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Proper time in classical and quantum mechanics ([1])

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In I, it will be shown that one can regard proper time τ as an independent variable in Hamilton’s action integral when one employs the expression that is given in (I.6) as a Lagrangian function. In II, proper time will be introduced into the Dirac equation and a method for integrating that equation that is based upon proper time will be developed. The Cauchy problem will be treated with the help of that method. Part II contains a further generalization (and simplification) of the application of the Wentzel-Brillouin method to the Dirac equation that Pauli gave. In III, the mixed densities that are considered in the theory of the positron will be expressed by the fundamental solution (*solution élémentaire*) and the Riemann function of the Dirac equation.

I. Classical mechanics.

1. Let L^0 be the usual Lagrangian function from which the relativistic equations of motion for a charged mass point in an external field are derived. It is known that L^0 has the form:

$$L^0 = -\,mc^2\sqrt{1-\beta^2} - \frac{e}{c}(x'A_x + y'A_y + z'A_z) + e\,\Phi, \tag{1}$$

with

$$\beta^2 = \frac{1}{c^2}(x'^2 + y'^2 + z'^2), \tag{2}$$

in which the prime denotes temporal derivatives. If one introduces the proper time:

$$\tau = \int_{t^0}^{t}\sqrt{1-\beta^2}\,dt \tag{3}$$

then the action integral:

$$S = \int_{t^0}^{t} L^0\,dt\,, \tag{4}$$

whose variation will yield the equations of motion, can be written in the form:

$$S = \int_{0}^{\tau} L^0\frac{dt}{d\tau}d\tau\,. \tag{5}$$

([1]) Presented on 14 March 1937 at the meeting of the Academy of Sciences of the USSR.

However, since the upper integration limit τ depends upon the form of the path, its variation must also be considered. As a result, proper time τ cannot be considered to be an independent (i.e., unvaried) variable in a variational principle with the Lagrangian function $L^0\, dt / d\tau$.

2. However, once can introduce another Lagrangian function when one sets:

$$L = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - c^2 \dot{t}^2) - \tfrac{1}{2} mc^2 - \frac{e}{c}(\dot{x} A_x + \dot{y} A_y + \dot{z} A_z) + e\dot{t}\,\Phi, \tag{6}$$

in which the dots signify derivatives with respect to an independent variable τ. (The quantity τ will later prove to be identical to proper time.)

The variation of the integral:

$$S = \int_0^\tau L\, d\tau \tag{7}$$

for a fixed integration limits τ yields the "equations of motion":

$$\begin{aligned} \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0, \text{ etc.,} \\ \frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} &= 0. \end{aligned} \tag{8}$$

Since L does not depend upon τ explicitly, those equations possess the integral:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - c^2 \dot{t}^2 = \text{const.} \tag{9}$$

If one requires that the constant has the value $-c^2$:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - c^2 \dot{t}^2 = -c^2 \tag{10}$$

then one will get an equation, according to which, the independent variable τ will coincide with proper time. Equations (8) then reduce to the usual relativistic equations of motion.

If one considers the relation (10) then one will have:

$$L = L^0 \frac{dt}{d\tau}, \tag{11}$$

with the previous meaning (1) of L^0. The action integral (7) is then numerically equal the usual expression (4).

3. With the Lagrangian function (6), one easily constructs Hamilton's equations of motion, as well as the Hamilton-Jacobi partial differential equation. The latter has the form:

$$\frac{\partial S}{\partial \tau} + \frac{1}{2m}\left[\left(\text{grad } S + \frac{e}{c}\mathbf{A}\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial t} - e\Phi\right) + m^2c^2\right] = 0. \tag{12}$$

In the theory in question, the quantities x, y, z, t play the role of coordinates and the quantity τ plays the role of time in classical, non-relativistic mechanics. We can then employ known formulas from classical mechanics.

If one expresses the action integral (7) in terms of the variables x, y, z, t; x^0, y^0, z^0, t^0; τ then one will get a function:

$$S = S\,(x, y, z, t; x^0, y^0, z^0, t^0;\ \tau), \tag{13}$$

in which τ can be considered to be one of the independent variables. The partial derivative $\partial S / \partial t$ is constant, as a result of the equations of motion. If one sets that constant equal to zero:

$$\frac{\partial S}{\partial \tau} = 0 \tag{14}$$

then one will get the condition (10) for the proper time.

The impulse variables that are conjugate to the "coordinates" x, y, z, t, as well as their initial values, will be expressed in terms of the partial derivatives of S as follows:

$$p_x = \frac{\partial S}{\partial x}, \qquad p_y = \frac{\partial S}{\partial y}, \qquad p_z = \frac{\partial S}{\partial z}, \qquad p_t = -H = \frac{\partial S}{\partial t}, \tag{15}$$

$$p_x^0 = -\frac{\partial S}{\partial x^0}, \qquad p_y^0 = -\frac{\partial S}{\partial y^0}, \qquad p_z^0 = -\frac{\partial S}{\partial z^0}, \qquad p_t^0 = -H^0 = -\frac{\partial S}{\partial t^0}. \tag{16}$$

If equation (14) is solved for τ and the value of τ substituted in the expression (13) for S then one will get the usual action function:

$$S = S^*\,(x, y, z, t; x^0, y^0, z^0, t^0). \tag{17}$$

One will then obviously have:

$$\frac{\partial S^*}{\partial x} = \frac{\partial S}{\partial x} + \frac{\partial S}{\partial \tau}\frac{\partial \tau}{\partial x} = \frac{\partial S}{\partial x}, \tag{18}$$

as a result of (14).

