# The place of the relativity principle in the systems of mechanics and electrodynamics 

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## Introduction

In theoretical physics, one can distinguish essentially two different ways of deriving the fundamental equations for physical processes. The one starts from intuitive, but hypothetical, pictures (viz., the mechanistic theory), while the other places certain general relations between very general abstract concepts at its focus. Those concepts are non-intuitive, but not hypothetical. Rather, they correspond to observable quantities in nature. The second path would probably be referred to most aptly as the conceptual theory $\left({ }^{1}\right)$. The laws from which the conceptual theory started were laws of energy or minimal principles up to the most-recent times. By the work of Einstein, and most recently Minkowski, the relativity principles were added to them, which state (when we temporarily express it somewhat imprecisely) that the processes in a system will not change the system as a whole suffers a uniform rectilinear translatory motion. Mathematically, those principles correspond to invariant (covariance, respectively) under certain transformations, and indeed when we restrict ourselves to two variables (one spatial and one temporal), in Newtonian mechanics the group:

$$
\begin{align*}
x^{\prime} & =x-a t, \\
t^{\prime} & =t \tag{1}
\end{align*}
$$

will come under consideration, and for the Lorentzian electrodynamics, it will be the group:

$$
\begin{equation*}
x^{\prime}=\frac{x-a t}{\sqrt{1-a^{2}}}, \quad t^{\prime}=\frac{-a x+t}{\sqrt{1-a^{2}}} \tag{2}
\end{equation*}
$$

Now, Einstein and Minkowski have already constructed theories of mechanics in which the group (2) enters in place of the group (1). However, the construction still has something arbitrary left in it.

Now, the goal of this article is to consistently implement a standpoint that moves the relativity principle to the center point of the construction of mechanics and electrodynamics.

In § 1, the implications of the theory of relativity to Newtonian mechanics will be formulated precisely, and in particular, the concept of absolute translatory motion. That formulation will also be important in the later electrodynamical problems.

In § 2, we will start from the law of energy, in its usual formulation, and show all of what can follow from it with the help of the relativity principle. This investigation, which has some partial points of contact with an older one of J. R. Schütz $\left({ }^{2}\right)$, is also interesting due to the fact that it will shed a certain light on the derivability of the fundamental mechanical equations from the law of energy that was asserted by the proponents of energetics and denied by its opponents $\left({ }^{3}\right)$.
${ }^{(1)}$ That terminology goes back to A. Rey (Die Theorie der Physik bei den modernen Physikern, German by R. Eisler, Leipzig, 1909) and refers to the counterpoint to the mechanistic picture more precisely with the frequentlyemployed term "energetics," which is too specialized.
$\left(^{2}\right)$ Göttinger Nachrichten, mathem.-phys, Kl. (1897).
$\left({ }^{3}\right)$ E.g., L. Boltzmann, "Ein Wort der Mathematik an die Energetik," Ann. Phys. (Leipzig) 57 (1896). Also, Populäre Schriften, pp. 104, et seq.

In § 3, the concepts of "arc-length" and "curvature" will be defined as invariants under the group of orthogonal transformations:

$$
\begin{align*}
x^{\prime} & =x \cos a+t \sin a,  \tag{3}\\
t^{\prime} & =-x \sin a+t \cos a
\end{align*}
$$

and a concept of "vis viva" will be derived from them.
In § 4, the one-parameter projective group (3) will be replaced by the group (1) with the same property and concepts that are analogous to the ones in $\S \mathbf{3}$ will be defined. Those concepts will then refer to the motion of a point along the $x$-axis. Using the axiom of the independence of vis viva from the direction of the velocity, we will then go on to motion in space, and in § $\mathbf{5}$, we will systematically construct Newtonian mechanics by making a special use of the relativity principle.

In § 6, it will be shown that one can deduce Hertz's equations for the processes in moving bodies by applying the Newtonian principle of relativity. By the experimental refutation of Hertz's equations, the validity of Newton's principle of relativity would also be shown to be impossible in the realm of electromagnetic phenomena.

In § 7, the electrodynamics of moving bodies will be constructed on the basis of some simpler assumptions. I believe that this construction has gained a certain amount of clarity in comparison to the presentations up to now by starting from the concept of refraction-free bodies. Lorentz's relativity theorem will be proved, and equations (2) will be derived with the most-practicable specification of assumptions. The entire presentation will be intended to make the deduction of Minkowski's derivation of the equations for arbitrary ponderable bodies seem logical.

In § 8, the definitions in $\S \S \mathbf{3}$ and $\mathbf{4}$ will be modified in such a way that we can introduce the one-parameter projective group (2) in place of (3) [(1), respectively].

Finally, in $\S \S 9$ and 10, it will be shown that electromagnetic mechanics (so a theory of mechanics for which the Lorentz-Einstein law of relativity is valid rather than the Newtonian one) can be constructed simply in such a way that one absorbs all definitions and assumptions from the structure of classical mechanics verbatim and only replace the group (1) with the group (2) everywhere. That theorem can be considered to be the main result of this article. In order to make it emerge more clearly, I have already posed the greatest-possible number of arguments within the context of classical mechanics ( $\S 2$ to 5) that might seem perhaps trivial there, due to the simplicity and popularity of the subject, but which will then make it possible to present electromagnetic mechanics at one stroke and in a completely "mechanical" way.

Let it be further mentioned that the impetus for this entire investigation goes back to the Appendix to Minkowski’s $\left({ }^{1}\right)$ paper with the title "Mechanik und Relativitätstheorie," which was rich in ideas, but fragmentary.

[^0]
## Section I

## § 1. - The content of the law of relativity. Invariance and covariance

The observation of mechanical processes in nature always shows us incontrovertibly only the relative motion of bodies with respect to each other. However, since their earliest founding by Newton up to their most recent one by G. Hamel ( ${ }^{1}$ ), the basic laws or basic equations of rational mechanics have almost never $\left(^{2}\right)$ described motions of bodies with respect to each other, but always relative to an absolute space, or what amounts to the same thing, relative to a merely-fictitious reference system whose position with respect to empirically-observed bodies is not given ( ${ }^{3}$ ).

Newton already exhibited the connection between those two facts in a theorem that he deduced as an immediate corollary to his three celebrated laws of motion $\left({ }^{4}\right)$ : "Corporum dato spatio inclusorum iidem sunt motus inter se, sive spatium illud quiescat, sive moveatur idem uniformiter in directum sine motu circulari." $\left(^{\dagger}\right)$ Naturally, the state of rest or motion of the "given" space is understood to mean relative to absolute space. This Newtonian form of the law of relativity of mechanics is not sufficiently precise for our present purposes due to the fact that we must ask what it even actually means to say: "a body at rest in absolute space."

