"Über die Streuung von Licht an Licht nach der Diracschen Theorie," Ann. Phys. (Leipzig) **26** (1936), 398-448.

On the scattering of light by light in Dirac's theory [1])

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(With 3 figures)

Contents

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[1]) Dissertation presented to the philosophy faculty at the University of Leipzig. The present work is the detailed treatment of a notice of Euler and Kockel in "Naturwissenschaften" **23**, pp. 246, 1935. The work done in Parts II and III was done jointly with Herrn Kockel, while § 5 was mainly due to Herrn Kockel.

Part III

Introduction

Halpern [1]) and Debye [2]) have remarked that one must expect a *scattering of light by light* in Dirac's theory. The two light quanta can create a pair – a positron and an electron – and this pair can, moreover, immediately radiate; two light quanta can therefore spontaneously convert into two other light quanta (under the conservation of total energy and impulse).

In this process, one must distinguish two cases:

Either the energies cg^1 and cg^2 of the two light quanta and the angle between their impulses $\mathfrak{g}^1$ and $\mathfrak{g}^2$ are so large that the law of energy and impulse allows the creation of a *virtual* pair ($g^1g^2 - (\mathfrak{g}^1\mathfrak{g}^2) > 2(mc)^2$). One then obtains the probability for the scattering of the light quanta by each other when one multiplies the probabilities for pair creation and re-radiation and sums over all possibilities. This was carried out by Breit and Wheeler [3]).

Or the energy and impulse of the two light quanta does not attain the magnitude that is necessary for the creation of a virtual pair:

$$\left(\begin{array}{c} g^1g^2 - (\mathfrak{g}^1\mathfrak{g}^2) < 2(mc)^2, \\ \text{i.e., in a particular reference system: } g^1 < mc, g^2 < mc \end{array}\right). \tag{0.1}$$

The light quanta g^1, g^2 can then go over to two other light quanta with the *virtual* possibility of pair creation, and in this case as well (visible light, say) there must still be the scattering of light by light. Its interaction cross-section shall be computed here. (§ 10, formulas 9 and 10).

Part I

The probability for the transition of two light quanta $\mathfrak{g}_1$, $\mathfrak{g}_2$ into two other – $\mathfrak{g}_3$, – $\mathfrak{g}_4$ will be given by the square of the matrix element H_{in}^4 in Dirac's theory (which, as will later be shown, is of fourth order in the electric charge).

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[1]) O. Halpern, Phys. Rev. **44**, pp. 885, 1934.
[2]) P. Debye, in a verbal discussion with Herrn Prof. Heisenberg.
[3]) G. Breit and J. Wheeler, Phys. Rev. **46**, pp. 1087, 1934.

The direct computation of this matrix H_{in}^4 in Dirac's theory [i.e., the matrix element for the *general* case of arbitrary scattering and polarization directions] would be very tedious. One can however revert to a simpler problem of the computation of two matrix elements [i.e., the calculation of H_{in}^4 for two *special* scattering and polarization directions] by the following general considerations (Part I)

§ 1. **Provisional statement of an intuitive expression for the interaction $\bar{U}_1$ of light with light that leads to the transition from two light quanta g_1, g_2 into two others $-g_3$, $-g_4$:**

$$((g_1\, g_2\, |\bar{U}_1| - g_3 - g_4) = H_{in}^4).$$

When two light waves scatter off of each other, instead of passing through each other undisturbed, that implies a violation of the superposition principle. The optical superposition principle is given its expression in the linearity of the vacuum Maxwell equations. The scattering of light by light can also be described by a nonlinear contribution to the vacuum Maxwell equations, in which case an intuitive description is possible. This intuitive description whose possibility will be proved later (§ 7), is suggested by the following analogy that exists in Dirac's theory between light quanta and electrons:

Two electrons can create light quanta and thus introduce a mutual interaction, which perhaps expresses the scattering of electrons from each other, and which gives an intuitive expression for the Coulomb law in a certain approximation.

Likewise, two light quanta create a set of virtual pairs and there thus exists an interaction between them that leads to the scattering of light by light. One should also expect a simple, intuitive expression for this interaction of light quanta with each other that is analogous to the Coulomb law.

The Coulomb interaction in a matter field, which will be described by a density operator $\psi^* \psi$, is:

$$\bar{U}' = \frac{e^2}{2} \iint \frac{\psi^*(\xi)\psi(\xi)\psi^*(\xi')\psi(\xi')}{(\xi - \xi')}\, dV\, dV' . \tag{1.1}$$

One obtains the interaction cross-section for the scattering of an electron by an electron from the square of the matrix element for (1.1) for the transition that takes the form of the scattering of two electrons from each other in a matter field.

In order to find an interaction for light quanta that is analogous to (1.1), one must look for a function $\bar{U}_1$ of the degree of freedom the radiation field represents, hence, the field strength F_{ik} , whose matrix element for the transition into a radiation field, which the scattering of two light quanta from each other represents, will be equal to the one mentioned above and later to the matrix element H_{in}^4 calculated from Dirac's theory for this process.

Concerning this interaction $\bar{U}_1$ for the light quanta as a function of the field strengths, one may state the following:

Since it shall lead to processes in which two light quanta go in while two come out, it must include the field strengths or their derivatives to the fourth power:

$$\bar{U}' = \text{const.} \int \left[FFFF + \text{const.}' \frac{\partial F}{\partial x} \frac{\partial F}{\partial x} FF + \cdots \right] dV \, .$$

(Here, and in what follows, indices for vectors and tensors will be omitted or represented by special indices that make their connection with a scalar obvious).

However, since the interaction $\bar{U}_1$ has the dimension of energy (as a fourth order term in Dirac's theory), the electron charge must appear to the fourth power (and there as only a dimensionless number that can be constructed out of the four universal units *e*, *m*, *c*, *h*, namely, the Sommerfeld fine-structure constant $\frac{e^2}{\hbar c} \sim \frac{1}{137}$); the constant is determined up to a numerical factor:

$$\text{(1.2)} \qquad \left\{ \begin{aligned} &\text{const.} = \frac{\hbar c}{e^2} \frac{1}{E_0^2} \\ &\text{with } E_0 = \frac{e}{\left(\frac{e^2}{mc^2} \right)^2} = \text{"field strength at the electron radius."} \end{aligned} \right.$$

On the same grounds, the terms in the derivatives of the field strengths must include a length that is independent of the electron charge, hence, the Compton wavelength *h*/*mc* as an additional factor.

One next wonders whether the electron mass shall figure in vacuum electrodynamics, since it is assumed that only light quanta and absolutely no electrons are present. However, whether the terms considered here are valid only as long as no actual pairs are created, nonetheless, they come into being only through the virtual possibility of pair creation, and that expresses itself by the introduction of the electron mass.

One thus expects that along with the Maxwellian energy of the individual light quanta there is a mutual interaction between the light quanta of the form:

$$\text{(1.3)} \qquad \bar{U}_1 = \frac{\hbar c}{e^2} \frac{1}{E_0^2} \int \left[FFFF + \left(\frac{\hbar}{mc} \frac{\partial}{\partial x} F \right) \left(\frac{\hbar}{mc} \frac{\partial}{\partial x} F \right) FF + \cdots \right] dV \, .$$

It will later be shown that the matrix element H_{in}^4 that was mentioned above, and which follows from Dirac's theory, can also actually be converted into the matrix element of an expression such as (1.3).

Since we would like to restrict (0.1) for soft light ($|g| < mc$), hence, for slowly varying fields $\left(\left|\frac{\hbar}{mc}\frac{\partial F}{\partial x}\right| < |F|\right)$, we can neglect the terms in (1.3) that involve derivatives of the field strengths.

We thus assume, in advance of the later proof (§ 7), that the scattering of weak light by light can be described through an additional (to the Maxwellian) energy density in the radiation field that has the form:

$$U_1 = \frac{\hbar c}{e^2}\frac{1}{E_0^2} F\,F\,F\,F, \tag{1.4}$$

namely:

$$H_{in}^4 = \left(g_1 g_2 \left| \int U_1 dV \right| - g_3 - g_4\right). \tag{1.5}$$

F_{ik} are the field strengths, V is the volume of the radiation space, $E_0 = \frac{e}{\left(\frac{e^2}{mc^2}\right)}$, and:

$$\begin{array}{rll} g_1 g_2 & \text{light quanta before the collision} & \\ -g_3 - g_4 & \text{light quanta after the collision} & \\ (g_1 g_2 \,|\, O \,|\, -g_3 - g_4) & \text{matrix element of the operator } O & \\ H_{in}^4 & \text{matrix element of Dirac's theory} & \end{array} \left\{\begin{array}{l} \text{for} \\ \text{this} \\ \text{tran -} \\ \text{sition.} \end{array}\right.$$

§ 2. Approximate determination of the interaction $\bar{U}_1$ of light by light from the invariance of the associated corrected Maxwell equations:

$$\left(\bar{U}_1 = \frac{\hbar c}{e^2}\frac{1}{E_0^2}\int [\alpha(\mathfrak{B}^2 - \mathfrak{D}^2)^2 + \beta(\mathfrak{B}\mathfrak{D})^2] dV\right)^{1}).$$

The form of this interaction U_1 (1.4) of light by light shall now be determined approximately by the requirement of relativistic invariance.

In the *general* quantum theory of light and matter [2]) the tensor of electric field strength and magnetic induction, which shall be denoted by $\mathfrak{E}$, $\mathfrak{B}$, satisfies the equations:

$$\boxed{\frac{1}{c}\dot{\mathfrak{B}} + \text{rot}\,\mathfrak{E} = 0, \qquad \text{div}\,\mathfrak{B} = 0} \tag{2.1}$$

[1]) The mathematical proofs of this paragraph are identical with the ones that were used by Born (M. Born, M. Born and L. Infeld, Proc. Roy. Soc. London, A**143**, pp. 410, 1933; A**144**, pp. 425, 1934; A**147**, pp. 522, 1934). They will be repeated because here we will make other physical assumptions. Cf., also pp. (?).

[2]) W. Heisenberg and W. Pauli, Zeit. f. Phys. **56**, pp. 1, 1930; pp. 168, 1930.

which means the same thing as the existence of a potential $\mathfrak{A}$:

$$\boxed{\mathfrak{E} = -\frac{1}{c}\dot{\mathfrak{A}}, \qquad \mathfrak{B} = \text{rot}\,\mathfrak{A}} \tag{2.2}$$

and the equations:

$$\boxed{-\frac{1}{c}\dot{\mathfrak{E}} + \text{rot}\,\mathfrak{B} = \frac{4\pi i}{c}, \qquad \text{div}\,\mathfrak{E} = 4\pi\rho} \tag{2.3}$$

which couples the field $\mathfrak{E}$, $\mathfrak{B}$ to the matter with density ρ and current i. The evolution of the matter ρ, i and its reaction to the field is, in its own right, further determined through the Dirac equation.

This general connection (2.1, 2.2, 2.3) exists before one introduces the hole theory.

However, something new appears in the hole theory when one tries the following *specialization:*

When no electrons are present, one can, *before introducing hole theory*, omit ρ and i and obtain the Maxwell equations of the vacuum: (2.1), (2.2), and:

$$-\frac{1}{c}\dot{\mathfrak{E}} + \text{rot}\,\mathfrak{B} = 0, \qquad \text{div}\,\mathfrak{B} = 0. \tag{2.4}$$

(2.5) $\left\{ \begin{array}{l} \quad \textit{In the hole theory,} \text{ however, also when no electrons are present and} \\ \text{also when the energy of the radiation field is not sufficient for the} \\ \text{creation of electrons and positrons,} \end{array} \right.$

as we saw, there is the possibility of the creation of matter finds, which finds its expression in the behavior of the field.

The equations for this special case (2.5) must, on the one hand, be in agreement with the general equations (2.1, 2.2, 2.3), and, on the other hand, must include only the field strengths. Thus, all that can emerge from (2.1, 2.2, 2.3) is the fact that the current ρ, i will be replaced by certain functions of the field strengths $\mathfrak{E}$, $\mathfrak{B}$ that one can think of as "the virtual matter created by the field $\mathfrak{E}$, $\mathfrak{B}$."

In other words: For our special case (2.5), equations (2.1, 2.2) remain the same, but the vacuum Maxwell equations (2.4) are corrected by certain supplementary terms that can be neglected only for small fields (compared to E_0).

We assume that the altered field equations can be described by a *Hamilton function* $\bar{U}$ and its canonical equations. We can (from 2.2) choose the *coordinates* of the system to be the (negative) four-potential $-\mathfrak{A}$. The *impulse* that is canonically conjugate to $-\mathfrak{A}$ shall be called $\mathfrak{D}/4\pi c$. Hence, it shall be defined by:

$$\mathfrak{D}_i(\xi)\,\mathfrak{A}_k(\xi') - \mathfrak{A}_k(\xi')\,\mathfrak{D}_i(\xi) = 2hci\,\delta(\xi - \xi')\,\delta_{ik} \tag{2.6}$$

or:

(2.7) $$\boxed{\mathfrak{D}_i(\xi)\,\mathfrak{B}_k(\xi')-\mathfrak{B}_k(\xi')\mathfrak{D}_i(\xi)=2hci\,\frac{\partial}{\partial\xi'_l}\,\delta(\xi-\xi')}$$

(with cyclic *ikl*).

The energy $\bar{U}$ is then a function of all the coordinates and the impulse:

(2.8) $$\boxed{\bar{U}=\int U\,dV}$$

which shall, in general, include only the field strengths, but not their derivatives:

(2.9) $$U=U(\mathfrak{B},\mathfrak{D}).$$

The *canonical equations* of the Hamilton function $\bar{U}$ will now be:

$$\begin{aligned}\dot{\mathfrak{B}}_k(\xi')&=\frac{i}{\hbar}\int\left[U(\mathfrak{B}(\xi),\mathfrak{D}(\xi))\,\mathfrak{B}_k(\xi')-\mathfrak{B}_k(\xi')\,U(\mathfrak{B}(\xi),\mathfrak{D}(\xi))\right]d\xi\\&=-4\pi c\,\mathrm{rot}_k\,\frac{\partial U}{\partial\mathfrak{D}},\end{aligned}$$

or, with (2.1):

(2.10) $$\boxed{\frac{\partial U}{\partial\mathfrak{D}}=\frac{\mathfrak{E}}{4\pi}}$$

one has [1]):

(2.10′): $$\boxed{\frac{1}{c}\dot{\mathfrak{B}}+\mathrm{rot}\,\mathfrak{E}=0}$$

and:

$$\begin{aligned}\dot{\mathfrak{D}}_k(\xi')&=\frac{i}{\hbar}\int\left[U(\mathfrak{B}(\xi),\mathfrak{D}(\xi))\,\mathfrak{D}_k(\xi')-\mathfrak{D}_k(\xi')\,U(\mathfrak{B}(\xi),\mathfrak{D}(\xi))\right]d\xi\\&=-4\pi c\,\mathrm{rot}_k\,\frac{\partial U}{\partial\mathfrak{B}},\end{aligned}$$

or, with the definition:

(2.11) $$\boxed{\frac{\partial U}{\partial\mathfrak{B}}=\frac{\mathfrak{H}}{4\pi}}$$

one has:

(2.12) $$\boxed{-\frac{1}{c}\dot{\mathfrak{D}}+\mathrm{rot}\,\mathfrak{H}=0}$$

which further implies:

$$\boxed{\mathrm{div}\,\mathfrak{D}=0}$$

[1]) Translators note: This form of the first canonical equation was not included in the original paper, although it is inserted here for completeness, as it complements (2.12).

With that, the field equations for any energy U are established: (2.1) and (2.2) give the time evolution of the field, while (2.10) and (2.11) couple the field strengths $\mathfrak{E}$, $\mathfrak{B}$ with the field functions $\mathfrak{D}$, $\mathfrak{H}$. As equations (2.12) and (2.1) show, $\mathfrak{D}$ means the electric displacement, while $\mathfrak{B}$ is the magnetic induction and, as such, the force on the true current [1]).

The general schema (2.1, 2.2, 2.12, 2.10, 2.11), which rests upon only the induction law (2.1) and the dependence of energy on the field strengths, first gains significance when one is given a particular Hamilton function U.

If $U = \frac{\mathfrak{B}^2 + \mathfrak{D}^2}{8\pi} = U_0$ then (2.10) becomes $\mathfrak{D} = \mathfrak{E}$ and (2.11) becomes $\mathfrak{H} = \mathfrak{B}$ and (2.12) becomes eq. (2.4) for the uncorrected Maxwell vacuum field, which is true only in the first approximation. In the next order, (1.4) gives the Hamilton function:

$$U = \frac{\mathfrak{B}^2 + \mathfrak{D}^2}{8\pi} + \frac{\hbar c}{e^2}\frac{1}{E_0^2} f(\mathfrak{B}\ \mathfrak{D}) = U_0 + U_1 \tag{2.13}$$

in which f is a function of degree four in $\mathfrak{B}$ and $\mathfrak{D}$.

However, only certain particular f will agree with the relativity principle. We determine them when we show that the field equations (2.1, 2.2, 2.12, 2.10, 2.11) can also be derived from a variational principle and arrive at the conclusion that the Lagrange function L that is extremized in this variational principle is a Lorentz invariant.

Thus, in line with a general mechanical procedure we define the function:

$$\boxed{\frac{L}{4\pi} = \frac{(\mathfrak{E}\mathfrak{D})}{4\pi} - U} \tag{2.14}$$

and calculate its partial derivatives with respect to $\mathfrak{B}$ and $\mathfrak{E}$: We find (after varying the fields by $\delta\mathfrak{E}$, $\delta\mathfrak{B}$, $\delta\mathfrak{D}$, $\delta\mathfrak{H}$):

$$\frac{\delta L}{4\pi} = \frac{\mathfrak{E}}{4\pi}\delta\mathfrak{D} + \frac{\mathfrak{D}}{4\pi}\delta\mathfrak{E} - \frac{\partial U(\mathfrak{B}\mathfrak{D})}{\partial\mathfrak{B}}\delta\mathfrak{B} - \frac{\partial U(\mathfrak{B}\mathfrak{D})}{\partial\mathfrak{D}}\delta\mathfrak{D}$$

or, from (2.10):

$$\frac{\delta L}{4\pi} = \frac{\mathfrak{D}}{4\pi}\delta\mathfrak{E} - \frac{\partial U(\mathfrak{B}\mathfrak{D})}{\partial\mathfrak{B}}\delta\mathfrak{B},$$

hence:

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[1]) The addition of a true current to eq. (2.1, 2.12), i.e., real electrons such as one might see in a Wilson chamber, as opposed to the virtual ones (2.3) that are considered here, but which do not contribute to the radiation of the field and are introduced into this theory only as test particles, would show that:

$\mathfrak{D}$ describes the streamlines of true charges,

$\mathfrak{H}$ describes the vortex lines of true currents,

and would confirm that $\mathfrak{E}$ refers to the force on the true charge while $\mathfrak{B}$ is the force on a true current. Cf., also, C. F. v. Weizsäcker, Ann. d. Phys. **17**, pp. 869, 1933.

(2.15)
$$\boxed{\frac{\partial L(\mathfrak{B},\mathfrak{E})}{\partial \mathfrak{E}} = \mathfrak{D}}$$

and, due to (2.11):

(2.16)
$$\boxed{\frac{\partial L(\mathfrak{B},\mathfrak{E})}{\partial \mathfrak{B}} = -\,\mathfrak{H}}$$

and see that these partial derivatives of L are coupled by eq. (2.12) to a differential equation for L:

(2.17)
$$\frac{1}{c}\frac{\partial}{\partial t}\frac{\partial L}{\partial \mathfrak{E}} + \operatorname{rot}\frac{\partial L}{\partial \mathfrak{B}} = 0,$$

which is equivalent to the variational principle:

(2.18)
$$\boxed{\iint L(\mathfrak{B},\,\mathfrak{E})\,dV\,dt = \text{extremum}}$$

for the *Lagrange function* $L = L(\mathfrak{B},\,\mathfrak{E})$ with the associated conditions (2.1) or (2.2). The Lagrange equations (2.1, 2.15, 2.16, 2.18), which, like the Hamilton equations (2.1, 2.10, 2.11, 2.12) determine the evolution of the field, shall now take on their significance through the choice of a Lagrange function $L = L(\mathfrak{B},\,\mathfrak{E})$, which must be a Lorentz and parity invariant.

All Lorentz invariants of the anti-symmetric tensors $\mathfrak{B}$, $\mathfrak{E}$ must be functions of the Lorentz invariants $\mathfrak{E}^2 - \mathfrak{B}^2$ and $(\mathfrak{E}\ \mathfrak{B})$, the second of which is not, however, parity invariant.

Thus, to lowest degree – viz., second – the only Lorentz and parity invariant expression is $\mathfrak{E}^2 - \mathfrak{B}^2$, which, from (2.15, 2.16, 2.18), leads to the well-known linear vacuum Maxwell equations $\mathfrak{D} = \mathfrak{E}$, $\mathfrak{H} = \mathfrak{B}$ and (2.4) when it is used as a Lagrange function.

