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On the defining equations of continuous transformation groups

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Introduction

A lengthy sojourn in Christiania afforded me the opportunity to collaborate with Sophus Lie in person in order to study his theory of continuous transformation groups in detail. The goal of our colleague was to produce a coherent presentation of this theory, while I addressed essentially only the editorial side of the problem. Along the way, I also concerned myself with independent investigations in the realm of transformation groups. My results shall be set down in the course of this treatise.

At the present time, a coherent presentation of the theory has been by no means achieved, much less published. For an understanding of the following, it would be therefore necessary to understand the quite numerous treatises of Lie on transformation groups. In order to ameliorate this drawback, the main ideas and theorems of the theory of transformation groups shall first be summarized briefly, to the extent that they will be used later on. Naturally, we must omit the proofs in this summary.

A. A family of transformations:

$$x_i' = f_i(x_1, \ldots, x_n; a_1, \ldots, a_r) \qquad (i = 1, \ldots, n)$$

of the variables $x_1, \ldots, x_n$ defines a *finite continuous transformation group* when two transformations of the family, carried out in sequence, again yield a transformation of the family. Analytically, this means that the transformation above, when composed with:

$$x_i'' = f_i(x_1', \ldots, x_n'; b_1, \ldots, b_r),$$

yields a transformation of the form:

$$x_i'' = f_i(x_1, \ldots, x_n; c_1, \ldots, c_r),$$

in which the c depend upon only the a and b:

$$c_\kappa = \varphi_\kappa(a_1, \ldots, a_r; b_1, \ldots, b_r).$$

The quantities $a_1, \ldots, a_r$ are called the *parameters* of the group. It is clear that it makes no difference to the group as such if one introduces r independent functions of the a_κ as new parameters.

Here, we will restrict ourselves to those groups that include the identity transformation $x_i' = x_i$, and are generated by infinitesimal transformations.

By the term *infinitesimal transformation*, Lie understood this to mean a transformation of the form:

$$x_i' = x_i + \xi_i(x_1, \ldots, x_n)\,\delta t \qquad (i = 1, \ldots, n),$$

where δt denotes an infinitely small quantity. He introduced the symbol:

$$\sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i} = X(f)$$

for this infinitesimal transformation, from which, their transformation equations can be derived by inspection. Namely, if f means an arbitrary function of $x_1, \ldots, x_n$ then the equation means:

$$f(x_1', \ldots, x_n') = f(x_1, \ldots, x_n) + X(f)\,\delta t,$$

from which, $x_i' = x_i + \xi_i\,\delta t$ follows with no further assumptions. It is also clear, and likewise easy to verify, that *the symbol $X(f)$ is independent of the choice of variables.*

If one imagines that the infinitesimal transformation $X(f)$ has been repeated infinitely often then one obtains ∞^1 finite transformations, namely, the following ones:

$$x_i' = x_i + t\,\xi_i + \frac{t^2}{1\cdot 2}\sum_{\kappa=1}^{n} \xi_\kappa \frac{\partial \xi_i}{\partial x_\kappa} + \ldots \qquad (i = 1, \ldots, n),$$

or, more briefly:

$$x_i' = x_i + t\,X(x_i) + \frac{t^2}{1\cdot 2} X\,X(x_i) + \ldots$$

As one can easily show, these ∞^1 transformations define a finite continuous transformation group with one parameter t. Lie called this group *one-parameter* and said that it is *generated* by the infinitesimal transformation $X(f)$.

Several, – say, r – infinitesimal transformations:

$$X_\kappa f = \sum_{i=1}^{n} \xi_{i\kappa} \frac{\partial f}{\partial x_i} \qquad (\kappa = 1, \ldots, r)$$

are said to be *independent* of each other when it is not possible to choose r constants $c_1, \ldots, c_r$ such that expression $c_1 X_1 f + \ldots + c_r X_r f$ vanishes identically.

If $X_1 f, \ldots, X_r f$ are independent of each other then one may easily show that the totality of all infinitesimal transformations:

$$\sum_{\kappa=1}^{r} c_\kappa X_\kappa f$$

generates a family of one-parameter groups:

(1) $$x_i' = x_i + t\sum_{\kappa=1}^{n} c_\kappa X_\kappa(x_i) + \ldots$$

that includes ∞^r different transformations. Lie has further proved that *this family of transformations defines a finite continuous transformation group when, but only when, relations of the form:*

$$X_i(X_\kappa(f)) - X_\kappa(X_i(f)) \equiv \sum_{j=1}^{n} \{X_i(\xi_{j\kappa}) - X_j(\xi_{i\kappa})\}\frac{\partial f}{\partial x_j} = \sum_{s=1}^{r} c_{i\kappa s} X_s f$$

exist, where the $c_{i\kappa s}$ are numerical constants. If this condition is fulfilled then (1) represents the finite equations of a continuous transformation group with r parameters or, more briefly, an r-*parameter* group.

The family of ∞^{r-1} infinitesimal transformations that belong to such an r-parameter group is obviously characterized by the fact that it includes all infinitesimal transformations of the form:

$$c_i X_i f + c_\kappa X_\kappa f, \qquad X_i(X_\kappa(f)) - X_\kappa(X_i(f)) = (X_i X_\kappa)(f)$$

along with $X_i f$ and $X_\kappa f$.

The general projective group of the one-dimensional manifold will serve as an example. Its finite transformations are:

$$x' = \frac{x + a_1}{a_2 x + a_3},$$

but the infinitesimal transformations are:

$$X_1 f = \frac{df}{dx}, \qquad X_2 f = x\frac{df}{dx}, \qquad X_3 f = x^2\frac{df}{dx},$$

and indeed one has the following relations:

$$(X_1 X_2) = X_1 f, \qquad (X_1 X_3) = 2 X_2 f, \qquad (X_2 X_3) = X_3 f.$$

B. Let:

$$X f = \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i} = c_1 X_1 f + \ldots, + c_r X_r f$$

be the general infinitesimal transformation of an r-parameter group.

If one differentiates the equations:

$$\xi_i = \sum_{\kappa=1}^{r} c_\kappa \xi_{i\kappa} \qquad (i = 1, \ldots, n)$$

sufficiently often with respect to $x_1, \ldots, x_n$ then one can eliminate the $c_1, \ldots, c_r$ and obtain a series of linear, homogeneous, differential equations:

$$(2) \qquad \sum_{i=1}^{n} A_{\lambda i}(x_1, \cdots, x_n)\xi_i + \sum_{i=1}^{n}\sum_{\kappa=1}^{n} B_{\lambda i\kappa}(x_1, \cdots, x_n)\frac{\partial \xi_i}{\partial x_\kappa} + \ldots = 0 \qquad (\lambda = 1, 2, \ldots)$$

for $x_1, \ldots, x_n$. Now, one can exhibit as many such equations as it takes to define the $\xi_1, \ldots, \xi_n$, and therefore the entire group, completely. Lie therefore called them the *defining equations* of the finite continuous group in question. Considering what was said in **A**, we can now state the theorem:

A system of linear, homogeneous, differential equations of the form (2) *defines a finite continuous transformation group when its most general solutions* $\xi_1, \ldots, \xi_n$ *depend upon only arbitrary constants, and when from any two systems of solutions* $\xi_{11}, \ldots, \xi_{n1}$, $\xi_{12}, \ldots, \xi_{n2}$, *one can always compose them into a third one of the form:*

$$(3) \qquad \sum_{i=1}^{n}\left(\xi_{i1}\frac{\partial \xi_{j2}}{\partial x_i} - \xi_{i2}\frac{\partial \xi_{j1}}{\partial x_i}\right) \qquad (j = 1, \ldots, n).$$

C. The *finite*, continuous transformations, with their *arbitrary parameters*, are to be contrasted with the *infinite* ones, with *arbitrary parameters* and *arbitrary functions.* Naturally, we consider only such infinite groups whose infinitesimal transformations may be defined by linear, homogeneous, differential equations of the form (2). These differential equations will then possess all of the properties that were given in the last theorem, except that their most general system of solutions depends upon not merely arbitrary constants, but also on arbitrary functions.

Here, a complication appears: The family of infinitesimal transformations that are defined by any system of differential equations generates a family of one-parameter groups; however, the proof that this family of one-parameter groups defines a transformation group and that it is identical with the original infinite group has still not been accomplished.

In order to avoid this complication, we will – at least, for the infinite groups – restrict ourselves to infinitesimal transformations, and the finite transformations will almost always be left out of play. Indeed, in the theory of continuous transformation groups the infinitesimal transformations are more important than the finite ones, anyway.

To that end, we introduce the expression: *group of infinitesimal transformations*. We understand this to means a family of infinitesimal transformations that have the property that, along with X_1f and X_2f, they also include the transformations $c_1\, X_1f + c_2\, X_2f$ and $(X_1\, X_2)$, and are, in addition, defined by differential equations. It is self-evident that these differential equations are linear and homogeneous, and that one can always derive a third system of solutions of the form (3) from any two systems of solutions ξ_{i1} and ξ_{i2} . We call these differential equations the *defining equations of our group of infinitesimal transformations.* *)

If the most general system of solutions of the aforementioned differential equations contains only *arbitrary constants* then we have a *finite* group of infinitesimal transformations before us, and indeed, from the above, these transformations generate a finite continuous transformation group. By contrast, if the most general system of solutions of some differential equations also contains *arbitrary functions* then we are dealing with an *infinite* group of infinitesimal transformations. Whether it does or does not generate an infinite transformation group is still undecided.

D. The concept of *similarity* is of great importance.

Two r-parameter transformation groups are called *similar* when one of them becomes identical with the other one by the introduction of appropriate variables.

Therefore, let, e.g.:

$$x' = \varphi(x;\, a_1,\, a_2), \qquad x' = \psi(x;\, b_1,\, b_2)$$

be two two-parameter groups of the simple manifold, and assume that under the introduction of new variables $x_1 = \omega(x)$ the second one assumes the form:

$$x_1' = \varphi(x_1\,;\, a_1(b_1\, b_2),\, a_2(b_1\, b_2)).$$

If we now introduce a_1 and a_2 as new parameters then the second group becomes identical with the first one; therefore, the two are similar to each other.

Therefore, two r-parameter transformation groups are also obviously similar to each other when the infinitesimal transformations of the one go over to the infinitesimal transformations of the other one by a suitable choice of variables. It is then also clear what we mean by the similarity of two finite or infinite groups of infinitesimal transformations.

E. When an analytical expression is not changed by a transformation – so it then remains invariant –.Lie said that the expression *admits* or *allows* the transformation in question.

A function $\varphi(x_1, \ldots, x_n)$ *admits the infinitesimal transformation* Xf *when the expression* $X(\varphi)$ *vanishes identically,* or, what amounts to the same thing, when φ is a

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*) The introduction of these relations is in accord with Lie. Indeed, he has still not used the expression "group of infinitesimal transformations," but only the notion that he made use of in his treatise "Ueber unendliche continuirliche Gruppen" (Christiania Videnskabeselskabs Forh. 1882, no. 12).

solution of the equation $Xf = 0$. Along with the infinitesimal transformation Xf, φ then also admits all finite transformations:

$$x_i' = x_i + t\,X(x_i) + \frac{t^2}{1\cdot 2} X\,X(x_i) + \dots$$

of the associated one-parameter group.

If there is a function $\varphi(x_1, \dots, x_n)$ that admits r independent *infinitesimal* transformations $X_1f, \dots, X_r f$ of an r-parameter group then it also admits all *finite* transformations of the group. Therefore, for our group, any one of the ∞^1 manifolds φ = const. remains invariant – i.e., any point in the space $(x_1, \dots, x_n)$ moves on such a manifold. In this case, one calls the group *intransitive.* By contrast, our group is called *transitive* when it takes any point of the space $(x_1, \dots, x_n)$ to another one; analytically speaking, when the equations:

$$X_1f = 0, \dots, \quad X_rf = 0$$

have no common solution.

A group of infinitesimal transformations, which may be finite or infinite now, is called *transitive* when there is no function that admits the most general infinitesimal transformation of the group, while in the contrary case, it is called *intransitive.*

A system of equations:

$$\varphi_i(x_1, \dots, x_n) = 0 \qquad (i = 1, \dots, m),$$

or, geometrically speaking, a manifold in the space $(x_1, \dots, x_n)$ *admits the infinitesimal transformation Xf as long as all expressions* $X(\varphi_i)$ *vanish as a result of the system of equations.*

If a function of a system of equations admits the two infinitesimal transformations X_1f and X_2f then it likewise also admits the transformations $c_1X_1f + c_2X_2f$ and $(X_1\ X_2)$. *Therefore, if one has a group of infinitesimal transformations then the totality of all of the transformations included in it that leave invariant a well-defined function, or a well-defined system of equations, is itself once more a group of infinitesimal transformations.*

Here, we shall pass over the concept of primitivity, as important as it is, because it will have no application in what follows.

F. We would now like to briefly speak about the concepts of *composition* and *isomorphism*, and all that is connected with them. These are concepts that have meaning only for the finite transformation groups (at least temporarily).

Let $X_1f, \dots, X_r f$ be independent infinitesimal transformations of an r-parameter transformation group, so:

$$(X_i\, X_\kappa) = \sum_{s=1}^{r} c_{i\kappa s} X_s f\ .$$

Any $m < r$ independent infinitesimal transformations of our group:

$$Y_j f = \sum_{\kappa=1}^{r} l_{\kappa j} X_\kappa f \qquad (j = 1, \ldots, m)$$

possess the property that if any $(Y_i\ Y_j)$ may be expressed in terms of only $Y_1 f, \ldots, Y_m f$ linearly with constant coefficients:

$$(Y_i\ Y_j) = \sum_{s=1}^{m} g_{ijs} Y_s f$$

then the infinitesimal transformations $Y_1 f, \ldots, Y_m f$ generate an m-parameter *subgroup* of the first one. In particular, this subgroup is called *invariant* (distinguished) when any $(X_i\ Y_j)$ can also be expressed in terms of $Y_j f$ alone:

$$(X_i\ Y_j) = \sum_{s=1}^{m} h_{ijs} Y_s f\ ,$$

in which the h_{ijs} are understood to mean constants.

If one knows the infinitesimal transformations $X_\kappa f$, and therefore also the constants h_{ijs}, then one can find all subgroups of the group $X_\kappa f$ by algebraic operations. Namely, the determination of the *invariant* subgroups takes a very simple form; one needs only to observe that in an invariant subgroup in which the infinitesimal transformation Yf is present, all transformation of the form $(X_\kappa\ Y)$ must also be included.

Following Lie, we call the system of constants $c_{i\kappa s}$ the *composition* of the group $X_1 f, \ldots, X_r f$.

The Jacobi identity:

$$((X_i\ X_j)\ X_\kappa) + ((X_j\ X_\kappa)\ X_i) + ((X_\kappa\ X_i)\ X_j) = 0,$$

which exists between three arbitrary infinitesimal transformations $X_i f, X_j f, X_k f$, delivers the following relations between the $c_{i\kappa s}$:

$$\sum_{\kappa=1}^{r} (c_{ijs} c_{s\kappa\nu} + c_{j\kappa s} c_{si\nu} + c_{\kappa is} c_{sj\nu}) = 0 \qquad (i, j, \kappa; \nu = 1, \ldots, r).$$

Therefore, should a system of $c_{i\kappa s}$ represent the composition of a group, it must necessarily satisfy the relations that were just written down. Conversely, if these equations are fulfilled then, as Lie showed, there are also always r-parameter groups that have the composition in question. Moreover, Lie gave methods of presenting all transitive groups with a given composition.

If one has two r-parameter groups $X_1 f, \ldots, X_r f$, and $Y_1 f, \ldots, Y_r f$ that are associated with one and the same system of $c_{i\kappa s}$:

$$(4) \qquad (X_i\ X_\kappa) = \sum_{s=1}^{r} c_{i\kappa s} X_s f\ , \qquad (Y_i\ Y_\kappa) = \sum_{s=1}^{r} c_{i\kappa s} Y_s f$$

then one calls the two groups *equally composed*, or *holomorphically isomorphic.*

On the other hand, let the group $X_1 f, \ldots, X_r f$ be r-parameter, while the group $Y_1 f, \ldots, Y_r f$ is merely m-parameter, such that among the $Y_\kappa f$, only m mutually independent transformations are present. If equations (4) are true then one says that the group $Y_\kappa f$ is *meromorphically isomorphic* to the group $X_\kappa f$. Among the assumptions that were just made, $Y_1 f, \ldots, Y_r f$ are coupled with each other by $r - m$ mutually independent relations of the form:

(5) $$\sum_{\kappa=1}^{r} g_{\kappa\mu} Y_\kappa f = 0 \qquad (\mu = 1, \ldots, r - m).$$

The expressions:

$$\left(Y_i, \sum_{\kappa=1}^{r} g_{\kappa\mu} Y_\kappa f \right) = \sum_{s=1}^{r} \sum_{\kappa=1}^{r} g_{\kappa\mu} c_{i\kappa s} Y_s f$$

must then vanish as a result of equations (5); i.e., equations must exist of the form:

$$\left(Y_i, \sum_{\kappa=1}^{r} g_{\kappa\mu} Y_\kappa f \right) = \sum_{\nu=1}^{r} h_{i\mu\nu} \sum_{\kappa=1}^{r} g_{\kappa\nu} Y_\kappa f ,$$

and indeed they must be a direct consequence of equations (4). This yields the fact that the infinitesimal transformations:

$$\sum_{\kappa=1}^{r} g_{\kappa\mu} X_\kappa f \qquad (\mu = 1, \ldots, r - m),$$

which are obviously independent of each other, generate an $(r - m)$-parameter invariant subgroup of the r-group $X_1 f, \ldots, X_r f$.

From what was said, one easily recognizes *how one finds the composition of all groups that are meromorphically isomorphic to a given r-parameter group* $X_1 f, \ldots, X_r f$.

Let:

$$\sum_{\kappa=1}^{r} g_{\kappa\mu} X_\kappa f \qquad (\mu = 1, \ldots, r - m)$$

be independent infinitesimal transformations of an $(r - m)$-parameter invariant subgroup of the group $X_1 f, \ldots, X_r f$. One establishes the relations:

$$\sum_{\kappa=1}^{r} g_{\kappa\mu} Y_\kappa f \qquad (\mu = 1, \ldots, r - m),$$

which allow one to express – say – $Y_{m+1} f, \ldots, Y_r f$ in terms of the remaining ones. If one substitutes the values found for $Y_{m+1} f, \ldots, Y_r f$ in the relations:

$$(Y_i \, Y_\kappa) = \sum_{s=1}^{r} c_{i\kappa s} Y_s f$$

then one obtains all ($Y_i\ Y_\kappa$) expressed in terms of $Y_1f, \ldots, Y_rf$, and thus obtains the composition of an m-parameter group $Y_1f, \ldots, Y_rf$, which is obviously meromorphically isomorphic with the group $X_1f, \ldots, X_rf$.

In this way, one goes through all invariant subgroups of the group $X_1f, \ldots, X_rf$, step-by-step.

If a transformation group contains no invariant subgroups at all then it is called *simple*. A group that is meromorphically isomorphic to it can therefore contain no infinitesimal transformation; i.e., it consists of only the identity transformation $x_i' = x_i$.

Along with these prefatory remarks, which were certainly introduced in the theory of transformation groups, above all, we interpolate some general remarks on the foregoing treatise.