In order to make the sense of such a statement more precise, we shall start from the concept of the geometry of Euclidian space. However, that is nothing but the embodiment of certain things that are defined by the system of axioms for geometry, namely, the points in space. Geometric space has those properties that we would demand of absolute space. The words "rest" and "motion" cannot be applied to either space as a whole nor to its parts or points since the concepts of rest and motion do not occur in the system of axioms for geometry. Now, we can distinguish three mutuallyperpendicular lines in geometric space, which we shall call the reference system of absolute rest. Furthermore, we can fabricate some other imaginary things along with the points of space, namely, the material points, which we associate with certain points of space at each point in time. (Here, we shall understand "time" to simply mean a real variable that can take all values from $-\infty$ to $+\infty$.) We shall call every such law of association a motion of the material point in question. When the association is specialized in such a way that a material point is associated with the same point in space for a certain interval of values for time, we say: The material point in question has come to a state of absolute rest during that time interval. Now, the basic equations of rational mechanics do not describe the motions of observable bodies in absolute space, but only the absolute motions of material points.

[^1]Now, if the sense of the statement "a material point is in a state of absolute rest in geometric space" has also been clarified, nonetheless, we still do not know what it means to say that "a certain observable body is in a state of absolute rest." Theoretical physics not only concerns itself with proposing laws for the variations of the state of bodies, but also with depicting observable bodies in geometric space. The basic equations of rational mechanics are then treated within that picture. When I am given a system of empirical bodies and a law of depiction for all times in question, I can say of each body whether its image in absolute (i.e., geometric) space is or is not at rest. That therefore depends not merely upon the observable (thus relative) motion of the body, but also upon the basic law of depiction, and therein lies the non-relative element in rational mechanics. Ever since the law of depiction first appeared in modern mechanics by way of Galilei and Newton, it was presented as follows: We imagine $n$ material points in geometric space. Let the coordinates of the $h^{\text {th }}$ point relative to the system of absolute rest be $\xi_{h}, \eta_{h}, \zeta_{h}$. Furthermore, let each point be assigned a positive constant $m_{h}$. Moreover, let $V$ be a function that depends upon only the mutual distances between the material points, and let $\tau$ be time. The motion of the material points might then result according to equations of the following form:

$$
\begin{equation*}
m_{h} \frac{d^{2} \xi_{h}}{d \tau^{2}}=-\frac{\partial V}{\partial \xi_{h}}, \quad m_{h} \frac{d^{2} \eta_{h}}{d \tau^{2}}=-\frac{\partial V}{\partial \eta_{h}}, \quad m_{h} \frac{d^{2} \zeta_{h}}{d \tau^{2}}=-\frac{\partial V}{\partial \zeta_{h}} \quad(h=1,2, \ldots, n) \tag{1}
\end{equation*}
$$

We now have a system of observable bodies. We would like to call a coordinate system that is rigidly coupled with an observable body a substantial one. If the dimensions of the bodies in question are small enough in comparison to the distances between them that their motions no longer depend upon the form of the bodies then it can happen that we can give a substantial coordinate system $S$ relative to which the bodies will perform motions that can be represented by equations of the form (1). Let $x_{h}, y_{h}, z_{h}$ be the coordinates of one such small body in the system $S$, let $m_{h}$ be a constant that is assigned to it, and let $t$ be the time is shown by a clock that is appropriate to the system $S$. From our assumption, the body moves according to the equations:

$$
\begin{equation*}
m_{h} \frac{d^{2} x_{h}}{d t^{2}}=-\frac{\partial V}{\partial x_{h}}, \quad m_{h} \frac{d^{2} y_{h}}{d t^{2}}=-\frac{\partial V}{\partial y_{h}}, \quad m_{h} \frac{d^{2} z_{h}}{d t^{2}}=-\frac{\partial V}{\partial z_{h}} \quad(h=1,2, \ldots, n) . \tag{1.a}
\end{equation*}
$$

The $n$ bodies can then be represented by the mapping equations:

$$
\begin{equation*}
x_{h}=\xi_{h}, \quad y_{h}=\eta_{h}, \quad z_{h}=\zeta_{h}, \quad t=\tau \quad(h=1,2, \ldots, n), \tag{2}
\end{equation*}
$$

so to geometric space, in such a way that the images of the bodies will continually coincide with the fictitious material points. The image of $S$ and the bodies that are rigidly coupled with it are therefore in a state of absolute rest. We would like to say of a body whose image is at rest under such a mapping that "it can be regarded as being in a state of absolute rest." We would like to say the same thing of a body that is rest relative to another one that can be regarded as being in a state of absolute rest in its own right. With the latter convention, one can also speak of bodies that cannot themselves be regarded as part of a system in which equations (1.a) are true.

Thus, let $S$ be a substantial coordinate system that can be regarded as being in a state of absolute rest, and let $S^{\prime}$ be another substantial system that moves rectilinearly relative to $S$ with constant velocity $a$. The axes of the two systems might therefore be parallel and remain that way. Let the coordinates of our small body in $S^{\prime}$ be $x_{h}^{\prime}, y_{h}^{\prime}, z_{h}^{\prime}$, and let the clock in $S^{\prime}$ show the time $t^{\prime}$, which coincides with the one in $S$. When the motion results in the direction of the $x$-axis, the coordinates are connected by the equations:

$$
\begin{equation*}
x_{h}^{\prime}=x_{h}-a t, \quad y_{h}^{\prime}=y_{h}, \quad z_{h}^{\prime}=z_{h}, \quad t^{\prime}=t \quad(h=1,2, \ldots, n) . \tag{3}
\end{equation*}
$$

If the motion results in a direction that is given by the direction cosines $\alpha, \beta, \gamma$ then the following relations will exist:

$$
\begin{equation*}
x_{h}^{\prime}=x_{h}-\alpha a t, \quad y_{h}^{\prime}=y_{h}-\beta a t, \quad z_{h}^{\prime}=z_{h}-\gamma a t, \quad t^{\prime}=t \quad(h=1,2, \ldots, n) . \tag{3.a}
\end{equation*}
$$

However, there is no loss of generality if we always lay the $x$-axis along the direction of motion. We therefore need to deal with only equations (3). If we introduce the variables $x_{h}^{\prime}, y_{h}^{\prime}$, $z_{h}^{\prime}$, etc., into equations (1.a) then with their help, we will get:

$$
\begin{equation*}
m_{h} \frac{d^{2} x_{h}^{\prime}}{d t^{\prime 2}}=-\frac{\partial V}{\partial x_{h}^{\prime}}, \quad m_{h} \frac{d^{2} y_{h}^{\prime}}{d t^{\prime 2}}=-\frac{\partial V}{\partial y_{h}^{\prime}}, \quad m_{h} \frac{d^{2} z_{h}^{\prime}}{d t^{\prime 2}}=-\frac{\partial V}{\partial z_{h}^{\prime}} \quad(h=1,2, \ldots, n) \tag{1.b}
\end{equation*}
$$

Those equations represent the motions of our body relative to the substantial coordinate system $S^{\prime}$. If we map them to geometric space by the equations:

$$
\begin{equation*}
x_{h}^{\prime}=\xi_{h}, \quad y_{h}^{\prime}=\eta_{h}, \quad z_{h}^{\prime}=\zeta_{h}, \quad t^{\prime}=\tau \quad(h=1,2, \ldots, n) \tag{2.a}
\end{equation*}
$$

then the images of the bodies will once more continually coincide with the material points, i.e., with the previously-established terminology: The substantial system $S^{\prime}$ can be regarded as being in a state of absolute rest. Naturally, under the mapping (2.a), the image of $S$ in geometric space will move with velocity $-a$.

