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On the dynamics of a rigid system of electric charges in translatory motion

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§ 1. – When a system of electric charges moves with an arbitrary motion, the electric field that it generates will be different from the one that obeys Coulomb's law. Now, the electric field that is produced by the entire system will exert forces on all of the elements of charges in the system. The resultant of that force, i.e., the resultant of the internal electric force, will obviously be zero if Coulomb's law is valid, but that will no longer be true, at least in general, when the system moves, since that law will not be valid in that case.

That resultant gives the reaction of electromagnetic inertia, and the scope of the present work is precisely to calculate it in the case of an arbitrary system in translatory motion. In the case where the system is a spherical distribution, as is assumed in most electron models, it is known that one can find ([1]) that the resultant is given, at least in the first approximation, by:

$$-\frac{2e^2}{3Rc^3}\Gamma+\frac{2e^2}{3Rc^2}\dot{\Gamma}, \tag{1}$$

in which e, R indicate the total charge and radius of the system, resp., while c is the speed of light, and Γ and $\dot{\Gamma}$ are the acceleration and its derivative with respect to time, resp. For quasi-stationary motion, the second term in (1) will become negligible, and (1) will reduce to:

$$-m\,\Gamma, \tag{2}$$

in which m is the electromagnetic mass.

In **§ 2**, one will find the generalization of (1) to the case of arbitrary systems, which might correspond to molecular models, for example, while always supposing that the velocity is negligible compared to that of light. If F_i (i = 1, 2, 3) are the components of the resultant in question then one will find that:

([1]) RICHARDSON, *Electron Theory of Matter*, Chap. XIII. The difference between my formula and that of Richardson comes from the fact that the latter uses Heaviside units.

(3) $$F_i = \sum_k m_{ik}\,\Gamma_k + \sum_k \sigma_{ik}\,\dot{\Gamma}_k\,,$$

in which m_{ik}, σ_{ik} are quantities that depend upon the constitution of the system. One can no longer speak of a scalar electromagnetic mass in the general case, but one must consider the tensor m_{ik} in its place.

§ 3 is dedicated to the dynamical study of the law of quasi-stationary motion:

(4) $$K_i = \sum_k m_{ik}\,\Gamma_k\,,$$

in which K_i are the components of the external force. It will be shown that the fundamental principles of *vis viva* and Hamilton will continue to be valid with that law.

Finally, in **§ 4**, the law of quasi-stationary motion (4), which is valid only for small velocities, will be generalized to the case of arbitrary velocities by means of the principle of relativity in the strict sense.

With that, what will remain is to complete the study of the electromagnetic mass as the inertial mass. In later work, the electromagnetic mass will be considered to be the gravitational mass from the standpoint of the general theory of relativity.

§ 2. – It is known ([2]) that the electric force that is due to a point-like unit charge in motion is the sum of two terms. When one supposes that the velocity v of the particle is negligible with respect to the speed of light c, they will be: Firstly, $\mathbf{E}_1$, which is the force that is given by Coulomb's law, and secondly $\mathbf{E}_2$ has the expression:

(5) $$\mathbf{E}_2 = \frac{\Gamma^* \cdot \mathbf{a}}{c^2\,r}\,a - \frac{1}{c^2\,r}\,\Gamma^*\,.$$

In that formula, r represents the distance between the particle M and the point P at which one calculates the force, $\mathbf{a}$ is a vector of unit magnitude and orientation MP. Finally, Γ^* is the acceleration of the particle at time $t - (r / c)$. However, if one has a charge $\rho\,dt$ (ρ is the electric density, $d\tau$ is the volume element) at M instead of 1 then the force at P will be $\rho\,d\tau\,(\mathbf{E}_1 + \mathbf{E}_2)$, and then the force that is exerted at P by all of the charges will be $\int_\tau \rho\,(\mathbf{E}_1 + \mathbf{E}_2)\,d\tau$, where the integration must extend over all of the space τ that is occupied by the charge. Now, if one has the charge $\rho'\,d\tau'$ at the point P then the force that acts upon it will be $\rho'\,d\tau' \int_\tau \rho\,(\mathbf{E}_1 + \mathbf{E}_2)\,d\tau$.