In this treatise, a new method shall be developed, by means of which, the defining equations of groups of infinitesimal transformations can be determined.

In several places in his cited treatise "Ueber unendlichen Gruppen," Lie calculated some defining equations in the plane directly. He simply stipulated that along with ξ_1, η_1 and ξ_2, η_2, simultaneously, also:

$$\xi_1 \frac{\partial \xi_2}{\partial x} + \eta_1 \frac{\partial \xi_2}{\partial y} - \xi_2 \frac{\partial \xi_1}{\partial x} - \eta_2 \frac{\partial \xi_1}{\partial y},$$
$$\xi_1 \frac{\partial \eta_2}{\partial x} + \eta_1 \frac{\partial \eta_2}{\partial y} - \xi_2 \frac{\partial \eta_1}{\partial x} - \eta_2 \frac{\partial \eta_1}{\partial y}$$

is a system of solutions of the differential equations in question, and in this way, obtained all relations from which the coefficients of the differential equations can be obtained. In general, this process leads to truly laborious calculations (cf., "Ueber unendliche Gruppen," § 6).

The method that shall be set down here is more general. For some time, I had conjectured that it delivers all defining equations of groups, so Lie surprised me one day with a proof of my conjecture. Unfortunately, I must omit a reproduction of this proof, because it would take up too much space.

Naturally, we will find only those groups that are already known from Lie's investigations. The interest of the following developments can, however, lie principally in just the convenience of the method and the form of the results, moreover.

In fact, Lie considered all similar groups as being on an equal footing. It occurred to him to present normal forms for the various types of groups, only to the extent that they did not go to each other under changes of variables. He always sought the representatives of each such type; the entire family is, indeed, similar to the representative, so all groups that belong to the type are implicitly given.

However, there can also be some advantage to moving one or another viewpoint to the foreground. We will therefore seek no normal form, since the groups that are similar to Lie's normal forms shall all be given explicitly. We do not wish to know the simplest forms for the defining equations, but rather we demand the general form of the defining equations for any family of equivalent groups, from which each individual group of the family can be obtained by specialization.

So much for the forms in which the groups can present themselves to us. By the way, we will, however, arrive at some results for the integration of defining equations. Among them one recognizes, in fact, the *a priori* existence of certain integrals; the means to exhibit them will be given. We thus arrive at several of the same ways of simplifying the integration that Lie achieved along another path.

On easily-recognized grounds, we begin with the simplest case, so we first consider the simple manifold. The development of this theory encounters the fewest difficulties and may also be brought to a certain state of completion.

§ 1.

Generalities on the simple manifold.

1. The line of reasoning that originally led me to the present developments was briefly this:

We imagine any group of the simple manifold is given, and in addition, exhibit all groups that are similar to it.

The original group will be defined by a defining equation of the form:

$$\frac{d^n\xi}{dx^n}+a_1(x)\frac{d^{n-1}\xi}{dx^{n-1}}+\cdots+a_{n-1}(x)\frac{d\xi}{dx}+a_n(x)\xi = 0.$$

Likewise, the totality of all groups that are similar to the original one can be defined by a differential equation:

(1) $$\frac{d^n\xi}{dx^n}+B_1\frac{d^{n-1}\xi}{dx^{n-1}}+\cdots+B_{n-1}\frac{d\xi}{dx}+B_n\xi = 0,$$

except that the B coefficients in it depend upon an arbitrary function – say, $\alpha(x)$ – and its differential quotients.

It just so happens that the family of groups in question remains invariant with the addition of a new variable $x_1 = F(x)$; in fact, any group of the family goes to one that is similar to it, and which likewise belongs to the family. Now, since the value that the function $\alpha(x)$ possesses determines the individual groups in (1), and on the other hand, the individual groups must be permuted amongst themselves by the introduction of new variables x_1 , $\alpha(x)$ must go to a new function $\alpha_1(x_1)$. Analytical relations must then exist between α, α_1, F, and their differential quotients, by means of which α_1 can be determined from α and F.

In particular, the case can arise in which α_1 is expressed analytically in terms of α and F with its differential quotients – perhaps up to order n:

$$\alpha_1 = \chi(\alpha, F, F', \ldots, F^{(s)});$$

in addition, we would like to assume that the function χ contains no arbitrary constants. With this assumption, α_1 is determined completely when α and F are given, and likewise, α is, conversely, determined completely when one knows α_1 and F.

Now, the equations:

$$x_1 = F(x), \qquad \alpha_1 = \chi(\alpha, F, F', \ldots, F^{(s)}) \tag{2}$$

define an *infinite transformation group*. Namely, if one has:

$$x_2 = \Phi(x_1), \qquad \alpha_2 = \chi(\alpha_1, \Phi, \Phi', \ldots, \Phi^{(s)})$$

then one gets $x_2 = \Phi(F(x))$, and thus:

$$\begin{aligned}\alpha_2 &= \chi(\alpha, \Phi(F), \Phi'(F)\, F, \ldots) \\ &= \chi(\alpha_1, \Phi, \Phi', \ldots, \Phi^{(n)}),\end{aligned}$$

from which, one sees precisely that equations (2) represent an infinite group.

Likewise, the equations:

$$x_1 = F(x), \qquad \xi_1 = \xi\, F'(x), \quad \alpha_1 = \chi(\alpha, F, F', \ldots, F^{(n)}); \tag{3}$$

determine an infinite group, under which, the differential equation (3) of the aforementioned family of groups obviously remains invariant.

We obtain the identity transformation of the infinite group (2) for $x_1 = x$, and then each and every group of the family (1) indeed remains invariant; for $x_1 = x$ one must then also have $\alpha_1 = \alpha$. Furthermore, this is true for the infinitesimal transformations of the group (2) when one sets $x_1 = x + \varphi(x)\, \delta t$, if one understands δt to mean an infinitesimal quantity. One thus obtains, perhaps:

$$\alpha_1 = \alpha + \Omega(\alpha, \varphi, \varphi', \ldots, \varphi^{(n)})\, \delta t,$$

where the terms of second and higher order have been omitted. The function Ω obviously vanishes when the arbitrary function φ is set equal to zero. Applying Lie's notation, the infinitesimal transformations of the group (2) can be written as follows:

$$\varphi(x)\frac{\partial f}{\partial x} + \Omega(\alpha, \varphi, \varphi', \ldots, \varphi^{(n)})\frac{\partial f}{\partial \alpha}. \tag{4}$$

If $x_1 = x + \varphi\, \delta t$ then one has:

$$\xi_1 = \xi + \xi\varphi'\, \delta t;$$

the infinitesimal transformations of the infinite group (3) will therefore be:

$$\varphi(x)\frac{\partial f}{\partial x} + \xi\varphi'\frac{\partial f}{\partial x} + \Omega(\alpha, \varphi, \varphi', \ldots, \varphi^{(n)})\frac{\partial f}{\partial \alpha}. \tag{5}$$

Naturally, the expressions (4) and (5) represent two infinite groups of infinitesimal transformations, and indeed, the second of these two groups leaves the differential equation (1) invariant.

As far as the form of the function Ω is concerned, it is easy to see that Ω must be linear and homogeneous in the $\varphi, \varphi', \ldots, \varphi^{(n)}$. Namely, if:

$$\psi(x)\frac{\partial f}{\partial x}+\Omega(\alpha,\psi,\psi',\ldots,\psi^{(n)})\frac{\partial f}{\partial \alpha}$$

is a second infinitesimal transformation of the group (4) then every transformation of the form:

$$(a\varphi+b\psi)\frac{\partial f}{\partial x}+(a\,\Omega(\alpha,\varphi,\ldots)+b\,\Omega(\alpha,\psi,\cdots))\frac{\partial f}{\partial \alpha}$$

also belongs to the group. This transformation must therefore possess the form:

$$(a\varphi+b\psi)\frac{\partial f}{\partial x}+\Omega(\alpha,a\,\varphi+b\,\psi,\cdots,a\,\varphi^{(n)}+b\,\psi^{(n)}))\frac{\partial f}{\partial \alpha},$$

and since $\Omega(\alpha, \varphi, \ldots)$ simultaneously vanishes with φ, one finds that $\Omega(\alpha, \varphi, \ldots)$ has the form:

$$\Omega(\alpha, \varphi, \ldots, \varphi^{(n)}) = a_0\,\varphi + a_1\,\varphi' + \ldots + a_n\,\varphi^{(n)},$$

where the coefficients a_i depend upon only α, but also contain no arbitrary constants.

2. I recently remarked that any infinite group whatsoever of infinitesimal transformations that possesses the form (4) belongs to a differential equation of the form (1), which defines an infinite family of groups of simple manifolds. Lie recognized the intrinsic basis for this result, which I found by computation. We next repeat Lie's line of reasoning.

The infinitesimal transformations:

$$\varphi(x)\frac{\partial f}{\partial x}+\sum_{i=0}^{n}a_i(\alpha)\varphi^{(i)}\frac{\partial f}{\partial \alpha}, \tag{6}$$

with the arbitrary function $\varphi(x)$, might define an infinite group. Furthermore, let $\alpha(x)$ be an arbitrarily-chosen function of x. Now, among the infinitesimal transformations (6), Lie looked for the ones that left the equation $\alpha - \alpha(x) = 0$ invariant. The totality of all these transformations must define a group of infinitesimal transformations (cf., the introduction, subsection E).

If one applies the infinitesimal transformation (6) to the expression $\alpha - \alpha(x)$ then one obtains:

$$\sum_{i=0}^{n}a_i(\alpha)\varphi^{(i)}-\alpha'(x)\varphi.$$

Now, should the equation $\alpha - \alpha(x) = 0$ remain invariant under a transformation of the form (6), the expression that we just found must vanish along with $\alpha - \alpha(x)$ (cf., introduction, E); i.e., the function φ must be a solution of the differential equation:

$$\sum_{i=0}^{n} a_i(\alpha(x))\varphi^{(i)} - \alpha'(x)\varphi = 0.$$

Thus, if one substitutes the general solution of the differential equation that we just obtained for φ in (6) then one obtains the desired group of infinitesimal transformations.

The group that we found includes the two variables x and α. Even more important is a closely-connected group in the just the variable x. Namely, the differential equation for φ obviously possesses the property that for any two of its solutions – say, φ and ψ – one can always combine them into a third solution of the form $\varphi\,\psi' - \psi\,\varphi'$. Thus, if one understands φ to mean the general solution of the differential equation above then $\varphi\, df / dx$ also represents a group of infinitesimal transformations. This group is obviously finite, and therefore also generates a finite transformation group. Thus, with no further assumptions, Lie obtained the theorem:

Theorem 1. *If the expression:*

$$\varphi(x)\frac{\partial f}{\partial x} + \sum_{i=0}^{n} a_i(\alpha)\varphi^{(i)}\frac{\partial f}{\partial \alpha}$$

represents an infinite group of infinitesimal transformations then the differential equation:

$$\sum_{i=0}^{n} a_i(\alpha)\xi^{(i)} - \alpha'(x)\xi = 0 \tag{7}$$

defines a finite group of transformations on the simple manifold when the α in it is set to any function of x.

As one sees, the process by which Lie arrived at this theorem leaves nothing to be desired in terms of simplicity and brevity.

My original proof started by proving that equation (7) defines a group by calculation. The proof also has a certain interest now, namely, due to the ultimate formula that it produced. For that reason, it might find its place here.

To abbreviate, we write $U(\alpha, \varphi)$ for $\sum a_i(\alpha)\,\varphi^{(i)}$. The infinite group (6) then assumes the form:

$$\varphi\frac{\partial f}{\partial x} + U(\alpha, \varphi)\frac{\partial f}{\partial \alpha}.$$

Furthermore, if:

$$\psi\frac{\partial f}{\partial x} + U(\alpha, \psi)\frac{\partial f}{\partial \alpha}$$

is any other infinitesimal transformation of the group then:

$$(\varphi\psi'-\psi\varphi')\frac{\partial f}{\partial x}+\left\{\varphi\sum_{i=0}^{n}\frac{\partial U(\alpha,\psi)}{\partial\psi^{(i)}}\psi^{(i+1)}-\psi\sum_{i=0}^{n}\frac{\partial U(\alpha,\varphi)}{\partial\varphi^{(i)}}\varphi^{(i+1)}\right.$$
$$\left.+\,U(\alpha,\psi)\frac{\partial U(\alpha,\psi)}{\partial\alpha}-U(\alpha,\psi)\frac{\partial U(\alpha,\varphi)}{\partial\alpha}\right\}\frac{\partial f}{\partial\alpha}$$

also belongs to the group (introduction to C); the coefficient of $\partial f/\partial\alpha$ in this transformation must therefore have the form $U(\alpha,\ \varphi\,\psi'-\psi\,\varphi')$.

In order to be able to write $U(\alpha,\ \varphi\,\psi'-\psi\,\varphi')$ as concisely as possible, we introduce the following relations. We set:

$$\sum_{i=0}^{n}\frac{\partial U(\alpha,\varphi)}{\partial\varphi^{(i)}}\varphi^{(i+1)} = \frac{d'U(\alpha,\varphi)}{dx}$$

and:

$$\sum_{i=0}^{n}\frac{\partial U(\alpha,\varphi)}{\partial\varphi^{(i)}}\varphi^{(i+1)}+\alpha'\frac{\partial U(\alpha,\varphi)}{\partial\alpha} = \frac{dU(\alpha,\varphi)}{dx},$$

such that one then has:

$$\frac{d'U(\alpha,\varphi)}{dx}=\frac{dU(\alpha,\varphi)}{dx}-\alpha'\frac{\partial U(\alpha,\varphi)}{\partial\alpha}.$$

We then understand α to mean the differential quotient $d\alpha\,/\,dx$ of α when we regard x as the independent variable and α as the dependent variable.

With these preparations, we obtain the identity:

$$U(\alpha,\ \varphi\,\psi'-\psi\,\varphi')$$
$$\equiv\ \varphi\frac{d'U(\alpha,\psi)}{dx}-\psi\frac{d'U(\alpha,\varphi)}{dx}+U(\alpha,\varphi)\frac{\partial U(\alpha,\psi)}{\partial\alpha}-U(\alpha,\psi)\frac{\partial U(\alpha,\varphi)}{\partial\alpha}$$

or, after a simple conversion:

(8) $$U(\alpha,\ \varphi\,\psi'-\psi\,\varphi')$$
$$\equiv\ \varphi\frac{d'U(\alpha,\psi)}{dx}-\psi\frac{d'U(\alpha,\varphi)}{dx}+\frac{\partial U(\alpha,\psi)}{\partial\alpha}\{U(\alpha,\varphi)-\alpha'\varphi\}-\frac{\partial U(\alpha,\varphi)}{\partial\alpha}\{U(\alpha,\psi)-\alpha'\psi\}.$$

If we finally subtract the identity:

$$\alpha'(\varphi\,\psi'-\psi\,\varphi')\equiv\alpha'\,\varphi\,\psi'+\alpha''\,\psi\,\varphi-\alpha'\,\psi\,\varphi'-\alpha''\,\psi\,\varphi$$

from this then this yields the important formula:

(9) $$U(\alpha,\ \varphi\,\psi'-\psi\,\varphi')-\alpha'(\varphi\,\psi'-\psi\,\varphi')$$

$$\equiv \varphi\frac{d}{\partial x}(U(\alpha,\psi)-\alpha'\psi)-\psi\frac{d}{dx}(U(\alpha,\varphi)-\alpha'\varphi)$$
$$+\frac{\partial U(\alpha,\psi)}{\partial\alpha}\{U(\alpha,\varphi)-\alpha'\varphi\}-\frac{\partial U(\alpha,\varphi)}{\partial\alpha}\{U(\alpha,\psi)-\alpha'\psi\}.$$

The proof of theorem 1 now lies in this formula. Namely, if we set α equal to an arbitrary function of x and assume that $\xi = \varphi$ and $\xi = \psi$ are two solutions of the differential equation (7) then the right-hand side of (9) vanishes identically. Since the left-hand side likewise vanishes as a result of this, $\varphi\,\psi' - \psi\varphi'$ is also a solution of (7), and this was to be proved.

3. *We will now show that the differential equation:*

(10) $$U(\alpha, \xi) - \alpha'\xi = 0$$

admits the infinite group of infinitesimal transformations:

(1) $$\varphi(x)\frac{\partial f}{\partial x}+\xi\varphi'\frac{\partial f}{\partial \xi}+U(\alpha,\varphi)\frac{\partial f}{\partial\alpha}.$$

From Lie's standpoint, the proof of this takes the brief form: As we saw above, equation (10) defines those of the infinitesimal transformations:

$$\chi(x)\frac{\partial f}{\partial x}+U(\alpha,\chi)\frac{\partial f}{\partial\alpha}$$

that leave invariant the equation $\alpha - \alpha(x) = 0$ when $\alpha(x)$ is understood to mean any function of x. Now, if, by performing the infinitesimal transformation (11) or:

(11′) $$x_1 = x + \varphi(x)\,\delta t, \qquad \xi_1 = \xi + \xi\,\varphi'\,\delta t, \qquad \alpha_1 = \xi + U(\alpha, \varphi)\,\delta t,$$

$\alpha - \alpha(x) = 0$ assumes the form $\alpha_1 - \alpha_1(x_1) = 0$ then the infinitesimal transformations that are defined by (10) must be converted into those transformations:

$$\chi(x_1)\frac{\partial f}{\partial x_1}+U(\alpha_1,\chi)\frac{\partial f}{\partial\alpha_1}$$

that leave $\alpha_1 - \alpha_1(x_1) = 0$ invariant. Therefore, by performing the infinitesimal transformation (11′), equation (10) goes to:

$$U(\alpha_1, \xi_1) - \alpha_1'\xi_1 = 0.$$

With that, our assertion is proved.

My older proof was perhaps more satisfying to some readers than the one that was just developed. My procedure was simply the following:

The variables x, ξ, and α were conferred the increments:

$$\delta x = \varphi\,\delta t, \qquad \delta\xi = \xi\,\varphi'\,\delta t, \quad \delta\alpha = U(\alpha, \varphi)\,\delta t,$$

and one then calculated the increment in $U(\alpha, \xi) - \alpha'\xi$.

From the fact that $d\alpha - \alpha' dx = 0$, this yields:

$$\delta\alpha' = \frac{d}{dx}\delta\alpha - \alpha'\frac{d}{dx}\delta x = \left(\frac{d}{dx}U(\alpha,\varphi) - \alpha'\varphi'\right)\delta t.$$

Likewise, one would have:

$$\frac{\delta\xi'}{\delta t} = \xi\,\varphi'', \qquad \frac{\delta\xi''}{\delta t} = \xi\,\varphi''' + \xi'\,\varphi'' - \xi''\,\varphi', \text{ etc.}$$

By the remark that one writes as:

$$\frac{\delta\xi}{\delta t} = \xi'\,\varphi + (\xi\,\varphi' - \varphi\,\xi'), \qquad \frac{\delta\xi'}{\delta t} = \xi\,\varphi'' + \frac{d}{dx}(\xi\,\varphi' - \varphi\,\xi'),$$

$$\frac{\delta\xi''}{\delta t} = \xi''\,\varphi' + \frac{d^2}{dx^2}(\xi\,\varphi' - \varphi\,\xi')$$

one would then be led to conjecture that in general the following formula might be true:

$$\frac{\delta\xi^{(i)}}{\delta t} = \xi^{(i+1)}\varphi + \frac{d^i}{dx^i}(\xi\,\varphi' - \varphi\,\xi').$$

In fact, from the equation:

$$d\xi^{(i)} - \xi^{(i)}\,dx = 0,$$

one again obtains:

$$\frac{\delta\xi^{(i+1)}}{\delta t} = \xi^{(i+2)}\varphi + \xi^{(i+1)}\varphi' + \frac{d^{i+1}}{dx^{i+1}}(\xi\,\varphi' - \varphi\,\xi') - \xi^{(i+1)}\varphi'$$

$$= \xi^{(i+2)}\varphi + \frac{d^{i+1}}{dx^{i+1}}(\xi\,\varphi' - \varphi\,\xi'),$$

with which the general validity of the conjectured formula is proved.

