all expressions $(\Phi_\mu\,\Phi_\nu)$ must vanish for the system of values x_i, p_i that is considered here.

This condition is now not merely necessary, but also sufficient. Namely, if it is fulfilled then obviously for arbitrary λ equations (19) represent a system of values that satisfies the equations $d\Phi_\nu = 0$, and indeed, the most general system of values of this type; however, by means of (19), one will have:

$$\sum_{i=1}^{n}(dx_i\delta p_i - dp_i\delta x_i) = \sum_{\mu=1}^{n}\lambda_\mu\delta\Phi_\mu\cdot dt\,,$$

which, due to the equations $\delta\Phi_\nu = 0$, must vanish.

Thus, we have the well-known theorem:

If x_i^0, p_i^0 *is an element that satisfies the n independent equations* (16) *and does not make all of the n-rowed determinants in the matrix* (17) *vanish then the manifold of* ∞^n *elements that is represented by* (16) *in the neighborhood of the element* x_i^0, p_i^0 *is a union when and only when all of the expressions* $(\Phi_\mu\,\Phi_\nu)$ $(\mu, \nu = 1, \ldots, n)$ *also vanish for each element* x_i, p_i *that fulfills* (16) *and lies in a certain neighborhood of* x_i^0, p_i^0, *or, more briefly, when the equations* $(\Phi_\mu\,\Phi_\nu) = 0$ *are a consequence of* (16).

Since we are stuck with the Poisson bracket symbol here, it is likewise advisable to add the important relationship that exists between this symbol and the bilinear covariant of the Pfaffian expression, a relationship that likewise seems to have first been established by S. Kantor, or something close to it.

Namely, if one interprets the quantities dx_i, dp_i as homogeneous point coordinates in a plane space R_{2n-1} of $2n - 1$ dimensions, and if one defines homogeneous plane coordinates by the equation:

$$\sum_{i=1}^{n}(u_i dx_i + v_i dp_i) = 0$$

then the covariant form that belongs to the bilinear alternating form:

$$\sum_{i=1}^{n}(dx_i\delta p_i - dp_i\delta x_i)$$

reads, in plane coordinates:

$$\sum_{i=1}^{n}(v_i u_i' - u_i v_i').$$

Now, since the derivatives of two functions φ and χ with respect to x_i, p_i are nothing but two such systems of coordinates, the Poisson bracket expression is simply the covariant constructed from the plane coordinates to the bilinear covariant of the Pfaffian expression. Furthermore, since the equation:

$$(9) \qquad \sum_{i=1}^{n}(dx_i\delta p_i - dp_i\delta x_i) = 0$$

represents a linear complex in R_{2n-1}, the demand that the equations $\Phi_\nu = 0$ should determine a union obviously says that all of the lines in the $(n - 1)$-fold extended manifold:

$$(21) \qquad \sum_{i=1}^{n}\left(\frac{\partial\Phi_\nu}{\partial x_i}dx_i + \frac{\partial\Phi_\nu}{\partial p_i}dp_i\right) = 0 \qquad (n = 1, \dots, n)$$

in R_{2n-1} belong to this complex. However, this means the same thing as saying that the n planar $(2n - 2)$-fold extended manifolds in R_{2n-1} whose intersection is (21) fulfill the equations $(\Phi_\mu\,\Phi_\nu) = 0$.

The importance of this relationship between the bilinear covariants and the Poisson bracket expressions rests especially upon the fact that, with no further assumptions, it can be carried over to any Pfaffian expression in $2n$ variables that includes the normal form $p_1\,dx_1 + \dots + p_n\,dx_n$.

We have seen that a system of n independent equations:

$$(16) \qquad \Phi_\nu(x_1, \dots, x_n, p_1, \dots, p_n) = 0 \qquad (\nu = 1, \dots, n)$$

represents a union of ∞^n elements when and only when all expressions $(\Phi_\mu\,\Phi_\nu)$ vanish due to (16). Since each system of equations that is equivalent to (16) represents the same union, it must possess this property, and, in particular, it must then be true for every system of equations that follows by solving (16).

We would like to assume that (16) can be solved for exactly m of the p_i, so it can take on the form:

$$p_{i_\mu} + \varphi_\mu(x_1, \cdots, x_n, p_{i_{m+1}}, \cdots, p_{i_n}) = 0 \qquad (\mu = 1, \dots, m),$$
$$\chi_k(x_1, \dots, x_n) = 0 \qquad (k = 1, \dots, n - m),$$

in which we understand $i_1, \dots, i_n$ to mean any permutation of the numbers $1, \dots, n$. If this now lets us derive a relation between just the $x_{i_1}, \dots, x_{i_m}$ from the equations $\chi_k = 0$ – perhaps:

$$x_{i_m} + \omega(x_{i_1}, \cdots, x_{i_{m-1}}) = 0,$$

then the equations of the union can take on such a form that the two equations $p_{i_m} + \varphi_m = 0$, $x_{i_m} + \omega = 0$ emerge. However, from the equations of the union, the left-hand side of the expression:

$$(p_{i_m} + \varphi_m,\ x_{i_m} + \omega) = 1$$

must then vanish, which is impossible. Thus, the quantities $x_{i_{m+1}}, \ldots, x_{i_n}$ cannot be eliminated from the $n - m$ equations $\chi_k = 0$, and the equations of our union – or, indeed, any union of ∞^n elements – can then be put into the form:

$$(22) \qquad \begin{cases} p_{i_\mu} + \varphi_\mu(x_{i_1}, \cdots, x_{i_m}, p_{i_{m+1}}, \cdots, p_{i_n}) = 0 \\ \qquad\qquad x_{i_{m+1}} + \chi_k(x_{i_1}, \cdots, x_{i_m}) = 0 \end{cases} \qquad (\mu = 1, \ldots, m;\ k = 1, \ldots, n - m).$$

Here, however, the bracket expressions on the left-hand sides are functions of only the $x_{i_1}, \ldots, x_{i_m}$, $p_{i_{m+1}}, \ldots, p_{i_n}$ and must then be identically zero, since they must vanish due to (22).

When two functions φ and χ make the bracket expression $(\varphi\,\chi)$ vanish identically, one says that they *lie in involution.* We can then also express our result in the form:

The equations of a union of ∞^n *elements can always be put into such a form:*

$$\Omega_\nu(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0 \quad (\nu = 1, \ldots, n)$$

that the functions $\Omega_1, \ldots, \Omega_n$ *lie pairwise in involution.*

This shows that there are systems of n independent functions of x, p that lie pairwise in involution. If $X_1, \ldots, X_n$ is such a system then the equations:

$$(23) \qquad X_\nu(x_1, \ldots, x_n, p_1, \ldots, p_n) = a_\nu \qquad (\nu = 1, \ldots, n)$$

always represent a union of ∞^n elements whose values are also given by $a_1, \ldots, a_n$. One then has a family of ∞^n unions and ∞^n elements, and it is clear that the space of ∞^{2n} elements x, p is divided into ∞^n unions of ∞^n elements by means of equations (23), such that any element x, p belongs to one, and generally only one, of these unions.

Conversely, if one knows that equations (23), in which the X_ν are independent functions, represent nothing but unions then one can infer that the X_ν lie pairwise in involution if the expressions:

$$(X_\mu - a_\mu, X_\nu - a_\nu) = (X_\mu\, X_\nu)$$

must always vanish for arbitrary a_ν, due to (23), which is possible only when they vanish identically.

One can, moreover, easily form the most general system of equations (23) that represents ∞^n unions of ∞^n elements. One needs only to choose the functions φ_i and Ω in equations (13) and (14) to be functions of l variables $u_1, \ldots, u_l$ and n parameters $a_1, \ldots, a_n$ in the most general way that makes equations (13), (14) soluble in terms of $u_1, \ldots, u_\nu, a_1, \ldots, a_n$.

If equations (23) represent unions of ∞^n elements for arbitrary choices of the constants a_ν, and if $\Phi_1, \ldots, \Phi_m$ ($m < n$) are arbitrary functions that are independent only of each other and the X_ν then obviously the equations:

$$X_1 = a_1, \ldots, X_n = a_n, \quad \Phi_1 = b_1, \ldots, \Phi_m = b_m$$

also represent unions for arbitrary values of the a, b, and indeed, unions of ∞^{n-m} elements. Thus, there are certain systems of equation of the form:

$$F_\nu(x_1, \ldots, x_n, p_1, \ldots, p_n) = a_\nu \qquad (\nu = 1, \ldots, n + m, 0 \leq m \leq n) \tag{24}$$

that determine unions for arbitrary values of the values a_ν. The space of ∞^{2n} elements x, p will be subdivided by such a system of equations into a family of ∞^{n+m} unions of ∞^{n-m} elements in such a way that each element x, p belongs to one, and generally only one, of these unions.

Let $\psi_1, \ldots, \psi_{n-m}$ be functions of the x, p that are independent of each other and the $F_1, \ldots, F_{n+m}$. If we then set:

$$\psi_k(x_1, \ldots, x_n, p_1, \ldots, p_n) = u_k \qquad (k = 1, \ldots, n-m) \tag{25}$$

then equations (24), (25) may be solved for the x, p, and we obtain a new representation of our ∞^{n+m} unions from this solution:

$$\begin{cases} x_i = \Phi_i(u_1, \cdots, u_{n-m}, a_1, \cdots, a_{n+m}) \\ p_i = X_i(u_1, \cdots, u_{n-m}, a_1, \cdots, a_{n+m}) \end{cases} \qquad (i = 1, \ldots, n), \tag{26}$$

in which the $u_1, \ldots, u_{n-m}$ are to be regarded as independent variables. The system of equations (26) may thus be obviously solved for the u and a, and thus again delivers equations (24) and (25).

Since equations (24) represent unions for arbitrary values of the a, the expression $\sum X_i \, d\Phi_i$ represents a complete differential in the variables u, so one has:

$$\sum_{i=1}^{n} X_i \sum_{k=1}^{n-m} \frac{\partial \Phi_i}{\partial u_k} du_k \equiv \sum_{k=1}^{n-m} \frac{\partial \Omega(u_1, \cdots, u_{n-m}, a_1, \cdots, a_{n+m})}{\partial u_k} du_k . \tag{27}$$

If we make the substitution $a_\nu = F_\nu$, $u_k = y_k$ in this identity, which we would like to suggest by enclosing them in square brackets, and imagine that $[\Phi_i] \equiv x_i$, $[X_i] \equiv p_i$ then we obtain an identity in the variables x, p of the form:

$$\sum_{i=1}^{n} p_i dx_i - \sum_{i=1}^{n} p_i \sum_{\nu=1}^{n+m} \left[\frac{\partial \Phi_i}{\partial a_\nu}\right] dF_\nu \equiv d[\Omega] - \sum_{\nu=1}^{n+m} \left[\frac{\partial \Omega}{\partial a_\nu}\right] dF_\nu ,$$

or, when we set:

$$(28) \qquad \begin{cases} [\Omega] = -\omega(x_1, \cdots, x_n, p_1, \cdots, p_n) \\ \displaystyle\sum_{i=1}^{n} p_i \left[\frac{\partial \Phi_i}{\partial a_\nu}\right] - \left[\frac{\partial \Omega}{\partial a_\nu}\right] = f_\nu (x_1, \cdots, x_n, p_1, \cdots, p_n), \end{cases}$$

an identity:

$$(29) \qquad \sum_{\nu=1}^{n+m} f_\nu dF_\nu \equiv \sum_{i=1}^{n} p_i dx_i + d\omega ,$$

which clearly expresses the fact that equations (24) represent a family of ∞^{n+m} unions.

Here, the function Ω is determined by (27) up to an arbitrary, additive function ϑ of $a_1, \ldots, a_{n+m}$, so one can replace ω with $\omega + \vartheta(F_1, \ldots, F_{n+m})$, from which (29) assumes the form:

$$\sum_{\nu=1}^{n-m} \left(f_\nu + \frac{\partial \vartheta}{\partial F_\nu} \right) dF_\nu \equiv \sum_{i=1}^{n} p_i dx_i + d(\omega + \vartheta).$$

It is also easy to see that in this we have found the most general form for this identity in the form of (29). Namely, if:

$$\sum_{\nu=1}^{n+m} \bar{f}_\nu \, dF_\nu \equiv \sum_{\nu=1}^{n} p_i dx_i + d\varpi$$

then:

$$\sum_{\nu=1}^{n+m} (\bar{f}_\nu - f_\nu) \, dF_\nu \equiv d(\varpi - \omega)$$

is therefore equal to a complete differential, and since the F_ν are independent functions of the x, p, $\varpi - \omega$ is a function ϑ of $F_1, \ldots, F_{n+m}$; hence:

$$\bar{f}_\nu - f_\nu = \frac{\partial \vartheta(F_1, \cdots, F_{n+m})}{\partial F_\nu} \qquad (\nu = 1, \ldots, n+m).$$

If one has found ω by the aforementioned quadrature, to which certain eliminations must generally be added, then one finds the f_ν from the $2n$ linear equations:

$$\sum_{\nu=1}^{n+m} f_\nu \frac{\partial F_\nu}{\partial x_i} = p_i + \frac{\partial \omega}{\partial x_i},$$
$$\sum_{\nu=1}^{n+m} f_\nu \frac{\partial F_\nu}{\partial p_i} = \frac{\partial \omega}{\partial p_i},$$

into which (29) separates. Among these equations, which are certainly compatible with each other, there are exactly $n + m$ mutually independent ones, due to the independence of the F_ν .

Thus, if equations (24) *represent unions of* ∞^{n-m} *elements for an arbitrary choice of the* a_ν *then there always exists an identity of the form* (29), *where* ω *must be found by a quadrature, while the* f_ν *are determined after discovering* ω *by linear equations.* [1])

If we construct the bilinear covariants from the two sides of the identity (29), which are furthermore identically equal, for self-explanatory reasons, then we obtain the new identity:

(30) $$\sum_{\nu=1}^{n+m}(df_\nu \delta F_\nu - \delta f_\nu dF_\nu) \equiv \sum_{i=1}^{n}(dp_i \delta x_i - dx_i \delta p_i) .$$

In this, if we set:

$$\delta x_i = \frac{\partial \varphi}{\partial p_i}\delta t , \qquad \delta p_i = -\frac{\partial \varphi}{\partial x_i}\delta t ,$$

in which φ is understood to mean an arbitrary function, then this yields:

(31) $$\sum_{\nu=1}^{n+m}\big((\varphi F_\nu)\, df_\nu - (\varphi f_\nu)\, dF_\nu\big) \equiv d\varphi,$$

and from this, one further obtains by the substitution:

$$dx_i = \frac{\partial \chi}{\partial p_i}dt , \qquad dp_i = -\frac{\partial \chi}{\partial x_i}dt ,$$

the identity:

(32) $$\sum_{\nu=1}^{n+m}\big\{\big((\varphi F_\nu)(\chi f_\nu) - (\varphi f_\nu)(\chi F_\nu)\big)\big\} \equiv d\varphi.$$

Conversely, if (32) is true identically for all functions φ and χ then obviously (30) is also fulfilled identically, and there thus likewise exists an identity of the form (29).

If we now assume, in particular, that $m = 0$ then we consider the case in which equations (24), or, as we would like to now write them:

(24′) $$X_\nu(x_1, \dots, x_n , p_1, \dots, p_n) = a_\nu \qquad (\nu = 1, \dots, n),$$

represent a family of ∞^n unions of ∞^n elements then all $(X_\mu\, X_\nu) \equiv 0$. If we then write the identity (29) in the form:

(29′) $$\sum_{\nu=1}^{n} P_\nu dX_\nu \equiv \sum_{\nu=1}^{n} p_\nu dx_\nu + d\omega$$

[1]) Cf., Lie, Math. Ann., Bd. XI, pp. 465, *et seq.*

then for $\varphi = X_\mu$ and $\varphi = P_\mu$ the identity (31) delivers this one:

$$\begin{cases} \sum_{\nu=1}^{n} (P_\nu X_\mu)\, dX_\nu \equiv dX_\mu \\ \sum_{\nu=1}^{n} \{(P_\nu X_\mu)\, dP_\nu - (P_\nu P_\mu)\, dX_\nu\} \equiv dP_\mu \end{cases} \qquad (\mu = 1, \ldots, n). \tag{33}$$

From this, however, it next follows from the independence of the X_ν that:

$$(P_\nu\, X_\mu) = \varepsilon_{\mu\nu}\,,$$

where $\varepsilon_{\mu\nu} = 0$ or 1, according to whether $\mu \neq \nu$ or $\mu = \nu$, resp., so one has, however:

$$(P_\mu\, P_\nu) \equiv 0.$$

Finally, if we replace of the dx_i, dp_i in (29′) with the expressions that we just employed then for any arbitrary function χ, one has:

$$\sum_{\nu=1}^{n} P_\nu (\chi X_\nu) \equiv \sum_{\nu=1}^{n} p_\nu \frac{\partial \chi}{\partial p_\nu} + (\chi\, \omega).$$

We then have the well-known:

Theorem: *If $X_1, \ldots, X_n$ are independent functions of $x_1, \ldots, x_n$, $p_1, \ldots, p_n$ that are pairwise in involution or, what amounts to the same thing, if the equations $X_\nu = a_\nu$ ($\nu = 1, \ldots, n$) represent unions of ∞^n elements for arbitrary a_ν, then there exists an identity of the form:*

$$\sum_{\nu=1}^{n} P_\nu dX_\nu \equiv \sum_{\nu=1}^{n} p_\nu dx_\nu + d\omega, \tag{29′}$$

where ω is found by a quadrature, while the P_i were obtained by solving linear equations. Between the functions X_i, P_i, and ω there thus exist the relations:

$$(X_i\, X_\nu) = 0, \qquad (P_i\, X_\nu) = \varepsilon_{i\nu}\,, \quad (P_i\, P_\nu) = 0, \qquad (i, \nu = 1, \ldots, n) \tag{34}$$

and:

$$\begin{cases} (\omega X_i) = \sum_{\mu=1}^{n} p_\mu \frac{\partial X_i}{\partial p_\mu} \\ (\omega P_i) = \sum_{\mu=1}^{n} p_\mu \frac{\partial P_i}{\partial p_\mu} - P_i \end{cases} \qquad (i = 1, \ldots, n), \tag{35}$$

and in addition, there is the identity:

(32′) $$\sum_{\nu=1}^{n}\{(P_\nu\varphi)(X_\nu\chi)-(X_\nu\varphi)(P_\nu\chi)\}\equiv(\varphi\,\chi),$$

in which the φ and χ may also be functions of the x, p. [1])

From the existence of the relations (34), it follows, moreover, that all $2n$ functions X_1, …, X_n, P_1, …, P_n are independent of each other. Namely, if one forms the square of the functional determinant of the X, P relative to the x, p, in which one writes these determinants in the two forms:

$$\begin{pmatrix} P_1\cdots P_n & X_1\cdots X_n \\ p_1\cdots p_n & x_1\cdots x_n \end{pmatrix}$$

$$\begin{pmatrix} X_1\cdots X_n & -P_1\cdots -P_n \\ x_1\cdots x_n & p_1\cdots p_n \end{pmatrix}$$

and multiplies the two together then one obtains a determinant that possesses the value 1, due to (34).