However, the elimination of τ is quite inconvenient in practice. Hence, the function S (which contains t) can be expressed in an elementary way in, e.g., the problem of the motion of an electron in constant electric and magnetic fields, while no closed expression will exist for S^*.

II. Proper time in the Dirac equation.

4. The Dirac wave equation for the electron in an electromagnetic field can be written:

$$\left\{(\boldsymbol{\alpha}\cdot\mathbf{P}) + mc\,\alpha_4 - \frac{T}{c}\right\}\psi = 0. \tag{1}$$

Here, $\boldsymbol{\alpha}$ denotes a vector with the Dirac matrices α_1, α_2, α_3 as components, $\mathbf{P}$ is the operator:

$$\mathbf{P} = -\,ih\ \mathrm{grad} + \frac{e}{c}\mathbf{A} \tag{2}$$

for the quantities of motion, and T is the operator:

$$T = ih\,\frac{\partial}{\partial t} + e\,\Phi \tag{3}$$

for the kinetic energy of the electron.

One can represent a solution ψ of the Dirac equation in the form:

$$\psi = \left\{(\boldsymbol{\alpha}\cdot\mathbf{P}) + mc\,\alpha_4 - \frac{T}{c}\right\}\Psi, \tag{4}$$

in which Ψ satisfies the second-order differential equation:

$$\left\{\mathbf{P}^2 + m^2c^2 - \frac{T^2}{c^2} + \frac{eh}{c}(\boldsymbol{\sigma}\cdot\mathfrak{H}) - \frac{ieh}{c}(\boldsymbol{\alpha}\cdot\mathfrak{E})\right\}\Psi = 0, \tag{5}$$

which can also be written in the form:

$$h^2\,\Lambda\Psi = 0. \tag{6}$$

The operator Λ will be defined by the following equation:

$$\begin{aligned}\Lambda\Psi = &-\Box\Psi - \frac{2ie}{hc}\left(\mathbf{A}\cdot\mathrm{grad}\,\Psi + \frac{1}{c}\frac{\partial\Psi}{\partial t}\right)\\ &+ \left\{-\frac{ie}{hc}\left(\mathrm{div}\,\mathbf{A} + \frac{1}{\mathrm{c}}\frac{\partial\Phi}{\partial t}\right) + \frac{e^2}{h^2c^2}(\mathbf{A}^2 - \Phi^2) + \frac{m^2c^2}{h^2}\right\}\Psi\\ &+ \{(\boldsymbol{\sigma}\cdot\mathfrak{H}) - i\,(\boldsymbol{\alpha}\cdot\mathfrak{E})\}\,\Psi.\end{aligned} \tag{7}$$

The solution of equation (5) can be put into the form of a definite integral over an auxiliary variable τ:

$$\Psi = \int_C F\,d\tau . \tag{8}$$

The equation for Ψ will be satisfied in the event that F is subject to the differential equation:

$$\frac{h^2}{2m}\Lambda F = ih\,\frac{\partial F}{\partial \tau}, \tag{9}$$

and in the event that the integration limits (or the integration path in the complex τ-plane) is chosen such that the condition:

$$\int_C \frac{\partial F}{\partial \tau} d\tau = F|_C = 0 \tag{10}$$

is fulfilled.

The variable τ plays the role of proper time, which would emerge from the following. Equation (9) can then be considered to be the *Dirac equation with proper time.*

We set:

$$F = e^{iS/h} f, \qquad \Psi = \int_C e^{iS/h} d\tau , \tag{11}$$

in which S is the classical action function, which satisfies the differential equation (I.12). In order to get the equation for f, we remark that:

$$\Lambda F = e^{iS/h}\,\Lambda' f, \qquad ih\,\frac{\partial F}{\partial \tau} = e^{iS/h}\left(ih\frac{\partial f}{\partial \tau} - \frac{\partial S}{\partial \tau} f\right), \tag{12}$$

in which Λ' emerges from Λ when one replaces $\mathbf{A}$ with $\mathbf{A} + \frac{c}{e}\operatorname{grad} S$ and Φ with $\Phi - \frac{1}{e}\frac{\partial S}{\partial t}$. As a result of the differential equation for S, the terms in the expression (9) that do not contain any factor of h cancel out, and one will get an equation for f that can be written in the form:

$$2m\frac{df}{d\tau} + \left\{\Box S + \frac{e}{c}\left(\operatorname{div}\mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right)\right\} f + \frac{e}{c}\{i(\boldsymbol{\sigma}\cdot\mathfrak{H}) + (\boldsymbol{\alpha}\cdot\mathfrak{E})\} f = ih\Box f . \tag{13}$$

In this, df / dt denotes the "complete differential quotient" with respect to proper time τ; i.e., the quantity:

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} + \dot{z}\frac{\partial f}{\partial z} + \dot{t}\frac{\partial f}{\partial t}, \tag{14}$$

in which one understands $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{t}$ to mean the classical expressions:

$$\dot{x} = \frac{1}{m}\left(\frac{\partial S}{\partial x} + \frac{e}{c}A_x\right), \text{ etc.,} \qquad \dot{t} = -\frac{1}{mc^2}\left(\frac{\partial S}{\partial t} - e\Phi\right). \tag{15}$$

5. There is some advantage to treating equation (13) with the Brillouin-Wentzel method. One can also find an exact solution to this equation in that way in some cases, such as, e.g., the case of constant electric and magnetic fields. The constant h appears only on the right-hand side of the equation. If one were to neglect the right-hand side then one would get an equation, namely:

$$2m\frac{df}{d\tau} + \left\{\Box S + \frac{e}{c}\left(\operatorname{div}\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right)\right\} f + \frac{e}{c}\{i(\boldsymbol{\sigma}\cdot\mathfrak{H}) + (\boldsymbol{\alpha}\cdot\mathfrak{E})\} f = 0, \tag{16}$$

whose solution is equivalent to the solution of a system of ordinary (and not partial) differential equations.

