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On the electrostatics of a uniform gravitational field and the weight of the electromagnetic mass

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Introduction.

The goal of this article is the search for the alteration of electrostatic phenomena that is produced by a uniform gravitational field that comes about on the basis of the general theory of relativity. That will involve establishing the differential equation that would link the electric potential to the charge density and correspond to the Poisson equation in classical electrostatics, which would amount to an integration, at least in the case where the intensity of the gravitational field is sufficiently small (and the terrestrial gravitational field satisfies that condition), and then finding the correction that must be added to Coulomb's law in the presence of the field of gravity.

In a first application, a study of the distribution of electricity over a conducting sphere will show that the sphere becomes polarized under the influence of the field.

The second application is dedicated to the search for the weight of an electromagnetic mass, i.e., the force that is exerted upon a system of rigid electric charges (for example, ones that are suspended in a rigid dielectric) as a result of being in a field of gravity.

One finds that the weight is given by the product of the acceleration of gravity with u / c^2, where u represents the electrostatic energy of the charge of the system, and c is the speed of light. One then finds that the gravitational mass, i.e., the ratio between the weight and the acceleration of gravity, for our system does not coincide with the inertial mass, at least in general, since in the example of a system that has spherical symmetry, the latter will be given by $(4 / 3) u / c^2$, using the same symbols as in the former.

Moreover, it is known that relativity, in the strict sense, leads one to take $\Delta u / c^2$ to be the increase in the *inertial* mass of a system to which the energy Δu was communicated, and that can be easily obtained from the aforementioned result.

Finally, one shows how one can find a point that enjoys the same property with respect to the weight of our system of charges that the center of gravity enjoys with respect to the weight of an ordinary system of material masses.

PART ONE

ELECTROSTATICS IN A FIELD OF GRAVITY

§ 1. – Consider a region in space that is the site of a uniform field of gravity and suppose that the electrostatic phenomena that one imagines to be taking place in it are sufficiently-small in intensity that one can regard the alteration that they produce on the metric in the region in question to be negligible. With that hypothesis, the metric element of the space-time manifold relative to that region can be put into the form ([1]):

$$ds^2 = a\,dt^2 - dx^2 - dy^2 - dz^2, \tag{1}$$

in which a is a function of only z.

The variables t, x, y, z will also be denoted by x_0, x_1, x_2, x_3, and the coefficients of the quadratic form (1), by g_{ik} . Let φ_i be the vector potential, and let F_{ik} be the electromagnetic field. Let:

$$F_{ik} = \varphi_{ik} - \varphi_{ki}, \tag{2}$$

when one refers to the fundamental formula (1).

If we confine our considerations to electrostatic fields then we can set $\varphi_1 = \varphi_2 = \varphi_3 = 0$, and $\varphi_0 = \varphi$, for brevity. We will then have:

$$F_{ik} = \varphi_{ik} - \varphi_{ki} = \frac{\partial \varphi_i}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_i},$$

i.e.:

$$\left\{\begin{array}{lll} F_{01} = \dfrac{\partial \varphi}{\partial x}, & F_{02} = \dfrac{\partial \varphi}{\partial y}, & F_{03} = \dfrac{\partial \varphi}{\partial z}, \\ F_{23} = F_{31} = F_{12} = 0, & F_{ik} = -F_{ki}, & F_{ii} = 0. \end{array}\right. \tag{3}$$

If one has, in addition:

$$F^{(ij)} = \sum_{h,k} g^{(ih)}\, g^{(jk)} F_{hk} = g^{(ii)}\, g^{(jj)} F_{ij},$$

in which one observes that:

$$g^{(00)} = \frac{1}{a}, \qquad g^{(11)} = g^{(22)} = g^{(33)} = -1,$$

then one will get:

([1]) T. LEVI-CIVITA, Nota II, "Sui ds^2 einteiniani," Rend. Acc. Lincei, **27**, 1st sem., no. 7.

