# On the phenomena that happen in the vicinity of a timeline (") 

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NOTE I.
§ 1. - In order to study the phenomena that happen in the vicinity of a time-line (i.e., in nonrelativistic language, in a portion of space that possibly varies in time, but always very slightly in comparison to the divergence from Euclidean geometry of the spacetime manifold), we first agree to look for an opportune reference system such that the $d s^{2}$ of the manifold will take a simple form in the vicinity of the line under study. In order to find that reference system, we must first address some geometric considerations.

Let a line $L$ be given in a Riemannian manifold $V_{n}$, or also in a metrically-connected manifold, in the sense of Weyl $\left({ }^{1}\right)$. Associate a direction $y$ that is perpendicular to $L$ at each point $P$ of $L$ with the law that the direction $y+d y$ that relates to the point $P+d P$ is deduced from the $y$ that relates to $P$ in the following way: Let $\eta$ be the tangent direction to $L$ at $P$. Parallel translate $\left({ }^{2}\right) y, \eta$ from $P$ to $P+d P$ and let $y+\delta y, \eta+\delta \eta$ be the directions thus-obtained, which will again be orthogonal, from the fundamental property of parallel transport. If $L$ is not a geodetic then $\eta+\delta \eta$ will not coincide with the direction $\eta+d \eta$ of the tangent to $L$ at $P+d P$, and those two directions will specify a plane at $P+d P$. Consider the element of $S_{n-2}$ at $P+d P$ that is perpendicular to it and rotate all of a particle $P+d P$ rigidly around that $S_{n-2}$ until $\eta+\delta \eta$ does not overlap with $\eta+d \eta . y$ $+\delta y$ will then change into a final position that takes the direction of the $y+d y$ that relates to the point $P+d P$. That means the same thing as saying that the $y$ is fixed arbitrarily at a point on $L$, which is a process of integration that will permit one to know it for all points of $L$.

We now look for the analytical expressions that the indicated operations translate into for a Riemannian manifold, which are identical to the ones that are valid for a Weyl metric manifold, as long as you are careful to choose the "gauge" in such a way that the measure of a segment that moves rigidly in the vicinity of $L$ is constant. Let:

[^0]\[

$$
\begin{equation*}
d s^{2}=\sum_{i . k} g_{i k} d x_{i} d x_{k} \quad(i, k=1,2, \ldots, n) \tag{1}
\end{equation*}
$$

\]

and let $y_{i}, y^{(i)}$; let $\eta_{i}, \eta^{(i)}=d x_{i} / d s$ be the covariant and contravariant systems of components of the direction $y, \eta$. Meanwhile, we have:

$$
\frac{\delta \eta^{(i)}}{d s}=-\sum_{h, l}\left\{\begin{array}{c}
h l \\
i
\end{array}\right\} \eta^{(h)} \frac{d x_{l}}{d s}=-\sum_{h, l}\left\{\begin{array}{c}
h l \\
i
\end{array}\right\} \frac{d x_{h}}{d s} \frac{d x_{l}}{d s},
$$

and in addition: $\frac{d \eta_{i}}{d s}=\frac{d}{d s} \frac{d x_{i}}{d s}=\frac{d^{2} x_{i}}{d s^{2}}$. We will then find that:

$$
\frac{\delta \eta^{(i)}-d \eta^{(i)}}{d s}=-\left(\frac{d^{2} x_{i}}{d s^{2}}+\sum_{h, l}\left\{\begin{array}{c}
h l \\
i
\end{array}\right\} \frac{d x_{h}}{d s} \frac{d x_{l}}{d s}\right)=-C^{i}
$$

The $C^{i}$ are the contravariant components of the vector $\mathbf{C}$, which is the geodetic curvature, i.e., a vector that has the orientation of the geodetic principal normal to $L$ and a magnitude that is equal to the geodetic curvature.