One can drop the term with $\Box S$ from equation (16). Let ρ denote the absolute value of the fourth-order determinant that is constructed from the second derivatives of S with respect to x, y, z, t, and with respect to x^0, y^0, z^0, t^0 (or with respect to corresponding integration constants):

$$\rho = \operatorname{Det}\left\| \frac{\partial^2 S}{\partial x\,\partial y^0} \right\|. \tag{17}$$

The quantity ρ satisfies the "continuity equation":

$$\frac{\partial\rho}{\partial\tau} + \frac{\partial}{\partial x}(\rho\dot{x}) + \frac{\partial}{\partial y}(\rho\dot{y}) + \frac{\partial}{\partial z}(\rho\dot{z}) + \frac{\partial}{\partial t}(\rho\dot{t}) = 0, \tag{18}$$

in which $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{t}$ have the previous meanings (15). It follows from this that the quantity $\sqrt{\rho}$ will satisfy the equation:

$$2m\frac{d\sqrt{\rho}}{d\tau} + \left\{\Box S + \frac{e}{c}\left(\operatorname{div}\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right)\right\}\sqrt{\rho} = 0. \tag{19}$$

If one then sets:

$$f = \sqrt{\rho}\, f^0 \tag{20}$$

then f^0 (in the approximation considered) will satisfy the differential equation:

$$2m\frac{df^0}{d\tau} + \frac{e}{c}\{i\,(\boldsymbol{\sigma}\cdot\mathfrak{H}) + (\boldsymbol{\alpha}\cdot\mathfrak{E})\} f^0 = 0. \tag{21}$$

In the case of a constant field, one can assume that f^0 depends upon only τ, but not upon x¸ y, z, t. (21) will then give a system of ordinary differential equations with constant coefficients. Since ρ also depends upon τ in this case, one will have $\Box f = 0$, and the approximation equation (16) will coincide with the exact equation (13). One will get an exact solution in that way with the help of the method considered, and its exact form will be given below (no. **8**).

In the general case of an arbitrary field, one must proceed as follows: x, y, z, t are expressed in terms of τ with the help of equation (I.16) for the classical path and then substituted in $\mathfrak{E}$ and $\mathfrak{H}$. The coefficients in the system of equations (21) will then be functions of τ alone. p_x^0, p_y^0, p_z^0, p_t^0 are then replaced with their expressions in (I.16) in the solution to the resulting system of ordinary differential equations with variable coefficients. The result will yield the desired solution of the system of partial differential equations.

When we replace S with S^* in our formulas [equation (I.17)] and consider f to be independent of τ, we will find a generalization of **Pauli**'s result ([1]), which was the first application of the Brillouin-Wentzel method to the Dirac equation. However, the Pauli formulas are quite complicated, since he based his investigation upon the first-order Dirac equation, and not the second one.

Let us make the following remark: If we calculate the integral (11) by the saddle-point method then we will have to take the value of S at the point where $\partial S / \partial t = 0$ under the integral sign; however, that is the usual action function S^* (which does not contain proper time τ).

6. The form for the solution of the Dirac equation that we obtained (a definite integral over proper time τ) is especially suited to the investigation of the Cauchy problem (initial-value problem for ψ).

Let ψ be a function that satisfies the first-order Dirac equation, along with the condition:

$$\psi = \psi^0 \qquad \text{for} \qquad t = t^0. \tag{22}$$

In order to determine ψ, it will suffice to find a solution Ψ to the second-order Dirac equation $\Lambda\Psi = 0$ that satisfies the initial conditions:

$$\Psi = 0, \quad \frac{\partial \Psi}{\partial t} = -\frac{ic}{h}\psi^0 = \dot{\Psi}^0 \qquad \text{for } t = t^0. \tag{23}$$

The function Ψ can be put into the form of an integral:

$$\Psi = \int Q\,\dot{\Psi}^0\, dV, \tag{24}$$

in which one sets:

$$dV = dx^0\, dy^0\, dz^0. \tag{25}$$

([1]) **W. Pauli**, Helvetica Physica Acta **5** (1932), 179.