The desired increment can then be calculated. Since x and its differential quotients enter into $U(\alpha, \xi)$ only linearly and homogeneously, this yields:

$$\frac{\delta}{\delta t}(U(\alpha, \xi) - \alpha'\xi) = \frac{d'}{dx}U(\alpha, \xi)\cdot\varphi + U(\alpha, \xi\varphi' - \varphi\,\xi')$$

$$- \alpha'\,\xi\,\varphi' + U(\alpha, \xi)\frac{\partial}{\partial\alpha}U(\alpha, \xi) - \xi\left(\frac{d}{dx}U(\alpha,\varphi) - \alpha'\varphi'\right).$$

Here, the right-hand side can be converted by means of the identity (8), and one comes to the simple formula:

$$\frac{\delta}{\delta t}(U(\alpha,\,\xi) - \alpha'\xi) = \frac{\partial}{\partial \alpha}U(\alpha,\,\xi)\{U(\alpha,\,\xi\varphi' - \varphi\,\xi')\}. \tag{12}$$

This formula clearly shows that the equation:

$$U(\alpha,\,\xi) - \alpha'\xi = 0$$

admits the infinitesimal transformations (11).

We then have the theorem:

Theorem 2. *If the infinitesimal transformations:*

$$\varphi(x)\frac{\partial f}{\partial x} + \sum_{i=0}^{n} a_i(\alpha)\varphi^{(i)}\frac{\partial f}{\partial \alpha}$$

define an infinite group then the equation:

$$\sum_{i=0}^{n} a_i(\alpha)\,\xi^{(i)} - \alpha'\xi = 0$$

admits the infinite group of transformations:

$$\varphi(x)\frac{\partial f}{\partial x} + \xi\varphi'\frac{\partial f}{\partial \xi} + \sum_{i=0}^{n} a_i(\alpha)\varphi^{(i)}\frac{\partial f}{\partial \alpha}\,.$$

4. As we have seen, the differential equation (10) admits the infinitesimal transformations:

$$\begin{cases} \dfrac{\delta x}{\delta t}\dfrac{\partial f}{\partial x} + \dfrac{\delta \xi}{\delta t}\dfrac{\partial f}{\partial \xi} + \cdots + \dfrac{\delta \xi^{(n)}}{\delta t}\dfrac{\partial f}{\partial \xi^{(n)}} + \dfrac{\delta \alpha}{\delta t}\dfrac{\partial f}{\partial \alpha} + \dfrac{\delta \alpha'}{\delta t}\dfrac{\partial f}{\partial \alpha'} \\ \qquad \dfrac{\delta x}{\delta t} = \varphi(x),\ \dfrac{\delta \xi}{\delta t} = \xi\varphi',\ \dfrac{\delta \alpha}{\delta t} = U(\alpha, \varphi). \end{cases} \tag{13}$$

In the space of the variables x, ξ, …, $\xi^{(n)}$, α, α', we have before us the equation (10), presented as a manifold that remains invariant under the transformations (13), while its points will be permuted amongst themselves. Now, it is quite conceivable that the points of this manifold can be organized into infinitely many manifolds, in their own right, that individually remain invariant under the transformations (13). Analytically expressed, this takes the form: There can exist a function of x, ξ, …, $\xi^{(n)}$ that admits the transformations (13) when one considers equation (10). If one imagines that – say – $\xi^{(n)}$ has been eliminated by means of equation (10) then one is dealing with simply the existence of a function:

$$\Omega(x, \xi, \ldots, \xi^{(n-1)}; \alpha, \alpha')$$

that admits the transformations (13).

If such a function Ω *exists then it always represents an integral of the equation* (10) *when it is set equal to a constant.* We will now prove this.

The function $\Omega(x, \xi, \ldots, \alpha')$ is, by assumption, a solution of the equation:

$$(14)\qquad \frac{\delta x}{\delta t}\frac{\partial f}{\partial x}+\frac{\delta \xi}{\delta t}\frac{\partial f}{\partial \xi}+\cdots+\frac{\delta \xi^{(n-1)}}{\delta t}\frac{\partial f}{\partial \xi^{(n-1)}}+\frac{\delta \alpha}{\delta t}\frac{\partial f}{\partial \alpha}+\frac{\delta \alpha'}{\delta t}\frac{\partial f}{\partial \alpha'}=0.$$

Due to the arbitrariness of $\delta x/\delta t = \varphi(x)$, this equation decomposes into a series of differential equations, so the coefficients of $\varphi, \varphi', \ldots, \varphi^{(n+1)}$ must indeed vanish individually. Since $\varphi^{(n+1)}$ enters into $\delta\alpha / \delta t$ by itself, this immediately yields that $\partial f / \partial \alpha' = 0$, so α is not included in the possible solutions of (14). By comparison, α must necessarily be present. Namely, if one had $\partial f / \partial\alpha = 0$ then $\varphi^{(n)}$ would only enter into $\delta\xi^{(n)} / \delta t$, so $\partial f / \partial \xi^{(n-1)}$ would have to vanish. Likewise, one would have $\partial f / \partial \xi^{(n-2)} = 0$, etc., so, briefly, equation (14) would have only the unspoken solution f = const.

Now, let $\Omega(x, \xi, \ldots, \xi^{(n-1)}, \alpha)$ be a solution of (14), so $\delta\Omega / \delta t$ is identically zero. With the use of the previously-found expression for $\delta\xi^{(i)} / \delta t$, we then have:

$$\frac{\delta\Omega}{\delta t}=\varphi\frac{d\Omega}{dx}-\varphi\alpha'\frac{\partial\Omega}{\partial\alpha}+\sum_{i=0}^{n-1}\frac{\partial\Omega}{\partial\xi^{(i)}}\frac{d^i}{dx^i}(\xi\varphi'-\varphi\xi')+U(\alpha,\varphi)\frac{\partial\Omega}{\partial\alpha}\equiv 0.$$

If we set $\varphi = \xi$ in this then we get:

$$\xi\frac{d\Omega}{dx}+\frac{\partial\Omega}{\partial\alpha}\{U(\alpha,\xi)-\alpha'\xi\}\equiv 0,$$

such that equation $d\Omega / dx = 0$ differs from equation (10) only by a non-vanishing factor. In fact, Ω = const. is therefore an integral of the differential equation (10):

Theorem 3. *If the infinitesimal transformations:*

$$\varphi\frac{df}{dx}+\xi\varphi'\frac{\partial f}{\partial\xi}+\sum_{i=0}^{n}a_i(\alpha)\varphi^{(i)}\frac{\partial f}{\partial\alpha}$$

define an infinite group, and if $\Omega(x, \xi, \xi', \ldots, \xi^{(n-1)}, \alpha)$ *is a solution of the differential equation:*

$$(15)\qquad \frac{\delta x}{\delta t}\frac{\partial f}{\partial x}+\frac{\delta \xi}{\delta t}\frac{\partial f}{\partial \xi}+\cdots+\frac{\delta \xi^{(n-1)}}{\delta t}\frac{\partial f}{\partial \xi^{(n-1)}}+\frac{\delta \alpha}{\delta t}\frac{\partial f}{\partial \alpha}=0,$$

$$\frac{\delta x}{\delta t} = \varphi, \qquad \frac{\delta \xi}{\delta t} = \xi \varphi', \qquad \frac{\delta \alpha}{\delta t} = \sum_{i=0}^{n} a_i(\alpha)\,\varphi^{(i)}$$

then:

$$\Omega(x, \xi, \xi', \ldots, \xi^{(n-1)}, \alpha) = \text{const.}$$

is an integral of the n^{th}*-order differential equation of:*

$$\sum_{i=0}^{n} a_i(\alpha)\,\xi^{(i)} = 0.$$

In conclusion, we make a remark that is important in what follows: Namely, we will show that *in the infinite group:*

$$\varphi \frac{\partial f}{\partial x} + U(\alpha, \varphi) \frac{\partial f}{\partial \alpha}$$

one can introduce the new variable:

(16) $$\alpha_1 = \omega(\alpha) + \pi(x)$$

in place of α *without this having an essential effect on the differential equation:*

$$U(\alpha, \xi) - \alpha' \xi = 0.$$

The functions ω and π thus mean arbitrary functions of their arguments.

By the introduction of the new variables α_1, our infinite group assumes the form:

$$\varphi \frac{\partial f}{\partial x} + \{\varphi \cdot \pi'(x) + U(\alpha, \varphi)\omega'(\alpha)\} \frac{\partial f}{\partial \alpha_1}.$$

This now yields the differential equation:

$$\xi\, \pi'(x) + U(\alpha, \xi)\, \omega'(\alpha) - (\omega'(\alpha)\, \alpha' + \pi'(\alpha))\, \xi = 0,$$

so by omitting the factor $\omega'(\alpha)$, one again gets the earlier equation:

$$U(\alpha, \xi) - \alpha' \xi = 0.$$

We see from this that two infinite groups of the form:

(17) $$\varphi \frac{\partial f}{\partial x} + U(\alpha, \varphi) \frac{\partial f}{\partial \alpha}$$

always produce the same family of groups on the simple manifold when they are similar to each other under a transformation of the form (16). *In the following paragraphs, where we determine all of the various infinite groups* (17), *we will not need to exhibit all* *of these groups, but only the simplest normal form of all the different types that do not go to each other under transformations of the form* (16).

§ 2. The individual groups on the simple manifold.

5. We now turn our attention to *the determination of all infinite groups of infinitesimal transformations:*

$$\varphi\frac{\partial f}{\partial x}+U(\alpha,\varphi)\frac{\partial f}{\partial \alpha}.$$

The general infinitesimal transformation of such a group has the form:

$$Xf = \varphi\frac{\partial f}{\partial x}+(a_0(\alpha)\varphi+a_1(\alpha)\varphi'+\cdots+a_n(\alpha)\varphi)^{(n)})\frac{\partial f}{\partial \alpha},$$

where the coefficients are functions of α, although the contain no arbitrary constants. If we introduce the general symbol $A_i\, f$ for the infinitesimal transformation $a_i\, \partial f / \partial\alpha$ then we obtain:

$$Xf = \varphi\frac{\partial f}{\partial x}+\sum_{i=0}^{n} A_i\, f\cdot\varphi^{(i)}.$$

Let another infinitesimal transformation of the group be:

$$Yf = \psi\frac{\partial f}{\partial x}+\sum_{i=0}^{n} A_i\, f\cdot\psi^{(i)}.$$

We now define the transformation (X Y), or as we would like to say, more briefly, in the future: We *compose* the two transformations Xf and Yf with each other. The transformation (X Y) must then again belong to the group, so one must have:

(1) $$(X\ Y) = (\varphi\psi'-\psi\,\varphi')\frac{\partial f}{\partial x}+\sum_{i=0}^{n} A_i\, f\,\frac{d^i}{dx^i}(\varphi\psi'-\psi\,\varphi').$$

On the other hand, one obtains by direct calculation:

(2) $$X(Y(f)) - Y(X(f)) = (\varphi\psi'-\psi\,\varphi')\frac{\partial f}{\partial x}+\sum_{i=0}^{n} A_i\, f\,\frac{d^i}{dx^i}(\varphi\psi^{(i+1)}-\psi\,\varphi^{(i+1)})$$
$$+\sum_{i=0}^{n-1}\sum_{n=0}^{n}(A_i A_\kappa)\,(\varphi^{(i)}\,\psi^{(\kappa)}-\psi^{(i)}\,\varphi^{(\kappa)}).$$

The two equations (1) and (2) must agree. This requirement can be expressed in an equation of the form:

$$\sum_{i=0}^{n-1}\sum_{n=0}^{n} g_{i\kappa}\,(\varphi^{(i)}\,\psi^{(\kappa)} - \psi^{(i)}\,\varphi^{(\kappa)}) = 0,$$

in which the $g_{i\kappa}$ are independent of x. However, since φ and ψ are arbitrary, one convinces oneself with no difficulty that this equation can be true only when all of the $g_{i\kappa}$ vanish.

In order to compute the expressions $g_{i\kappa}$ for the present case, we must next develop the right-hand side of (1). With the use of the known formulas:

$$\frac{d^i}{dx^i}(\varphi\psi') = \varphi\,\psi^{(i+1)} + \frac{n}{1}\varphi'\psi^{(i)} + \frac{n(n-1)}{1\cdot 2}\,\varphi''\psi^{(i-1)} + \ldots,$$

$$\frac{d^i}{dx^i}(\psi\varphi') = \qquad + \quad \varphi'\psi^{(i)} + \quad \frac{n}{1}\quad \varphi''\psi^{(i-1)} + \ldots,$$

we obtain:

$$\frac{d^i}{dx^i}(\psi\varphi' - \psi\varphi') = \varphi\,\psi^{(i+1)} + (n-1)\varphi'\psi^{(i)} + \left(\frac{n(n-1)}{1\cdot 2} - n\right)\varphi''\psi^{(i-1)} + \ldots,$$

and from this, by an easy calculation, we obtain:

$$\frac{d^i}{dx^i}(\psi\varphi' - \psi\varphi') = \varphi\,\psi^{(i+1)} + (n-1)(\varphi'\psi^{(i)} - \psi'\varphi^{(i)}) + \left(\frac{n(n-1)}{1\cdot 2} - n\right)\varphi''\psi^{(i-1)} + \ldots$$

This series concludes with any i that has the form $2\nu - 1$ or 2ν, with a term that has the form $\varphi^{(\nu-1)}\,\psi^{(\nu+1)} - \varphi^{(\nu-1)}\,\psi^{(\nu+1)}$, or the form:

$$\varphi^{(\nu)}\,\psi^{(\nu+1)} - \varphi^{(\nu)}\,\psi^{(\nu+1)}.$$

If we now compare the two expressions for $X(Y(f)) - Y(X(f))$, after substituting the expression that we just found in (1), then we next find that the term:

$$\sum_{i=0}^{n} A_i\,f\,(\varphi\,\psi^{(i+1)} - \psi\,\varphi^{(i+1)})$$

drops out. We further obtain:

$$(3)\qquad \begin{cases} (A_0A_i) = 0 & (i = 1, \ldots, n), \\ (A_1A_i) = (i-1)A_i & (i = 2, \ldots, n), \\ (A_2A_{i-1}) = \left(\dfrac{i(i-1)}{1\cdot 2} - i\right)A_i & (i = 3, \ldots, n). \end{cases}$$

If we then consider that no term of the form $\varphi^{(\nu)}\,\psi^{(n)} - \psi^{(\nu)}\,\varphi^{(n)}$ appears in (1) when the sum $\nu + \pi$ exceeds the number $n + 1$ then we have:

$$(A_2\,A_n) = (A_3\,A_n) = \dots = (A_{n-1}\,A_n) = 0 \qquad (n > 2).$$

We have thereby in no way presented all of the relations between $A_0 f$, $A_1 f$, …, $A_n f$, so we will at least show that the ones presented suffice to determine all of the infinitesimal transformations $A_i f$.

6. *Naturally, the simplest case is that of* $n = 0$. Then, either $Af = 0$ or is it non-zero. With the first assumption, we obtain the infinite group:

$$\varphi\frac{\partial f}{\partial x},$$

and with the second one, we can always think of the variable α as being chosen such that $A_0 f$ assumes the form $\partial f / \partial\alpha$. In this way, we thus obtain the infinite group:

$$\varphi\frac{\partial f}{\partial x} + \varphi\frac{\partial f}{\partial \alpha}.$$

Since this is not essentially different from what we just found, we introduce $\alpha - x$ as a new α, so it also takes on the form:

$$\varphi\frac{\partial f}{\partial x}.$$

We can assume that *n is greater than zero* and that $A_n f$ does not vanish. *We will show that the number n cannot be greater than* 3.

For $n > 2$, we found above that $(A_{n-1}\,A_n) = 0$; however, since the infinitesimal transformations $A_i f$ depend upon only the one variable α, it follows from $(A_{n-1}\,A_n) = 0$ that $A_{n-1} = \lambda_{n-1}\,A_n$, where λ_{n-1} means a constant. Due to equations (3), one now has:

$$\begin{aligned}(A_1\,A_{n-1}) = (n-2)\,A_{n-1} &= \lambda_{n-1}\,(n-2)\,A_n, \\ = \lambda_{n-1}\,(A_1\,A_n) &= \lambda_{n-1}\,(n-1)\,A_n,\end{aligned}$$

from which, since $A_n \not\equiv 0$, it follows that $\lambda_{n-1} = 0$; that is, $A_{n-1} = 0$.

On the other hand, we have:

$$(A_2\,A_{n-1}) = 0 = \left(\frac{n(n-1)}{1\cdot 2} - n\right)A_n,$$

an equation that can only be valid for $n = 3$, since A_n must necessarily vanish for $n > 3$, which is excluded. *Therefore,* 3 *is, in fact, the largest value that the number n can have.*

Furthermore, if $(A_0\,A_n) = 0$ then $A_0 = \lambda_0\,A_n$, and since, in addition, one has the equations:

$$(A_1\, A_0) = 0 = \lambda_0\, (A_1\, A_n) = \lambda_0\, (n - 1)\, A_n\,,$$

λ_0 *always vanishes, and therefore* A_0*, as well, whenever* n *is greater than* 1.

We must now treat the cases n = 1, 2, 3 individually.

1. *Case* $n = 1$:

$A_0 = \lambda_0 A_1$: If we introduce a suitable function of α as a new α then we can bring $A_1 f$ into the form $\partial f / \partial\alpha$ and thus obtain the infinite group:

$$\varphi \frac{\partial f}{\partial x} + (\lambda_0 \varphi + \varphi') \frac{\partial f}{\partial \alpha}.$$

If we then introduce $\alpha - \lambda_0\, x$ as the new α then the group takes the form:

$$\varphi \frac{\partial f}{\partial x} + \varphi' \frac{\partial f}{\partial \alpha}.$$

2. *Case* $n = 2$:

As we saw above:

$$A_0 = 0, \qquad (A_1, A_2) = A_2\,.$$

If we then choose $A_2 f = \partial f / \partial\alpha$ then this gives:

$$A_1 f = -(\alpha + \text{const.}) \frac{\partial f}{\partial \alpha},$$

or, when we introduce α + const. as the new α:

$$A_2 f = \frac{\partial f}{\partial \alpha}, \qquad A_1 f = -\alpha \frac{\partial f}{\partial \alpha}.$$

The infinite group in question is therefore:

$$\varphi \frac{\partial f}{\partial x} + (\varphi'' - \alpha\, \varphi') \frac{\partial f}{\partial \alpha}.$$

3. *Case* $n = 3$.