In the next higher degree – viz., fourth – one constructs only the Lorentz and parity invariant expressions $(\mathfrak{E}^2 - \mathfrak{B}^2)^2$ and $(\mathfrak{E}\ \mathfrak{B})^2$. Therefore, corresponding to the most general corrected Hamilton function (2.13) to fourth order in the field strengths, there is a Lagrange function:

(2.19)
$$\boxed{\frac{L}{4\pi} = \frac{\mathfrak{E}^2 - \mathfrak{B}^2}{8\pi} + \frac{\hbar c}{e^2}\frac{1}{E_0^2}\left[-\alpha(\mathfrak{E}^2 - \mathfrak{B}^2)^2 - \beta(\mathfrak{E}\ \mathfrak{B})^2\right] = \frac{L_0 + L_1}{4\pi}}$$

in which $-\alpha$ and $-\beta$ are numerical coefficients.

For this Lagrange function the equations that couple the field strengths $\mathfrak{E}$, $\mathfrak{B}$ with the quantities $\mathfrak{D}$, $\mathfrak{H}$ (2.15, 2.16) become:

(2.20)
$$\boxed{\begin{aligned}\frac{\mathfrak{D}}{4\pi} &= \frac{\mathfrak{E}}{4\pi} + \frac{\hbar c}{e^2}\frac{1}{\mathfrak{E}_0^2}\left[-4\alpha(\mathfrak{E}^2 - \mathfrak{B}^2)\mathfrak{E} - 2\beta(\mathfrak{B}\mathfrak{E})\mathfrak{B}\right]\\ \frac{\mathfrak{H}}{4\pi} &= \frac{\mathfrak{B}}{4\pi} + \frac{\hbar c}{e^2}\frac{1}{E_0^2}\left[-4\alpha(\mathfrak{E}^2 - \mathfrak{B}^2)\mathfrak{B} + 2\beta(\mathfrak{B}\mathfrak{E})\mathfrak{E}\right]\end{aligned}}$$

whose inverse equations (as a consequence of neglecting powers of the field strengths that are higher than four) read:

(2.20′)
$$\begin{cases} \dfrac{\mathfrak{E}}{4\pi} = \dfrac{\mathfrak{D}}{4\pi} + \dfrac{\hbar c}{e^2}\dfrac{1}{E_0^2}[+4\alpha(\mathfrak{D}^2 - \mathfrak{H}^2)\mathfrak{D} + 4\beta(\mathfrak{D}\mathfrak{H})\mathfrak{H}] \\ \dfrac{\mathfrak{B}}{4\pi} = \dfrac{\mathfrak{H}}{4\pi} + \dfrac{\hbar c}{e^2}\dfrac{1}{E_0^2}[+4\alpha(\mathfrak{D}^2 - \mathfrak{H}^2)\mathfrak{H} - 4\beta(\mathfrak{D}\mathfrak{H})\mathfrak{D}]. \end{cases}$$

Therefore, from (2.14, 2.20) the Hamilton function that belongs with the Lagrange function (2.19) is:

(2.21)
$$\boxed{U = \frac{\mathfrak{D}^2 + \mathfrak{B}^2}{8\pi} + \frac{\hbar c}{e^2}\frac{1}{E_0^2}\,[\alpha(\mathfrak{D}^2 - \mathfrak{B}^2)^2 + \beta(\mathfrak{D}\mathfrak{B})^2] = U_0 + U_1}$$

With that, the interaction energy U_1 of the light quantum is determined up to two numerical constants α and β. These will be established in § 8 by computing the Dirac matrix element H_{in}^4 in two special, simplest possible cases and comparing with (2.21).

§ 3. **Discussion of the commutation relations for the field strengths in the system of corrected Maxwell equations**

Equations (2.20) lead to the noteworthy result that the electric field strength $\mathfrak{E}$ and quantity $\mathfrak{D}$ (2.6) that is conjugate to the potentials $-\frac{1}{4\pi c}\mathfrak{A}$ are different, while we still assume the general theory of light and matter [1]), in which they are the same. The self-evident contradiction that this represents warrants a thorough discussion. It has been suggested that the physical situation that we are presented with can be best clarified when one compares the system composed of a radiation field and a matter field with the mechanical system of two atoms. We shall follow through with this comparison by placing each property of the one system next to the corresponding property of the other system:

[1]) W. Heisenberg and W. Pauli, Zeit. f. Phys. **56**, pp. 1, 1930; **59**, pp. 168, 1930.

A light and matter field can be described by the potential of the radiation and the density of matter. When no actual electrons are present and the energy of the field is not sufficient for the creation of pairs, the field strengths will suffice for the characterization of the state.

There can, however, be virtual pairs created, and through these transitions there arises an interaction between light quanta.

If one now describes the field through equations that include only the field strengths then one must consider the interaction between the light quanta, i.e., the nonlinear corrections to the Maxwell equations.

The addition L_1 to the Maxwell Lagrange function L_0 includes the magnetic induction $\mathfrak{B}$ and the electric field strength $\mathfrak{E}$ (= time derivative of the potential):

$$(3.19)\qquad \begin{cases} L = L_0(\mathfrak{E}\mathfrak{B}) + L_1(\mathfrak{E}\mathfrak{B}), \\ L_0 = \int \dfrac{\mathfrak{E}^2 - \mathfrak{B}^2}{2}\, dV. \end{cases}$$

The quantity $\mathfrak{D}$ (2.6) that is conjugate to $-\dfrac{1}{4\pi c}$ is therefore not the electric field strength, but:

$$\begin{pmatrix} 3.15 \\ 3.20 \end{pmatrix}\qquad \begin{cases} \dfrac{\partial L}{\partial\left(-\dfrac{\dot{\mathfrak{A}}}{4\pi c}\right)} = \mathfrak{D} \\ = \mathfrak{E} + \dfrac{\partial L_1}{\partial\left(-\dfrac{\dot{\mathfrak{A}}}{4\pi c}\right)}; \quad \mathfrak{D} \neq \mathfrak{E}. \end{cases}$$

Two atoms can be described by the coordinates of their nuclei and the coordinates of their electrons. If it is assumed that the electrons are in the ground state then the coordinates of the nuclei alone succeed in characterizing the state.

The electrons of both atoms can thus be virtually excited and then return to the ground state, and as a result of this transition there will be an interaction between the nuclei: the van der Waals attraction.

If one now describes the system, perhaps by the calculation of the band spectra of the molecule, through only the degrees of freedom of the nuclei then the van der Waals force must be included in its equations of motion.

The van de Waals contribution L_1 to the Lagrange function depends upon the coordinates q_i and, by more precise computation, which we shall only assume here, also on the velocities $\dot{q}_i$ of the nuclei:

$$L = L_0(q_i\dot{q}_i) + L_1(q_i\dot{q}_i)$$

$$L_0 = \sum_i \frac{m}{2}\dot{q}_i^2 + \text{funct}(q_i).$$

The quantities p_i that are conjugate to the nuclear coordinates q_i are therefore not the velocities multiplied by mass, but:

$$\frac{\partial L}{\partial \dot{q}_i} = p_i = m\dot{q}_i + \frac{\partial L_1}{\partial \dot{q}_i}$$

$$p_i \neq \dot{q}_i m.$$

Therefore, between the field strengths there now exist, not the ordinary commutation relations:

$$\mathfrak{E}_i(\xi)\mathfrak{B}_k(\xi') - \mathfrak{B}_k(\xi')\mathfrak{E}_i(\xi) = 2hci \frac{\partial}{\partial \xi'_l}\delta(\xi - \xi'),$$

but the altered ones:

$$(3.7) \quad \begin{cases} \mathfrak{D}_i(\xi)\mathfrak{B}_k(\xi') - \mathfrak{B}_k(\xi')\mathfrak{D}_i(\xi) \\ \quad = 2hci \dfrac{\partial}{\partial \xi'_l}\delta(\xi - \xi'). \end{cases}$$

Thus, in the general case, in which the radiation field can also create pairs, one must describe the total field by the field strengths *and* the matter density, and the interaction of the light quanta does not appear explicitly in the equations. The Lagrange function includes the field strengths only in the form:

$$L = L_0(\mathfrak{E}\ \mathfrak{B}) + L'(\text{matter}).$$

However, the electric field that is conjugate to the potentials is again:

$$\frac{\partial L}{\partial\left(-\dfrac{\dot{\mathfrak{A}}}{4\pi c}\right)} = \mathfrak{D} = \mathfrak{E},$$

and commutation relations read:

$$\mathfrak{E}_i(\xi)\mathfrak{B}_k(\xi') - \mathfrak{B}_k(\xi')\mathfrak{E}_i(\xi) = 2hci \frac{\partial}{\partial \xi'_l}\delta(\xi - \xi'),$$

in contrast to the special case.

Therefore, between the coordinates and the velocities of the nuclei there now exist, not the ordinary commutation relations:

$$m\dot{q}_i q_i - q_i m\dot{q}_i = \frac{\hbar}{i}$$

but the altered ones:

$$p_i q_i - q_i p_i = \frac{\hbar}{i}.$$

Thus, in the general case, in which the electrons can also rise above the ground state, one must then describe the system by the degrees of freedom of the nuclei *and* the electrons, and the van der Waals forces do not appear explicitly in the equations. The Lagrange function includes the nuclear coordinates and velocities only in the form:

$$L = L_0(q_i \dot{q}_i) + L'(\text{electrons}).$$

However, the nuclear coordinates are again conjugate to the mass multiplied by the velocities:

$$\frac{\partial L}{\partial \dot{q}_i} = p_i = m\dot{q}_i$$

and the commutation relations read:

$$m\dot{q}_i q_i - q_i m\dot{q}_i = \frac{\hbar}{i},$$

in contrast to the special case.

Just as in Lagrangian mechanics, the phenomena that were described above can also be expressed in Hamilton mechanics (we denote the commutation $ab - ba$ by $[a, b]$).

In *general*, the Hamilton function H of the field includes the energy H_0 of the light the energy H' of the matter. The field potential $\mathfrak{A}$ commutes with the H' and therefore one has $\mathfrak{E} = \mathfrak{D}$:

$$H = H_0(\mathfrak{A}, \mathfrak{D}) + H'(\text{matter}),$$

$$H_0 = \int \frac{(\text{rot}\,\mathfrak{A})^2 + \mathfrak{D}^2}{8\pi}\, dV,$$

$\mathfrak{D}$ is conjugate to $\mathfrak{A}$ (2.6), $\mathfrak{D}$ and $\mathfrak{H}$ commute with H':

$$\mathfrak{E} = -\frac{1}{c}\dot{\mathfrak{A}} = -\frac{i}{c\hbar}[H\mathfrak{A}] = -\frac{i}{c\hbar}[H_0\mathfrak{A}] = \mathfrak{D};$$

hence, $\mathfrak{E} = \mathfrak{D}$.

However, in the *special* case in which no actual matter is created, the energy of the electrons can be replaced by an interaction $H_1(\mathfrak{A}\mathfrak{D})$ of the light quanta. However, it now no longer commutes with the field potentials $\mathfrak{A}$ and therefore $\mathfrak{E} \neq \mathfrak{D}$.

$$H = H_0(\mathfrak{A}\ \mathfrak{D}) + H_1(\mathfrak{A}\ \mathfrak{D}), \tag{2.22}$$

$\mathfrak{D}$ is conjugate to $\mathfrak{A}$ (2.6):

$$\begin{cases} \mathfrak{E} = -\frac{1}{c}\dot{\mathfrak{A}} = -\frac{i}{c\hbar}[H\mathfrak{A}] \\ \quad = -\frac{i}{c\hbar}[H_0\mathfrak{A}] - \frac{i}{c\hbar}[H_1\mathfrak{A}] \\ \quad = \mathfrak{D} + \cdots, \end{cases} \tag{3.21}$$

hence $\mathfrak{E} \neq \mathfrak{D}$.

In *general*, the Hamilton function H of the molecule includes the energy H_0 of the nucleus and the energy H' of the electrons. The nuclear coordinates q_i commute with H' and therefore one has $m\dot{q}_i = p_i$:

$$H = H_0(q_i\, p_i) + H'(\text{electrons})$$

$$H_0 = \sum_i \frac{p_i^2}{2m} + \text{funct}(q_i),$$

p_i is conjugate to q_i, i.e., $p_i\, q_i - q_i\, p_i = \hbar / i$, q_i commutes with H':

$$m\dot{q}_i = \frac{mi}{\hbar}[H\, q_i] = \frac{mi}{\hbar}[H_0\, q_i] = p_i\,;$$

hence, $m\dot{q}_i = p_i$.

However, in the *special* case in which the atoms are not actually excited the energy of the electrons can be replaced by the van der Waals force between the nuclei $H_1(q_i\, p_i)$. However, it now no longer commutes with the nuclear coordinates and therefore $m\dot{q} \neq p$:

$$H = H_0(p_i\, q_i) + H_1(p_i\, q_i),$$

p_i is conjugate to q_i, i.e.:

$$p_i\, q_i - q_i\, p_i = \frac{\hbar}{i}.$$

$$m\dot{q}_i = \frac{mi}{\hbar}[H\, q_i] = \frac{mi}{\hbar}[H_0\, q_i] + \frac{mi}{\hbar}[H_1\, q_i]$$
$$= p + \ldots,$$

hence, $m\dot{q}_i \neq p_i$.

This comparison once again makes it clear that the aforementioned alteration of the Maxwell equations of the vacuum is not a alteration of present-day field theory [1]), but only a particular example of it.

Furthermore, the following situation must be stressed: The interaction between the light quanta and the alteration of the Maxwell equations that was assumed here exists only as long as no actual pairs can be created, but only virtual pairs. Likewise, one can deal with the van der Waals forces only as long as the atoms are in the ground state (or at least in a well-defined state), whether or not the forces virtually excite the atoms out of the ground state.

One may present the mathematical process of altering of the commutation relations in a mechanical system by specialization in such a manner that the specialization that was assumed here to the case in which no actual pairs can be created refers to a choice of term in the total system of light and matter, hence, a restriction of all matrices to sub-matrices, a neglect of certain transition and occupation probabilities by deleting certain matrix boxes, such that the remaining sub-matrices have other commutation relations than the complete ones.

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline & 0 \\ \hline 0 & 0 \\ \hline\end{array}$$

[1]) Cf., footnote pp. (?).

Part II

After the foregoing considerations about the general form of the results, we now must exhibit the matrix element H_{in}^4 of Dirac's theory for the scattering of light by light, and to show that it is identical with the corresponding matrix element of the interaction energy of light with light:

$$\bar{U}_1 = \frac{\hbar c}{e^2}\frac{1}{E_0^2}\int [\alpha(\mathfrak{B}^2 - \mathfrak{D}^2)^2 + \beta(\mathfrak{B}\,\mathfrak{D})^2]\, dV, \tag{2.21}$$

namely:

$$(g_1\, g_2 \mid \bar{U}_1 \mid -g_3 - g_4) = H_{in}^4,$$

so ultimately one comes down to the calculation of two simple special cases that determine the two constants α, β in $\bar{U}_1$. We begin with a thorough representation of the perturbation schema that will be used for these calculations.

§ 4. General perturbation schema that will be used in the calculation of the scattering of light by light

Suppose that a closed system with the approximate stationary states i, k, l, m, n, μ, μ', which have the energies E_i, E_k, … and the occupation probabilities $|a_i|^2$, $|a_k|^2$, …, is subjected to a perturbation with the time-independent energy matrix V_{ik} that provokes the change of state:

$$i\hbar\frac{\partial}{\partial t}a'_\mu(t) = \sum_\mu e^{i(E_{\mu'} - E_\mu)t/\hbar}V_{\mu\mu'}a_\mu(t)\,. \tag{4.1}$$

(t: time, $h = 2\pi\hbar$: Planck's quantum of action). If one develops this perturbation V and the state a_μ in a small parameter:

$$\begin{cases} V = \quad V^1 + V^2 + V^3 + V^4 + \cdots \\ a_\mu = a_\mu^0 + a_\mu^1 + a_\mu^2 + a_\mu^3 + a_\mu^4 + \cdots, \end{cases} \tag{4.2}$$

and assumes that initially the state i is realized ($a_\mu^0(t) = \delta_{i\mu}$, $a_\mu^\alpha(t) = 0$, for $\alpha \geq 1$) then it follows in the first approximation that:

$$i\hbar\frac{\partial}{\partial t}a_k^1(t) = \sum_\mu e^{i(E_k - E_\mu)t/\hbar}V_{\mu k}a_\mu^0(t) = e^{i(E_k - E_i)t/\hbar}V_{ik},$$

and integration gives:

$$a_k^1(t) = \left(\frac{e^{i(E_k - E_i)t/\hbar} - 1}{E_i - E_k}\right)\cdot V_{ik}\,. \tag{4.3}$$

The second factor of this approximation is the matrix element of the perturbation V^1 for the transition $i \to k$. The first factor is considerable only inside the error domain $| E_k - E_i | \leq h/t$; i.e., $a^1(t)$ always has meaning for small t, but for large t, it is meaningful only in the case where the system can go from state i to state k while conserving energy $E_k = E_i$.

Therefore, a transition probability $i \to k$ of first order:

$$(| a^1(t) |^2 \neq 0 \text{ for large } t)$$

exists only when $V_{ik} \neq 0$ and $E_k = E_i$.

When these conditions are fulfilled, one calculates the transition probability $i \to k$ in a well-known way following Dirac:

$$\frac{1}{t}\sum_{k'\sim k} | a_{k'}(t) |^2 \approx \frac{1}{t}\left|V_{ik}^1\right|^2 \cdot \frac{1}{\Delta E} \cdot \int_{-\infty}^{+\infty} 2\frac{1-\cos(E_{k'}-E_i)t/\hbar}{(E_i - E_{k'})^2} dE_{k'} = \frac{2\pi}{\hbar} \cdot \frac{1}{\Delta E} \cdot \left|V_{ik}^1\right|^2, \tag{4.4}$$

in which $1/\Delta E$ means the number density in the energy spectrum of the system in the final state k.

However, when there is a vanishing transition probability in first order one must consider higher approximations. We now assume that all transition probabilities (from each state i to any other state of equal energy) up to order $\beta - 1$ vanish and that one first encounters transitions in order β, so:

$$\left|a_\mu^1(t)\right|^2 = \left|a_\mu^2(t)\right|^2 = \left|a_\mu^3(t)\right|^2 = \ldots = \left|a_\mu^{\beta-1}(t)\right|^2 = 0, \qquad \left|a_\mu^\beta(t)\right|^2 \neq 0 \tag{4.5}$$

for large t and all $\mu \neq i$ when $E_\mu = E_i$.

Under these assumptions, we make the *claim:* All approximations up to ultimately the β^{th} have the time dependency:

$$\begin{cases} a_{\mu(t)}^{\alpha'} = \left(\dfrac{e^{i(E_\mu - E_i)t/\hbar} - 1}{E_i - E_\mu}\right) \cdot H_{i\mu}^{\alpha'} + \displaystyle\sum_{\kappa \neq i}\left(\dfrac{e^{i(E_\mu - E_\kappa)t/\hbar} - 1}{E_\kappa - E_\mu}\right) \cdot K_{\kappa\mu}^{\alpha'} \\ \qquad\qquad \text{for } \alpha' = 1, 2, \cdots, \beta - 1, \beta. \end{cases} \tag{4.6}$$

In this expression:

- i is the index of the initial state,
- μ is the index of the states considered,
- κ is the index of an "intermediate" state that is different from i and μ.

The first summand leads from the initial state i to the state μ in question; i.e., for large t its time factor is large only for $E_i = E_\mu$.

The second summand leads from a state $\kappa \neq i$ that is different from the initial state to the state μ in question; i.e., for large t its time factor is significant only when $E_\kappa = E_\mu$.

$H_{i\mu}^{\alpha'}$ and $K_{\kappa\mu}^{\alpha'}$ shall not include time t.

