“Correzione di una contradizione tra la teoria elettrodinamica e quella relativistica delle masse elettromagnetiche,” Nouv. Cim. **25** (1923), 159-170; *Collected Papers (Nota e memorie)*, v. I, U. of Chicago Press, 1962, pp. 26-32.

Correction to a contradiction between the electrodynamical theory and the relativistic theory of electromagnetic mass (*)

By E. Fermi

Translated by D. H. Delphenich

---

§ 1. – The theory of electromagnetic mass was studied for the first time by M. Abraham ([1]) before the discovery of the theory of relativity. Thus, it was only natural for Abraham to consider the mass of a rigid system of electric charges in the sense of classical mechanics in his calculations and found that under the hypothesis that such a system had spherical symmetry, its mass would vary with the velocity and be equal to precisely ([2]) $\frac{4}{3}\frac{u}{c^2}$ (in which u is the electrostatic energy of the system and c is the speed of light) for a velocity that is zero or very small, while for a velocity that is comparable to c, some correction terms will intervene that are somewhat more complicated and have the order of magnitude of v^2 / c^2. Also before the theory of relativity, Fitzgerald introduced the hypothesis that solid bodies will submit to a contraction with a ratio of:

$$\sqrt{1-\frac{v^2}{c^2}}:1$$

in the direction of its motion, and Lorentz reprised Abraham’s theory of electromagnetic mass, but while considering the systems of electric charge that were subject to that contraction, rather than systems that were rigid in the sense of classical mechanics. It would then result that the rest mass, or the limit of the mass for zero velocity, would always be $\frac{4}{3}\frac{u}{c^2}$, while the correction terms that depended upon v^2 / c^2 would be altered. The experiments of Kaufmann, Bucherer, *et al*, regarding the mass of the β particle of radioactive particles, and that of cathode rays of large velocity, decided neatly in favor of Lorentz’s theory of the contractible electron, so to speak, and against Abraham’s

(*) See my notes on the same argument in Rend. Acc. Lincei (5) **81** (1922), pp. 184, 306.

([1]) **Abraham**, *Theorie der Electricität*; **Richardson**, *Electron Theory of Matter*, Chap. XI; **Lorentz**, *The Theory of Electrons*, pp. 37.

([2]) One ordinarily says that the electromagnetic mass of a homogeneous spherical electric layer of charge e and radius r is $\frac{2}{3}\frac{e^2}{rc^2}$. However, if one observes that the electrostatic energy is $u = \frac{1}{2}\frac{e^2}{r}$ then one will find that the mass is precisely $= \frac{4}{3}\frac{u}{c^2}$.

theory of the rigid electron. One can then interpret that as saying that it was the first proof that the mass of electrons is exclusively electromagnetic in nature, since otherwise one would have to think that their masses would have to be constant. When the theory of relativity was discovered, in turn, that led to the consequence that all mass, whether electromagnetic or not, must vary with the velocity like the Lorentz contraction of the electron in such a way that experiments indicated that the question of whether the electronic mass was or was not totally electromagnetic in nature was still left unresolved, since they only constituted a confirmation of the theory of relativity. On the other hand, that theory of relativity, in the strict sense (and even more so as a result of the general one), leads one to attribute a mass of $u : c^2$ to given system of energy u in such a way that a grave discrepancy arose between Lorentz's electrodynamical theory, which attributed a rest mass of $\frac{4}{3}\frac{u}{c^2}$ to a spherical distribution of electricity, and the theory of relativity, which attributed a mass of u / c^2 instead, and such a difference ([3]) proves to be particularly serious when one considers the great importance of the notion of electromagnetic mass as the basis for the electronic theory of matter.

I presented that discrepancy in a particularly strident way in two recent notes ([4]). In one of them, on the basis of the ordinary theory of electrodynamics, I considered the electromagnetic mass of a system with arbitrary symmetry and found that in general it is represented by a tensor, rather than a scalar, that naturally reduces to $\frac{4}{3}\frac{u}{c^2}$ in the case of spherical symmetry. However, in the other one, starting from the general theory of relativity, I considered the weight of that system, which was found to be equal to $\frac{u}{c^2}G$, in any case, where G is the acceleration of gravity.