(4) $$\begin{cases} F^{(01)} = -\dfrac{1}{a}\dfrac{\partial\varphi}{\partial x}, \qquad F^{(02)} = -\dfrac{1}{a}\dfrac{\partial\varphi}{\partial y}, \quad F^{(03)} = -\dfrac{1}{a}\dfrac{\partial\varphi}{\partial z}, \\ F^{(23)} = F^{(31)} = F^{(12)} = 0, \quad F^{(ik)} = -F^{(ki)}, \qquad F^{(ii)} = 0. \end{cases}$$

In the case that we are dealing with, we can then give the following form to the action:

(5) $$W = \int_\omega \sum_{i,k} F_{ik}\, F^{(ik)}\, d\omega + \int de \int \varphi\, dx_0 ,$$

in which:

$$d\omega = \sqrt{-\|g_{ik}\|}\, dx_0\, dx_1\, dx_2\, dx_3 = \sqrt{a}\, dx\, dy\, dz\, dt$$

is the element of hypervolume on the manifold, and the integration over $d\omega$ must be extended over a well-defined region in the manifold, while the integration relative to de, dx_0 must be extended over all elements of electric charge whose time-line passes through the region in question and the segments of that line that are found in it.

§ 2. – φ can vary arbitrarily under the variation of W, with the single condition that $\delta\varphi = 0$ on the contour of that region.

However, other than the conditions that $\delta x = \delta y = \delta z = 0$ on the contour, the variations δx, δy, δz can also be subject to other ones that are defined by the various special cases. For example, in the interior of a conducting body, they will be completely arbitrary, while in a rigid dielectric, they must represent the components of a rigid virtual displacement, and so on.

When one substitutes the values (3), (4) in (5), one will find that:

(6) $$W = -\tfrac{1}{2}\iiint\int \frac{1}{\sqrt{a}}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right\} dx\, dy\, dz\, dt + \int e \int \varphi\, dt ,$$

from which:

(7) $$\delta W = \iiint\int \delta\varphi \left[\frac{1}{\sqrt{a}}\Delta_2\varphi + \frac{\partial\varphi}{\partial z}\frac{d(1/\sqrt{a})}{dz} + \rho\right] dx\, dy\, dz\, dt + \iiint\int \rho\left(\frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z\right) dx\, dy\, dz\, dt ,$$

which one obtains immediately upon observing that from the hypotheses that were made, one will have $dx = dy = dz = 0$ along a time-line, and that if ρ is the electric charge density then $\rho\, dx\, dy\, dz = de$.

Meanwhile, since δW is identically zero, one will find that since $\delta\varphi$ is arbitrary in the interior of the domain of integration, one will have:

$$\Delta_2\varphi - \frac{d\log\sqrt{a}}{dz}\frac{\partial\varphi}{\partial z} = -\ \rho\sqrt{a}\ . \tag{8}$$

Other than that, one must have:

$$\iiint\int \rho\left(\frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z\right) dx\,dy\,dz\,dt\ = 0 \tag{9}$$

for any system of values of δx, δy, δz that conform to the supposed constraints.

Equation (8) contains the generalization of Poisson's law, to which (8) will reduce when a is constant, i.e., in the absence of the gravitational field.

§ 3. – If G denotes the acceleration of gravity in the field in question, i.e., the acceleration with which a free material point will begin to move, then one will have:

$$G = -\ \frac{1}{2}\frac{da}{dz}\ .$$

With that, (8) can be written:

$$\Delta_2\varphi + \frac{G}{a}\frac{\partial\varphi}{\partial z} = -\ \rho\sqrt{a}\ . \tag{11}$$

In order to find the solution to (11) when ρ is given at any point, imagine that the electric charge is contained within a small region about the coordinate origin. In addition, set $a = c^2$ (c = speed of light in the vicinity of the origin) at the origin, and suppose that the intensity of gravity is small enough that one can neglect the terms that contain the square of the ratio $l\,G / c^2$, in which l represents the maximum length that intervenes in the problem under consideration. Under those hypotheses, one can set:

$$\sqrt{a}\ =\ c + \frac{1}{2c}\frac{da}{dz}\,z\ =\ c\left(1 - \frac{G}{c^2}\,z\right)\ .$$