On the other hand, one has:

$$
\frac{\delta \eta^{(i)}}{d s}=-\sum_{h, l}\left\{\begin{array}{c}
h l  \tag{2}\\
i
\end{array}\right\} \frac{d x_{h}}{d s} \frac{d x_{l}}{d s}
$$

Now, since $y$ is perpendicular to $L$, the displacement that takes $y+\delta y$ to $y+d y$ will have a magnitude that is equal to the projection of $\delta \eta-d \eta$ onto $y$, i.e., since $y$ has length 1 , the scalar product of $\delta \eta-d \eta$ with $y$, so:

$$
\sum_{i}\left(\delta \eta^{(i)}-d \eta^{(i)}\right) y^{(i)}=-d s \sum_{i} C_{i} y^{(i)} .
$$

Its contravariant components are then obtained by multiplying its magnitude by the contravariant components of the tangent to $L$, i.e., $d x_{i} / d s$. In the final analysis, it will then be $-d x_{i} \sum_{i} C_{i} y^{(i)}$. Now, it will result immediately from (2) that:

$$
\frac{d y^{(i)}}{d s}=-\sum_{h, k}\left\{\begin{array}{c}
h k  \tag{3}\\
i
\end{array}\right\} y^{(h)} \frac{d x_{k}}{d s}-\frac{d x_{i}}{d t} \sum_{h} C_{h} y^{h} .
$$

When (3) it written out for $i=1,2, \ldots, n$, that will give a system of $n$ first-order differential equations between the $n y^{(1)}, y^{(2)}, \ldots, y^{(n)}$, which will then prove to be determinate once one has assigned initial values to them. It will also be easy to formally verify from (3) that if the initial
values of the $y^{(i)}$ satisfy the condition of being perpendicular to $L$ then that condition will remain verified along all of the line.
§ 2. - At a point $P_{0}$ of $L$, we then assign $n$ mutually-perpendicular directions $y_{1}, y_{2}, \ldots, y_{n}$ at will, with the condition that $y_{n}$ must be tangent to $L$. The directions $y_{1}, y_{2}, \ldots, y_{n-1}$ will be perpendicular to $L$ and can be transported along $L$ according to the law that was defined in the preceding section, which will preserve their orthogonality, as is obvious from their definition. In that way, we will associate any point of $L$ with $n$ mutually-orthogonal directions, the last of which is tangent to $L$. Now imagine that our $V_{n}$ is immersed in a Euclidian $S_{N}$ with a convenient number of dimensions. We can then set the coordinates of a point of $V_{n}$ equal to the orthogonal Cartesian coordinates of its projection onto the $S_{n}$ that is tangent to $V_{n}$ at a generic point $P$ of $L$, has its origin at $P$, and its directions are those of $y_{1}, y_{2}, \ldots, y_{n}$ that relate to the point $P$. With those coordinates, the metric element of $V_{n}$ at $P$ will take the form $d s^{2}=d y_{1}^{2}+d y_{2}^{2}+\cdots+d y_{n}^{2}$. That is to say, with the coordinates $y$, one can take $g_{i i}=1 ; g_{i k}=0(i \neq k)$ in a neighborhood of $P$, at least up to infinitesimals of order higher than one. It is obvious that such a reference frame will exist at any point of $L$. Now consider a point $Q_{0}$ of $V_{n}$ that has the coordinates $\bar{y}_{1}, \bar{y}_{2}, \ldots, \bar{y}_{n-1}, 0$ in the reference frame at the point $P_{0}$ of $L$. For any other point $P$ of $L$, we can then determine a point $Q$ that has the same coordinates relative to the reference frame at $P$ that $Q_{0}$ has in the reference frame at $P_{0}$. The point $Q$ will then traverse a path that is parallel to $L$. We would now like to find the relation that couples the $d s_{Q}$ to $d s_{P}$ under the hypothesis that $Q$ is infinitely-close to $P$. Therefore, observe that the displacement that takes $Q$ to $Q+d Q$ is composed of the displacements that were denoted by $\delta$ and $d-\delta$ in $\S 1$, and that since the former is a parallel displacement, it will make $\delta s_{Q}=d s Q$, at least up to higher-order infinitesimals. The latter is a rotation that, as we saw in $\S \mathbf{1}$, will give $(d-\delta) s_{Q}=$ $d s_{P} \mathbf{C} \cdot(Q-P)$, if $\cdot$ is the symbol for the scalar product, and $Q-P$ is the vector with its origin at $P$ and it terminus at $Q$. In addition, $d s_{Q}$ and $(d-\delta) s_{Q}$ both have the direction of the tangent to $L$. We will then have $d s_{Q}=\delta s_{Q}+(d-\delta) s_{Q}$, i.e.:

$$
\begin{equation*}
d s_{Q}=d s_{P}[1+\mathbf{C} \cdot(Q-P)] . \tag{4}
\end{equation*}
$$