The present work will show precisely that the difference between the two values of the mass that are obtained in the two ways has its origin in the concept of a rigid body, which contradicts the principle of relativity, as it applies to the theory of electrodynamics (as well as that of the contractible electron) and which leads to a mass of $\frac{4}{3}\frac{u}{c^2}$, while the notion of a rigid body that is better justified and conforms to the theory of relativity will lead to the value u / c^2.

One should further note that the relativistic dynamics of the electron that was developed by M. Born ([5]), which nonetheless started from a viewpoint that is not essentially different from the usual one, naturally found that the rest mass was $\frac{4}{3}\frac{u}{c^2}$.

We shall take Hamilton's principle to be the basis for our considerations, as it is more adapted to the study of a problem that is subject to constraints that are somewhat complicated. However, our system of electric charges must satisfy a constraint that is, by nature, different from the ones that are considered in ordinary mechanics, namely, that depending upon its velocity, it must exhibit

([3]) Naturally, the experiments of Kaufmann, *et al*, cannot serve to resolve the question of which of the two results is correct in this case, because they only permit one to measure the correction terms that depend upon velocity and are equal in both theories, while the difference exists between the rest masses.

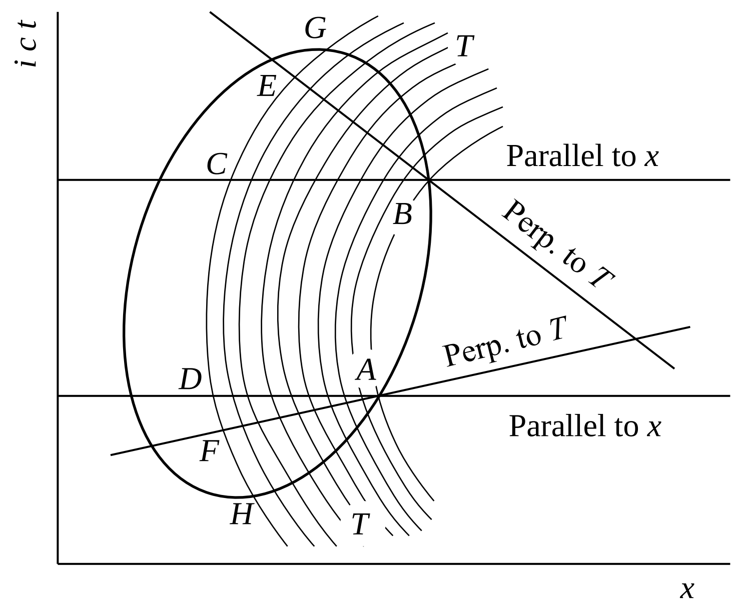
([4]) **E. Fermi**, Nouv. Cim. (6) **22** (1921), pp. 176, 192.

([5]) **Max Born**, Ann. Phys. (Leipzig) **30** (1909), pp. 1.

the Lorentz contraction as a consequence of the principle of relativity. However, from now on, in order to avoid confusion, note that whereas the Lorentz contraction has order v^2 / c^2, its influence on the electromagnetic mass concerns its principal terms, i.e., the rest mass, and therefore has much greater importance, because it is also appreciable for small velocities.

§ 2. – Now consider a system of electric charges that is suspended in a rigid dielectric that moves with a translatory motion that describes a time-tube in space-time ([6]) under the action of an electromagnetic field that is due, in part, to that system itself and in part to external cases.

More precisely, one sees that this implies a rigid translatory motion. Therefore, consider an arbitrary Lorentz-Einstein reference system and suppose that a point of the system of charges has velocity zero in it at a certain instant. That is equivalent to saying that the timelines of the points of our system are orthogonal trajectories to a family of linear spaces, and in fact, in a Lorentz-Einstein reference system in which space is one of the spaces in the family, and the time axis is naturally perpendicular to it, the entire system will be at rest at time zero, because space cuts the timeline orthogonally at all points of the system. With that definition of translatory motion, which is, in substance, the one that M. Born adopted, the rigidity of the system is expressed by the idea that its figure will remain invariable in those spaces that are perpendicular to the tube, or that all of the cross-sections of the tube are mutually equal.