(11) can then be written:

$$\Delta_2\varphi + \frac{G}{a}\frac{\partial\varphi}{\partial z} = -\ c\left(1 - \frac{G}{c^2}\,z\right)\rho\ . \tag{12}$$

As one can certainly verify when a material is given, the integral of that equation is given by:

$$\varphi = \frac{c}{4\pi}\int\left(1 - \frac{G}{c^2}\,z\right)z_M\,d\tau_M\left(1 - \frac{G}{2c^2}\,\frac{z_P - z_M}{r}\right) \tag{13}$$

$$= \frac{c}{4\pi} \int \rho_M \, d\tau_M \left(\frac{1}{r} - \frac{G}{2c^2} \frac{z_P + z_M}{r} \right)$$

to the indicated approximation, in which M is a generic point of the region τ_M that the electric charge occupies, P is the point at which one calculates φ, and r is the distance MP.

Given the linearity of equation (12), we can naturally add an arbitrary integral of the equation:

$$\Delta_2 \varphi + \frac{G}{a} \frac{\partial \varphi}{\partial z} = 0 \tag{12*}$$

to (13) that is obtained by taking $\rho = 0$ in (12). That integral represents the field that is due to sources external to τ_M . For the application that we have in mind, we agree to consider a particular solution to (12*) that is given by:

$$\varphi = - \, c \, E_x^* \, x - c \, E_y^* \, y + \frac{c^2}{G} E_z^* \, e^{-(G/c^2)z} , \tag{14}$$

in which E_x^*, E_y^*, E_z^* are constants.

If **E** is the electric force at the origin then one will have:

$$E_x = - \frac{1}{c} F_{01} , \quad E_y = - \frac{1}{c} F_{02} , \quad E_z = - \frac{1}{c} F_{03} .$$

If then result that the electric force of the external field (14) will have the components:

$$E_x^* , \; E_y^* , \; E_z^*$$

at the origin.

§ 4. – Let us now calculate the electric field that is due to a charge *e* that is concentrated at the origin of the coordinates. From (13), one has:

$$\varphi = \frac{c\,e}{4\pi} \left(\frac{1}{r} - \frac{G}{2c^2} \frac{z}{r} \right) , \tag{15}$$

and we will immediately recognize that this formula is the generalization of Coulomb's elementary law when we set $G = 0$. If we recall (3) then we will get:

(16)
$$\begin{cases} F_{01} = \dfrac{c\,e}{4\pi}\left(\dfrac{x}{r^3} - \dfrac{G}{2c^2}\dfrac{z\,x}{r^3}\right), \\[2ex] F_{02} = \dfrac{c\,e}{4\pi}\left(\dfrac{y}{r^3} - \dfrac{G}{2c^2}\dfrac{z\,y}{r^3}\right), \\[2ex] F_{03} = \dfrac{c\,e}{4\pi}\left(\dfrac{z}{r^3} - \dfrac{G}{2c^2}\dfrac{z^2}{r^3} + \dfrac{G}{2c^2}\dfrac{1}{r}\right). \end{cases}$$

We can summarize the preceding three formulas in a single vectorial one. Indeed, let $\mathbf{F}_0$ denote the vector whose components are F_{01}, F_{02}, F_{03}, let $\boldsymbol{a}$ be a vector of magnitude 1 and orientation *MP*, and finally let $\mathbf{G}$ be a vector of magnitude G and orientation z. (16) can then be written:

(17)
$$\mathbf{F}_0 = \frac{c\,e}{4\pi}\left\{\frac{\boldsymbol{a}}{r^2} - \frac{\mathbf{G}\cdot\boldsymbol{a}}{c^2\,r}\boldsymbol{a} + \frac{1}{2c^2\,r}\mathbf{G}\right\} .$$