The relations between $A_0 f, \ldots, A_3 f$ that were found above are:

$$A_0 = A_2 = 0, \qquad (A_1\, A_3) = 2\, A_3\,.$$

If we set $A_3 f = \partial f / \partial\alpha$ then it follows that:

$$A_1 f = -(2\alpha + \text{const.})\,\frac{\partial f}{\partial \alpha},$$

and when we discard the constant as we did in the second case, we get:

$$A_3 f = \frac{\partial f}{\partial \alpha}, \qquad A_1 f = -2\alpha\,\frac{\partial f}{\partial \alpha}.$$

We then have the following infinite group:

$$\varphi\frac{\partial f}{\partial x} + (\varphi''' - 2\alpha\varphi')\frac{\partial f}{\partial \alpha}.$$

7. In the foregoing number, we saw that *by a suitable choice of* α*, any infinite group of the form:*

$$\varphi\frac{\partial f}{\partial x} + U(\alpha, \varphi)\frac{\partial f}{\partial \alpha}$$

can be put into one of the four forms:

$$(4)\qquad \begin{cases} \varphi\dfrac{\partial f}{\partial x}, & \varphi\dfrac{\partial f}{\partial x} + \varphi'\dfrac{\partial f}{\partial \alpha}, \\ \varphi\dfrac{\partial f}{\partial x} + (\varphi'' - \alpha\varphi')\dfrac{\partial f}{\partial \alpha}, & \varphi\dfrac{\partial f}{\partial x} + (\varphi''' - 2\alpha\varphi')\dfrac{\partial f}{\partial \alpha}. \end{cases}$$

From theorem 1 that was proved in § 1, no. 2, when one replaces α with a well-defined function of x the equation $U(\alpha, \xi) - \alpha'\xi = 0$ always represents the defining equation of a group. If we apply this to the present four groups (2) then we next remark that the group $\varphi\,\partial f / \partial x$ yields only that $\alpha'\xi = 0$, so $\xi = 0$. By contrast, we obtain non-trivial groups in the remaining three cases, namely, the following ones:

$$(5)\qquad \begin{cases} \xi' - \alpha'\xi = 0, \\ \xi'' - \alpha\xi' - \alpha'\xi = 0, \\ \xi''' - 2\alpha\xi' - \alpha'\xi = 0. \end{cases}$$

Therefore, any defining equation $U(\alpha, \xi) - \alpha'\xi = 0$ *that belongs to an infinite group:*

$$\varphi\frac{\partial f}{\partial x} + U(\alpha, \varphi)\frac{\partial f}{\partial \alpha}$$

may be brought into one of these three forms.

As is known from Lie's investigations, the three group types (5) are *the only* ones on the simple manifold.

The equation $\xi' - \alpha'\xi = 0$ defines a one-parameter group that is similar to the group df / dx or $x_1 = x + a$. The equation $\xi'' - \alpha\,\xi' - \alpha'\xi = 0$ defines the groups that are similar to the group df / dx, $x\,df / dx$, or $x_1 = ax + b$. Finally, the equation $\xi''' - 2\alpha\,\xi' - \alpha'\xi = 0$ defines those groups that are similar to the general projective group $df / dx, x\,df / dx, x^2\,df / dx$.

Our method thus yields all of the groups on the simple manifold. This was by no means generally clear, *a priori*, and we must actually examine why we have obtained all of the groups. (Cf., what was said in the last part of the introduction.)

8. *We now examine whether we arrive at integrals of the defining equations* (5) *with the use of theorem* 3, § 1, no. 4.

We shall go through each of the three cases in turn, present the defining equation (15) for each, and look for its integrals.

Case 1. The equation:

$$\varphi\frac{\partial f}{\partial x} + \xi\varphi'\frac{\partial f}{\partial \xi} + \varphi'\frac{\partial f}{\partial \alpha} = 0$$

decomposes into the two:

$$\frac{\partial f}{\partial x} = 0, \qquad \xi\frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \alpha} = 0.$$

The only solution to it is $\xi\,e^{-\alpha}$, so $\xi\,e^{-\alpha}$ = const. is the integral of the differential equation $\xi' - \alpha'\xi = 0$.

Case 2. The equation:

$$\varphi\frac{\partial f}{\partial x} + \xi\varphi'\frac{\partial f}{\partial \xi} + \xi\varphi''\frac{\partial f}{\partial \xi'} + (\varphi'' - \alpha\varphi')\frac{\partial f}{\partial \alpha} = 0$$

decomposes into two, as follows:

$$\frac{\partial f}{\partial x} = 0, \qquad \xi\frac{\partial f}{\partial \xi} - \alpha\frac{\partial f}{\partial \alpha} = 0, \qquad \xi\frac{\partial f}{\partial \xi'} + \frac{\partial f}{\partial \alpha} = 0.$$

One obtains an integral of the equation $\xi'' - \alpha\,\xi' - \alpha'\xi = 0$ from this, namely, $\xi' - \alpha\,\xi$ = const.

Case 3.

$$\varphi\frac{\partial f}{\partial x} + \xi\varphi'\frac{\partial f}{\partial \xi} + \xi\varphi''\frac{\partial f}{\partial \xi'} + (\xi\varphi''' + \xi'\varphi'' - \xi''\varphi')\frac{\partial f}{\partial \xi''} + (\varphi'' - 2\alpha\varphi')\frac{\partial f}{\partial \alpha} = 0$$

yields:

$$\frac{\partial f}{\partial x} = 0, \qquad \xi\frac{\partial f}{\partial \xi} - \xi''\frac{\partial f}{\partial \xi''} - 2\alpha\frac{\partial f}{\partial \alpha} = 0,$$

$$\xi \frac{\partial f}{\partial \xi'} + \xi' \frac{\partial f}{\partial \xi''} = 0, \qquad \xi \frac{\partial f}{\partial \xi''} + \frac{\partial f}{\partial \alpha} = 0.$$

One finds an integral of the differential equation $\xi'' - 2\alpha\,\xi' - \alpha'\,\xi = 0$, namely:

$$\xi\,\xi'' - \tfrac{1}{2}\xi'^2 - \alpha\,\xi^2 = \text{const.}$$

In each individual case, one easily verifies that one, in fact, has an integral of the differential equation in question before one.

This would be a suitable place to go into the theory of integrating the equation $\xi''' - 2\alpha\,\xi' - \alpha'\,\xi' = 0$ *in a bit more detail.* By contrast, we do not need to add anything further to the two other defining equations, since from what we said, it already emerges that both of them can be solved by quadratures.

We thus imagine that we are given the differential equation:

$$\xi''' - 2\alpha\,\xi' - \alpha'\,\xi' = 0, \tag{6}$$

and have replaced α with any given function of x.

Indeed, we already know an integral of equation (6), namely:

$$\xi\,\xi'' - \tfrac{1}{2}\xi'^2 - \alpha\,\xi^2 = \text{const.},$$

but that, in itself, does not give us any advantage.

The group that is defined by (6) is simple, if it is composed the same way as, and is likewise similar to, the general projective group df / dx, $x\, df / dx$, $x^2\, df / dx$ of the simple manifold. The desired group thus includes ∞^1 two-parameter subgroups, which, from what was said above, are defined by differential equations of the form:

$$\xi'' - \beta(x)\xi' - \beta'(x)\,\xi = 0.$$

We would like to determine the general defining equation of these ∞^1 subgroups. To that end, we differentiate the equation $\xi'' - \beta\xi' - \beta'\,\xi = 0$ with respect to x:

$$\xi''' - \beta\,\xi'' - 2\beta'\,\xi - \beta''\,\xi = 0$$

and then eliminate ξ'':

$$\xi''' - (\beta^2 + 2\beta)\,\xi' - (\beta'\,\xi + \beta'')\xi = 0.$$

This equation must agree with (6), so β must then satisfy the differential equation:

$$\beta' + \tfrac{1}{2}\beta^2 = \alpha. \tag{7}$$

Now, if $\beta = \beta(x, a)$ is the general solution of this equation then $\xi'' - \beta(x, a)\,\xi' - \beta'(x)\,\xi = 0$ defines the aforementioned ∞^1 subgroups, and it is obviously likewise an integral of equation (6).

We can now find another integral with the help of the integral $\xi\xi'' - \frac{1}{2}\xi'^2 - \alpha\,\xi^2 =$ const.; it is, however, more convenient, and comes down to the same thing, if we make use of the previously-found integral of the equation $\xi'' - \beta\,\xi' - \beta'\,\xi = 0$. In this way, we obtain an integral of (6) with two arbitrary constants:

$$\xi' - \beta(x, a)\,\xi = \text{const.}$$

The integration of this equation ostensibly requires a quadrature, but it can be achieved without one. The following considerations show this to us, but we shall, however, touch upon them briefly, because they – or at least, ones that are entirely similar to them – have already been applied several times by Lie.

In the three-parameter group df / dx, $x\,df / dx$, $x^2\,df / dx$, and likewise in each of the ones that are equally-composed with it, two two-parameter subgroups always have a one-parameter subgroup in common, as one may show, and also any one-parameter subgroup is included in two two-parameter subgroups. We can account for this remark here.

If we imagine that a and b are fixed, but still arbitrarily-chosen, constants in the two equations:

$$\xi' - \beta(x, a)\,\xi = \text{const.}, \qquad \xi' - \beta(x, b)\,\xi = \text{const.}$$

then these two equations collectively define the infinitesimal transformation that is common to two well-defined two-parameter subgroups of the desired group.

We thus obtain the following expression for this infinitesimal transformation:

$$\xi = \frac{\text{const.}}{\beta(x,a) - \beta(x,b)}. \tag{8}$$

However, if we let a and b vary in this then we obtain all infinitesimal transformations that are common to two two-parameter subgroups, so from the above, we obtain all infinitesimal transformations of the desired group.

Equation (8) thus represents the general solution of equation (6). One can, moreover, verify this directly when one differentiates (8), and eliminates the functions β and the constant c by means of equation (7), which $\beta(x, a)$ and $\beta(x, b)$ both satisfy.

The integration of the third order differential equation (6) *is therefore immediately executable when the first-order differential equation* (7) *is completely integrated.*

9. It was once provisionally mentioned above (cf., no. 7) that *any of the three defining equations:*

$$\xi' - \alpha'\,\xi = 0, \qquad \xi'' - \alpha\,\xi' - \alpha'\,\xi = 0, \qquad \xi''' - 2\alpha\,\xi' - \alpha'\,\xi = 0 \tag{9}$$

define an infinite family of groups that are similar to each other. At the time, this was resolved by referring to Lie's determination of the groups on the simple manifold. Now, we would like to go into this matter in a bit more detail.

Should all groups that are defined by the defining equation:

(10) $$U(\alpha, \xi) - \alpha' \xi = 0$$

be similar to each other then it is next necessary that any group of this family should goes to another group of the family under the introduction of a new variable $x_1 = F(x)$, in place of x; in addition, however, any group of the family must go to another one.

If we make the substitution $x_1 = F(x)$, and correspondingly, $\xi_1 = \xi F'(x)$, then we obtain an equation of the form:

(11) $$\sum_{i=1}^{n} B_i(\alpha, F, F', \ldots)\xi_1^{(i)} - \frac{\alpha'}{F'}\xi_1 = 0,$$

where $\xi_1^{(i)}$ is written for $d^i\xi_1 / dx_i^i$. Now, should any group again go to a group of the family then when one expresses the coefficients in terms of the new variable x_1 this latter equation can take on the form:

$$U(\alpha_1, \xi_1) - \alpha_1' \xi_1 = 0,$$

in which α_1 means a function of x_1. This demand gives one or more equations for the determination of α_1 and α_1' as functions of $\alpha, \alpha', F, F', \ldots$

We now observe the form that things take for the three defining equations (9).

We first find:

$$\xi = \frac{\xi_1}{F'}, \qquad \xi' = \xi_1' - \xi_1 \frac{F''}{F'^2},$$

$$\xi'' = \xi_1'' F - \xi_1' \frac{F''}{F'} - \xi_1 \frac{F' F''' - 2F''^2}{F'^3},$$

$$\xi''' = \xi_1''' F'^2 - \xi_1' \frac{2F' F''' - 3F''^2}{F'^2} - \xi_1 \frac{d}{dx} \frac{F' F''' - 2F''^2}{F'^3}.$$

If we use these formulas as the basis for defining the equation (11) that corresponds to each of the three differential equations (9) and require that the three equations that are found in this way are identical with:

$$\xi_1' - \alpha_1' \xi_1 = 0, \qquad \xi_1'' - \alpha_1 \xi_1' - \alpha_1' \xi_1 = 0, \qquad \xi_1''' - 2\alpha_1 \xi_1' - \alpha_1' \xi_1 = 0$$

then we obtain the following:

Case 1.

$$\alpha_1' = \frac{\alpha'}{F'} + \frac{F''}{F'^2},$$

so, since one has $\alpha_1' F' = d\alpha_1 / dx$:

(12) $$x_1 = F(x), \qquad \xi_1 = \xi\, F'(x), \qquad \alpha_1 = \alpha + l\, F' + \text{const.}$$

In the two remaining cases, one derives two equations, of which, we will, however, only need to exhibit the first one, since the other one emerges from it by differentiation.

Case 2.

(13) $$x_1 = F(x), \qquad \xi_1 = \xi\, F'(x), \qquad \alpha_1 = \frac{\alpha}{F'} + \frac{F''}{F'^2}.$$

Case 3.

(14) $$x_1 = F(x), \qquad \xi_1 = \xi\, F'(x), \qquad \alpha_1 = \frac{\alpha}{F'^2} + \frac{F'F''' - \frac{3}{2}F''^2}{F'^4}.$$

If we understand $F(x)$ to mean an arbitrary function of x here then in each of these three cases we obviously have an infinite group of finite transformations that leaves invariant the differential equation (9) in question. The individual groups that such a differential equation define for the various values of the function $\alpha(x)$ will then be permuted amongst themselves.

In each of the three cases, we can, moreover, replace α with an arbitrary function $\alpha(x)$ and α_1 with an arbitrary function $\alpha_1(x_1)$, and then obtain, in each case, a differential equation for the determination of $F(x)$. From this, we conclude that – e.g., by means of the infinite group (13) – any arbitrary group $\xi'' - \alpha(x)\,\xi' - \alpha'(x)\,\xi = 0$ can go to any other arbitrary group $\xi_1'' - \alpha_1(x_1)\,\xi_1' - \alpha_1'(x_1)\,\xi_1 = 0$, and likewise in the remaining two cases. However, this was just what we wished to prove.

If we substitute a *well-defined* function $\alpha(x)$ for α in (13) and replace α_1 with the same function of x_1 – namely, $\alpha(x_1)$ – then we obtain the following differential equation for F:

(15) $$\alpha(x_1) = \frac{\alpha(x)}{F'} + \frac{F''}{F'^2}.$$

These define all transformations $x_1 = F(x)$ that convert the equation $\xi'' - \alpha(x)\,\xi' - \alpha'(x)\,\xi = 0$ into $\xi_1'' - \alpha_1(x_1)\,\xi_1' - \alpha_1'(x_1)\,\xi_1 = 0$. It is clear that all transformations $x_1 = F(x)$ with this property define a finite group. In order to find its infinitesimal transformations, we set $x_1 = F(x) = x + \xi\,\delta t$, so that:

$$\alpha(x_1) = \alpha(x) + \alpha'(x)\,\xi\,\delta t,$$

and from (15), one deduces the following differential equation for ξ:

(16) $$\xi'' - \alpha(x)\,\xi' - \alpha'(x)\,\xi = 0.$$

The differential equation:

$$\frac{d^2x_1}{dx^2}+\alpha(x)\frac{dx_1}{dx}-\alpha(x_1)\left(\frac{dx_1}{dx}\right)^2=0$$

thus defines the finite equations $x_1 = F(x)$ of the group whose infinitesimal transformations are defined by the equation (16).

We likewise recognize that the differential equation:

$$\frac{dx_1}{dx}\frac{d^3x_1}{dx^3}-\frac{3}{2}\left(\frac{d^2x_1}{dx^2}\right)^2+\alpha(x)\left(\frac{dx_1}{dx}\right)^2-\alpha(x_1)\left(\frac{dx_1}{dx}\right)^4=0$$

defines the finite transformations of the group:

(17) $$\xi''' - 2\alpha(x)\,\xi' - \alpha(x)\,\xi = 0.$$

We would not like to go into the relationship of the differential equation $\xi' - \alpha(x)\,\xi = 0$ to the finite group (12), in order to not go too far afield. However, we would like to make a brief remark on the two groups (13) and (14).

The equations of these groups are obtained from the above without integration. However, the knowledge of these equations allows us to find the integrals of the two differential equations (16) and (17) that were discussed in no. 8 without integration.

We will only hint at the proof. The integral of equation (17), for example, has the form:

$$\xi\,\xi'' - \tfrac{1}{2}\xi'^2 - \alpha\,\xi^2 = \text{const.},$$

where the function on the left admits the infinite group of infinitesimal transformations:

$$\varphi\frac{\partial f}{\partial x}+\xi\varphi'\frac{\partial f}{\partial\xi}+(\varphi'''-2\alpha\varphi')\frac{\partial f}{\partial\alpha}.$$

Now, this function must also admit the infinite group of finite transformations (14), and one must therefore obtain an equation of the form:

$$\xi_1'\,\xi_1''-\tfrac{1}{2}\xi_1'^2-\alpha_1\xi_1^2=\xi'\,\xi''-\tfrac{1}{2}\xi'^2-\alpha\xi^2$$

from equations (14), with the addition of the expression for $\xi_1'\,\xi_1''$, when one eliminates φ and its differential quotients, This is, in fact, also the case.

One likewise easily obtains from (13):

$$\xi_1'-\alpha_1\xi_1=\xi'-\alpha\,\xi.$$

The two integrals are, in fact, found without integration.

§ 3. General developments for the *n*-fold extended manifold.

10. We now turn to the task of working through the considerations of numbers **2** and **3** of § 1 for the *n*-fold extended manifold. Still, we can now make things somewhat more concise.

In the variables $x_1, \dots, x_n, \alpha_1, \dots, \alpha_m$:

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} \Omega_\kappa \left(\alpha_1, \dots, \alpha_n, \varphi_1, \dots, \varphi_n, \frac{\partial \varphi_1}{\partial x_1}, \dots, \frac{\partial^2 \varphi_1}{\partial x_1^2} \right) \frac{\partial f}{\partial \alpha_\kappa}$$

represents an infinite group of infinitesimal transformations. We thus understand the φ_i to mean arbitrary functions of $x_1, \dots, x_n$; in addition to the $\alpha_1, \dots, \alpha_m$, the Ω_κ include the φ_i , along with their differential quotients up to a certain order. In addition, we assume that the functions Ω_κ include no arbitrary constants, and are therefore determined completely when the φ_i are given; finally, one can have $\Omega_1 = \dots = \Omega_m = 0$ when all φ_i vanish.

If we add a second infinitesimal transformation to the above group – say:

$$\sum_{i=1}^{n} \psi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} \Omega_\kappa \left(\alpha_1, \dots, \alpha_n, \psi_1, \dots, \psi_n, \frac{\partial \psi_1}{\partial x_1}, \dots \right) \frac{\partial f}{\partial \alpha_\kappa},$$

then:

$$\sum_{i=1}^{n} (a\varphi_i + b\psi_i) \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} \big(a\Omega_\kappa(\alpha, \varphi) + b\Omega_\kappa(\alpha, \psi) \big) \frac{\partial f}{\partial \alpha_\kappa}$$

must also belong to the group; under the assumptions made, the coefficients of the $\partial f / \partial \alpha_\kappa$ must therefore be able to include the form $\Omega_\kappa(\alpha, a\varphi + b\psi)$. From this, we conclude that the functions Ω_κ are linear and homogeneous in the φ_i and their differential quotients.