In order to be able to apply Hamilton's principle to our case, we must perform a variation of the motion of our system that conforms to the constraints in the problem, i.e., rigidity, when properly interpreted. Now, we can show that if we add the value $\frac{4}{3}\frac{u}{c^2}$, or that of u / c^2, for the

([6]) In all of what follows, one regards space-time as being Euclidian, since it is intended that the electromagnetic fields that exist in it are considered to be sufficiently small in intensity that it will not alter the metric structure appreciably.

electromagnetic mass, according to whether one takes that variation to be one or the other of the two that are illustrated above and will be distinguished by the letters *A* and *B*, resp. However, the variation *A* must be discarded, as it contradicts the principle of relativity. Let *T* be the time-tube that is described by the system. In the figure, space (x, y, z) is represented as only one-dimensional by way of the *x*-axis, and the time *t* is replaced with *ict* in order to give a definite metric.

Variation A: One considers a variation that satisfies the constraint of rigidity to be an infinitesimal displacement (i.e., rigid in the usual kinematical sense) that is parallel to the space (x, y, z) of any section of the tube that is parallel to that space. In the figure, one will then obtain such a variation by displacing any section t = const. of the tube parallel to the *x*-axis by an arbitrary infinitesimal segment. If one confines oneself to the consideration of translatory displacements then one will have that δx, δy, δz are arbitrary functions of only time, and $\delta t = 0$.

Variation B: One considers a variation that satisfies the constraint of rigidity to be infinitesimal displacement of any normal section of the tube that is perpendicular to that tube and is rigid in the usual kinematical sense. In the figure, one will obtain that variation by displacing any normal section of the tube parallel to itself through an arbitrary segment.

Of those two variations, *A obviously contradicts the principle of relativity* and must then be discarded since it is not even invariant with respect to Lorentz transformations. Indeed, it is determined by the particular choice of reference frame (t, x, y, z), so it cannot express any physical notion such as that of rigidity. However, the variation *B*, other than obviously satisfying the aforementioned condition of invariance, because it is composed of only elements that are inherent to the tube *T* and things that are independent of the position of the reference axes, is the only one that spontaneously presents itself as the one that takes its basis to be a virtual displacement that is rigid in the reference system with respect to which the system of charges has zero velocity at the instant in question. Now, a superficial observation can nonetheless create the impression that the difference between the consequences of the two systems of variation *A* and *B* must become perceptible only for considerable velocities, i.e., when the tube *T* has a noticeable inclination with respect to the time axis. However, the calculations that we shall develop will show immediately that the difference is already perceptible for zero velocity, and that it is precisely *A* that will give the electromagnetic mass as $\frac{4}{3}\frac{u}{c^2}$, while *B* will give $\frac{u}{c^2}$.

§ 3. – For the sake of convenience, let (t, x, y, z) or (x_0, x_1, x_2, x_3) denote the coordinates of time and space, let φ_i be the quadri-potential, and let:

$$F_{ik} = \frac{\partial \varphi_i}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_i}$$

be the electromagnetic field, while **E** and **B** are the electric and magnetic force, resp., that are deduced from it.

Hamilton's principle, which subsumes the laws of Maxwell-Lorentz and those of mechanics, says that ([7]): The total action, or the sum of the actions of the electromagnetic field and those of the material and electric masses, will have zero variation under the effect of an arbitrary variation of the φ_i and the coordinates of the points of the timelines of the electric charges that conforms to the constraints and is annulled on the contour of the region of integration. In our case, there are no material masses, so the only elements that must be varied are the coordinates of the points of the timelines of the charges. It will therefore be enough to consider only the action of the electric charge, i.e.:

$$W = \sum_i \int de \int \varphi_i \, dx_i \ ,$$

in which *de* is the generic element of electric charge, and the second integral must be extended over the arc of the timeline that is described by *de* and is contained in the quadri-dimensional domain G of integration. For any system of variations δx_i that conforms to the constraints and *is annulled on the contour of G*, one must then have $\delta W = 0$, i.e.:

$$\sum_{i,k} \iint de \, F_{ik} \, \delta x_i \, dx_k = 0 \ . \tag{1}$$

We need to examine separately the results that are obtained replacing the δx_i with the given values of the systems of variation A or B.