$\dot{\Psi}^0$ is a given function of x^0, y^0, z^0, and Q is a function of x, y, z, t and of x^0, y^0, z^0, t^0. We set:

$$\xi = c^2 (t - t^0)^2 - (x - x^0)^2 - (y - y^0)^2 - (z - z^0)^2, \tag{26}$$

and define an auxiliary function $\gamma(\xi)$ whose derivative $\gamma'(\xi) = \delta(\xi)$ is the Dirac delta function by the equations:

$$\left.\begin{array}{lll} \gamma(\xi) = 1 & \text{for} & \xi > 0, \\ \gamma(\xi) = \frac{1}{2} & \text{for} & \xi = 0, \\ \gamma(\xi) = 0 & \text{for} & \xi < 0. \end{array}\right\} \tag{27}$$

The quantity Q in the integral (24) is a true function of the form:

$$Q = R\,\gamma(\xi) + R^*\delta(\xi), \tag{28}$$

in which R and R^* are continuous functions; hence, R is the so-called *Riemann function.*

If one substitutes the expression (28) for Q in (24) then Ψ will become a sum of two integrals:

$$\Psi = \int R\,\dot{\Psi}^0\,\gamma(\xi)\,dV + \int R^*\,\dot{\Psi}^0\,\delta(\xi)\,dV\,. \tag{29}$$

The first one is a volume integral over the volume V of the ball:

$$c^2 (t - t^0)^2 - (\mathbf{r} - \mathbf{r}^0)^2 \geq 0$$

of radius $r^* = | t - t^0 |$ with its center at $\mathbf{r}^0 = \mathbf{r}$. The second one is a surface integral over the surface:

$$c^2 (t - t^0)^2 - (\mathbf{r} - \mathbf{r}^0)^2 = 0;$$

i.e., over the outer surface S of that ball. If one eliminates the discontinuous (improper, resp.) functions $\gamma(\xi)$ and $\delta(\xi)$, resp., from (29) then one will, in fact, get:

$$\Psi = \int_V R\,\dot{\Psi}^0\,dV + \frac{1}{2r^*}\int_S R^*\,\dot{\Psi}^0\,dS\,. \tag{30}$$

Since the radius r^* of the ball tends to zero as $t \to t^0$, one will obviously have $\Psi = 0$ for $t = t^0$. Furthermore, the time derivative of the volume integral will also vanish for $t = t^0$. However, the surface integral will be equal to:

$$\frac{1}{2r^*}\int_S R^*\,\dot{\Psi}^0\,dS = 2\pi r^* (R^*\,\dot{\Psi}^0)_0 = 2\pi c\,(t - t^0)(R^*\,\dot{\Psi}^0)_0 \tag{31}$$

for small $t - t^0 > 0$. Therefore, one will have:

$$\left(\frac{\partial\Psi}{\partial t}\right)_{t=t^0+0} = 2\pi c R_0^* \dot{\Psi}^0 , \tag{32}$$

in which R_0^* denotes the value of R for $\mathbf{r} = \mathbf{r}^0$, $t = t^0$. The initial condition (23) will be fulfilled when we demand that:

$$R_0^* = 1 / 2\pi c, \tag{33}$$

independently of the coordinates and time.

However, the function Ψ must also satisfy the second-order Dirac equation. The function Q must therefore satisfy that equation, as well. We must then have:

$$\Lambda Q = \Lambda (R\,\gamma(\xi) + R^*\delta(\xi)) = 0. \tag{34}$$

If one observes the equations:

$$\Box\,\gamma(\xi) = -4\,\delta(\xi), \tag{35}$$

$$\Box\,\delta(\xi) = 0 \tag{36}$$

then performing the differentiation in (34) will yield terms that contain the factors $\gamma(\xi)$, $\delta(\xi)$, $\delta'(\xi)$. If one denotes the operator that is defined by:

$$\begin{aligned} MF &= (x-x^0)\frac{\partial F}{\partial x} + (y-y^0)\frac{\partial F}{\partial y} + (z-z^0)\frac{\partial F}{\partial z} + (t-t^0)\frac{\partial F}{\partial t} \\ &\quad + \frac{ie}{hc}\{(x-x^0)\,A_x + (y-y^0)\,A_y + (z-z^0)\,A_z - c\,(t-t^0)\,\Phi\}\,F \end{aligned} \tag{37}$$

by M, to abbreviate, then (34) can be written:

$$\begin{aligned} &\Lambda\,(R\,\gamma(\xi) + R^*\delta(\xi)) \\ &\qquad = (\Lambda R)\,\gamma(\xi) + \{\Lambda R^* + 4\,(M+1)\,R\}\,\delta(\xi) + 4\,(MR^*)\,\delta'(\xi). \end{aligned} \tag{38}$$

That expression will vanish when we require that:

$$\Lambda R = 0, \tag{39}$$

$$(M+1)\,R = -\tfrac{1}{4}\Lambda R^*, \tag{40}$$

$$MR^* = 0. \tag{41}$$

For that to be true, it will suffice that equation (40) is fulfilled on the light-cone $\xi = 0$.

A solution of equation (41) is easy to give. If one sets:

$$\chi = \int_{(r^0,t^0)}^{(r,t)} (A_x dx + A_y dy + A_z dz - c\,\Phi\,dt)\,, \tag{42}$$

in which the integral is taken along a line that connects the points ($\mathbf{r}^0$, t^0) and ($\mathbf{r}$, t), then one will have:

$$(\mathbf{r} - \mathbf{r}^0) \cdot \text{grad}\, \chi + (t - t^0) = (\mathbf{r} - \mathbf{r}^0)\, \mathbf{A} - c\,(t - t^0)\, \Phi. \tag{43}$$

Therefore, the function:

$$R^* = \frac{1}{2\pi c} e^{-\frac{ie}{hc}\chi} \tag{44}$$

will satisfy the differential equation $MR^* = 0$. The condition (33) will obviously be satisfied by (44).