We can then state the following theorem, which we would like to refer to as the law of relativity of Newtonian mechanics, and which is nothing but a more precise statement of the Newtonian law that was cited above:

If a body $S$ can be regarded as being in a state of absolute rest then every body that moves uniformly and rectilinearly relative to $S$ can also be regarded as being in a state of absolute rest.

The mathematical content can be formulated thus: Equations (1.a) remain invariant under transformations of the form (3). When we regard $a$ in them as a variable parameter, a one-parameter continuous group will be defined by equations (3). We would like to refer to it as the group of Galilei transformations.

The simplest example of this theorem is defined by the system that consists of the Sun and the Earth. The Sun $\left({ }^{1}\right)$ can be regarded as being in a state of absolute rest, and as a result, it can also be regarded as one that moves rectilinearly with arbitrary constant velocity. By contrast, the Earth can never be regarded as being in a state of absolute rest. When one often currently hears the opinion maintained that one can actually just as well say that the Earth stands still and the Sun moves as the opposite statement, that is correct only in the following sense: There is no criterion for absolute rest and motion to be found in the observable motion of bodies, because in order to have one, a mapping law must be given that will always be logically arbitrary. When one now drops the mapping law of Galilean-Newtonian mechanics and replaces it with a different one, one can naturally achieve the picture of the Earth being at rest in geometric space.

Euler had already expressed the law of relativity in a more precise form than Newton by initially replacing the concept of absolute space with an arbitrary reference system $\left({ }^{2}\right)$ and then proving the following theorem $\left({ }^{3}\right)$ :
"Quando spatium, ex quo motus relativus determinator, absolute vel quiescit vel movetur uniformiter in directum; tum leges datae de motu et quiete etiam in statu corporum relative valebunt." ${ }^{\dagger}$ )

Up to now, we have discussed the content of the law of relativity in the present complete system of mechanics. Now, that suggests the idea that this law can also be employed conversely in order to construct the system of mechanics. In that case, we would like to appeal to Minkowski's ${ }^{4}$ ) terminology and speak of the relativity principle. In the following subsections, we will put the demand of invariance under the group of Galilei transformations into the form of certain expressions. To that end, here we would like to extend the concept of invariant somewhat beyond the one that was employed in the foregoing.

Let the variables $x, y, z$, etc., be functions of the variable $t$. Now, when a transformation of those variables is performed:

$$
\begin{array}{ll}
x=\varphi\left(x^{\prime}, y^{\prime}, z^{\prime}, \ldots, t^{\prime}\right), & t=\chi\left(x^{\prime}, y^{\prime}, z^{\prime}, \ldots, t^{\prime}\right) \\
y=\psi\left(x^{\prime}, y^{\prime}, z^{\prime}, \ldots, t^{\prime}\right), & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .
\end{array}
$$

at the same time, a transformation of the derivatives $\frac{d x}{d t}, \frac{d^{2} y}{d t^{2}}$, etc., will come about; say:

[^2]$\left({ }^{4}\right)$ H. Minkowski, Göttinger Nachrichten (1908), pps. 54 and 55.
$$
\frac{d x}{d t}=\varphi_{1}\left(x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}, \frac{d x^{\prime}}{d t^{\prime}}, \frac{d y^{\prime}}{d t^{\prime}}, \ldots\right), \quad \quad \text { etc. }
$$

With the meaning of the word that we have used up to now, we shall refer to a system of equations of the form:

$$
\begin{aligned}
& f_{1}\left(x, y, \frac{d x}{d t}, \frac{d y}{d t}, \ldots\right)=0 \\
& f_{2}\left(x, y, \frac{d x}{d t}, \frac{d y}{d t}, \ldots\right)=0,
\end{aligned}
$$

as invariant under the given transformation when that system will go to:

$$
\begin{aligned}
& f_{1}\left(x, y, \frac{d x}{d t}, \frac{d y}{d t}, \ldots\right)=0 \\
& f_{2}\left(x, y, \frac{d x}{d t}, \frac{d y}{d t}, \ldots\right)=0,
\end{aligned}
$$

under the transformation.
However, we would now like to consider equations in which other variables $X, Y, \ldots, T, \ldots$ appear in addition to the ones that are subject to transformation; say:

$$
\begin{aligned}
& \psi_{1}\left(x, y, \frac{d^{2} x}{d t^{2}}, \ldots, X, \ldots, T, \ldots\right)=0 \\
& \psi_{2}\left(x, y, \frac{d^{2} x}{d t^{2}}, \ldots, X, \ldots, T, \ldots\right)=0, \quad \text { etc. }
\end{aligned}
$$

We would like to refer to the system as invariant in the broader sense, or as one usually says, covariant, under the given transformation of the $x, y, \ldots, t$ when a well-defined transformation of the $X, Y, Z, \ldots, T$ can be given:

$$
\begin{aligned}
& X=\chi_{1}\left(X^{\prime}, Y^{\prime}, \ldots, T^{\prime}\right), \\
& Y=\chi_{2}\left(X^{\prime}, Y^{\prime}, \ldots, T^{\prime}\right), \quad \text { etc. }
\end{aligned}
$$

such that our system of equations will go to the same system as the original one under it, but in the primed quantities:

$$
\begin{aligned}
& \psi_{1}\left(x^{\prime}, y^{\prime}, \frac{d^{2} x^{\prime}}{d t^{\prime 2}}, \ldots, X^{\prime}, \ldots, T^{\prime}, \ldots\right)=0 \\
& \psi_{2}\left(x^{\prime}, y^{\prime}, \frac{d^{2} x^{\prime}}{d t^{\prime 2}}, \ldots, X^{\prime}, \ldots, T^{\prime}, \ldots\right)=0
\end{aligned}
$$ etc.

