"Ein invarianter Variationssatz für die Bewegung mehrerer elektrischer Massenteilchen," Zeit. Phys. **58** (1929), 386-393.

# An invariant variational principle for the motion of several electric mass-points

By **A. D. Fokker** in Haarlem

With 2 Figures. (Received on 1 October 1929)

Translated by D. H. Delphenich

---

**Abstract:** A variational principle for the point mechanics of several electric mass-points is exhibited that is invariant under Lorentz transformations and into which the motions of the particles will enter in a completely-symmetric way by means of an application of retarded and advanced potentials. The form of the conservation laws for energy and impulse will be determined and their consequences for the definition of those quantities will be discussed.

Quantum mechanics correspondingly took its starting point from the classical theory of the dynamics of a single particle and formulated its laws and methods for the one-body problem in connection with **Hamilton**'s canonical equations. Wherever it brought the interaction of several particles under consideration, it nonetheless found no form that the demand of invariance under Lorentz transformation might take. In that case, even the previous work in the classical theory was absent.

It would therefore seem useful to make an attempt to arrive at a foundation for such a theory in the form of **Hamilton**'s variation principle and develop it further (*) when it is concerned exclusively with the motions and actions of the particles on each other but does without any consideration of a field completely (**).

If one would like to give a relativistic form to the usual conception of the variational principle for a mass-point:

$$\delta \int (T - U)\, dt = 0 ,$$

into which the kinetic and potential energy $T$ and $U$ enter, then, as one knows, one must write:

---

(*) Cf., Physica **9** (1929), pp. 33.

(**) **W. Heisenberg** and **W. Pauli** started from a consideration of the field recently, Zeit. Phys. **56** (1929), pp. 1.

$$- mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} dt = - m\, c\, ds$$

in place of $T\,dt$, in which $m$ represents the mass, $v$ represents the velocity of the particle, and $c$ is the speed of light.

In order to extend the expression $U\,dt$ to an invariant scalar, one must imagine that the potential energy is only the temporal component of a covariant four-vector whose spatial components represent the negative components $-A_x$, $-A_y$, $-A_z$ of an impulse, and by analogy with the energy component one might call it a "potential" impulse. Such an extension of the four-vector will make it possible for one to write the scalar:

$$- U\,dt + A_x\,dx + A_x\,dy + A_x\,dz$$

in place of $-U\,dt$. If one is dealing with the motion of an electrical particle then the covariant potential energy-impulse vector will given by the product of its charge and the field potential, in which the vector potential is divided by $-e$.

The potential at the reference point $X$ with the space-time coordinates:

$$x^0,\ x^1,\ x^2,\ x^3\ (=)\ t, x, y, z$$

might originate at a charge $e$ whose motion is given by representing its time and space coordinates $w^i$ ($i = 0, 1, 2, 3$) as functions of a parameter $u$. In order to find the potential, according to **Liénard** and **Wiechert**, we must look for the "effective" space-time point $W$ of the generating charge from which the reference point can be reached at the speed of light. Therefore, we should have:

$$c^2\,(x^0-w^0)^2-(x^1-w^1)^2-(x^2-w^2)^2-(x^3-w^3)^2 = R^2 = 0$$

or

$$x^0-w^0 = \frac{r}{c}, \qquad r = \sqrt{(x^1-w^1)^2+(x^2-w^2)^2+(x^3-w^3)^2}\,.$$

When the velocity component of the generating charge that points in the radial direction at that space-time point is $v_r$, the potentials $\varphi$ and $a_x$, $a_y$, $a_z$ will be:

$$\varphi = \frac{e}{4\pi}\,\frac{1}{r\left(1-\frac{v_r}{c}\right)}, \qquad a_x = \frac{e}{4\pi\,c}\,\frac{\frac{dw^1}{dw^0}}{r\left(1-\frac{v_r}{c}\right)}, \quad \text{etc.}$$

