The Mannheim curve of a space curve

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In the study of curves, the method of coordinate conversion has proved to be a fruitful principle for the discovery of new curves and the exhibition of connections between entire families of curves \(^{(1)}\). One of the most useful conversions is the one that takes natural coordinates to rectangular. For any curve, it will produce the so-called Mannheim curve \(^{(2)}\), which is the locus of the curvature centers that belong to the respective contact points when the base curve rolls on a line without slipping.

In what follows, the concept of Mannheim curve shall be carried over to space curves.

The form of a space curve is known to be determined completely by the values of the radii of its first and second curvatures \(\rho\) and \(T\), when expressed as functions of the arc length \(s\). One refers to the givens:

\[
\begin{align*}
\rho &= \varphi (s), \\
T &= F(s)
\end{align*}
\]  

(1)

as the *natural equations* for the space curve \(K\). The search for the equations of \(K\) in rectangular coordinates can come down to the integration of a differential equation of Ricatti type \(^{(3)}\).

*Let the curve with the equations:

\[
\begin{align*}
y &= \varphi (x), \\
z &= F(x)
\end{align*}
\]  

(2)

be referred to the Mannheim curve \(M\) of the curve \(M\).*

The validity of the following theorem can be seen immediately moreover:

*If a space curve \(K\) rolls along the x-axis of a rectangular coordinate system without slipping in such a way that the x-axis always coincides with the curve tangent and the xy-plane always coincides with the osculating plane then the curvature center that belongs to the respective contact point will describe the orthogonal projection \(M'\) of \(M\) onto the xy-plane, and the torsion center that belongs to the respective contact point will describe*


\(^{(2)}\) Wölffing, Zeit. Math. 44 (1899), 140; Mannheim, J. math. pure appl. (2) 4 (1859), 93-104.

\(^{(3)}\) See, perhaps, Bianchi, Vorlesung über Differenzialgeometrie, (German translation by Lukat), 2nd ed., Teubner, Leipzig, 1903.
the orthogonal projection $M''$ of $M$ onto the $xz$-plane, where $M$ means the Mannheim curve of $K$.

The simplest example is defined by the ordinary helix:

$$\rho = \gamma, \quad T = C.$$  \hfill (3)

Its Mannheim curve reads:

$$y = \gamma, \quad z = C,$$  \hfill (3')

so it is therefore a line that is parallel to the $x$-axis.

The equations:

$$\rho = as, \quad T = bs$$  \hfill (4)

represent the loxodrome of the circular cone. Its Mannheim curve:

$$y = ax, \quad z = bx$$  \hfill (4')

is also a line.

The general helix (v.z, the cylindrical loxodrome) the characterized by the fact that its curvatures have a constant ratio. Its equation is:

$$\frac{\rho}{T} = a,$$  \hfill (5)

and a curve:

$$\frac{y}{z} = a$$  \hfill (6)

that lies in a plane that is perpendicular to the $yz$-plane (so it is a plane curve) will be its Mannheim curve.

The skew circles, which are characterized by constant flexure, also produce plane curves as their Mannheim curves.

One also finds the asymptotic lines of the pseudo-sphere:

$$\rho = \frac{a}{4}(e^{\gamma a} + e^{-\gamma a}) T = a$$  \hfill (7)

amongst the lines of constant torsion.

Their Mannheim curves are planar and are closely related to the catenary:

$$y = \frac{1}{2} \cdot \frac{a}{2} (e^{\gamma a} + e^{-\gamma a}) z = a,$$  \hfill (7')

and they bear the name of vault lines (Ger. Gewölbelinie) \(^\dagger\).

As a final example, let the asymptotic lines to the catenoids be cited. Their natural equations are:

\[ \rho = s + \frac{2a^2}{s}, \quad T = a + \frac{s^2}{2a}, \quad (8) \]

which implies the equations:

\[ y = x + \frac{2a^2}{x}, \quad z = a + \frac{s^2}{2a}, \quad (8') \]

for the Mannheim curve, or more simply:

\[ x^2 - xy + 2a^2 = 0, \quad x^2 = 2a (z - a). \quad (8'') \]

It is then a space curve of order four and type one that is defined by the intersection of a hyperbolic cylinder that is perpendicular to the \(xy\)-plane and a parabolic one that is perpendicular to the \(xz\)-plane. As the elimination of \(x\) will show, its perpendicular projection onto the \(yz\)-plane is:

\[ y^2 (x - a) - 2a x^2 = 0, \quad (8^*) \]

which is a special tangent curve to the parabola \( (5) \).

The number of examples can be expanded arbitrarily \( (6) \).

In the same way as one does for plane curves \( (5) \), one can also examine space curves that arise from the Mannheim curve when the base curve rolls without slipping along an arbitrary curve, instead of a line, in such a way that the moving triad covers the fixed and moving curves. However, that shall remain a further topic to pursue.

\[ \text{———} \]

\( (5) \) L. Henkel, Über die aus einer Kurve \( y = f(x) \) abgeleitete Kurve \( y_1 = x \frac{dy}{dx} = x f'(x) \), etc. Dissertation Marburg 1882.

\( (6) \) In regard to the examples that were cited here, see: Cesàro, Natürliche Geometrie (German translation by Kowalewski), Teubner, Leipzig, 1901.

\( (7) \) L. Braude, Über einige Verallgemeinerungen des Begriffes der Mannheimschen Kurve, Dissertation, Neumann, Pirmasens, 1911.