A very simple example of such covariance is already defined by the equations for the motions of a material point in the plane:

$$
m \frac{d^{2} x}{d t^{2}}=X, \quad m \frac{d^{2} y}{d t^{2}}=Y
$$

in which $X, Y$ are the components of the force. If we perform a rotation of the coordinate system:

$$
\begin{aligned}
x^{\prime} & =x \cos \vartheta+y \sin \vartheta \\
y^{\prime} & =-x \sin \vartheta+y \cos \vartheta,
\end{aligned} \quad t=t^{\prime}, ~ ;
$$

and we then perform precisely the same transformation on $X, Y$, we will get:

$$
m \frac{d^{2} x^{\prime}}{d t^{\prime 2}}=X^{\prime}, \quad m \frac{d^{2} y^{\prime}}{d t^{\prime 2}}=Y^{\prime}
$$

In what follows, along with the demand of invariance, we would also occasionally like to employ covariance under the group of Galilei transformations.

## § 2. - Connection between the energy principle, the relativity principle, and the center of mass theorem.

The considerations in § 2 referred to motions of material points in geometric spaces. Here, we would like to examine the connections between some important general mechanical principles, such as the law of energy, the relativity principle, the center of mass theorem, etc. $\left(^{1}\right)$.

We have $n$ material points. Let the coordinates of the $h^{\text {th }}$ one be $x_{h}, y_{h}, z_{h}$, and let it be associated with a number $m_{h}$. Moreover, a function $V$ exists that depends upon the distances between the $n$ points, such that the quantity:

$$
\begin{equation*}
E_{0}=\sum_{h=1}^{n} \frac{1}{2} m_{h}\left[\left(\frac{d x_{h}}{d t}\right)^{2}+\left(\frac{d y_{h}}{d t}\right)^{2}+\left(\frac{d z_{h}}{d t}\right)^{2}\right]+V\left(x_{h}, y_{h}, z_{h}\right) \tag{1}
\end{equation*}
$$

will remain constant during the entire motion. As a second demand to impose upon the motion, we also require that the expression $E_{a}$ that is defined in precisely the same way as $E_{0}$, but referred to a system that moves along the $x$-axis with a velocity of $a$, will remain continually constant. Let $x_{h}^{\prime}, y_{h}^{\prime}, z_{h}^{\prime}$ be the coordinates in the new system. We will then have:

[^3]\[

$$
\begin{equation*}
E_{a}=\sum_{h=1}^{n} \frac{1}{2} m_{h}\left[\left(\frac{d x_{h}^{\prime}}{d t}\right)^{2}+\left(\frac{d y_{h}^{\prime}}{d t}\right)^{2}+\left(\frac{d z_{h}^{\prime}}{d t}\right)^{2}\right]+V\left(x_{h}^{\prime}, y_{h}^{\prime}, z_{h}^{\prime}\right) . \tag{2}
\end{equation*}
$$

\]

Those two demands $\left({ }^{1}\right)$, namely, the energy principle and the relativity principle, can then be written:

$$
\begin{equation*}
E_{0}=c, \quad E_{a}=c+c^{\prime} . \tag{3}
\end{equation*}
$$

$c$ and $c^{\prime}$ are arbitrarily-prescribed constants in that.
It follows from equations (3) in § $\mathbf{1}$ by differentiation that:

$$
\begin{equation*}
\frac{d x_{h}^{\prime}}{d t^{\prime}}=\frac{d x_{h}}{d t}-a, \quad \frac{d y_{h}^{\prime}}{d t^{\prime}}=\frac{d y_{h}}{d t}, \quad \frac{d z_{h}^{\prime}}{d t^{\prime}}=\frac{d z_{h}}{d t} \quad(h=1,2, \ldots, n), \tag{4}
\end{equation*}
$$

and substituting that in equation (3) of this subsection, while considering (1) and (2) will yield:

$$
E_{0}+\frac{1}{2} a^{2} \sum_{h=1}^{n} m_{h}-a \sum_{h=1}^{n} m_{h} \frac{d x_{h}}{d t}=c+c^{\prime} .
$$

When we denote the coordinates of the center of mass by $\mathfrak{x}, \mathfrak{h}, \mathfrak{z}$, that will give:

$$
\begin{equation*}
\frac{d \mathfrak{x}}{d t}=\frac{a}{2}-\frac{c^{\prime}}{a \sum_{h=1}^{n} m_{h}}=\frac{\frac{1}{2} a^{2} \sum m_{h}-c^{\prime}}{a \sum m_{h}}, \tag{5}
\end{equation*}
$$

i.e., the center of mass of the system moves rectilinearly with constant velocity. However, that velocity has the magnitude $a$, which is only the velocity of the second system, relative to which we have postulated the validity of the law of energy, and the arbitrary constant $c^{\prime}$ is the arbitrarilygiven difference between the two coordinate systems.

Now, that suggests that one must set $c^{\prime}=0$, i.e., one must demand that the energy relative to the moving system has the same value as it does relative to the system at rest. It would then follow that:

$$
\frac{d \mathfrak{x}}{d t}=\frac{a}{2} .
$$

The velocity of the center of mass would be one-half of that of the second reference system. We have a theory of mechanics in which a distinguished velocity appears, namely $a$, which is equal to

[^4]the measure of energy of the coordinate system and twice that of the center of mass. If I had a system that moves with the velocity $b$ then I would have:
\[

$$
\begin{equation*}
E_{b}=E_{0}-b \frac{d \mathfrak{x}}{d t} \sum_{h=1}^{n} m_{h}+\frac{1}{2} b^{2} \sum_{h=1}^{n} m_{h}=c+\frac{1}{2} \sum_{h=1}^{n} m_{h}\left(b^{2}-a b\right) . \tag{6}
\end{equation*}
$$

\]

It is only for $b=0$ and $b=a$ that one will have $E_{b}=c$, so the first two basic systems would also be the only ones for which the energy would have the same value $c$ then.

Now, instead of $c^{\prime}=0$, we would like to add a $c^{\prime}$ for which no distinguished value of the velocity exists. In order to do that, it is necessary that the velocity of the center of mass should not depend upon the quantity $a$ that we employed in order to derive the center of mass theorem. We then choose $c^{\prime}$ to be a function of $a$ such that the expression for $d \mathfrak{r} / d t$ that is given by equation (5) is independent of $a$. We set:

$$
c^{\prime}=f(a)
$$

and demand that:

$$
\frac{1}{2} a-\frac{f(a)}{a \sum m_{h}}
$$

should not depend upon $a$. That condition can be written as a differential equation for $f(a)$ in the form of:

$$
\frac{1}{2}-\frac{1}{a \sum m_{h}} \frac{d f(a)}{d a}+\frac{1}{a^{2} \sum m_{h}} f(a)=0
$$

or

$$
a \frac{d f(a)}{d a}-f(a)+\frac{1}{2} a^{2} \sum m_{h}=0 .
$$

If $k$ is an arbitrary constant then the general solution to this equation will read:

$$
f(a)=\frac{1}{2} a^{2} \sum_{h=1}^{n} m_{h}-k a \sum_{h=1}^{n} m_{h} .
$$

If we substitute that value for $c^{\prime}$ in (5) then we will get:

$$
\frac{d \mathfrak{x}}{d t}=k,
$$

i.e., the center of mass can move with any arbitrary velocity. With that convention, the basic assumptions (3) can be written in the form:

$$
\begin{equation*}
E_{0}=c, \quad E_{a}=c+\frac{1}{2} a^{2} \sum_{h=1}^{n} m_{h}-a k \sum_{h=1}^{n} m_{h} . \tag{3.a}
\end{equation*}
$$