In order to define the covariant vector from this, we write:

$$\varphi = \frac{e}{4\pi c}\,\frac{c^2\,dw^0}{c^2\,(x^0-w^0)^2\,dw^0-(x^1-w^1)^2\,dw^1-\cdots} = \frac{e}{4\pi c}\,\frac{dw^0}{(R\cdot dw)},$$

and correspondingly:

$$-\frac{\mathrm{a}_x}{c} = \frac{e}{4\pi c}\,\frac{-dw^1}{c^2\,(x^0-w^0)^2\,dw^0-(x^1-w^1)^2\,dw^1-\cdots} = \frac{e}{4\pi c}\,\frac{dw^1}{(R\cdot dw)}, \quad \text{etc.}$$

Let $(R \cdot dw)$ denote the scalar product of the *contact ray:*

$$R^i = x^i - w^i \qquad\qquad (i = 0, 1, 2, 3)$$

and let $dw^1$ denote the four-dimensional element of motion.

We have in that the *retarded* potential at $X$ that is generated by $e$ at $W$.

Now let the motions of the charges $e_2$, $e_3$, … be given by their time and space coordinates $y^i(u)$, $z^i(v)$, … as functions of the parameters $u$, $v$, …

If one would like to determine the motion $x^i$ of the mass-point $m_1$ with the charge $e_1$ under their influence then from what was said before, one must pose the variational principle:

$$0 = \delta\left[\int -m_1\,c\,ds_1 - \frac{e_1\,e_2}{4\pi c}\int\frac{(dx\cdot dy)}{(R\cdot dy)} - \frac{e_1\,e_2}{4\pi c}\int\frac{(dx\cdot dz)}{(S\cdot dz)} - \cdots\right],$$

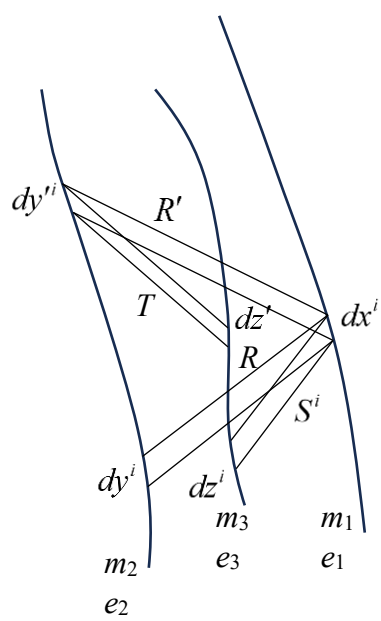


Figure 1.

in which $ds_1$ represent the magnitude of the arc-length element $dx^i$, and $R^i$, $S^i$, … are the contact rays that connect the element $dx^i$ with the corresponding effective elements $dy^i$, $dz^i$, … and have the magnitude zero (Fig. 1).

We would like to establish that we can always make a certain elementary division $dx^i$ of the motion of the charge $e_1$ correspond to an infinitesimal division of the other charges in such a way that we will continually have:

$$R^2 = 0\,, \qquad\qquad S^2 = 0\,, \qquad \text{etc.}$$

A consequence of that is that with our correlations, we will always have:

$$(R\cdot dy) = (R\cdot dx)\,, \qquad (S\cdot dy) = (S\cdot dx)\,, \qquad \text{etc.}$$

If the motions of the charges $e_1$, $e_3$, … are given, and we are supposed to determine the motion of the charge $e_2$ with mass $m_2$ then we would make the element $dy'^i$ of that motion (with the arc-length $ds_2$) correspond to the element $dx^i$, $dz'^i$, … of the other charge motion by means of the contact rays $R'$, $T$, … and exhibit the law:

$$0 = \delta\left[\int - m_2\, c\, ds_2 - \frac{e_1 e_2}{4\pi c}\int \frac{(dy'\cdot dy)}{(R'\cdot dy)} - \frac{e_2 e_3}{4\pi c}\int \frac{(dy'\cdot dz')}{(T\cdot dz')} - \cdots\right].$$

We must do something similar for the motion of the third charge, and likewise for the other charges.