The force that acts upon the entire system will therefore be:

([2]) See, for example, RICHARDSON, *loc. cit.*

$$\mathbf{F} = \iint \rho\,\rho'(\mathbf{E}_1 + \mathbf{E}_2)\,d\tau\,d\tau' ,$$

in which the two integrations must be extended over the same region. On the other hand, one obviously has:

$$\iint \rho\,\rho'\,\mathbf{E}_1\,d\tau\,d\tau' = 0 ,$$

and therefore:

$$\mathbf{F} = \iint \rho\,\rho'\,\mathbf{E}_2\,d\tau\,d\tau' .$$

If we now let Γ, $\dot{\Gamma}$ indicate the acceleration and its derivative with respect to time t, resp., then if r is sufficiently small, we can set:

$$\Gamma^* = \Gamma - \frac{r}{c}\dot{\Gamma}$$

and finally get:

$$\mathbf{F} = \iint \left(\frac{\Gamma \cdot \mathbf{a}}{c^2\,r} a - \frac{\Gamma}{c^2\,r} \right) \rho\,\rho'\,d\tau\,d\tau' - \iint \left(\frac{\dot{\Gamma} \cdot \mathbf{a}}{c^3} a - \frac{\dot{\Gamma}}{c^3} \right) \rho\,\rho'\,d\tau\,d\tau' \quad . \tag{6}$$

Let x_1, x_2, x_3 indicate the orthogonal Cartesian coordinates, and let (κ_i) be the coordinates of M, while (κ_i) are those of P. The components of $\mathbf{a}$ are then $a_i = (x_i' - x_i) / r$. If one writes (6) in scalar form and observes that from the hypotheses that were made on the translatory motion that Γ_i and $\dot{\Gamma}_i$ are constant during the integration then one will get:

$$F_i = - \sum_k m_{ik}\,\Gamma_k + \sum_k \sigma_{ik}\,\dot{\Gamma}_k \ . \tag{7}$$

In that, one has set:

$$\left\{ \begin{aligned} m_{ii} &= \frac{2U}{c^2} - \iint \frac{\rho\rho'(x_i' - x_i)^2}{c^2\,r^3}\,d\tau\,d\tau', \\ m_{ik} = m_{ki} &= - \iint \frac{\rho\rho'(x_i' - x_i)(x_k' - x_k)}{c^2\,r^3}\,d\tau\,d\tau' \quad i \neq k \end{aligned} \right. \tag{8}$$

$$\left\{ \begin{aligned} \sigma_{ii} &= \frac{e^2}{c^3} - \iint \frac{\rho\rho'(x_i' - x_i)^2}{c^3\,r^2}\,d\tau\,d\tau', \\ \sigma_{ik} = \sigma_{ki} &= - \iint \frac{\rho\rho'(x_i' - x_i)(x_k' - x_k)}{c^3\,r^2}\,d\tau\,d\tau' \quad i \neq k. \end{aligned} \right. \tag{9}$$

In those formulas, U represents the electrostatic energy of the system, which equals $\frac{1}{2} \iint \frac{\rho\rho'}{r}\,d\tau\,d\tau'$, and e is the total electric charge, which equals $\int \rho\,d\tau = \int \rho'\,d\tau'$.

It results immediately from the expressions (8), (9) that if one changes the axes (x_i) into other ones (y_i) by the orthogonal substitution:

$$y_i = \sum_k \alpha_{ik}\, x_k$$

then m_{ik} and σ_{ik} will be:

$$m'_{ik} = \sum_{r,s} \alpha_{ir}\, \alpha_{ks}\, m_{rs}\,,$$

$$\sigma'_{ik} = \sum_{r,s} \alpha_{ir}\, \alpha_{ks}\, \sigma_{rs}$$

relative to the new axes.

m_{ik}, as well as σ_{ik}, are symmetric covariant tensors then. For each of them, one will then have three orthogonal principal directions such that if one takes the axes to be parallel to them then one will have $m_{ik} = 0$ or $\sigma_{ik} = 0$ when $i \neq k$.

The principal axes of the tensors m, σ will then be different, in general. One can perform the integrations in (8), (9) in the case where the system have spherical symmetry, because one can replace $(x'_i - x_i)(x'_k - x_k)/r^2$ with the mean value of that expression over all possible direction MP as long as one has two points M, P that correspond to a pair of points at infinity that are distinct, except for their orientation. Now, if $i = k$ then that mean value will be $\frac{2\pi}{4\pi}\int_0^\pi \cos^2\theta \sin\theta\, d\theta$; however, it will be zero when $i \neq k$.

One will then have:

$$m_{11} = m_{22} = m_{33} = \frac{4U}{3c^2} \qquad ; \qquad m_{11} = m_{22} = m_{33} = 0\,;$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \frac{2}{3}\frac{e^2}{c^3} \qquad ; \qquad \sigma_{11} = \sigma_{22} = \sigma_{33} = 0\,.$$

If one substitutes those values in (7) then one will get well-known formulas in the case where the system is a homogeneous spherical shell.