Now, if we construct $E_{b}$ under those assumptions then we will find that when we substitute the value $k$ for $d \mathfrak{r} / d t$ in equation (6), it will be:

$$
E_{b}=E_{0}+\frac{1}{2} b^{2} \sum_{h=1}^{n} m_{h}-b k \sum_{h=1}^{n} m_{h},
$$

and it will follow that:

$$
E_{a}=E_{0}+\frac{1}{2} a^{2} \sum_{h=1}^{n} m_{h}-a k \sum_{h=1}^{n} m_{h}
$$

just as it did from (3.a).
The values of energy for every system that moves with constant velocity are calculated in the same from $E_{0}$ then; no velocity is distinguished in any way. The convention (3.a) is also the one that conventional mechanics is based upon. Naturally, we will get analogous equations for $\frac{d \mathfrak{h}}{d t}$ and $\frac{d \mathfrak{z}}{d t}$ when we assume that the energy equation is also valid for any coordinate system that moves rectilinearly with constant speed in the $y(z$, respectively) direction.

Nothing can be said here about an actual invariance of the energy equation under the group of Galilean transformations since the constants on the right-hand side can indeed change in value. That suggests that the energy equation should be established in its differentiated form.

Therefore, we would now like to assume that the motion of the material points fulfills the equation:

$$
\begin{equation*}
\sum_{h=1}^{n} m_{h}\left(\frac{d^{2} x_{h}}{d t^{2}} \frac{d x_{h}}{d t}+\frac{d^{2} y_{h}}{d t^{2}} \frac{d y_{h}}{d t}+\frac{d^{2} z_{h}}{d t^{2}} \frac{d z_{h}}{d t}\right)+\sum_{h=1}^{n}\left(\frac{\partial V}{\partial x_{h}} \frac{d x_{h}}{d t}+\frac{\partial V}{\partial y_{h}} \frac{d y_{h}}{d t}+\frac{\partial V}{\partial z_{h}} \frac{d z_{h}}{d t}\right)=0 \tag{7}
\end{equation*}
$$

relative to a system at rest.
Furthermore, an equation of the same form should exist relative to a system that moves in the $x$-direction with a velocity $a$, so:

$$
\begin{equation*}
\sum_{h=1}^{n} m_{h}\left(\frac{d^{2} x_{h}^{\prime}}{d t^{2}} \frac{d x_{h}^{\prime}}{d t}+\frac{d^{2} y_{h}^{\prime}}{d t^{2}} \frac{d y_{h}^{\prime}}{d t}+\frac{d^{2} z_{h}^{\prime}}{d t^{2}} \frac{d z_{h}^{\prime}}{d t}\right)+\sum_{h=1}^{n}\left(\frac{\partial V}{\partial x_{h}^{\prime}} \frac{d x_{h}^{\prime}}{d t}+\frac{\partial V}{\partial y_{h}^{\prime}} \frac{d y_{h}^{\prime}}{d t}+\frac{\partial V}{\partial z_{h}^{\prime}} \frac{d z_{h}^{\prime}}{d t}\right)=0 . \tag{8}
\end{equation*}
$$

Upon substituting that in equations (4) and the ones that arise from it by differentiation in equation (8) and subtracting from equation (7), one will get:

$$
a \sum_{h=1}^{n} m_{h} \frac{d^{2} x_{h}}{d t^{2}}+a \sum_{h=1}^{n} \frac{\partial V}{\partial x_{h}}=0
$$

and due to the fact that:

$$
\mathfrak{x}=\frac{\sum_{h=1}^{n} m_{h} x_{h}}{\sum_{h=1}^{n} m_{h}},
$$

one will get:

$$
\begin{equation*}
\frac{d^{2} \mathfrak{x}}{d t^{2}} \sum_{h=1}^{n} m_{h}=-\sum_{h=1}^{n} \frac{\partial V}{\partial x_{h}} \tag{9}
\end{equation*}
$$

Since, by assumption, $V$ depends upon only the differences between the coordinates, one has:

$$
\sum_{h=1}^{n} \frac{\partial V}{\partial x_{h}}=0
$$

so

$$
\frac{d^{2} \mathfrak{x}}{d t^{2}}=0 .
$$

The theorem that the center of mass moves rectilinearly and uniformly gives the necessary and sufficient condition for equation (7) to remain invariant under the group of Galilei transformations here.

Let it be pointed out here that in the derivation of equation (9), the fact that $\frac{\partial V}{\partial x_{h}}, \frac{\partial V}{\partial y_{h}}, \frac{\partial V}{\partial z_{h}}$ are derivatives of a function of distance was used only to the extent that one could derive the relation:

$$
\begin{equation*}
\sum_{h=1}^{n}\left[\left(\frac{\partial V}{\partial x_{h}}-\frac{\partial V}{\partial x_{h}^{\prime}}\right) \frac{d x_{h}}{d t}+\left(\frac{\partial V}{\partial y_{h}}-\frac{\partial V}{\partial y_{h}^{\prime}}\right) \frac{d y_{h}}{d t}+\left(\frac{\partial V}{\partial z_{h}}-\frac{\partial V}{\partial z_{h}^{\prime}}\right) \frac{d z_{h}}{d t}\right]=0 \tag{10}
\end{equation*}
$$

between the quantities $\frac{\partial V}{\partial x_{h}}$, etc., and the corresponding primed ones $\frac{\partial V}{\partial x_{h}^{\prime}}$, etc. When any other quantities that are coupled by a relation of the form (10) stand in place of the quantities $\frac{\partial V}{\partial x_{h}}$, etc., and the corresponding primed ones in equations (7) and (8), equation (9) can be deduced likewise.