Conversely, if there exists an identity of the form (29′) then it is clear that the equations:

$$X_1 = \text{const.},\qquad \ldots,\qquad X_n = \text{const.}$$

represent unions. Were the functions X_1, …, X_n not independent of each other then each of these unions would consist of more than ∞^n elements, which is impossible, so we can conclude that X_1, …, X_n are independent of each other and lie pairwise in involution. Then, however, it likewise follows that equations (34), (35), and (32′) are valid.

Finally, if one is given $2n$ functions X_1, …, X_n, P_1, …, P_n that satisfy the relations:

(34) $$(X_i\,X_\nu) = 0,\qquad (P_i\,X_\nu) = \varepsilon_{i\nu},\quad (P_i\,P_\nu) = 0,\qquad (i,\ \nu = 1, \ldots, n)$$

then, as we have seen, all of these functions are independent of each other.

One can, as a consequence, express any arbitrary function φ of x, p by X_1, …, X_n, P_1, …, P_n and obtain:

(36) $$(\varphi\,X_i) = \frac{\partial\varphi}{\partial P_i},\qquad (\varphi\,P_i) = -\frac{\partial\varphi}{\partial X_i},\qquad (i = 1, \ldots, n).$$

If one adds yet a second function χ then one obtains:

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[1]) Strangely enough, the identity (32′) seems to have not been noticed up to now.

(37)
$$\begin{cases} (\varphi\chi) = \displaystyle\sum_{i=1}^{n}\left\{(\varphi X_i)\frac{\partial\chi}{\partial X_i} + (\varphi P_i)\frac{\partial\chi}{\partial P_i}\right\} \\ \qquad = \displaystyle\sum_{i=1}^{n}\left\{\frac{\partial\varphi}{\partial P_i}\frac{\partial\chi}{\partial X_i} - \frac{\partial\varphi}{\partial X_i}\frac{\partial\chi}{\partial P_i}\right\}, \end{cases}$$

from which, we can also write:

(32′)
$$(\varphi\chi) \equiv \sum_{i=1}^{n}\{(P_i\varphi)(X_i\chi) - (X_i\varphi)(P_i\chi)\}.$$

However, as we recently remarked, the identity (30) follows from the existence of (32) for arbitrary functions φ, χ so it follows from (32′) that there is an identity:

$$\sum_{i=1}^{n}(dx_i\delta p_i - dp_i\delta x_i) = \sum_{i=1}^{n}(dX_i\delta P_i - dP_i\delta X_i).$$

In this, one sees that the two expressions $\sum p_i\, dx_i$ and $\sum P_i\, dX_i$ have the same bilinear covariants, so they differ only by a complete differential. As a result, there exists an identity of the form:

(29′)
$$\sum_{\nu=1}^{n} P_\nu dX_\nu \equiv \sum_{\nu=1}^{n} p_\nu dx_\nu + d\omega,$$

where the function ω satisfies the $2n$ equations:

(35)
$$\begin{cases} (\omega X_i) = \displaystyle\sum_{\mu=1}^{n} p_\mu \frac{\partial X_i}{\partial p_\mu}, \\ (\omega P_i) = \displaystyle\sum_{\mu=1}^{n} p_\mu \frac{\partial P_i}{\partial p_\mu} - P_i, \end{cases}$$

through which, its $2n$ derivatives are determined. Thus:

Theorem: *If the* $2n$ *functions* X_i, P_i *satisfy relations of the form* (34), *then they are independent of each other, and there exists an identity of the form* (29′), *where the function* ω *satisfies equations* (35).

The bilinear covariants have already shown us their great utility in the derivation of the equations (34) from the identity (29′), and it will become even clearer in the proof of the converse that an identity of the form (29′) can be deduced from equations (34). It was precisely this proof of the converse that led Lie to such rather circuitous considerations. [1])

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[1]) Cf., *Transformationsgruppen*, Bd. II, pp. 126-130.

§ 4. Contact transformations in the *x*, *p*.

We refer to any transformation:

(38) $x'_i = X_i(x_1, \ldots, x_n, p_1, \ldots, p_n)$, $p'_i = P_i(x_1, \ldots, x_n, p_1, \ldots, p_n)$, $(i = 1, \ldots, n)$

as a contact transformation in the x, p when the Pfaffian expression $\sum p_i\, dx_i$ remains invariant up to a complete differential, so there exists a relation of the form:

(39) $$\sum_{i=1}^{n} p'_i dx'_i = \sum_{i=1}^{n} p_i dx_i + d\omega(x, p).$$

From (39), it follows that due to (38) an equation of the form:

(40) $$\sum_{i=1}^{n} (dp'_i\, \delta x'_i - dx'_i\, \delta p'_i) = \sum_{i=1}^{n} (dp_i\, \delta x_i - dx_i\, \delta p_i)$$

exists, that, in turn, implies one of the form (39). The contact transformations in the x, p can therefore also be defined as the group of all transformations in the x, p that leave the bilinear covariant of the Pfaffian expression $\sum p\; dx$ invariant. Likewise, it is clear that our contact transformations take each element x, p, along with two infinitely close ones $x + dx$, $p + dp$ and $x + \delta x$, $p + \delta p$ that it is united with, to another such element, and each union of elements to a union, in addition.

If (38) is a contact transformation in the x, p then an identity of the form (29′) exists, and it follows that the functions X_i, P_i, ω are coupled by the relations (34) and (35). Conversely, however, from pp. ?, the equations (34) imply the independently of the $2n$ functions X_i, P_i and the existence of a relation of the form (29′), where ω satisfies equations (35). Thus, equations (38) represent a contact transformation in the x, p when and only when they satisfy equations (34). However, as we just said, the identity (37) follows from (34), and thus, the equation:

$$\sum_{i=1}^{n} \left(\frac{\partial \varphi}{\partial p_i} \frac{\partial \chi}{\partial x_i} - \frac{\partial \varphi}{\partial x_i} \frac{\partial \chi}{\partial p_i} \right) = \sum_{i=1}^{n} \left(\frac{\partial \varphi}{\partial p'_i} \frac{\partial \chi}{\partial x'_i} - \frac{\partial \varphi}{\partial x'_i} \frac{\partial \chi}{\partial p'_i} \right),$$

or, written more briefly:

(41) $$(\varphi\, \chi)_{xp} = (\varphi\, \chi)_{x'p'}\,.$$

The contact transformations in the x, p then leave the Poisson bracket expression invariant.

Conversely, any transformation in the x, p under which the Poisson bracket expression remains invariant is obviously a contact transformation in the x, p. The group of all contact transformations in the x, p can therefore also be defined by the invariance of this bracket expression, which, from the relationship of this expression to the bilinear covariant, would not be surprising.

Now, if:

$$\delta x_i = \xi_i\ \delta t, \qquad \delta p_i = \pi_i\ \delta t, \qquad (i = 1, \dots, n)$$

or:

$$Xf = \sum_{i=1}^{n}\left(\xi_i \frac{\partial f}{\partial x_i} + \pi_i \frac{\partial f}{\partial p_i}\right)$$

is an infinitesimal transformation the one has:

$$\delta(\sum p_i\ dx_i) = du(x, p) \cdot \delta t,$$

so:

$$\sum_{i=1}^{n}(p_i\ d\xi_i + \pi_i\ dx_i) = du,$$

or:

$$\sum_{i=1}^{n}(\pi_i\ dx_i - \xi_i\ dp_i) = d(u - \sum p_i\ \xi_i)\ .$$

If we then set $u = - \sum p_i\ \xi_i = - U$ then we will have:

$$\xi_i = \frac{\partial U}{\partial p_i}, \qquad \pi_i = -\frac{\partial U}{\partial x_i},$$

from which:

(42) $$Xf = (Uf).$$

The function U can obviously be chosen arbitrarily here, and one has:

(43) $$X(\sum p_i\ dx_i) = d\left(\sum p_i U_{p_i} - U\right).$$

From the invariance of the Poisson bracket expression, it then follows that the function U is invariantly connected with the infinitesimal transformation Xf with respect to any finite contact transformation.

We would like to call U the *characteristic* that belongs to the infinitesimal contact transformation and then remark that U will be found by a quadrature from a given Xf, so one has:

(44) $$dU = \sum_{i=1}^{n}(\xi_i\ dp_i - \pi_i\ dx_i)\ .$$

If we also choose the transformation (38) to be infinitesimal with the characteristic V then we have:

$$x_i' = x_i + \frac{\partial V}{\partial p_i}\delta t, \qquad p_i' = p_i - \frac{\partial V}{\partial x_i}\delta t,$$

so for any function $f(x, p)$ one will have:

$$f' = f(x', p') = f + (Vf)_{xp}\,\delta t, \qquad f = f' - (V'f')_{x'p'}\,\delta t.$$

Now, however, one has:

$$\begin{aligned}(Uf)_{xp} &= (Uf)_{x'p'}\,, \\ &= (U' - (V'\,U')\,\delta t, f' - (V'f')\,\delta t)_{x'p'} \\ &= (U'\,f')_{x'p'} - (U'(V'\,U')_{x'p'}\,\delta t - ((V'U')\,f')_{x'p'}\,\delta t \\ &= (U\,f) + \{(V(U\,f)) - (U(V\,f)) - ((V\,U)f)\}\,\delta t,\end{aligned}$$

which then gives the celebrated *Jacobi identity:*

(45) $$(U(V\,f)) - (V(U\,f)) = ((U\,V)f),$$

which is true for arbitrary functions U, V, f of the x, p.

This proof of the identity, which goes back to Lie, obviously cannot be replaced with a conceptually simpler one.

§ 5. Differential equations in $x_1, \ldots, x_n, p_1, \ldots, p_n$.

Now, let there be given a system of equations:

$$F_k(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0 \qquad (k = 1, \ldots, l),$$

and suppose that all of the unions of ∞^n elements that one finds will satisfy this system of equations, or, more briefly: all of the associated integral unions of ∞^n elements. Then, from pp. ?, one understands that these unions all satisfy equations of the form:

$$(F_k\,F_j) = 0 \qquad (k, j = 1, \ldots, l),$$

as well. If one finds no contradiction from the construction of these equations and the ones that follow from them then one ultimately finds that the problem that we posed always comes down to the other one, of finding all integral unions of ∞^n elements for a system of the form:

(46) $$F_\mu(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0 \qquad (\mu = 1, \ldots, l),$$

where the $(F_\mu\,F_\nu)$ all vanish, due to (46). However, one calls such a system of equations an *m-parameter system in involution.*

If equations (46) define an m-parameter system in involution in that sense then any equivalent system of equations $\Phi_1 = 0, \ldots, \Phi_m = 0$ has the same property.

In fact, in the R_{2n-1} of the dx_i, dp_i the equations $dF_1 = 0, \ldots, dF_m = 0$ represent a $(2n - m - 1)$-fold extended planar manifold E_{2n-m-1}, and indeed it represents the intersection of m $(2n - 2)$-fold extended planar manifolds. If we restrict ourselves now to such elements x_i, p_i that satisfy (46) then $(F_\mu\,F_\nu) = 0$ (m, $n = 1, \ldots, m$); that is, any two planar $(2n - 2)$-fold extended manifolds u_i, v_i and u_i', v_i' that go through E_{2n-m-1} always satisfy the equation: $\sum (v_i u_i' - u_i v_i') = 0$. On the other hand, the equations: $d\Phi_1 = 0, \ldots, d\Phi_m = 0$ represent the same manifold E_{2n-m-1} as only the intersection of m other $(2n - 2)$-fold extended planar manifolds; thus, along with the assumptions that one makes on x_i, p_i, one must also require that all $(\Phi_\mu \, \Phi_\mu)$ vanish.

If we now think of the system of equations (46) then one finds, as on pp. ?, that the solution can be obtained in the form:

$$(46') \qquad \begin{cases} x_{i_\lambda} + \varphi_\lambda(x_{i_{l+1}}, \cdots, x_{i_m}) = 0, \\ p_{i_{l+k}} + \chi_k(x_{i_{l+1}}, \cdots, x_{i_m}, p_{i_{l+1}}, \cdots, p_{i_m}, p_{i_{m+1}}, \cdots, p_{i_n}) = 0, \end{cases} \qquad (\lambda = 1, \ldots, l;\ k = 1, \ldots, m - l),$$

where $i_1, \ldots, i_n$ mean any permutation of the numbers $1, \ldots, n$. Here, however, the bracket expressions on the left-hand sides are free of $x_{i_1}, \ldots, x_{i_l}, p_{i_{l+1}}, \ldots, p_{i_m}$ and must vanish identically, since, by virtue of (46′) they cannot vanish. (?)

Thus, any m-parameter system in involution can be brought into the form:

$$(46'') \qquad \Omega_\mu(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0 \qquad (\mu = 1, \ldots, m),$$

such that the functions $\Omega_1, \ldots, \Omega_m$ lie pair-wise in involution.

If one replaces the system in involution (46) with its solved form (46′) then generally any possible integral union of ∞^n elements that makes the functional determinant:

$$D = \begin{pmatrix} F_1 & \cdots & F_l & F_{l+1} & \cdots & F_m \\ x_{i_1} & \cdots & x_{i_l} & p_{i_{l+1}} & \cdots & p_{i_m} \end{pmatrix}$$

drops away. Since, however, these integral unions satisfy the equations:

$$F_1 = 0, \ldots, F_m = 0, D = 0,$$

their determination comes down to the integration of an at least $(m + 1)$-parameter system in involution, and one can say, with no loss of generality, that the determination of the integral union of ∞^n elements of a given system of equation can always come down to the normal problem:

Integrate a system in involution of the form (46″) *when the functions* $\Omega_1, \ldots, \Omega_m$ *lie pair-wise in involution.*

What this normal problem addresses, we would like to satisfy ourselves here with proving that it possesses complete solutions so the ∞^{2n-m} elements that satisfy (46″) can always be arranged into ∞^{n-m} unions of ∞^n elements. All integral unions can be found from just such a complete solution without integration.

It then comes down to the addition of equations:

$$\Omega_{m+k}(x_1, \ldots, x_n, p_1, \ldots, p_n) = a_k \qquad (k = 1, \ldots, n - m)$$

to the equations (46″), such that the Ω_{m+k} are independent of $\Omega_1, \dots, \Omega_m$ and each other, and likewise lie in involution with the $\Omega_1, \dots, \Omega_m$ and each other. Now, however, the m equations:

$$A_\mu f = (\Omega_\mu f) = 0 \qquad (\mu = 1, \dots, m) \tag{47}$$

are obviously independent of each other, and from:

$$A_\mu A_\nu f - A_\nu A_\mu f = (\Omega_\mu (\Omega_\nu f)) - (\Omega_\nu (\Omega_\mu f)) = (\Omega_\mu \Omega_\nu) f) \equiv 0$$

they define an m-parameter complete system. One then finds a function Ω_{m+1} when one seeks a solution of this complete system that is independent of $\Omega_1, \dots, \Omega_m$, such that one obtains Ω_{m+1} by determining a solution of the $(m + 1)$-parameter complete system:

$$(\Omega_1 f) = 0, \dots, (\Omega_{m+1} f) = 0$$

that is independent of $\Omega_1, \dots, \Omega_m$, and so forth.

We further mention that the integration of the system in involution (46′) in the case of $l > 0$ can always be converted into the integration of an $(m - l)$-parameter system in involution in $2(n - l)$ variables.

If one sets:

$$x'_{i_\lambda} = x_{i_\lambda} + \varphi_\lambda \qquad (\lambda = 1, \dots, l) \tag{48}$$

then one converts the Pfaffian expression $\sum p_i \, dx_i$ into:

$$\sum_{\lambda=1}^{l} p_{i_\lambda} (dx'_{i_\lambda} - d\varphi_\lambda) + \sum_{k=1}^{n-l} p_{i_{l+k}} dx_{i_{l+k}} .$$

If one then sets:

$$\begin{cases} x'_{l+k} = x_{i_{l+k}} & (k = 1, \cdots, n-l) \\ p'_{i_\lambda} = p_{i_\lambda} & (\lambda = 1, \cdots, l) \\ p'_{i_{l+j}} = p_{i_{l+j}} - \displaystyle\sum_{\mu=1}^{l} p_{i_\mu} \frac{\partial \varphi_\mu}{\partial x_{i_{l+j}}} & (j = 1, \cdots, m-l) \\ p'_{i_{m+\tau}} = p_{i_{m+\tau}} & (\tau = 1, \cdots, n-m) \end{cases} \tag{49}$$

then equations (48) and (49) collectively represent a transformation under which the Pfaffian expression $\sum p_i \, dx_i$ indeed remains invariant, and is certainly a contact transformation. Since the bracket expression $(\varphi \chi)$ remains invariant under it, it is clear that the new form:

$$x'_{i_\lambda} = 0, \qquad p'_{i_{l+k}} + \bar{\chi}_k \left(x'_{i_{l+1}}, \cdots, x'_{i_n}, p'_{i_1}, \cdots, p'_{i_l}, p'_{i_{m+l}}, \cdots, p'_{i_n} \right) = 0$$
$$(\lambda = 1, \dots, l, k = 1, \dots, m - l),$$

which includes the involutive system (46′), is again an m-parameter system in involution. From this, however, it follows that the $\bar{\chi}_k$ are free of $p'_{i_1}, \ldots, p'_{i_n}$ such that the equations:

$$p'_{i_{l+k}} + \bar{\chi}_k\left(x'_{i_{l+1}}, \cdots, x'_{i_n}, p'_{i_{m+1}}, \cdots, p'_{i_n}\right) = 0 \qquad (k = 1, \ldots, m - l),$$

which define an $(m - l)$-parameter system in involution in the $2(n - l)$ variables $x'_{i_{l+k}}$, $p'_{i_{l+k}}$, whose integration can be inferred from that of the involutive system (46′).

