"Géométrie quantique linéaire et déplacement parallèle," C. R. Acad. Sci. Paris **188** (1929), 1470-1472.

THEORETICAL PHYSICS. – *Linear quantum geometry and parallel displacement*, Note ([1]) by **V. FOCK** and **D. IVANENKO**, presented by Maurice de Broglie.

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1. Recent developments in the theory of quanta lead one to believe that Riemannian geometry, with its fundamental form ds^2, mostly explains gravitational phenomena, while quantum and electric phenomena demand the introduction of new geometric notions that are foreign to Riemannian geometry. Such notions were introduced for the first time (if only implicitly) by Dirac in his theory of the electron. The geometric character of Dirac's operators α_k was pointed out by the authors of this note ([2]), who proposed the introduction of operators that are analogous to the Dirac matrices into geometry and the consideration of the linear differential form:

$$ds = \sum_{\nu} \gamma_\nu dx_\nu \,, \tag{1}$$

whose square gives the usual ds^2 of Riemann. That modification of geometry was called *linear quantum geometry*.

2. In the sequel, it will be useful to introduce, with Ricci and Levi-Civita, an orthogonal *n*-hedron of directions that is defined at each point of space. By means of that *n*-hedron, one can define a geometric quantity whose components transform under an arbitrary rotation of the *n*-hedron like the Dirac Ψ-functions. That quantity will be referred to by the name of "semi-vector" (Landau).

Following Levi-Civita, the geometry of ds^2 can be based upon the study of the infinitesimal parallel displacement of a vector. In an analogous manner, the notion of the infinitesimal displacement of a semi-vector can serve as the point of departure for the study of linear geometry. We write the increase in the components of a semi-vector as:

$$\delta\Psi = \sum_{i} C_i ds_i \Psi \,. \tag{2}$$

The C_i are operators (i.e., matrices) that operate on the components of Ψ; the ds_i are the components of the displacement along the directions of the *n*-hedron. The equation for the quantity Ψ^+ that is adjoint to Ψ will be written:

([1]) Session on 22 May 1929.
([2]) "Ueber eine mögliche geometrische Deutung der relativistischen Quantentheorie" (in press).

(2*) $$\delta\Psi^+ = \Psi^+ \sum_i C_i^+ ds_i \, .$$

If one introduces the Dirac matrices α_i , which satisfy the relations:

$$\alpha_i \, \alpha_k + \alpha_k \, \alpha_i = 2 \, \delta_{ik} \, ,$$

and defines the vector $A_i = \Psi^+ \alpha_i \Psi$ then if formulas (2) and (2*) are given, we can calculate the change that the vector A undergoes:

(3) $$\delta A_i = \delta(\psi^+ \alpha_i \, \psi) \, \psi = \psi^+ \sum_l (C_l^+ \alpha_i + \alpha_i C_l) ds_l \, \psi \, .$$

That change must be a linear function of the A_i, namely:

(4) $$\delta A_i = \sum_{kl} \gamma_{ikl} A_k ds_l \, ,$$

where the γ_{ikl} are the Ricci coefficients. Upon comparing (3) and (4), we obtain:

(5) $$C_l^+ \alpha_i + \alpha_i C_l = \sum_k \gamma_{ikl} \alpha_k \, .$$

Upon multiplying (5) by α_i on the right and on the left, adding the results, and taking into account the identity $\gamma_{iil} = 0$, one obtains:

(6) $$\alpha_i (C_l + C_l^+) + (C_l + C_l^+) \alpha_i = 0.$$

One verifies that the identities $\gamma_{ikl} + \gamma_{kil} = 0$ are satisfied. The relation (6) shows that the operator C_l is of the form:

(7) $$C_l = g_l + i\Phi_l \, ,$$

where g_l and Φ_l are Hermitian operators (i.e., they are identical to their adjoints) and g_l further verifies the relation $g_l \, \alpha_i + g_i \, \alpha_l = 0$. Upon introducing (7) into (5), one obtains:

(5*) $$i(\alpha_i \, \Phi_l - \Phi_l \, \alpha_i) = \sum_k \gamma_{ikl} \alpha_k \, .$$

3. Formula (2) immediately provides the law of of covariant differentiation of a semi-vector, namely:

(8) $$\nabla_l \psi = \left(\frac{\partial}{\partial s_l} - C_l \right) \psi \, .$$

Set $g_l = 0$, $\Phi_l = \dfrac{2\pi e}{hc} \varphi_l$, where φ_l are the components of the vector potential, and introduce the coordinates x_α ; formula (8) then gives:

(9) $$\nabla_a \psi = \left(\frac{\partial}{\partial x_\alpha} - \frac{2\pi i}{h} \frac{e}{c} \varphi_\alpha \right) \psi \, .$$

This is precisely the expression that appears in the Dirac equations. Upon introducing φ_α into formula (2), one obtains:

(10) $$\delta \psi = \frac{2\pi ie}{hc} \sum_\alpha \varphi_\alpha dx_\alpha \psi \, .$$

Therefore, Weyl's linear differential form must appear in the law of displacement for a semi-vector.

4. A complete theory must provide equations for the C_l and the γ_{ikl} that are analogous to those of Maxwell and Einstein.

It is important to point out an aspect of this theory that distinguishes the ideas that were presented in this note from those of Einstein and Levi-Civita: the intervention of matrices-operators into the equations for purely geometric quantities. Thanks to that, one can indeed imagine an electromagnetic field in Euclidian space, which is impossible in the other theories.