"Sopra i fenomeni che avcegono in vicinza di una linea oraria," Atti Acc. Naz. Lincei, Rend. **31** (1922), 21-23.

Relativity. – *On the phenomena that occur in the neighborhood of a time-line,* Note I by ENRICO FERMI, presented by the correspondent G. ARMELLINI.

Translated by D. H. Delphenich

§ 1. In order to study the phenomena that occur in the neighborhood of a time-line (i.e., in non-relativistic language, in a portion of space in the space-time manifold that possibly varies with time and is always very small with respect to all Euclidian distances), one is led to the search for an opportune reference frame such that the ds^2 of the manifold will take on a simple form in the neighborhood of the line being studied. In order to find such a reference frame, we first discuss a few geometric considerations.

Let a line L be given in a Riemannian manifold V_n that is also a metrically-connected manifold, in the sense of Weyl ([1]). Associate a direction y to any point P of L that is perpendicular to L, according to the rule that the way that the direction $y + dy$ relates to the point $P + dP$ is deduced from the way that y relates to P in the following manner: Let η be the direction that is tangent to L at P. One parallel-transports ([2]) y, η from P to $P + dP$ and lets $y + \delta y$, $\eta + \delta\eta$ be the directions thus-obtained, which will still be orthogonal, by a fundamental property of parallel transport. If L is not geodetic then $\eta + \delta\eta$ will not coincide with the direction $\eta + \delta\eta$ of the tangent to L at $P + dP$, and these two directions will characterize a plane at $P + dP$. Consider the element of S_{n-2} ([†]) at $P + dP$ that is perpendicular to that plane, and let all of the particles that neighbor $P + dP$ rotate rigidly around that S_{n-2} until $\eta + \delta\eta$ does not coincide with $\eta + d\eta$. $y + \delta y$ will then end up in a position that will have $y + dy$ for its direction relative to the point $P + dP$. If one fixes the direction y at one point of L at will then a process of integration will permit one to know it for all points of L.

We now look for the analytical expressions that translate the operations that were indicated for a Riemannian manifold, which will be identical to the ones that are valid for a Weyl metric manifold, provided that one is careful to choose the "Eichung" ([††]) in the form of the length of a segment that one moves rigidly in the neighborhood of L, and which is assumed constant. Let:

$$ds^2 = \sum_{ik} g_{ik}\,dx_i\,dx_j \qquad (i, k = 1, 2, \ldots, n), \tag{1}$$

([1]) Weyl, *Raum, Zeit, Materie*, Berlin, Springer, 1921, pp. 109.

([2]) T. Levi-Civita, Rend. Circ. Palermo, **42** (1917), 173.

([†]) Translator's note: S_n was a standard notation in those days for $\mathbb{R}\mathrm{P}^n$, although they also refer to it as "Euclidian space."

([††]) Translator's note: i.e., the gauge.

and let y_i, $y^{(i)}$; η_i, $\eta^{(i)} = dx_i / ds$ be the covariant and contravariant systems of components for the directions y, η, resp. Meanwhile, we have:

$$\frac{\delta\eta^{(i)}}{ds} = -\sum_{h,l}\begin{Bmatrix} h\ l \\ i \end{Bmatrix}\eta^{(h)}\frac{dx_l}{ds} = -\sum_{h,l}\begin{Bmatrix} h\ l \\ i \end{Bmatrix}\frac{dx_h}{ds}\frac{dx_l}{ds},$$

along with $\frac{d\eta^i}{ds} = \frac{d}{ds}\frac{dx_i}{ds} = \frac{d^2x_i}{ds^2}$. One thus finds that:

$$\frac{\delta\eta^{(i)} - d\eta^{(i)}}{ds} = -\left(\frac{d^2x_i}{ds^2} + \sum_{h,l}\begin{Bmatrix} h\ l \\ i \end{Bmatrix}\frac{dx_h}{ds}\frac{dx_l}{ds}\right) = -C^i.$$

C^i are the contravariant components of the vector C of geodetic curvature; i.e., it is a vector that has the orientation of the principal geodetic normal to L and a magnitude that is equal to its geodetic curvature.