Proof: The claim is already proved in first order by (4.3), and one has, in fact:

$$\boxed{H_{i\mu}^1 = V_{i\mu}^1} \quad K_{\lambda\mu}^1 = 0. \tag{4.7}$$

We now assume that the claim has been proved for all solutions up to ultimately the $(\alpha - 1)^{\text{th}}$, with:

$$(\alpha' = 1, 2, \ldots (\alpha - 1);\ \alpha - 1 < \beta),$$

and show that it is also true for the α^{th} ($\alpha' = \alpha \leq \beta$). The α^{th} approximation will be (when we regard the initial state i as being realized and consider the temporary final state to be μ'):

$$i\hbar \frac{\partial}{\partial t} a_{\mu'}^{\alpha}(t) = \sum_{\mu} e^{i(E_{\mu'} - E_\mu)t/\hbar} \cdot \left[a_{i\mu}^{\alpha-1} V_{\mu\mu'}^1 + \cdots + a_{i\mu}^1 V_{\mu\mu'}^{\alpha-1} + a_{i\mu}^0 V_{\mu\mu'}^{\alpha} \right] \tag{4.8}$$

and, with (4.6):

$$= \sum_{\mu} e^{i(E_{\mu'} - E_i)t/\hbar} \cdot \left[\sum_{\mu} \frac{H_{i\mu}^{\alpha-1} V_{\mu\mu'}^1}{E_i - E_\mu} + \cdots + \sum_{\mu} \frac{H_{i\mu}^1 V_{\mu\mu'}^1}{E_i - E_\mu} + V_{\mu\mu'}^{\alpha} \right] + \sum_{\kappa' \neq i} e^{i(E_{\mu'} - E_{\kappa'})t/\hbar} \cdot K_{\kappa'\mu}^{\alpha}. \tag{4.8′}$$

In this expression, the first part encompasses all terms whose time factor has the energy difference $E_i - E_{\mu'}$ between the initial state and temporary final state μ' in its exponent (and which comes about by substituting the first part of the first summands (4.6) in (4.8)). The second part encompasses all terms that do not have this property, and which can be written out more explicitly as:

$$\left\{ \begin{aligned} \sum_{\kappa' \neq i} e^{i(E_{\mu'} - E_{\kappa'})t/\hbar} \cdot K_{\kappa\mu'}^{\alpha} = -\sum_{\mu} \quad & e^{i(E_{\mu'} - E_\mu)t/\hbar} \cdot \left[\frac{H_{i\mu}^{\alpha-1} V_{\mu\mu'}^1}{E_i - E_\mu} + \cdots + \frac{H_{i\mu}^1 V_{\mu\mu'}^{\alpha-1}}{E_i - E_\mu} \right] \\ & + \sum_{\substack{\mu \neq i \\ \kappa \neq i}} \left(e^{i(E_{\mu'} - E_\kappa)t/\hbar} - e^{i(E_{\mu'} - E_\mu)t/\hbar} \right) \cdot \left[\frac{K_{i\mu}^{\alpha-1} V_{\mu\mu'}^1}{E_\kappa - E_\mu} + \cdots + \frac{H_{\kappa\mu}^2 V_{\mu\mu'}^{\alpha-2}}{E_\kappa - E_\mu} \right]. \end{aligned} \right. \tag{4.9}$$

Time integration of (4.8′) yields, in the α^{th} approximation, an expression of the form (4.6) when one sets:

$$\boxed{H_{i\mu'}^{\alpha} = \sum_{\mu} \frac{H_{i\mu}^{\alpha-1} V_{\mu\mu'}^1}{E_i - E_\mu} + \cdots + \sum_{\mu} \frac{H_{i\mu}^1 V_{\mu\mu'}^{\alpha-1}}{E_i - E_\mu} + V_{i\mu'}^{\alpha}} \tag{4.10}$$

with which the claim is proved.

With the assumption (4.5) the first and second summands of (4.6) are small to all orders of approximation up to the $(\beta - 1)^{\text{th}}$ for large t:

$$\left\{\begin{matrix} \text{i.e.} & E_\mu \neq E_i & \text{when} & H^\alpha_{i\mu} \neq 0 \\ \text{and} & E_\mu \neq E_\kappa & \text{when} & K^\alpha_{\kappa\mu} \neq 0 \end{matrix}\right\} \quad \text{for } \alpha = 1, 2, \ldots, \beta - 1. \tag{4.6}$$

Furthermore, the second summand (4.6) of the β^{th} approximation is small for large t (i.e., $E_\mu \neq E_\kappa$ when $K^\beta_{\mu\kappa} \neq 0$). In the other case, as one sees from (4.9), one is already given a transition $\kappa \to \mu'$ in less than the β^{th} approximation, which contradicts the assumption. On the same grounds, all of the terms that appear in (4.10) are non-zero.

However, since there must exist a transition probability $i \to \mu'$ in the β^{th} approximation it must come from the first summands in (4.6). Hence, it must be determined by:

$$\left|a^\beta_{i\mu'}(t)\right|^2 = \left|H^\beta_{i\mu'}\right|^2 2\frac{1-\cos(E_{\mu'} - E_i)t/\hbar}{(E_i - E_{\mu'})^2},$$

from which, with the final Dirac expression (4.4) above, the transition probability follows:

$$\boxed{\frac{2\pi}{\hbar}\frac{1}{\Delta E}\cdot\left|H^\beta_{i\mu'}\right|^2} \tag{4.11}$$

(4.11), (4.10), and (4.7) include the *result* of perturbation theory: The transition probability in the smallest non-vanishing order (viz., the β^{th}) is the product (multiplied by $2\pi/\hbar$) of the number of states $1/\Delta E$ per energy interval in the final state with the square of a "matrix element."

The matrix element $H^\beta_{i\mu'}$ of β^{th} order (4.10), which leads from the initial state i to the final state μ', is composed of the matrix elements $V^{\beta' \leq \beta}_{ik}$ of the perturbation energy, e.g.:

$$H^4_{in} = \frac{V^1_{ik}V^1_{kl}V^1_{lm}V^1_{mn}}{(E_i - E_k)(E_i - E_l)(E_i - E_m)} + \cdots + V^4_{in} \quad [1]). \tag{4.12}$$

Thus, some parts of the perturbation energy (V^4) can lead directly from the initial state to the final state in first order, but, by contrast, other perturbations (V^1) can do this only through sub-processes:

$$(i \to k,\ k \to l,\ l \to m,\ m \to n)$$

---

[1]) The matrix (4.12) is symmetric in the initial and final state: ($H^*_{in} = H_{in}$), which follows from the symmetry of the perturbation $V_{ik} = V^*_{ki}$ and conservation of energy $E_i = E_k$.

involving intermediate states (k, l, m) of higher (viz., 4th) order. These sub-processes do not imply any energy conservation for the "virtual" intermediate states, but only the total process ($i \rightarrow n$) to the *actual* final state. Therefore, whether the states k, l, m cannot, by any means, be assumed to be actual system states insofar as the energy law is concerned (or else there would be transitions in lower order approximations that would contradict the assumption on the formula that was given here), their virtual possibility brings about the transition $i \rightarrow n$ in question. The scattering of light by light rests upon these circumstances.

§ 5. **Presentation of the matrix H_{in}^4 for the scattering of light by light in Dirac's theory**

We now apply this perturbation schema to the *system*: radiation field and matter field. For its approximate stationary states, we choose plane light waves and plane matter waves. The *perturbation energy* then exists as the coupling between light and matter, along with certain subtraction terms that Heisenberg [1]) would have us combine with the Hamilton function of ordinary hole theory in order to arrive at finite results that are compatible with the conservation laws.

In the following we denote by:

(5.1)

$\mathfrak{A}$	the vector potential of the radiation field,
$\psi(\xi, s), (s = 1, 2, 3, 4)$	the wavefunction of the matter field,
$\alpha = \alpha_{ss'}$	the Dirac operator,
ξ, dV	position and volume element in the radiation field,
V	the volume of a cube in which the field is assumed to be periodic,
$\mathfrak{g}, \mathfrak{e}$	impulse and polarization of a light quantum,
$\mathfrak{p}, \sigma \rightarrow \pm 1, \lambda \rightarrow \pm 1$	impulse, spin, and sign of the energy of an electron,
$d\mathfrak{p}$	the impulse space element,
$M_{\mathfrak{g}\mathfrak{e}}, B_{\mathfrak{g}\mathfrak{e}}$	occupation number and amplitude of the plane light wave $g, \mathfrak{e}$,
$N_{\mathfrak{p}\sigma\lambda}, A_{\mathfrak{p}\sigma\lambda}$	occupation number and amplitude for the plane matter wave $\mathfrak{p}, \lambda, \sigma$,
$c, h = 2\pi\hbar$	velocity of light and Planck quantum of action,
e, m	electron charge and mass.

The decomposition of the field into plane waves is:

[1]) W. Heisenberg, Zeit. f. Phys. **90**, pp. 209, 1934.

$$(5.2)\qquad \begin{cases} \mathfrak{A} = \sum_{g\mathfrak{e}} \mathfrak{A}_g (B_{g\mathfrak{e}} + B^*_{-g\mathfrak{e}}), & \mathfrak{A}_g = \mathfrak{e}\sqrt{\dfrac{ch\hbar}{|g|V}} e^{i(\mathfrak{g}\xi)/\hbar} \\ \psi = \sum_{\mathfrak{p}\lambda\sigma} \dfrac{e^{i(\mathfrak{g}\xi)/\hbar}}{\sqrt{V}} \cdot a_{\mathfrak{p}\lambda\sigma}(s) \cdot A_{\mathfrak{p}\lambda\sigma}, \quad (\mathfrak{e}\mathfrak{e}) = 1, & (\mathfrak{e}\mathfrak{g}) = 0, \\ & \sum_{s=1}^{4} \left| a_{\mathfrak{p}\lambda\sigma}(s) \right|^2 = 1. \end{cases}$$

The potentials and densities are operators that are characterized by the matrix properties of their Fourier amplitudes:

B^*_g (B_g, resp.) refers to the creation (annihilation, resp.) of a light quantum,

$A^*_{\mathfrak{p}}$ ($A_{\mathfrak{p}}$, resp.) refers to the creation (annihilation, resp.) of an electron;

i.e., the matrix element of B^*_g is non-zero for a transition in which a light quantum is created:

$$(5.3)\qquad \begin{cases} (\cdots M_g \cdots | B^*_g | \cdots M_g + 1 \cdots) = \sqrt{M_g + 1}, \\ (\cdots M_g \cdots | B_g | \cdots M_g - 1 \cdots) = \sqrt{M_g}, \\ (\cdots N_{\mathfrak{p}} \cdots | A^*_{\mathfrak{p}} | \cdots N_{\mathfrak{p}} + 1 \cdots) = \sqrt{1 - N_{\mathfrak{p}}} \cdot J, \\ (\cdots N_{\mathfrak{p}} \cdots | A_{\mathfrak{p}} | \cdots N_{\mathfrak{p}} - 1 \cdots) = \sqrt{N_{\mathfrak{p}}} \cdot J, \end{cases}$$

with the Jordan-Wigner sign function:

$$J = (-1)^{\sum_{\mathfrak{p}\lambda} N_{\mathfrak{p}'\lambda'}}.$$

A process of light scattering from light will be described by:

$$(5.4)\qquad \begin{cases} \mathfrak{g}^1, \mathfrak{g}^2, \quad g^1 = |\mathfrak{g}^1|, \quad g^2 = |\mathfrak{g}^2|, & \text{the impulse and energy of the primary, absorbed light quanta,} \\ -\mathfrak{g}^3, -\mathfrak{g}^4, -g^3 = |\mathfrak{g}^3|, -g^4 = |\mathfrak{g}^4|, & \text{the impulse and energy of the secondary, emitted light quanta,} \\ \mathfrak{e}^1, \mathfrak{e}^2, \mathfrak{e}^3, \mathfrak{e}^4, \quad (|\mathfrak{e}^1| = 1; \mathfrak{e}^1 \ll \mathfrak{g}^1 \cdots) & \text{the associated polarizations,} \\ (g^1, g^2 | O | -g^3, -g^4) & \text{the matrix element of an operator } O \text{ for the scattering of light by light, i.e., for the transition} \\ & \text{between two light quanta } g^1, g^2 \text{ into two others } -g^3, -g^4 \text{ (instead of the detailed notation:} \\ & [\cdots N_{g^1}, N_{g^2} \cdots N_{-g^3}, N_{-g^4} \cdots | O | \cdots N_{g^1} - 1, N_{g^2} - 1, \\ & \cdots N_{-g^3} + 1, N_{-g^4} + 1 \cdots]). \end{cases}$$

The perturbation energy of the field that is approximated by the plane waves includes, when developed in powers of the electron charge e:

The coupling of light and matter, which is determined by the current and potential:

$$V^1 = e \int \psi^*(\alpha\mathfrak{A})\ \psi\, dV \tag{5.5}$$

and the subtraction terms [1]):

$$\begin{cases} V^2 = e^2 \int dV \text{(function of second order in the field strengths)}, \\ V^3 = e^3 \int dV \text{(function of third order in the field strengths)}, \\ V^4 = e^4 \left(-\dfrac{1}{12\pi^2}\right)\left(\dfrac{1}{\hbar c}\right)^3 \lim\limits_{\mathfrak{r}\to 0} \displaystyle\int \dfrac{(\mathfrak{A}(\xi)\mathfrak{r})^4}{|\mathfrak{r}|^4} dV. \end{cases} \tag{5.6}$$

V^1 is of first order in the potentials and second order in the matter waves, and thus, from (5.3), gives rise to transitions in which a light quantum g is created (or annihilated) and an electron $\mathfrak{p}$ jumps from one state into another one $\mathfrak{p}'$. The matrix element of V_1 for this transition is (5.2, 5.3):

$$V_{ik} = e \int \mathfrak{A}_{\mp\mathfrak{g}} \begin{pmatrix} \overset{\lambda\sigma}{\mathfrak{p}} \,|\, \boldsymbol{\alpha} \,|\, \overset{\lambda'\sigma'}{\mathfrak{p}'} \end{pmatrix} \frac{e^{i(\mathfrak{p}-\mathfrak{p}')\xi/\hbar}}{V} dV\,, \tag{5.7}$$

$$= e\sqrt{\frac{ch\hbar}{|g|V}} \begin{pmatrix} \overset{\lambda\sigma}{\mathfrak{p}} \,|\, \alpha\mathfrak{e} \,|\, \overset{\lambda'\sigma'}{\mathfrak{p}'} \end{pmatrix}, \tag{5.8}$$

in the case where the impulse is conserved under the transition ($\mathfrak{p} - \mathfrak{p}' = \pm\,\mathfrak{g}$) and is 0 in the case where it is not.

In the latter expression, $\begin{pmatrix} \overset{\lambda\sigma}{\mathfrak{p}} \,|\, \alpha\mathfrak{e} \,|\, \overset{\lambda'\sigma'}{\mathfrak{p}'} \end{pmatrix}$ refers to the matrix element of the four-rowed Dirac matrix ($\alpha\,\mathfrak{e}$) for the pair of electron states $\overset{\lambda\sigma}{\mathfrak{p}}$ and $\overset{\lambda'\sigma'}{\mathfrak{p}'}$. Here and in the sequel, for the sake of brevity the factors $\sqrt{M_g+1}$ and $\sqrt{M_g}$ are omitted.

V^2 (V^3, resp.) is of second (third, resp.) order in the field strengths and therefore leads to matrix elements that combine two (three, resp.) light quanta with an impulse sum of zero.

Finally, V^4 includes the potential to fourth order and can therefore take two light quanta $+\,\mathfrak{g}^1 + \mathfrak{g}^2$ to two other ones $-\,\mathfrak{g}^3 - \mathfrak{g}^4$ with the same impulse sum. Its matrix element for this transition is (5.2), 5.3):

[1]) W. Heisenberg and, Zeit. f. Phys. **90**, pp. 209, 1934, formula 59, 60, 61; Zeit. Phys. **92**, pp. 692, 1934.

(5.9) $$(g^1\, g^2 \mid V^4 \mid -g^3 - g^4) = \lim_{\mathfrak{r}\to 0} \frac{64\pi}{3} C \sum_{\text{perm}} \frac{(\mathfrak{e}^1\mathfrak{r})(\mathfrak{e}^2\mathfrak{r})(\mathfrak{e}^3\mathfrak{r})(\mathfrak{e}^4\mathfrak{r})}{|\mathfrak{r}|^4} = V_{in}^4 .$$

In this, $\sum_{\text{perm}}$ refers to the sum over all 24 permutations of the indices 1, 2, 3, 4 in the vectors $\mathfrak{e}^1$, $\mathfrak{e}^2$, $\mathfrak{e}^3$, $\mathfrak{e}^4$, and one has:

(5.10) $$C = -\frac{1}{32\cdot(2\pi)^3}\left(\frac{e^2}{\hbar c}\right)^2 \frac{1}{\hbar c}\left(\frac{ch\hbar}{V}\right)^2 \frac{V}{\sqrt{g^1 g^2 g^3 g^4}} .$$

For the scattering of light by light, V_1 can therefore matter only in fourth order, while V_4 can matter only in first order. The other terms V_2 and V_3 give no contribution. (Whereas V_3 must be combined with V_1, which would amount to a source electron, V_2, however, must be combined with two V_1 terms or itself, which would place special conditions on the incoming light quanta.)

In other words: The scattering of light by light is a *4th order* process in Dirac's theory. Its matrix element will be composed of the ordinary perturbation V_1 in the fourth order and the Heisenberg subtraction term V^4 in the first order: (4.12)

(5.11) $$H_{in}^4 = \sum_{klm} \frac{V_{ik}^1 V_{kl}^1 V_{lm}^1 V_{mn}^1}{(E_i - E_k)(E_i - E_l)(E_i - E_m)} + V_{in}^4 .$$

Therefore, one has conservation of impulse for the sub-process ($i \to k$, $k \to l$, $l \to m$, $m \to n$) and therefore for the entire process, but there is conservation of energy only for the total process ($i \to n$):

(5.11′) $$\begin{cases} \mathfrak{g}^1 + \mathfrak{g}^2 + \mathfrak{g}^3 + \mathfrak{g}^4 = 0 \\ g^1 + g^2 + g^3 + g^4 = 0. \end{cases}$$

The sub-processes that the coupling V_1 implies for the scattering of light by light are then:

Light quanta / Electrons	Absorption of g^1	Absorption of g^2	Emission of $-g^3$	Emission of $-g^4$
$\mu = 1$	pair creation	source electron	source electron	pair annihilation
$\mu = 2$	pair creation	source electron	source positron	pair annihilation
$\mu = 3$	pair creation	source positron	source electron	pair annihilation
$\mu = 4$	pair creation	source positron	source positron	pair annihilation
$\mu = 5$	pair creation	pair creation	radiation of the electron from pair 1 with the positron from pair 2	radiation of the positron from pair 1 with the electron from pair 2
$\mu = 6$	pair creation	pair creation	radiation of the positron from pair 1 with the electron from pair 2	radiation of the electron from pair 1 with the positron from pair 2

In this, the first pair creation, instead of by the absorption of g^1, as assumed in the figure (and as represented in the following calculations), there can also be an emission of $-g^3$ from it, etc. I.e., the four headings of the columns in the table in the upper row (and which will be given the light quanta indices 1, 2, 3, 4 in all of the following formulas) can be permuted arbitrarily while preserving all of the remaining table (and formula) rows.

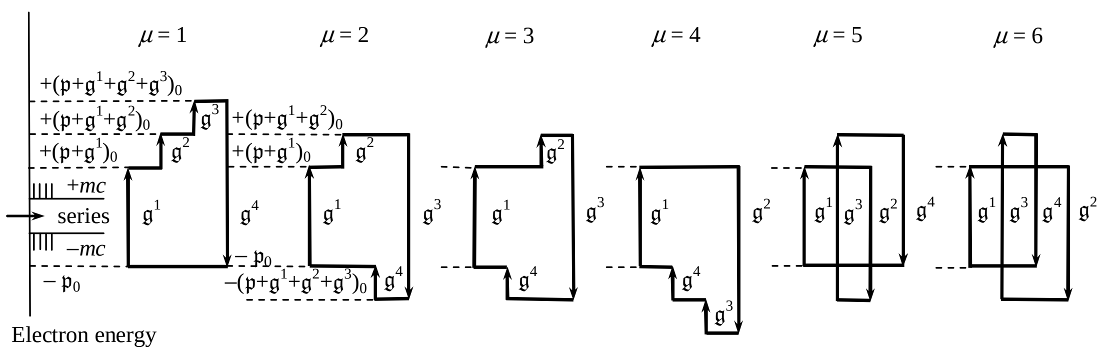

Fig. 1. The six possible transitions

Indeed, from the behavior of the created pair there are six different possible transitions, which are denoted by $\mu = 1$ to 6, and each of these six transitions can be combined with all 24 permutations of the light quanta.

In the following, we denote (cf., figure):

(5.12)

$\mathfrak{p}^4 = \mathfrak{p}$	the (negative) impulse of the first positrons created,
$p = \lvert \mathfrak{p} \rvert$	its magnitude,
$\left.\begin{aligned} \mathfrak{p}^1 &= \mathfrak{p} + \mathfrak{g}^1 \\ \mathfrak{p}^2 &= \mathfrak{p} + \mathfrak{g}^1 + \mathfrak{g}^2 \\ \mathfrak{p}^3 &= \mathfrak{p} + \mathfrak{g}^1 + \mathfrak{g}^2 + \mathfrak{g}^3 = \mathfrak{p} - \mathfrak{g}^4 \end{aligned}\right\}$	the impulse of the electrons in the intermediate states,
$\left.\begin{aligned} &p_0^i = +\sqrt{(mc)^2 + (\mathfrak{p}^i \mathfrak{p}^i)}, \quad (i = 1 \cdots 4) \\ &\text{e.g.}: p_0^i = (\mathfrak{p} + \mathfrak{g}^1)_0, \cdots \end{aligned}\right\}$	the corresponding energy magnitude (divided by c),
$\lambda_\mu^1, \lambda_\mu^2, \lambda_\mu^3, \lambda_\mu^4$	the signs of these electron energies for the μ^{th} transition,
$Z_\mu = V_{ik}^1 V_{kl}^1 V_{lm}^1 V_{mn}^1$	(up to a factor) the product of the matrix elements for the coupling V^1 in the *numerator* of (5.11) for the μ^{th} of the six transition types,
$N_\mu = -\dfrac{1}{8c^3}(E_i - E_k)(E_i - E_l)(E_i - E_m)$	the product of the energy differences in the *denominator* of the first term of (5.11) for the μ^{th} of the six transition types.