If $l = m$ then the determination of the n-fold extended integral union of (46′) requires no integration whatsoever. In the new variables, (46′) actually takes on the form: $x'_{i_\lambda} = 0$, $(\lambda = 1, \ldots, l)$. The Pfaffian expression $\sum p'_i dx'_i$ thus reduces to:

$$\sum_{k=1}^{n-l} p'_{i_{l+k}} dx'_{i_{l+k}} ,$$

and all that remains is to determine all unions of ∞^{n-l} elements in the residual $2n - 2l$ variables. However, that is a feasible operation.

§ 6. The invariant theory of contact transformations in the x, p.

We have seen that the integration of a system of equations in the x, p can always be converted into the integration of a system in involution, but then, in turn, this can lead to one looking for solutions of a sequence of complete systems. Thus, each of these complete systems has the form:

$$(47) \qquad (\Omega_\mu f) = 0 \qquad (\mu = 1, \ldots, m),$$

where the functions $\Omega_1, \ldots, \Omega_m$ are independent of each other and lie pair-wise in involution.

If one now happens to find not merely one solution to one of these complete systems, but several of them, then this raises the question of how one can best exploit this situation for the resolution of the integration problem. By the very fact that he posed this question, Lie was induced to develop his invariant theory of contact transformations.

From the form (47) of the complete solutions, and from the invariance of the Poisson bracket symbols under contact transformations in the x, p, it emerges that all of the complete systems that appear in (47) are invariantly linked with the original system of equations in x, p that is to be integrated by means of contact transformations. If one then knows several solutions of a such a system (47) then the question arises of what properties the totality of all the known solutions of the system (47) might possess with respect to all contact transformations in the x, p.

If one knows for the system (47), not just the self-explanatory solutions $\Omega_1, \ldots, \Omega_m$, but also a number of other ones $u_1, \ldots, u_l$ that are independent of each other and the Ω_μ then, first of all, absolutely any arbitrary function of $\Omega_1, \ldots, \Omega_m, u_1, \ldots, u_l$ is likewise a solution, and secondly, the Jacobi identity:

$$((\Omega_\mu\, \varphi) f) - ((\Omega_\mu\, f)\varphi) \equiv (\Omega_\mu\, (\varphi\, f)),$$

shows that along with φ and f, $(\varphi\, f)$ is likewise always a solution. That is, in fact, the celebrated *Poisson-Jacobi theorem*. Therefore, all of the expressions:

$$(\Omega_\mu \Omega_\nu), \qquad (\Omega_\mu\, u_k), \qquad (u_k\, u_j)$$

are also solutions of the complete system. Of these solutions, clearly the $(\Omega_\mu \Omega_\nu)$ and $(\Omega_\mu\, u_k)$ are identically zero, but the other ones $(u_k\, u_j)$ are possibly new. If one adds the new solutions that included among the expressions $(u_k\, u_j)$ – i.e., the ones that are independent of $\Omega_1, \ldots, \Omega_m$, $u_1, \ldots, u_l$ and each other – to $u_1, \ldots, u_l$, once again applies the Poisson-Jacobi theorem, and proceeds in that manner then only two cases are possible: Either one finds $2n - m$ independent solutions of (47), and therefore, all of the ones that are present, or one finds so many new solutions $u_{l+1}, \ldots, u_r$ that indeed $m + r < 2n - m$, but all of the $(u_k\, u_j)$ $(k, j = l, \ldots, r)$ can be expressed in terms of $\Omega_1, \ldots, \Omega_m$, $u_1, \ldots, u_r$.

In the first case, the integration of the system in involution: $\Omega_1 = a_1, \ldots, \Omega_m = a_m$, requires only feasible operations, which have generally been known for a long time for $m = 1$, but were first exhibited by Lie in a theorem, upon which, the extension that he gave of the Cauchy integration method rests. In the second case, a number of solutions of (47) are still unknown, and one then tries to take advantage of the solutions that one finds as much as is possible; for that, it is even necessary to establish the invariant properties that the totality of all known solutions, and therefore the totality of all functions of $\Omega_1, \ldots, \Omega_m$, $u_1, \ldots, u_r$, possess under all contact transformations of the x, p.

The system of functions $\Omega_1, \ldots, \Omega_m$, $u_1, \ldots, u_r$ that one arrives at here has the property that the bracket expression of any two functions of the system is expressible in terms of functions of the system alone. However, it is a completely special system of this type, because $\Omega_1, \ldots, \Omega_m$ and $u_1, \ldots, u_r$ are in involution with each other. It is a closely related problem then for us to likewise consider completely general systems of r independent functions $u_1, \ldots, u_r$ of x, p that are arranged so that relations of the form:

(50) $$(u_i\, u_k) = \omega_{ik}(u_1, \ldots, u_r) \qquad (i, k = 1, \ldots, r)$$

exist. The totality of all functions of the functions $u_1, \ldots, u_r$ of such a system is what Lie called an *r-parameter function group in* $\Omega_1, \ldots, \Omega_m$, $u_1, \ldots, u_r$. The significance of his brief paper of 1872 that was mentioned in the introduction consists of the fact that all invariant properties that such an r-parameter function group possesses relative to the group of all contact transformations were established in it.

This is not the place to develop the invariant theory of an r-parameter function group, since that would be essentially a repetition of the presentation that Lie gave in the second volume of *Transformationsgruppen*. I thus content myself with the following remarks:

The fundamental theorem of the theory is that the r mutually independent equations:

(51) $$(u_k\, f) = 0 \qquad (k = 1, \ldots, r)$$

define an r-parameter complete system with $2n - r$ independent solutions and that the totality of all solutions of (51), and thus, the totality of all functions of $v_1, .., v_{2n-r}$, define a $(2n - r)$-parameter function group, namely, the group $v_1, \ldots, v_{2n-r}$ that is reciprocal to the group $u_1, .., u_r$. The two function groups of combined functions are inside of each group of functions of the group that is in involution with all of the functions of the group; they are called the *distinguished functions* of each group.

The number of parameters r of a function group and the number m of mutually independent distinguished functions that the group includes are the only two invariant properties of the group under all contact transformations. Two function groups that are both associated with the same numbers r and m can always be mapped to each other by contact transformations in the x, p. The proof of this theorem led Lie to show that any r-parameter function group can be brought to a canonical form. If we, with S. Kantor, call any system of r independent functions of an r-parameter function group a *basis* for the function group then we can also say: One can determine a *canonical basis* for any r-parameter function group, which is then r independent functions:

$$X_1 \ldots X_i , P_1 \ldots P_i , \quad X_{i+1} \ldots X_{i+m} \qquad (2i + m = r)$$

that belong to the group and satisfy the canonical relations:

$$(X_i\, X_k) = 0, \quad (P_i\, X_k) = \varepsilon_{ik} , \quad (P_i\, P_k) = 0. \tag{52}$$

Therefore, the functions of $X_{i+1} , \ldots, X_{i+m}$ are distinguished functions of the group; it then happens that the difference between the number of parameters r and the number of independent distinguished functions is always even.

By pursuing the invariant theory of function groups in x, p, Lie was then in a position to establish which invariant properties an arbitrary given system of functions in the x, p:

$$\varphi_k(x_1, \ldots, x_n , p_1, \ldots, p_n) \qquad (k = 1, \ldots, m)$$

possesses under all contact transformations. One can briefly express the result that one arrives at as follows:

To $\varphi_1, \ldots, \varphi_m$, one adds all functions:

$$(\varphi_i\ \varphi_k), ((\varphi_i\ \varphi_k), \varphi_j), ((\varphi_i\ \varphi_k)\ (\varphi_j\ \varphi_l)), \ldots,$$

such that when one forms the bracket expression of any two given functions, it gives only those functions that can be expressed in terms of only the given functions. In this way, one arrives at a system:

$$\varphi_1, \ldots, \varphi_m , \varphi_{m+1}, \ldots, \varphi_r ,$$

whose functions do not need to be mutually independent, but has the property that all $(\varphi_i\ \varphi_k)$ are expressible in terms of the $\varphi_1, \ldots, \varphi_r$ alone. All of the invariant properties of the system $\varphi_1, \ldots, \varphi_m$ will then be represented by the totality of all relations that exist between the $\varphi_1, \ldots, \varphi_m$ and $(\varphi_i\ \varphi_k)$ $(i, k = 1, \ldots, r)$. In other words:

If $\chi_1, \ldots, \chi_m$ is a second function system then there is a contact transformation in the x, p that takes the $\chi_1, \ldots, \chi_r$ in the sequence to $\varphi_1, \ldots, \varphi_r$ if and only if the following requirement is fulfilled:

To the $\chi_1, \ldots, \chi_m$, one adds the expressions $(\chi_i\,\chi_k)$, $((\chi_i\,\chi_k)\;\chi_j)$, …, in the same sequence that one defines for the φ_i, and denotes the corresponding numbers as $\chi_{m+1}, \ldots, \chi_r$. Therefore, the same relations must exist between $\chi_1, \ldots, \chi_m$ and all $(\chi_i\,\chi_k)$, $(i, k = 1, \ldots, r)$ that exist between the corresponding quantities $\varphi_1, \ldots, \varphi_r$ and all $(\varphi_i\,\varphi_k)$.

§ 7. Other treatments of the theory of function groups. Kantor's generalization of the problem.

Since an r-parameter function group with the basis $u_1, \ldots, u_r$ consists of the totality of all functions of $u_1, \ldots, u_r$, this suggests that instead of defining the group in terms of such a basis, one regards, the $(2n - r)$-parameter complete system whose most general solution is an arbitrary function of $u_1, \ldots, u_r$ as given. The difference between these two viewpoints is precisely the same as when one, on the one hand, operates with the roots of an algebraic equation, while, on the other hand, one regards only the algebraic equation as given. In any event, it seems desirable to also treat the theory of function groups from this new viewpoint.

Lie himself has occasionally assumed this viewpoint. For example, he already showed in 1877 that when an arbitrary complete system is present, one can always present the complete system in such a way that its solutions consist of all functions that are in involution with the solutions of the given complete system. [1]) In particular, when a function group is defined by a complete system, one can then always present a complete system that defines the reciprocal function group. By contrast, Lie did not generally go into the question that he posed in more detail anywhere. S. Kantor first placed himself at the viewpoint of the previously-mentioned papers as a foundation, and defined the function groups through complete systems, and then took that as an excuse to generalize the entire theory in an extraordinary way. We must content ourselves with just a few remarks here.

It is known that there exists a correspondence between systems of Pfaffian equations and systems of linear, homogeneous, partial differential equations of first order. In n variables, any m-parameter system of the first type is conversely associated with an $(n - m)$-parameter of the second type, and indeed this is likewise equivalent to whether the system in question is or is not an integrable or complete system.

In the space $x_1, \ldots, x_n, p_1, \ldots, p_n$, one can add another type of reciprocity to this correspondence that is determined by the bilinear covariant:

$$\sum_{i=1}^{n}(dx_i\,\delta p_i - dp_i\,\delta x_i)$$

of the Pfaffian expression $\sum p_i\,dx_i$, or also through the associated covariant:

[1]) Neue Integrations-Methode der Monge-Ampèreschen Gleichung, Archiv for Math. og Naturvid. Band II, pp. 4.

$$\sum_{i=1}^{n}(v_i u_i' - u_i v_i')$$

in plane coordinates. In fact, let:

(53) $$\sum_{i=1}^{n}(\alpha_{ki}dx_i + \beta_{ki}dp_i) = 0 \qquad (k = 1, \ldots, m)$$

be an m-parameter Pfaffian system, and let:

(54) $$\sum_{i=1}^{n}\left(\rho_{ji}\frac{\partial f}{\partial x_i} + \sigma_{ji}\frac{\partial f}{\partial p_i}\right) = 0 \qquad (j = 1, \ldots, 2n - m)$$

be the associated $(2n - m)$-parameter system of linear, partial differential equations, such that between the function α, β, ρ, σ, there exist the $m(2n - m)$ identities:

(55) $$\sum_{i=1}^{n}(\alpha_{ki}\rho_{ji} + \beta_{ki}\sigma_{ji}) = 0 \qquad (k = 1, \ldots, m, j = 1, \ldots, 2n - m).$$

If one now imagines that the ρ_{ji} , σ_{ji} are transformed like the point coordinates dx_i , dp_i , the α_{ki}, β_{ki}, and the derivatives of f with respect to the x_i and p_i are transformed like the plane coordinates u_i, v_i then one recognizes that the form $\sum(v_i u_i' - u_i v_i')$ of the system (53) is associated with a covariant m-parameter system of linear, partial differential equations:

(53) $$\sum_{i=1}^{n}\left(\beta_{ki}\frac{\partial f}{\partial x_i} - \alpha_{ki}\frac{\partial f}{\partial p_i}\right) = 0 \qquad (k = 1, \ldots, m),$$

and the form $\sum (dx_i\, \delta p_i - dp_i\, \delta x_i)$ of the system (54) is associated with a covariant $(2n - m)$-parameter Pfaffian system:

(54′) $$\sum_{i=1}^{n}(\sigma_{ji}dx_i - \rho_{ji}dp_i) = 0 \qquad (j = 1, \ldots, 2n - m).$$

Likewise, it is clear that one also obtains the system (53′) when one subjects the derivatives of φ in the equation:

$$\sum_{i=1}^{n}\left(\frac{\partial\varphi}{\partial p_i}\frac{\partial f}{\partial x_i} - \frac{\partial\varphi}{\partial x_i}\frac{\partial f}{\partial p_i}\right) = 0$$

to the relations:

(54) $$\sum_{i=1}^{n}\left(\rho_{ji}\frac{\partial\varphi}{\partial x_i} + \sigma_{ji}\frac{\partial\varphi}{\partial p_i}\right) = 0 \qquad (j = 1, \ldots, 2n - m),$$

while still regarding these derivatives as arbitrary. One also sees, in a corresponding way, that the system (54′) emerges from (53) by the use of the equation $\sum (dx_i\ \delta p_i - dp_i\ \delta x_i) = 0$.

That is the general reciprocity between the systems (53) and (54′) and the associated systems (54) and (53′) that S. Kantor first proved.

If the system (54) possesses a solution u such that u is likewise an integral function of the Pfaffian system (53) then there is a multiplier χ_k such that:

$$\sum_{k=1}^{m} \chi_k \sum_{i=1}^{n} (\alpha_{ki} dx_i + \beta_{ki} dp_i) \equiv du.$$

Then, however, one will obviously have:

$$\sum_{k=1}^{m} \chi_k \sum_{i=1}^{n} \left(\beta_{ki} \frac{\partial f}{\partial x_i} - \alpha_{ki} \frac{\partial f}{\partial p_i} \right) \equiv (u\ f);$$

that is, the system (53′) includes the equation $(u\ f) = 0$. Conversely, if (53′) includes an equation of the form $(u\ f) = 0$ then u is an integral function of (53), and therefore a solution of (54). If the system (54) possesses two solutions u_1, u_2 then (53′) includes the two equations $(u_1\ f) = 0$, $(u_2\ f) = 0$, and, when it is, moreover, complete, it includes the equation:

$$(u_1\ (u_2\ f)) - (u_2\ (u_1\ f)) \equiv ((u_1\ u_2\ )\ f) = 0.$$

as well. This comes from the fact that the solutions of (54), in any case, define a function group when the reciprocal system (53′) is complete.

Should the system (54) define an m-parameter function group in particular, then it must have m independent solutions $u_1, \ldots, u_m$, so it must be complete, and, in addition, every $(u_\mu\ ,\ u_\nu)$ $(\mu,\ \nu = 1, \ldots, m)$ must be a solution. Then, however, the m-parameter reciprocal system (53′) includes the m independent equation $(u_\mu\ f) = 0$ $(\mu = 1, \ldots, m)$, and likewise, every equation:

$$((u_\mu\ u_\nu)\ f) = (u_\mu\ (u_\nu\ f)) - (u_\mu\ (u_\nu\ f)) = 0 \qquad (\mu,\ \nu = 1, \ldots, m),$$

so it is likewise complete. One can, however, conclude that (54) defines an m-parameter function group when and only when the reciprocal systems (54) and (53′) are both complete.

These criteria were already found by S. Kantor.

If (54) has a solution u and (53′) has a solution v then (53′) includes the equation $(u\ f) = 0$, and (54) includes the equation $(v\ f) = 0$, and one then has $(u\ v) = 0$. Therefore, (53′) is the system of equations that Lie already taught us to address, whose solutions are all functions that are in involution with (54). Thus, it will generally be assumed that (54) is a $(2n - m)$-parameter complete system.