On the other hand, one has:

$$\frac{\delta y^{(i)}}{ds} = -\sum_{h,k}\begin{Bmatrix} h\ k \\ i \end{Bmatrix} y^{(h)}\frac{dx_k}{ds}. \tag{2}$$

Now, since y is perpendicular to L, the motion by which one deduces $y + dy$ from $y + \delta y$ will be parallel to the tangent to L and will have a magnitude that is equal to the projection of $\delta\eta - d\eta$ onto y; that is to say, since y has length 1, it will be equal to the scalar product of $\delta\eta - d\eta$ with y, i.e.:

$$\sum_i(\delta\eta_i - d\eta_i)\,y^{(i)} = -\,ds\,\sum_i C_i y^{(i)}.$$

The magnitudes of the contravariant components that one obtains will then multiply the contravariant coordinates of the tangent to L, namely, dx_i / ds. In the final analysis, one will have $-\,dx_i \sum_i C_i y^{(i)}$. It will then result immediately from (2) that:

$$\frac{dy^{(i)}}{ds} = -\sum_{h,k}\begin{Bmatrix} h\ k \\ i \end{Bmatrix} y^{(h)}\frac{dx_k}{ds} - \frac{dx_i}{ds}\sum_h C_h y^h. \tag{3}$$

If (3) is written for $i = 1, 2, \ldots, n$ then that will give a system of n first-order differential equations in the n unknowns $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$, which will be determined once one assigns an initial value to them. One also easily verifies formally from (3) that if the initial values of the $y^{(i)}$ satisfy the condition that they must be perpendicular to L then that condition will remain valid along all of the line.

§ 2. We shall now arbitrarily assign n mutually-orthogonal directions $y_1, y_2, \ldots, y_n$ to a point P_0 of L with the condition that y_n must be tangent to L. The directions $y_1, y_2, \ldots, y_n$ will be perpendicular to L and can be transported along L by the law that was described in the preceding §, which will preserve their orthogonality, as should be obvious from that same definition. In that way, we proceed to associate n mutually-orthogonal direction with any point of L, such that the last one is tangent to L.

Now, suppose that our V_n is immersed in a Euclidian S_n of a convenient dimension We can take the coordinates of a point of V_n to be the Cartesian coordinates of its orthogonal projection onto S_n that are tangent to S_n at a generic point P of L that has P for its origin, and take the directions $y_1, y_2, \ldots, y_n$ that relate to the point P to be its directions. The metric element of V_n at P will take the form $ds^2 = dy_1^2 + dy_2^2 + \cdots + dy_n^2$ in these coordinates. Moreover, as we can recognize immediately, we will have that it is geodetic at P. That is, one will have $g_{ii} = 1$, $g_{ik} = 0$ $(i \neq k)$, up to infinitesimals of order greater than one, for the coordinate system y that one constructs around P. It is obvious that one can find such a reference system at any point of L.

Now, consider a point Q_0 of V_n that is referred to the point P_0 of L that has the coordinate $\dot{y}_1, \dot{y}_2, \ldots, \dot{y}_{n-1}, 0$. For any other point P of L, we can then determine a point Q that has the same coordinates when referred to P_0 that Q_0 has when referred to P. The point Q will describe a line that is parallel to L. One now wishes to find the relation that couples ds_Q to ds_P, under the hypothesis that Q is infinitely close to P. In order to do that, observe that the motion that takes Q to $Q + dQ$ is composed of the motions that were denoted by δ and $d - \delta$ in § 1, of which, the first one will give $\delta s_Q = ds_P$, up to first-order infinitesimals, since it is a parallel motion. The second one is a rotation, as we saw in § 1 and will give $(d - \delta)s_Q = ds_P\, C \times Q - P$, in which we have denoted the scalar product symbol by $\times$ and the vector of origin P and terminus Q by $Q - P$. Furthermore, ds_Q and $(d - \delta)s_Q$ will both have the direction of the tangent to L. One will then have $ds_Q = \delta s_Q + (d - \delta)s_Q$; i.e.:

$$ds_Q = ds_P\,[1 + C \times Q - P]. \tag{4}$$

The trajectory of the point Q defines an $(n - 1)$-fold infinitude of lines, and therefore there will pass one such line through any point M of V_n ; at least, with some suitable limitations. One can characterize M by this fact using the coordinates $\dot{y}_1, \dot{y}_2, \ldots, \dot{y}_{n-1}, 0$ of the point Q that correspond to the line that passes through M and the arc s_Q of the line L, when it is considered to be an origin that is arbitrarily close to the point P that corresponds to Q and is coincident with M.