In the fourth order matrix element, the following summations are to be carried out:

$\sum_{\mu=1}^{6}$ over the six transition types,

$\sum_{\mathfrak{p}} = \dfrac{V}{h^3}\int d\mathfrak{p}$ over all possibilities for the first pair creation,

$\sum_{\sigma}$ over the spins of the electrons in the initial and intermediate states,

$\sum_{\text{perm}}$ over the 24 series of light quanta, i.e., the 24 permutations of the four indices i in $\mathfrak{g}^i$, g^i, and $\mathfrak{e}^i$.

(By exchanging the emission of light quantum $-g^3$ with the absorption of the light quantum g^1 the impulse of the electrons $\mathfrak{p}^1 = \mathfrak{p} + \mathfrak{g}^1$ will be exchanged with $\mathfrak{p}^{1'} = \mathfrak{p} + \mathfrak{g}^3$, while the energy of the intermediate state $E_k / c =$ const. $+ g^1$ will be exchanged with $E_k' / c =$ const. $+ g^3$. Thus, exchanging absorption and emission energies, as well as impulses of the light quanta, in formulas (5.11), (5.12) will change the sign, which, under our notation (5.4), will be altered by permutations of the indices i in $\mathfrak{g}^i$.)

The matrix element can thus be written:

(5.13)
$$\boxed{H_{in}^{4} = C \int d\mathfrak{p} \sum_{\text{perm}} \sum_{\mu=1}^{6} \frac{Z_\mu}{N_\mu} + V_{in}^{1}}$$

In this, the denominators are:

$$\boxed{\begin{array}{llll}
8N_1 = (p_0^4 + p_0^1 - g^1) & (p_0^4 + p_0^2 - g^1 - g^2) & (p_0^4 + p_0^3 + g^4) \\
8N_2 = (p_0^4 + p_0^1 - g^1) & (p_0^4 + p_0^2 - g^1 - g^2) & (p_0^2 + p_0^3 + g^3) \\
8N_3 = (p_0^4 + p_0^1 - g^1) & (p_0^1 + p_0^3 - g^1 - g^4) & (p_0^2 + p_0^3 + g^3) \\
8N_4 = (p_0^4 + p_0^1 - g^1) & (p_0^1 + p_0^3 - g^1 - g^4) & (p_0^1 + p_0^2 + g^2) \\
\multicolumn{3}{l}{8N_5 = (p_0^4 + p_0^1 - g^1)(p_0^4 + p_0^1 + p_0^2 + p_0^3 - g^1 - g^3)(p_0^1 + p_0^2 + g^2)} \\
\multicolumn{3}{l}{8N_6 = (p_0^4 + p_0^1 - g^1)(p_0^4 + p_0^1 + p_0^2 + p_0^3 - g^1 - g^3)(p_0^4 + p_0^3 + g^4)}
\end{array}}$$

In the *numerators* Z_μ, one must sum over the spins in the intermediate states (1, 2, 3) and the final (= end) state (4).

The spin summation in the state of an electron with impulse $\mathfrak{p}'$ and energy sign λ' can be carried out with the help of the operator:

$$\frac{1}{2}\left(1 - \lambda' \frac{(\alpha\,\mathfrak{p}') + \beta mc}{p_0'}\right),$$

which yields 1 when applied to the state with impulse $\mathfrak{p}'$ and energy λ_0', and 0 when applied to the state with impulse $\mathfrak{p}'$ and energy – $\lambda' p_0$.

The signs of the energies λ', λ^2, λ^3, λ^4 of the intermediate electron states and the products $\mathfrak{p}'$ of the Jordan-Wigner sign function for the six possible transitions are:

sign / case	λ_μ^1	λ_μ^2	λ_μ^3	λ_μ^4	J_μ'
$\mu=1$	+	+	+	−	+
$\mu=2$	+	+	−	−	−
$\mu=3$	+	+	−	−	−
$\mu=4$	+	−	−	−	+
$\mu=5$	+	−	+	−	−
$\mu=6$	+	−	+	−	−

from which, one sees that $J_\mu' = \lambda_\mu^2 \lambda_\mu^3$.

Therefore, the numerators Z_μ will be:

$$
\boxed{
\begin{aligned}
Z_\mu = \frac{\mathrm{Spur}}{4}\lambda_\mu^2\lambda_\mu^3(\alpha\mathfrak{e}^1)&\left(1-\frac{(\alpha\mathfrak{p}^1)+\beta mc}{p_0^1}\right)\\
(\alpha\mathfrak{e}^2)&\left(1-\lambda_\mu^2\frac{(\alpha\mathfrak{p}^2)+\beta mc}{p_0^2}\right)\\
(\alpha\mathfrak{e}^3)&\left(1-\lambda_\mu^3\frac{(\alpha\mathfrak{p}^3)+\beta mc}{p_0^3}\right)\\
(\alpha\mathfrak{e}^4)&\left(1+\frac{(\alpha\mathfrak{p}^4)+\beta mc}{p_0^4}\right)\\
\text{with:}\quad
&\begin{array}{r|cccccc}
\mu = & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\lambda_\mu^2 = & + & + & + & - & - & - \\
\lambda_\mu^3 = & + & - & - & - & + & +
\end{array}
\end{aligned}
}
\tag{5.15}
$$

(one has $Z_2 = Z_3$ and $Z_5 = Z_6$).

§ 6. **Development of the matrix H_{in}^4 to order zero in light frequency and comparison with Heisenberg's subtraction term**

In (5.13, 5.9, 5.10, 5.14, 5.15) in the previous section the matrix element H_{in}^4 was examined in Dirac's theory for the scattering of light by light. It is a function of the four light quanta that take part in the scattering and, since the light is assumed to be soft, (0.1) shall be *developed* in terms of the impulses $\mathfrak{g}^1/mc$, $\mathfrak{g}^2/mc$, $-\mathfrak{g}^3/mc$, $-\mathfrak{g}^4/mc$ and energies g^1/mc, g^2/mc, $-g^3/mc$, $-g^4/mc$, of the light quanta.

Along with the development in the electron charge e, for whose fourth-order term we have already distinguished between the types of perturbation calculations, we now also carry out a development in the light frequencies:

The zero-order term in the development of the matrix element H_{in}^4 in light frequencies g^i/mc vanishes, because the part of it that is constructed from the ordinary coupling V^1 will be cancelled by the Heisenberg subtraction term V_{in}^4, which we show as follows:

In order zero, the denominator (5.14) is:

$$N_1 = N_2 = N_3 = N_4 = p_0^3, \qquad N_5 = N_6 = 2\,p_0^3. \tag{6.1}$$

The numerator (5.15), when divided by this, namely:

$$(6.2)\qquad \begin{cases} \sum_{\text{perm}}\sum_{\mu}\frac{Z_\mu}{N_\mu}=\sum_{\text{perm}}\sum_{\mu}\frac{1}{N_\mu}\frac{\text{spur}}{4}\lambda_\mu^2\lambda_\mu^3 \\ \cdot(\alpha\mathfrak{e}^1)\left(1-\frac{(\alpha\mathfrak{p})+\beta mc}{p_0}\right)(\alpha\mathfrak{e}^2)\left(1-\lambda^2\,\frac{(\alpha\mathfrak{p})+\beta mc}{p_0}\right) \\ \cdot(\alpha\mathfrak{e}^3)\left(1-\lambda^3\,\frac{(\alpha\mathfrak{p})+\beta mc}{p_0}\right)(\alpha\mathfrak{e}^4)\left(1-\lambda^4\,\frac{(\alpha\mathfrak{p})+\beta mc}{p_0}\right) \end{cases}$$

will be, after taking the spur and summing over all six cases of μ:

$$(6.3)\qquad = -\sum_{\text{perm}}\frac{S}{p_0^3}\left[(\mathfrak{e}^1\mathfrak{e}^2)(\mathfrak{e}^3\mathfrak{e}^4)-6\frac{(\mathfrak{p}\mathfrak{e}^1)(\mathfrak{p}\mathfrak{e}^2)(\mathfrak{e}^3\mathfrak{e}^4)}{p_0^2}+5\frac{(\mathfrak{p}\mathfrak{e}^1)(\mathfrak{p}\mathfrak{e}^2)(\mathfrak{p}\mathfrak{e}^3)(\mathfrak{p}\mathfrak{e}^4)}{p_0^4}\right].$$

In order to carry out the comparison of this ordinary term with the subtraction term (5.9), one converts it, with Heisenberg, into a total derivative:

$$(6.4)\qquad \begin{cases} =\frac{S}{3}\sum_{\text{perm}}\left(\mathfrak{e}^1\frac{\partial}{\partial\mathfrak{p}}\right)\left(\mathfrak{e}^2\frac{\partial}{\partial\mathfrak{p}}\right)\left(\mathfrak{e}^3\frac{\partial}{\partial\mathfrak{p}}\right)\left(\mathfrak{e}^4\frac{\partial}{\partial\mathfrak{p}}\right)p_{0,} \\ \left(\left(\mathfrak{e}^1\frac{\partial}{\partial\mathfrak{p}}\right)=\mathfrak{e}_x^1\frac{\partial}{\partial\mathfrak{p}_x}+\mathfrak{e}_y^1\frac{\partial}{\partial\mathfrak{p}_y}+\mathfrak{e}_z^1\frac{\partial}{\partial\mathfrak{p}_z}\right), \end{cases}$$

and one goes from the matrix element (6.4) to the corresponding (in $\xi\pm\mathfrak{r}$) mixed energy density:

$$\begin{aligned} &C\int d\mathfrak{p}\sum_{\text{perm}}\sum_{\mu}\frac{Z_\mu}{N_\mu}\to C\int d\mathfrak{p}\,e^{i(\mathfrak{p}\mathfrak{r})/\hbar}\sum_{\text{perm}}\sum_{\mu}\frac{Z_\mu}{N_\mu} \\ &=\frac{8}{3}C\sum_{\text{perm}}\int d\mathfrak{p}\,e^{i(\mathfrak{p}\mathfrak{r})/\hbar}\left(\mathfrak{e}^1\frac{\partial}{\partial\mathfrak{p}}\right)\left(\mathfrak{e}^2\frac{\partial}{\partial\mathfrak{p}}\right)\left(\mathfrak{e}^3\frac{\partial}{\partial\mathfrak{p}}\right)\left(\mathfrak{e}^4\frac{\partial}{\partial\mathfrak{p}}\right)p_0 \\ &=-\frac{64\pi}{3}C\sum_{\text{perm}}\frac{(\mathfrak{e}^1\mathfrak{r})(\mathfrak{e}^2\mathfrak{r})(\mathfrak{e}^3\mathfrak{r})(\mathfrak{e}^4\mathfrak{r})}{|\mathfrak{r}|^4}=-\,(g^1g^2\,|V^4\,|-g^3-g^4). \end{aligned}$$

$\left[\int d\mathfrak{p}\;p_0\,e^{i(\mathfrak{p}\mathfrak{r})/\hbar}=-8\pi\hbar^4/|\mathfrak{r}|^4\right.$ if one artificially makes the integral convergent and calculates for $|\mathfrak{r}|\ll\hbar/mc$.$\left.\right]$ The claim is thus proved: In order zero the development in light frequencies the subtraction term V^4 gives, as a consequence, no scattering of light by light.

If the Heisenberg term (5.9) were not added to the matrix element then for a sufficiently large wavelength Dirac's theory would give arbitrarily large scattering of light by light, in contradiction to the result (5.9, 5.10).

§ 7. Proof of the identity of the matrix H_{in}^4 that follows from Dirac's theory with the aforementioned interaction energy $\bar{U}_1$ of the light quanta

It can be easily shown that the matrix H_{in}^4 that was just derived in Dirac's theory (5.13), whose transition elements describe the scattering of light by light, is identical with the previously described interaction of light quanta (2.21).

a) The matrix element H_{in}^4 (5.13, 5.14, 5.15) of Dirac's theory can be described as a simple integral $\int dV$ over a product of four plane light waves (5.2):

$$\underset{g^1}{\mathfrak{A}}\,\underset{g^2}{\mathfrak{A}}\,\underset{g^3}{\mathfrak{A}}\,\underset{g^4}{\mathfrak{A}} = \mathfrak{e}^4 e^{i(\mathfrak{g}^4\xi)/\hbar} \cdot \sqrt{\frac{ch\hbar}{|g^4|V}} .$$

Then, due to the conservation of impulse $\mathfrak{g}^1 + \mathfrak{g}^2 + \mathfrak{g}^3 + \mathfrak{g}^4 = 0$ one has:

$$(7.1)\qquad \left\{\begin{aligned} H_{in}^4 &= \frac{e^4}{16c^3h^3}\sum_{\text{perm}}\sum_{\mu}\text{spur}\int d\mathfrak{p}\int dV \\ &\cdot(\alpha\underset{g^1}{\mathfrak{A}})\left(\frac{1-\dfrac{(\alpha,\mathfrak{p}+\mathfrak{g}^1)+\beta mc}{(\mathfrak{p}+\mathfrak{g}^1)_0}}{p_0+(\mathfrak{p}+\mathfrak{g}^1)_0-g^1}\right)(\alpha\underset{g^2}{\mathfrak{A}})\left(\frac{1-\dfrac{(\alpha,\mathfrak{p}+\mathfrak{g}^1+\mathfrak{g}^2)+\beta mc}{(\mathfrak{p}+\mathfrak{g}^1+\mathfrak{g}^2)_0}}{p_0+(\mathfrak{p}+\mathfrak{g}^1+\mathfrak{g}^2)_0-g^1-g^2}\right) \\ &\cdot(\alpha\underset{g^3}{\mathfrak{A}})\left(\frac{1-\dfrac{(\alpha,\mathfrak{p}-\mathfrak{g}^4)+\beta mc}{(\mathfrak{p}-\mathfrak{g}^4)_0}}{p_0+(\mathfrak{p}-t)_0+g^4}\right)(\alpha\underset{g^4}{\mathfrak{A}})\left(1-\frac{(\alpha\mathfrak{p})+\beta mc}{p_0}\right)+V_{in}^4. \end{aligned}\right.$$

b) This total matrix H_{in}^4 can be regarded as a simple matrix element that is a function $\bar{U}$ of the radiation field that is the integral of an expression in the potentials and their derivatives (multiplied by $\hbar/mc$). The *order of the derivative* corresponds to the *order of the development in the light quantum energies:*

If one then carried out the first part of (7.1), viz., taking the spur, integrating over $\mathfrak{p}$, and summing over μ, and develops in the light quantum energies g^i/mc then what remains is an expression of the form:

$$
(7.2)\qquad \left\{
\begin{aligned}
H^4_{in} &= \frac{e^4}{16c^3h^3}\sum_{\text{perm}}\int dV\left[\mathfrak{A}_{g^1}\mathfrak{A}_{g^2}\mathfrak{A}_{g^3}\mathfrak{A}_{g^4} + \frac{\mathfrak{g}^1}{mc}\mathfrak{A}_{g^1}\mathfrak{A}_{g^2}\mathfrak{A}_{g^3}\mathfrak{A}_{g^4} + \cdots + \frac{\mathfrak{g}^1}{mc}\frac{\mathfrak{g}^2}{mc}\frac{\mathfrak{g}^3}{mc}\frac{\mathfrak{g}^4}{mc}\mathfrak{A}_{g^1}\mathfrak{A}_{g^2}\mathfrak{A}_{g^3}\mathfrak{A}_{g^4} + \cdots\right] + V^4_{in} \\
&= \frac{e^4}{16c^3h^3}\sum_{\text{perm}}\int dV\Bigg[\mathfrak{A}_{g^1}\mathfrak{A}_{g^2}\mathfrak{A}_{g^3}\mathfrak{A}_{g^4} + \left(\frac{\hbar}{mc}\frac{\partial}{\partial x}\mathfrak{A}_{g^1}\right)\mathfrak{A}_{g^2}\mathfrak{A}_{g^3}\mathfrak{A}_{g^4} + \cdots \\
&\qquad \left(\frac{\hbar}{mc}\right)^4\left(\frac{\partial}{\partial x}\mathfrak{A}_{g^1}\right)\left(\frac{1}{c}\frac{\partial}{\partial t}\mathfrak{A}_{g^2}\right)\left(\frac{\partial}{\partial x}\mathfrak{A}_{g^3}\right)\left(\frac{1}{c}\frac{\partial}{\partial t}\mathfrak{A}_{g^4}\right) + \cdots\Bigg] + V^4_{in} \\
&= (g^1 g^2 \,|\int U_1\, dV\,| -g^3 - g^4) \\
&\text{with:} \\
U_1 &= \frac{e^4}{16c^3h^3}\left[\mathfrak{A}\mathfrak{A}\mathfrak{A}\mathfrak{A} + \frac{\hbar}{mc}\frac{\partial\mathfrak{A}}{\partial x}\mathfrak{A}\mathfrak{A}\mathfrak{A} + \cdots + \left(\frac{\hbar}{mc}\right)^4\left(\frac{\partial\mathfrak{A}}{\partial x}\right)\left(\frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t}\right)\left(\frac{\partial\mathfrak{A}}{\partial x}\right)\left(\frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t}\right) + \cdots\right].
\end{aligned}
\right.
$$

In this, the Heisenberg term is included as:

$$
V^4 = -\frac{1}{12\pi^2}\cdot\left(\frac{e^2}{\hbar c}\right)^2\frac{1}{\hbar c}\lim_{\mathfrak{r}\to 0}\int\frac{(\mathfrak{A}(\xi)\cdot\mathfrak{r})^4}{|\,\mathfrak{r}\,|^4}dV \to -\frac{1}{12\pi^2}\cdot\frac{e^4}{\hbar^3c^3}\cdot\frac{1}{5}\int(\mathfrak{A}\mathfrak{A})(\mathfrak{A}\mathfrak{A})\,dV
$$

when one takes the limit $\mathfrak{r}\to 0$ of the mean-square over $\mathfrak{r}$.

c) The terms that include the potentials $\mathfrak{A}$ directly, and cannot be expressed in terms of the fields strengths F_{ik}, therefore (*inter alia*) the *development terms of order* 0, 1, 2, 3 *in the light frequencies*, must *vanish.*

Then, since the assumptions from which these calculations proceed are gauge invariant this must also be a result; i.e., the only combinations of the derivatives of the potentials that can appear must refer to field strengths or derivatives of the field strengths.

The vanishing of the null-order terms in the development of H^4_{in} in g^i/mc, which can come about by means of the Heisenberg subtraction term V^4_{in}, is therefore understandable in terms of the requirement of gauge invariance, which was indeed also the basis for the term V^4. The vanishing of the terms of order 1, 2, and 3 in the development in g^i/mc shall be confirmed later (§ 9) for the first order in general, and by direct computation in some special cases for order 2 and 3.

Thus, all that remains of our expression (7.2) is:

$$
\begin{aligned}
U_1 = \frac{e^4}{16c^3h^3}\left(\frac{\hbar}{mc}\right)^4\Bigg[&\left(\frac{\partial\mathfrak{A}}{\partial x}\right)\left(\frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t}\right)\left(\frac{\partial\mathfrak{A}}{\partial x}\right)\left(\frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t}\right) \\
&+\left(\frac{\hbar}{mc}\frac{\partial}{\partial x}\frac{1}{c}\frac{\partial\mathfrak{A}}{\partial x}\right)\left(\frac{\hbar}{mc}\frac{\partial}{\partial x}\frac{\partial\mathfrak{A}}{\partial t}\right)\left(\frac{\partial\mathfrak{A}}{\partial x}\right)\left(\frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t}\right) + \cdots\Bigg]
\end{aligned}
$$

or:

$$U_1 = \frac{\hbar c}{c^2}\frac{1}{E_0^2}\frac{1}{16\cdot(2\pi)^3}\left[FFFF + \frac{\hbar}{mc}\frac{\partial F}{\partial x}\frac{\hbar}{mc}\frac{\partial F}{\partial x}FF + \cdots\right], \tag{7.3}$$

and one has:

$$H_{in}^4 = \left(g^1 g^2 \left|\int U_1 dV\right| - g^3 - g^4\right), \tag{7.4}$$

i.e., the interaction of the light quanta, in fact, can be described by an intuitive law and this is true for soft light of the form (1.4) that was expected above, thus, on the grounds of invariance, of the form (2.21).

We therefore do without terms of order higher than four in the development of in light quantum energies, hence, to the terms with the derivatives of the field strengths, and furthermore, as is already suggested by the type of perturbative calculations, to terms of order higher than four in the development of the field equations in the electron charge – i.e., to powers higher than four in the field strengths. That is, we restrict ourselves to fields that are not too strong and not too quickly varying, in which no pairs can be created and for which, moreover, it is assumed that they include no electrons that take part in the radiation.

$$\left\{\begin{array}{l}\text{Field strengths small compared to the field strength at the boundary of the}\\ \text{electron:} \qquad\qquad |F_{ik}| \ll E_0,\\ \text{Wavelengths large compared to the Compton wavelength; i.e., the invariant}\\ \text{conditions (0.1, 5.2) take the form:}\\ \left(\dfrac{\partial F_{ik}}{\partial x}\right)^2 + \left(\dfrac{\partial F_{ik}}{\partial y}\right)^2 + \left(\dfrac{\partial F_{ik}}{\partial z}\right)^2 - \dfrac{1}{c^2}\left(\dfrac{\partial F_{ik}}{\partial t}\right)^2 \ll 2\left(\dfrac{mc}{\hbar}\right)(F_{ik})^2.\end{array}\right. \tag{7.5}$$

Part III

§ 8. **Computation of the matrix element for the scattering of light by light (to order four in the development in light frequencies) for two special cases of the determination of the numerical coefficients α, β in the interaction of light with light**

$$\left(\alpha = -\frac{1}{360\pi^2}, \quad \beta = -\frac{7}{360\pi^2}\right)$$

Up to now, the interaction of the light quanta was determined up to two constants α, β (2.21). In order to calculate these two constants we now consider a special process in which two light quanta of equal energy and opposite impulse collide with each other and exchange impulses (or, which cannot be distinguished at this point, pass through each other undisturbed).

