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The Mannheim curve of a space curve

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In the study of curves, the method of coordinate conversion has proved to be a fruitful principle for the discovery of new curves and the exhibition of connections between entire families of curves ([1]). One of the most useful conversions is the one that takes natural coordinates to rectangular. For any curve, it will produce the so-called Mannheim curve ([2]), which is the locus of the curvature centers that belong to the respective contact points when the base curve rolls on a line without slipping.

In what follows, the concept of Mannheim curve shall be carried over to space curves.

The form of a space curve is known to be determined completely by the values of the radii of its first and second curvatures ρ and T, when expressed as functions of the arc length s. One refers to the givens:

$$\rho = \varphi(s), \qquad T = F(s) \tag{1}$$

as the *natural equations* for the space curve K. The search for the equations of K in rectangular coordinates can come down to the integration of a differential equation of Ricatti type ([3]).

Let the curve with the equations:

$$y = \varphi(x), \qquad z = F(x) \tag{2}$$

be referred to the Mannheim curve M of the curve M.

The validity of the following theorem can be seen immediately moreover:

If a space curve K rolls along the x-axis of a rectangular coordinate system without slipping in such a way that the x-axis always coincides with the curve tangent and the xy-plane always coincides with the osculating plane then the curvature center that belongs to the respective contact point will describe the orthogonal projection M' of M onto the xy-plane, and the torsion center that belongs to the respective contact point will describe

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([1]) Cf., e.g., **B. Wieleitner**, *Spezielle ebene Kurven*, Göschen, Leipzig, 1908. (Sammlung **Schubert 56**), pp. 313, *et seq.*

([2]) **Wölffing**, Zeit. Math. **44** (1899), 140; **Mannheim**, J. math. pure appl. (2) **4** (1859), 93-104.

([3]) See, perhaps, **Bianchi**, *Vorlesung über Differenzialgeometrie*, (German translation by **Lukat**), 2nd ed., Teubner, Leipzig, 1903.

the orthogonal projection M″ of M onto the xz-plane, where M means the Mannheim curve of K.

The simplest example is defined by the *ordinary helix:*

$$\rho = \gamma, \quad T = C. \tag{3}$$

Its Mannheim curve reads:

$$y = \gamma, \quad z = C, \tag{3′}$$

so it is therefore a *line* that is parallel to the *x*-axis.

The equations:

$$\rho = as, \qquad T = bs \tag{4}$$

represent the *loxodrome of the circular cone.* Its Mannheim curve:

$$y = ax, \qquad z = bx \tag{4′}$$

is also a *line*.

The *general helix (v.z, the cylindrical loxodrome)* the characterized by the fact that its curvatures have a constant ratio. Its equation is:

$$\frac{\rho}{T} = a, \tag{5}$$

and a curve:

$$\frac{y}{z} = a \tag{6}$$

that lies in a plane that is perpendicular to the *yz*-plane (so it is a *plane curve*) will be its Mannheim curve.

The *skew circles*, which are characterized by constant flexure, also produce *plane curves* as their Mannheim curves.

One also finds the *asymptotic lines of the pseudo-sphere:*

$$\rho = \frac{a}{4}(e^{s/a} + e^{-s/a})T = a \tag{7}$$

amongst the lines of constant torsion.

Their Mannheim curves are planar and are closely related to the *catenary:*

$$y = \frac{1}{2} \cdot \frac{a}{2}(e^{x/a} + e^{-x/a})\, z = a\,, \tag{7′}$$

and they bear the name of *vault lines* (Ger. *Gewölbelinie*) ([4]).

As a final example, let the *asymptotic lines to the catenoids* be cited. Their natural equations are:

([4]) **Schlömilch**, *Übungsbuch zum Studium höheren Analysis,* I. T., 3rd ed., Leipzig, 1879, pp. 101.

$$\rho = s + \frac{2a^2}{s}, \qquad T = a + \frac{s^2}{2a}, \tag{8}$$

which implies the equations:

$$y = x + \frac{2a^2}{x}, \qquad z = a + \frac{s^2}{2a} \tag{8′}$$

for the Mannheim curve, or more simply:

$$x^2 - xy + 2a^2 = 0, \qquad x^2 = 2a\,(z - a). \tag{8″}$$

It is then a *space curve of order four and type one* that is defined by the intersection of a hyperbolic cylinder that is perpendicular to the *xy*-plane and a parabolic one that is perpendicular to the *xz*-plane. As the elimination of x will show, its perpendicular projection onto the *yz*-plane is:

$$y^2\,(x - a) - 2a\,x^2 = 0, \tag{8*}$$

which is a special *tangent curve* to the *parabola* ([5]).

The number of examples can be expanded arbitrarily ([6]).

In the same way as one does for plane curves ([7]), one can also examine space curves that arise from the Mannheim curve when the base curve rolls without slipping along an arbitrary curve, instead of a line, in such a way that the moving triad covers the fixed and moving curves. However, that shall remain a further topic to pursue.

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([5]) **L. Henkel**, *Über die aus einer Kurve* $y = f(x)$ *abgleitete Kurve* $y_1 = x\frac{dy}{dx} = x f'(x)$, etc. Dissertation Marburg 1882.

([6]) In regard to the examples that were cited here, see: **Cesàro**, *Natürliche Geometrie* (German translation by **Kowalewski**), Teubner, Leipzig, 1901.

([7]) **L. Braude**, *Über einige Verallgemeinerungen des Begriffes der Mannheimschen Kurve*, Dissertation, Neumann, Pirmasens, 1911.