One might wish to have a single variational principle for the interaction of the whole system instead of one for each particle. However, that contradicts the fact that the action integral:

$$-\frac{e_1 e_2}{4\pi c}\int \frac{(dx\cdot dy)}{(R\cdot dy)}$$

indeed accounts for the influence of the motion of $e_1$ by its retarded action on $e_2$, but does not agree with the corresponding integral that reproduces the reciprocal influence of the motion of $e_2$ by the retarded action of $e_1$. However, one can observe that due to the fact that $(R \cdot dy) = (R \cdot dx)$, when the integral is written out in the form:

$$-\frac{e_2 e_1}{4\pi c}\int \frac{(dy\cdot dx)}{(R\cdot dx)},$$

it will reproduce the action of $e_2$ by way of the *advanced* potential that $e_1$ creates.

One might get used to the idea there is something arbitrary about calculating only the retarded effects (and this is not the first time that this idea has been emphasized). With an eye towards achieving a complete reciprocity of the interaction, it would be appropriate to consider *half* of the action of $e_2$ on $e_1$ *as retarded, while the other half is advanced.* That would have the consequence that in the variational principle for the motion of the first particle, the effect of the second particles will be represented by precisely the integral that gives the effect of the first particles on the second one in its law of motion. That would make it further possible to exhibit *a single variational principle for the collective motion*, and indeed in the form of:

$$0 = \delta\left[\sum\int - m_i\, c\, ds_i - \sum \frac{e_i e_j}{8\pi c}\left\{\int \frac{(dx\cdot dy)}{(R'\cdot dy)} + \int \frac{(dy'\cdot dx)}{(R'\cdot dx)}\right\}\right],$$

in which the first sum extends over the world-lines of the corresponding individual particles, and the second sum extends over the interaction integrals that correspond to the particle-pairs.

That law is completely invariant under Lorentz transformations. It makes no reference to any field. The symmetry and reciprocity in the interactions of the motions are complete. One must say that it refers more to the system of motions than to the system of particles.

The concept of a system of particles would demand a certain ordering of its elements of motion. One could indeed define that ordering uniquely, but not invariantly, or invariantly, but not uniquely. However, it is impossible to simultaneously define the particle system invariantly and uniquely. It is also unavoidable that one must regard the phenomenon as a system of motions and not a moving system of particles.

We would now like to perform a variation of the motion of the charge $e_1$ by displacing each space-time point $x^i$ of that motion by an infinitesimal space-time segment $\delta x^i$, and in that way, calculate the variation of the interaction integral:

$$-\frac{e_1 e_2}{8\pi c}\delta\int\frac{(dx\cdot dy)}{(R\cdot dy)}$$

that corresponds to the retarded action of $e_2$ on $e_1$. The correspondence that is established between the elements $dx^i$ and $dy^i$ must remain preserved under the variation. The contact ray $R^i$ between them shall represent a light signal ($R = 0$), and in order to do that, the variation $\delta x^i$ shall take the motion of the charge $e_2$ to itself by a displacement:

$$\delta y^i = \frac{dy^i}{(R\cdot dy)}(R\cdot\delta x).$$

Nothing about that motion will be changed by it, except that the necessary correspondence between the elements of motion of the particle in the varied action integral will be guaranteed by $(R\cdot\delta y) = (R\cdot\delta x)$, and therefore $\delta R^2 = 0$. It must also remain true that $(R\cdot dy) = (R\cdot dx)$ for the corresponding elements such that we can take the variation of $(R\cdot dx)$ to be the variation of denominator.

We can now write:

$$\delta\frac{(dx\cdot dy)}{(R\cdot dy)} = \frac{(\delta dx\cdot dy)+(dx\cdot\delta dy)}{(R\cdot dy)} - \delta\frac{(dx\cdot dy)\,\delta(R\cdot dx)}{(R\cdot dy)(R\cdot dx)}.$$