We would like to employ the last remark in order to generalize our considerations up to now in a certain sense. We will no longer assume the existence of a function $V$ of the previouslyconsidered form now, but start from a law of energy in a more general form, and indeed we will initially restrict ourselves to a single material point. We assume that there exists a real constant $m$ and three quantities $X, Y, Z$ that can possibly depend upon $x, y, z, t$, the derivatives of the coordinates with respect to time, and any other type of variables, but nothing else will be assumed about them. Those quantities $m, X, Y, Z$ shall be arranged only in such a way that the equation:

$$
\begin{equation*}
\frac{d}{d t} \frac{1}{2} m\left[\left(\frac{d x}{d t}\right)^{2}+\left(\frac{d y}{d t}\right)^{2}+\left(\frac{d z}{d t}\right)^{2}\right]=X \frac{d x}{d t}+Y \frac{d y}{d t}+Z \frac{d z}{d t} \tag{11}
\end{equation*}
$$

is always fulfilled during the entire motion. Moreover, quantities $X^{\prime}, Y^{\prime}, Z^{\prime}$ should exist for a coordinate system that moves with a velocity of $a$ such that the same motion relative to that system will fulfill the equation:

$$
\begin{equation*}
\frac{d}{d t} \frac{1}{2} m\left[\left(\frac{d x^{\prime}}{d t}\right)^{2}+\left(\frac{d y^{\prime}}{d t}\right)^{2}+\left(\frac{d z^{\prime}}{d t}\right)^{2}\right]=X^{\prime} \frac{d x^{\prime}}{d t}+Y^{\prime} \frac{d y^{\prime}}{d t}+Z^{\prime} \frac{d z^{\prime}}{d t} \tag{12}
\end{equation*}
$$

The connection between $X, Y, Z$, and the corresponding primed quantities shall be assumed to be the following one:

$$
\begin{equation*}
\left(X^{\prime}-X\right) \frac{d x}{d t}+\left(Y^{\prime}-Y\right) \frac{d y}{d t}+\left(Z^{\prime}-Z\right) \frac{d z}{d t}=0 \tag{13}
\end{equation*}
$$

as equation (10) would suggest.
It would be simplest for us to fulfill equation (13) by setting:

$$
X^{\prime}=X, Y^{\prime}=Y, Z^{\prime}=Z
$$

Upon subtracting (12) and (13), we will then get:

$$
\begin{equation*}
m \frac{d^{2} x^{\prime}}{d t^{\prime 2}}=X^{\prime}, \quad m \frac{d^{2} x}{d t^{2}}=X \tag{14}
\end{equation*}
$$

That derivation can be understood even more abstractly. Namely, in the derivation of equation (14), we did not use the detailed form of the right-hand side of equations (11) and (12) at all, but only the following property: If we denote the right-hand side of equation (11) by $T$ and that of (12) by $T^{\prime}$ then the following relation will exist between them:

$$
\begin{equation*}
T-T^{\prime}=-a X \tag{14}
\end{equation*}
$$

due to equation (13). If we combine that relation with the relation:

$$
\begin{equation*}
X^{\prime}=X \tag{15.a}
\end{equation*}
$$

then we can argue as follows: Let $X, X^{\prime}, T, T^{\prime}$ be any quantities that are connected by equations (15) and (15.a). The motion of a material point relative to system of axes at rest will fulfill the equation:

$$
\begin{equation*}
\frac{d}{d t} \frac{1}{2} m\left[\left(\frac{d x}{d t}\right)^{2}+\left(\frac{d y}{d t}\right)^{2}+\left(\frac{d z}{d t}\right)^{2}\right]=T \tag{16}
\end{equation*}
$$

and when referred to a system that moves with velocity $a$ in the $x$-direction, it will fulfill:

$$
\begin{equation*}
\frac{d}{d t} \frac{1}{2} m\left[\left(\frac{d x^{\prime}}{d t}\right)^{2}+\left(\frac{d y^{\prime}}{d t}\right)^{2}+\left(\frac{d z^{\prime}}{d t}\right)^{2}\right]=T^{\prime} \tag{17}
\end{equation*}
$$

It follows from equations (17) and (15) that:

$$
m\left[\frac{d^{2} x}{d t^{2}}\left(\frac{d x}{d t}-a\right)+\frac{d^{2} y}{d t^{2}} \frac{d y}{d t}+\frac{d^{2} z}{d t^{2}} \frac{d z}{d t}\right]=T-a X
$$

and subtracting that from (16) will give:

$$
\begin{equation*}
m \frac{d^{2} x}{d t^{2}}=X \tag{16.a}
\end{equation*}
$$

The system that consists of equations (16) and (16.a) is covariant under the group of Galilei transformations in the sense that was defined in $\S \mathbf{1}$, so the second series of variables $X, T$ will be given by (15) and (15.a).

If we denote the force component in the $x$-direction by $X$ then equation (16.a) will give us the first fundamental equation of mechanics. We can derive the three basic equations of mechanics analogously from the following assumptions:

We have three directions in space that are not all parallel to the same plane. Let the directions cosines of the $i^{\text {th }}$ direction be given by $\alpha_{i}, \beta_{i}, \gamma_{i}$. Moreover, equation (16) exists for a coordinate system at rest, and equation (17) exists for one that moves with a velocity of $a$ in the direction that is given by $\alpha_{i}, \beta_{i}, \gamma_{i}$, except that we must then set:

$$
\begin{aligned}
T^{\prime} & =T-a\left(\alpha_{i} X+\beta_{i} Y+\gamma_{i} Z\right) . \\
x^{\prime} & =x-a \alpha_{i} t, \quad X^{\prime}=X, \\
y^{\prime} & =y-a \beta_{i} t, \quad Y^{\prime}=Y, \\
z^{\prime} & =z-a \gamma_{i} t, \quad Z^{\prime}=Z \quad(i=1,2,3) .
\end{aligned}
$$

It follows from this, as above, that:

$$
m\left(\alpha_{i} \frac{d^{2} x}{d t^{2}}+\beta_{i} \frac{d^{2} y}{d t^{2}}+\gamma_{i} \frac{d^{2} z}{d t^{2}}\right)=\alpha_{i} X+\beta_{i} Y+\gamma_{i} Z \quad(i=1,2,3)
$$

or

$$
\alpha_{i}\left(m \frac{d^{2} x}{d t^{2}}-X\right)+\beta_{i}\left(m \frac{d^{2} y}{d t^{2}}-Y\right)+\gamma_{i}\left(m \frac{d^{2} z}{d t^{2}}-Z\right)=0 \quad(i=1,2,3)
$$

Due to the assumptions about the three directions considered, the determinant will satisfy:

$$
\left|\begin{array}{lll}
\alpha_{1} & \beta_{1} & \gamma_{1} \\
\alpha_{2} & \beta_{2} & \gamma_{2} \\
\alpha_{3} & \beta_{3} & \gamma_{3}
\end{array}\right| \neq 0
$$

Therefore:

$$
m \frac{d^{2} x}{d t^{2}}=X, \quad m \frac{d^{2} y}{d t^{2}}=Y, \quad m \frac{d^{2} z}{d t^{2}}=Z .
$$