The polarizations of both primary quanta shall be equal to each other, as well as those of the secondary ones. In order to compute two constants we need two specializations: In the first one a) the polarizations of the primary light quanta are perpendicular to the polarizations of the secondary ones, and in the second case, b) all four light quanta have the same polarization.

With the notation:

$$\mathfrak{g}^1 = \mathfrak{g}, \qquad g^1 = g$$

one shall thus have:

$$\text{(8.1)} \qquad \begin{array}{c|c} \multicolumn{2}{c}{\mathrm{I}} \\ \mathfrak{g}^1 = +\mathfrak{g} & \mathfrak{g}^2 = -\mathfrak{g} \\ \mathfrak{g}^3 = +\mathfrak{g} & \mathfrak{g}^4 = -\mathfrak{g} \end{array}\ \mathrm{II} \qquad\qquad \begin{array}{c|c} \multicolumn{2}{c}{\mathrm{I}} \\ \mathfrak{g}^1 = +\mathfrak{g} & \mathfrak{g}^2 = +\mathfrak{g} \\ \mathfrak{g}^3 = -\mathfrak{g} & \mathfrak{g}^4 = -\mathfrak{g} \end{array}\ \mathrm{II},$$

and in a coordinate system along whose x axis the vector $\mathfrak{g}$ lies one shall have:

(8.2)

in the *special case* a):				in the *special case* b):			
$\parallel$ impulse and $\perp$ polarizations				$\parallel$ impulse and $\parallel$ polarizations			
	(x)	(y)	(z)		(x)	(y)	(z)
$\mathfrak{g} =$	$(g,$	$0,$	$0)$	$\mathfrak{g} =$	$(g,$	$0,$	$0)$
$\mathfrak{e}^1 = \mathfrak{e}^2 =$	$(0,$	$1,$	$0)$	$\mathfrak{e}^1 = \mathfrak{e}^2 =$	$(0,$	$1,$	$0)$
$\mathfrak{e}^3 = \mathfrak{e}^4 =$	$(0,$	$0,$	$1)$	$\mathfrak{e}^3 = \mathfrak{e}^4 =$	$(0,$	$1,$	$0)$

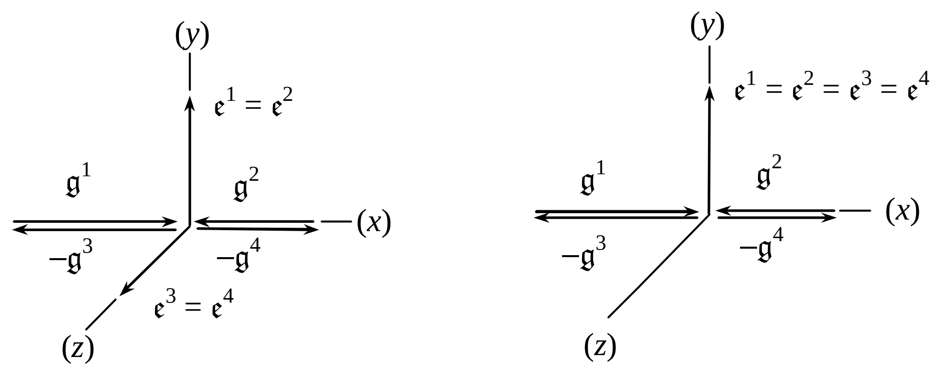

Fig. 2

We now calculate the Dirac matrix element H_{in}^4 [(5.13), (5.14), (5.15)] for these two special cases (in terms of fourth order in the development in light frequency g^1/mc) and the matrix element of the field function $\bar{U}_1$ (2.21), set both of them equal to each other:

$$H_{in}^4 = \frac{\hbar c}{e^2}\frac{1}{E_0^2}\left(g^1 g^2 \left| \int [\alpha(\mathfrak{D}^2 - \mathfrak{B}^2)^2 + \beta(\mathfrak{D}\mathfrak{B})^2]\, dV \right| -g^3 -g^4\right), \tag{8.3}$$

and thus obtain two linear equations for the determination of the constants α and β.

(Therefore, the definition of the special case in which the final state $-g^3$, $-g^4$ is equal to the initial state g^1, g^2, in which one therefore finds no actual radiation at all, is generally not worth discussing. It must, moreover, be specified that the final light quanta $-g^3$, $-g^4$ deviate from the initial light quanta only relatively little, since the development is in terms of these deviations and the terms of order zero in this development will be equated.)

We next compute the matrix element of the field function $\bar{U}_1$ (2.21) for both special cases (8.2a), (8.2b):

$$\begin{pmatrix} [\cdots] & \text{vector product} \\ (\cdots) & \text{scalar product} \\ |\cdots| & \text{determinant} \end{pmatrix}.$$

Therefore, (8.6), (8.7) are calculated to be the matrix elements of the two field functions (2.21) in the two special cases (8.2a) and (8.2b). As one sees, the transition a) (viz., parallel polarization) is determined by only the interaction term $(\mathfrak{D}^2 - \mathfrak{B}^2)^2$. The other term $(\mathfrak{D}\mathfrak{B})^2$ gives no contribution here, because $\mathfrak{D}\perp\mathfrak{B}$ for a plane wave. However, for the transition b) (viz., perpendicular polarization) both of the interaction terms $(\mathfrak{D}\mathfrak{B})^2$ and $(\mathfrak{D}^2 - \mathfrak{B}^2)^2$ contribute, because perpendicular, as well as parallel, polarizations can be combined in it.

We must now compute the matrix element H_{in}^4 [(5.13), (5.14), (5.15)] of Dirac's theory for the same two transitions (8.2a, b), and next treat the summation $\sum\limits_{\text{perm}}$ over the 24 sequences of light quanta that can take part in them.

A symmetry between the light quanta comes about through the specialization (8.2a, b) that allows us to easily sum over only some of the permutations:

As (8.1), (8.2) shows, the matrix element depends upon only the vector $\mathfrak{g}$, the number g, and the exchange:

I of $\mathfrak{g}^1$ with $\mathfrak{g}^2$ and $\mathfrak{g}^3$ with $\mathfrak{g}^4$ in the matrix element
refers to a change of signs for $\mathfrak{g}$........................ $\mathfrak{g} \to -\mathfrak{g}$,
but while keeping the same signs for g................ $g \to +g$,
and the same polarizations.................................. $p_y \to p_y, p_z \to p_z$.

The exchange:

II of $\mathfrak{g}^1$ with $\mathfrak{g}^2$ and $\mathfrak{g}^3$ with $\mathfrak{g}^4$ in the matrix element,
by contrast, refers to keeping the same signs for $\mathfrak{g}$ $\mathfrak{g} \to +\mathfrak{g}$,
but while changing the signs for g........................ $g \to -g$,
and switching the polarizations........................... $p_y \to p_z, p_z \to p_y$.

The four-group of exchanges that is generated by the permutations I, II (I, II, product I · II, identity) can thus be schematically applied to the formulas. The six remaining "classes" of all 24 permutations in these four, however, must be specially computed. They can be represented by the following six sequences of light quanta, which are associated with particular polarizations and intermediate electron states.

(8.4) By $\begin{matrix}(5.2) & (5.3)\\ (2.2) & (2.6)\end{matrix}$, the matrix element of the field function:	$\frac{1}{D}\left(g^1 g^2 \left\lvert \int (\mathfrak{B}\mathfrak{D})^2 dV \right\rvert -g^3-g^4\right)$	$\frac{1}{D}\left(g^1 g^2 \left\lvert \int (\mathfrak{B}^2-\mathfrak{D}^2)^2 dV \right\rvert -g^3-g^4\right)$
(8.5) is equal to: in which, D refers to the constant: $D=\left(\frac{ch\hbar}{V}\right)^2 \frac{1}{\hbar^4}\frac{V}{\sqrt{g^1g^2g^3g^4}}$	$=\sum_{\text{perm}} \lvert \mathfrak{e}^1\mathfrak{e}^2\mathfrak{g}^1\rvert \lvert \mathfrak{e}^2\mathfrak{e}^4\mathfrak{g}^4\rvert g^1 g^3$	$=\sum_{\text{perm}}(\mathfrak{e}^1\mathfrak{e}^2)(\mathfrak{e}^3\mathfrak{e}^4)g^1g^2g^3g^4 -2\sum_{\text{perm}}(\mathfrak{e}^1\mathfrak{e}^2)g^1g^2([\mathfrak{e}^3\mathfrak{e}^4][g^3g^4]) +\sum_{\text{perm}}([\mathfrak{e}^1\mathfrak{g}^1][\mathfrak{e}^2\mathfrak{g}^2][\mathfrak{e}^3\mathfrak{g}^3][\mathfrak{e}^4\mathfrak{g}^4])$
(8.5) In the case of $\perp$ *polarizations*, this becomes: $\begin{matrix} & (x) & (y) & (z)\\ \mathfrak{e}^1=\mathfrak{e}^2= & (0, & 1, & 0)\\ \mathfrak{e}^3=\mathfrak{e}^4= & (0, & 0, & 1)\end{matrix}$	$=2\begin{bmatrix} 2\mathfrak{g}_c^1\mathfrak{g}_x^2 g^3g^4+2\mathfrak{g}_c^3\mathfrak{g}_c^4g^1g^2\\ -\mathfrak{g}_x^1\mathfrak{g}_c^3g^2g^4-\ \mathfrak{g}_c^1\mathfrak{g}_c^4g^2g^3\\ -\mathfrak{g}_x^2\mathfrak{g}_c^3g^1g^4-\ \mathfrak{g}_c^2\mathfrak{g}_c^4g^1g^3\end{bmatrix}$	$=\sum_{\text{perm}}(\mathfrak{e}^1\mathfrak{e}^2)(\mathfrak{e}^3\mathfrak{e}^4)\begin{bmatrix}(\mathfrak{g}^1\mathfrak{g}^2)(\mathfrak{g}^3\mathfrak{g}^4)\\ -2(\mathfrak{g}^1\mathfrak{g}^2)g^3g^4\\ +g^1g^2g^3g^4\end{bmatrix}$
(8.6) and by further specialization to $\Vert$ and opposing impulses, hence, in the *special case* (8.2a):	$= 8 \mid \mathfrak{g} \mid 4$	$= 32 \mid \mathfrak{g} \mid 4$
(8.6′) By contrast, in the case of $\Vert$ *polarizations*: $\begin{matrix} & (x) & (y) & (z)\\ \mathfrak{e}^1=\mathfrak{e}^2=\mathfrak{e}^3=\mathfrak{e}^4= & (0, & 1, & 0)\end{matrix}$	$= 0$	$=\sum_{\text{perm}}\begin{bmatrix}(\mathfrak{g}^1\mathfrak{g}^2)(\mathfrak{g}^3\mathfrak{g}^4)-2(g^1g^2)g^3g^4\\ +g^1g^2g^3g^4\end{bmatrix}$
(8.7) and by further specialization to $\Vert$ and opposing impulses, hence, in the *special case* (8.2b):	$= 0$	$= 64 \mid g \mid 4$

8.8α	8.8β				8.8γ				8.8δ				8.8ε			
Sequence of *light quanta*	*Polarizations* in case a) used in (5.15)				*Polarizations* in case b) used instead				*Intermediate states* used in (5.14), (5.15)							
	$\mathfrak{e}^1$	$\mathfrak{e}^2$	$\mathfrak{e}^3$	$\mathfrak{e}^4$	$\mathfrak{e}^1$	$\mathfrak{e}^2$	$\mathfrak{e}^3$	$\mathfrak{e}^4$	$\mathfrak{p}^1$	$\mathfrak{p}^2$	$\mathfrak{p}^3$	$\mathfrak{p}^1$	g^1	g^2	g^3	g^4
$g^1g^2g^3g^4$	y	y	z	z	y	y	y	y	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}$	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}$	g	g	$-g$	$-g$
$g^1g^2g^3g^4$	y	y	z	z	y	y	y	y	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}$	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}$	g	g	$-g$	$-g$
$g^1g^2g^3g^4$	y	z	z	y	y	y	y	y	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}+2\mathfrak{g}$	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}$	g	$-g$	$-g$	$+g$
$g^1g^2g^3g^4$	y	z	z	y	y	y	y	y	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}$	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}$	g	$-g$	$-g$	$+g$
$g^1g^2g^3g^4$	y	z	y	z	y	y	y	y	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}$	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}$	g	$-g$	g	$-g$
$g^1g^2g^3g^4$	y	z	y	z	y	y	y	y	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}+2\mathfrak{g}$	$\mathfrak{p}+\mathfrak{g}$	$\mathfrak{p}$	g	$-g$	g	$-g$

The matrix element (5.13) is a sum of products of the numerators Z_μ, which refer to the collective matrix elements (5.15) and the reciprocal denominators $1/N_\mu$, which refer to the energy differences (5.14).

Certainly, in the denominators one can sum over the permutations of the polarizations (II: $p_y \to p_z$, $p_z \to p_y$), because the polarization directions p_y and p_z appear only in the numerator and not in the denominator, and because the subsequent mean over the angle of $\mathfrak{p}$ does not distinguish between these two polarization directions ($\overline{p_y^2 p_x^n} = \overline{p_z^2 p_x^n}$). Therefore, one can indeed set $p_y^2 = p_z^2$ in the numerator (in expressions of second order in y).

One can sum over the permutations of the light quantum energies in the reciprocal denominators $1/N_\mu$, because they do not appear in the numerators (5.15), i.e., one can omit all odd powers of g from the computation of the reciprocal denominators.

However, the permutations of the impulses (I: $\mathfrak{g} \to -\mathfrak{g}$) must be carried out by first multiplying the numerator by the reciprocal denominators, because $\mathfrak{g}$ appears in both; i.e., under this multiplication, all odd powers of $\mathfrak{g}_x$, … drop out.

After these simplifications, all that remains in the sum $\sum_{\text{perm}}$ over the 24 permutations of the light quanta is just the summation of the matrix element over the six sequences (8.8) and the multiplication of the results by the factor 4.

The reciprocal denominators $1/N_\mu$ will be, as one finds by substituting the specialization (8.8ε) in (5.14), developing in light frequencies g/mc, $\mathfrak{g}/mc$ up to order four and omitting all of the odd powers of g (cf., eq. (8.9)).

(The number in the i^{th} row and the j^{th} column of the table refers to the magnitude of the term in the development that is at the top of the k^{th} column for the expression that is at the left end of the i^{th} row)

For the numerator of the matrix element in the case a), one obtains, when one substitutes the specialization to $\perp$ polarizations (8.8β) in (5.15), takes the spur, and replaces the p quadratic terms p_z^2 with p_y^2:

For the *sequences* $g^1g^2g^3g^4$ and $g^1g^2g^4g^3$:

$$Z_\mu = S_1 + 2p_y^2\left[\frac{\lambda_\mu^2}{p_0^1 p_0^2} - \frac{\lambda_\mu^2}{p_0^3 p_0^4} - \frac{\lambda_\mu^2\lambda_\mu^3}{p_0^1 p_0^4} + \frac{1}{p_0^2 p_0^3} - \frac{2\lambda_\mu^2}{p_0^1 p_0^3}\right]$$
$$- \frac{1}{p_0^1 p_0^2 p_0^3 p_0^4}\left[\begin{matrix} S_2 + 8p_y^2 p_z^2 \\ +2p_y^2[-(\mathfrak{p}^1\mathfrak{p}^2) - (\mathfrak{p}^3\mathfrak{p}^4) - (\mathfrak{p}^2\mathfrak{p}^3) - (\mathfrak{p}^1\mathfrak{p}^4) + 2(\mathfrak{p}^3\mathfrak{p}^4) - 2(mc)^2].\end{matrix}\right]$$

Eq. (8.9)

Sequence	Reciprocal denominator	$\frac{1}{\mathfrak{p}_0^3}$	$\frac{(\mathfrak{p}\mathfrak{g})}{2p_0^5}$	$\frac{g^2}{4p_0^5}$	$\frac{(\mathfrak{p}\mathfrak{g})^2}{4p_0^7}$	$\frac{(\mathfrak{p}\mathfrak{g})g^2}{8p_0^7}$	$\frac{(\mathfrak{p}\mathfrak{g})^3}{8p_0^9}$	$\frac{g^4}{16p_0^7}$	$\frac{(\mathfrak{p}\mathfrak{g})^2g^2}{16p_0^9}$	$\frac{(\mathfrak{p}\mathfrak{g})^4}{16p_0^{11}}$
$g^1g^2g^3g^4$	$\frac{1}{N_1}=$	1	– 2	9	5	– 22	– 14	30	56	42
	$\frac{1}{N_2}+\frac{1}{N_3}=$	2	– 6	8	20	– 10	–70	32	– 28	252
	$\frac{1}{N_4}=$	1	– 4	– 3	15	24	– 56	10	– 140	210
	$\frac{1}{N_5}+\frac{1}{N_6}=$	1	– 3	– 1	9	8	– 28	2	– 44	90
$g^1g^2g^4g^3$	$\frac{1}{N_1}=$	1	0	9	3	0	0	30	28	22
	$\frac{1}{N_2}+\frac{1}{N_3}=$	2	0	8	8	0	0	32	0	64
	$\frac{1}{N_4}=$	1	– 2	– 3	7	10	– 18	10	– 60	66
	$\frac{1}{N_5}+\frac{1}{N_6}=$	1	– 1	– 1	5	4	– 8	2	– 28	42
$g^1g^3g^4g^2$	$\frac{1}{N_1}=$	1	– 4	– 5	17	48	– 80	38	– 372	414
	$\frac{1}{N_2}+\frac{1}{N_3}=$	2	– 12	– 4	68	72	– 392	44	– 840	2332
	$\frac{1}{N_4}=$	1	– 6	3	33	– 18	– 182	– 18	28	1030
	$\frac{1}{N_5}+\frac{1}{N_6}=$	1	– 5	– 5	25	56	– 132	34	– 484	738
$g^1g^4g^3g^2$	$\frac{1}{N_1}=$	1	– 2	– 1	5	6	– 14	2	– 28	42
	$\frac{1}{N_2}+\frac{1}{N_3}=$	2	– 6	8	20	– 50	– 70	– 8	252	252
	$\frac{1}{N_4}=$	1	– 4	7	15	– 48	– 56	– 6	252	210
	$\frac{1}{N_5}+\frac{1}{N_6}=$	1	– 3	– 1	9	8	– 28	2	– 44	90
$g^1g^4g^2g^3$	$\frac{1}{N_1}=$	1	0	1	3	0	0	– 2	– 4	22
	$\frac{1}{N_2}+\frac{1}{N_3}=$	2	0	– 4	8	0	0	12	– 60	64
	$\frac{1}{N_4}=$	1	– 2	– 1	7	2	– 18	2	– 28	66
	$\frac{1}{N_5}+\frac{1}{N_6}=$	1	– 1	3	5	– 4	– 8	6	12	42
$g^1g^3g^2g^4$	$\frac{1}{N_1}=$	1	– 4	– 3	17	36	– 80	26	– 312	414
	$\frac{1}{N_2}+\frac{1}{N_3}=$	2	– 12	– 16	68	192	– 392	144	– 1740	2332
	$\frac{1}{N_4}=$	1	– 6	– 5	33	62	– 182	30	– 564	1030
	$\frac{1}{N_5}+\frac{1}{N_6}=$	1	– 5	– 1	25	24	– 132	2	– 276	738

For the *sequences* $g^1g^2g^3g^4$ and $g^1g^4g^3g^2$:

$$Z_\mu = S_1 + 2p_y^2\left[\frac{\lambda_\mu^2}{p_0^1p_0^2} - \frac{\lambda_\mu^2}{p_0^3p_0^4} - \frac{\lambda_\mu^2\lambda_\mu^3}{p_0^1p_0^4} + \frac{1}{p_0^2p_0^3} + \frac{2\lambda_\mu^2}{p_0^2p_0^4}\right]$$
$$- \frac{1}{p_0^1p_0^2p_0^3p_0^4}\left[\begin{array}{l} S_2 + 8p_y^2p_z^2 \\ +2p_y^2[-(\mathfrak{p}^1\mathfrak{p}^2) - (\mathfrak{p}^3\mathfrak{p}^4) - (\mathfrak{p}^2\mathfrak{p}^3) - (\mathfrak{p}^1\mathfrak{p}^4) + 2(\mathfrak{p}^1\mathfrak{p}^3) - 2(mc)^2]. \end{array}\right]$$

In this, S_1, S_2 are abbreviations for:

$$S_1 = \lambda_\mu^2\lambda_\mu^3 - \lambda_\mu^3\frac{(\mathfrak{p}^1\mathfrak{p}^2) + (mc)^2}{p_0^1p_0^2} + \lambda_\mu^2\frac{(\mathfrak{p}^3\mathfrak{p}^4) + (mc)^2}{p_0^3p_0^4} - \lambda_\mu^3\frac{(\mathfrak{p}^2\mathfrak{p}^4) + (mc)^2}{p_0^2p_0^4}$$
$$+\lambda_\mu^2\lambda_\mu^3\frac{(\mathfrak{p}^1\mathfrak{p}^4) + (mc)^2}{p_0^1p_0^4} - 1\cdot\frac{(\mathfrak{p}^2\mathfrak{p}^3) + (mc)^2}{p_0^2p_0^3} + \lambda_\mu^2\frac{(\mathfrak{p}^1\mathfrak{p}^3) + (mc)^2}{p_0^1p_0^3},$$

$$S_2 = \left[\begin{array}{l} [(\mathfrak{p}^1\mathfrak{p}^2) + (mc)^2][(\mathfrak{p}^3\mathfrak{p}^4) + (mc)^2] + [(\mathfrak{p}^2\mathfrak{p}^3) + (mc)^2][(\mathfrak{p}^1\mathfrak{p}^4) + (mc)^2] \\ -[(\mathfrak{p}^1\mathfrak{p}^3) + (mc)^2][(\mathfrak{p}^2\mathfrak{p}^4) + (mc)^2] \end{array}\right].$$

Under further specialization to parallel impulses, as one finds by substituting in (8.8δ) and developing in g/mc up to order four, these expressions become (cf., eq. (8.10)).