The trajectory of the point $Q$ will form an $(n-1)$-fold infinitude of lines, and therefore one such line will pass through any point $M$ of $V_{n}$, at least with suitable limiting conditions. Therefore, $M$ can be characterized by means of the coordinates $\bar{y}_{1}, \bar{y}_{2}, \ldots, \bar{y}_{n-1}$ of the point $Q$ that corresponds to the line that passes through $M$ and the arc-length $s_{P}$ of the line $L$ of contact from an arbitrary origin to the point $P$ that corresponds to $Q$, which is coincident with $M$.

If $M$ is infinitely close to $L$ then $d s_{Q}$ will be perpendicular to the hypersurface $s_{P}=$ constant. One will then have:

$$
d s_{M}^{2}=d s_{Q}^{2}+d \bar{y}_{1}^{2}+d \bar{y}_{2}^{2}+\cdots+d \bar{y}_{n-1}^{2},
$$

and if one takes (4) into account:

$$
\begin{equation*}
d s_{M}^{2}=[1+\mathbf{C} \cdot(M-P)]^{2} d s_{P}^{2}+d \bar{y}_{1}^{2}+d \bar{y}_{2}^{2}+\cdots+d \bar{y}_{n-1}^{2} . \tag{5}
\end{equation*}
$$

We have then found a very simple expression for $d s^{2}$ in the vicinity of $L$.

## NOTE II.

§ 3. - Before we move on to the physical applications of the results that were obtained, we would like to make a few more geometric observations. Meanwhile, it is obvious that the preceding considerations [and therefore formula (5), as well, which is their conclusion], which are valid for an arbitrary manifold only in the vicinity of $L$, are nonetheless completely rigorous for Euclidian spaces. We then associate the line $L$ in $V_{n}$ with a line $L^{*}$ in a Euclidian space $S_{n}$ in which $x_{i}^{*}$ denote the orthogonal Cartesian coordinates. If the asterisk denotes the symbols that refer to the line $L^{*}$ then one can write the following formula for $S_{n}$, which is analogous to (5):

$$
\begin{equation*}
d s_{M^{*}}^{2}=\left[1+\mathbf{C}^{*} \cdot\left(M^{*}-P^{*}\right)\right]^{2} d s_{P^{*}}^{2}+d \bar{y}_{1}^{* 2}+d \bar{y}_{2}^{* 2}+\cdots+d \bar{y}_{n-1}^{* 2} . \tag{*}
\end{equation*}
$$

As in (5), $\mathbf{C}$ is a function of $s_{P}$, so $\mathbf{C}^{*}$ is a function of $s_{P}$ in (5).
Let $K^{(1)}, K^{(2)}, \ldots, K^{(n-1)}$ be the contravariant components of $\mathbf{C}$ relative to $\bar{y}_{1}, \bar{y}_{2}, \ldots, \bar{y}_{n-1}$, and let $K^{(1) *}, K^{(2) *}, \ldots, K^{(n-1) *}$ be those of $\mathbf{C}^{*}$ relative to the $\bar{y}^{*}$. Now see if one can determine $L^{*}$ in such a manner that the functions $K^{(r) *}\left(s_{P}\right)$ will be equal to the $K^{(r)}\left(s_{P}\right)$. One then begins by setting $s_{P}=s_{P^{*}}$, i.e., by establishing a bijective e correspondence between the points of $L$ and those of $L^{*}$ that will preserve the arc-lengths. Next, observe that $K^{(r) *}$ is the projection of $\mathbf{C}^{*}$ onto the $r^{\text {th }}$ direction $y^{*}$. That is, one has:

$$
\begin{equation*}
K^{(r) *}=\sum_{i=1}^{n} y_{i / r}^{*} \frac{d^{2} x_{i}^{*}}{d s_{P}^{2}} \quad(r=1,2, \ldots, n-1) \tag{6}
\end{equation*}
$$