We shall now move on to consider a system of $n$ points for which we will carry out analogous considerations. We set:

$$
\begin{equation*}
\frac{d}{d t} \sum_{h=1}^{n} \frac{1}{2} m_{h}\left[\left(\frac{d x_{h}}{d t}\right)^{2}+\left(\frac{d y_{h}}{d t}\right)^{2}+\left(\frac{d z_{h}}{d t}\right)^{2}\right]=T . \tag{18}
\end{equation*}
$$

Moreover, for a system that moves with a velocity $a$ in the direction whose direction cosines are $\alpha_{i}, \beta_{i}, \gamma_{i}$, we will have:

$$
\begin{equation*}
\frac{d}{d t} \sum_{h=1}^{n} \frac{1}{2} m_{h}\left[\left(\frac{d x_{h}^{\prime}}{d t}\right)^{2}+\left(\frac{d y_{h}^{\prime}}{d t}\right)^{2}+\left(\frac{d z_{h}^{\prime}}{d t}\right)^{2}\right]=T^{\prime} \tag{19}
\end{equation*}
$$

We therefore let:

$$
\begin{align*}
& T^{\prime}=T-a\left(\alpha_{i} X+\beta_{i} Y+\gamma_{i} Z\right), \\
& X^{\prime}=X, \quad Y^{\prime}=Y, \quad Z^{\prime}=Z . \tag{20}
\end{align*}
$$

It follows from this, as above, that:

$$
\sum_{h=1}^{n} m_{h} \frac{d^{2} x_{h}}{d t^{2}}=X, \quad \sum_{h=1}^{n} m_{h} \frac{d^{2} y_{h}}{d t^{2}}=Y, \quad \sum_{h=1}^{n} m_{h} \frac{d^{2} z_{h}}{d t^{2}}=Z
$$

or

$$
\begin{equation*}
\frac{d^{2} \mathfrak{x}}{d t^{2}} \sum_{h=1}^{n} m_{h}=X, \quad \frac{d^{2} \mathfrak{y}}{d t^{2}} \sum_{h=1}^{n} m_{h}=Y, \quad \frac{d^{2} \mathfrak{z}}{d t^{2}} \sum_{h=1}^{n} m_{h}=Z . \tag{21}
\end{equation*}
$$

Thus, the energy equation and the relativity principle, which appears here in the form of a demand of covariance on the energy equation, will imply the center of mass theorem for a system.

All of that might serve as only a provisional overview of the connections between the most important principles of mechanics. In the next subsection, we will go on to erect the edifice of Newtonian mechanics in a systematic and logical way, in which some concepts that were posed in
a seemingly-arbitrary way up to now will also be analyzed in more detail from the standpoint of the relativity principle, so in particular the concept of vis viva and the transformation of the quantities $X$ and $T$ that is given by equation (15) [(20), respectively].

## § 3. - The basic differential-geometric concepts as invariants under orthogonal transformations.

We consider curves in a $t x$-plane that are given by equations of the form:

$$
\begin{equation*}
x=x(t) \tag{1}
\end{equation*}
$$

or in parametric form:

$$
\begin{equation*}
x=x(u), \quad y=y(u) . \tag{2}
\end{equation*}
$$

Furthermore, we fix our attention on the group of orthogonal transformations, which is a oneparameter group that is given by:

$$
\begin{align*}
t^{\prime} & =t \cos a+x \sin a,  \tag{3}\\
x^{\prime} & =-t \sin a+x \cos a .
\end{align*}
$$

One understands a finite invariant of the group (3) to mean a function of the quantities $x^{\prime}$ and $y^{\prime}$ that goes to the same function of $x$ and $t$ identically under equations (3).

The infinitesimal transformation $\left({ }^{1}\right)$ that generates the group (3) is given by:

$$
\begin{equation*}
\delta t=x \delta a, \quad \delta x=-t \delta a \tag{4}
\end{equation*}
$$

A function $F$ is an invariant of the group if and only if:

$$
\delta F=\frac{\partial F}{\partial x} \delta x+\frac{\partial F}{\partial t} \delta t=0
$$

or, due to (4):

$$
\begin{equation*}
x \frac{\partial F}{\partial x}-t \frac{\partial F}{\partial t}=0 \tag{5}
\end{equation*}
$$

$F$ must then be a function of $x^{2}+t^{2}$, and any function of $x^{2}+t^{2}$ will then be an invariant of the group.

We now define the concept of the element of arc-length $d s$ in the following way:
A) Let the square of the element of arc-length be a quadratic differential form with coefficients that can depend upon $x$ and $t$ :
( ${ }^{1}$ ) See, perhaps, Enzykl. d. math. Wiss., II, A. 6 ("Kontinuierliche Transformationsgruppen"), § 4.

$$
d s^{2}=E(x, t) d t^{2}+2 F(x, t) d x d t+G(x, t) d x^{2} .
$$

B) That form shall be invariant under the group of orthogonal transformations [equation (3)]. The integral over the element of arc-length over a curve segment is called the length of the arc in question.
C) Congruent curve segments that also have the same distances from the $t$-axis shall have equal lengths.
$D)$ A curve segment that coincides with a piece of the $t$-axis shall have a length that equals the difference between the abscissas of its endpoints.

We can deduce from $B$ ) that: Since $d t, d x$ transform just like $t$ and $x$, in order for $d s^{2}$ to remain invariant, it is necessary that:

$$
E(x, t) t^{2}+2 F(x, t) x t+G(x, t) x^{2}
$$

should be a finite invariant of the group (3). Therefore, that expression must be a function of $x^{2}+t^{2}$ alone, which would be possible only if:

$$
F=0, \quad E=G=f\left(x^{2}+t^{2}\right)
$$

so:

$$
d s^{2}=f\left(x^{2}+t^{2}\right)\left(d x^{2}+d t^{2}\right) .
$$

It follows from $C$ ) that $f$ must be a constant, and from $D$ ), that this constant must have the value 1. It then follows from our definition that:

$$
\begin{equation*}
d s^{2}=d t^{2}+d x^{2} \tag{6}
\end{equation*}
$$

We now choose the arc-length to be the parameter $u$ in the parametric representation of the curves in question that is given by equation (2):

$$
\begin{equation*}
u=\int \sqrt{d t^{2}+d x^{2}} \tag{7}
\end{equation*}
$$

The parameter $u^{\prime}$ that emerges from the transformation will then be:

$$
\begin{equation*}
u^{\prime}=\int \sqrt{d t^{\prime 2}+d x^{\prime 2}}=\int \sqrt{d t^{2}+d x^{2}}=u . \tag{8}
\end{equation*}
$$