It is interesting to compare that formula with the one that gives the electric force that is exerted on an electric charge e that has an acceleration Γ in the absence of a gravitational field when it is in quasi-stationary motion with a velocity that is negligible compared to that of light. With the same notation, that force is expressed by:

(18)
$$\mathbf{E} = \frac{e}{4\pi}\left\{\frac{\boldsymbol{a}}{r^2} + \frac{\mathbf{G}\cdot\boldsymbol{a}}{c^2\,r}\boldsymbol{a} - \frac{1}{c^2\,r}\Gamma\right\} .$$

One sees from this that when one sets:

(19)
$$\Gamma = -\,\frac{\mathbf{G}}{2}$$

in (18), one will get:

$$\mathbf{F}_0 = c\,\mathbf{E} .$$

If one observes that $c\,\mathbf{E}$ is the electric part of the electromagnetic field that is generated by the charge in accelerated motion then that result can be stated in words as:

The electric part (F_{01}, F_{02}, F_{03}) of the electromagnetic field (F_{ik}) that is generated by a stationary electric charge in a uniform field of intensity G is equal to the electric part of the electromagnetic field that the same charge would produce if it were to move in the absence of the gravitational field under indicated conditions with an acceleration G / 2 in the opposite direction to the gravitational field.

§ 5. – Let us now study how the distribution of electricity on a conductor will be altered by the gravitational field. Thus, we observe that since δx, δy, δz are arbitrary inside of the conductor, from (9), we must have φ = constant inside of it, and therefore from (8), we must have $\rho = 0$. The electricity is therefore all on the surface. Therefore, suppose that our conductor is a sphere with its center O at the coordinate origin and a radius of R.

We shall try to satisfy the condition φ = constant inside of it while assuming that the surface density of electricity at a generic point M of the surface has the expression:

$$\frac{e}{4\pi R^2} + \frac{e}{R} a \cos\theta , \tag{20}$$

in which θ represents the angle that the radius vector OM forms with the z-axis, and a is a constant that must be determined and is supposed to have the same order of magnitude as G / c^2. The potential at an interior point P will be given by (13) as:

$$\varphi_P = \frac{c}{4\pi} \int_\sigma \left(\frac{e}{4\pi R^2} + \frac{e}{R} a \cos\theta \right) \left(\frac{1}{r} - \frac{G}{2c^2} \frac{z_P + z_M}{r} \right) d\sigma ,$$

in which the integration must be extended over all of the surface σ of the sphere. If one neglects terms of order higher than G / c^2 then one will get:

$$\varphi_P = \frac{c\,e}{16\pi^2 R^2} \int \frac{d\sigma}{r} + \frac{c\,e\,a}{4\pi R} \int \frac{\cos\theta \, d\sigma}{r} - \frac{c\,e\,G\,z_P}{32\pi^2 R^2 c^2} \int \frac{d\sigma}{r} - \frac{c\,e\,G}{32\pi^2 R^2 c^2} \int \frac{z_M \, d\sigma}{r} . \tag{21}$$

If P is internal then one will have:

$$\int \frac{d\sigma}{r} = 4\pi R , \qquad \int \frac{\cos\theta}{r} d\sigma = \frac{4}{3} \pi z_P , \qquad \int \frac{z_M}{r} d\sigma = \frac{4}{3} \pi R z_P .$$

One will then find that:

$$\varphi_P = \frac{c\,e\,a}{4\pi R} + \frac{c}{3} \left(\frac{e}{R} a - \frac{e\,G}{2\pi R c^2} \right) z_P . \tag{22}$$

If one then wants φ_P to be constant then one must set:

$$a = \frac{1}{2\pi} \frac{G}{c^2} .$$

If one substitutes that value in (20) then one will find the following expression for the surface density:

(23) $$\frac{e}{4\pi R^2}\left(1+\frac{2G}{c^2}R\cos\theta\right).$$

The fact that it is found in a gravitational field will then produce a polarization in the sphere with a moment of:

$$\frac{2}{3}\frac{G}{c^2}eR^2 .$$

PART TWO

WEIGHT OF THE ELECTROMAGNETIC MASS

§ 6. – Suppose that we have a system of charges that are suspended in a rigid substance in such a way that the δx, δy, δz in **§ 2** need to take the form of the components of a rigid displacement. If we leave any consideration of the rotational displacement until later then we will first consider the translational ones, i.e., we will suppose that δx, δy, δz are arbitrary functions of time, but do not depend upon x, y, z.