If the system (54) is complete and defines an m-parameter function group then one can, as Lie showed, determine a canonical basis $X_1, \ldots, X_{l+h}, P_1, \ldots, P_l$ ($2l + k = m$) for this function group, for which the canonical relations:

$$(52) \qquad (X_i\, X_k) = 0, \qquad (P_i\, X_k) = \varepsilon_{ik}\,, \quad (P_i\, P_k) = 0$$

exist. The complete system (53′) that the reciprocal group defines can then take the form:

$$(X_{k+l}\, f) = 0, \qquad (P_{l+h+j}\, f) = 0, \quad (k = 1, \ldots, n - l;\, j = 1, \ldots, n - l - h).$$

S. Kantor generalized this to the case where the system (54) is completely arbitrary. He called two equations:

$$\sum\left(\rho_i \frac{\partial f}{\partial x_i} + \sigma_i \frac{\partial f}{\partial p_i}\right) = 0, \qquad \sum\left(\rho_i' \frac{\partial f}{\partial x_i'} + \sigma_i' \frac{\partial f}{\partial p_i'}\right) = 0$$

conjugate when the covariant $\sum(\rho_i \sigma_i' - \sigma_i \rho_i')$ vanishes. Thus, if the equations (54) are not pair-wise conjugate then one can choose an equation:

$$(56) \qquad \sum\left(\rho_i \frac{\partial f}{\partial x_i} + \sigma_i \frac{\partial f}{\partial p_i}\right) = 0$$

from this system that is not conjugate to all equations of the system, and can then determine an equation:

$$(57) \qquad \sum\left(\rho_i' \frac{\partial f}{\partial x_i} + \sigma_i' \frac{\partial f}{\partial p_i}\right) = 0$$

of the system in such a way that one has $\sum(\rho_i \sigma_i' - \sigma_i \rho_i') = 1$. If one has chosen (56) and (57) in that way then one easily convinces oneself that the system (54) includes precisely $2n - m - 2$ independent equations that are conjugate to (56), as well as (57). If one treats this new system just like the original one (54) and then proceeds, then one ultimately obtains a representation of (54) in the form:

$$(58) \qquad X_1 f = 0, \ldots, X_{l+h}\, f = 0, \;\; P_1\, f = 0, \ldots, P_l\, f = 0 \qquad\qquad (2l + h = 2n - m),$$

where the covariant $\sum(\rho_i \sigma_i' - \sigma_i \rho_i')$ vanishes for any two equations $X_i\, f = 0$ and $X_k\, f = 0$ and any two equations $P_i\, f = 0$ and $P_k\, f = 0$, while for any two equations $X_i\, f = 0$ and $P_k\, f = 0$ they have the value ε_{ik}. With S. Kantor, we call (58) a *canonical basis* for the system (54), and we call $X_{i+l}\, f = 0, \ldots, X_{i+h}\, f = 0$ the *distinguished equations* of the system.

The system (53′), which is reciprocal to (54), consists of all equations that are conjugate to all equations of (54). As a consequence, the distinguished equations $X_{i+l}\, f =$

0, …, $X_{i+h}\,f = 0$ are the only equations that belong to both systems (54) and (53′), and (53′) thus includes a canonical basis:

(59) $X_{i+l}\,f = 0, \ldots, X_n\,f = 0, P_{l+h+1}\,f = 0, \ldots, P_n\,f = 0, \qquad (n - l + n - n - l - h = m)$

in such a way that the $2n - m$ equations:

(60) $X_1\,f = 0, \ldots, X_n\,f = 0, P_1\,f = 0, \ldots, P_l\,f = 0, P_{l+k+1}\,f = 0, \ldots, P_n\,f = 0,$

are independent of each other.

One can ultimately extend the system (60) by the addition of h equations $P_{l+1}\,f = 0$, …, $P_{l+h}\,f = 0$ to a system of $2n$ independent equations:

$$(61) \qquad \begin{cases} X_i\,f = \sum_{\nu=1}^{n}\left(\rho_{i\nu}\frac{\partial f}{\partial x_\nu} + \sigma_{i\nu}\frac{\partial f}{\partial p_\nu}\right) = 0 \\ P_i\,f = \sum_{\nu=1}^{n}\left(\tau_{i\nu}\frac{\partial f}{\partial x_\nu} + \upsilon_{i\nu}\frac{\partial f}{\partial p_\nu}\right) = 0 \end{cases} \qquad (i = 1, \ldots, n),$$

which represents a canonical basis for the $2n$-parameter system:

$$\frac{\partial f}{\partial x_1} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0, \quad \frac{\partial f}{\partial p_1} = 0, \ldots, \frac{\partial f}{\partial p_n} = 0,$$

for which, the following relations then exist:

$$(62) \qquad \begin{cases} \sum_{\nu=1}^{n}(\rho_{i\nu}\sigma_{k\nu} - \sigma_{i\nu}\rho_{k\nu}) = 0, \\ \sum_{\nu=1}^{n}(\tau_{i\nu}\upsilon_{k\nu} - \upsilon_{i\nu}\tau_{k\nu}) = 0, \\ \sum_{\nu=1}^{n}(\rho_{i\nu}\upsilon_{k\nu} - \sigma_{i\nu}\tau_{k\nu}) = \varepsilon_{ik}. \end{cases} \qquad (i, k = 1, \ldots, n).$$

Obviously, one can also apply precisely the same considerations to the Pfaffian systems (53) and (54′) when one calls two Pfaffian equations:

$$\sum_{i=1}^{n}(\alpha_i dx_i + \beta_i dp_i) = 0, \qquad \sum_{i=1}^{n}(\alpha_i' dx_i' + \beta_i' dp_i') = 0$$

conjugate as long as the expression $\sum(\alpha_i\beta_i' - \beta_i\alpha_i')$ vanishes. However, one reaches this conclusion more quickly, and likewise more completely, when one adds $2n$ mutually independent Pfaffian expressions D_i, E_i such that for any function f the equation:

(63) $$\sum_{i=1}^{n}(D_i P_i f - E_i X_i f) = df$$

is satisfied identically to the $2n$ expressions $X_i f$ and $P_i f$ in (61), which is always possible, and in only one way.

Namely, if, in this identity, which is true for all dx_i, dp_i and for all values of the derivatives of f, one sets:

$$\frac{\partial f}{\partial x_\nu} = \sigma_{k\nu}\,, \qquad \frac{\partial f}{\partial p_\nu} = -\rho_{k\nu}\,,$$

then, from (62), all $X_i f$ vanish, and likewise all $P_i f$ will be equal to zero, except for $P_k f$, which takes on the value – 1. This then gives:

(64) $$D_k = \sum_{\nu=1}^{n}(\rho_{k\nu}dp_\nu - \sigma_{k\nu}dx_\nu) \qquad (k = 1, \ldots, n),$$

and, in a corresponding way:

(64′) $$E_k = \sum_{\nu=1}^{n}(\tau_{k\nu}dp_\nu - \upsilon_{k\nu}dx_\nu) \qquad (k = 1, \ldots, n).$$

That is, the D_k and E_k emerge from the $X_k f$ and $P_k f$ when one sets the derivatives f_{x_ν} and f_{p_ν} with respect to the sequence equal to dp_ν and – dx_ν, resp., and by the opposite substitution, one obtains the $X_k f$ and $P_k f$ from the D_k and E_k .

On the one hand, it happens that the $2n$ equations: $D_i = 0$, $E_i = 0$ define a canonical basis for the system: $dx_i = 0$, $dp_i = 0$ ($i = 1, .., n$). Moreover, one finds that when one substitutes the expressions (64), (64′), and (61) in (63), the quantities ρ_{ki} , …, also satisfy:

(62′) $$\begin{cases} \sum_{\nu=1}^{n}(\rho_{\nu i}\tau_{\nu k} - \tau_{\nu i}\rho_{\nu k}) = 0, \\ \sum_{\nu=1}^{n}(\sigma_{\nu i}\upsilon_{\nu k} - \upsilon_{i\nu}\sigma_{\nu k}) = 0, \\ \sum_{\nu=1}^{n}(\rho_{\nu i}\upsilon_{\nu k} - \sigma_{\nu i}\tau_{\nu k}) = \varepsilon_{ik}, \end{cases}$$

which then follow from the relations as long as the $2n$ equations (61) are independent of each other.

If one replaces the dx_ν , dp_ν with $-\varphi_{p_\nu}$ and φ_{x_ν} in (63), and inverts the sign on both sides of the resulting equation then this yields:

(65) $$(\varphi f) \equiv \sum_{i=1}^{n}\{P_i\varphi\, X_i f - X_i\varphi\, P_i f\}\,.$$

On the other hand, if one replaces the f_{x_ν}, f_{p_ν} in (63) with δp_ν, $-\delta x_\nu$, and one denotes the result of writing δ for d in D_k and E_k by Δ_k and E_k then it happens that:

$$(66) \qquad \sum_{i=1}^{n}(dx_i\delta p_i - dp_i\delta x_i) \equiv \sum_{i=1}^{n}(D_i\mathsf{E}_i - E_i\Delta_i).$$

The two systems of expressions $X_i f$, $P_i f$ and D_i, E_i are characterized, as the canonical basis, by the existence of both identities (65) and (66). Namely, since, e.g., (65) is true for arbitrary values of the derivatives of φ and f, one can replace φ_{p_i} with $X_k x_i$ and φ_{x_i} with $-X_k p_i$, with which (φf) is converted into $X_k f$, and since $X_k f$ must likewise appear on the right-hand side of (65), this implies that all $X_i \varphi$ vanish under the substitution that was performed and likewise all $P_i \varphi$, with the exception of $P_k \varphi$, which equals 1. Corresponding statements are true when one replaces φ_{p_i} and φ_{x_i} with $P_k x_i$ and $-P_k p_i$, resp.

The identity (65) shows immediately that the system that is reciprocal to the system (58) consists of the equations (59). Namely, for:

$$(67) \qquad X_k \varphi = 0, \qquad P_i \varphi = 0 \qquad (k = 1, \ldots, l + h,\ i = 1, \ldots, l)$$

(65) is converted into:

$$(\varphi f) = \sum_{i=1}^{n-l} P_{l+i}\varphi \cdot X_{l+i} f - \sum_{i=1}^{n-l-h} X_{l+h+i}\varphi \cdot P_{l+h+i} f,$$

so (59) is the system of equations that one obtains when one demands that (φf) must vanish, while the derivatives of φ are only linked by the relations (67).

On the other hand, one recognizes from (63) that the Pfaffian system that belongs to (58) has the form:

$$(58') \qquad E_{l+h+1} = 0, \ldots, E_n = 0, D_{l+1} = 0, \ldots, D_n = 0,$$

and the Pfaffian system that belongs to (59) has the form:

$$(59') \qquad E_1 = 0, \ldots, E_l = 0, D_1 = 0, \ldots, D_{l+h} = 0;$$

(66), however, shows that the systems (58′) and (59′) are reciprocal to each other relative to the equation $\sum (dx_i\, \delta p_i - dp_i\, \delta x_i) = 0$.

The argument that was developed here concerning reciprocal systems and canonical bases for systems of linear partial differential equations and Pfaffian systems was already found, in essence, by S. Kantor, only his presentation is less clear and, in particular, Kantor did not obtains the identities (63), (65), (66), while it is precisely the first of these that makes all of the relations between the systems seem obvious.

Kantor coupled this with some remarks on a classification of the system (38) under the group of all contact transformations, in which he started with the number of independent solutions that the systems (58) and (59) possess. However, what he accomplished was obviously only a primitive starting point for the invariant theory of an arbitrary system of linear partial differential equations under any group, and since the development of this invariant theory in full generality certainly requires entirely new lemmas, we would not like to go further into the Kantor argument here. Let it be nonetheless mentioned that Lie already concerned himself very early on with the invariant theory of a complete system under the group of all contact transformations: Namely, he carried out investigations into complete systems that are invariant under any group that is linked with a given function group. [1]) It would certainly be profitable to excerpt and present these investigations again.

We now turn our attention to the function groups once more.

Let there be given a $(2n - m)$-parameter complete system in the x, p and let (58) be a canonical basis that is associated with it. From the aforementioned theorem of Kantor, this complete system defines an m-parameter function group if and only if the reciprocal m-parameter system (59) is also complete. However, in the proof of this theorem we employed the solutions of the system (58), as Kantor also did. It is obviously desirable to avoid this, and thus, to prove Kantor's theorem without the use of the solutions. However, in order to do this, we must first derive some general properties of the $2n$-parameter canonical basis (61).

Since the $2n$ equations (61) are mutually independent, they determine relations of the form:

$$(68)\qquad \begin{cases} (X_i X_k)\, f = \sum_{\mu=1}^{n} (a_{ik\mu} X_\mu f + b_{ik\mu} P_\mu f), \\ (X_i P_k)\, f = \sum_{\mu=1}^{n} (a'_{ik\mu} X_\mu f + b'_{ik\mu} P_\mu f), \\ (P_i P_k)\, f = \sum_{\mu=1}^{n} (a''_{ik\mu} X_\mu f + b''_{ik\mu} P_\mu f), \end{cases}$$

where $(X_i\, X_k)f$ is written for $X_i\, X_k\, f - X_k\, X_i\, f$ and the $a_{ik\mu}$, … are certain functions of the x, p.

If one now forms the expression $((\varphi\,\chi)\;\psi)$ by a double application of the identity (65) then one obtains:

$$\begin{aligned} ((\varphi\,\chi)\;\psi) &= \sum_{i\nu}^{1\cdots n} \{P_\nu P_i\varphi \cdot X_i\chi + P_\nu X_i\chi \cdot P_i\varphi - P_\nu P_i\chi \cdot X_i\varphi - P_\nu X_i\varphi \cdot P_i\chi\} X_\nu\psi \\ &\quad - \sum_{i\nu}^{1\cdots n} \{X_\nu P_i\varphi \cdot X_i\chi + X_\nu X_i\chi \cdot P_i\varphi - X_\nu P_i\chi \cdot X_i\varphi - X_\nu X_i\varphi \cdot P_i\chi\} P_\nu\psi\,, \end{aligned}$$

and the Jacobi identity between φ, χ, ψ then delivers the equation:

[1]) "Diskussion aller Integrationsmethoden der partiellen Differentialgleichungen 1. O." Ges. d. Wiss. zu Kristiania 1875, pp. 16, *et seq*. The theorems that this paper contained on the invariant theory of complete systems are only partially excerpted in the great treatise in Bd. XI of the Annalen.

$$\sum \sum_{i\nu}^{1\cdots n}\{(P_\nu P_i)\varphi\cdot X_i\chi\cdot X_\nu\psi+(X_\nu X_i)\varphi\cdot P_i\chi\cdot P_\nu\psi$$
$$-(P_\nu P_i)\ \varphi\cdot P_i\,\chi\cdot X_\nu\,\psi-(X_\nu P_i)\ \varphi\cdot X_i\,\chi\cdot P_\nu\,\psi\} = 0,$$

where the first $\sum$ refers to the cyclic sum over the φ, χ, ψ. If one substitutes the values of the bracket expressions that follow from (68) and considers that the signs of a, b, a'', b'' change when one exchanges the first two indices, which does not need to be the case for the a', b', then after multiplying by 2 one comes to:

$$\begin{aligned}&\sum_{si\nu}^{1\cdots n} a''_{\nu is}\left|X_s\varphi X_i\chi X_\nu\psi\right|\\ &+\sum_{si\nu}^{1\cdots n} b_{\nu is}\left|P_s\varphi P_i\chi P_\nu\psi\right|\\ &+\sum_{si\nu}^{1\cdots n}(a_{\nu is}-b'_{is\nu}+b'_{\nu si})\left|X_s\varphi P_i\chi P_\nu\psi\right|\\ &+\sum_{si\nu}^{1\cdots n}(b''_{\nu is}+a'_{si\nu}-a'_{s\nu i})\left|P_s\varphi X_i\chi X_\nu\psi\right|\equiv 0.\end{aligned}$$

Here, the functions φ, χ, ψ are entirely arbitrary, so due to the independence of equations (61) this identity is therefore equivalent to the following relations:

$$(69)\qquad \begin{cases} b_{\nu is}+b_{is\nu}+b_{si\nu}=0\\ a''_{\nu is}+a''_{is\nu}+a''_{s\nu i}=0\\ a_{\nu is}-b'_{is\nu}+b'_{si\nu}=0\\ b''_{\nu is}+a'_{is\nu}-a'_{si\nu}=0\end{cases}\qquad (\nu,\ i,\ s = 1,\ \ldots,\ n).$$

For each canonical basis $X_\nu f$, $P_\nu f$ are then coupled with the coefficients of the equations (68) by the relations (69); the relations (69) then obviously say nothing more than the fact that the alternating bilinear differential quotient form:

$$(70)\qquad \sum_{i=1}^{n}(P_i\varphi\cdot X_i\chi-P_i\chi\cdot X_i\varphi) = \{\varphi\,\chi\}$$

implies an identity of the form:

$$\{\{\varphi\,\chi\}\,\psi\} + \{\{\chi\,\psi\}\,\varphi\} + \{\{\psi\,\varphi\}\,\chi\} \equiv 0.$$

Now if Zf is an arbitrary infinitesimal transformation in the x, p then the bracket expression $Z(\varphi\,\chi) - (\varphi,\,Z\chi)$ that is formed from the two infinitesimal transformations $Z\,\chi$ and $(\varphi\,\chi)$ contains no derivatives of second order of χ, and the same is true for φ, so the expression:

$$Z(\varphi\,\chi) - (Z\varphi\,,\,\chi) - (\varphi,\,Z\chi)$$

contains only derivatives of first order of χ, as well as φ. If one evaluates this expression with the help of (65) then one finds:

(71) $$\sum_{k=1}^{n}\{(Z\,P_k)\varphi\cdot X_k\chi+(Z\,X_k)\chi\cdot P_k\varphi-(Z\,X_k)\varphi\cdot P_k\chi+(Z\,P_k)\chi\cdot X_k\varphi\},$$

which, from (68), will become:

$$\begin{aligned}
X_i(\varphi\,\chi)-(X_i\,\varphi,\chi)-(\varphi,X_i\,\chi)
&=\sum_{k\mu}^{1\cdots n}a'_{ik\mu}(X_\mu\varphi\cdot X_k\chi-X_k\varphi\cdot X_\mu\chi)\\
&+\sum_{k\mu}^{1\cdots n}(b'_{ik\mu}+a_{ik\mu})(P_\mu\varphi\cdot X_k\chi-X_k\varphi\cdot P_\mu\chi)\\
&+\sum_{k\mu}^{1\cdots n}b_{ik\mu}(P_k\varphi\cdot P_\mu-P_\mu\varphi\cdot P_k\chi)
\end{aligned}$$

$$\begin{aligned}
P_i(\varphi\,\chi)-(P_i\,\varphi,\chi)-(\varphi,P_i\,\chi)
&=\sum_{k\mu}^{1\cdots n}a''_{ik\mu}(X_\mu\varphi\cdot X_k\chi-X_k\varphi\cdot X_\mu\chi)\\
&+\sum_{k\mu}^{1\cdots n}(b''_{ik\mu}-a'_{\mu ik})(X_\mu\varphi\cdot X_k\chi-X_k\varphi\cdot X_\mu\chi)\\
&-\sum_{k\mu}^{1\cdots n}b'_{ik\mu}(P_k\varphi\cdot P_\mu-P_\mu\varphi\cdot P_k\chi),
\end{aligned}$$

which, by employing (69), can be written in the simpler form:

(72) $$\begin{cases}
X_i(\varphi\,\chi)-(X_i\varphi,\chi)-(\varphi,X_i\chi)=-\displaystyle\sum_{k\mu}^{1\cdots n}b''_{k\mu i}X_k\varphi\cdot X_\mu\chi\\
\qquad+\displaystyle\sum_{k\mu}^{1\cdots n}b'_{k\mu i}(P_k\varphi\cdot X_\mu\chi-P_k\chi\cdot X_\mu\varphi)\\
\qquad+\displaystyle\sum_{k\mu}^{1\cdots n}b_{k\mu i}P_k\varphi\cdot P_\mu\chi,\\
P_i(\varphi\,\chi)-(P_i\varphi,\chi)-(\varphi,P_i\chi)=-\displaystyle\sum_{k\mu}^{1\cdots n}a''_{k\mu i}X_k\varphi\cdot X_\mu\chi\\
\qquad-\displaystyle\sum_{k\mu}^{1\cdots n}a'_{k\mu i}(P_k\varphi\cdot X_\mu\chi-P_k\chi\cdot X_\mu\varphi)\\
\qquad+\displaystyle\sum_{k\mu}^{1\cdots n}a_{k\mu i}P_k\varphi\cdot P_\mu\chi.
\end{cases}$$

The formulas (72) tell us how the bracket expression $(\varphi\,\chi)$ behaves when one performs the infinitesimal transformations $X_i f$, $P_i f$ on it. However, we would also like to establish how the Pfaffian expressions D_k, E_k behave under these operations. This leads us to the identity:

$$df \equiv \sum_{k=1}^{n} (D_k P_k f - E_k X_k f)\,. \tag{63}$$

We find, in fact:

$$\begin{aligned} X_i\, df - d\, X_i f &= \sum_{k=1}^{n} (D_k P_k X_i f - E_k X_k X_i f) \\ &= \sum_{k=1}^{n} (X_k D_k \cdot P_i f - X_k E_k \cdot X_i f) + \sum_{k=1}^{n} (D_k \cdot X_k P_i f - E_k \cdot X_i X_k f)\,, \end{aligned}$$

or:

$$\sum_{k=1}^{n} (X_k D_k \cdot P_i f - X_k E_k \cdot X_i f) = -\sum_{k=1}^{n} \{D_k (X_k P_i) f - E_k (X_i X_k) f\}\,,$$

an equation that, due to (68), decomposes into the following ones:

$$\text{(73)} \qquad \begin{cases} X_i D = -\sum_{k=1}^{n} (b'_{iks} D_k - b_{iks} E_k) \\ X_i E_s = \sum_{k=1}^{n} (a'_{iks} D_k - a_{iks} E_k) \end{cases} \qquad (i, s = 1, \ldots, n).$$

In precisely the same way, we obtain:

$$\text{(73}'\text{)} \qquad \begin{cases} P_i D = -\sum_{k=1}^{n} (b''_{iks} D_k + b'_{iks} E_k) \\ P_i E_s = \sum_{k=1}^{n} (a''_{iks} D_k + a'_{iks} E_k) \end{cases} \qquad (i, s = 1, \ldots, n).$$

All of the equations (69), (72), (73), (73′) that were found are true when the $X_i f$, $P_i f$ define a canonical basis and the D_i, E_i are the associated canonical basis of Pfaffian expressions. It would be desirable to know whether the $X_i f$, $P_i f$ could be characterized as a canonical basis by these equations; however, the response to this question does not seem to be so simple.

We can, moreover, go on to the treatment of function groups.

The equations:

$$X_1 f = 0, \ldots, X_{l+h} f = 0,\; P_1 f = 0, \ldots, P_l f = 0 \tag{58}$$

define a complete $(2l + h)$-parameter system. Should this system define a $(2n - 2l - h)$-parameter function group then it would be necessary and sufficient that whenever φ and χ are solutions of (58), $(\varphi\,\chi)$ is also a solution. However, since no linear, homogeneous relations can exist between $X_{l+h+1}\,\varphi, \ldots, X_n\,\varphi$, $P_{l+1}\,\varphi, \ldots, P_n\,\varphi$ if φ is a completely arbitrary solution of the complete system (58), on account of (72), this demand emerges from the equations:

$$(74)\qquad \begin{cases} b''_{k\mu i} = 0 & (k, \mu = l + h + 1, \cdots, n; \quad i = 1, \cdots, l + h) \\ b'_{k\mu i} = 0 & (k = l + 1, \cdots, n; \quad \mu = l + h + 1, \cdots, n; \quad i = 1, \cdots, l + h) \\ b_{k\mu i} = 0 & (k, \mu = l + 1, \cdots, n; \quad i = 1, \cdots, l + h) \\ a''_{k\mu i} = 0 & (k, \mu = l + h + 1, \cdots, n; \quad i = 1, \cdots, l + h) \\ a'_{k\mu i} = 0 & (k = l + 1, \cdots, n; \quad \mu = l + h + 1, \cdots, n; \quad i = 1, \cdots, l + h) \\ a_{k\mu i} = 0 & (k, \mu = l + 1, \cdots, n; \quad i = 1, \cdots, l + h) \end{cases}$$

which express nothing but the fact that the system that is reciprocal to (58):

$$(59)\qquad X_{l+1} = 0, \ldots, X_n f = 0, \qquad P_{l+h+1} f = 0, \ldots, P_n f = 0$$

is a complete $(2n - 2l - h)$-parameter system.

Thus, we again have the Kantor theorem that a complete $(2n - m)$-parameter system in the x, p determines an m-parameter function group if and only if the reciprocal system is also complete. The desired proof, in which the solutions were not employed, is then achieved.

The fact that the complete system (59) defines a function group is also implicit. If φ is a solution of (58) and χ is a solution of (59) then, from (65), one obviously has $(\varphi\chi) \equiv 0$, so the two function groups are mutually reciprocal.

However, our argument proves even more. Namely, if all we know of the system (58) is that its $(2n - 2l - h)$-parameter reciprocal system is complete then we likewise know that the equations (74) are valid, but then from (72), this implies that the expressions:

$$X_i(\varphi\chi) \qquad (i = 1, \ldots, l+h), \qquad P_k(\varphi\chi) \qquad (k = 1, \ldots, l)$$

always vanish when φ and χ are solutions of the system (58). We then obtain Kantor's theorem once more:

If the system (58) *has solutions then the totality of these solutions defines a function group whenever the reciprocal system* (59) *is complete.*

Once again, let the system (59) be complete, so the equations (74) are valid. If we link them with (69) then this yields the following:

From (74), one has, in particular:

$$b'_{ik\mu} = 0 \qquad (i, m + l + 1, \ldots, l + h; k = l + h + 1, \ldots, n),$$

from which, due to the penultimate equation (69), one also has $a_{ik\mu} = 0$ for all of these values of i, k, μ. On the other hand, from (74), one has:

$$b_{k\mu i} = 0 \qquad (k = l + 1, \ldots, n; i = l + 1, \ldots, l + h)$$

and therefore also $b_{ik\mu} = - b_{ki\mu} = 0$. From the first equation (69), it then follows that $b_{\mu ik} = 0$ for the values of μ, i, k in question. In this, and when one recalls (74), moreover, one sees that the distinguished equations:

$$X_{l+1} f = 0, \ldots, X_{l+h} f = 0 \tag{75}$$

of the system (59) define an h-parameter complete system, in their own right.

For that reason, we are only starting from the reciprocal system (59), because we have the conditions (74) for the completeness of this system at hand. Now, since this reciprocal system is just as general as the system (58), one obviously has the theorem:

If a complete system in the x, p includes distinguished equations then the totality of them again defines a complete system.

If the system (58) and (59) are both complete, so they define two reciprocal function groups, then, from the theorem that was just proved, equations (75) likewise define a complete system; this is actually self-explanatory here, since (75) determines the totality of all equations that are common to two complete systems. This system (75) possesses $2n - h$ independent solutions, and on the other hand, will be satisfied by all solutions of (58) and all solutions of (59). One may show that it possesses no other solutions, so all of its solutions are expressible in terms of (58) and (59).

One sees this most quickly when one goes over to the Pfaffian systems that correspond to our complete systems. In fact, the complete systems (58) and (59) correspond to the two unrestricted, integrable Pfaffian systems:

$$D_{l+1} = 0, \ldots, D_n = 0, \quad E_{l+h+1} = 0, \ldots, E_n = 0, \tag{58$'$}$$

and:

$$D_1 = 0, \ldots, D_{l+h} = 0, \quad E_1 = 0, \ldots, E_l = 0, \tag{59$'$}$$

and the integral functions of (58′) are, for example, the solutions of (58) – that is, the functions of the function group that is defined by (58). On the other hand, the complete system (75) corresponds to the unrestricted, integrable Pfaffian system:

$$\begin{cases} D_1 = 0, & \cdots \quad D_n = 0, \\ E_1 = 0, & \cdots \quad E_l = 0, \quad E_{l+h+1} = 0, \quad \cdots \quad E_n = 0, \end{cases} \tag{75$'$}$$

that arises from the union of (58′) and (59′). Now since, of the $2n - 2l - h + (2l + h) = 2n$ equations (58′) and (59′), precisely $2n - h$ mutually independent ones are present – namely, the equations (75′) – so it is clear that the system that consists of the solutions of (58) and (59) contains precisely $2n - h$ mutually independent functions, and therefore exactly as many as there are mutually independent solutions of (75).

The solutions of (75) are then all of the functions that can be expressed in terms of the functions of the two function groups that are defined by (58) and (59). However, the totality of these functions, in turn, obviously defines a function group so this implies that the complete system (75), in turn, defines a $(2n - h)$-parameter function group. From this, it finally follows that the system reciprocal to (75):

$$(76) \qquad \begin{cases} X_1 f = 0, & \cdots \quad X_n f = 0, \\ P_1 f = 0, & \cdots \quad P_l f = 0, \quad P_{l+h+1} f = 0, \quad \cdots \quad P_n f = 0 \end{cases}$$

is complete and, in turn, defines a function group.

(75) is the smallest function group that includes both of our reciprocal function groups (58) and (59), and (76) is the function group that consists of all of the functions that are common to each reciprocal function group; otherwise expressed: (76) consists of the distinguished functions of each of the two reciprocal function groups. The solutions of (76) are therefore pair-wise in involution.

The fact that (76) is also a complete system has only the consequence that a complete sequence of the coefficients $a_{ik\mu}$, … in (68) vanishes. I managed to prove the vanishing of these coefficients in yet another manner from the one described. Namely, it also follows from equations (69) and the conditions for the completeness of the systems (58) and (59).

In order to bring the theory of function groups in the new treatment to a complete conclusion, we must still show that two complete systems that define function groups with an equal number of parameters, and which likewise include equally many equations, can always be converted into each other by a contact transformation in the x, p. We would therefore not like to go into this examination, but only add some remarks on the invariant theory of arbitrary complete systems.

If the system (58) is complete, while its solutions do not, however, define a function group, then one can define a complete system that is covariant under all contact transformations in the x, p as follows: One seeks the complete system that has for its solutions, first, all solutions of (58) and second, all expressions $(\varphi\,\chi)$, where φ and χ are arbitrary solutions of (58).

Any equation of the new system belongs to the system (58) and thus has the form:

$$Z f = \sum_{i=1}^{l+h} \alpha_i X_i f + \sum_{k=1}^{l} \beta_i P_i f = 0.$$

We now need only to determine the functions α_i, β_i in the most general way such that the expression defined with the help of (72):

$$Z(\varphi\,\chi) - (Z\varphi, \chi) - (\varphi, Z\chi)$$

always vanishes identically, as long as φ and χ are completely arbitrary solutions of (58). That gives a number of linear, homogeneous equations for the α_i, β_i , so the extended complete system can always be constructed. If no other function groups are defined then one can treat it similarly, and thus ultimately arrive at the complete system that is defined by the function group that is generated by the solutions of (58). The associated reciprocal system subsumes the reciprocal system (59) to (58), and is the smallest complete system of the form $(u_\mu f) = 0$ ($\mu = 1, \ldots, r$) in which (59) is included.

In the treatise "Neue Grundlagen, etc." Wiener Berichte, Bd. CXII, Abt. IIa, pp. 782, Kantor spoke of the largest function group that contains the totality of all solutions of a complete system. It is then in no way certain that such a largest function group always exists.

Of especial interest are the complete systems whose solutions represent a family of unions when equal to arbitrary constants. Should (58) be such a complete system, it is necessary and sufficient that the associated Pfaffian system (58′), together with the system:

(58″) $$\Delta_{l+1} = 0, \ldots, \Delta_n = 0, \qquad \mathrm{E}_{l+h+1} = 0, \ldots, \mathrm{E}_n = 0,$$

make the bilinear covariants $\sum (dx_i\, \delta p_i - dp_i\, \delta x_i)$ vanish. Due to (66), however, one will have, by means of (58′) and (58″):

$$\sum_{i=1}^{n} (dx_i \delta p_i - dp_i \delta x_i) = \sum_{i=1}^{l} (D_i \mathrm{E}_i - E_i \Delta_i),$$

which vanishes only when $l = 0$. The general form for a complete system of the given character is then:

(77) $$X_1 f = 0, \ldots, X_h f = 0 \qquad (h \leq n),$$

in which we understand $X_1 f, \ldots, X_n f, P_1 f, \ldots, P_n f$ to be a particular canonical basis.

The associated unconstrained integrable Pfaffian system has the form:

(77′) $$D_1 = 0, \ldots, D_n = 0, \qquad E_{h+1} = 0, \ldots, E_n = 0,$$

and its reciprocal system is:

(78) $$D_1 = 0, \ldots, D_h = 0.$$

Thus, (78) will be integrable if and only if the solutions of (77) define a function group, as well.

In the invariant theory of the family of unions defined by (77), as is self-explanatory, the Pfaffian system (78) plays a significant role.

Since our family of unions can also be defined by the Pfaffian system (77′), it must be possible to determine certain functions α_i , β_k such that there exists an identity of the form:

(79) $$\sum_{i=1}^{n} p_i dx_i = \sum_{i=1}^{n} \alpha_i D_i + \sum_{k=1}^{n-h} \beta_k E_{h+k} + d\omega.$$

If one replaces the arbitrary differentials dx_i , dp_i here with the increment that x_i and p_i experience under $X_\nu f$ and $P_\nu f$ then one finds:

(80)
$$\begin{cases} \sum_{i=1}^{n} p_i \rho_{\mu i} = X_\mu \omega & (\mu = 1, \cdots, h), \\ \sum_{i=1}^{n} p_i \rho_{h+k,i} = -\beta_k + X_{h+k}\omega & (k = 1, \cdots, n-h), \\ \sum_{i=1}^{n} p_i \tau_{\nu i} = \alpha_\nu + P_\nu \omega & (\nu = 1, \cdots, n). \end{cases}$$

Thus, ω is to be determined from the equations:

(81)
$$X_\mu \, \omega = \sum_{i=1}^{n} p_i \rho_{\mu i} \qquad (\mu = 1, \ldots, h)$$

that emerge from a complete h-parameter system in the $2n + 1$ variables x_i, p_i, ω. If ω is determined then the α_ν, β_ν are known with no further assumptions. If one knows the solutions of (77) then one finds ω by a quadrature.

The theorem that was proved on pp. ? is therewith derived in a new way.

Actually, we would now have to treat the homogeneous contact transformations and their invariants, but we would like to forego that, and turn to another generalization of the invariant theory of contact transformations.

§ 8. The invariant theory of contact transformations, as carried over to Pfaffian expressions in 2*n* variables.

It was known to Lie that for any Pfaffian expression in 2*n* variables:

(82)
$$\sum_{i=1}^{2n} \alpha_1(x_1, \cdots, x_{2n}) \, dx_i \ ,$$

which can take on the normal form $p_1 \, dx_1 + \ldots + p_n \, dx_n$, an entirely similar theory of invariants can be developed, but he did not arrive at the actual demonstration of this. S. Kantor has indeed already treated this generalization, so it seems appropriate for me to briefly present it.

First, let the Pfaffian expression (82) be completely arbitrary and let:

(83)
$$\sum_{i,\nu}^{1\cdots 2n} \alpha_{i\nu} dx_i \, \delta x_\nu$$

be its bilinear covariant, where one sets:

(84)
$$\alpha_{i\nu} = \frac{\partial \alpha_i}{\partial x_\nu} - \frac{\partial \alpha_\nu}{\partial x_i} .$$

We further employ the notations that Jacobi introduced, when we understand $i_1, \ldots, i_{2m}$ to mean any numbers from the sequence 1, …, $2n$, namely, $\alpha_{i\nu} = (i\,\nu)$ and the symbol $(i_1, \ldots, i_{2m})$, which is generally defined as:

(85)
$$\begin{cases} (i_1 \cdots i_{2m}) = \dfrac{1}{(2m)!}\sum \pm (i_1 i_2)(i_3 i_4)\cdots(i_{2n-1} i_{2n}), \\ \qquad\qquad = \displaystyle\sum_{\nu=2}^{2n} (i_1 i_\nu)(i_{\nu+1} \cdots i_{2m} i_2 \cdots i_{\nu-1}); \end{cases}$$

therefore, the sum $\sum \pm$ is defined in such a way that one permutes $i_1, \ldots, i_{2m}$ in all possible ways and gives each even permutation the + sign, while each odd one gets the – sign. In particular, we would like to set:

(86)
$$(1, 2, \ldots, 2n) = A$$

and would like to define the quantities $A_{i\nu}$ through the formulas:

$$A = \sum_{\nu=1}^{2n} \alpha_{i\nu} A_{i\nu} \qquad (i = 1, \ldots, 2n)$$

such that $A_{i\nu} + A_{\nu i} = 0$ and one has, in general:

(87)
$$\sum_{\nu=1}^{2n} \alpha_{i\nu} A_{k\nu} = \sum_{\nu=1}^{2n} \alpha_{\nu i} A_{\nu k} = \varepsilon_{ik} A. \qquad (i, k = 1, \ldots, 2n).$$

For $i < \nu$, one then has, in particular:

$$A_{i\nu} = (-1)^{i+\nu-1} \qquad (1, \ldots, i-1, i+1, \ldots, \nu-1, \nu+1, \ldots, 2n)$$

We then remark that the expression that is defined from A_{ik} in precisely the same as A is defined from the α_{ik} has the value A^{n-1}.