Formulas (39) and (40) yield equations for the determination of R with that value of R^*. These equations will simplify by means of the Ansatz:

$$R = \frac{1}{2\pi c} e^{-\frac{ie}{hc}\chi} R'. \tag{45}$$

This substitution has the effect that one replaces the potentials $\mathbf{A}$, Φ with:

$$\mathbf{A}' = \mathbf{A} - \text{grad}\, \chi, \qquad \Phi' = \Phi + \frac{1}{c}\frac{\partial \chi}{\partial t}. \tag{46}$$

If $\mathbf{A}$ and Φ satisfy the equations:

$$\Box \mathbf{A} = 0, \qquad \Box\, \Phi = 0, \qquad \text{div}\, \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0 \tag{47}$$

then they will also be satisfied by $\mathbf{A}'$, Φ'. Furthermore, one has the relation:

$$(\mathbf{r} - \mathbf{r}^0)\, \mathbf{A}' - c\,(t - t^0)\, \Phi' = 0, \tag{48}$$

which will follow from (43).

The new potentials will be determined by the field uniquely. If one lets a double overbar denote the mean that is taken between the points ($\mathbf{r}^0$, t^0) and ($\mathbf{r}$, t) according to the formula:

$$\overline{\overline{f}} = 2\int_0^1 f(\mathbf{r}^0 + (\mathbf{r} - \mathbf{r}^0)u, t^0 + (t - t^0)u)\, u\, du \tag{49}$$

then one will have:

$$\mathbf{A}' = -\tfrac{1}{2}[(\mathbf{r} - \mathbf{r}^0) \times \overline{\overline{\mathfrak{H}}}] - \tfrac{1}{2} c\,(t - t^0)\, \overline{\overline{\mathfrak{E}}}, \quad \Phi' = -\tfrac{1}{2}(\mathbf{r} - \mathbf{r}^0) \cdot \overline{\overline{\mathfrak{E}}}. \tag{50}$$

After the substitutions (45) – i.e., after the introduction of new potentials – equations (39) and (40) will assume the following form:

$$\Lambda' R' = 0, \tag{51}$$

$$(L + 1)\, R' = -\tfrac{1}{4}\Lambda'\, 1 = -\frac{1}{4}\left\{\frac{m^2c^2}{h^2} + \frac{e^2}{h^2c^2}(\mathbf{A}'^2 - \Phi'^2)\right\} - \frac{1}{4}\frac{e}{hc}\{(\boldsymbol{\sigma}\cdot \mathrm{H}) - i\,(\boldsymbol{\alpha}\cdot \mathrm{E})\}, \quad (52)$$

in which L denotes the operator that is defined by:

$$Lf = (\mathbf{r} - \mathbf{r}^0)\cdot \operatorname{grad} f + (t - t^0)\,\frac{\partial f}{\partial t}. \quad (53)$$

(One can also denote it by M', since it emerges from M by introducing new potentials.)

Not only does equation (52) yield the value of $(L + 1)\, R'$ for $\xi = 0$, but it also allows one to calculate the value of R' for $\xi = 0$. Namely, if one considers a function $f(x, y, z, t)$ to be a function of the quotient $(x - x^0) : (y - y^0) : (z - z^0) : c\,(t - t^0)$ and the quantity ξ and assumes that one also has $\xi \dfrac{\partial f}{\partial \xi} = 0$ for $x \to 0$ then one can determine the value of $(L + 1)\, f$ for $\xi = 0$ from the value of f for $\xi = 0$, and conversely.

We consider the equation:

$$(L + p)\, f = \varphi(\mathbf{r}, t), \quad (54)$$

in which p is a positive whole number. One solution of that equation is:

$$f(\mathbf{r}, t) = \int_0^1 \varphi(\mathbf{r}^0 + (\mathbf{r} - \mathbf{r}^0)\,u, t^0 + (t - t^0)\,u)\, u^{p-1} du\,. \quad (55)$$

However, the only solution of the homogeneous equation $(L + p)\, f = 0$ that is regular in the neighborhood of $\mathbf{r} = \mathbf{r}^0$, $t = t^0$ is $f = 0$. Therefore, for a positive p, the function f will be determined uniquely by equations (54) and (55). If one sets φ equal to the right-hand side of (52) in (54) then one will get the value of the Riemann function on the light-cone by an application of formula (55).

7. It emerges from the classical investigations of **Hadamard** ([1]) on the Cauchy problem that the Riemann function of a differential equation of hyperbolic type is closely related to the fundamental solution (*solution élémentaire*) of that equation. The function $1 / r$ can serve as an example of a fundamental solution for the Laplace equation and:

$$\frac{1}{\sqrt{c^2(t-t^0)^2 - (x-x^0)^2 - (y-y^0)^2 - (z-z^0)^2}}$$

can serve as example of the fundamental solution for the equation:

([1]) **J. Hadamard**, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Paris, Hermann, 1932.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

The fundamental solution of an equation of hyperbolic type possesses a singularity on the characteristic cone that depends upon the form of the equation. In the case of an odd number of independent variables, the fundamental solution will be determined uniquely by the form of the equation. In the case of an even number, there will exist infinitely many fundamental solutions. Those solutions will possess a logarithmic singularity, in which the coefficient of the logarithm will be precisely the Riemann function. A fundamental solution can also be constructed for equations of parabolic type. One obtains them from a fundamental solution in the elliptic or hyperbolic case by passing to the limit. The function:

$$u = \frac{1}{\sqrt{y}} e^{-x^2/4y}$$

gives an elementary example of the fundamental solution of the parabolic equation $\partial^2 u / \partial x^2 = \partial u / \partial y$.