§ 4. Consequences of the system of variations *A*. – In this case, the domain of integration reduces to simply *ABCD*, and indeed, the regions *BCG*, *ADH* give zero contributions, since all of the δx_i are annulled in them by being zero on the contour of G, and therefore along the line segments *BG*, *AH*, and will have constant values for constant t, or only parallels to the x-axis. If t_1 and t_2 denote the times of A and B, resp., then (1) can be written as:

$$\sum_{i,k} \int_{t_1}^{t_2} dt \, \delta x_i \int de \, F_{ik} \frac{dx_k}{dt} \qquad (i = 1, 2, 3 \ ; \ k = 0, 1, 2, 3)$$

when $\delta t = 0$ and δx, δy, δz are functions of only time.

Since δx_i are arbitrary functions of t, one will then get the three equations:

$$\int de \sum_k F_{ik} \frac{dx_k}{dt} = 0$$

or

([7]) WEYL, *Raum, Zeit, Materie*, Berlin, Springer, 1921, pp. 194-196.

$$\int de\left[E_x + \frac{dy}{dt}H_z - \frac{dz}{dt}H_y\right] = 0 ,$$

and two analogous ones.

If our system has zero velocity in the reference frame (t, x, y, z) at the instant considered then the three equations can be summarized in the single vectorial one:

$$\int \mathbf{E}\, de = 0 . \tag{2}$$

We could have arrived at that equation without calculation if (as one does in the ordinary treatments and, in substance, as M. Born did in the cited paper) we suppose *a priori* that the total force that acts on the system is zero. We specifically wished to deduce it from Hamilton's principle in order to show the flaw in its origin, since it follows from the system of variations A that it contradicts the principle of relativity. The value $\frac{4}{3}\frac{u}{c^2}$ for the electromagnetic mass follows immediately from (2). Indeed, suppose that $\mathbf{E}$ is the sum of a part $\mathbf{E}^{(i)}$ that is due to the system itself and a uniform field $\mathbf{E}^{(s)}$ that is due to external causes. (2) will then give:

$$\int \mathbf{E}^{(i)}\, de + \mathbf{E}^{(s)} \int de = 0 .$$

Now, $\int de = e$ = charge, and therefore $\mathbf{E}^{(s)} \int de = \mathbf{F}$ = external force. On the other hand, in the case of spherical symmetry, either direct calculation or the well-known consideration of the electromagnetic moment ([8]) will show that:

$$\int \mathbf{E}^{(i)}\, de = -\frac{4}{3}\frac{u}{c^2}\boldsymbol{\Gamma} ,$$

in which $\boldsymbol{\Gamma}$ is the acceleration.

The preceding equation will then become:

$$\mathbf{F} = \frac{4}{3}\frac{u}{c^2}\boldsymbol{\Gamma} ,$$

and when that is compared to the fundamental law of the dynamical of the point, namely, $\mathbf{F} = m\boldsymbol{\Gamma}$, that will give:

$$m = \frac{4}{3}\frac{u}{c^2} .$$

([8]) **Richardson**, *loc. cit.*

§ 5. Consequences of the system of variations *B*. – In that case, the same considerations as in the preceding § will show that the domain of integration reduces to *ABEF*, or the region that is found between two normal sections of the tube *T*. One decomposes it into an infinitude of layers of infinitesimal thickness by means of an infinitude of normal sections, and in order to calculate the contribution of one of them to the integral (1), one refers it to the rest frame and takes the space of (x, y, z) be parallel to the layer. From that, one will then have $\delta t = 0$, while δx, δy, δz will be arbitrary constants. In addition, one will have $dx = dy = dz = 0$, because the velocity of any of its points is zero, while dt = thickness of the layer, which varies from point to point, because the layer is based upon two normal sections that are not generally parallel. If O is a point of the layer that is fixed, but generic (for example, the coordinate origin) at which dt has the value dt_0, and $\mathbf{K}$ is the vector that is oriented along the principal normal to the timeline that passes through O and has a magnitude that equals the curvature of the timeline then one will obviously have:

$$dt = dt_0\,[1 - \mathbf{K}\cdot(P - O)]\,,$$

if dt is the thickness of the layer at the generic point P.