We then inquire about the integral manifolds of (82), and thus, the manifolds on which (82) becomes a complete differential, and vanishes identically as a result of (83).

If x_i is a point of such a manifold and $x_i + dx_i$ an infinitely neighboring point then any other infinitely neighboring point must satisfy the equation:

(88)
$$\sum_{\nu=1}^{2n} \left\{ \sum_{i=1}^{2n} \alpha_{i\nu} dx_i \right\} \delta x_\nu = 0.$$

Therefore, if the dx_i satisfy the equations:

(89)
$$\sum_{i=1}^{2n} \alpha_{i\nu} dx_i = 0 \qquad (\nu = 1, \ldots, n)$$

then this yields no condition on the δx_ν. However, since:

$$\sum_{\nu=1}^{2n} A_{k\nu} \sum_{i=1}^{2n} \alpha_{i\nu} dx_i \equiv A\, dx_k ,$$

equations (89) can be true without all of the dx_i vanishing only when $A = 0$. On the other hand, if (82) can be given the normal form $p_1\, dx_1 + \ldots + p_n\, dx_n$ then the bilinear covariant $\sum (dx_i\, \delta p_i - dp_i\, \delta x_i)$ will vanish for arbitrary δx_i, δp_i, only when all dx_i, dp_i are set equal to zero, so it is clear that this normal form is certainly not possible for $A \equiv 0$.

Therefore, let $A \not\equiv 0$, moreover, so we would like to restrict ourselves to those integral manifolds of (82) on which A generally possesses a value that is different from zero.

Such a manifold includes, in addition to the points x_i at which A does not vanish, also m infinitely neighboring points $x_i + dx_i$ that do not belong to any flat $(m - 1)$-fold extended manifold through x_i, in such a way that not all of the m-rowed determinants in the matrix:

(90) $$| d_k x_1 , \ldots, d_k x_{2n} | \qquad (k = 1, \ldots, m)$$

vanish. Then, one first obtains the equations:

(91) $$\sum_{\nu=1}^{2n} \left\{ \sum_{i=1}^{2n} \alpha_{i\nu} d_k x_i \right\} d_j x_\nu = 0 \qquad (k, j = 1, \ldots, m),$$

and then any other point $x_i + \delta x_i$ that is infinitely close to the point x_i of the manifold must satisfy the m equations:

(92) $$\sum_{\nu=1}^{2n} \left\{ \sum_{i=1}^{2n} \alpha_{i\nu} d_k x_i \right\} \delta x_\nu = 0 \qquad (k = 1, \ldots, m).$$

However, if we set:

$$\sum_{i=1}^{n} \alpha_{i\nu} d_k x_i = d_k u_\nu \qquad (n = 1, \ldots, 2n)$$

here then these equations can be solved for $d_k x_1, \ldots, d_k x_\nu$, and since not all m-rowed determinants of the matrix (90) vanish, certainly not all m-rowed determinants of the matrix:

$$| d_k u_1, \ldots, d_k u_{2n} | \qquad (k = 1, \ldots, m)$$

vanish either. As a consequence, the m equations (92) for the dx_i are independent of each other, and possess exactly $2n - m$ linearly independent solutions. From (91), however, they already possess m of them, so one has $m \leq 2n - m$ and $m \leq n$. That is, the Pfaffian expression (82) certainly possesses no integral manifold of dimension greater than n that makes A vanish.

Now, let:

(93) $$F_\nu(x_1, \ldots, x_{2n}) = 0 \qquad (n = 1, \ldots, n)$$

be an n-fold extended integral manifold of (82) that does not make A vanish, if such a thing is possible. Every system of values x_i, dx_i, δx_i that satisfies the equations: $F_\nu = 0$, $dF_\nu = 0$, $\delta F_\nu = 0$ must also satisfy (88).

However, if $d_k x_i$ ($k = 1, \ldots, n$) are n linearly independent systems of values that satisfy the equations $dF_1 = 0, \ldots, dF_n = 0$ then the n equations:

$$\sum_{\nu=1}^{2n} \left\{ \sum_{i=1}^{2n} \alpha_{i\nu} d_k x_i \right\} \delta x_\nu = 0 \qquad (k = 1, \ldots, m)$$

are linearly independent and therefore must define a system that is equivalent to the system of n equations $\delta F_1 = 0, \ldots, \delta F_n = 0$. From this, for any system of values $dx_1 = 0, \ldots, dx_n = 0$ that fulfills $dF_1 = 0, \ldots, dF_n = 0$ there are n multipliers $\lambda_1, \ldots, \lambda_m$ such that:

(94) $$\sum_{i=1}^{2n} \alpha_{i\nu} dx_i = \sum_{\mu=1}^{n} \lambda_\mu \frac{\partial F_\mu}{\partial x_\nu} dt \qquad (\nu = 1, \ldots, 2n),$$

and since these equations are soluble for the dx_i :

(95) $$dx_k = \sum_{\mu=1}^{n} \lambda_\mu \sum_{\nu=1}^{2n} \frac{A_{k\nu}}{A} \frac{\partial F_\mu}{\partial x_\nu} dt \qquad (k = 1, \ldots, 2n),$$

it is clear that when one regards the λ_μ as parameters (95) represents the most general system of values dx_k that satisfies $dF_1 = 0, \ldots, dF_n = 0$. However, from this it follows that since the λ_μ are arbitrary:

(96) $$\sum_{k,\nu=1}^{2n} \frac{A_{k\nu}}{A} \frac{\partial F_i}{\partial x_k} \frac{\partial F_\mu}{\partial x_\nu} = 0 \qquad (i, \mu = 1, \ldots, n).$$

These equations must then be a consequence of (93) if (93) is to be an n-fold extended integral manifold of (82).

Conversely, if (96) is a consequence of (93) then obviously for arbitrary λ_μ (95) represents a system of values that satisfies $dF_1 = 0, \ldots, dF_n = 0$, and indeed, the most general system of values of this type, then conversely (94) follows from (95), and from equations (94), there follow exactly n, and no more, equations that are free of $\lambda_1, \ldots, \lambda_\nu$, if, as is self-explanatory, we assume that not all of the n-rowed determinants in the matrix of derivatives of $F_1, \ldots, F_n$ with respect to $x_1, \ldots, x_{2n}$ vanish by means of (93). From (95), one finally obtains:

$$\begin{aligned}\sum_{k,\pi=1}^{2n} \alpha_{k\pi} dx_k \delta x_\pi &= \sum_{\mu=1}^{n} \lambda_\mu \sum_{\nu=1}^{2n} \sum_{k,\pi=1}^{2n} \frac{A_{k\nu}\alpha_{k\pi}}{A} \frac{\partial F_\mu}{\partial x_\nu} \delta x_\pi dt \\ &= \sum_{\mu=1}^{n} \lambda_\mu \sum_{\nu=1}^{2n} \frac{\partial F_\mu}{\partial x_\nu} \delta x_\nu dt ,\end{aligned}$$

which vanishes for all systems of values δx_ν that satisfy $\delta F_1 = 0, \ldots, \delta F_n = 0$.

Should the n mutually independent equations (93), *by means of which, A does not vanish, represent an integral manifold of* (82) *then it is necessary and sufficient that the equations* (96) *should follow from them.*

In particular, should the n mutually independent equations:

(97) $$F_\nu(x_1, \ldots, x_{2n}) = a_\nu \qquad (n = 1, \ldots, n)$$

represent integral manifolds of (82) *for arbitrary values of the constants* a_ν *then it is necessary and sufficient that the expressions:*

$$\sum_{k,\nu=1}^{n} \frac{A_{k\nu}}{A} \frac{\partial F_i}{\partial x_k} \frac{\partial F_\mu}{\partial x_\nu} \qquad (i, m = 1, \ldots, n)$$

vanish identically.

In this, we freely admit that the existence of such n-fold extended integral manifolds has still not been shown.

Therefore, let:

(98) $$x_i' = \Phi_i(x_1, \ldots, x_{2n}) \qquad (i = 1, \ldots, n)$$

be a transformation that leaves our Pfaffian expression $\sum \alpha_i \, dx_i$ invariant, up to an additive complete differential, such that, by means of (98), an equation of the form:

(99) $$\sum_{i=1}^{2n} \alpha_i' dx_i' = \sum_{i=1}^{2n} \alpha_i dx_i + d\omega(x_1, \cdots, x_{2n}) .$$

Then, from § 1, one likewise has:

(100) $$\sum_{i,\nu=1}^{2n} \alpha_{i\nu}(\Phi_1, \cdots, \Phi_{2n}) \, d\Phi_i \delta\Phi_\nu \equiv \sum_{i,\nu=1}^{2n} \alpha_{i\nu} dx_i \delta x_\nu$$

for all values of the dx_i, δx_i .

If one regards the dx_i , δx_ν in the equation $\sum \alpha_{i\nu} \, dx_i \, \delta x_\nu = 0$ as homogeneous point coordinates of an R_{2n-1} then one has a duality that associates each point dx_i with a $(2n-2)$-fold extended plane in R_{2n-1} . If one considers the u_i as plane coordinates in the equation $\sum u_i \, \delta x_i = 0$ then one has equations of the form:

(101) $$u_\nu \, dt = \sum_{i,\nu=1}^{2n} \alpha_{i\nu} dx_i$$

for the transition from planes to point coordinates, or, when solved:

(101′) $$dx_\mu = \sum_{\nu=1}^{2n} \frac{A_{\mu\nu}}{A} u_\nu dt .$$

From this, it follows that for any function Φ:

(102) $$d\Phi = \sum_{\mu,\nu=1}^{2n} \frac{A_{\mu\nu}}{A} \frac{\partial \Phi}{\partial x_\mu} u_\nu dt ,$$

and, in addition:

(103) $$\sum_{i,\nu=1}^{2n} \alpha_{i\nu} dx_i \delta x_\nu = \sum_{\nu=1}^{2n} u_\nu \delta x_\nu dt ,$$

or, when sets:

$$\delta x_\mu = \sum_{\nu=1}^{2n} \frac{A_{\mu\nu}}{A} v_\nu \delta t ,$$

one has:

(104) $$\sum_{i,\nu=1}^{2n} \alpha_{i\nu} dx_i \delta x_\nu = \sum_{\nu=1}^{2n} \frac{A_{\mu\nu}}{A} u_\nu u_\nu dt \, \delta t .$$

From the identity (100), it now follows that:

(105) $$\sum_{i,\nu=1}^{2n} \alpha_{i\nu} (\Phi_1, \cdots, \Phi_{2n}) \sum_{k,j=1}^{2n} \frac{A_{kj}}{A} \frac{\partial \Phi_i}{\partial x_k} u_j \delta \Phi_\nu \equiv \sum_{\nu=1}^{2n} u_\nu \delta x_\nu ,$$

for arbitrary u_ν and δx_i, although, in addition:

(106) $$\sum_{i,\nu=1}^{2n} \alpha_{i\nu} (\Phi_1, \cdots, \Phi_{2n}) \sum_{k,j=1}^{2n} \frac{A_{kj}}{A} \frac{\partial \Phi_i}{\partial x_k} u_j \sum_{\lambda,\tau=1}^{2n} \frac{A_{\lambda\tau}}{A} \frac{\partial \Phi_\nu}{\partial x_\lambda} v_\tau \equiv \sum_{\mu,\nu=1}^{2n} \frac{A_{\mu\nu}}{A} u_\mu v_\nu ,$$

for arbitrary u_μ and v_μ . Conversely, the validity of (106) implies (105) and (100).

If one sets the u and v in (105) and (106) equal to the derivatives of two arbitrary functions of $x_1, \ldots, x_{2n}$, and employs the abbreviation:

(107) $$\sum_{i,\nu=1}^{2n} \frac{A_{i\nu}}{A} \frac{\partial \varphi}{\partial x_i} \frac{\partial \chi}{\partial x_\nu} = | \varphi\chi |$$

then one obtains the following identities from (105) and (106):

(105′) $$\sum_{i,\nu=1}^{2n} \alpha_{i\nu} (\Phi_1 \cdots \Phi_{2n}) | \Phi_i \varphi | \delta \Phi_\nu \equiv \delta\varphi$$

(106′) $$\sum_{i,\nu=1}^{2n} \alpha_{i\nu} (\Phi_1 \cdots \Phi_{2n}) | \Phi_i \varphi | \Phi_\nu \chi \equiv | \varphi\chi | .$$

Conversely, if (106′) is true for arbitrary φ, χ then (106) is true for arbitrary u_μ, v_μ, and therefore (100) is also true for arbitrary dx_i , δx_i .

If one sets $\varphi = \Phi_k$ in (105′) and considers that $\Phi_1, \ldots, \Phi_{2n}$ are independent functions then this yields:

(108) $$\sum_{i,\nu=1}^{2n} \alpha_{i\nu}(\Phi_1 \cdots \Phi_{2n}) \,|\, \Phi_i \Phi_\nu \,| \equiv \varepsilon_{\nu k} \qquad (\nu, k = 1, \ldots, 2n)$$

from which:

(109) $$|\, \Phi_i\, \Phi_k \,| = \frac{A_{ik}(\Phi_1, \cdots, \Phi_{2n})}{A(\Phi_1, \cdots, \Phi_{2n})} \qquad (i, k = 1, \ldots, 2n),$$

where the denominator on the right certainly does not vanish identically. Finally, if one thinks of φ and χ in (106′) as being expressed in terms of $\Phi_1, \ldots, \Phi_{2n}$ then it becomes:

$$|\, \Phi_i\, \varphi \,| = \sum_{\nu=1}^{2n} |\, \Phi_i \Phi_k \,| \frac{\partial \varphi}{\partial \Phi_\nu} = \sum_{\nu=1}^{2n} \frac{A_{i\nu}(\Phi)}{A(\Phi)} \frac{\partial \varphi}{\partial \Phi_\nu},$$

from which:

(106″) $$|\, \varphi\, \chi \,| \equiv \sum_{i,\nu=1}^{2n} \frac{A_{i\nu}(\Phi_1 \cdots \Phi_{2n})}{A(\Phi_1 \cdots \Phi_{2n})} \frac{\partial \varphi}{\partial \Phi_i} \frac{\partial \chi}{\partial \Phi_\nu}.$$

That is, under the transformation (98), there exists the equation:

(110) $$|\, \varphi\, \chi \,|_x = |\, \varphi\, \chi \,|_{x'},$$

which states that the bracket symbol $|\, \varphi\, \chi \,|$ remains invariant.

Ultimately, if one sets the expressions (101′) in place of the dx_ν and the derivatives of Φ_k for the u_ν in the identity:

$$\sum_{i=1}^{2n} \alpha_i(\Phi_1, \cdots, \Phi_{2n})\, d\Phi_i \equiv \sum_{i=1}^{2n} \alpha_i\, dx_i + d\omega$$

then one obtains the $2n$ identities:

$$|\, \omega \Phi_k \,| \equiv \sum_{i=1}^{2n} \alpha_i(\Phi_1, \cdots, \Phi_{2n}) |\, \Phi_i \Phi_k \,| - \sum_{i,\nu=1}^{2n} \frac{\alpha_i A_{i\nu}}{A} \frac{\partial \Phi_k}{\partial x_\nu}$$

or:

(111) $$|\, \omega \Phi_k \,| = \sum_{i=1}^{2n} \alpha_i(\Phi) \frac{A_{ik}(\Phi)}{A(\Phi)} - \sum_{i,\nu=1}^{2n} \frac{\alpha_i(x) A_{i\nu}(x)}{A(x)} \frac{\partial \Phi_k}{\partial x_\nu} \qquad (k = 1, \ldots, 2n)$$

which determine the $2n$ derivatives of ω.

Conversely, now let $2n$ functions $\Phi_1, \ldots, \Phi_{2n}$ be given, of which we assume that they satisfy relations of the form (109), although it is self-explanatory that $A(\Phi_1, \ldots, \Phi_{2n})$ vanishes identically. The functions $A(x)$ and $A_{ik}(x)$ shall thus be the expressions that we just now derived from the $\alpha_{ik}(x)$.

We define the expression:

$$\frac{1}{(2n)!}\sum \pm \mid \Phi_1\,\Phi_2 \mid \mid \Phi_3\,\Phi_4 \mid \ldots \mid \Phi_{2n-1}\,\Phi_{2n} \mid,$$

in which we think of the numbers 1, …, $2n$ as being permuted in all possible ways and the ± as being chosen in the well-known way. This becomes:

$$\sum_{\mu_1\cdots\mu_{2n}=1}^{2n} \frac{1}{(2n)!\,A^n} A_{\mu_1\mu_2} A_{\mu_3\mu_4}\cdots A_{\mu_{2n-1}\mu_{2n}} \sum \pm \frac{\partial\Phi_1}{\partial x_{\mu_1}}\frac{\partial\Phi_2}{\partial x_{\mu_2}}\cdots\frac{\partial\Phi_{2n-1}}{\partial x_{\mu_{2n-1}}}\frac{\partial\Phi_{2n}}{\partial x_{\mu_{2n}}}.$$

Here, the only terms that remain in the inner sum $\sum \pm$ are the ones in which $\mu_1, \ldots, \mu_{2n}$ take on all of the numbers 1, …, $2n$, so we can write our expression:

$$\begin{pmatrix}\Phi_1\cdots\Phi_{2n}\\ x_1\cdots x_{2n}\end{pmatrix}\cdot\frac{1}{(2n)!\,A^n}\sum \pm A_{12}A_{34}\cdots A_{2n-1,2n}\ .$$

On other hand, due to (109) our expression will be equal to:

$$\frac{1}{(2n)!}\frac{\sum \pm A_{12}(\Phi)\cdots A_{2n-1,2n}(\Phi)}{[A(\Phi)]^n}.$$

However, as we mentioned on pp. ?, the expression:

$$\frac{1}{(2n)!}\sum \pm A_{12}A_{34}\cdots A_{2n-1,2n}$$

has the value A^{n-1}, so this finally yields:

(112) $$\begin{pmatrix}\Phi_1\cdots\Phi_{2n}\\ x_1\cdots x_{2n}\end{pmatrix} = \frac{A(x_1\cdots x_{2n})}{A(\Phi_1\cdots\Phi_{2n})}.$$

With that, it is proved that $2n$ functions $\Phi_1, \ldots, \Phi_{2n}$ that satisfy the equations (109) and do not make $A(\Phi_1, \ldots, \Phi_{2n})$ vanish are always independent of each other [1]), such that the equations:

(113) $$x_i' = \Phi_i(x_1, \ldots, x_{2n}) \qquad (i = 1, \ldots, n)$$

then represent a transformation.