With those prefatory remarks, we now go on to the Dirac equation. The Riemann function of the second-order Dirac equation can be represented in the form of an integral over the proper time τ:

$$R = \int F \, d\tau , \tag{56}$$

in which F represents the fundamental solution of the Dirac equation with proper time:

$$\frac{h^2}{2m} \Lambda F = ih \frac{\partial F}{\partial \tau} . \tag{57}$$

The independent variables in this equation are the five quantities x, y, z, t, τ. Only one fundamental solution will exist since their number is odd.

We would like to determine the character of the fundamental solution in the vicinity of the essential singular point $\tau = 0$. To that end, we employ our previous Ansatz:

$$F = e^{iS/h} f \tag{58}$$

and develop S, as well as f, in powers of τ. The function S satisfies the Hamilton-Jacobi equation (I.12) with proper time. If one substitutes the development:

$$S = \frac{S_{-1}}{\tau} + S_0 + S_1 \tau + S_2 \tau^2 + \ldots \tag{59}$$

then one will get:

$$(\text{grad } S_{-1})^2 - \frac{1}{c^2}\left(\frac{\partial S_{-1}}{\partial t}\right)^2 = 2m \, S_{-1} . \tag{60}$$

We can then set:

$$S_{-1} = \tfrac{1}{2} m\, [(\mathbf{r} - \mathbf{r}^0)^2 - c^2\, (t - t^{\,0})^2] = -\tfrac{1}{2} m\, \xi. \tag{61}$$

With this value of S_{-1}, the equation for S_0 reads:

$$(\mathbf{r} - \mathbf{r}^0) \cdot \operatorname{grad} S_0 + (t - t^{\,0}) \frac{\partial S_0}{\partial t} = -\frac{e}{c} [(\mathbf{r} - \mathbf{r}^0) \cdot \mathbf{A} - c\, (t - t^{\,0})\, \Phi]. \tag{62}$$

That equation agrees with equation (43) for χ, up to the factor $- e / c$ on the right. We then have:

$$S_0 = -\frac{e}{c} \chi = -\frac{e}{c} \int_{(r^0, t^0)}^{(r,t)} (\mathbf{A} \cdot d\mathbf{r} - c \Phi\, dt)\,. \tag{63}$$

The equation for S_1 finally reads:

$$(L + 1)\, S_1 = -\frac{1}{2m} \left\{ m^2 c^2 + \frac{e^2}{c^2} (\mathbf{A}'^2 - \Phi'^2) \right\}, \tag{64}$$

with the meaning for L in (53). One gets the solution to this equation with the help of formula (55). The further coefficients in the development (59) are determined uniquely from equations of the form:

$$(L + p)\, S_p = \varphi_p \qquad (p = 2, 3, \ldots), \tag{65}$$

in which φ_p is known once the foregoing approximations are given. In that way, we will get:

$$S = -\frac{m\xi}{2\tau} - \frac{e}{c} \chi - \frac{1}{2} m c^2 \tau - \frac{e^2 \tau}{2 m c^2} \int_0^1 (\mathbf{A}'^2 - \Phi'^2)\, du + \ldots \tag{66}$$

This formula will yield the exact value for S in the field-free case.

Equation (16) can also be solved for f analogously. One gets:

$$f = \frac{C}{\tau^2} \left\{ 1 - \frac{e\,\tau}{2mc} \int_0^1 [i(\boldsymbol{\sigma} \cdot \mathfrak{H}) + (\boldsymbol{\alpha} \cdot \mathfrak{E})]\, du + \cdots \right\}. \tag{67}$$

We must now investigate how the integration path in the integral:

$$R = \int e^{iS/h} f\, d\tau \tag{68}$$

must be chosen in order for it to yield the Riemann function. The integral (68) obviously satisfies the Dirac equation. In order for it to coincide with the Riemann function, the condition (40) or (52) must still be satisfied on the light-cone $\xi = 0$. We would like to show that that will be the case when one chooses a small circle around the point $\tau = 0$ in the complex τ-plane to be the integration path. One then observes that the action function S no longer has a pole for $\xi = 0$, such that the point $\tau = 0$ is not an essential singularity of the integrand. The integral can then be evaluated simply by finding the residue at the pole for $\tau = 0$.

In the neighborhood of $\tau = 0$, the integrand is equal to (for $\xi = 0$):

$$F = e^{-\frac{ie}{hc}\chi}\left[1 - \frac{imc^2}{2h}\tau - \frac{ie^2\tau}{2hmc^2}\int_0^1 (\mathbf{A}'^2 - \Phi'^2)\,du - \frac{e\tau}{2mc}\int_0^1 (i(\boldsymbol{\sigma}\cdot\mathfrak{H}) + (\boldsymbol{\alpha}\cdot\mathfrak{E}))\,du + \cdots\right]. \quad (69)$$

Hence, from (45), we will have:

$$R = \frac{1}{2\pi c} e^{-\frac{ie}{hc}\chi} R', \quad (70)$$

with:

$$R' = 2\pi^2 C\,\frac{hc}{m}\int_0^1 \left[\frac{m^2c^2}{h^2} + \frac{e^2}{h^2c^2}(\mathbf{A}'^2 - \Phi^2) + \frac{e}{hc}(\boldsymbol{\sigma}\cdot\mathfrak{H}) - i\,(\boldsymbol{\alpha}\cdot\mathfrak{E})\right] du. \quad (70^*)$$

Equation (52) will be satisfied when we set the constant C equal to:

$$C = -\frac{m}{8\pi^2 hc}. \quad (71)$$

The integral (68) is actually the Riemann function with that value of C.