In order to divide the numerators (5.15) by the denominators (5.14) and sum over the six transitions μ and the 24 sequences (8.8*a*) one must multiply the columns of table (8.10) with the corresponding columns of table (8.9) and by the factor four. One then obtains, in terms of fourth order in g/mc, after taking the mean over the positron direction $\mathfrak{p}$: [cf., § 9.7]:

$$\sum_{\text{perm}}\sum_\mu \frac{Z_\mu}{N_\mu} = \frac{4|g|^4}{p_0^7}\cdot\left[\begin{array}{l} -63 + 1002\dfrac{p^2}{3p_0^2} - 7734\dfrac{p^4}{3\cdot5p_0^4} - 315\dfrac{p^4}{5p_0^4} \\ +22800\dfrac{p^6}{2\cdot5\cdot7p_0^6} + 7524\dfrac{p^6}{5\cdot7p_0^6} - \dfrac{47606}{5\cdot7\cdot9\cdot11\cdot13} \end{array}\right], \tag{8.11}$$

and after integration [1]) $4\pi\int p^2\,dp$ over the positron energy p_0 :

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[1]) $\int_0^\infty \frac{p^m}{p_0^{m+7}}p^2dp = \frac{2}{(m+3)(m+5)}\frac{1}{(mc)^4}$.

Eq. (8.10)

Sequence	Numerator	1: 1	1: $\frac{2p_y^2}{p_0^2}$	1: $\frac{8p_y^2p_z^2}{p_0^4}$	$\frac{(\mathfrak{p}\mathfrak{g})}{p_0^2}$: 1	$\frac{(\mathfrak{p}\mathfrak{g})}{p_0^2}$: $\frac{2p_y^2}{p_0^2}$	$\frac{(\mathfrak{p}\mathfrak{g})}{p_0^2}$: $\frac{8p_y^2p_z^2}{p_0^4}$	$\frac{\lvert\mathfrak{g}\rvert^2}{p_0^2}$: 1	$\frac{\lvert\mathfrak{g}\rvert^2}{p_0^2}$: $\frac{2p_y^2}{p_0^2}$	$\frac{\lvert\mathfrak{g}\rvert^2}{p_0^2}$: $\frac{8p_y^2p_z^2}{p_0^4}$	$\frac{(\mathfrak{p}\mathfrak{g})^2}{p_0^4}$: 1	$\frac{(\mathfrak{p}\mathfrak{g})^2}{p_0^4}$: $\frac{2p_y^2}{p_0^2}$	$\frac{(\mathfrak{p}\mathfrak{g})^2}{p_0^4}$: $\frac{8p_y^2p_z^2}{p_0^4}$
$g^1g^2g^3g^4$	$z_1 =$	0	0	− 1	0	4	2	2	0	1	− 2	− 8	
	$z_2 = z_3 =$	0	0	− 1	0	4	2	2	0	1	− 2	− 8	
	$z_4 =$	0	4	− 1	0	− 4	2	2	− 4	1	− 2	8	
	$z_5 = z_6 =$	− 8	8	− 1	0	− 8	2	4	− 6	1	− 4	14	
$g^1g^2g^4g^3$	$z_1 =$	0	0	− 1	0	0	0	− 2	0	1	2	0	
	$z_2 = z_3 =$	0	0	− 1	0	0	0	− 2	0	1	2	0	
	$z_4 =$	0	4	− 1	0	4	0	2	− 4	1	− 2	8	
	$z_5 = z_6 =$	− 8	8	− 1	0	0	0	4	− 6	1	− 4	14	
$g^1g^3g^4g^2$	$z_1 =$	0	4	− 1	0	− 12	4	2	− 12	3	− 2	40	
	$z_2 = z_3 =$	0	0	− 1	0	0	4	− 2	0	3	2	0	
	$z_4 =$	0	0	− 1	0	0	4	− 2	0	3	2	0	
	$z_5 = z_6 =$	− 8	8	− 1	0	− 16	4	4	− 14	3	− 4	46	
$g^1g^4g^3g^2$	$z_1 =$	0	4	− 1	0	− 4	2	2	− 4	1	− 2	8	
	$z_2 = z_3 =$	0	0	− 1	0	− 4	2	2	− 4	1	− 2	8	
	$z_4 =$	0	0	− 1	0	− 4	2	2	− 4	1	− 2	8	
	$z_5 = z_6 =$	− 8	8	− 1	0	− 8	2	4	− 6	1	− 4	14	
$g^1g^4g^2g^3$	$z_1 =$	0	0	− 1	0	0	0	2	− 4	1	− 2	4	
	$z_2 = z_3 =$	0	4	− 1	0	0	0	2	− 4	1	− 2	4	
	$z_4 =$	0	0	− 1	0	− 4	0	− 2	0	1	2	− 4	
	$z_5 = z_6 =$	8	− 8	− 1	0	0	0	− 4	2	1	4	− 10	
$g^1g^3g^2g^4$	$z_1 =$	0	0	− 1	0	4	4	− 2	4	3	2	− 20	
	$z_2 = z_3 =$	0	4	− 1	0	− 8	4	2	− 8	3	− 2	20	
	$z_4 =$	0	0	− 1	0	0	4	2	− 4	3	− 2	4	
	$z_5 = z_6 =$	8	− 8	− 1	0	16	4	− 4	10	3	4	− 42	

$$(8.12)\quad \left\{ \begin{aligned} C\int d\mathfrak{p}\sum_{\text{perm}}\sum_{\mu} &= C\cdot 4\pi\cdot 4\cdot 2\left(\frac{g}{mc}\right)^4 \\ &\cdot\left[-\frac{63}{3\cdot 5}+\frac{1002}{3\cdot 5\cdot 7}-\frac{7734}{3\cdot 5\cdot 7\cdot 9}-\frac{315}{5\cdot 7\cdot 9} \right. \\ &\quad \left. +\frac{22800}{3\cdot 5\cdot 7\cdot 9\cdot 11}+\frac{7524}{5\cdot 7\cdot 9\cdot 11}-\frac{47606}{5\cdot 7\cdot 9\cdot 11\cdot 13}\right]. \end{aligned} \right.$$

$$(8.13)\quad = 4\pi\cdot 4\cdot 2\left(-\frac{8}{3\cdot 5}\right)\cdot\left(\frac{g}{mc}\right)^4\cdot C = H_\perp^4 .$$

Eq. (8.10) (cont.)

$\frac{(\mathfrak{p}\mathfrak{g})\lvert g\rvert^2}{p_0^4}$			$\frac{(\mathfrak{p}\mathfrak{g})^3}{p_0^6}$			$\frac{\lvert\mathfrak{g}\rvert^4}{p_0^4}$			$\frac{(\mathfrak{p}\mathfrak{g})^2\lvert g\rvert^2}{p_0^6}$			$\frac{(\mathfrak{p}\mathfrak{g})^4}{p_0^8}$		
1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$
	−4	−4	4	16	8	−2	0	−1	10	16	12	−8	−2	−8
	−4	−4	4	16	8	−2	0	−1	10	16	12	−8	−2	−8
	12	−4	4	−16	8	−2	4	−1	10	−32	12	−8	−2	8
	18	−4	−8	−26	8	−7/2	11/2	−1	19	−47	12	−31/2	−4	14
0	0	0	0	0	0	2	0	−1	−6	0	6	4	0	−6
0	0	0	0	0	0	2	0	−1	−6	0	6	4	0	−6
4	−6	0	−4	10	0	−2	4	−1	10	−24	6	−8	24	−6
0	0	0	0	0	0	−7/2	11/2	−1	19	−39	6	−31/2	83/2	−6
	78	−22	12	−138	48	−10	38	−9	66	−402	118	−56	488	−166
	0	−22	−8	0	48	6	0	−9	−34	0	118	28	0	−166
	0	−22	−8	0	48	6	0	−9	−34	0	118	28	0	−166
	84	−22	16	−148	48	−23/2	79/2	−9	75	−417	118	−127/2	1011/2	−166
	12	−4	4	−16	8	−2	4	−1	10	−32	12	−8	32	−16
	12	−4	4	−16	8	−2	4	−1	10	−32	12	−8	32	−16
	12	−4	4	−16	8	−2	4	−1	10	−32	12	−8	32	−16
	18	−4	8	−16	8	−7/2	11/2	−1	10	−47	12	−31/2	99/2	−16
	0	0	0	0	0	−2	4	−1	6	−16	6	−4	12	−6
	0	0	0	0	0	−2	4	−1	6	−16	6	−4	12	−6
	6	0	4	−10	0	2	0	−1	−10	8	6	8	−12	−6
	0	0	0	0	0	7/2	−3/2	−1	−19	23	6	31/2	−59/2	−6
	−38	−22	−12	82	48	10	−18	−9	−66	230	118	56	−316	−166
	40	−22	8	−56	48	−6	20	−9	34	−172	118	−28	172	−166
	24	−22	8	−24	48	−6	16	−9	34	−124	118	−28	108	−166
	−60	−22	−12	124	48	23/2	−47/2	−9	−75	293	118	127/2	−795/2	−166

Likewise, one carries out the calculations for the special case (8.2b): When one takes the spur in the numerator (5.15) for parallel polarizations (8.8γ), one obtains:

Eq. (8.15)

Sequence	Numer-ator	1			$\frac{(\mathfrak{p}\mathfrak{g})}{p_0^2}$			$\frac{\lvert\mathfrak{g}\rvert^2}{p_0^2}$			$\frac{(\mathfrak{p}\mathfrak{g})^2}{p_0^4}$		
		1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$
$g^1g^2g^3g^4$	$-z_1 =$	0	− 4	1	0	4	− 2	− 2	4	− 1	2	− 8	
$g^1g^4g^3g^2$	$-z_2 = -z_3 =$	0	− 4	1	0	4	− 2	−2	4	− 1	2	− 8	
	$-z_4 =$	0	− 4	1	0	4	− 2	− 2	4	−1	2	− 8	
	$-z_5 = -z_6 =$	8	− 8	1	0	8	− 2	− 4	6	−1	4	−14	
$g^1g^2g^4g^3$	$-z_1 =$	0	− 4	1	0	0	0	2	4	− 1	− 2	− 8	
$g^1g^4g^2g^3$	$-z_2 = -z_3 =$	0	− 4	1	0	0	0	2	4	− 1	− 2	− 8	
	$-z_4 =$	0	− 4	1	0	− 4	0	− 2	4	− 1	2	− 8	
	$-z_5 = -z_6 =$	8	− 8	1	0	0	0	−4	6	− 1	4	− 14	
$g^1g^3g^4g^2$	$-z_1 =$	0	− 4	1	0	12	− 4	− 2	12	− 3	2	− 40	
$g^1g^3g^2g^4$	$-z_2 = -z_3 =$	0	− 4	1	0	8	− 4	2	8	− 3	− 2	− 24	
	$-z_4 =$	0	− 4	1	0	8	− 4	2	8	− 3	− 2	− 24	
	$-z_5 = -z_6 =$	8	− 8	1	0	16	− 4	− 4	14	− 3	4	− 46	

$$
(8.14)\qquad \left\{
\begin{aligned}
Z_\mu &= S_1 + 2p_y^2\left[\frac{\lambda_\mu^2}{p_0^1 p_0^2} - \frac{\lambda_\mu^2}{p_0^3 p_0^4} - \frac{\lambda_\mu^2\lambda_\mu^3}{p_0^1 p_0^4} + \frac{1}{p_0^2 p_0^3}\right] \\
&\quad - \frac{1}{p_0^1 p_0^2 p_0^3 p_0^4}\left[\begin{matrix} S_2 + 8p_y^4 \\ -2p_y^2[(\mathfrak{p}^1\mathfrak{p}^2) + (\mathfrak{p}^3\mathfrak{p}^4) + (\mathfrak{p}^2\mathfrak{p}^3) + (\mathfrak{p}^1\mathfrak{p}^4) + 4(mc)^4\end{matrix}\right] \\
&\quad (S_1, S_2 \quad \text{cf., pp (?).}
\end{aligned}
\right.
$$

When one substitutes the further specialization to parallel impulse (8.8*d*) and develops in light quanta energies g/mc this becomes (cf., eq. (8.15)).

Multiplication of these numerators (8.15) with the corresponding denominators (8.9), addition over the six sequences (8.8) and the six cases μ, multiplication by four, and taking the mean over the angle of $\mathfrak{p}$ gives, to terms of fourth order in the development in light frequencies:

Eq. (8.15) (cont.)

$\frac{(\mathfrak{p}\mathfrak{g})\lvert g\rvert^2}{p_0^4}$			$\frac{(\mathfrak{p}\mathfrak{g})^3}{p_0^6}$			$\frac{\lvert\mathfrak{g}\rvert^4}{p_0^4}$			$\frac{(\mathfrak{p}\mathfrak{g})^2\lvert g\rvert^2}{p_0^6}$			$\frac{(\mathfrak{p}\mathfrak{g})^4}{p_0^8}$		
1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$	1	$\frac{2p_y^2}{p_0^2}$	$\frac{8p_y^2p_z^2}{p_0^4}$
4	– 12	4	– 4	16	– 8	– 2	– 4	1	– 10	32	– 12	8	– 32	16
4	– 12	4	–4	16	–8	2	– 4	1	– 10	32	– 12	8	– 32	16
4	– 12	4	–4	16	–8	2	– 4	1	– 10	32	– 12	8	– 32	16
8	– 18	4	– 8	26	–8	7/2	–11/2	1	– 19	47	– 12	31/2	–99/2	16
0	0	0	0	0	0	– 2	– 4	1	6	0	– 6	– 4	0	6
0	0	0	0	0	0	– 2	– 4	1	6	0	– 6	–4	0	6
4	6	0	4	– 10	0	2	– 4	1	–10	–24	– 6	8	24	6
0	0	0	0	0	0	7/2	–11/2	1	–19	– 39	– 6	31/2	83/2	6
	– 78	22	– 12	138	–48	10	–38	9	– 66	– 402	–118	56	488	166
	– 48	22	8	80	–48	– 6	– 24	9	+ 34	0	–118	– 28	0	166
	– 48	22	8	80	–48	– 6	– 24	9	+ 34	0	–118	– 28	0	166
	– 84	22	–16	148	–48	23/2	–79/2	9	– 75	– 417	–118	127/2	1011/2	166

$$
(8.16)\qquad \begin{cases} -\dfrac{1}{2}\displaystyle\sum_{\text{perm}}\sum_{\mu}\frac{Z_\mu}{N_\mu} = \\[2ex] \qquad = 4\dfrac{\lvert g\rvert^4}{p_0^7}\left[\begin{array}{l} \dfrac{41}{2} - 917\dfrac{p^2}{3p_0^2} + \dfrac{2379}{2}\dfrac{p^4}{5p_0^4} + 8400\dfrac{p^4}{3\cdot 5p_0^4} \\[2ex] \qquad -26316\dfrac{p^6}{5\cdot p_0^6} + 23803\dfrac{p^8}{3\cdot 5\cdot 7p_0^8} \end{array}\right] \end{cases}
$$

and this becomes, after integrating over p:

$$
(8.17)\qquad \begin{cases} C\displaystyle\int d\mathfrak{p}\sum_{\text{perm}}\sum_{\mu}\frac{Z_\mu}{N_\mu} \\[2ex] \qquad = C\cdot 4\pi(-2)\cdot 4\cdot 2\left(\dfrac{g}{mc}\right)^4\left[\begin{array}{l} \qquad\dfrac{41}{2\cdot 3\cdot 5} - \dfrac{917}{3\cdot 5\cdot 7} + \dfrac{2379}{2\cdot 3\cdot 5\cdot 7\cdot 9} \\[2ex] + \dfrac{8400}{3\cdot 5\cdot 7\cdot 9} - \dfrac{26316}{5\cdot 7\cdot 9\cdot 11} + \dfrac{23803}{3\cdot 5\cdot 7\cdot 11\cdot 13} \end{array}\right] \end{cases}
$$

$$
(8.18)\qquad = -4\pi\cdot 16\cdot\left(-\frac{32}{5\cdot 9}\right)\left(\frac{g}{mc}\right)^4\cdot C = H_\parallel^4 .
$$

The matrix elements of Dirac's theory (8.13, 8.18) are now calculated to fourth order terms, as well as the matrix elements of the light quanta interaction (8.6, 8.7), for both of the transitions considered (8.2a, b).

Setting them equal to each other (8.3):

$$\text{a)} \qquad \perp: \quad (32\alpha - 8\beta)\,|\,g\,|^4\, D\,\frac{\hbar c}{e^2}\frac{1}{E_0^2} = -\frac{4\pi\cdot 6\cdot 32}{5\cdot 9}\left(\frac{g}{mc}\right)^4 C$$

$$\text{b)} \qquad \|: \quad (64\alpha \qquad)\,|\,g\,|^4\, D\,\frac{\hbar c}{e^2}\frac{1}{E_0^2} = +\frac{4\pi\cdot 16\cdot 32}{5\cdot 9}\left(\frac{g}{mc}\right)^4 C$$

determines the two constants α, β:

$$\boxed{\begin{aligned} \alpha &= -\frac{1}{360\pi^2} \\ \beta &= -\frac{7}{360\pi^2} \end{aligned}} \tag{8.19}$$

The simplicity of the results makes one suspect that it must be possible to arrive at the results that were derived here by a simpler path. This simpler path is described in a work that appeared in the meantime [1]).

§ 9. **Confirmation of the method**

In the foregoing sections two constants α, β were determined in such a manner that the Dirac matrix element H_{in}^4 for the scattering of light by light (in terms of fourth order in the development in light frequencies g^i/mc) in two special cases (8.2α, β) would be represented by the expression (2.21):

$$\frac{\hbar c}{e^2}\frac{1}{E_0^2}\int[\alpha(\mathfrak{B}\mathfrak{D})^2 + \beta(\mathfrak{B}^2 - \mathfrak{D}^2)^2]\,dV\,. \tag{9.1}$$

The possibility of making this determination rests on the fact that both expressions give a non-vanishing matrix element to the two transitions in question.

However, the assertion that it comes down to, that (9.1) completely represents the scattering of light by light in all cases (up to fourth order in g^i/mc), includes the assumption that there is a corresponding expression for the scattering of light by light whose development terms of order 1, 2, and 3 in the Dirac matrix element H_{in}^4 in light frequencies always vanish (due to gauge invariance) and (due to its Lorentz invariance) the term of order four in this development of the Dirac matrix has the form (9.1).

[1]) W. Heisenberg and H. Euler, Zeit. f. Phys. **98**, pp. 714, 1936.

The assumption and the assertion that follows from it, that we also obtained the result (9.1) without specialization by carrying out the computation of the Dirac matrix element for four arbitrary light quanta, shall now be confirmed by some numerical calculations.

In the development of the Dirac matrix element in light frequencies, we calculate:

1. in the first approximation in the general case,

2. in the second approximation under restricting assumptions on the polarizations,

3. in the third approximation for the two special cases a) and b) that were treated above, and determine:

4. the constant α, once again, in a manner that is completely independent of the one above.

When 1, 2, and 3 yield the result 0, and 4 again gives the result $\alpha = -1/360\pi^2$, we can see a direct confirmation of our method in this.

1. The first order term in the development of the Dirac matrix element in g^i/mc must always vanish since it is linear in the impulses and energies of the light quanta as first-order terms, symmetric in the four light quanta, due to the summation over the permutations of the light quanta, hence 0, as a result of the conservation laws $\sum_{\text{perm}} g^1 = 0$, $\sum_{\text{perm}} \mathfrak{g}^1 = 0$.

2. The second-order term in g^i/mc in the Dirac matrix element shall be calculated in the special case in which all four light quanta have parallel polarizations, but arbitrary coplanar impulses $\mathfrak{g}^1$, $\mathfrak{g}^2$, $\mathfrak{g}^3$, $\mathfrak{g}^4$.