Moreover, it follows from (109) that for two arbitrary functions φ, χ of $x_1, \ldots, x_{2n}$:

(114) $$\mid \varphi\,\chi \mid = \sum_{i,k=1}^{2n} \frac{A_{ik}(\Phi_1\cdots\Phi_{2n})}{A(\Phi_1\cdots\Phi_{2n})}\frac{\partial\varphi}{\partial\Phi_i}\frac{\partial\chi}{\partial\Phi_\nu};$$

[1]) The argument that was just carried out shows that the functional determinant that was considered on pp. ? has the value + 1.

that is, our transformation leaves the expression $|\varphi\chi|$ invariant. In addition, it follows that:

(108)
$$\sum_{i=1}^{2n} \alpha_{i\nu}(\Phi_1 \cdots \Phi_{2n}) \, | \Phi_i \Phi_k | \equiv \varepsilon_{\nu k} \, ,$$

and thus:

$$\sum_{i=1}^{2n} \alpha_{i\nu}(\Phi_1 \cdots \Phi_{2n}) \, | \Phi_i \Phi_k | \equiv \frac{\partial \chi}{\partial \Phi_\nu} \, ,$$

which yields:

$$|\varphi\chi| \equiv \sum_{\mu,\nu=1}^{2n} \frac{A_{\mu\nu}(\Phi)}{A(\Phi)} \frac{\partial \varphi}{\partial \Phi_\mu} \sum_{i=1}^{2n} \alpha_{i\nu}(\Phi) \, | \Phi_i \chi | ,$$

that is:

$$|\varphi\chi| \equiv \sum_{\mu=1}^{2n} \frac{\partial \varphi}{\partial \Phi_\mu} | \Phi_\mu \chi | ,$$

and therefore:

(106′)
$$|\varphi\chi| \equiv \sum_{\mu,\nu=1}^{2n} \alpha_{i\nu}(\Phi) \, | \Phi_\mu \varphi | | \Phi_\nu \chi | .$$

From this it follows, as we recently remarked, that:

$$\sum_{i=1}^{2n} \alpha_{i\nu}(x) \, dx_i \delta x_\nu \equiv \sum_{i=1}^{2n} \alpha_i(x) \, dx_i + d\omega(x),$$

where ω satisfies equations (111).

The equations $x_i' = \Phi_i(x)$ *thus represent a transformation that leaves the Pfaffian expression* $\sum \alpha_i \, dx_i$ *invariant, up to a complete differential, when and only when* $A(\Phi_1, \ldots, \Phi_{2n}) \not\equiv 0$ *and the equations* (109) *are valid.*

We can also characterize these transformations as the ones that leave the expression $|\varphi\chi|$ invariant.

We still need to determine the infinitesimal transformations of the type that are considered here.

If:

$$Xf = \sum_{i=1}^{2n} \xi_i \frac{\partial f}{\partial x_i}$$

is an infinitesimal transformation for which there exists an identity of the form:

$$X\left(\sum_{i=1}^{2n} \alpha_i dx_i\right) = du(x_1, \ldots, x_{2n})$$

then one must have:

$$\sum_{i=1}^{2n} X\alpha_i dx_i + \sum_{i=1}^{2n} \alpha_i d\xi_i = du,$$

so:

$$\sum_{i=1}^{2n} (X\alpha_i dx_i - \xi_i d\alpha_i) = d\left(u - \sum_{i=1}^{2n} \alpha_i \xi_i\right),$$

or:

$$\sum_{i,\nu=1}^{2n} \alpha_{i\nu} \xi_\nu dx_i = d\left(u - \sum_{i=1}^{2n} \alpha_i \xi_i\right).$$

If we then set:

(115) $$u - \sum_{i=1}^{2n} \alpha_i \xi_i = U$$

then this yields:

$$\sum_{\nu=1}^{2n} \alpha_{i\nu} \xi_\nu = \frac{\partial U}{\partial x_i},$$

in which:

$$\xi_\mu = \sum_{i=1}^{2n} \frac{A_{i\mu}}{A} \frac{\partial U}{\partial x_i}$$

and:

(116) $$Xf = |\, Uf\,|,$$

where U remains completely arbitrary, and where:

(117) $$X\sum_{i=1}^{2n} \alpha_i dx_i = d\left(U + \sum_{i,\mu=1}^{2n} \frac{A_{i\mu}}{A} \frac{\partial U}{\partial x_i} \alpha_\mu\right).$$

We call the function U the *characteristic* of the infinitesimal transformation Xf. If we introduce new variables x_i' into Xf by means of a finite transformation of the aforementioned type then this gives:

$$|\, Uf\,|_x = |\, Uf\,|_{x'},$$

then the characteristic U is invariantly linked with the infinitesimal transformation Xf relative to any finite transformation of any sort.

If we introduce new variables x_i' into Xf by means of an infinitesimal transformation:

$$x_i' = x_i + \sum_{\mu=1}^{2n} \frac{A_{\mu i}}{A} \frac{\partial V}{\partial x_\mu} \delta t \qquad (i = 1, \ldots, 2n)$$

with the characteristic V then this makes:

$$f' = f + |\, Vf\,|\, \delta t, \qquad f = f' - |\, V' f'\,|_{x'}\, \delta t,$$

so:

$$| Uf | = | U'f' |_{x'} + \Big| |U'f' | V' \Big|_{x'} \delta t.$$

However, on the other hand, one has:

$$\begin{aligned} | Uf | = | Uf |_{x'} &= | U' + | U'V' |_{x'} \delta t,\ f' + | f'V' |_{x'} \delta t |_{x'} \\ &= | U'f' |_{x'} + \{\big| \, | U'\, V' | f'\big|_{x'} + \big| U' \, | f'\, V' |\big|_{x'}\} \delta t, \end{aligned}$$

from which, it emerges that:

(118) $$| U \,| Vf \,|| - | V \,| Uf \,|| \equiv || UV \,| f \,|$$

or:

(119) $$|| UV \,| W \,| + || UV \,| W \,| + || UV \,| W \,| \equiv 0,$$

which is the generalization of the Jacobi identity. [1])

The symbol $| UV |$ then has the important property that the identity (119) is true. With Kantor, we remark that any alternating bilinear expression:

(120) $$\{\varphi\,\chi\} = \sum_{i,k=1}^{n} \omega_{ik}(x_1, \cdots, x_n) \frac{\partial \varphi}{\partial x_i} \frac{\partial \chi}{\partial x_k} \qquad (\omega_{ik} + \omega_{ki} = 0)$$

possesses a trilinear covariant:

(121) $$\{\varphi\,\{\chi\,\psi\}\} + \{\chi\,\{\psi\,\varphi\}\} + \{\psi\,\{\varphi\,\chi\}\}.$$

In this:

$$\{\varphi\,\{\chi\,\psi\}\} - \{\chi\,\{\varphi\,\psi\}\}$$

is free of the second derivatives of ψ, and since $\{\psi\,\{\varphi\,\chi\}\}$ includes nothing but first derivatives of ψ, then only the first derivatives of ψ enter into (121), and naturally, only the first derivatives of χ and φ, as well. The symbol $| \varphi\chi |$ is therefore distinguished by the fact that its trilinear covariant (121) vanishes identically.

It is, moreover, trivial to prove that our Pfaff expression $\sum \alpha_i\, dx_i$ actually possesses a family of n-fold extended integral manifolds of the form (97). Namely, the functions F_1, …, F_n must be independent of each other and pair-wise satisfy the relation $| F_i\, F_\nu | \equiv 0$. If we then choose F_1 arbitrarily and set F_2 equal to an arbitrary solution of the equation $| F_1 f | = 0$ that is independent of F_1 then F_3 must satisfy the two equations:

$$A_1 f = | F_1 f | = 0, \qquad A_2 f = | F_2 f | = 0.$$

However, these are obviously independent of each other, and since:

$$A_1A_2 f - A_2A_1 f = | F_1 \,| F_2\, f \,|| - | F_2 \,| F_1 f \,||$$

---

[1]) This identity (119) was already found in Clebsch's second treatise on the Pfaff problem. Crelle, Bd. 61 (1863).

$$= \| F_1 F_2 f \| \equiv 0$$

is a complete two-parameter system with $2n - 2$ independent solutions, of which two − namely, F_1 and F_2 − are already known. We therefore choose F_3 equal to a solution of this complete system that is independent of F_1 and F_3 and proceed in that way until we have found n independent functions $F_1, \ldots, F_n$ with the desired characteristics. The only solutions to the complete n-parameter system $| F_i f | = 0$ ($i = 1, \ldots, n$) can then be expressed in terms of $F_1, \ldots, F_n$ alone.

With this, we have shown how one can find the most general family of ∞^n n-fold extended integral manifolds that fills up the entire space $x_1, \ldots, x_{2n}$ exactly once.

One knows one such family of ∞^n integral manifolds (97) and understands $\Psi_1, \ldots, \Psi_n$ to mean two arbitrary functions that are independent of each other and the F_i. If one then sets:

(122) $$\Psi_i(x_1, \ldots, x_{2n}) = u_i \qquad (i = 1, \ldots, n)$$

and thinks of the equations (97) and (122) as having been solved for the x then one has:

(123) $$x_i = \varphi_i(u_1, \ldots, u_n, a_1, \ldots, a_n) \qquad (i = 1, \ldots, 2n),$$

and these equations represent integral manifolds for arbitrary values of the a_k, so under the substitution (123), $\sum \alpha_i \, dx_i$ becomes a complete differential in the u:

(124) $$\sum_{i=1}^{2n} \alpha_i(\varphi_1, \cdots, \varphi_{2n}) \sum_{i=1}^{2n} \frac{\partial \varphi_i}{\partial u_\nu} du_\nu \equiv \sum_{i=1}^{2n} \frac{\partial \Omega(u, a)}{\partial u_\nu} du_\nu .$$

If one makes the substitution (97), (122) then one gets:

$$\sum_{i=1}^{2n} \alpha_i(x_1, \cdots, x_n)\, dx_i \equiv d\Omega(\Psi, F) - \sum_{\nu=1}^{n} \left\{ \frac{\partial \Omega(u, a)}{\partial a_\nu} - \sum_{i=1}^{n} \alpha_i(x) \frac{\partial \varphi_i}{\partial a_\nu} \right\}_{\substack{a=F \\ u=\Psi}} dF_\nu ,$$

so there exists an identity of the form:

(125) $$\sum_{i=1}^{2n} \alpha_i(x)\, dx_i \equiv \sum_{i=1}^{n} f_i(x) dF_i(x) + d\omega(x),$$

where ω can be found by a quadrature.

From (125), it now follows that:

$$\sum_{i,\nu=1}^{2n} \alpha_{i\nu}\, dx_\nu \delta x_i \equiv \sum_{i=1}^{n} (df_i \delta F_i - dF_i \delta f_i)$$

or, when one sets (cf. (101′) and (103)):

$$\delta x_i = \sum_{\mu=1}^{2n} \frac{A_{i\mu}}{A} \frac{\partial \varphi}{\partial x_\mu} \delta t \,,$$

one gets the identity:

(126) $$d\varphi \equiv \sum_{i=1}^{n} \{| F_i \varphi | \, df_i - | f_i \varphi | \, dF_i \} \,,$$

from which, in turn, when one sets:

$$dx_i = \sum_{\mu=1}^{2n} \frac{A_{i\mu}}{A} \frac{\partial \chi}{\partial x_\mu} dt \,,$$

it follows that:

(127) $$| \varphi \chi | \equiv \sum_{i=1}^{n} \{| F_i \varphi || f_i \chi | - | f_i \varphi || F_i \chi |\} \,.$$

The identity (126) yields, when one sets $\varphi = 0$ and imagines that $F_1, \ldots, F_n$ are independent of each other, and that all $| F_i \, F_k | = 0$:

$$| F_i \, f_k | = \varepsilon_{ik} \,,$$

and then, when $\varphi = f_k$:

$$| f_i \, f_k | = 0.$$

Thus, the $2n$ functions F_i , f_i in (125) are coupled together by the relations:

(128) $$| F_i \, F_k | = 0, \quad | F_i \, f_k | = \varepsilon_{ik} \,, \quad | f_i \, f_k | = 0 \qquad (i, k = 1, \ldots, n),$$

from which, in the same way as on pp. ?, *et seq.*, we can conclude that they are mutually independent.

Conversely, if $2n$ functions $f_1, \ldots, f_n, F_1, \ldots, F_n$ are present that satisfy the relations (128) then they are certainly independent of each other. Furthermore, if (126) is true for any function φ then one has for arbitrary u_ν and dx_ν :

$$\sum_{\nu=1}^{2n} u_\nu dx_\nu \equiv \sum_{i=1}^{n} \sum_{\mu,\nu=1}^{2n} \frac{A_{\mu\nu}}{A} \left(\frac{\partial F_i}{\partial x_\mu} df_i - \frac{\partial f_i}{\partial x_\mu} dF_i \right) u_\nu \,.$$

If one sets:

$$u_\nu \, \delta t = \sum_{k=1}^{2n} \alpha_{k\nu} \delta x_k$$

then one gets:

$$\sum_{k,\nu=1}^{2n} \alpha_{k\nu} dx_\nu \delta x_k \equiv \sum_{i=1}^{n} (df_i \delta F_i - dF_i \delta f_i)$$

for all dx_i, δx_i, so the existence of the relations (128) implies an identity of the form (125).

If one substitutes the expression:

$$dx_\mu = \sum_{\nu=1}^{2n} \frac{A_{\mu\nu}}{A} \frac{\partial \varphi}{\partial x_\nu} dt$$

in (125) then one obtains the identity:

(129) $$\sum_{\mu,\nu=1}^{2n} \frac{A_{\mu\nu}}{A} \frac{\partial \varphi}{\partial x_\nu} \equiv \sum_{i=1}^{n} f_i \,|\, F_i \varphi \,| + |\, \omega\varphi \,|,$$

which delivers the following for $\varphi = F_k$ and $\varphi = f_k$:

(130) $$\begin{cases} |\, \omega F_k \,| = \mathsf{A} F_k \\ |\, \omega f_k \,| = \mathsf{A} f_k - f_k \end{cases} \qquad (k = 1, \ldots, n),$$

where the expression:

(131) $$\mathsf{A} f = \sum_{\mu,\nu=1}^{2n} \frac{\alpha_\mu A_{\mu\nu}}{A} \frac{\partial f}{\partial x_\nu} \, ,$$

which, under the assumptions that we made here, certainly does not vanish identically, is the symbol of an infinitesimal transformation.

Since the F_i, f_i are mutually independent, the same is true for equations (130); they thus determine the function ω by a quadrature. The existence of relations (128) and (130) is thus necessary and sufficient for the existence of the identity (125).

As one learned on pp. ?, *et seq.*, the expression:

(132) $$\mathsf{A} \,|\, \varphi\chi \,| - |\, \mathsf{A}\varphi, \chi \,| - |\, \varphi, \mathsf{A}\chi \,|$$

includes only the first derivatives of φ and χ. From (129), however, one has:

$$\mathsf{A}\varphi \equiv \sum_{i=1}^{n} f_i \,|\, F_i \,|\, \varphi\chi \,\|\, + |\, \omega \,|\, \varphi\chi \,\|,$$

$$|\, \mathsf{A}\varphi, \chi \,| \equiv \sum_{i=1}^{n} \{ f_i \,|\, F_i \varphi \,|\, \chi \,| + |\, f_i \chi \,|\, |\, F_i \varphi \,| \} + \|\, \omega\varphi \,|\, \chi \,|,$$

$$|\, \varphi, \mathsf{A}\chi \,| \equiv \sum_{i=1}^{n} \{ f_i \,|\, \varphi \,|\, F_i \chi \,|| + |\, \varphi f_i \,|\, |\, \chi F_i \,| \} + |\, \varphi \,|\, \omega\chi \,\|.$$

From this, by the use of the identity (119), (132) yields the value:

$$\sum_{i=1}^{n} \{ |\, f_i \varphi \,\|\, F_i \chi \,| - |\, f_i \chi \,|\, |\, F_i \varphi \,| \} = - |\, \varphi\chi \,|,$$

from (127). Therefore, the infinitesimal transformation $\mathsf{A}f$ has the following relationship with respect to the symbol $|\varphi\chi|$:

$$\mathsf{A}|\varphi\chi| \equiv |\mathsf{A}\varphi, \chi| + |\varphi, \mathsf{A}\chi| - |\varphi\chi|, \tag{133}$$

which can also be written:

$$\mathsf{A}\,|Uf| - |U, \mathsf{A}f| \equiv |\mathsf{A}U - U, f|, \tag{133′}$$

and from this, it emerges that $\mathsf{A}f$ leaves the totality of all infinitesimal transformations $|Uf|$.

The fact that things must be this way can be obtained more quickly in another way.