If M is infinitely close to L then ds_Q will be perpendicular to the hypersurface s_P = constant. One will then get:

$$ds_M^2 = ds_Q^2 + d\dot{y}_1^2 + d\dot{y}_2^2 + \cdots + d\dot{y}_{n-1}^2,$$

and upon taking (4) into account:

$$ds_M^2 = (1 + C\times M - P)^2\, ds_P^2 + d\dot{y}_1^2 + d\dot{y}_2^2 + \cdots + d\dot{y}_{n-1}^2. \tag{5}$$

We will get a simple expression for ds^2 in the neighborhood of L from this.

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§ 3. Before getting to the physical applications of the results that were obtained, we first make a few more geometric observations. It is immediately obvious that the preceding considerations are rigorously complete for Euclidian space, and therefore so is formula (5), which is their conclusion, and which is valid only in a neighborhood of L for arbitrary manifolds. We then associate a line L in V_n with a line L^* in a Euclidian space S_n, in which we shall denote orthogonal Cartesian coordinates by x_i^*. If we denote any symbols that refer to the line L^* by an asterisk then we can write the formula that is analogous to (5) for S_n :

$$(5)^* \qquad ds_{M^*}^2 = (1 + C \times M^* - P)\ ds_{P^*}^2 + d\dot{y}_1^2 + d\dot{y}_2^2 + \cdots + d\dot{y}_{n-1}^2 \ ;$$

C^* is a function of s_{P^*} in $(5)^*$, just as C is a function of s_P in (5).

Let $K^{(1)}, K^{(2)}, \ldots, K^{(n-1)}$ be the contravariant components of C relative to $\dot{y}_1, \dot{y}_2, \ldots, \dot{y}_{n-1}$, and let $K^{(1)*}, K^{(2)*}, \ldots, K^{(n-1)*}$ be those of C^* relative to the $\dot{y}^*$. We wish to know whether it is possible to determine L' in such a way that the function $K^{(r)*}(s_P)$ will be equal to $K^{(r)}(s_P)$. For that reason, we shall commence by setting $s_P = s_{P^*}$; i.e., by establishing a one-to-one correspondence between the points of L and those of L^* that preserves all arc lengths. Therefore, observe that $K^{(r)*}$ is the projection of C^* onto the r^{th} direction y^*. In fact, one has:

$$(6) \qquad K^{(r)*} = \sum_{i=1}^{n} y_{i/r}^* \frac{d^2 x_i^*}{ds_P^2} \qquad (r = 1, 2, \ldots, n-1).$$

The $K^{(r)}$ will then be known functions of s_P. The condition $K^{(r)} = K^{(r)*}$ will then lead to the $n - 1$ equations:

$$(7) \qquad K^{(r)}(s_P) = \sum_{i=1}^{n} y_{i/r}^* \frac{d^2 x_i^*}{ds_P^2} \qquad (r = 1, 2, \ldots, n-1).$$

On the other hand, when (3) is written for S_n, it will give us $n\,(n - 1)$ more equations. If we add the other one to them, namely:

$$(8) \qquad ds_P^2 = dx_1^{*2} + dx_2^{*2} + \cdots + dx_n^{*2},$$

then we will get a system of $n - 1 + n\,(n - 1) + 1 = n^2$ equations in the n^2 unknowns x_i^*, $y_{i/r}^*$ that will serve to express them as functions of s_P. We can then determine the parametric equations $x_i^* = x_i^*(s_P)$ of L'. With that, formula $(5)^*$ will become identical with (5). In other words, we have represented a neighborhood of the line L' as a neighborhood of the line L for the sake of applications. Since L' is in a Euclidian space, we can also say that we have extended a neighborhood of L into a Euclidian space. To be precise, we have found coordinates that are instantaneously geodetic at all points of L.

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Relativity. – *On the phenomena that occur in the neighborhood of a time-line,* Note III by ENRICO FERMI, presented by the correspondent G. ARMELLINI.