In this case, one obtains, by developing the numerators Z_μ (5.15), which include the products of four ordinary matrix elements for the sub-process, up to order two in g^i/mc: (instead of for the six transitions $\mu = 1, \ldots, 6$, of which only four are distinct, one has four equations, which give four individual equations, and are summarized into one equation for the columns $\begin{vmatrix} Z_1 \\ Z_2 = Z_3 \\ Z_4 \\ Z_5 = Z_6 \end{vmatrix}$ of four numbers):

$$
(9.2)\quad \begin{cases}
\left|\begin{matrix} Z_1 \\ Z_2 = Z_3 \\ Z_4 \\ Z_5 = Z_6 \end{matrix}\right| = -8\left|\begin{matrix}0\\0\\0\\1\end{matrix}\right| + 8\dfrac{p_y^2}{p_0^2}\left|\begin{matrix}1\\1\\1\\1\end{matrix}\right| - 80\dfrac{p_y^4}{p_0^4}\left|\begin{matrix}1\\1\\1\\1\end{matrix}\right| - 4\dfrac{p_y^2}{p_0^4}\left[(\mathfrak{p}^1\mathfrak{g}^1)\left|\begin{matrix}3\\2\\1\\4\end{matrix}\right| + (\mathfrak{p}\mathfrak{g}^2)\left|\begin{matrix}2\\1\\1\\2\end{matrix}\right| + (\mathfrak{p}\mathfrak{g}^4)\left|\begin{matrix}-1\\-1\\-2\\-2\end{matrix}\right|\right] \\[2ex]
\qquad + 8\dfrac{p_y^4}{p_0^4}(\mathfrak{p}, 2\mathfrak{g}^1 + \mathfrak{g}^2 - \mathfrak{g}^4)\left|\begin{matrix}1\\1\\1\\1\end{matrix}\right| - \dfrac{(\mathfrak{g}^1\mathfrak{g}^1)}{6p_0^2}\left[4\left|\begin{matrix}-1\\-1\\-1\\6\end{matrix}\right| + 8\dfrac{p_y^2}{p_0^2}\left|\begin{matrix}10\\8\\8\\13\end{matrix}\right| + 80\dfrac{p_y^4}{p_0^4}\left|\begin{matrix}-1\\-1\\-1\\-1\end{matrix}\right|\right] \\[2ex]
\qquad - \dfrac{(\mathfrak{p}\mathfrak{g}^1)^2}{6p_0^4}\left[4\left|\begin{matrix}1\\-1\\1\\6\end{matrix}\right| + 8\dfrac{p_y^2}{p_0^2}\left|\begin{matrix}-28\\-20\\-20\\-37\end{matrix}\right| + 320\left|\begin{matrix}1\\1\\1\\1\end{matrix}\right|\right].
\end{cases}
$$

In this, one has already taken the mean over the 24 permutations of the light quanta in the second-order terms, while observing the following relation, which follows from the conservation laws:

$$
(9.3)\qquad \ldots = \sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^1) = \sum_{\text{perm}}(\mathfrak{g}^2\mathfrak{g}^2) = -3\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^2) = -3\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^3) = \ldots
$$

The development of the reciprocal denominators $1/N_\mu$ (5.14), which represent energy differences between the intermediate states and the final state, up to the same degree becomes (in the notation above):

$$
(9.4)\quad \begin{cases}
\left|\begin{matrix} N_1^{-1} \\ N_2^{-1} + N_3^{-1} \\ N_4^{-1} \\ N_5^{-1} + N_6^{-1} \end{matrix}\right| = \dfrac{1}{p_0^3}\left|\begin{matrix}1\\2\\1\\1\end{matrix}\right| + \dfrac{1}{2p_0^5}\left[(\mathfrak{p}\mathfrak{g}^1)\left|\begin{matrix}-2\\-6\\-4\\-3\end{matrix}\right| + (\mathfrak{p}\mathfrak{g}^2)\left|\begin{matrix}-1\\-3\\-1\\-1\end{matrix}\right| + (\mathfrak{p}\mathfrak{g}^4)\left|\begin{matrix}1\\3\\1\\1\end{matrix}\right|\right] \\[2ex]
\qquad + \dfrac{(\mathfrak{g}^1\mathfrak{g}^2)}{12p_0^5}\left|\begin{matrix}5\\0\\-1\\-3\end{matrix}\right| + \dfrac{(\mathfrak{p}\mathfrak{g}^2)}{2p_0^7}\left|\begin{matrix}25\\90\\55\\39\end{matrix}\right|,
\end{cases}
$$

in which the mean over the permutations of light quanta was likewise already taken in the highest order term.

Multiplication of these numerators (9.2) and reciprocal denominators (9.4) yields, in its second-order terms in g^i/mc, after summing over the six transition types $\mu = 1, \ldots, 6$, taking the mean over $\mathfrak{p}$, and summing over the permutations of the light quanta, and considering (9.3):

$$\sum_{\text{perm}}\sum_{\mu}\frac{\bar{Z}_\mu}{N_\mu}=-\sum_{\text{perm}}\frac{(\mathfrak{g}^1\mathfrak{g}^1)}{p_0^5}\left[-36+\frac{564}{3}\frac{p^2}{p_0^2}-\frac{3876}{3\cdot 5}\frac{p^4}{p_0^4}+\frac{3564}{5\cdot 7}\frac{p^6}{p_0^6}\right]$$

and after integration [1]) over p:

$$\int d\mathfrak{p}\sum_{\text{perm}}\sum_{\mu}\frac{Z_\mu}{N_\mu}=-4\pi\sum_{\text{perm}}\left(\frac{g^1}{mc}\right)^2\left[-\frac{36}{3}+\frac{564}{3\cdot 5}-\frac{3876}{3\cdot 5\cdot 7}+\frac{3564}{5\cdot 7\cdot 9}\right]$$
$$= 0.$$

In fact, the second order term in g^i/mc in the Dirac matrix element for the scattering of light by light therefore vanishes for arbitrary impulses and parallel polarizations.

The corresponding second-order term for non-parallel polarizations shall be examined only in the special case (8.2b) of perpendicular polarization and parallel impulses of equal magnitudes.

Here, one obtains for the second-order term of the Dirac matrix element, when one multiplies the numerators in the columns of ((8.15) with the reciprocal denominators in the columns of (8.9), which leads to terms of second order in g^i/mc, and takes the mean over the positron direction $\mathfrak{p}$:

$$\sum_{\text{perm}}\sum_{\mu}\frac{\bar{Z}_\mu}{N_\mu}=4\sum_{\text{perm}}\frac{|g|^2}{p_0^5}\left[44-\frac{332}{3}\frac{p^2}{p_0^2}+\frac{1180}{3\cdot 5}\frac{p^4}{p_0^4}-\frac{3564}{3\cdot 5\cdot 7}\frac{p^6}{p_0^6}\right],$$

and after integration over the positron energy p_0 this becomes:

$$\int d\mathfrak{p}\sum_{\text{perm}}\sum_{\mu}\frac{Z_\mu}{N_\mu}=4\pi\cdot 4\sum_{\text{perm}}\left(\frac{g}{mc}\right)^2\left[\frac{44}{3}-\frac{332}{3\cdot 5}+\frac{1180}{3\cdot 5\cdot 7}-\frac{3564}{3\cdot 5\cdot 7\cdot 9}\right]$$
$$= 0$$

in agreement with our general assertion.

3. The vanishing of the third-order terms in the development of the Dirac matrix element in light frequencies shall, in any case, be carried out only in the two special cases that were treated above (8.2a, b) of parallel impulses with equal magnitudes, along with either parallel or perpendicular polarizations; we have already confirmed this, though. Then, from pp. (?) all of the odd powers of the light quanta impulses and energies drop out, because one sums over all permutations of the equal (opposite, resp.) light quanta.

4. The constant α, which is definitive for the scattering of light with light, shall now be calculated in a new way. Instead of considering, as we did above, a special case (8.2b) of equal (opposite, resp.) impulses and equating the matrix element (5.13) of Dirac's theory with the field function (8.4), we now consider the general case of arbitrary impulse

---

[1]) $\int_0^\infty \frac{p^n}{p_0^{n+5}}\,p^2 dp=\frac{1}{(mc)^2}\frac{1}{n+3}.$

(with parallel polarization), but restrict ourselves to equating with one of the terms in (8.4) that refers to the transition considered.

From (8.4), the matrix element of:

$$\alpha \frac{\hbar c}{e^2} \frac{1}{E_0^2} \int (\mathfrak{B}^2 - \mathfrak{D}^2)^2 \, dV$$

for a transition with || polarizations includes the three terms:

$$D \cdot \alpha \cdot \frac{\hbar c}{e^2} \frac{1}{E_0^2} \sum_{\text{perm}} [(\mathfrak{g}^1 \mathfrak{g}^2)(\mathfrak{g}^3 \mathfrak{g}^4) - 2(\mathfrak{g}^1 \mathfrak{g}^2) g^3 g^4 + g^1 g^2 g^3 g^4]. \tag{9.5}$$

We focus on the middle term and determine the constant α by equating this term:

$$\alpha D \frac{\hbar c}{e^2} \frac{1}{E_0^2} (-2) \sum_{\text{perm}} (\mathfrak{g}^1 \mathfrak{g}^2) g^3 g^4 \tag{9.5a}$$

with the corresponding term of the Dirac matrix element, which we now compute.

The matrix element H_{in}^4 in Dirac's theory takes the form (in its terms of fourth order in the development in g^i/mc) of a symmetric form of degree four in the four light quantum impulses and energies. Due to the conservation laws, however, they are not independent, and furthermore, they may all be linearly composed of the four forms:

$$\sum_{\text{perm}} (\mathfrak{g}^1 \mathfrak{g}^2)(\mathfrak{g}^3 \mathfrak{g}^4), \qquad \sum_{\text{perm}} (\mathfrak{g}^1 \mathfrak{g}^2) g^3 g^4, \qquad \sum_{\text{perm}} g^1 g^2 g^3 g^4, \quad \sum_{\text{perm}} (\mathfrak{g}^1 \mathfrak{g}^1)(\mathfrak{g}^2 \mathfrak{g}^2),$$

which are mutually linearly independent.

As a simple application of the conservation laws:

$$\sum_{\text{perm}} \mathfrak{g}^1 = 0, \qquad \sum_{\text{perm}} g^1 = 0$$

shows, these linear relations between the differing terms of fourth order in g/mc of the Dirac matrix element read:

Eq. (9.6)

	$\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^2)(\mathfrak{g}^3\mathfrak{g}^4)$	$\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^2)g^3g^4$	$\sum_{\text{perm}}g^1g^2g^3g^4$	$\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^1)(\mathfrak{g}^2\mathfrak{g}^2)$
$\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^1)g^2g^3 = \sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^1)(\mathfrak{g}^2\mathfrak{g}^3) =$	0	0	$-\frac{1}{3}$	0
$\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^1)g^1g^2 = \sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^1)(\mathfrak{g}^1\mathfrak{g}^2) =$	0	0	$+\frac{2}{3}$	-1
$\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^1)^2 =$	0	0	-2	3
$\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^2)g^1g^3 =$	0	$-\frac{1}{2}$	$+\frac{1}{6}$	0
$\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^2)g^1g^2 =$	0	$+1$	-1	1
$\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^2)(\mathfrak{g}^1\mathfrak{g}^3) =$	$-\frac{1}{2}$	0	$+\frac{1}{6}$	0
$\sum_{\text{perm}}(\mathfrak{g}^1\mathfrak{g}^2)^2 =$	$+1$	0	-1	1

The linear relations between terms like:

$$\sum_{\text{perm}}(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)g^3g^4\,,$$

which includes the positron impulse – $\mathfrak{p}$, when integrated, follow with the help of their properties relative to the mean over the angles of $\mathfrak{p}$:

$$(9.7)\quad \begin{cases} \overline{(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)(\mathfrak{p}\mathfrak{g}^3)(\mathfrak{p}\mathfrak{g}^4)} = \dfrac{p^4}{3\cdot 5}[(\mathfrak{g}^1\mathfrak{g}^2)(\mathfrak{g}^3\mathfrak{g}^4)+(\mathfrak{g}^1\mathfrak{g}^3)(\mathfrak{g}^2\mathfrak{g}^4)+(\mathfrak{g}^1\mathfrak{g}^4)(\mathfrak{g}^2\mathfrak{g}^3)] \\ \overline{p_y^2\cdot(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)(\mathfrak{p}\mathfrak{g}^3)(\mathfrak{p}\mathfrak{g}^4)} = \dfrac{p^6}{3\cdot 5\cdot 7}[(\mathfrak{g}^1\mathfrak{g}^2)(\mathfrak{g}^3\mathfrak{g}^4)+(\mathfrak{g}^1\mathfrak{g}^3)(\mathfrak{g}^2\mathfrak{g}^4)+(\mathfrak{g}^1\mathfrak{g}^4)(\mathfrak{g}^2\mathfrak{g}^3)] \\ \overline{p_y^4\cdot(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)(\mathfrak{p}\mathfrak{g}^3)(\mathfrak{p}\mathfrak{g}^4)} = \dfrac{p^8}{5\cdot 7\cdot 9}[(\mathfrak{g}^1\mathfrak{g}^2)(\mathfrak{g}^3\mathfrak{g}^4)+(\mathfrak{g}^1\mathfrak{g}^3)(\mathfrak{g}^2\mathfrak{g}^4)+(\mathfrak{g}^1\mathfrak{g}^4)(\mathfrak{g}^2\mathfrak{g}^3)] \\ \overline{(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)} = \dfrac{p^2}{3}(\mathfrak{g}^1\mathfrak{g}^2) \\ \overline{p_y^2\cdot(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)} = \dfrac{p^4}{3\cdot 5}(\mathfrak{g}^1\mathfrak{g}^2) \\ \overline{p_y^4\cdot(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)} = \dfrac{p^6}{5\cdot 7}(\mathfrak{g}^1\mathfrak{g}^2). \end{cases}$$

The demand of gauge invariance now says that in the table (9.6) the linear relations do not appear in the four columns (in the sum over all terms of the Dirac matrix elements) and the demand of Lorentz invariance says that the first three columns (summed over all terms), which have the ratios 1: – 2 : 1, are coupled (9.5).

We now arrange (9.5a) for only the terms of the second column to matter – i.e., (9.6), (9.7) in the Dirac matrix element – and compute only the terms:

$$
(9.8)\qquad \begin{cases}
\displaystyle\sum_{\text{perm}} (\mathfrak{g}^1\mathfrak{g}^2)g^3g^4 = \sum_{\text{perm}} (\mathfrak{g}^1\mathfrak{g}^2)g^3g^4, \\
\displaystyle\sum_{\text{perm}} (\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)g^3g^4 = \sum_{\text{perm}} (\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)g^3g^4, \\
\displaystyle\sum_{\text{perm}} (\mathfrak{g}^1\mathfrak{g}^2)g^1g^2 = \sum_{\text{perm}} (\mathfrak{g}^1\mathfrak{g}^2)g^3g^4, \\
\displaystyle\sum_{\text{perm}} (\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)g^1g^2 = \sum_{\text{perm}} (\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)g^3g^4, \\
\displaystyle\sum_{\text{perm}} (\mathfrak{g}^1\mathfrak{g}^2)g^1g^3 = -\frac{1}{2}\cdot\sum_{\text{perm}} (\mathfrak{g}^1\mathfrak{g}^2)g^3g^4, \\
\displaystyle\sum_{\text{perm}} (\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)g^1g^3 = -\frac{1}{2}\cdot\sum_{\text{perm}} (\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)g^3g^4,
\end{cases}
$$

and the ones that can lead to them, and omit all others.

With this simplification, one obtains the numerators of the Dirac matrix element by developing (5.15) in g^i/mc (in the same notations as the transition types $\mu = 1, \ldots, 6$ by putting them in columns, as in (9.2)):

$$
(9.9)\qquad \begin{cases}
\left|\begin{matrix} Z_1 \\ Z_2 = Z_3 \\ Z_4 \\ Z_5 = Z_6 \end{matrix}\right| = -8\left|\begin{matrix}0\\0\\0\\1\end{matrix}\right| + \dfrac{8p_y^2}{p_0^2}\left|\begin{matrix}1\\1\\1\\2\end{matrix}\right| - \dfrac{8p_y^4}{p_0^4}\left|\begin{matrix}1\\1\\1\\1\end{matrix}\right| \\[2ex]
\qquad -\dfrac{4p_y^2}{p_0^4}\left[(\mathfrak{p}\mathfrak{g}^1)\left|\begin{matrix}3\\2\\1\\4\end{matrix}\right| + (\mathfrak{p}\mathfrak{g}^2)\left|\begin{matrix}2\\1\\1\\2\end{matrix}\right| + (\mathfrak{p}\mathfrak{g}^4)\left|\begin{matrix}-1\\-1\\-2\\-2\end{matrix}\right|\right] - \dfrac{8p_y^4}{p_0^6}(\mathfrak{p}, -2\mathfrak{g}^1 - \mathfrak{g}^2 + \mathfrak{g}^4)\left|\begin{matrix}1\\1\\1\\1\end{matrix}\right| \\[2ex]
\qquad -2\left[\dfrac{(\mathfrak{g}^1\mathfrak{g}^2)}{p_0^2} - \dfrac{(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)}{p_0^4}\right]\left|\begin{matrix}-1\\0\\0\\-1\end{matrix}\right| + 2\left[\dfrac{(\mathfrak{g}^1\mathfrak{g}^4)}{p_0^2} - \dfrac{(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^4)}{p_0^4}\right]\left|\begin{matrix}0\\0\\1\\1\end{matrix}\right| + 2\left[\dfrac{(\mathfrak{g}^2\mathfrak{g}^4)}{p_0^2} - \dfrac{(\mathfrak{p}\mathfrak{g}^2)(\mathfrak{p}\mathfrak{g}^4)}{p_0^4}\right]\left|\begin{matrix}1\\1\\1\\1\end{matrix}\right| \\[2ex]
\qquad -2\dfrac{p_y^2}{p_0^4}\left[(\mathfrak{g}^1\mathfrak{g}^2)\left|\begin{matrix}5\\3\\3\\5\end{matrix}\right| + (\mathfrak{g}^1, \mathfrak{g}^1 + \mathfrak{g}^2)\left|\begin{matrix}1\\1\\1\\1\end{matrix}\right| - \dfrac{(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)}{p_0^2}\left|\begin{matrix}15\\7\\7\\15\end{matrix}\right| + \dfrac{(\mathfrak{p}\mathfrak{g}^4)(\mathfrak{p}, \mathfrak{g}^1 + \mathfrak{g}^2)}{p_0^2}\left|\begin{matrix}1\\1\\1\\1\end{matrix}\right|\right] \\[2ex]
\qquad -8\dfrac{p_y^4}{p_0^6}\left[-(\mathfrak{g}^1\mathfrak{g}^2) + \dfrac{4(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2)}{p_0^2} - 2\dfrac{(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^4)}{p_0^2} - \dfrac{(\mathfrak{p}\mathfrak{g}^2)(\mathfrak{p}\mathfrak{g}^4)}{p_0^2}\right]\left|\begin{matrix}1\\1\\1\\1\end{matrix}\right|.
\end{cases}
$$

Here, the development goes only up to terms of order two, since the terms, which will be summed over, will include two powers of the light quantum energies $g^3 g^4$ …, which appear only in the denominators (5.14).

In a similar way, the development of the reciprocal denominators (5.14) simplifies into:

$$
(9.10)\quad \left\{
\begin{aligned}
\begin{vmatrix} N_1^{-1} \\ N_2^{-1}+N_3^{-1} \\ N_4^{-1} \\ N_5^{-1}+N_6^{-1} \end{vmatrix}
&= \frac{g^1 g^2}{p_0^5}\begin{vmatrix} 6 \\ 12 \\ -4 \\ -4 \end{vmatrix} + \frac{g^1 g^4}{p_0^5}\begin{vmatrix} -4 \\ 12 \\ 6 \\ -4 \end{vmatrix} + \frac{g^2 g^4}{p_0^5}\begin{vmatrix} -2 \\ 0 \\ -2 \\ 0 \end{vmatrix} \\
&+ \frac{g^1 g^2}{p_0^7}\left[(\mathfrak{p}\mathfrak{g}^1)\begin{vmatrix} -12 \\ -30 \\ 14 \\ 10 \end{vmatrix} + (\mathfrak{p}\mathfrak{g}^2)\begin{vmatrix} -8 \\ -22 \\ 4 \\ \frac{7}{2} \end{vmatrix} + (\mathfrak{p}\mathfrak{g}^4)\begin{vmatrix} 3 \\ 8 \\ -3 \\ -\frac{7}{2} \end{vmatrix}\right] \\
&+ \frac{g^1 g^4}{p_0^7}\left[(\mathfrak{p}\mathfrak{g}^1)\begin{vmatrix} 6 \\ -30 \\ -18 \\ 8 \end{vmatrix} + (\mathfrak{p}\mathfrak{g}^2)\begin{vmatrix} 3 \\ -12 \\ -3 \\ \frac{5}{2} \end{vmatrix} + (\mathfrak{p}\mathfrak{g}^4)\begin{vmatrix} -4 \\ 18 \\ 8 \\ -\frac{9}{2} \end{vmatrix}\right] \\
&+ \frac{g^2 g^4}{p_0^7}\left[(\mathfrak{p}\mathfrak{g}^1)\begin{vmatrix} 3 \\ 0 \\ 7 \\ -1 \end{vmatrix} + (\mathfrak{p}\mathfrak{g}^2)\begin{vmatrix} 2 \\ -2 \\ 2 \\ -\frac{1}{2} \end{vmatrix} + (\mathfrak{p}\mathfrak{g}^4)\begin{vmatrix} -2 \\ -2 \\ -2 \\ -\frac{1}{2} \end{vmatrix}\right] \\
&+ \frac{(\mathfrak{g}^1\mathfrak{g}^2) g^2 g^4}{2 p_0^7}\begin{vmatrix} -21 \\ -30 \\ 9 \\ 5 \end{vmatrix} + \frac{(\mathfrak{p}\mathfrak{g}^1)(\mathfrak{p}\mathfrak{g}^2) g^3 g^4}{2 p_0^9}\begin{vmatrix} 91 \\ 70 \\ -119 \\ -21 \end{vmatrix},
\end{aligned}\right.
$$

in which the terms of order 0 and 1 can be omitted.