We then try to satisfy (9) while imagining that the potential φ_P at a generic point P is the sum of the potential that is given by (13) and one of the type (14). Let φ'_P and φ''_P denote those two additional terms and suppose that the ratio of the derivatives of φ''_P and φ'_P with respect to any direction has the order of magnitude $l\,G/c^2$, whose square we have agreed to neglect. With that, (9) will be written:

$$\int dt\left\{\delta x\int_{\tau_P}\left(\frac{\partial\varphi'}{\partial x}+\frac{\partial\varphi''}{\partial x}\right)\rho_P\,d\tau_P+\delta y\int_{\tau_P}\left(\frac{\partial\varphi'}{\partial y}+\frac{\partial\varphi''}{\partial y}\right)\rho_P\,d\tau_P+\delta z\int_{\tau_P}\left(\frac{\partial\varphi'}{\partial z}+\frac{\partial\varphi''}{\partial z}\right)\rho_P\,d\tau_P\right\}=0 .$$

When one is given that δx, δy, δz are arbitrary functions of time that are mutually independent, that equation will give rise to three equivalent ones:

(24) $$\int_{\tau_P}\left(\frac{\partial\varphi'}{\partial x}+\frac{\partial\varphi''}{\partial x}\right)\rho_P\,d\tau_P=\int_{\tau_P}\left(\frac{\partial\varphi'}{\partial y}+\frac{\partial\varphi''}{\partial y}\right)\rho_P\,d\tau_P=\int_{\tau_P}\left(\frac{\partial\varphi'}{\partial z}+\frac{\partial\varphi''}{\partial z}\right)\rho_P\,d\tau_P=0 .$$

Now, φ' is obtained immediately from the expression (13) upon observing that:

$$\frac{\partial r}{\partial x_P}=\frac{x_P-x_r}{r},$$

so

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial x} \rho_P \, d\tau_P = - \frac{c}{4\pi} \int_{\tau_P} \int_{\tau_M} \rho_P \, \rho_M \, d\tau_P \, d\tau_M \left\{ \frac{x_P - x_M}{r^3} - \frac{G}{2c^2} \frac{(x_P - x_M)(z_P + z_M)}{r^3} \right\},$$

in which all of the integrals must be extended over the region that the charge occupies. If one switches P with M in the right-hand side (which will not change anything) then one will get:

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial x} \rho_P \, d\tau_P = - \frac{c}{4\pi} \int_{\tau_M} \int_{\tau_P} \rho_M \, \rho_P \, d\tau_M \, d\tau_P \left\{ \frac{x_M - x_P}{r^3} - \frac{G}{2c^2} \frac{(x_M - x_P)(z_M + z_P)}{r^3} \right\},$$

and finally, when one takes one-half the sum, one will get:

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial x} \rho_P \, d\tau_P = 0 \,. \tag{25}$$

In a completely analogous way:

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial y} \rho_P \, d\tau_P = 0 \,. \tag{26}$$

On the other hand, we similarly have:

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial z} \rho_P \, d\tau_P = - \frac{c}{4\pi} \int_{\tau_P} \int_{\tau_M} \rho_P \, \rho_M \, d\tau_P \, d\tau_M \left\{ \frac{z_P - z_M}{r^3} - \frac{G}{2c^2} \frac{(z_P - z_M)(z_P + z_M)}{r^3} + \frac{G}{2c^2} \frac{1}{r} \right\},$$

and switching M with P will give:

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial z} \rho_P \, d\tau_P = - \frac{c}{4\pi} \int_{\tau_M} \int_{\tau_P} \rho_M \, \rho_P \, d\tau_M \, d\tau_P \left\{ \frac{z_M - z_P}{r^3} - \frac{G}{2c^2} \frac{(z_M - z_P)(z_M + z_P)}{r^3} + \frac{G}{2c^2} \frac{1}{r} \right\},$$

so taking one-half the sum will give:

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial z} \rho_P \, d\tau_P = - \frac{c}{4\pi} \frac{G}{2c^2} \int_{\tau_P} \int_{\tau_M} \frac{\rho_P \, \rho_M}{r} d\tau_P \, d\tau_M = - G \frac{u}{c^2} c \,, \tag{27}$$

in which u denotes the electrostatic energy of the system (up to gravitational correction terms). Under the hypotheses that were made on the derivatives of φ'', we can certainly write, with our approximation:

(28)
$$\begin{cases} \int_\tau \frac{\partial \varphi''}{\partial x} \rho\, d\tau = -c\, E_x^* e, \\ \int_\tau \frac{\partial \varphi''}{\partial y} \rho\, d\tau = -c\, E_y^* e, \\ \int_\tau \frac{\partial \varphi''}{\partial z} \rho\, d\tau = -c\, E_z^* e, \end{cases}$$

in which $e = \int_\tau \rho\, d\tau$ denotes the total charge of the system. If we substitute the expression that was just obtained in (24) then we will find that:

(29)
$$\begin{cases} e\, E_x^* = 0, \\ e\, E_y^* = 0, \\ e\, E_z^* = -G \frac{u}{c^2}. \end{cases}$$

Our result is contained in those formulas. We can say, in fact, that in order to maintain equilibrium in our system, it will be necessary for an external field $(\mathbf{E}^*)$ to exert a given force (in the first approximation) on the system, since $e\,\mathbf{E}^*$ must counterbalance the weight of the system, which is then given by $-\,e\,\mathbf{E}^*$, and therefore has the components:

(30)
$$0,\ 0,\ G \frac{u}{c^2}.$$

With that, we reach the conclusion that *the weight of an electromagnetic mass always has the direction of the vertical and a magnitude that is equal to weight of a material mass of* u / c^2.

§ 7. – In the preceding paragraph, we took δx, δy, δz to be the components of a translatory displacement. However, if we take the components to be those of a rotatory virtual displacement whose axis passes through the coordinate origin, i.e.:

(31)
$$\delta x = q\, z - r\, y, \qquad \delta y = r\, x - p\, z, \qquad \delta z = p\, y - q\, x$$

then the integral (9) will become:

(32)
$$\int d\tau \left\{ p \int_\tau \rho \left(y \frac{\partial \varphi}{\partial z} - z \frac{\partial \varphi}{\partial y} \right) d\tau + q \int_\tau \rho \left(z \frac{\partial \varphi}{\partial x} - x \frac{\partial \varphi}{\partial z} \right) d\tau + r \int_\tau \rho \left(x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} \right) d\tau \right\},$$

up to the part that is due to the external field φ''.

The integrals in parentheses are calculated easily on the basis of (13) using devices that are similar to the ones that were used in the preceding section. They have the values:

$$(33) \qquad -\frac{G}{8\pi c}\int\int \frac{y_P}{r}\rho_P\,\rho_M\,d\tau_P\,d\tau_M\,, \qquad +\frac{G}{8\pi c}\int\int \frac{x_P}{r}\rho_P\,\rho_M\,d\tau_P\,d\tau_M\,, \qquad 0.$$

If we take the origin to be the point O' that is defined by the point O and the vector:

$$O' - O = \frac{1}{2u}\int\int \frac{P-O}{r}\rho_P\,\rho_M\,d\tau_P\,d\tau_M$$

then we will see immediately that the three integrals will be annulled *for any* orientation of the system around O'. It will then follow that the integral (9) is identically zero with respect to the new origin, i.e., the moment of the weight with respect to O' is zero for any orientation of the system: O' will then enjoy the property of the center of gravity.

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