§ 3. – Recall the general case and observe that for quasi-stationary motion, one can replace (5) with:

$$F_i = -\sum_k m_{ik}\, \Gamma_k\,.$$

If one imagines that an external force (X_i) acts upon the system then the total force will be ($X_i + F_i$). If one then supposes that the system is devoid of material mass then one must have $X_i + F_i = 0$, and therefore:

$$X_i = \sum_k m_{ik}\, \Gamma_k\,. \tag{10}$$

It is easy to show that the principles of *vis viva* and that of Hamilton will be preserved with the law of motion (10). Indeed, if $\mathbf{V} = (V_1, V_2, V_3)$ indicates the velocity and one multiplies (10) by V_i then when one sums over i, one will get:

$$\sum_i X_i V_i = \sum_{i,k} m_{ik} V_i \frac{dV_k}{dt}.$$

If one switches i with k in the second sum and observes that $m_{ik} = m_{ki}$ then:

$$\sum_i X_i V_i = \sum_{i,k} m_{ik} V_k \frac{dV_i}{dt}.$$

Adding the last two equations will give:

$$2\sum_i X_i V_i = \sum_{i,k} m_{ik} \left(V_i \frac{dV_k}{dt} + V_k \frac{dV_i}{dt} \right) = \frac{d}{dt} \sum_{i,k} m_{ik} V_i V_k .$$

The left-hand side is twice the power P that is exerted by the external force. One will then have:

(11) $$P = \frac{dT}{dt}, \qquad \text{in which} \qquad T = \tfrac{1}{2} \sum_{i,k} m_{ik} V_i V_k .$$

However, if one multiplies the two sides of (10) by δx_i and sums then one will similarly get:

$$\begin{aligned} \sum_i X_i \,\delta x_i &= \tfrac{1}{2} \sum_{i,k} m_{ik} \left(\frac{d^2 x_k}{dt^2} \delta x_i + \frac{d^2 x_i}{dt^2} \delta x_k \right) \\ &= \frac{d}{dt} \left\{ \tfrac{1}{2} \sum_{i,k} m_{ik} \left(\dot{x}_k \,\delta x_i + \dot{x}_i \,\delta x_k \right) \right\} - \tfrac{1}{2} \sum_{i,k} m_{ik} \left(\dot{x}_k \,\delta \dot{x}_i + \dot{x}_i \,\delta \dot{x}_k \right) \\ &= \frac{d}{dt} \left\{ \tfrac{1}{2} \sum_{i,k} m_{ik} \left(\dot{x}_k \,\delta x_i + \dot{x}_i \,\delta x_k \right) \right\} - \delta T . \end{aligned}$$

If one multiplies by dt and integrates between two limits t', t'' at which one supposes that the variations δx_i are zero then one will get:

(12) $$\int_{t'}^{t''} \left(\delta T + \sum_i X_i \,\delta x_i \right) dt = 0 ,$$

which expresses Hamilton's principle.

However, if one refers to the principal axes of the tensor m_{ik}, instead of arbitrary axes, then (10) will take the simple form:

$$X_i = m_{ii}\,\Gamma_i\,. \tag{13}$$

§ 4. – That formula is valid only when V / c is negligible. In order to generalize it to an arbitrary velocity, let $S = (x_1, x_2, x_3, t)$ denote the indicated reference system. Let $S^* \equiv (x, y, z, t)$ be a system that is fixed with respect to S with its x-axis oriented along the velocity of the system at a certain time $\bar{t}$ that is generic, but fixed. Finally, let $S' = (x', y', z', t')$ be a system whose spatial axes are parallel to xyz and which moves relative to S^* with a uniform motion whose velocity is equal to that of the moving point at the time $\bar{t}$ and has a magnitude of v. One will have:

$$t' = \beta\left(t - \frac{v}{c^2}\,x\right), \qquad x' = \beta\,(x - v\,t)\,, \qquad y' = y\,, \qquad z' = z\,, \qquad \beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,, \tag{14}$$

in which $\bar{t}$ is fixed, so v and b will be constants.

Suppose that the force that acts upon our system is derived from an external electromagnetic field (**E**, **H**). Since the system has zero velocity with respect to S' at the instant t, (10) will be valid, and one will then have:

$$\begin{aligned} e\,E'_x &= m_{xx}\,\Gamma'_x + m_{xy}\,\Gamma'_y + m_{xz}\,\Gamma'_z\,, \\ e\,E'_y &= m_{yx}\,\Gamma'_x + m_{yy}\,\Gamma'_y + m_{yz}\,\Gamma'_z\,, \\ e\,E'_z &= m_{zx}\,\Gamma'_x + m_{zy}\,\Gamma'_y + m_{zz}\,\Gamma'_z\,, \end{aligned}$$

with an obvious meaning for the symbols.