As before, we set, to abbreviate:

$$\Omega_\kappa \left(\alpha_1, \dots, \alpha_m, \varphi_1, \dots, \varphi_n, \frac{\partial \varphi_1}{\partial x_1}, \dots \right) = U_\kappa(\alpha, \varphi).$$

Among the infinitesimal transformations:

$$(1) \qquad \sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} U_\kappa(\alpha, \varphi) \frac{\partial f}{\partial \alpha_\kappa},$$

we seek the ones that leave invariant the system of equations:

$$(2) \qquad \alpha_1 - \alpha_1(x_1, \dots, x_n) = 0, \dots, \alpha_m - \alpha_m(x_1, \dots, x_n) = 0.$$

As we know, the totality of all these transformations – if there are any such things, at all – then defines a group of infinitesimal transformations.

Should the system of equations (2) admit the infinitesimal transformations (1) then the expressions:

$$U_\kappa(\alpha,\ \gamma) - \sum_{i=1}^{n} \varphi_i \frac{\partial}{\partial x_i} \alpha_\kappa(x_1, \ldots, x_n)$$

must vanish under the substitutions $\alpha_\kappa = \alpha_\kappa(x_1, \ldots, x_n)$; i.e., $\varphi_1, \ldots, \varphi_n$ must satisfy the m differential equations:

(3) $$U_\kappa(\alpha,\ \gamma) - \sum_{i=1}^{n} \alpha_{\kappa i}\varphi_i = 0 \qquad (\kappa = 1, \ldots, m).$$

In this, the function $\alpha_\kappa(x_1, \ldots, x_n)$ is supposed to replace the α_κ; $\alpha_{\kappa i}$ is written as an abbreviation for $\partial\alpha_\kappa / \partial x_i$. For the time being, we assume that equations (3) actually possess solutions. If we then substitute the most general solutions of (3) in place of the $\varphi_1, \ldots, \varphi_n$ in (1) then we obtain a group of infinitesimal transformations in the variables $x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m$.

However, equations (3) also define a group in the variables $x_1, \ldots, x_n$ alone, and the latter is the one that interests us here. If we have two systems of solutions $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$, of these equations then obviously:

$$\sum_{i=1}^{n}\left(\varphi_i \frac{\partial \psi_j}{\partial x_i} - \psi_i \frac{\partial \varphi_j}{\partial x_i}\right), \qquad (j = 1, \ldots, n)$$

is also a system of solutions.

The system of defining equations:

(3′) $$U_\kappa(\alpha,\ \xi) - \sum_{i=1}^{n} \alpha_{\kappa i}\xi_i = 0 \qquad (\kappa = 1, \ldots, m)$$

now admits the infinite group of infinitesimal transformations:

(4) $$\sum_{i=1}^{n}\varphi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_j \frac{\partial \varphi_i}{\partial x_j}\frac{\partial f}{\partial \xi_i} + \sum_{\kappa=1}^{m} U_\kappa(\alpha, \varphi)\frac{\partial f}{\partial \alpha_\kappa},$$

which reads, when written more completely:

(4′) $$x_i' = x_i + \varphi_i(x_1, \ldots, x_n)\ \delta t, \qquad \xi_i' = \xi_i + \sum_{j=1}^{n}\xi_j \frac{\partial \varphi_i}{\partial x_j}\ \delta t,$$

$$\alpha_\kappa' = \alpha_\kappa + U_\kappa(\alpha,\ \varphi)\ \delta t.$$

In order to prove this, we next recall the fact that under the substitution $\alpha_\kappa = \alpha_\kappa(x_1, \dots, x_n)$ equations (3′) define those infinitesimal transformations of (1) that leave the system of equations $\alpha_\kappa - \alpha_\kappa(x_1, \dots, x_n) = 0$ invariant. Now, if one goes from this system of equations to $\alpha'_\kappa - \alpha'_\kappa(x'_1, \dots, x'_n) = 0$ by performing the transformations (4′) then the transformations that are defined by (3′) must be converted into those transformations (1) that leave the system of equations $\alpha'_\kappa - \alpha'_\kappa(x'_1, \dots, x'_n) = 0$ invariant. In other words, by means of the transformations (4′), equations (3′) assume the form:

$$U_\kappa(\alpha', \xi') - \sum_{i=1}^{n} \alpha'_{\kappa i} \xi'_i = 0 \qquad (\kappa = 1, \dots, m).$$

With that, our assertion is proved.

We can summarize the results up to now in the following theorem:

Theorem 4. *The expression:*

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} U_\kappa(\alpha, \varphi) \frac{\partial f}{\partial \alpha_\kappa}$$

determines an infinite group of infinitesimal transformations, and thus the U_κ mean functions of $\alpha_1, \dots, \alpha_m$ that, in addition, depend up the $\varphi_1, \dots, \varphi_n$ and their differential quotients. It is further possible to choose $\alpha_1, \dots, \alpha_m$ to be functions of $x_1, \dots, x_n$ such that the m differential equations for $\xi_1, \dots, \xi_n$:

$$U_\kappa(\alpha, \xi) - \sum_{i=1}^{n} \alpha_{\kappa\nu} \xi_\nu = 0 \qquad (\kappa = 1, \dots, m)$$

define an integrable system then this system defines a group of infinitesimal transformations in the variables $x_1, \dots, x_n$. Moreover, the system of differential equations in question admits the infinite group of infinitesimal transformations:

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_j \frac{\partial \varphi_i}{\partial x_j} \frac{\partial f}{\partial \xi_i} + \sum_{\kappa=1}^{m} U_\kappa(\alpha, \varphi) \frac{\partial f}{\partial \alpha_\kappa}.$$

11. Whereas we were following the train of thought that was given by Lie in the previous number, we would now like to briefly pause to mention my original proof of theorem 4.

I prove the fact that equations (3′) define a group of infinitesimal transformations in the following way:

The two infinitesimal transformations:

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} U_\kappa(\alpha, \varphi) \frac{\partial f}{\partial \alpha_\kappa},$$

$$\sum_{i=1}^{n}\psi_i\frac{\partial f}{\partial x_i}+\sum_{\kappa=1}^{m}U_\kappa(\alpha,\psi)\frac{\partial f}{\partial \alpha_\kappa}$$

are composed with each other. The resulting transformation must then again belong to the infinite group (1) and thus possesses the form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\varphi_j\frac{\partial\psi_i}{\partial x_j}-\psi_j\frac{\partial\varphi_i}{\partial x_j}\right)\frac{\partial f}{\partial x_i}+\sum_{\kappa=1}^{m}U_\kappa\left(\alpha,\varphi\frac{\partial\psi}{\partial x}-\psi\frac{\partial\varphi}{\partial x}\right)\frac{\partial f}{\partial \alpha_\kappa};$$

in this, we understand $U_\kappa\left(\alpha,\varphi\frac{\partial\psi}{\partial x}-\psi\frac{\partial\varphi}{\partial x}\right)$ to mean the expression that $U_\kappa(\alpha,\varphi)$ turns into when one substitutes:

$$\sum_{j=1}^{n}\left(\varphi_j\frac{\partial\psi_i}{\partial x_j}-\psi_j\frac{\partial\varphi_i}{\partial x_j}\right)$$

in place of φ_i. Furthermore, $\frac{\partial}{\partial x_i}U_\kappa(\alpha,\varphi)$ means the partial differential quotients of $U_\kappa(\alpha,\varphi)$ with respect to x_i, as long as the x_i enter into the φ_j and the α_κ, while, by contrast, $\frac{\partial'}{\partial x_i}U_\kappa(\alpha,\varphi)$ means the differential quotients with respect to x_i, as long as the x_i enter into the φ_j, such that the following equation comes about:

$$\frac{\partial}{\partial x_i}U_\kappa(\alpha,\varphi)=\sum_{\mu=1}^{m}\alpha_{\mu i}\frac{\partial}{\partial\alpha_\mu}U_\kappa(\alpha,\varphi)+\frac{\partial'}{\partial\alpha_\mu}U_\kappa(\alpha,\varphi).$$

With these assumptions, we can calculate the expression that the aforementioned composition yields for $U_\kappa\left(\alpha,\varphi\frac{\partial\psi}{\partial x}-\psi\frac{\partial\varphi}{\partial x}\right)$. It becomes identically:

$$\begin{aligned}U_\kappa\left(\alpha,\varphi\frac{\partial\psi}{\partial x}-\psi\frac{\partial\varphi}{\partial x}\right)&\equiv\sum_{i=1}^{n}\left(\varphi_i\frac{\partial'}{\partial x_i}U_\kappa(\alpha,\psi)-\psi_i\frac{\partial'}{\partial x_i}U_\kappa(\alpha,\varphi)\right)\\&+\sum_{j=1}^{m}\left\{U_j(\alpha,\varphi)\frac{\partial}{\partial\alpha_j}U_\kappa(\alpha,\psi)-U_j(\alpha,\psi)\frac{\partial}{\partial\alpha_j}U_\kappa(\alpha,\varphi)\right\}.\end{aligned}$$

In a somewhat different form, this equation reads:

$$(5)\quad \begin{cases} U_\kappa\left(\alpha, \varphi\dfrac{\partial\psi}{\partial x} - \psi\dfrac{\partial\varphi}{\partial x}\right) \equiv \displaystyle\sum_{i=1}^{n}\left(\varphi_i \frac{\partial U_\kappa(\alpha,\psi)}{\partial x_i} - \psi_i \frac{\partial U_\kappa(\alpha,\varphi)}{\partial x_i}\right) \\ \qquad + \displaystyle\sum_{j=1}^{m}\left\{\frac{\partial U_\kappa(\alpha,\psi)}{\partial \alpha_j}\left(U_j(\alpha,\varphi) - \sum_{\nu=1}^{n}\alpha_{j\nu}\varphi_\nu\right)\right. \\ \qquad \left. - \dfrac{\partial U_\kappa(\alpha,\varphi)}{\partial \alpha_j}\left(U_j(\alpha,\psi) - \displaystyle\sum_{\nu=1}^{n}\alpha_{j\nu}\psi_\nu\right)\right\}. \end{cases}$$

If we subtract the identity

$$\sum_{\nu=1}^{n}\alpha_{\kappa\nu}\sum_{\lambda=1}^{n}\left(\varphi_\lambda\frac{\partial\psi_\nu}{\partial x_\lambda} - \psi_\lambda\frac{\partial\varphi_\nu}{\partial x_\lambda}\right) \equiv \sum_{i=1}^{n}\left\{\varphi_i\frac{\partial}{\partial x_i}\left(\sum_{\nu=1}^{n}\alpha_{\kappa\nu}\psi_\nu\right) - \psi_i\frac{\partial}{\partial x_i}\left(\sum_{\nu=1}^{n}\alpha_{\kappa\nu}\varphi_\nu\right)\right\}$$

from this then we get the identity equation:

$$(5')\quad \begin{cases} U_\kappa\left(\alpha, \varphi\dfrac{\partial\psi}{\partial x} - \psi\dfrac{\partial\varphi}{\partial x}\right) - \displaystyle\sum_{\nu=1}^{n}\alpha_{\kappa\nu}\sum_{\lambda=1}^{n}\left(\varphi_\lambda\frac{\partial\psi_\nu}{\partial x_\lambda} - \psi_\lambda\frac{\partial\varphi_\nu}{\partial x_\lambda}\right) \\ \equiv \displaystyle\sum_{i=1}^{n}\left\{\varphi_i\frac{\partial}{\partial x_i}\left(U_\kappa(\alpha,\psi) - \sum_{\nu=1}^{n}\alpha_{\kappa\nu}\psi_\nu\right) - \psi_i\frac{\partial}{\partial x_i}\left(U_\kappa(\alpha,\varphi) - \sum_{\nu=1}^{n}\alpha_{\kappa\nu}\varphi_\nu\right)\right\} \\ \qquad + \displaystyle\sum_{j=1}^{m}\left\{\frac{\partial}{\partial \alpha_j}U_\kappa(\alpha,\psi)\left(U_\kappa(\alpha,\varphi) - \sum_{\nu=1}^{n}\alpha_{\kappa\nu}\varphi_\nu\right)\right. \\ \qquad \left. - \dfrac{\partial}{\partial \alpha_j}U_\kappa(\alpha,\varphi)\left(U_\kappa(\alpha,\psi) - \displaystyle\sum_{\nu=1}^{n}\alpha_{\kappa\nu}\psi_\nu\right)\right\}. \end{cases}$$

Equation (5′) shows clearly that the system of equations (3′) represents a system of defining equations. It would be superfluous to go into this further.

It still remains for us to show that the system (3′) admits the infinitesimal transformations (4). In order to prove this, we endow the variables x_i, ξ_i, and α_κ with the increments:

$$\delta x_i = \varphi_i\,\delta t, \qquad \delta\xi_i = \sum_{j=1}^{n}\xi_j\frac{\partial\varphi_i}{\partial x_j}\delta t, \qquad \delta\alpha_\kappa = U_\kappa(\alpha,\varphi)\,\delta t,$$

and we now have to calculate the corresponding increment of $U_\kappa(\alpha, \varphi) - \sum \alpha_{\kappa i}\,\xi_i$.

If we set:

$$\frac{\partial^{\nu_1+\cdots+\nu_n}}{\partial x_1^{\nu_1}\cdots\partial x_n^{\nu_n}}\xi_i = \xi_i^{(\nu_1,\cdots,\nu_n)}$$

then this gives the general formula:

$$\frac{\delta\xi_i^{(\nu_1,\cdots,\nu_n)}}{\delta t} = \sum_{j=1}^{n}\frac{\partial\xi_i^{(\nu_1,\cdots,\nu_n)}}{\partial t}\varphi_j + \sum_{j=1}^{n}\frac{\partial^{\nu_1+\cdots+\nu_n}}{\partial x_1^{\nu_1}\cdots\partial x_n^{\nu_n}}\left(\xi_j\frac{\partial\varphi_i}{\partial x_j} - \varphi_j\frac{\partial\psi_i}{\partial x_j}\right),$$

whose very simple proof we would not like to stop for. From the equation:

$$d\alpha_\kappa - \sum_{\nu=1}^{n}\alpha_{\kappa\nu}dx_\nu = 0,$$

one gets, in addition, the expression for the increment of $\alpha_{\kappa\nu}$:

$$\frac{\delta\alpha_{\kappa\nu}}{\delta t} = \frac{\partial}{\partial x_\nu}U_\kappa(\alpha,\varphi) - \sum_{\pi=1}^{n}\alpha_{\kappa\pi}\frac{\partial\varphi_\pi}{\partial x_\nu}.$$

We can now calculate the desired increment. Since $U_\kappa(\alpha, \xi)$ is linear and homogeneous in the ξ and their differential quotients, we obtain:

$$\frac{\delta}{\delta t}\left(U_\kappa(\alpha,\xi) - \sum_{\nu=1}^{n}\alpha_{\kappa\nu}\xi_\nu\right) = \sum_{\pi=1}^{n}\varphi_\pi\frac{\partial'}{\partial x_\pi}U_\kappa(\alpha,\xi) + U_\kappa\left(\alpha, \xi\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial\xi}{\partial x}\right)$$

$$+ \sum_{j=1}^{m}U_j(\alpha,\xi)\frac{\partial}{\partial\alpha_j}U_\kappa(\alpha,\xi) - \sum_{\nu=1}^{n}\alpha_{\kappa\nu}\sum_{\pi=1}^{n}\xi_\pi\frac{\partial\varphi_\nu}{\partial x_\pi} - \sum_{\nu=1}^{n}\xi_\nu\left(\frac{\partial}{\partial x_\nu}U_\kappa(\alpha,\varphi) - \sum_{\pi=1}^{n}\alpha_{\kappa\pi}\frac{\partial\varphi_\pi}{\partial x_\nu}\right).$$

However, by the use of the identity (5), this yields the simple formula:

$$(6)\qquad \frac{\delta}{\delta t}\left(U_\kappa(\alpha,\xi) - \sum_{\nu=1}^{n}\alpha_{\kappa\nu}\xi_\nu\right) = \sum_{j=1}^{m}\frac{\partial}{\partial\alpha_j}U_\kappa(\alpha,\xi)\left\{U_\kappa(\alpha,\xi) - \sum_{\nu=1}^{n}\alpha_{j\nu}\xi_\nu\right\}.$$

This equation expresses precisely the fact that the system of equations (3′) admits the infinitesimal transformations (4).

12. A system that consists of differential equations of order s and lower is *unrestrictedly* integrable when no new differential equations of order s or less can be obtained by one or more differentiations and subsequent combinations.

If a given system of differential equations is not unrestrictedly integrable then new differential equations must be added by means of the operations that we referred to, until one finally completes an unrestrictedly integrable system; when this first happens, one can begin the actual examination of the system. For that reason, we would always like to think of the defining equations of a group in the form of an unrestrictedly integrable system.

Naturally, in general, the system:

$$(7) \qquad U_\kappa(\alpha,\xi) - \sum_{\nu=1}^{n} \alpha_{\kappa\nu}\xi_\nu = 0 \qquad (\kappa = 1, \dots, m),$$

which might include, perhaps, differential quotients of the ξ up to order s, is not unrestrictedly integrable, as long as we understand the $\alpha_1, \dots, \alpha_m$ to mean entirely arbitrary functions of $x_1, \dots, x_n$. In most cases, one obtains new equations of order s and lower by differentiation and combination. However, if we demand that our system (7) should be unrestrictedly integrable then these new equations must be a consequence of the system (7). It must be true identically when any m of the quantities ξ_i , $\partial\xi_i / \partial x_\kappa$ are eliminated by means of equations (7). The coefficients of all the ξ_i , $\partial\xi_i / \partial x_\kappa$ that still appear in the equations that we obtain by carrying out this elimination must therefore vanish.

This gives a number of relations between the α_κ and their differential quotients:

$$(8) \qquad \chi_\kappa(\alpha_1, \dots, \alpha_m, \alpha_{11}, \alpha_{12}, \dots, \alpha_{111}, \dots) = 0 \qquad (\kappa = 1, 2, \dots).$$

We imagine that precisely as many such relations have been exhibited as it takes to define an unrestrictedly integrable system, by means of which the system (7) is unrestrictedly integrable. Therefore, it will be assumed that this is indeed possible, at all.

The differential equations (8) define all functions $\alpha_1, \dots, \alpha_m$, of the variables $x_1, \dots, x_n$ that yield an unrestrictedly integrable system when they are substituted in (7); equations (8) are thus the integrability conditions for the system (7).

This strongly suggests the conjecture that the integrability conditions (8) remain invariant under the infinitesimal transformations:

$$(9) \qquad \sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} U_\kappa(\alpha,\varphi)\frac{\partial f}{\partial \alpha_\kappa},$$

just as the system (7) remains invariant under the infinitesimal transformations:

$$(10) \qquad \sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n}\sum_{\nu=1}^{n} \xi_\nu \frac{\partial \varphi_i}{\partial x_\nu}\frac{\partial f}{\partial \xi_i} + \sum_{\kappa=1}^{m} U_\kappa(\alpha,\varphi)\frac{\partial f}{\partial \alpha_\kappa}.$$

We will, at least, suggest a proof of the fact that this is actually the case.