One can partially integrate that expression and establish that the variation should vanish at the limits of the integral. When one considers the value of $\delta y^i$, one will get:

$$\sum_{i,m}\delta x^i\left[-d\left\{\frac{dy_i}{(R\cdot dy)} - R_i\frac{(dx\cdot dy)}{(R\cdot dy)(R\cdot dy)}\right\}\right.$$
$$\left. - R_i\frac{dy^m}{(R\cdot dy)}d\frac{dx_m}{(R\cdot dy)} - dx_i\frac{(dx\cdot dy)}{(R\cdot dx)(R\cdot dy)} + R_i\frac{(dx\cdot dy)^2}{(R\cdot dx)(R\cdot dy)^2}\right]$$

in the integrand. The expression in the bracket will give the increase in kinetic energy and momentum (for $i = 0$ or $i = 1, 2, 3$, resp.), so we say briefly that it is the force that is exerted upon the first particle. One sees that except for their dependency on velocity, which is neglected in the usual derivations, part of the forces will depend upon the acceleration of the active charge and part of them will not. That corresponds to the electrostatic action (i.e., the last term) and the so-called radiation reaction (i.e., the term in curly brackets, with the following ones).

In our Ansatz, we have also included the advanced action of $e_2$ in the form of the integral:

$$-\frac{e_1 e_2}{8\pi c}\delta\int\frac{(dx\cdot dy')}{(R'\cdot dy)} = -\frac{e_1 e_2}{8\pi c}\delta\int\frac{(dx\cdot dy')}{(R'\cdot dy')}.$$

If one treats the variation in the same way here and observes that here one has $\delta R'^i = \delta y'^i - \delta x^i$ then one can soon write down the requirement for the vanishing of the variation of the total integral in the form:

$$\begin{aligned}0 = {} & d\left(m_1 c\frac{dx_i}{ds_1}\right) + \frac{e_1 e_2}{8\pi c}\Bigg[d\left\{\frac{dy_i}{(R\cdot dy)} - R_i\frac{(dx\cdot dy)}{(R\cdot dy)(R\cdot dy)}\right\} + d\left\{\frac{dy'_i}{(R'\cdot dy')} - R'_i\frac{(dx\cdot dy)}{(R'\cdot dy)(R'\cdot dy)}\right\} \\ & + R_i\sum\frac{dy^m}{(R\cdot dy)}d\frac{dx_m}{(R\cdot dy)} + dx_i\frac{(dx\cdot dy)}{(R\cdot dx)(R\cdot dy)} - R_i\frac{(dx\cdot dy)^2}{(R\cdot dx)(R\cdot dy)^2} \\ & + R'_i\sum\frac{dy'^m}{(R'\cdot dy)}d\frac{dx_m}{(R'\cdot dy)} - dx_i\frac{(dx\cdot dy')}{(R'\cdot dx)(R'\cdot dy')} + R'_i\frac{(dx\cdot dy')^2}{(R'\cdot dx)(R'\cdot dy')^2}\Bigg].\end{aligned}$$

One finds some total differentials in the outer bracketed expression, which is minus the covariant vector of the increment of kinetic energy and momentum (for $i = 0$ and $i = 1, 2, 3$) that is transferred from $e_2$ to $e_1$.

One can interpret that as the potential energy and impulse of the charge $e_1$ that is required by the presence and motion of $e_2$ .

One has a similarly-constructed equation for the motion of the second particle under the influence of the charge $e_1$, into which two elements of the motion of $e_1$ will enter. If we write out that equation for the element $dy'^i$ (Fig. 2), which is coupled with the previously-considered element $dx^i$ by the correlation $R' = 0$, and coupled with a second element by the correlation $R'' = 0$, then we will have:

$$0 = d\left(m_2 c\frac{dy'_i}{ds_2}\right) + \frac{e_1 e_2}{8\pi c}\Bigg[d\left\{\frac{dx_i}{(R'\cdot dy)} - R'_i\frac{(dy'\cdot dy)}{(R'\cdot dy)(R'\cdot dy')}\right\}$$

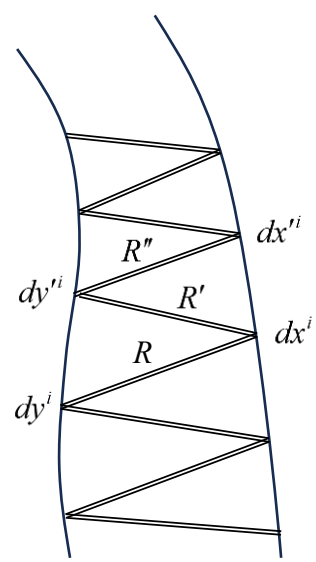


Figure 2.