One will then have:

$$e\,E'_x = e\,E_x\,, \quad e\,E'_y = e\,\beta\left(E_y - \frac{v}{c}H_z\right), \qquad e\,E'_z = e\,\beta\left(E_z + \frac{v}{c}H_y\right).$$

If one then sets:

$$\mathbf{k} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{H}\right) \tag{15}$$

then one will find that:

$$e\,E'_x = k_x\,, \qquad e\,E'_y = e\,\beta\,k_y\,, \qquad e\,E'_z = e\,\beta\,k_z\,.$$

On the other hand:

$$\Gamma'_x = \frac{d^2x'\,dt' - d^2t'\,dx'}{dt'^3}\,,$$

but $\frac{dx'}{dt'} = 0$ at time $\bar{t}$, so $\Gamma'_x = \frac{d^2x'}{dt'^2}$. Take t to be the independent variables and observe that $\frac{dx}{dt} = v$, so:

$$\Gamma'_x = \beta^3\,\Gamma_x . \qquad \text{Likewise} \qquad \Gamma'_y = \beta^3\,\Gamma_y , \quad \Gamma'_z = \beta^3\,\Gamma_z .$$

Make the substitutions:

$$\left\{\begin{aligned} k_x &= m_{xx}\,\beta^3\,\bar{x} + m_{xy}\,\beta^3\,\bar{y} + m_{xz}\,\beta^3\,\bar{z} , \\ k_y &= m_{yx}\,\beta^3\,\bar{x} + m_{yy}\,\beta^3\,\bar{y} + m_{yz}\,\beta^3\,\bar{z} , \\ k_z &= m_{zx}\,\beta^3\,\bar{x} + m_{zy}\,\beta^3\,\bar{y} + m_{zz}\,\beta^3\,\bar{z} . \end{aligned}\right. \tag{16}$$

Let α_{xi} denote the angle between the x-axis and the x_i-axis, so one will have:

$$k_1 = \alpha_{xi}\,k_x + \alpha_{yi}\,k_y + \alpha_{zi}\,k_z .$$

On the other hand, since m_{i0} is covariant, one will have, for example:

$$m_{xy} = \sum_r m_{rr}\,\alpha_{xr}\,\alpha_{yr} .$$

Likewise:

$$\ddot{x} = \sum_j \ddot{x}_j\,\alpha_{xj} .$$

If one then multiplies (16) by α_{xi}, α_{yi}, α_{zi} and sums then one will find that:

$$k_i = \sum_{r,j} m_{rr}\,\ddot{x}_j \begin{bmatrix} \beta^3\alpha_{xr}^2\alpha_{xj}\,\alpha_{xi} & +\beta^2\alpha_{xr}\,\alpha_{yr}\alpha_{yj}\,\alpha_{xi} & +\beta^2\alpha_{xr}\,\alpha_{zr}\alpha_{zj}\,\alpha_{xi} \\ +\beta^2\alpha_{yr}\,\alpha_{xr}\alpha_{xj}\,\alpha_{yi} & +\beta\,\alpha_{yr}^2\alpha_{yj}\,\alpha_{yi} & +\beta^2\alpha_{yr}\,\alpha_{zr}\alpha_{zj}\,\alpha_{yi} \\ +\beta^2\alpha_{zr}\,\alpha_{xr}\alpha_{xj}\,\alpha_{zi} & +\beta^2\alpha_{zr}\,\alpha_{yr}\alpha_{yj}\,\alpha_{zi} & +\beta\,\alpha_{zr}^2\alpha_{zj}\,\alpha_{zi} \end{bmatrix} .$$

One will then have $\alpha_{xi} = \dot{x}_i / v$. Upon taking into account the relations between the α, one will ultimately find the desired generalization of (13):

$$k_i = \beta\sum_{j,r}\ddot{x}_j\,m_{rr}\left\{(\beta-1)^2\,\frac{\dot{x}_i\,\dot{x}_j\,\dot{x}_r^2}{v^4} + (\beta-1)\left[(j\,r)\frac{\dot{x}_i\,\dot{x}_r}{v^2} + (ir)\frac{\dot{x}_j\,\dot{x}_r}{v^2}\right] + (ir)(jr)\right\} , \tag{17}$$

in which:

$$(j\,r) = 1 \quad \text{if } j = r \qquad ; \qquad (j\,r) = 0 \quad \text{if} \quad j \neq r .$$

In the case of spherical symmetry, one sets $m_{11} = m_{22} = m_{33} = m$. If one performs the sum in (17) then one will find that:

$$k_i = \beta m \ddot{x}_i + m\beta(\beta^2 - 1)\frac{\dot{x}_i}{v^2}\sum_j \dot{x}_j \ddot{x}_j ,$$

from which, if one recalls that:

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

then one will recover the known formula from electronic dynamics:

$$k_i = \frac{d}{dt}\frac{m\dot{x}_i}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

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