Since the velocity is zero, one will have simply:

$$\mathbf{K} = -\,\mathbf{\Gamma} : c^2\,,$$

and therefore:

$$dt = dt_0\left(1 + \frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}\right).$$

If we substitute those values then we will find that the contribution of our layer to the integral (1) is:

$$-\,dt_0\left\{\delta x\int\left(1+\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}\right)E_x\,de+\delta y\int\left(1+\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}\right)E_y\,de+\delta z\int\left(1+\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}\right)E_z\,de\right\}.$$

That expression must be annulled for all values of δx, δy, δz, so we will get three equations from it that can be summarized in the single vectorial one:

(3) $$\int\left(1+\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}\right)\mathbf{E}\,de = 0\,.$$

A correct application of Hamilton's principle has then led from (3) to (2), instead. It is then easy to examine the consequences. Indeed, if one sets:

$$\mathbf{E} = \mathbf{E}^{(i)} + \mathbf{E}^{(s)}$$

then one will find that:

$$\int \mathbf{E}^{(i)} de + \int \mathbf{E}^{(i)} \frac{\mathbf{\Gamma} \cdot (P-O)}{c^2} de + e\,\mathbf{E}^{(s)} + \mathbf{E}^{(s)} \int \frac{\mathbf{\Gamma} \cdot (P-O)}{c^2} de = 0 .$$

In that case of spherical symmetry, one will have:

$$\int \mathbf{E}^{(i)} de = -\frac{4}{3}\frac{u}{c^2}\mathbf{\Gamma} ,$$

as above. If one substitutes that in the preceding equation then one will find that $\mathbf{E}^{(s)}$ is composed of only terms that contain $\mathbf{\Gamma}$. If one then neglects the terms ([9]) in Γ^2 then one can neglect the last integral, and one will get:

$$-\frac{4}{3}\frac{u}{c^2}\mathbf{\Gamma} + \int \mathbf{E}^{(i)} \frac{\mathbf{\Gamma} \cdot (P-O)}{c^2} de + \mathbf{F} = 0 . \tag{4}$$

In order to calculate the integral that once more figures in (4), observe that $\mathbf{E}^{(i)}$ is the sum of the Coulomb force, which is:

$$= \int \frac{P-P'}{r^3} de'$$

(P' is the point where the charge de' is located, and $r = \overline{PP'}$), and a term that contains $\mathbf{\Gamma}$, which can be neglected because it will give a contribution that contains Γ^2. Our integral will then give:

$$\iint \frac{P-P'}{r^3} \frac{\mathbf{\Gamma} \cdot (P-O)}{c^2} de\, de' ,$$

or if we switch P with P' (which will change nothing) and takes one-half the sum of the two values thus-obtained:

$$\tfrac{1}{2} \iint \frac{P-P'}{c\, r^3} [\mathbf{\Gamma} \cdot (P-P')]\, de\, de' .$$

Observe that in our approximation, $\mathbf{\Gamma}$ is constant for all points, so it can be taken outside of the integrals. Therefore, the x-component of the preceding integral will be:

$$\frac{1}{2c^2} \left\{ \Gamma_x \iint \frac{(x-x')^2}{r^3} de\, de' + \Gamma_y \iint \frac{(y-y')(x-x')}{r^3} de\, de' + \Gamma_z \iint \frac{(z-z')(x-x')}{r^3} de\, de' \right\} .$$

Now, since the system has spherical symmetry, any segment PP' will correspond to an infinitude of other ones that are distinguished only by their orientations. We can then replace:

([9]) Properly speaking, the magnitude of the squares that one neglects is $\Gamma l / c^2$, in which l is the maximum length that enters into the problem. It is obvious that such an approximation is more than justified in the common cases.

$$(x-x')^2, \qquad (x-x')(y-y'), \qquad (x-x')(z-z')$$

in the three integrals with their mean values over all possible orientations of PP', which are: $\frac{1}{3}r^2$, 0, 0.

With that, the x-component will become:

$$\frac{\Gamma_x}{3c^2}\frac{1}{2}\iint\frac{de\,de'}{r} .$$

Now observe that the expression:

$$\frac{1}{2}\iint\frac{de\,de'}{r}$$

is nothing but the electrostatic energy u. If we then revert to the vectorial notation, we will find that the integral that figures in (4) is expressed by: $\frac{u}{3c^2}\mathbf{\Gamma}$. (4) will then become:

$$\frac{u}{c^2}\mathbf{\Gamma} = \mathbf{F} , \tag{5}$$

which expresses precisely the idea that the electromagnetic mass is u/c^2.