For a suitable choice of the integration path, one can also obtain the fundamental solution of the second-order Dirac equation from (68).

That fundamental solution U of the equation $\Lambda U = 0$ has the form:

$$U = \frac{1}{2i}\left[R \ln|\xi| + \frac{R^*}{\xi}\right] + U^*, \quad (72)$$

in which R and R^* have the previous meaning. [R is the Riemann function, R^* is the expression (44).] The function U^* in (72) is regular in the neighborhood of $\xi = 0$. It is merely subject to the condition that the total expression (72) must satisfy the equation $\Lambda U = 0$. That condition obviously does not suffice to establish U^* uniquely. Therefore, there are also different fundamental solutions that differ from each other by the values of their regular parts, and which can be derived from the integral (68) by a corresponding choice of the integration path. The multi-valuedness of the fundamental solution corresponds to the fact that the number of independent variables in the Dirac equation without proper time is odd.

8. As an example of this, we consider the motion of an electron in a constant electric and magnetic field and calculate the Riemann function for that case. In order to simplify the calculations, we restrict ourselves to the case of parallel fields whose direction we choose to be the z-axis.

If one sets the potentials equal to:

$$A_x = -\tfrac{1}{2}Hy, \quad A_y = -\tfrac{1}{2}Hx, \quad A_z = 0, \qquad \Phi = -Ez \tag{73}$$

then the classical equations of motion that correspond to the Lagrangian function (I.6) can be solved easily. If one forms the expression for the action integral (I.7) then one will find:

$$S = S_0 - \frac{1}{2}mc^2\tau + \frac{eE}{4c}[(z-z^0)^2 - c^2(t-t^0)^2]\coth\frac{eE\tau}{2mc} + \frac{eE}{4c}[(x-x^0)^2 + (y-y^0)^2]\cot\frac{eH\tau}{2mc}, \tag{74}$$

in which S_0 denotes the quantity:

$$S_0 = -\frac{e}{c}\chi = -\frac{eE}{2}(z+z^0)(t-t^0) - \frac{eH}{2c}(x^0\,y - y^0\,z), \tag{74*}$$

which is independent of t.

If one considers equation (13) for f then one will easily see that this equation can be fulfilled by function f that independent of only τ, such that one will have $\Box f = 0$. However, equation (13) will then reduce to (16), the latter of which was already transformed into (21). In our case, the determinant ρ is equal to:

$$\rho = \frac{\text{const.}}{\sin^2\frac{eH\tau}{2mc}\sinh^2\frac{eE\tau}{2mc}}, \tag{75}$$

and the solution of (21) reads:

$$f^0 = \exp\left[-\frac{ie}{2mc}\sigma_z H\tau - \frac{e}{2mc}\alpha_z E\tau\right]. \tag{76}$$

If one determines the constant factor in f by way of (71) then one will get the following expression for f:

$$f = -\frac{m}{8\pi^2 hc}\cdot\left(\frac{eH}{2mc}\right)\left(\frac{eE}{2mc}\right)\cdot\frac{f^0}{\sin\frac{eH\tau}{2mc}\sinh\frac{eE\tau}{2mc}}. \tag{77}$$

With that value of f, the integral:

$$R = \int e^{iS/h} f\,d\tau, \tag{68}$$

which is taken around a small circle surrounding $\tau = 0$, will yield the Riemann function of the problem in question.

In the special case where no field is present, one will get:

$$f = -\frac{m}{8\pi^2 hc}\cdot\frac{1}{\tau^2}, \qquad S = -\frac{m\xi}{2\tau} - \frac{1}{2}mc^2\tau, \tag{78}$$

and as a result:

$$R = -\frac{m}{8\pi^2 hc}\int e^{-\frac{im\xi}{2h\tau}-\frac{imc^2\tau}{2h}}\frac{d\tau}{\tau^2} = -\frac{m}{4\pi h\sqrt{\xi}}J_1\left(\frac{mc}{h}\sqrt{\xi}\right), \tag{79}$$

while the quantity R^* has the constant value $1 / 2\pi c$. In the absence of a field, the function Q of the general theory is then equal to:

$$Q = -\frac{m}{4\pi h\sqrt{\xi}}J_1\left(\frac{mc}{h}\sqrt{\xi}\right)\gamma(\xi) + \frac{1}{2\pi c}\delta(\xi). \tag{80}$$

III. Application to the theory of positrons.

The foundations of the theory of positrons in the form that it was given by **Dirac** ([1]) point to the consideration of the "mixed density" for the distribution of electrons in states of negative and positive energy.