§ 4. For most of the applications of the preceding results to the theory of relativity, we shall assume that V_n is space-time – i.e., V_4 – and that L is a time-line in whose neighborhood we propose to study phenomena. If we set $ds_M = ds$ in (5) for brevity then we will find that in that case:

$$ds^2 = (1 + C\times M - P)^2 \; ds_P^2 + d\dot{y}_1^2 + d\dot{y}_2^2 + d\dot{y}_3^2 .$$

In order to avoid the appearance of imaginary numbers and to restore homogeneity, we agree to make the following substitutions of variables:

$$s_P = vt, \qquad \dot{y}_1 = ix, \qquad \dot{y}_2 = iy, \qquad \dot{y}_3 = iz,$$

in which v is a constant with the dimensions of a velocity, so t will have the dimensions of time. From that, one will get:

(9) $$ds^2 = a\, dt^2 - dx^2 - dy^2 - dz^2,$$

in which:

(10) $$a = v^2(1 + C\times M - P)^2.$$

From now on, one will demand that all of the ordinary symbols of vector calculus refer to the space of x, y, z. It is in that sense that one will refer to the scalar products that appear in (10), provided that one intends that C should mean the vector whose components are the covariant components of the geodetic curvature of the line $x = y = z = 0$, and that $M - P$ means the vector whose components are x, y, z. We call x, y, z the spatial coordinates and t, the time coordinate. For the sake of uniformity, we shall sometimes write x_0, x_1, x_2, x_3, in place of t, x, y, z, and we shall also denote the coefficients of the quadratic form (9) by g_{ik} .

§ 5. Let F_{ik} be the electromagnetic field ([1]), and let $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ be the first-order "potential" tensor of F_{ik}, in such a way that $F_{ik} = \varphi_{i,k} - \varphi_{k,i}$. Set $\varphi_0 = \varphi$, and let u denote the vector whose components are $\varphi_1, \varphi_2, \varphi_3$. First of all, one will have:

([1]) For the notation and Hamiltonian formulation of the laws of physics, see Weyl, *op. cit.*, pp. 186 and 208.

$$\left.\begin{matrix} F_{01} \\ F_{02} \\ F_{03} \end{matrix}\right\} = \text{grad}\ \varphi - \frac{\partial u}{\partial t}, \qquad \left.\begin{matrix} F_{23} \\ F_{31} \\ F_{12} \end{matrix}\right\} = -\ \text{rot}\ u, \qquad F_{ii} = 0,\ F_{ik} = -F_{ki}\ ,$$

and likewise:

$$\left.\begin{matrix} F_{01} \\ F_{02} \\ F_{03} \end{matrix}\right\} = \frac{1}{a}\left(-\text{grad}\,\varphi + \frac{\partial u}{\partial t}\right), \qquad \left.\begin{matrix} F_{23} \\ F_{31} \\ F_{12} \end{matrix}\right\} = -\ \text{rot}\ u, \qquad F^{(ii)} = 0,\ F^{(ik)} = -F^{(ki)}\ ,$$

and therefore:

$$\tfrac{1}{4}\sum_{ik} F_{ik}F^{ik} = \frac{1}{2}\left\{\text{rot}^2 u - \frac{1}{a}\left(\text{grad}\,\varphi - \frac{\partial u}{\partial t}\right)^2\right\}.$$

Let $d\omega$ be the hyper-volume element of V_4 . We have:

$$d\omega = \sqrt{1 - \| g_{ik} \|}\ dx_0\ dx_1\ dx_2\ dx_3 = \sqrt{a}\ dt\ d\tau,$$

in which $d\tau = dy\ dy\ dz$ is the spatial volume element.