First, we establish that an unrestrictedly integrable system of differential equations goes over to such a system under any change of variables. If we therefore introduce the new variables into (7):

$$\bar{x}_i = \omega_i(x_1, \dots, x_n), \qquad \bar{\xi}_i = \sum_{\nu=1}^{n} \xi_\nu \frac{\partial \omega_i}{\partial x_\nu} \qquad (i = 1, \dots, n),$$

and then construct the integrability conditions for the differential equations thus obtained in the same way as above then we obtain an unrestrictedly integrable system of differential equations $\alpha_1, \ldots, \alpha_m$, and this system must be algebraically equivalent to the integrability conditions (8) under the reintroduction of the original variables x_i.

We now introduce the new variables:

$$\bar{x}_i = x_i + \varphi_i\, \delta t, \qquad \bar{\xi}_i = \xi_i + \sum_{\nu=1}^{n} \xi_\nu \frac{\partial \varphi_i}{\partial x_\nu} \cdot \delta t \qquad (i = 1, \ldots, n).$$

In addition, if we define the functions $\bar{\alpha}_\kappa$ by the equations:

$$\bar{\alpha}_\kappa = \alpha_\kappa + U_\kappa(\alpha, \varphi)\, \delta t \qquad (\kappa = 1, \ldots, m),$$

and recall the fact that the system (7) admits the infinitesimal transformations (10) then we recognize that in the new variables this system assumes the form:

$$U_\kappa(\bar{\alpha}, \bar{\xi}) - \sum_{\nu=1}^{n} \bar{\alpha}_{\kappa\nu} \bar{\xi}_\nu = 0 \qquad (\kappa = 1, \ldots, m).$$

The new integrability conditions then become:

$$\chi_\kappa(\bar{\alpha}_1, \ldots, \bar{\alpha}_m, \bar{\alpha}_{11}, \ldots, \bar{\alpha}_{111}, \ldots) = 0 \qquad (\kappa = 1, 2, \ldots),$$

or by reverting to the old variables:

$$\chi_\kappa(\alpha_1, \ldots, \alpha_m, \alpha_{11}, \ldots) + \frac{\delta}{\delta t} \chi_\kappa(\alpha_1, \ldots, \alpha_m, \alpha_{11}, \ldots)\, \delta t = 0.$$

From what was said above, this must be algebraically equivalent to the system of equations $\chi_\nu = 0$, so all the expressions:

$$\frac{\delta}{\delta t} \chi_\kappa(\alpha_1, \ldots, \alpha_m, \alpha_{11}, \ldots)$$

must vanish as a result of $\chi_1 = 0$, $\chi_2 = 0$, … However, this says nothing but the fact that the integrability conditions (8) admit the infinitesimal transformations (9), and that was just what we asserted.

The differential equations:

$$(11) \qquad \begin{cases} U_\kappa(\alpha, \xi) - \sum_{\nu=1}^{n} \alpha_{\kappa\nu} \xi_\nu = 0, & (\kappa = 1, \ldots, m), \\ \chi_j(\alpha_1, \ldots, \alpha_m, \alpha_{11}, \ldots) = 0, & (j = 1, 2, \ldots) \end{cases}$$

thus collectively define an unrestrictedly integrable system that admits the infinite group (10). As we known, *this system defines an infinite family of groups* if any system of functions α_κ that satisfy the equations $\chi_j = 0$ yields the defining equations of a group of infinitesimal transformations when substituted into (11).

If we interpret equations (11) in the space of variables x_i, ξ_i, $\partial\xi_i / \partial x_\kappa$, …, α_κ, $\alpha_{\kappa\nu}$, … then we obtain a manifold that remains invariant under the infinitesimal transformations (10), assuming that it can be extended by the addition of the differential quotients $\partial\xi_i / \partial x_\kappa$, …, $\alpha_{\kappa\nu}$, … The points of the manifold in question will be, by comparison, permuted amongst themselves, and it can therefore happen that these points can be arranged into families of infinitely many individually invariant manifolds.

Analytically speaking, this means: It is conceivable that one can add one or more differential equations of order *s* or lower to the differential equations (11):

$$\Omega\left(x_i, \xi_i, \frac{\partial\xi_i}{\partial x_\kappa}, \frac{\partial^2\xi_i}{\partial x_\kappa \partial x_j}, \ldots, \alpha_\kappa, \alpha_{\kappa\nu}, \ldots\right) = \text{const.},$$

in such a way that all of them collectively define a system of differential equations that remains invariant under the infinite group (10).

If we recall the analogous developments that we carried out in § 1, no. 4, for the simple manifold then we immediately come to the notion that each such equation Ω = const. is an integral of the system (11). However, the method of proof that served for us in the cited place lets us down here; it does not allow us to show that Ω = const. is actually an integral. One will see this immediately when one generalizes the formulas there, which is not difficult, at all.

I have still not arrived at a proof of the validity of the aforementioned conjecture. Still, I would be remiss in not remarking that in all of the examples that I have worked through the equations Ω = const. are integrals of the system (11).

Now, we would like to discuss somewhat more thoroughly a situation that we mentioned briefly at the conclusion of § 1 for the simple manifold. It is the fact that it makes no essential difference to the differential equations (3′) if one introduces the new variables:

(12) $$\alpha'_\kappa = \omega_\kappa(\alpha_1, \ldots, \alpha_m) + \pi_\kappa(x_1, \ldots, x_n) \qquad (\kappa = 1, \ldots, m),$$

in place of $\alpha_1, \ldots, \alpha_m$ in the infinite group (1), in which we understand the ω_κ and π_κ to mean arbitrary functions of their arguments. It is hardly necessary to also prove this rigorously here. Still, one must point out that fact, because in the next paragraphs we will always imagine that the infinite group (1) can be brought into the simplest possible form by a transformation of the form (12). Moreover, in the cases considered the functions π_κ will always be equal to zero.

§ 4. Some special considerations for the *n*-fold extended manifold.

13. *First, we will consider those infinite groups:*

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} U_\kappa(\alpha, \varphi) \frac{\partial f}{\partial \alpha_\kappa},$$

in which only differential quotients of first order in the φ_i appear in the U_κ. For the sake of simplicity, we would like assume that $\varphi_1, \ldots, \varphi_n$ themselves do not enter in.

If we understand $A_{i\kappa} f$ to mean infinitesimal transformations in only the $\alpha_1, \ldots, \alpha_m$ then we can write the groups in question:

(1)
$$Xf = \sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} \sum_{\kappa=1}^{n} A_{i\kappa} f \frac{\partial \varphi_i}{\partial x_\kappa}.$$

In composition with:

$$Yf = \sum_{i=1}^{n} \psi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} \sum_{\kappa=1}^{n} A_{i\kappa} f \frac{\partial \psi_i}{\partial x_\kappa},$$

we thus obtain the infinitesimal transformation:

(2)
$$\begin{cases} (XY) = \displaystyle\sum_{i=1}^{n} \sum_{\nu=1}^{n} \left(\varphi_\nu \frac{\partial \psi_i}{\partial x_\nu} - \psi_\nu \frac{\partial \varphi_i}{\partial x_\nu} \right) \frac{\partial f}{\partial x_i} \\ \quad + \displaystyle\sum_{i=1}^{n} \sum_{\kappa=1}^{n} \sum_{\nu=1}^{n} A_{i\kappa} f \left(\varphi_\nu \frac{\partial^2 \psi_i}{\partial x_\nu \partial x_\kappa} - \psi_\nu \frac{\partial^2 \varphi_i}{\partial x_\nu \partial x_\kappa} \right) \\ \quad + \displaystyle\sum_{i=1}^{n} \sum_{\kappa=1}^{n} \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} (A_{i\kappa}, A_{\mu\nu}) \frac{\partial \varphi_i}{\partial x_\kappa} \frac{\partial \psi_\mu}{\partial x_\nu}. \end{cases}$$

However, this transformation must again belong to the group (1), and as a result of this, it can be brought into the form:

(3)
$$(X\,Y) = \sum_{i=1}^{n} \sum_{\nu=1}^{n} \left(\varphi_\nu \frac{\partial \psi_i}{\partial x_\nu} - \psi_\nu \frac{\partial \varphi_i}{\partial x_\nu} \right) \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} \sum_{\kappa=1}^{n} \sum_{\nu=1}^{n} A_{i\kappa} f \frac{\partial}{\partial x_\kappa} \left(\varphi_\nu \frac{\partial \psi_i}{\partial x_\nu} - \psi_\nu \frac{\partial \varphi_i}{\partial x_\nu} \right).$$

If we compare the two expressions with each other then we observe that a whole series of terms drops out with no further assumptions. The comparison of the remaining expressions will be, however, complicated by the fact that three summation signs appear in (3), while four of them appear in (2).

We can alleviate this shortcoming by also introducing four summation signs in (3). Namely, if we assume that $\varepsilon_{\nu\nu}$ has the value 1, while $\varepsilon_{\mu\nu}$ always vanishes as long as μ and ν are different from each other then we can represent the term of (3) in question in the form:

$$\sum_{i=1}^{n} \sum_{\kappa=1}^{n} \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \varepsilon_{\mu\nu} A_{i\kappa} \left(\frac{\partial \varphi_\mu}{\partial x_\kappa} \frac{\partial \psi_i}{\partial x_\nu} - \frac{\partial \psi_\mu}{\partial x_\kappa} \frac{\partial \varphi_i}{\partial x_\nu} \right),$$

or then again:

$$\sum_{i=1}^{n}\sum_{\kappa=1}^{n}\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}(\varepsilon_{\mu\nu}A_{\mu\kappa}-\varepsilon_{\mu\kappa}A_{i\nu})\frac{\partial\varphi_i}{\partial x_\kappa}\frac{\partial\psi_\mu}{\partial x_\nu}.$$

We can now actually perform the comparison with the corresponding expression in (2). In it, if we observe that the φ_i and ψ_i actually appear and that every product $\frac{\partial\varphi_i}{\partial x_\kappa}\frac{\partial\psi_\mu}{\partial x_\nu}$ appears under the four summations only a single time then we find the relations:

(4) $$(A_{i\kappa}, A_{\mu\nu}) = \varepsilon_{i\nu}A_{\mu\kappa} - \varepsilon_{\mu\kappa}A_{i\nu}.$$

It is clear, *a priori*, that the values for the expressions $(A_{i\kappa}, A_{\mu\nu})$ that were just obtained satisfy the Jacobi identity:

(5) $$((A_{i\kappa}, A_{\mu\nu})\, A_{\pi\rho}) + ((A_{\mu\nu}, A_{\pi\rho}), A_{i\kappa}) + ((A_{\pi\rho}, A_{i\kappa}), A_{\mu\nu}) = 0.$$

Namely, if we form, from the three infinitesimal transformations Xf, Yf, and:

$$Zf=\sum_{i=1}^{n}\chi_i\frac{\partial f}{\partial x_i}+\sum_{i=1}^{n}\sum_{\kappa=1}^{n}A_{i\kappa}\frac{\partial\chi_i}{\partial x_\kappa},$$

the expression:

(6) $$((X\,Y)\,Z) + ((Y\,Z)\,X) + ((Z\,X)\,Y),$$

which must vanish identically, then we see that it vanishes only by means of equations (5).

However, if we further observe that – e.g., $(X\,Y)$ – takes on the form (3) due to (4), and that $((X\,Y)\,Z)$ is expressed in a similar way then we see that the expression (6) also vanishes identically by means of (4), which, from the foregoing, is possible only when equations (5) are a consequence of (4).

From this, it follows (cf., Intro. F) that the n^2 infinitesimal transformations $A_{i\kappa} f$ generate a finite group where composition is determined by the relations (4). We briefly saw that there are, moreover, groups that satisfy the relations (4) in the foregoing simple cases. Namely, if we set $A_{i\kappa} f = - x_i\, \partial f / \partial x_\kappa$ then equations (4) will be fulfilled, and we then find that the infinitesimal transformations $A_{i\kappa} f$ must generate a group that is isomorphic to the group $x_i\, \partial f / \partial x_\kappa$. The group $x_i\, \partial f / \partial x_\kappa$, however, includes n^2 parameters, and is the general linear homogeneous group of the n-fold extended manifold: The finite equations of this group are:

$$x_i' = \sum_{\kappa=1}^{n} a_{i\kappa}x_\kappa \qquad (i = 1, \ldots, n).$$

The group of $A_{i\kappa} f$ is either holomorphically or meromorphically isomorphic to the group $x_i\, \partial f / \partial x_\kappa$. We would like to set aside the case of holomorphic isomorphism, since its treatment would become too lengthy. *By contrast, we would like to determine those* *groups that are meromorphically isomorphic to the group* $x_i\, \partial f / \partial x_\kappa$ *, and therefore include the smallest possible number of variables.*

The general linear homogeneous group $x_i\, \partial f / \partial x_\kappa$ contains only the two invariant subgroups:

$$x_i \frac{\partial f}{\partial x_\kappa}, \quad x_i \frac{\partial f}{\partial x_i} - x_\kappa \frac{\partial f}{\partial x_\kappa} \qquad (i \neq k); \qquad x_1 \frac{\partial f}{\partial x_1} + \ldots + x_n \frac{\partial f}{\partial x_n}.$$

From the introduction under *F*, one thus also finds only two different types of groups $A_{i\kappa} f$ that are meromorphically isomorphic to the group $x_i\, \partial f / \partial x_\kappa$. The following relations exist between the infinitesimal transformations $A_{i\kappa} f$ of these two types of groups, respectively:

$$A_{i\kappa} = 0 \ (i \neq \kappa), \qquad A_{11} = A_{22} = \ldots = A_{nn}; \qquad A_{11} + \ldots + A_{nn} = 0.$$

In the former case, only a single infinitesimal transformation then remains; the simplest group is obtained when a single variable α is transformed. One can then set:

$$A_{11} = A_{22} = \ldots = A_{nn} = \frac{\partial f}{\partial \alpha},$$

and obtain the following form for the infinite group (1):

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_i} \frac{\partial f}{\partial \alpha},$$

and from this one gets the defining equation:

$$\text{(7)} \qquad \frac{\partial \xi_1}{\partial x_1} + \cdots + \frac{\partial \xi_n}{\partial x_n} - \frac{\partial \alpha}{\partial x_1} \xi_1 - \cdots - \frac{\partial \alpha}{\partial x_n} \xi_n = 0.$$

This differential equation, in which α is regarded as a completely arbitrary function of $x_1, \ldots, x_n$, defines a family of infinite groups that are similar to each other (cf., Chr. Videnskabss. Forh. 1883, no. 12, page 18 *et seq.*, where Lie proved this for two variables *x* and *y*). In particular, if one sets $\alpha = 0$ in (7) then one comes to the infinite groups that leave all volumes in the space $x_1, \ldots, x_n$ invariant.

We would now like to treat the second case only casually. Of the n^2 infinitesimal transformations $A_{i\kappa}$, only $n^2 - 1$ of them are independent – say:

$$A_{i\kappa} \quad (i \neq \kappa), \qquad A_{11}, \ldots, A_{n-1,n-1}.$$

This shows that this $(n^2 - 1)$-parameter group has the same composition as the general projective group:

(8) $$\frac{\partial f}{\partial y_i}, \quad y_i \frac{\partial f}{\partial y_\kappa}, \qquad y_i \sum_{j=1}^{n-1} y_j \frac{\partial f}{\partial y_j} \quad (i, \kappa = 1, \ldots, n-1)$$

of the $(n - 1)$-fold extended manifold $y_1, \ldots, y_{n-1}$. However, if one considers a theorem of Lie (Ann., Bd. XXV, page 132) then one recognizes that there are no $(n^2 - 1)$-parameter groups with this composition on less than $n - 1$ variables, while, by comparison, all such groups in $n - 1$ variables are similar to the group (8). A closer examination reveals that the group $A_{i\kappa}$ $(i \neq \kappa)$, $A_{11}, \ldots, A_{n-1,n-1}$ in $n - 1$ variables $\alpha_1, \ldots, \alpha_{n-1}$ can be obtained in two essentially different ways: Namely, either:

$$A_{in} = \frac{\partial f}{\partial \alpha_i}, \quad A_{ni} = -\alpha_i \sum_{j=1}^{n-1} \alpha_j \frac{\partial f}{\partial \alpha_j}, \ A_{i\kappa} = \alpha_\kappa \frac{\partial f}{\partial \alpha_i},$$

or one has:

$$A_{in} = \alpha_i \sum_{j=1}^{n-1} \alpha_j \frac{\partial f}{\partial \alpha_j}, \quad A_{ni} = -\frac{\partial f}{\partial \alpha_i}, \quad A_{i\kappa} = -\alpha_i \frac{\partial f}{\partial \alpha_\kappa}.$$

For A_{nn}, one therefore finds the values:

$$A_{nn} = -\sum_{j=1}^{n-1} \alpha_j \frac{\partial f}{\partial \alpha_j}, \quad A_{nn} = \sum_{j=1}^{n-1} \alpha_j \frac{\partial f}{\partial \alpha_j},$$

respectively.

The two infinite groups (1) that correspond to these two possibilities are:

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n-1} \frac{\partial \varphi_i}{\partial x_n} \frac{\partial f}{\partial \alpha_i} + \sum_{i=1}^{n-1} \sum_{\kappa=1}^{n-1} \frac{\partial \varphi_i}{\partial x_\kappa} \alpha_\kappa \frac{\partial f}{\partial \alpha_i} - \sum_{i=1}^{n-1} \frac{\partial \varphi_n}{\partial x_i} \alpha_i \sum_{\kappa=1}^{n-1} \alpha_\kappa \frac{\partial f}{\partial \alpha_\kappa} - \frac{\partial \varphi_n}{\partial x_n} \sum_{i=1}^{n-1} \alpha_\kappa \frac{\partial f}{\partial \alpha_\kappa},$$

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} - \sum_{i=1}^{n-1} \frac{\partial \varphi_n}{\partial x_i} \frac{\partial f}{\partial \alpha_i} - \sum_{i=1}^{n-1} \sum_{\kappa=1}^{n-1} \frac{\partial \varphi_i}{\partial x_\kappa} \alpha_i \frac{\partial f}{\partial \alpha_\kappa} + \sum_{i=1}^{n-1} \frac{\partial \varphi_i}{\partial x_n} \alpha_i \sum_{\kappa=1}^{n-1} \alpha_\kappa \frac{\partial f}{\partial \alpha_\kappa} + \frac{\partial \varphi_n}{\partial x_n} \sum_{i=1}^{n-1} \alpha_\kappa \frac{\partial f}{\partial \alpha_\kappa}.$$

From this, one finally gets the two systems of defining equations:

(9) $$\frac{\partial \xi_i}{\partial x_n} + \sum_{\kappa=1}^{n-1} \alpha_\kappa \frac{\partial \xi_i}{\partial x_\kappa} - \alpha_i \left(\frac{\partial \xi_n}{\partial x_n} + \sum_{\kappa=1}^{n-1} \alpha_\kappa \frac{\partial \xi_n}{\partial x_\kappa} \right) - \sum_{\nu=1}^{n} \alpha_{i\nu} \xi_\nu = 0 \qquad (i = 1, \ldots, n-1),$$

(10) $$\frac{\partial \xi_n}{\partial x_i} + \sum_{\kappa=1}^{n-1} \alpha_\kappa \frac{\partial \xi_\kappa}{\partial x_i} - \alpha_i \left(\frac{\partial \xi_n}{\partial x_n} + \sum_{\kappa=1}^{n-1} \alpha_\kappa \frac{\partial \xi_\kappa}{\partial x_n} \right) + \sum_{\nu=1}^{n} \alpha_{i\nu} \xi_\nu = 0 \qquad (i = 1, \ldots, n-1).$$

Equations (9) define the infinite group of all infinitesimal transformations that leave the Pfaffian system of equations:

$$dx_i - \alpha_i \, dx_n = 0 \qquad (i = 1, \ldots, n-1).$$

Equations (10) define the infinite group of all infinitesimal transformations that leave invariant the Pfaffian equation:

$$dx_n + \sum_{i=1}^{n-1} \alpha_i dx_i = 0.$$

14. *We now consider such groups:*

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{m} U_\kappa(\alpha, \varphi) \frac{\partial f}{\partial \alpha_\kappa},$$

in which the U_κ include differential quotients of first and second order in the φ_i, while the φ_i themselves do not enter in.