The $K^{(r)}$ are known functions of $s_{P}$ then. The condition $K^{(r)}=K^{(r) *}$ will then lead to the ( $n-$ 1) equations:

$$
\begin{equation*}
K^{(r)}\left(s_{P}\right)=\sum_{i=1}^{n} y_{i / r}^{*} \frac{d^{2} x_{i}^{*}}{d s_{P}^{2}} \quad(r=1,2, \ldots, n-1) \tag{7}
\end{equation*}
$$

On the other hand, when (3) is written for $S_{n}$, that will give $n(n-1)$ other equations. If one adds this other one to them:

$$
\begin{equation*}
d s_{P}^{2}=d x_{1}^{* 2}+d x_{2}^{* 2}+\cdots+d x_{n}^{* 2} \tag{8}
\end{equation*}
$$

then one will find a system of $n-1+n(n-1)+1=n^{2}$ equations in the $n^{2}$ unknowns $x_{i}^{*}, y_{i / r}^{*}$ that serve to express them as functions of $s_{P}$. One can then determine the parametric equations $x_{i}^{*}$ $=x_{i}^{*}\left(s_{P}\right)$ of $L^{*}$. With that, formula ( $5^{*}$ ) will become identical to (5), i.e., one will have represented the things that apply to the neighborhood of $L^{*}$ as things that apply to the neighborhood of $L$. Since $L^{*}$ is in a Euclidian space, one can also say that one has stretched out the neighborhood of $L$ into a Euclidian space or that one has found coordinates that are geodetic for all points of $L$ simultaneously.

## NOTE III.

§ 4. - In order to show how the preceding results apply to the theory of relativity, suppose that $V_{n}$ is the spacetime $V_{4}$ and that $L$ is a time-time in the neighborhood of which one proposes to study the phenomena. If one sets $d s_{M}=d s$ in (5), for brevity, then one will that in that case:

$$
d s^{2}=[1+\mathbf{C} \cdot(M-P)]^{2} d s_{P}^{2}+d \bar{y}_{1}^{2}+d \bar{y}_{2}^{2}+\cdots+d \bar{y}_{n-1}^{2} .
$$

In order to avoid the appearance of imaginaries and to establish homogeneity, we agree to make the following substitution of variables:

$$
s_{P}=v t, \quad \bar{y}_{1}=i x, \quad \bar{y}_{2}=i y, \quad \bar{y}_{3}=i z,
$$

in which $v$ is a constant with the dimensions of a velocity, in such a way that $t$ will have the dimensions of a time. One will then get:

$$
\begin{equation*}
d s^{2}=a d t^{2}-d x^{2}-d y^{2}-d z^{2} \tag{9}
\end{equation*}
$$

in which:

$$
\begin{equation*}
a=v^{2}[1+\mathbf{C} \cdot(M-P)]^{2} . \tag{10}
\end{equation*}
$$

From now on, the ordinary symbols of the vector calculus are meant to be referred to the space of $x, y, z$, and the scalar product that figures in (10) should be interpreted in that sense, provided that $\mathbf{C}$ means the vector whose components are the covariant components of the geodetic curvature of the line $x=y=z=0$, and $M-P$ means the vector whose components are $x, y, z$. Call $x, y, z$ the spatial coordinates and call $t$ the time. For uniformity, we sometimes write $x_{0}, x_{1}, x_{2}, x_{3}$ instead of $t, x, y, z$, resp., and call $g_{i k}$ the coefficients of the quadratic form (9).
§ 5. - Let $\left({ }^{3}\right) F_{i k}$ be the electromagnetic field, and let $\left(\varphi_{0}, \varphi_{1}, \varphi_{2}, \varphi_{3}\right)$ be the first-order tensor "potential" of $F_{i k}$, in such a way that $F_{i k}=\varphi_{i k}-\varphi_{k i} . \operatorname{Set} \varphi_{0}=\varphi$, and call $\boldsymbol{u}$ the vector with components $\varphi_{1}, \varphi_{2}, \varphi_{3}$. Meanwhile, one will have:

$$
\left.\left.\begin{array}{l}
F_{01} \\
F_{02} \\
F_{03}
\end{array}\right\}=\operatorname{grad} \varphi-\frac{\partial \boldsymbol{u}}{\partial t}, \quad \begin{array}{l}
F_{23} \\
F_{32} \\
F_{12}
\end{array}\right\}=-\operatorname{rot} \boldsymbol{u}, \quad F_{i i}=0, \quad F_{i k}=-F_{k i},
$$

and likewise:

$$
\left.\left.\begin{array}{l}
F^{(01)} \\
F^{(02)} \\
F^{(03)}
\end{array}\right\}=\frac{1}{a}\left(-\operatorname{grad} \varphi+\frac{\partial \boldsymbol{u}}{\partial t}\right), \quad \begin{array}{l}
F^{(23)} \\
F^{(31)} \\
F^{(12)}
\end{array}\right\}=-\operatorname{rot} \boldsymbol{u}, \quad F^{(i i)}=0, \quad F^{(i k)}=-F^{(k i)}
$$

and therefore:

$$
\frac{1}{4} \sum_{i, k} F_{i k} F^{(i k)}=\frac{1}{2}\left\{\operatorname{rot}^{2} \boldsymbol{u}-\frac{1}{a}\left(-\operatorname{grad} \varphi+\frac{\partial \boldsymbol{u}}{\partial t}\right)^{2}\right\} .
$$

Let $d \omega$ be the hypervolume element on $V_{4}$. One has:

$$
d \omega=\sqrt{-\left\|g_{i k}\right\|} d x_{0} d x_{1} d x_{2} d x_{3}=\sqrt{a} d t d \tau
$$

in which $d \tau=d x d y d z$ is the volume element on space.
One also has:

$$
\sum \varphi_{i} d x_{i}=\varphi d t+\boldsymbol{u} \cdot d M, \quad d M=(d x, d y, d z)
$$

If one overlooks the action of the metric field, whose variation is zero since one regards it as given a priori in (9), then the action will take the following form:

$$
\begin{gathered}
W=\frac{1}{4} \int_{\omega} \sum_{i, k} F_{i k} F^{(i k)} d \omega+\int_{e} d e \int \sum_{i} \varphi_{i} d x_{i}+\int_{m} d m \int d s \\
\binom{d e=\text { element of electric charge }}{d m=\text { element mass }} .
\end{gathered}
$$

When one introduces the notations that were defined, one will find that:

[^1]\[

$$
\begin{equation*}
W=\frac{1}{2} \iint\left\{\operatorname{rot}^{2} \boldsymbol{u}-\frac{1}{a}\left(-\operatorname{grad} \varphi+\frac{\partial \boldsymbol{u}}{\partial t}\right)^{2}\right\} \sqrt{a} d t d \tau+\iint\left(\varphi+\boldsymbol{u} \cdot \mathbf{V}_{L}\right) \rho d \tau d t+\iint \sqrt{a-V_{M}^{2}} k d \tau d t \tag{11}
\end{equation*}
$$

\]

in which $\rho, k$ are the densities of electricity and matter, respectively, in such a way that $d e=\rho d \tau$, $d m=k d \tau$, while $\mathbf{V}_{L}$ is the velocity of the electric charge, and $\mathbf{V}_{M}$ is that of the mass.

The integrals in the right-hand side can be extended over an arbitrary region $\tau$ between two arbitrary times $t_{1}, t_{2}$. One then has the constraint that all of the variations must be zero on the contour of the region $t$, and for both times $t_{1}, t_{2}$.

Aside from those conditions, the variations of $\varphi$ and $\boldsymbol{u}$ are completely arbitrary. By contrast, further conditions can be imposed upon the variations of $x, y, z$, which are considered to be the coordinates of an element of charge or mass, and those conditions are translations of the constraints in the particular problem under study. Meanwhile, if one writes that $\delta W$ is zero for an arbitrary variation $\delta \varphi$ of $\varphi$ then one will find:

$$
0=-\iint\left(\operatorname{grad} \varphi-\frac{\partial \boldsymbol{u}}{\partial t}\right) \cdot \delta \operatorname{grad} \varphi \frac{d t d \tau}{\sqrt{a}}+\iint \delta \varphi \rho d t d \tau
$$