We also have:

$$\sum_i \varphi_i\, dx_i = \varphi\, dt + u\ dM, \qquad dM = (dx, dy, dz).$$

If we omit the action for the metric field (whose variation will be zero, because we regard it as having been given *a priori* by (9)) then the action will take on the following form:

$$W = \tfrac{1}{4}\int_\omega \sum_{ik} F_{ik}F^{(ik)}d\omega + \int_e de \int \sum_i \varphi_i\, dx_i + \int_m dm \int ds$$

$$\left(\begin{matrix} de = \text{the element of electric charge,} \\ dm = \text{the element of mass.} \end{matrix}\right)$$

If one introduces the notations that were indicated above then one will find that:

$$W = \tfrac{1}{2}\iint\left\{\text{rot}^2 u - \tfrac{1}{2}\left(grad\ \varphi - \frac{\partial u}{\partial t}\right)^2\right\}\sqrt{a}\ dt\, d\tau + \iint(\varphi + u \times V_L)\rho\, dt\, d\tau \tag{11}$$
$$\iint \sqrt{a - V_M^2}\ k\, dt\, d\tau\ ,$$

in which ρ, k are the densities of electricity and matter, respectively, in such a way that $de = \rho\, dt$, $dm = k\, d\tau$, V_L is the velocity of the electric charges, and V_M is that of the masses.

The integrals on the right-hand side can be taken over an arbitrary region τ between two arbitrary time points t_1, t_2. One will then have the constraint that all of the variations must be zero on the contour of the region τ and at the two times t_1, t_2.

Except for that constraint, the variations of φ and u are completely arbitrary. By contrast, the variations of x, y, z, when they are considered to be the coordinates of an element of charge and mass, can be subject to further conditions that translate into the constraints of the particular problems that one is studying. Meanwhile, if one writes down that δW is zero for an arbitrary variation $\delta\varphi$ of φ then one will find that:

$$0 = -\iint \left(\operatorname{grad}\varphi - \frac{\partial u}{\partial t}\right) \times \delta \operatorname{grad}\varphi \frac{dt\, d\tau}{\sqrt{a}} + \iint \delta\varphi\, \rho\, dt\, d\tau\,,$$

and since $\delta\varphi$ is arbitrary, one will get the equation:

$$\rho + \operatorname{div}\left\{\frac{1}{\sqrt{a}}\left(\operatorname{grad}\varphi - \frac{\partial u}{\partial t}\right)\right\} = 0. \tag{12}$$

In an analogous fashion, if one varies **u** then one will find that:

$$\rho V_L + \operatorname{rot}\,(\sqrt{a}\ \operatorname{rot} u) - \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{a}}\left(\operatorname{grad}\varphi - \frac{\partial u}{\partial t}\right)\right] = 0. \tag{13}$$

These last two equations permit us to determine the electromagnetic field once we have specified the charge and its motion.

Another group of equations can be obtained by varying the trajectories of the charges and masses in W. Let δP_M be the variation of the trajectories of the masses and let δP_L be that of the masses. In addition, we assume that u is a vector function of the point and V is a vector, such that $\frac{\partial u}{\partial P}(V)$ is the vector whose components are $\frac{\partial u_x}{\partial x}V_x + \frac{\partial u_x}{\partial y}V_y + \frac{\partial u_x}{\partial z}V_z$, and analogous expressions. If we write down that the variation of W is zero then we will find, with the usual tricks, that:

$$\begin{aligned}\iint\left(\delta P_L \times \operatorname{grad}\varphi - \delta P_L + \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial P}(V_L)\right) + V_L \times \frac{\partial u}{\partial P}(\delta V_L)\right) \rho\, dt\, d\tau \\ + \iint \delta P_M \times \left\{\frac{dt}{ds}\frac{\operatorname{grad} a}{2} + \frac{d}{dt}\left(\frac{dt}{ds}V_M\right)\right\} k\, dt\, d\tau = 0.\end{aligned} \tag{14}$$

If the δP do not depend upon their values at one time point then the coefficient of dt in (14) must be zero for all other times. One will then get:

$$\int \left\{ \delta P_L \times \operatorname{grad} \varphi - \delta P_L \left[\frac{\partial u}{\partial t} + \frac{\partial u}{\partial P}(V_L) \right] + V_L \times \frac{\partial u}{\partial P}(\delta P_L) \right\} \rho \, d\tau$$
$$+ \int \delta P_M \times \left\{ \frac{1}{2} \frac{dt}{ds} \operatorname{grad} a + \frac{d}{dt}\left(\frac{dt}{ds} V_M \right) \right\} k \, d\tau, \tag{15}$$

which must be satisfied for all systems of δP that satisfy the constraints.