Dirac considered mixed densities of two types, namely, R_1 and R_F, for which one will have:

$$(\mathbf{r}, t, \zeta \,|\, R_1 \,|\, \mathbf{r}^0, t^0, \zeta^0) = \sum_{occ.}\psi(\mathbf{r},t,\zeta)\bar{\psi}(\mathbf{r}^0,t^0,\zeta^0) - \sum_{unocc.}\psi(\mathbf{r},t,\zeta)\bar{\psi}(\mathbf{r}^0,t^0,\zeta^0), \tag{1}$$

$$(\mathbf{r}, t, \zeta \,|\, R_F \,|\, \mathbf{r}^0, t^0, \zeta^0) = \sum_{occ.}\psi(\mathbf{r},t,\zeta)\bar{\psi}(\mathbf{r}^0,t^0,\zeta^0) + \sum_{unocc.}\psi(\mathbf{r},t,\zeta)\bar{\psi}(\mathbf{r}^0,t^0,\zeta^0). \tag{2}$$

In these formulas, $\psi(\mathbf{r}, t, \zeta)$ refers to the wave function of an electron that depends upon the coordinates $\mathbf{r}$, time t, and the spin quantities (component numbers) ζ. The summations will be made over an index (viz., the number of states) that we have suppressed. The first sum is taken over all occupied states, while the second one is taken over all unoccupied ones.

Dirac calculated the expressions (1) and (2) by direct summation for the field-free case and examined their singularities on the light-cone. He then constructed analogous expressions for the case of an arbitrary field and determined their singularities from the requirement that the aforementioned expressions must satisfy the wave equation and should go over to the previously-calculated expressions for a vanishing field.

We would now like to show that the quantities (1) and (2) are precisely the ones that appear in the theory of the Cauchy problem for hyperbolic differential equations.

We first consider the expression (2). The quantity R_F, as a matrix in the variables ζ and ζ^0, can be written:

$$(\mathbf{r}, t \,|\, R_F \,|\, \mathbf{r}^0, t^0) = R_F(\mathbf{r}, t). \tag{3}$$

([1]) **P. A. M. Dirac**, Proc. Camb. Phil. Soc. **30** (1934), 150.

This expression, when considered to be a function of **r** and t, satisfies the Dirac wave equation:

$$\left\{(\boldsymbol{\alpha}\cdot\mathbf{P}) + mc\,\alpha_4 - \frac{T}{c}\right\} R_F = 0 \tag{4}$$

and for $t = t^0$ it will reduce to the kernel of the identity operator:

$$R_F = \delta(\mathbf{r} - \mathbf{r}^0) \qquad \text{for } t = t^0. \tag{5}$$

However, the function R_F will be determined uniquely by (4) and (5). One therefore does not need to calculate it by direct summation at all; the function R_F can also be determined by solving the Cauchy problem, moreover.

The function Q [equation (II.26)] that we examined in Part II satisfies the second-order Dirac equation and the initial conditions:

$$Q = 0, \qquad \frac{\partial Q}{\partial t} = \delta(\mathbf{r} - \mathbf{r}^0) \qquad \text{for } t = t^0. \tag{6}$$

However, it will follow from this that the expression:

$$R_F = -\frac{ic}{h}\left\{(\boldsymbol{\alpha}\cdot\mathbf{P}) + mc\,\alpha_4 + \frac{T}{c}\right\} Q, \tag{7}$$

with

$$Q = R\,\gamma(\xi) + R^*\delta(\xi), \tag{8}$$

satisfies equations (4) and (5).

The Dirac function R_F is then expressed in terms of the Riemann function R.

As far as the other Dirac function R_1 is concerned, it can be expressed in terms of the fundamental solution U in the same way that R_F is expressed in terms of Q. We have:

$$R_1 = -\frac{ic}{h}\left\{(\boldsymbol{\alpha}\cdot\mathbf{P}) + mc\,\alpha_4 + \frac{T}{c}\right\} U, \tag{9}$$

with:

$$U = \frac{1}{\pi i}\left[R\ln|\xi| + \frac{R^*}{\xi}\right] + U^*. \tag{10}$$

One can refer to the expression (9) as the fundamental solution to the first-order Dirac equation.

A unique determination of the function R_1 is possible only with the help of initial conditions that must express certain physical assumptions. For example, one can demand that all negative states of the electron should be occupied for $t = t^0$ and all positive ones should be free. As one can see in the original definition (1) of the function R_1, that function must go to the kernel (taken to be negative) of the operator for the sign of the kinetic energy for $t = t^0$. When one solves the Dirac equation with those initial conditions, one will get the values of the mixed density R_1 for all $t > t^0$.

In conclusion, let us briefly touch upon the physical meaning of the mixed density. Which terms in the expression considered have any physical sense? Dirac, as well as **Heisenberg** ([1]), proposed fixing the singularities of R_1, subtracting them from the total expression for R_1, and interpreting the singularity-free remainder as the physically observable density. However, that process can hardly be regarded as correct, since it contains considerable arbitrariness. We believe that only the correspondence principle can yield a definite criterion for resolving that question. For the problem in question, the demands of the correspondence principle can perhaps be understood as follows: The only terms that can have any physical meaning are the ones that remain finite everywhere as $h \to 0$, and therefore on the light-cone $\xi = 0$, as well (i.e., uniformly in ξ). The remaining terms are to be dropped as physically meaningless. That criterion was confirmed by the investigations of vacuum polarization that were carried out by various authors ([2]). The supplementary terms in the Lagrangian function of the electromagnetic field that these authors calculated, in fact, defined simply a series in increasing powers of h. That situation also gives one a hint of the applicability of the method of Brillouin-Wentzel that we considered in Part II to the problem considered.

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([1]) **W. Heisenberg**, Zeit. Phys. **90** (1934), 209.
([2]) Cf., **V. Weisskopf**, "Über die Elektrodynamik des Vacuums auf Grund der Quantentheorie des Elektrons," Copenhagen, 1936.