If one transforms the first integral with an appropriate application of Gauss's theorem and takes into account the fact that $\delta \varphi$ is annulled on the contour then one will find that:

$$
0=\iint \delta \varphi\left\{\rho+\operatorname{div}\left[\frac{1}{\sqrt{a}}\left(\operatorname{grad} \varphi-\frac{\partial \boldsymbol{u}}{\partial t}\right)\right]\right\} d t d \tau
$$

and since $\delta \varphi$ is arbitrary, one will have, at the same time, the equation:

$$
\begin{equation*}
\rho+\operatorname{div}\left[\frac{1}{\sqrt{a}}\left(\operatorname{grad} \varphi-\frac{\partial \boldsymbol{u}}{\partial t}\right)\right]=0 . \tag{12}
\end{equation*}
$$

In an analogous fashion, if one varies $\boldsymbol{u}$ then one will find that:

$$
\begin{equation*}
\rho V_{L}+\operatorname{rot}(\sqrt{a} \operatorname{rot} \mathbf{u})-\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{a}}\left(\operatorname{grad} \varphi-\frac{\partial \boldsymbol{u}}{\partial t}\right)\right]=0 \tag{13}
\end{equation*}
$$

The last two equations will permit one to determine the electromagnetic field once one has assigned the charges and their motion.

Another group of equations can be easily obtained by varying the trajectories of the charge and mass in $W$. Let $\delta P_{M}$ be the variation of the trajectory of the mass and let $\delta P_{L}$ be that of the charge. In addition, if $\boldsymbol{u}$ be a vector function of the point and $\mathbf{V}$ is a vector then let $(\partial \boldsymbol{u} / \partial P)(\mathbf{V})$ denote
the vector whose components are $\frac{\partial u_{x}}{\partial x} V_{x}+\frac{\partial u_{x}}{\partial y} V_{y}+\frac{\partial u_{x}}{\partial z} V_{z}$, and analogously. If one writes that the variation of $W$ is zero then one will find, with the usual devices, that:

$$
\begin{gather*}
\iint\left[\delta P_{L} \cdot \operatorname{grad} \varphi-\delta P_{L}\left(\frac{\partial \boldsymbol{u}}{\partial t}+\frac{\partial \boldsymbol{u}}{\partial P}\left(\mathbf{V}_{L}\right)\right)+\mathbf{V}_{L} \cdot \frac{\partial \boldsymbol{u}}{\partial P}\left(\delta P_{L}\right)\right] \rho d t d \tau  \tag{14}\\
\quad+\iint \delta P_{M} \cdot\left[\frac{d t}{d s} \frac{\operatorname{grad} a}{2}+\frac{d}{d t}\left(\frac{d t}{d s} \mathbf{V}_{M}\right)\right] k d t d \tau=0
\end{gather*}
$$

If the value of $\delta P$ at one time does not depend upon its values at other times then the coefficient of $d t$ in (14) must necessarily be zero. One will then find that:

$$
\begin{gather*}
\int\left[\delta P_{L} \cdot \operatorname{grad} \varphi-\delta P_{L}\left(\frac{\partial \boldsymbol{u}}{\partial t}+\frac{\partial \boldsymbol{u}}{\partial P}\left(\mathbf{V}_{L}\right)\right)+\mathbf{V}_{L} \cdot \frac{\partial \boldsymbol{u}}{\partial P}\left(\delta P_{L}\right)\right] \rho d \tau  \tag{15}\\
+\int \delta P_{M} \cdot\left[\frac{d t}{d s} \frac{\operatorname{grad} a}{2}+\frac{d}{d t}\left(\frac{d t}{d s} \mathbf{V}_{M}\right)\right] k d \tau=0
\end{gather*}
$$

which must be verified for all systems of $\delta P$ that satisfy the constraints.


[^0]:    (*) Presented by Correspondent G. Armellini at the session on 22 January 1922.
    $\left.{ }^{( }{ }^{1}\right)$ WEYL, Raum, Zeit, Materie, Berlin, Springer, 1921, pp. 109.
    $\left(^{2}\right)$ T. LEVI-CIVITA, Rend. Circ. Palermo 42 (1917), pp. 173.

[^1]:    $\left(^{3}\right)$ For the notations and the Hamiltonian deduction of the laws of physics, see WEYL, loc. cit., pp. 186 and 208.