$$+\ d\left\{\frac{dx_i''}{(R''\cdot dx'')}-R_i''\frac{(dy'\cdot dx'')}{(R''\cdot dx'')(R''\cdot dy')}\right\}$$

$$+\ R_i'\sum\frac{dx^m}{(R'\cdot dx)}d\frac{dy_m'}{(R'\cdot dy')}+dy_i'\frac{(dy'\cdot dy)}{(R'\cdot dy')(R'\cdot dx)}-R_i'\frac{(dx\cdot dy')^2}{(R'\cdot dy')(R'\cdot dx)^2}$$

$$+\ R_i''\sum\frac{dx''^m}{(R''\cdot dx'')}d\frac{dy_m'}{(R''\cdot dy')}-dy_i'\frac{(dy'\cdot dx'')}{(R''\cdot dy')(R''\cdot dx'')}+R_i''\frac{(dy'\cdot dx'')^2}{(R''\cdot dy')(R''\cdot dx'')^2}\Bigg]\ .$$

It is now important to note that the third line in that expression and the last line of the previous one collectively represent a total differential. They both refer to the reciprocal action that one finds along the contact radius $R_i'$ between the elements of motion $dx^i$ and $dy'^i$. They collectively yield:

$$\frac{e_1 e_2}{8\pi c}d\left\{R_i'\frac{(dx\cdot dy')}{(R'\cdot dx)(R'\cdot dy')}\right\}\ .$$

That shows us the path that leads to the form that the *law of conservation of energy and impulse* ($i$ = 0, 1, 2, 3) will take here. For the sake of simplicity, we shall consider the interaction of only two particles.

We must draw a zigzag chain of contact radii (Fig. 2) that constantly runs back and forth and write out the equations for all elements of motion that are coordinated with two neighboring chains of that type and combine everything.

Every turning point of the zigzag chain will yield a (kinetic) contribution to that sum:

$$d\left(m_1 c\frac{dx_i}{ds_1}\right)\quad\text{or}\qquad d\left(m_2 c\frac{dy_i}{ds_2}\right)$$

according to whether it lies in the motion of $e_1$ or $e_2$, resp., and every contact ray $R'^i$ between $dx^i$ and $dy'^i$ will yield a (potential) contribution:

$$\frac{e_1 e_2}{8\pi c}d\left\{\frac{dx_i}{(R'\cdot dx)}+\frac{dy_i'}{(R'\cdot dy')}-R_i'\frac{(dx\cdot dy')}{(R'\cdot dx)(R'\cdot dy')}\right\}.$$

When the motion is periodic, the chain will also be periodic or at least almost-periodic, and one will extend the aforementioned sum over one period, so it must close upon itself. It will then produce a total differential, and there will exist a quantity that will not change when the zigzag chain is displaced along the motion. If one divides that constant quantity by the number of pairs of contact radii that one can count in the closed period then one will get the energy for $i$ = 0 and minus the momentum for $i$ = 1, 2, 3.

That corresponds completely to the remark that was made above that energy and momentum cannot be defined for the system of particles here, but only for the system of two complete motions.

The situation can then arise in which the motions do not exhibit any periodicity, so one cannot assimilate the aforementioned sum into a total differential. There will always be small segments that are missing at the limits of the zigzag chain. If one neglects them and once more divides by the number of pairs of radii in the chain then one will indeed always be able to define the energy and momentum with ever greater precision the longer that one makes that chain, i.e., as one pursues the motion of a bigger spacetime, but will mainly still be true that the energy is not definable at a particular moment.

That is the downside to the fact that the field is excluded from point-mechanical formulation of the variational principle. In general, such behavior for energy and impulse does not contradict the quantum-mechanical way of looking at things.

Natuurkundig Labortorium
van Teyler's Stichting.