Namely, if Xf is an arbitrary infinitesimal transformation then one has the equation:

$$X\sum_{\nu=1}^{2n}\alpha_\nu dx_\nu = \sum_{i,\nu=1}^{2n}\alpha_{\nu i}\xi_i dx_\nu + d\sum_{\nu=1}^{2n}\alpha_\nu\xi_\nu\,, \tag{134}$$

so, in particular, for $Xf = \mathsf{A}f$:

$$\mathsf{A}\sum_{\nu=1}^{2n}\alpha_\nu dx_\nu = \sum_{i,\nu=1}^{2n}\frac{\alpha_{\nu i}\alpha_\mu A_{\mu i}}{A}dx_\nu = \sum_{\nu=1}^{2n}\alpha_\nu dx_\nu\,, \tag{135}$$

and one realizes that, in fact, $\mathsf{A}f$ is the only infinitesimal transformation Xf for which $\sum \alpha_\nu x_\nu$ vanishes identically, and verifies the relation (135), as well.

If one sets $|Uf| = Xf$ from now on then, from (117), one has:

$$X\sum_{\nu=1}^{2n}\alpha_\nu dx_\nu = d(U - \mathsf{A}u), \tag{117′}$$

and thus, when one sets, for the moment:

$$\mathsf{A}\,|Uf| - |U, \mathsf{A}f| = \mathsf{A}\,Xf - X\,\mathsf{A}f = (\mathsf{A}\,X) = Zf$$

one gets:

$$\begin{aligned} Z\sum_{\nu=1}^{2n}\alpha_\nu dx_\nu &= d(\mathsf{A}U - \mathsf{A}\mathsf{A}u) - d(U - \mathsf{A}u), \\ &= d(\mathsf{A}U - U - \mathsf{A}(\mathsf{A}U - U)); \end{aligned}$$

that is, Zf has the form $|Vf|$. By comparison, in this way, one generally deduces only that the characteristic V of Zf satisfies the equation:

$$V - \mathsf{A}V = \mathsf{A}U - U - \mathsf{A}(\mathsf{A}U - U),$$

but not that it has the value $\mathsf{A}U - U$, as we just saw.

We now ask, in particular, whether an identity of the form (125) can also exist when ω is equal to zero.

For this, it is necessary and sufficient that, in addition to equations (128), one also has these:

(130′) $\mathsf{A}F_k = 0, \qquad \mathsf{A}f_k = f_k \qquad (k = 1, \ldots, n).$

However, it already suffices if one can determine n independent functions $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ that satisfy the equations:

(136) $|\,\mathfrak{F}_i\;\mathfrak{F}_k\,| = 0, \quad \mathsf{A}\mathfrak{F}_i = 0 \qquad (i, k = 1, \ldots, n).$

Namely, if one has n such functions then, as we saw on pp. ?, et seq., there exists an identity of the form:

(137) $$\sum_{i=1}^{2n} \alpha_i dx_i \equiv \sum_{\nu=1}^{n} \mathfrak{f}_\nu d\mathfrak{F}_\nu + d\vartheta$$

and one has:

$$|\,\mathfrak{F}_i\;\mathfrak{F}_k\,| = 0, \qquad |\,\mathfrak{F}_i\;\mathfrak{f}_k\,| = \varepsilon_{ik}, \qquad |\,\mathfrak{f}_i\;\mathfrak{f}_k\,| = 0,$$
$$|\,\vartheta\,\mathfrak{F}_k\,| = \mathsf{A}\mathfrak{F}_k = 0, \qquad |\,\vartheta\,\mathfrak{f}_k\,| = \mathsf{A}\mathfrak{f}_k - \mathfrak{f}_k .$$

Since $\mathfrak{F}_i$, $\mathfrak{f}_i$ are mutually independent here, one can think of ϑ as being expressed in terms of these $2n$ functions, and obtain:

$$\frac{\partial\vartheta}{\partial\mathfrak{f}_k} = 0, \qquad \frac{\partial\vartheta}{\partial\mathfrak{F}_k} = \mathsf{A}\mathfrak{f}_k - \mathfrak{f}_k ,$$

such that ϑ becomes a function of only $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$. The identity (137) thus possesses the form:

(137′) $$\sum_{i=1}^{2n} \alpha_i dx_i \equiv \sum_{\nu=1}^{n} \mathsf{A}\mathfrak{f}_\nu \cdot d\mathfrak{F}_\nu ,$$

which is then the desired one.

It still remains for us to show that one can satisfy equations (136) with n independent function $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$. However, one can now, at least, when $m = 1$, always determine m independent functions $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ such that the equations:

$$|\,\mathfrak{F}_\mu\,\mathfrak{F}_\nu\,| = 0, \qquad \mathsf{A}F_\mu = 0 \qquad (\mu, \nu = 1, \ldots, m)$$

are fulfilled. The m equations:

$$|\,\mathfrak{F}_\mu\,\mathfrak{F}\,| = 0, \qquad (\mu = 1, \ldots, m)$$

are then certainly independent of each other and, due to the identity (119), define a complete m-parameter system. However, the $m + 1$ equations:

(138) $|\,\mathfrak{F}_\mu\,\mathfrak{F}\,| = 0, \quad (\mu = 1, \ldots, m) \qquad \mathsf{A}\mathfrak{F} = 0,$

are also independent of each other, as long as $m < n$. Namely, if they were not then there would exist $2n$ identities of the form:

$$\sum_{i=1}^{2n} \frac{\alpha_i A_{i\nu}}{A} \equiv \sum_{\mu=1}^{m} \rho_\mu \sum_{i=1}^{2n} \frac{A_{i\nu}}{A} \frac{\partial \mathfrak{F}_\mu}{\partial x_i},$$

from which, it would follow that:

$$\alpha_k \equiv \sum_{\mu=1}^{m} \rho_\mu \frac{\partial \mathfrak{F}_\mu}{\partial x_k},$$

so:

$$\sum_{k=1}^{2n} \alpha_k dx_k \equiv \sum_{\mu=1}^{m} \rho_\mu d\mathfrak{F}_\mu,$$

and the equations $\mathfrak{F}_\mu$ = const. ($\mu = 1, \ldots, m$) would represent a family of $(2n - m)$-fold extended integral manifolds of the Pfaffian expression $\sum \alpha_\nu dx_\nu$, which is impossible for $m < n$. Finally, it follows from the identities (119) and (133′) that the $m + 1$ equations (138) define a complete $(m + 1)$-parameter system that possesses $2n - m - 1$ independent solutions. Now, since $\mathfrak{F}_1, \ldots, \mathfrak{F}_m$ are independent solutions of (138), under the assumptions that we made, the system (138) certainly possesses a solution that is independent of $\mathfrak{F}_1, \ldots, \mathfrak{F}_m$, as long as $2n - m - 1 > m$; that is, as long as $m < n$.

If one applies this theorem repeatedly, after one has first determined a solution $\mathfrak{F}_1$ to the equation $\mathsf{A}F = 0$, then one ultimately arrives at n independent functions $\mathfrak{F}_1, \ldots, \mathfrak{F}_m$ that satisfy (136), which was to be shown.

We then have the theorem:

In order for an identity of the form:

(139) $$\sum_{i=1}^{2n} \alpha_i dx_i \equiv \sum_{\nu=1}^{n} f_\nu dF_\nu$$

to exist, it is then necessary and sufficient that the equations (128) *and* (130′) *must be true. If one has n independent functions* $F_1, \ldots, F_n$ *that satisfy the equations:*

(140) $$|\, F_i \, F_k \,| = 0, \qquad \mathsf{A}F_i = 0 \qquad (i, k = 1, \ldots, n)$$

then there always exists an identity of the form (139), *and one finds* $f_1, \ldots, f_n$ *by solving linear equations.*

Since the $2n$ functions f_i, F_i are independent of each other, it is likewise shown that if A does not vanish identically then the expression $\sum \alpha_i dx_i$ can be brought to the normal form $p_1' dx_1' + \ldots + p_n' dx_n'$ by a transformation:

$$x_i' = F_i, \qquad p_i' = f_i \qquad (i = 1, \ldots, n).$$

It is now very easy to draw upon the theory of function groups in the case of a Pfaffian expression $\sum \alpha_i\, dx_i$ with non-vanishing A.

We say that m independent functions $u_1, \ldots, u_m$ of $x_1, \ldots, x_{2n}$ determine an m-parameter function group when relations of the form:

$$| u_i\, u_k | = \omega_{ik}(u_1, \ldots, u_m) \qquad (i, k = 1, \ldots, m)$$

exist. All of the discussion pertaining to reciprocal function groups, distinguished functions, and the construction of a canonical basis takes exactly the same form as it did for the function groups in the x, p. The same thing is also true of the theorem that two m-parameter function groups can be converted into each other by a transformation that leaves $\sum \alpha_i\, dx_i$ invariant, up to a complete differential, if and only if they have the same number of parameters and same number of distinguished functions.

It is not necessary to go through everything in detail. It suffices to refer to chapter 13 of the second volume of *Transformationsgruppen*, where the required developments of almost everything were, in fact, carried out, although clearly a completely different problem was being treated there.

We only mention that $2n$ independent functions $\Phi_1, \ldots, \Phi_{2n}$ that have the relationship (109) determine a $2n$-parameter function group. In order to find the most general function system $\Phi_1, \ldots, \Phi_{2n}$ of this type, one must first construct a canonical basis for this function group. If $F_1, \ldots, F_n, f_1, \ldots, f_n$ are such functions of x that obey the canonical relations (128) then the expressions $F_i(\Phi_1, \ldots, \Phi_{2n})$, $f_i(\Phi_1, \ldots, \Phi_{2n})$ are such a canonical basis. Finally, if $\mathfrak{F}_1, \ldots, \mathfrak{F}_n, \mathfrak{f}_1, \ldots, \mathfrak{f}_n$ is the most general system of functions of x that fulfill the canonical equations (128) then the equations:

$$F_i(\Phi) = \mathfrak{F}_i, \qquad f_i(\Phi) = \mathfrak{f}_i \qquad (i = 1, \ldots, n)$$

determine the most general function group $\Phi_1, \ldots, \Phi_{2n}$ that satisfies (109).

Now, some suggestions might be made, as Kantor did in his extension of the theory of function groups for the present case.

We call two Pfaffian equations:

$$\sum_{i=1}^{2n} \lambda_i dx_i = 0, \qquad \sum_{i=1}^{2n} \lambda_i' dx_i = 0$$

conjugate when the equation:

$$\sum_{i,\nu=1}^{2n} \frac{A_{i\nu}}{A} \lambda_i \lambda_\nu' = 0 \tag{141}$$

is fulfilled. Likewise, we call two linear partial differential equations:

$$\sum_{i=1}^{2n} \rho_i \frac{\partial f}{\partial x_i} = 0, \qquad \sum_{i=1}^{2n} \rho_i' \frac{\partial f}{\partial x_i} = 0$$

conjugate when the equation:

(142) $$\sum_{i,\nu=1}^{2n} \alpha_{i\nu} \rho_i \rho_\nu' = 0$$

is verified.

If we now have a system of m independent linear partial differential equations then we can always determine a canonical basis for this system:

(143) $$\begin{cases} X_i f = \sum_{\nu=1}^{2n} \rho_{i\nu} \frac{\partial f}{\partial x_\nu} = 0 & (i = 1, \cdots, l+h), \\ P_k f = \sum_{\nu=1}^{2n} \sigma_{k\nu} \frac{\partial f}{\partial x_\nu} = 0 & (k = 1, \cdots, l), \end{cases} \qquad (2l + h = m),$$

such that any two of these equations are always conjugate, if one assumes only $X_k f$ and $P_k f$ (k = 1, …, l) are known, for which one always has:

(144) $$\sum_{i,\nu=1}^{2n} \alpha_{i\nu} \sigma_{ki} \rho_{k\nu} = 1 \qquad (k = 1, \ldots, l).$$

The totality of all equations that are conjugate to the equations (143) defines the reciprocal system to (143), which has ($2n - m$) parameters, and for which we can determine a canonical basis of the form:

(143′) $$X_{l+1} f = 0, \ldots, \quad X_n f = 0, \qquad P_{l+h+1} f = 0, \qquad P_n f = 0.$$

Finally, we can choose $P_{l+1} f, \ldots, P_{l+k} f$ in such a way that all $2n$ equations $X_i f = 0$, $P_i f = 0$ are mutually independent and define a canonical basis.

From now on, there are $2n$ uniquely determined Pfaffian expressions D_i , E_i such that

(145) $$df \equiv \sum_{i=1}^{n} (D_i P_i f - E_i X_i f) .$$

If one replaces the f_{x_μ} in this identity with:

$$\sum_{\nu=1}^{2n} \rho_{k\nu} \alpha_{\mu\nu} \qquad (\mu = 1, \ldots, 2n)$$

then all of the $X_i f$ vanish, and likewise all of the $P_i f$, except for $P_k f$, which equals 1, and one obtains:

(146) $$D_i = \sum_{\nu=1}^{2n} \alpha_{\mu\nu}\rho_{k\nu}dx_\mu \qquad (k = 1, \ldots, n),$$

and one similarly finds:

(147) $$E_i = \sum_{\nu=1}^{2n} \alpha_{\mu\nu}\sigma_{k\nu}dx_\mu \qquad (k = 1, \ldots, n),$$

where from now on the D_k , E_k , in turn, can be regarded as a canonical basis, so any two of them will be conjugate, with the exception of any pair D_k , E_k (k = 1, …, n) for which one has:

$$\sum_{\mu,\nu=1}^{2n} \frac{A_{\mu\nu}}{A} \sum_{\pi,\tau=1}^{2n} \alpha_{\mu\pi}\sigma_{k\pi}\alpha_{\nu\tau}\rho_{k\tau} = \sum_{\pi,\tau} \alpha_{\pi\tau}\sigma_{k\pi}\rho_{k\tau} = 1.$$

If one then sets:

$$dx_\nu = \sum_{\mu=1}^{2n} \frac{A_{\mu\nu}}{A} \frac{\partial\varphi}{\partial x_\mu} dt$$

in equation (145) then one gets:

(148) $$|\,\varphi f\,| \equiv \sum_{i=1}^{n} (P_i\varphi \cdot X_i f - X_i\varphi \cdot P_i f)\,,$$

and finally obtains, when one substitutes:

$$\sum_{\nu=1}^{2n} \alpha_{\mu\nu}\delta x_\nu$$

in (145) for f_{x_μ} , the identity:

(149) $$\sum_{\mu,\nu=1}^{2n} \alpha_{\mu\nu} dx_\mu \delta x_\nu \equiv \sum_{i=1}^{n} (E_i\Delta_i - D_i\mathrm{E}_i)\,.$$

Everything now takes exactly the same form as it did on pp.?-?, except that the symbol ( ) must be replaced with | | everywhere. For example, this yields that a complete m-parameter system (143) defines a (2n – m)-parameter function group when the reciprocal system (143′) is also complete, and so on. Briefly, it behaves in such a way that for an arbitrary Pfaffian expression for which A does not vanish identically the entire theory is completely analogous to what it implies for the expression $\sum p_i\, dx_i$, and just as simple.

Appendix

On pp. ?, *et seq.*, I proved that, in general, as long as (58) and (59) are complete systems, (76) is also such a system, but the method of proof that was employed there does not apply in the context of the newer foundations of the theory of function groups. Technically, it must be shown that when (58) and (59) are complete systems the equations that exist between the coefficients of (68) are linked to equations (69) in such a way that (76) is also a complete system as a result. I have shown this just recently, so I would thus like to take up the argument that led me to this proof.

If (58), as well as (59), is a complete system then one first has $b_{ik\mu} = 0$ for:

$$k, \mu = 1, \ldots, l + h; \qquad i = l + 1, \ldots, n,$$

and

$$k, \mu = l + 1, \ldots, n; \qquad i = 1, \ldots, l + h,$$

so one has $b_{k\mu\, i} = b_{i\mu k} = 0$ for:

$$k = 1, \ldots, l; \qquad \mu = l + 1, \ldots, l + h; \qquad i = l + 1, \ldots, n,$$

since, however, $b_{i\mu k} = - b_{i\mu\, k}$, so one obtains from (69) that for the same values of i, k, m, $b_{ik\mu}$ also vanishes.

One further has $a'_{k\mu i} = 0$ for:

$$k = 1, \ldots, l + h; \qquad \mu = 1, \ldots, l; \qquad i = l + h + 1, \ldots, n$$

and for:

$$k = l +1, \ldots, n; \qquad \mu = l + h + 1, \ldots, n; \qquad i = 1, \ldots, l,$$

from which:

$$a'_{k\mu i} = a'_{ki\mu} = 0$$

for:

$$k = l + 1, \ldots, l + h; \qquad \mu = l + h + 1, \ldots, n; \qquad i = 1, \ldots, l,$$

and then, due to (69):

$$b''_{\mu ik} = a'_{k\mu i} - a'_{ki\mu} = 0$$

for the same values of μ, i, k.

Finally, $a_{k\mu i} = 0$ for:

$$k, \mu = 1, \ldots, l + h; \qquad i = l + h + 1, \ldots, n$$

and:

$$k, \mu = l + 1, \ldots, n; \qquad i = 1, \ldots, l,$$

so it follows from (69) that for the same values of k, μ, i, one likewise has $b'_{\mu ik} = b'_{ki\mu}$. However, one has $b'_{\mu ik} = 0$, moreover, for:

$$\mu = 1, \ldots, l + h; \qquad i = 1, \ldots, l; \qquad k = l + 1, \ldots, n$$

and for:

$$\mu = l + 1, \ldots, n; \qquad i = l + h + 1, \ldots, n; \qquad k = 1, \ldots, l + h,$$

so $b'_{ki\mu} = 0$ for:

$$i = 1, \ldots, l; \qquad k = l + 1, \ldots, n; \qquad \mu = l + 1, \ldots, l + h,$$

and for:

$$i = l + h + 1, \ldots, n; \qquad k = 1, \ldots, l + h; \qquad \mu = l + 1, \ldots, l + h.$$

In this lies the fact that (76) is also a complete system.

Now, since the system (75) that is reciprocal to (76) is likewise complete, (75) and (76) define two reciprocal function groups, and indeed, (76) consists of all functions that are common to both function groups (58) and (59), while (75) subsumes both function groups, but only includes such functions that are expressible in terms of both function groups.

Giessen, 25 November 1913.