We write these infinitesimal transformation in the form:

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n}\sum_{\kappa=1}^{n} A_{i\kappa} f \frac{\partial \varphi_i}{\partial x_\kappa} + \sum_{i=1}^{n}\sum_{\kappa=1}^{n}\sum_{j=1}^{n} A_{i\kappa j} f \frac{\partial^2 \varphi_i}{\partial x_\kappa \partial x_j}. \tag{11}$$

Thus, we understand the $A_{i\kappa} f$ and $A_{i\kappa j} f$ to mean infinitesimal transformations in the just the variables $\alpha_1, \ldots, \alpha_m$, and additionally establish that $A_{i\kappa j} f$ shall be equal to $A_{ij\kappa} f$, such that any differential quotient $\frac{\partial^2 \varphi_i}{\partial x_\kappa \partial x_i}$ with the factor 2 $A_{i\kappa j} f$ appears whenever κ and j are different from each other.

If we now compose the infinitesimal transformation:

$$\sum_{i=1}^{n} \psi_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n}\sum_{\kappa=1}^{n} A_{i\kappa} f \frac{\partial \psi_i}{\partial x_\kappa} + \sum_{i=1}^{n}\sum_{\kappa=1}^{n}\sum_{j=1}^{n} A_{i\kappa j} f \frac{\partial^2 \psi_i}{\partial x_\kappa \partial x_j}$$

with the one above then we must obtain:

$$\begin{cases} \sum_{i=1}^{n}\sum_{\nu=1}^{n} \left(\varphi_\nu \frac{\partial \psi_i}{\partial x_\nu} - \psi_\nu \frac{\partial \varphi_i}{\partial x_\nu} \right) \frac{\partial f}{\partial x_i} \\ + \sum_{i=1}^{n}\sum_{\kappa=1}^{n}\sum_{\nu=1}^{n} A_{i\kappa} f \frac{\partial}{\partial x_\kappa} \left(\varphi_\nu \frac{\partial \psi_i}{\partial x_\nu} - \psi_\nu \frac{\partial \varphi_i}{\partial x_\nu} \right) \\ + \sum_{i=1}^{n}\sum_{\kappa=1}^{n}\sum_{j=1}^{n}\sum_{\kappa=1}^{n} A_{i\kappa j} f \frac{\partial^2}{\partial x_\kappa \partial x_j} \left(\varphi_\nu \frac{\partial \psi_i}{\partial x_\nu} - \psi_\nu \frac{\partial \varphi_i}{\partial x_\nu} \right). \end{cases} \tag{12}$$

By direct composition, one comes to another expression that must be identical to the one that was just written down. If we compare both of them then it next occurs to us that all terms in which φ_ν or ψ_ν themselves appear drop out. Furthermore, this yields, as in the previous number:

$$(A_{i\kappa} A_{\mu\nu}) = \varepsilon_{i\nu} A_{\mu\kappa} - \varepsilon_{\mu\kappa} A_{i\nu}.$$

Finally, we conclude that one always has:

$$(A_{i\kappa j}\,A_{\mu\nu\pi}) = 0$$

if no terms of the form:

$$\frac{\partial^2\varphi_i}{\partial x_\kappa \partial x_j}\frac{\partial^2\psi_\mu}{\partial x_\nu \partial x_\pi} - \frac{\partial^2\psi_i}{\partial x_\kappa \partial x_j}\frac{\partial^2\varphi_\mu}{\partial x_\nu \partial x_\pi}$$

enter into (12).

What remains is the equation:

$$(13)\quad \begin{cases} \displaystyle\sum_{i=1}^n\sum_{\kappa=1}^n\sum_{\mu=1}^n\sum_{\nu=1}^n\sum_{\pi=1}^n (A_{i\kappa}A_{\mu\nu\pi})\left(\frac{\partial\varphi_i}{\partial x_\kappa}\frac{\partial^2\psi_\mu}{\partial x_\nu\partial x_\pi} - \frac{\partial\psi_i}{\partial x_\kappa}\frac{\partial^2\varphi_\mu}{\partial x_\nu\partial x_\pi}\right) \\ \displaystyle = \sum_{i=1}^n\sum_{\kappa=1}^n\sum_{j=1}^n\sum_{\nu=1}^n A_{i\kappa j}\left\{\frac{\partial\varphi_\nu}{\partial x_\kappa}\frac{\partial^2\psi_i}{\partial x_\nu\partial x_j} - \frac{\partial\psi_\nu}{\partial x_\kappa}\frac{\partial^2\varphi_i}{\partial x_\nu\partial x_j} + \frac{\partial\varphi_\nu}{\partial x_j}\frac{\partial^2\psi_i}{\partial x_\nu\partial x_\kappa} - \frac{\partial\psi_\nu}{\partial x_j}\frac{\partial^2\varphi_i}{\partial x_\nu\partial x_k}\right. \\ \displaystyle \left. - \left(\frac{\partial\varphi_i}{\partial x_\nu}\frac{\partial^2\psi_\nu}{\partial x_\kappa\partial x_j} - \frac{\partial\psi_i}{\partial x_\nu}\frac{\partial^2\varphi_\nu}{\partial x_\kappa\partial x_j}\right)\right\}. \end{cases}$$

In order to ease the comparison, we again introduce the quantities $\varepsilon_{i\kappa}$, and can thus write the right-hand side of (13) in a somewhat different form as follows:

$$(14)\qquad \sum_{i=1}^n\sum_{\kappa=1}^n\sum_{\mu=1}^n\sum_{\nu=1}^n\sum_{\pi=1}^n \{\varepsilon_{i\nu}A_{\mu\kappa\pi} + \varepsilon_{i\pi}A_{\mu\nu\pi} - \varepsilon_{\mu\kappa}A_{i\nu\pi}\}\left(\frac{\partial\varphi_i}{\partial x_\kappa}\frac{\partial^2\psi_\mu}{\partial x_\nu\partial x_\pi} - \frac{\partial\psi_i}{\partial x_\kappa}\frac{\partial^2\varphi_\mu}{\partial x_\nu\partial x_\pi}\right).$$

Regarding the expression (14), we remark that if ν and π are different then every term:

$$\frac{\partial\varphi_i}{\partial x_\kappa}\frac{\partial^2\psi_\mu}{\partial x_\nu\partial x_\pi} - \frac{\partial\psi_i}{\partial x_\kappa}\frac{\partial^2\varphi_\mu}{\partial x_\nu\partial x_\pi}$$

appears twice, but both times with the same factor; by contrast, the term appears only once when ν and π are equal. The same remark may be made about the left-hand side of (13). Now, since the φ_i and ψ_i are completely arbitrary, we find the following formula from (13) and (14):

$$(A_{i\kappa}\,A_{\mu\nu\pi}) = \varepsilon_{i\nu}\,A_{\mu\kappa\pi} + \varepsilon_{i\kappa}\,A_{\mu\nu\kappa} - \varepsilon_{\kappa\mu}\,A_{i\nu\pi}\,,$$

which is generally true, regardless of whether ν and π are different from or equal to each other.

The infinitesimal transformations $A_{i\kappa}f$, $A_{i\kappa j}f$ must then satisfy the following relations:

$$(15)\qquad \begin{cases} (A_{i\kappa}A_{\mu\nu}) = \varepsilon_{i\nu}A_{\mu\kappa} - \varepsilon_{\mu\kappa}A_{i\nu}, \\ (A_{i\kappa j}A_{\mu\nu\pi}) = 0, \\ (A_{i\kappa}A_{\mu\nu\pi}) = \varepsilon_{i\nu}A_{\mu\kappa\pi} + \varepsilon_{i\pi}A_{\mu\nu\kappa} - \varepsilon_{\pi\mu}A_{i\nu\pi}. \end{cases}$$

On the grounds of reasoning like what we followed in the previous number, it is clear *a priori* here that the relations (15) satisfy the Jacobi identity and that there are thus groups that have the composition (15).

Just as in the previous number, one once again treats the groups that, when presented in terms of the variables $\alpha_1, \ldots, \alpha_n$, possess the composition (15) or a composition that is meromorphically isomorphic to it.

Later (§ 5), we will present the simplest groups with this behavior completely, at least for the plane. By contrast, we would now like to treat the problem in full generality of determining the compositions of all groups that are meromorphically isomorphic to the composition (15). As we remarked before (cf., Intro., F), we must therefore first present the invariant subgroups of the group $A_{i\kappa} f$, $A_{i\kappa j} f$ with the composition (15).

15. Before we do anything else, we shall exclude those invariant subgroups of the group $A_{i\kappa} f$, $A_{i\kappa j} f$ that include all of the transformations $A_{i\kappa j} f$. These subgroups will, in fact, obviously deliver only those meromorphically isomorphic groups that we have already determined in no. 13.

If an invariant subgroup includes transformations of the form:

$$\sum_{i=1}^{n}\sum_{\kappa=1}^{n}\alpha_{i\kappa}A_{i\kappa} + \sum_{i=1}^{n}\sum_{\kappa=1}^{n}\sum_{j=1}^{n}\alpha_{i\kappa j}A_{i\kappa j}\ ,$$

moreover, and not all $\alpha_{i\kappa}$ vanish then obviously all of the expressions $\sum\sum\ \alpha_{i\kappa}A_{i\kappa}$ that appear in the group of $A_{i\kappa}$ define an invariant subgroup. These expressions thus possess one of the two forms:

$$A_{11} + \ldots + A_{nn}\ ; \qquad A_{i\kappa},\ A_{ii} - A_{\kappa\kappa}\ (i \neq \kappa),$$

if, perhaps, not all n^2 of the $A_{i\kappa}$ are present.

In each of these three cases, there are one or more transformations of the form:

$$S = \sum_{i=1}^{n}\alpha_i A_{ii} + \sum_{i=1}^{n}\sum_{\kappa=1}^{n}\sum_{j=1}^{n}\beta_{i\kappa j}A_{i\kappa j}\ ,$$

so the invariant subgroups also include the following transformations:

$$(S, A_{\mu\nu\pi}) = (a_\nu + a_\pi - a_\mu)\, A_{\mu\nu\pi}\,.$$

However, since one can choose the factors a_ν, a_π, a_μ in each of the three cases in such a manner that the expression $a_\nu + a_\pi - a_\mu$ does not vanish, there are none of these transformations whatsoever in the invariant subgroup of the transformation $A_{\mu\nu\pi}$, since μ, ν, π are arbitrary. However, such subgroups belong to the excluded sequence.

Therefore, all that remains are those invariant subgroups whose transformations are formed from only the $A_{\mu\nu\pi}$.

Let:

(16) $$\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}\sum_{\pi=1}^{n}\alpha_{\mu\nu\pi}A_{\mu\nu\pi}$$

be a transformation of a group of this type. When we compose it with A_{11} twice, we obtain the following two infinitesimal transformations, which likewise belong to the invariant subgroup:

$$2\sum_{\mu=1}^{n}\sum_{\pi=1}^{n}\alpha_{\mu1\pi}A_{\mu1\pi}-\sum_{\nu=1}^{n}\sum_{\pi=1}^{n}\alpha_{1\nu\pi}A_{1\nu\pi}\ ,$$

$$2\sum_{\mu=1}^{n}\sum_{\pi=1}^{n}\alpha_{\mu1\pi}A_{\mu1\pi}+2\sum_{\mu=1}^{n}\alpha_{\mu11}A_{\mu11}-4\sum_{\pi=1}^{n}\alpha_{11\pi}A_{11\pi}+\sum_{\nu=1}^{n}\sum_{\pi=1}^{n}\alpha_{1\nu\pi}A_{1\nu\pi}\ .$$

By subtracting the upper transformation from the lower one, we find, ignoring the factor of 2:

$$\sum_{\nu=1}^{n}\sum_{\pi=1}^{n}\alpha_{1\nu\pi}A_{1\nu\pi}+\sum_{\mu=1}^{n}\alpha_{\mu11}A_{\mu11}-2\sum_{\pi=1}^{n}\alpha_{11\pi}A_{11\pi}\ ,$$

or, when consolidated:

$$\sum_{\nu=2}^{n}\sum_{\pi=2}^{n}\alpha_{1\nu\pi}A_{1\nu\pi}+\sum_{\mu=2}^{n}\alpha_{\mu11}A_{\mu11}\ ,$$

and by repeated composition with A_{11}:

$$-\sum_{\nu=2}^{n}\sum_{\pi=2}^{n}\alpha_{1\nu\pi}A_{1\nu\pi}+2\sum_{\mu=2}^{n}\alpha_{\mu11}A_{\mu11}\ .$$

As a result, our subgroup also includes the transformations:

$$\sum_{\mu=2}^{n}\alpha_{\mu11}A_{\mu11}\ ,\qquad \sum_{\nu=2}^{n}\sum_{\pi=2}^{n}\alpha_{1\nu\pi}A_{1\nu\pi}\ .$$

In composition with A_{22}, this yields:

$$-\,\alpha_{211}\,A_{211},\qquad 2\sum_{\pi=2}^{n}\alpha_{12\pi}A_{12\pi}\ ,$$

and finally, by composing the last expression with A_{33} :

$$2\,\alpha_{123}\,A_{123}\,.$$

It is clear that one can derive all terms $\alpha_{\mu\nu\pi}\,A_{\mu\nu\pi}$ in which μ is equal to either ν or π.

Therefore, if there is a transformation (16) *in one of the desired invariant subgroups in which not all* $\alpha_{\mu\nu\pi}$ (ν, $\pi < \mu$ or $> \mu$) *equal zero then at least one transformation* $A_{\mu\nu\pi}$ (ν, $\pi < \mu$ or $> \mu$) *appears in this subgroup,* and we assert, *all of them do.*

First, let the number n – which, by the way, is always assumed to be equal to at least 2 – be greater than 2, and also let $\nu \neq \pi$ in the $A_{\mu\nu\pi}$ that are present – say, A_{123} . We define:

$$(A_{i\kappa}\,A_{123}) = \varepsilon_{i2}\,A_{1\kappa 3} + \varepsilon_{i3}\,A_{12\kappa} - \varepsilon_{\kappa 1}\,A_{\kappa 23}\,,$$

and set $i = 3$, $\kappa = 2$ in it; this yields A_{122}, which likewise belongs to the invariant subgroup. In any event, we can therefore assume that ν equals π in the $A_{\mu\nu\pi}$ that appear; for that reason, we call it A_{122} , and no longer need to regard the restriction $n > 2$ as being valid.

Now, along with A_{122}, the transformation:

$$(A_{i\kappa}\,A_{122}) = 2\varepsilon_{i2}\,A_{12\kappa} - \varepsilon_{\kappa 1}\,A_{i22}$$

belongs to the invariant subgroup. If we therefore choose $\kappa = 1$, $i > 2$ or $i < 2$ then we obtain all $A_{12\kappa}$ ($\kappa > 2$ or $\kappa < 2$). Furthermore, $\kappa > 1$, $i = 2$ yields all $A_{12\kappa}$ ($\kappa > 1$) and, since we have all A_{i22} ($i > 2$ or $i < 2$) we obtain all $A_{i2\kappa}$ ($i > \kappa$ or $i < \kappa$; $i > 2$ or $i < 2$). Just as we derived A_{i22} from A_{123} above, we can now derive all $A_{j2\kappa}$ ($j > \kappa$ or $j < \kappa$) from $A_{i2\kappa}$. Finally, the equation:

$$(A_{i\kappa},\,A_{\mu\nu\nu}) = 2\;2\varepsilon_{i2}\,A_{12\kappa} - \varepsilon_{\kappa 1}\,A_{i22} \qquad (\mu > \nu \text{ or } \mu < \nu),$$

with the substitution $i = \nu$, $\kappa > \mu$ or $\kappa < \mu$, gives us all $A_{\mu\nu\kappa}$ ($\mu > \nu$ or $\mu < \nu$, κ), as we asserted; however, with the substitution $i = \nu$, $\kappa = \mu$, we obtain:

$$2A_{\mu\mu\nu} - A_{\nu\nu\nu} \quad (\mu > \nu \text{ or } \mu < \nu).$$

One can verify immediately that the infinitesimal transformation:

$$(17) \qquad A_{\mu\nu\pi}, \quad A_{\mu\mu\mu} - 2A_{\nu\nu\mu} \quad (\nu,\,\pi > \nu \text{ or } \pi < \nu)$$

defines an invariant subgroup of the desired type. Now, if there is an invariant subgroup in which all transformations (17) are present, and still others, in addition, then it must include a transformation:

$$\sum_{\mu=1}^{n} \gamma_\mu A_{\mu\mu\mu}\,,$$

in which not all γ_μ vanish. If, say, $\gamma_1 \neq 0$ here then one immediately obtains A_{111} by composing with A_{11}, so all $A_{\mu\mu 1}$ appear, and since:

$$(A_{1\mu}\, A_{\mu\mu}) = A_{\mu\mu\mu} - A_{1\mu 1} \qquad (\mu > 1)$$

the group contains all $A_{\mu\mu\mu}$, etc.; more briefly, all $A_{\mu\nu\pi}$, which is excluded.

We shall now determine all invariant subgroups that contain only those transformations (16) *in which all* $\alpha_{\mu\nu\pi}$ (ν, $\pi > \mu$ or $\pi < \mu$) *are equal to zero.* Any transformation of such a group has the form:

$$\sum_{\mu=1}^{n}\sum_{\pi=1}^{n}\alpha_{\mu\mu\pi}A_{\mu\mu\pi} .$$

When composed with $A_{\kappa\kappa}$, we obtain:

$$\sum_{\pi=1}^{n}\alpha_{\mu\mu\pi}A_{\mu\mu\pi} + \sum_{\mu=1}^{n}\alpha_{\mu\mu\kappa}A_{\mu\mu\kappa} - \sum_{\pi=1}^{n}\alpha_{\kappa\kappa\pi}A_{\kappa\kappa\pi} ,$$

or, when consolidated:

$$\sum_{\mu=1}^{n}\alpha_{\mu\mu\kappa}A_{\mu\mu\kappa} .$$

The composition with $A_{i\kappa}$ ($i > \kappa$ or $i < \kappa$) yields:

$$\alpha_{ii\kappa}A_{i\kappa\kappa} - \alpha_{\kappa\kappa\kappa}A_{i\kappa\kappa} ,$$

but since $A_{i\kappa\kappa}$ does not enter in, one must have:

$$\alpha_{ii\kappa} = \alpha_{\kappa\kappa\kappa} = \alpha_\kappa .$$

Obviously, the α_κ may not all vanish, so our subgroup certainly includes a transformation of the form:

$$\sum_{\mu=1}^{n}A_{\mu\mu\pi} .$$

If we now form ($\kappa > \pi$ or $\kappa < \pi$):

$$\left(A_{\pi\kappa}, \sum_{\mu=1}^{n}A_{\mu\mu\pi}\right) = A_{\pi\kappa\pi} + \sum_{\mu=1}^{n}A_{\mu\mu\kappa} - A_{\pi\kappa\pi}$$

then we see that our subgroup includes the n transformations:

$$\text{(18)} \qquad \sum_{\mu=1}^{n}A_{\mu\mu\pi} \qquad (\nu = 1, \ldots, n).$$

From the foregoing, it cannot contain any more. In fact, one easily confirms that the infinitesimal transformations (18) define an invariant subgroup.