Multiplication of these numerators (9.9) and reciprocal denominators (9.10) yields the fourth order terms by taking the mean over the positron direction $\mathfrak{p}$:

$$
\sum_{\text{perm}}\sum_{\mu}\frac{\overline{Z}_\mu}{N_\mu} = -\frac{1}{2}\left[5 + \frac{25}{3}\frac{p^2}{p_0^2} - \frac{231}{3\cdot 5}\frac{p^4}{p_0^4} + \frac{99}{5\cdot 7}\frac{p^6}{p_0^6}\right]\sum_{\text{perm}}\frac{(\mathfrak{g}^1\mathfrak{g}^2) g^3 g^4}{p_0^7}
$$

and by integrating over the positron energy p_0:

(9.11)
$$\begin{cases} H_{in}^4 = \sum\limits_{\text{perm}} \sum\limits_{\mu} C \int d\mathfrak{p} \dfrac{Z_\mu}{N_\mu} \\ \quad = -4\pi C \sum\limits_{\text{perm}} \dfrac{(\mathfrak{g}^1\mathfrak{g}^2)g^3g^4}{(mc)^4} \left[\dfrac{5}{3\cdot 5} + \dfrac{25}{3\cdot 5\cdot 7} - \dfrac{231}{3\cdot 5\cdot 7\cdot 9} + \dfrac{99}{5\cdot 7\cdot 9\cdot 11} \right] \\ \quad = -4\pi C \left(\dfrac{16}{5\cdot 9} \right) \sum\limits_{\text{perm}} \dfrac{(\mathfrak{g}^1\mathfrak{g}^2)g^3g^4}{(mc)^4}. \end{cases}$$

With this, the definitive term of the Dirac matrix element for scattering of light by light for parallel polarization is calculated.

The matrix element for the corresponding terms in the light quantum interaction (2.21) is, from (9.5):

(9.12)
$$\alpha \frac{\hbar c}{e^2} \frac{1}{E_0^2} D(-2) \sum_{\text{perm}} (\mathfrak{g}^1\mathfrak{g}^2) g^3 g^4 .$$

Setting both expressions (9.11), (9.12) equal to each other determines the coefficient α:

$$\alpha = -\frac{1}{360\pi^2}$$

in agreement with the earlier calculation.

§ 10. **Discussion of the results**

We can regard our method as having been confirmed and summarize the result in the following way:

Just as in the ordinary Maxwell theory an electron is surrounded by an electromagnetic field, so is the light quantum in hole theory surrounded by a matter field. The Hamilton function of such a field is thus the sum over the energies of light and matter, so it includes the degrees of freedom: field strengths *and* charge waves.

However, just as in the special case in which no transversal light quanta are present and the electron moves so slowly that nothing can be created, this Hamilton function, includes only the degrees of freedom of the electrons (whose energy then replaces the electromagnetic field and whose creation will be indicated by the Coulomb interaction of the electrons): likewise, in the special case considered here, in which no actual electrons are present and the energy of the light quanta is not sufficient for the creation of electron pairs (0.1), one can approximately replace the energy of the total field with a Hamilton function that depends on the field strengths $\mathfrak{E}$, $\mathfrak{B}$ alone:

(10.1) $$\bar{U} = \int U\,dV \qquad \text{for} \left\{ \begin{array}{l} |\mathfrak{D}| \ll E_0, \quad (\text{grad}\,\mathfrak{D}_i)^2 - \left(\dfrac{1}{c}\dfrac{\partial \mathfrak{D}_i}{\partial t}\right)^2 \ll 2\left(\dfrac{\hbar}{mc}\right)^{-2} \mathfrak{D}_i^2 \\ |\mathfrak{B}| \ll E_0, \quad (\text{grad}\,\mathfrak{B}_i)^2 - \left(\dfrac{1}{c}\dfrac{\partial \mathfrak{B}_i}{\partial t}\right)^2 \ll 2\left(\dfrac{\hbar}{mc}\right)^{-2} \mathfrak{B}_i^2 \end{array} \right\}$$

(10.2) $$U = \frac{\mathfrak{D}^2 + \mathfrak{B}^2}{8\pi} - \frac{1}{360\pi^2}\frac{\hbar c}{e^2}\frac{1}{E_0^2}[(\mathfrak{B}^2 - \mathfrak{D}^2)^2 + 7(\mathfrak{B}\mathfrak{D})^2] \qquad \left(E_0 = \frac{e}{\left(\dfrac{e^2}{mc^2}\right)^2} \right)$$

(10.3) $$\mathfrak{D}_i(\xi)\mathfrak{B}_k(\xi') - \mathfrak{B}_k(\xi')\mathfrak{D}_i(\xi) = 2hci\,\delta(\xi - \xi').$$

This Hamilton function $\bar{U}$ is to be regarded as the beginning of a development that corresponds to the powers of the field strengths up to order four (corresponding to the development of the Dirac theory in the electron charge up to fourth order) and is carried out in the degree of the derivative of the field strengths up to order zero (corresponding to the development of the Dirac matrix element up to order four in the light frequencies g^i/mc).

The addition to the Maxwell energy in (10.2) is an interaction between the light quanta, which refers to the creation of virtual matter and replaces the energy of the matter field that surrounds the light quanta. The approximation that is considered here (in which the derivatives of the field strengths are neglected) describes a local interaction of the light quanta. The canonical equations that correspond to this Hamilton function (10.2) read:

(10.4) $$\left\{ \begin{array}{l} \dfrac{1}{c}\dot{\mathfrak{B}} + \text{rot}\,\mathfrak{E} = 0, \quad \text{div}\,\mathfrak{B} = 0; \\ -\dfrac{1}{c}\dot{\mathfrak{D}} + \text{rot}\,\mathfrak{h} = 0, \quad \text{div}\,\mathfrak{D} = 0; \\ \quad \mathfrak{D} = \mathfrak{E} + \dfrac{1}{90\pi}\dfrac{\hbar c}{e^2}\dfrac{1}{E_0^2}[4(\mathfrak{E}^2 - \mathfrak{B}^2)\mathfrak{E} - 14(\mathfrak{E}\mathfrak{B})\mathfrak{B}] \\ \quad \mathfrak{h} = \mathfrak{B} + \dfrac{1}{90\pi}\dfrac{\hbar c}{e^2}\dfrac{1}{E_0^2}[4(\mathfrak{E}^2 - \mathfrak{B}^2)\mathfrak{B} - 14(\mathfrak{E}\mathfrak{B})\mathfrak{E}] \end{array} \right.$$

or, in other notation:

$$(10.5)\quad \begin{cases} \frac{1}{c}\dot{\mathfrak{B}} + \text{rot}\,\mathfrak{E} = 0, \quad \text{div}\,\mathfrak{B} = 0; \\ -\frac{1}{c}\dot{\mathfrak{E}} + \text{rot}\,\mathfrak{B} = 0, \quad \text{div}\,\mathfrak{E} = 4\pi\rho; \\ \frac{4\pi i}{c} = \frac{1}{90\pi}\frac{\hbar c}{e^2}\frac{1}{E_0^2}\begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t}[4(\mathfrak{E}^2 - \mathfrak{B}^2)\mathfrak{E} + 14(\mathfrak{E}\mathfrak{B})\mathfrak{B}] \\ -\text{rot}\,[4(\mathfrak{E}^2 - \mathfrak{B}^2)\mathfrak{B} - 14(\mathfrak{E}\mathfrak{B})\mathfrak{E}] \end{bmatrix} \\ 4\pi\rho = \frac{1}{90\pi}\frac{\hbar c}{e^2}\frac{1}{E_0^2}\text{div}\,[-4(\mathfrak{E}^2 - \mathfrak{B}^2)\mathfrak{E} - 14(\mathfrak{E}\mathfrak{B})\mathfrak{B}]. \end{cases}$$

They can also be derived from the variational principle $\iint L\,dV\,dt$ = extremum for the Lagrange function:

$$(10.6)\qquad L = \frac{\mathfrak{E}^2 - \mathfrak{B}^2}{2} + \frac{1}{90\pi}\frac{\hbar c}{e^2}\frac{1}{E_0^2}[(\mathfrak{E}^2 - \mathfrak{B}^2)^2 + 7(\mathfrak{E}\mathfrak{B})^2]$$

under the associated condition $\mathfrak{E} = -\frac{1}{c}\dot{\mathfrak{A}}$, $\mathfrak{B}$ = rot $\mathfrak{A}$. In the first form (10.4), these equations will refer to the coupling of the field with the virtual matter by a coupling of the electrical field strength $\mathfrak{E}$ with the electrical displacement $\mathfrak{D}$ and the magnetic induction $\mathfrak{B}$ with the quantity $\mathfrak{H}$, just as in the electrodynamics of polarizable bodies this will represent the coupling of the actual matter with the field.

In the second notation (10.5), the virtual matter that is created by the field $\mathfrak{E}$, $\mathfrak{B}$ enters directly in the form of the apparent density $\rho = \rho(\mathfrak{E}, \mathfrak{B})$ and the apparent current $i = i(\mathfrak{E}, \mathfrak{B})$. Moreover, this nomenclature (10.5) shows that the equations (10.2, …, 10.6) that were assumed here are in agreement with the general equations [1]) for light and matter, except with the matter ρ, i being replaced with particular functions of the field strengths that produce it.

As one easily deduces from (10.2), (10.3), or (10.4), one has the conservation laws:

$$(10.8)\quad \begin{cases} \frac{1}{c}\frac{\partial U}{\partial t} + \text{div}\frac{[\mathfrak{E}\mathfrak{B}]}{4\pi} = 0; \\ \qquad\qquad\qquad T_{xx} = U - \frac{1}{4\pi}(\mathfrak{E}_y\mathfrak{D}_y + \mathfrak{E}_z\mathfrak{D}_z + \mathfrak{H}_y\mathfrak{B}_y + \mathfrak{H}_z\mathfrak{B}_z) \\ \frac{1}{c}\frac{\partial}{\partial t}\frac{[\mathfrak{B}\mathfrak{D}]_k}{4\pi} + \frac{\partial}{\partial x_i}T_{ik} = 0; \quad T_{xy} = \frac{1}{4\pi}(\mathfrak{E}_y\mathfrak{D}_y + \mathfrak{H}_x\mathfrak{B}_y) \\ \qquad\qquad\qquad T_{ik} = T_{ki}, \quad [\mathfrak{E}\mathfrak{H}] = [\mathfrak{D}\mathfrak{B}], \end{cases}$$

[1]) W. Heisenberg and W. Pauli, Zeit. f. Phys. **56**, pp. 1, 1930; **59**, pp. 168, 1930.

which shows that $[\mathfrak{E}\mathfrak{B}]/4\pi = [\mathfrak{D}\mathfrak{B}]/4\pi$, the energy current and impulse density, are in agreement with the general formulas of quantum dynamics of wave fields [1]).

The equations are nonlinear, i.e., there is no corresponding superposition principle for them and they describe a scattering of light by light that becomes large when the fields have sufficiently strong field strengths (compared to E_0) and sufficiently short wavelengths (compared to $\hbar/mc$).

One obtains the interaction cross-section dQ for the scattering of light by light when one takes the square of the matrix element H_{in}^4 of the interaction in (10.2) for the transition of two light quanta with impulses $\mathfrak{g}^1$, $\mathfrak{g}^2$, energies cg^1, cg^2, and polarizations $\mathfrak{e}^1$, $\mathfrak{e}^2$ into two other ones with impulses $-g^3$, $-g^4$, energies $-cg^3$, $-cg^4$, and polarizations $\mathfrak{e}^3$, $\mathfrak{e}^4$, and multiplies this by the number $1/\Delta W$, which is the number of secondary light quantum pairs $-g^3$, $-g^4$ per energy interval $c(|\mathfrak{g}^3| + |\mathfrak{g}^4|)$ for the spatial angle $d\Omega_4$ around $\mathfrak{g}^4$, and finally multiplies it by $\frac{2\pi}{\hbar}\cdot\frac{V}{c}$:

$$dQ = \frac{V}{c}\frac{2\pi}{\hbar}\frac{1}{\Delta W}\left|H_{in}^4\right|^2,$$

$$= d\Omega_4\frac{V}{c\hbar^3}\frac{|\mathfrak{g}^4|^2}{1-\cos\mathfrak{g}^3\mathfrak{g}^4},$$

$$= -\frac{1}{360\pi^2}\frac{\hbar c}{e^2}\frac{1}{E_0^2}\left(\frac{ch\hbar}{V}\right)^2\frac{1}{\hbar^4}\frac{V}{\sqrt{g^1g^2g^3g^4}}\sum_{\substack{\text{perm}\\1,2,3,4}}\left\{\begin{matrix}(\mathfrak{e}^1\mathfrak{e}^2)(\mathfrak{e}^3\mathfrak{e}^4)g^1g^2g^3g^4\\ -2([\mathfrak{e}^1\mathfrak{g}^1][\mathfrak{e}^2\mathfrak{g}^2])(\mathfrak{e}^3\mathfrak{e}^4)g^3g^4\\ +([\mathfrak{e}^1\mathfrak{g}^1][\mathfrak{e}^2\mathfrak{g}^2])([\mathfrak{e}^3\mathfrak{g}^3][\mathfrak{e}^4\mathfrak{g}^4])\\ +7|\mathfrak{e}^1\mathfrak{e}^2\mathfrak{g}^2||\mathfrak{e}^3\mathfrak{e}^4\mathfrak{g}^4|g^1g^2\end{matrix}\right\}$$

$$\left(\begin{matrix}\mathfrak{g}^1+\mathfrak{g}^2+\mathfrak{g}^3+\mathfrak{g}^4=0\\ g^1+g^2+g^3+g^4=0\end{matrix}\right)$$

$$\left\{\begin{aligned} dQ &= \frac{d\Omega_4}{(180\pi)^2}\left(\frac{e^2}{mc^2}\right)^2\left(\frac{e^2}{\hbar c}\right)^2\frac{1}{(mc)^2}\\ &\cdot\frac{g^4}{g^1g^2g^3\left(1-\cos(\mathfrak{g}^3\mathfrak{g}^4)\right)}\left|\sum_{\substack{\text{perm}\\1,2,3,4}}\left\{\begin{matrix}(\mathfrak{e}^1\mathfrak{e}^2)(\mathfrak{e}^3\mathfrak{e}^4)g^1g^2g^3g^4\\ -2([\mathfrak{e}^1\mathfrak{g}^1][\mathfrak{e}^2\mathfrak{g}^2])(\mathfrak{e}^3\mathfrak{e}^4)g^3g^4\\ +([\mathfrak{e}^1\mathfrak{g}^1][\mathfrak{e}^2\mathfrak{g}^2])([\mathfrak{e}^3\mathfrak{g}^3][\mathfrak{e}^4\mathfrak{g}^4])\\ +7|\mathfrak{e}^1\mathfrak{e}^2\mathfrak{g}^2||\mathfrak{e}^3\mathfrak{e}^4\mathfrak{g}^4|g^1g^3\end{matrix}\right\}\right|^2.\end{aligned}\right. \tag{10.9}$$

In order to give an example, we calculate the interaction cross-section for the case in which two light quanta of equal energies, opposite impulses, and the same polarizations collide with each other and turn into the same light quanta with the same polarizations. For the wavelength λ and the scattering angle φ one then has:

$$dQ = \frac{2d\Omega}{5^2 \cdot 3^4}[3+\cos^2\varphi]^2 \left(\frac{e^2}{mc^2}\right)^4 \left(\frac{\hbar}{mc}\right)^4 \frac{1}{\lambda^6}. \tag{10.10}$$

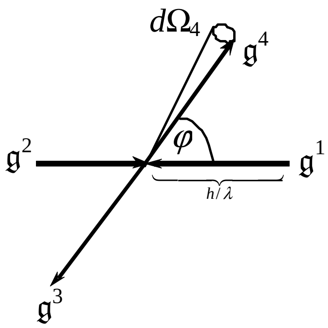

Fig. 3

The interaction radius for the scattering of light by light is then of the order of magnitude 10^{-15} cm for γ-rays, 10^{-24} for Röntgen rays, and 10^{-36} for visible light, and is therefore quite difficult to establish experimentally.

The fact that (despite the representation by classical field functions) it will be treated here as a purely quantum mechanical effect has already been suggested by the fact that the additional term in (10.2) is proportional to *h*.

It is now interesting to compare the deviation calculated here for the Maxwell equations due to the quantum-theoretic possibility of pair production with the one given by Born [1]) as a consequence of the classical theory.

As is well known, Born, on the grounds of the fact that the classical Maxwell equations give an infinite field energy to the electron, set down alternative field equations in which he could determine a certain constant in such a manner that the field of a point charge e possessed an energy of mc^2, and then quantized these equations in such a manner that they are described by their properties as a canonical system. Born's theory, when developed in field strengths, has for its first term, the Hamilton function:

$$\bar{U} = \int \frac{\mathfrak{B}^2 + \mathfrak{D}^2}{8\pi} dV - \frac{(1.236)^4}{32\pi}\frac{1}{E_0^2}\int [(\mathfrak{B}^2 - \mathfrak{D}^2)^2 + 4(\mathfrak{B}\mathfrak{D})^2]\, dv. \tag{10.11}$$

On the grounds of invariance, the additional terms in the Maxwellian energy in Born's theory (10.11) and in Dirac's theory (10.2) agree up to numerical coefficients. The numerical coefficients differ in the two theories by the factors:

$$\frac{4}{45\pi \cdot (1.236)^4}\frac{\hbar c}{e^2} \text{ for the term } (\mathfrak{B}^2 - \mathfrak{D}^2)^2$$

and:

$$\frac{7}{45\pi \cdot (1.236)^4}\frac{\hbar c}{e^2} \text{ for the term } (\mathfrak{B}\mathfrak{D})^2 .$$

Due to the actual value for the Sommerfeld fine-structure constant, the numerical values of these factors are 1.7 and 2.9, resp. Moreover, it is noteworthy that the quantum-theoretic deviation from the Maxwell equations is, in any event, of the order of magnitude of the classical expression that one would expect on the basis of self-energy.

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[1]) M. Born, M. Born and L. Infeld, Proc. Roy. Soc. London [A] **143**, pp. 410, 1933; [A] **144**, pp. 425, 1934; [A] **147**, pp. 522, 1934.

This agreement in the order of magnitude naturally means that in Dirac's theory the question of self-energy will really be solved when one goes to sufficiently high-order terms in its development in electron charge.

However, regarding the question of the convergence, one knows that the consideration of higher-order terms in this development can lead to another situation, as one finds in the lower-order approximations.

As eq. (10.2), (10.4) shows, it does not, by any means, always need to be permissible to truncate the usual development that one writes down for the field theory in the electron charge e (i.e., here, in the field strength $\mathfrak{E}/E_0 \cdot \hbar / mc$) after the first non-vanishing term. If the fourth-order terms that were written down here can, (as is true, as Weisskopf has remarked, for the terms in the derivatives of the field strengths, which are no longer written-down, for sufficiently short wavelengths), for sufficiently strong fields, be given quite nicely by a contribution that is of the same order of magnitude as the previous terms of order 1, 2, and 3 that have sufficed for all problems up till now, i.e., the development in the coupling e of light and matter need not converge when (say, for fast particles or in the neighborhood of an electron) this coupling becomes too intimate through matter creation.

One almost gets the impression [1]), in the development of scattering theory in the electron charge, that something similar is going on, as when one would calculate a finite expression for:

$$\int_0^1 \sqrt{1+\frac{e}{r}}\,dr$$

through a development:

$$\int_0^1 \sqrt{1+\frac{e}{r}}\,dr = \int_0^1 dr + e\int_0^1 \frac{dr}{2r} - e^2 \int_0^1 \frac{dr}{8r^2} + \cdots,$$

whose individual terms diverge with increasing order.

The results of this work thus mean that in order to go further into the principal difficulties of scattering theory one must next seek to replace the development in $e^2 / \hbar c$ with something else.

I would like to express my heartfelt thanks to Herrn Prof. Heisenberg for his essential assistance and his continuing interest in my work. Moreover, I thank Herrn Kockel for his collaboration, without which the execution of the calculations would have been completely impossible.

Leipzig, Physical Institute of the University, on 21 June 1935.

(Received 28 January 1936)

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[1]) Following Prof. W. Heisenberg.