With that, all invariant subgroups are found that do not belong to the excluded ones; there are just the two of them: (17) and (18). *For that reason, we also obtain only two types of groups that have a meromorphically isomorphic connection with* (15). *The first type is characterized by the equations:*

$$(19) \qquad A_{\mu\nu\pi} = 0, \qquad A_{\mu\mu\mu} - 2\, A_{\nu\nu\mu} = 0 \qquad (\nu, \pi \neq \mu),$$

while the second one is characterized by the equations:

$$(20) \qquad A_{11\pi} + A_{22\pi} + \ldots + A_{\pi\pi\pi} = 0 \qquad (p = 1, \ldots, n).$$

As we said, we would like to present completely the simplest group that corresponds to these compositions only for the *plane* (cf. § 5), *although at least the case* (19) *will be treated in general.* Of the infinitesimal transformations $A_{\mu\nu\pi}$, only n of them are independent – say, $A_{111}, \ldots, A_{nnn}$ – and from (15), the following equation exists:

$$(A_{i\kappa}\, A_{\pi\pi\pi}) = 2\varepsilon_{i\pi}\, A_{\pi\kappa\pi} - \varepsilon_{\kappa\pi}\; A_{i\pi\pi}.$$

Since we can set $A_{i\pi\pi} = \varepsilon_{i\pi}\, A_{\pi\pi\pi}$, we have:

$$(A_{i\kappa}\, A_{\pi\pi\pi}) = \varepsilon_{i\pi}\, (2\, A_{\pi\kappa\pi} - \varepsilon_{\kappa\pi}\; A_{\pi\pi\pi}),$$

and upon closer consideration of the two cases $\kappa \neq \pi$ and $\kappa = \pi$, this yields the formula:

$$(A_{i\kappa}\, A_{\pi\pi\pi}) = \varepsilon_{i\pi}\, A_{\pi\kappa\pi}.$$

Now, the simplest case is the one in which the number m of variables α is equal to n and the n-parameter group $A_{111}, \ldots, A_{nnn}$ is transitive. We can then choose the variables $\alpha_1, \ldots, \alpha_n$ such that one has:

$$A_{\pi\pi\pi} = 2\, A_{\nu\nu\pi} = \frac{\partial f}{\partial \alpha_\pi} \qquad (\pi = 1, \ldots, n).$$

From the equations:

$$\left(\sum_{i=1}^{n} A_{ii}, A_{\pi\pi\pi} \right) = A_{\pi\pi\pi},$$

we obtain, moreover:

$$\sum_{i=1}^{n} A_{ii} = - \sum_{i=1}^{n} (\alpha_i + c_i) \frac{\partial f}{\partial \alpha_i},$$

or, when we introduce the $\alpha_i + c_i$ as new α_i :

$$\sum_{i=1}^{n} A_{ii} = S = -\sum_{i=1}^{n} \alpha_i \frac{\partial f}{\partial \alpha_i}.$$

On the other hand, one comes to:

$$A_{i\kappa} = -\alpha_i \frac{\partial f}{\partial \alpha_\kappa} + \sum_{\nu=1}^{n} h_{i\kappa\nu} \frac{\partial f}{\partial \alpha_\nu}.$$

However, since $(A_{i\kappa}, S) = 0$, all $h_{i\kappa\nu}$ vanish, so this simply becomes:

$$A_{i\kappa} = -\alpha_i \frac{\partial f}{\partial \alpha_\kappa}.$$

The infinite group (1) in question has the form:

$$\sum_{i=1}^{n} \varphi_i \frac{\partial f}{\partial x_i} + \sum_{\kappa=1}^{n} \left\{ \sum_{\nu=1}^{n} \frac{\partial^2 \varphi_\nu}{\partial x_\nu \, \partial x_\kappa} - \sum_{i=1}^{n} \alpha_i \frac{\partial \varphi_i}{\partial x_\kappa} \right\} \frac{\partial f}{\partial \alpha_\kappa}; \tag{21}$$

it delivers the defining equations:

$$\sum_{\nu=1}^{n} \frac{\partial^2 \varphi_\nu}{\partial x_\nu \, \partial x_\kappa} - \sum_{i=1}^{n} \alpha_i \frac{\partial \varphi_i}{\partial x_\kappa} - \sum_{\pi=1}^{n} \alpha_{\pi\pi} \xi_\pi = 0 \qquad (\kappa = 1, \dots, n). \tag{22}$$

One obtains $\frac{n(n-1)}{1 \cdot 2}$ equations for the unrestricted integrability conditions for this system, namely, $\alpha_{i\kappa} = \alpha_{\kappa i}$, or $\frac{\partial \alpha_i}{\partial x_\kappa} = \frac{\partial \alpha_\kappa}{\partial x_i}$, so a function $\alpha(x_1, \dots, x_n)$ exists such that α_i can be expressed in the form $\alpha_i = \frac{\partial \alpha}{\partial x_i}$. It may be verified easily that the integrability conditions $\alpha_{i\kappa} = \alpha_{\kappa i}$ $(i, \kappa = 1, \dots, n)$ admit the infinite group (21).

§ 5. Three important groups in the plane.

16. We write the infinite group (1) of § 4 in the plane as follows:

$$\begin{cases} \varphi(x,y)\frac{\partial f}{\partial x} + \psi(x,y)\frac{\partial f}{\partial y} + Af\,\varphi_x + Bf\,\varphi_y + Cf\,\psi_x + Df\,\psi_y + \\ \quad + Ef\,\varphi_{xx} + 2Ff\,\varphi_{xy} + Gf\,\varphi_{yy} + Hf\,\psi_{xx} + 2If\,\psi_{xy} + Kf\,\psi_{yy}. \end{cases} \tag{1}$$

Here, from § 4, formulas (15), the following relations exist between the infinitesimal transformations *Af*, …, *Kf*:

$$
(2) \quad \begin{cases}
(AB) = (BD) = -B, \quad (AC) = (CD) = C, \\
(AD) = 0, \quad (BC) = D - A, \\
E = (AE), \quad 2F = (BE) = 2(DF), \\
2G = 2(BF) = (DG) = -2(AG) = -2(BK), \\
2H = 2(CJ) = (AH) = -2(DH) = -2(CE), \\
2J = (CK) = 2(AJ), \quad K = (DK), \\
(CF) = E - J, \quad (BJ) = K - F, \quad (BH) = 2J - E, \quad (CG) = 2F - K, \\
(AF) = (AK) = (BG) = (CH) = (DE) = (DJ) = 0.
\end{cases}
$$

In addition, *E*, *F*, *G*, *H*, *J*, *K* are pair-wise commutable. For what follows, it is ultimately to one's advantage to point out the relations:

$$
(3) \qquad (A + D, E) = E, \ \ldots, \ (A + D, K) = K.
$$

From the investigations in no. 15 of the foregoing paragraphs, we have to distinguish three cases. Of the six infinitesimal transformations *E*, …, *K*, which obviously themselves generate a group, either two or four of them, and ultimately, all six of them can, in fact, be independent of each other. In the first case, the following relations exist:

$$
G = 0, \qquad H = 0, \qquad 2J - E = 0, \qquad 2F - K = 0,
$$

and in the second case, the relations:

$$
E + J = 0, \qquad F + K = 0
$$

(cf., § 4, formula (19) and (20)).

For the first three cases, we assume that there are just as many variables $\alpha_1, \ldots, \alpha_m$ as independent ones among the infinitesimal transformations *E*, …, *K*. We further assume that these independent infinitesimal transformations define a transitive group.

Case I. We have already disposed of the first of the three given cases for *n* variables in the conclusion to the previous paragraphs. For that reason, we can restrict ourselves to giving the defining equations in question in the plane. They read:

$$
(4) \qquad \begin{cases}
\xi_{xx} + \eta_{xy} - \alpha\eta_x - \beta\eta_x - \alpha_x\xi - \alpha_y\eta = 0, \\
\xi_{xy} + \eta_{yy} - \alpha\xi_y - \beta\eta_y - \beta_x\xi - \beta_y\eta = 0,
\end{cases}
$$

where the functions α and β must satisfy the integrability condition $a_y - \beta_x = 0$. One can find an integral to the differential equations (4) with no further assumptions, namely:

$$\xi_x + \eta_y - \alpha\xi - \beta\eta = \text{const.};$$

moreover, our considerations in no. 12 of § 3 can lead to them, as well. Regarding the groups (4), one can further confer Lie, “Unendliche continuirliche Gruppen,” § 6, no. 14 and § 8, no. 22.

Case II. If we eliminate E and K by means of the relations $E + J = 0$, $F + K = 0$ then we obtain from (2):

$$(5)\qquad \begin{cases} (AF) = 0, \quad (AG) = -G, \quad (AH) = 2H, \quad (AJ) = J, \\ (BF) = G, \quad (BG) = 0, \quad (BH) = 3J, \quad (BJ) = -2F, \\ \qquad\qquad \text{etc.} \end{cases}$$

The four independent variables in the infinitesimal transformations F, G, H, J may be called λ, μ, ν, ρ. We choose them in such a way that:

$$F = -K = \frac{\partial f}{\partial\lambda}, \qquad G = \frac{\partial f}{\partial\mu}, \qquad H = \frac{\partial f}{\partial\nu}, \qquad J = -E = \frac{\partial f}{\partial\rho}.$$

Relations (3) show that by a suitable choice of variables, $A + D$ can take the form:

$$A + D = -\lambda\frac{\partial f}{\partial\lambda} - \mu\frac{\partial f}{\partial\mu} - \nu\frac{\partial f}{\partial\nu} - \rho\frac{\partial f}{\partial\rho};$$

however, equations (5) further yield:

$$(6)\qquad \begin{cases} A = \quad \mu\dfrac{\partial f}{\partial\mu} - 2\nu\dfrac{\partial f}{\partial\nu} - \rho\dfrac{\partial f}{\partial\rho}, \quad B = 2\rho\dfrac{\partial f}{\partial\lambda} - \lambda\dfrac{\partial f}{\partial\mu} - 3\nu\dfrac{\partial f}{\partial\rho}, \\ C = -3\mu\dfrac{\partial f}{\partial\lambda} - \rho\dfrac{\partial f}{\partial\nu} + 2\lambda\dfrac{\partial f}{\partial\rho}, \quad D = -\lambda\dfrac{\partial f}{\partial\lambda} - 2\mu\dfrac{\partial f}{\partial\mu} + \nu\dfrac{\partial f}{\partial\nu}. \end{cases}$$

From the usual rules (cf., § 3, no. 10, theorem 4), we thus find the four defining equations:

$$(7)\qquad \begin{cases} 2\xi_{xy} - \eta_{yy} + 2\rho\xi_y - 3\mu\eta_x - \lambda_x\xi - \lambda_y\eta = 0, \\ \xi_{yy} + \mu\xi_x - \lambda\xi_y - 2\mu\eta_y - \mu_x\xi - \mu_y\eta = 0, \\ \eta_{yy} - 2\nu\xi_x - \rho\eta_x + \nu\eta_y - \nu_x\xi - \nu_y\eta = 0, \\ 2\eta_{xy} - \xi_{xx} - \rho\xi_x - 3\nu\xi_y + 2\lambda\eta_x - \rho_x\xi - \rho_y\eta = 0. \end{cases}$$

One first obtains the integrability conditions for this system after two differentiations. If one has a system of functions λ, μ, ν, ρ that satisfy the integrability conditions then equations (7) define a group that is similar to the projective group in the plane. One obtains the defining equations of the latter from (7) by the substitution $\lambda = \mu = \nu = \rho = 0$, namely:

$$2\xi_{xy} - \eta_{yy} = 0, \qquad \xi_{yy} = 0, \qquad \eta_{xx} = 0, \qquad 2\eta_{xy} - \xi_{xx} = 0.$$

Case III. We now have six independent variables – say, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ – and can set:

$$E = \frac{\partial f}{\partial \alpha}, \qquad F = \frac{1}{2}\frac{\partial f}{\partial \beta}, \qquad G = \frac{\partial f}{\partial \gamma}, \qquad H = \frac{\partial f}{\partial \delta}, \qquad J = \frac{1}{2}\frac{\partial f}{\partial \varepsilon}, \qquad K = \frac{\partial f}{\partial \zeta}.$$

By a suitable choice of variables, we can further arrive at the fact that:

$$A + D = -\left(\alpha\frac{\partial f}{\partial \alpha} + \cdots + \zeta\frac{\partial f}{\partial \zeta}\right),$$

and then find from equations (2):

$$A = -\alpha\frac{\partial f}{\partial \alpha} + \gamma\frac{\partial f}{\partial \gamma} - 2\delta\frac{\partial f}{\partial \delta} - \varepsilon\frac{\partial f}{\partial \varepsilon},$$

$$B = \delta\frac{\partial f}{\partial \alpha} + (\varepsilon - \alpha)\frac{\partial f}{\partial \beta} + (\zeta - 2\beta)\frac{\partial f}{\partial \gamma} - \delta\frac{\partial f}{\partial \varepsilon} - 2\varepsilon\frac{\partial f}{\partial \zeta},$$

$$C = -\beta\frac{\partial f}{\partial \alpha} - \gamma\frac{\partial f}{\partial \beta} + (\alpha - 2\varepsilon)\frac{\partial f}{\partial \delta} + (\beta - \zeta)\frac{\partial f}{\partial \varepsilon} + \gamma\frac{\partial f}{\partial \zeta},$$

$$D = -\beta\frac{\partial f}{\partial \beta} - 2\gamma\frac{\partial f}{\partial \gamma} + \delta\frac{\partial f}{\partial \delta} - \zeta\frac{\partial f}{\partial \zeta}.$$

This yields the six defining equations:

$$(8) \qquad \begin{cases} \xi_{xy} - \alpha\xi_x + \delta\xi_y - 2\beta\eta_x - \alpha_x\xi - \alpha_y\eta = 0, \\ \xi_{xy} + (\varepsilon - \alpha)\xi_y - \gamma\eta_x - \beta_x\xi - \beta_y\eta = 0, \\ \xi_{yy} + \gamma\xi_x + (\xi - 2\beta)\xi_x - 2\gamma\eta_y - \delta_x\xi - \delta_y\eta = 0, \\ \qquad\qquad \text{etc.} \end{cases}$$

Since all of the differential quotients in (8) of a certain order – here, the second – are determined, one merely needs to differentiate once in order to be able to exhibit the integrability conditions. One obtains the following four:

$$(9) \qquad \begin{cases} \alpha_y - \beta_x = \zeta_x - \varepsilon_y = \gamma\delta - \beta\varepsilon, \\ \beta_y - \gamma_x = \beta(\beta - \zeta) + \gamma(\varepsilon - \alpha), \\ \varepsilon_x - \delta_y = \delta(\beta - \zeta) + \varepsilon(\varepsilon - \alpha). \end{cases}$$

These define an unrestrictedly integrable system that, as we know, admits the infinite group (1).

Any system of functions $\alpha, \ldots, \zeta$ that satisfies equations (9), when substituted in (8), delivers the defining equations of a group that is similar to the general linear group of the plane. The defining equations of the latter are given by $\alpha = \beta = \ldots = \zeta = 0$, namely:

$$\xi_{xx} = \xi_{xy} = \xi_{yy} = 0, \qquad \eta_{xx} = \eta_{xy} = \eta_{yy} = 0.$$

Equations (8) and (9) collectively define an unrestrictedly integrable system with the unknowns $\xi, \eta, \alpha, \ldots, \zeta$. The following two integrals of this system can be given:

$$(10) \qquad \begin{cases} \xi_x + \eta_y - (\alpha+\varepsilon)\xi - (\beta+\zeta)\eta = \text{const.}, \\ (\xi_x - \alpha\xi - \beta\eta)(\eta_y - \varepsilon\xi - \zeta\eta) - (\xi_y - \beta\xi - \gamma\eta)(\eta_x - \delta\xi - \varepsilon\eta) = \text{const.} \end{cases}$$

The reasoning of § 3, no. 12 leads to them.

17. *In conclusion, we make some remarks regarding the integration of the system* (7).

We imagine that λ, μ, ν, ρ are given functions of x, y, and assume that by substituting these values, the system (7) becomes unrestrictedly integrable; one then defines a certain group that is similar to the general projective group of the plane.

If we subject the $\xi, \beta, \gamma, \delta, \varepsilon, \zeta$ to the equations:

$$(11) \qquad \gamma = \mu, \qquad \delta = \nu, \qquad 2\beta = \zeta = \lambda, \qquad 2\varepsilon - \alpha = \rho$$

then (7) becomes a consequence of equations (8). If we then determine $\alpha, \ldots, \zeta$ in the most general way such that equations (11) are fulfilled and the system (8) becomes unrestrictedly integrable, then equations (8) must represent a system of integral equations of (7).

In order for the system (8) to become unrestrictedly integrable $\alpha, \ldots, \varepsilon$ must satisfy the conditions (9); if we then eliminate $\alpha, \gamma, \delta, \zeta$ from (9) by means of equations (11) then we obtain the following differential equations for the determination of β and ε:

$$(12) \qquad \begin{cases} 2\varepsilon_y - \beta_x - \rho_y = 2\beta_x - \varepsilon_y - \lambda_x = \mu\nu - \beta\varepsilon, \\ \beta_x = \mu_x + \beta(\lambda - \beta) + \mu(\rho - \varepsilon), \\ \varepsilon_x = \nu_y + \nu(\lambda - \beta) + \varepsilon(\rho - \varepsilon). \end{cases}$$

Lie arrived at the same differential equations in his paper "Classification und Integration, etc., III" Archiv for Math. og Naturvid. Bd. VIII, page 373, *et seq*. (cf., in particular, equations (3) and (4) themselves). However, the formulation of the entire integration problem is essentially different in Lie, since his derivation of the differential equations (12) also has no direct point of contact with the one that was given by us.

The system (12) is equivalent to a system of linear ordinary differentiations of third order (cf., Lie, *loc. cit.*, page 376); if the latter is integrated completely then one obtains $\alpha, \ldots, \zeta$, expressed as functions of x, y, and the substitution of these values in (8) yields a system of integral equations of (7). The complete resolution of the integration problem now introduces no difficulties, and may be accomplished without integration. We would not like to go into this, but only remark that the integral (10) proves very advantageous in the treatment of that problem. –

Leipzig, in June 1885.