We shall also consider the first derivatives $\frac{d x}{d u}, \frac{d t}{d u}$, and the expressions that they will transform into under (3), namely, $\frac{d x^{\prime}}{d u^{\prime}}, \frac{d t^{\prime}}{d u^{\prime}}$. Due to (8), the connection between then is given by simply:

$$
\begin{align*}
& \frac{d t^{\prime}}{d u^{\prime}}=\frac{d t}{d u} \cos a+\frac{d x}{d u} \sin a,  \tag{3.a}\\
& \frac{d x^{\prime}}{d u^{\prime}}=-\frac{d t}{d u} \sin a+\frac{d x}{d u} \cos a .
\end{align*}
$$

One calls (3) and (3.a) collectively the first extended group ( ${ }^{1}$ ).
If we focus upon the second derivatives then we will have:

$$
\begin{align*}
& \frac{d^{2} t^{\prime}}{d u^{\prime 2}}=\frac{d^{2} t}{d u^{2}} \cos a+\frac{d^{2} x}{d u^{2}} \sin a,  \tag{3.b}\\
& \frac{d^{2} x^{\prime}}{d u^{\prime 2}}=-\frac{d^{2} t}{d u^{2}} \sin a+\frac{d^{2} x}{d u^{2}} \cos a .
\end{align*}
$$

(3), (3.a), (3.b) collectively define the second extended group, which we can regard as a group in six variables when we consider the derivatives to be proper variables and denote them by $t_{u}, x_{u}$, $t_{u u}, x_{u u}, t_{u}^{\prime}$, etc., for brevity, or when we also include $u$ itself, which is not transformed, a group in seven variables.

One calls the finite invariants of the first extended group first-order differential invariants of the group (3), and those of the second extended group, second-order differential invariants. The infinitesimal transformation that generates the second extended group reads:

$$
\begin{array}{llll}
\delta t & =x \delta a, & \delta u=0, & \delta t_{u}=x_{u} \delta a, \\
\delta x & \delta t_{u u}=x_{u u} \delta a,  \tag{4}\\
\delta t \delta a, & \delta x_{u}=-t_{u} \delta a, & \delta x_{u u}=-t_{u u} \delta a .
\end{array}
$$

We would now like to define the concept of curvature $1 / r$, and indeed by the following postulate:
A) The curvature $1 / r$ shall be a second-order differential invariant under the group of orthogonality transformations (3).
B) The curvature shall be first-order in $x_{u u}$ and $t_{u u}$, so in a certain sense, it is the simplest differential invariant.

[^5]C) Congruent curve segments that also have the same distance from the $t$-axis shall possess the same curvatures at homologous locations.
$D)$ There must be at least one point on a circle of radius 1 where the curvature is $1 / r=1$.

We then set:

$$
\frac{1}{r}=F\left(u, t, t_{u}, x_{u}, t_{u u}, x_{u u}\right)
$$

Due to (9), the function $F$ must satisfy the following partial differential equations:

$$
\begin{equation*}
x \frac{\partial F}{\partial t}-t \frac{\partial F}{\partial x}+x_{u} \frac{\partial F}{\partial t_{u}}-t_{u} \frac{\partial F}{\partial x_{u}}+x_{u u} \frac{\partial F}{\partial t_{u u}}-t_{u u} \frac{\partial F}{\partial x_{u u}}=0 . \tag{10}
\end{equation*}
$$

That differential equation is equivalent to the following system of ordinary differential equations:

$$
\begin{equation*}
\frac{d t}{x}=-\frac{d x}{t}=\frac{d t_{u}}{x_{u}}=-\frac{d x_{u}}{t_{u}}=\frac{d t_{u u}}{x_{u u}}=-\frac{d x_{u u}}{t_{u u}}=\frac{d u}{0} . \tag{11}
\end{equation*}
$$

The integrals of that system are: $x^{2}+t^{2}, x x_{u}+t t_{u}, x_{u}^{2}+t_{u}^{2}, x_{u} x_{u u}+t_{u} t_{u u}, x_{u u}^{2}+t_{u u}^{2}, u$. Due to the meaning of $u$, one now has [equation (7)]:

$$
\begin{aligned}
x_{u}^{2}+t_{u}^{2} & =1, \\
x_{u} x_{u u}+t_{u} t_{u u} & =0 .
\end{aligned}
$$

It then follows from $A$ ) that $1 / r$ can depend upon only $x^{2}+t^{2}, x x_{u}+t t_{u}, x_{u u}^{2}+t_{u u}^{2}$. From $\left.B\right)$, that dependency can only take the form:

$$
\frac{1}{r}=f\left(x^{2}+t^{2}, x x_{u}+t t_{u}, u\right) \sqrt{x_{u u}^{2}+t_{u u}^{2}} .
$$

From $C$ ), the function $f$ must reduce to a constant, and from $D$ ), that constant must have the value 1. Thus:

$$
\begin{equation*}
\frac{1}{r}=\sqrt{x_{u u}^{2}+t_{u u}^{2}}=\sqrt{\left(\frac{d^{2} x}{d u^{2}}\right)^{2}+\left(\frac{d^{2} t}{d u^{2}}\right)^{2}}=\frac{d^{2} x}{d t^{2}}\left[1+\left(\frac{d x}{d t}\right)^{2}\right]^{-3 / 2} . \tag{12}
\end{equation*}
$$

We now define the expression:

$$
\begin{equation*}
\frac{1}{r} \frac{d x}{d t}=\frac{\frac{d^{2} x}{d t^{2}} \frac{d x}{d t}}{\left(\sqrt{1+\left(\frac{d x}{d t}\right)^{2}}\right)^{3}}=-\frac{d}{d t} \frac{1}{\sqrt{1+\left(\frac{d x}{d t}\right)^{2}}} \tag{13}
\end{equation*}
$$

If we set:

$$
\frac{d x}{d t}=w
$$

to abbreviate, then we see that due to equation (13), the expression:

$$
\int \frac{d x}{r},
$$

when extended from a fixed point of the curve up to a variable one, will depend upon only the value of $w$ at that variable point. We can then define a function $L$ of the variable $w$ by the following conventions:
A) $\frac{d L}{d x}=\frac{m}{r}$, where $m$ is a constant.
$B)$ For $w=0$, one will also have $L=0$.

It follows uniquely from those two postulates that:

$$
\begin{equation*}
L=m\left(1-\frac{1}{\sqrt{1+w^{2}}}\right) . \tag{14}
\end{equation*}
$$

We would like to call the quantity $L$ the vis viva of the curve at the point in question, due to an analogy that will be explained in the next subsection. Let $x_{1}, t_{1}$ and $x_{1}, t_{1}$ be two points on the curve, and let $L_{1}$ and $L_{2}$ be the values of the vis vivas at those points. Due to the condition $A$ ), we will then have the equation:

$$
\begin{equation*}
m \int_{x_{1}, t_{1}}^{x_{1}, t_{1}} \frac{d x}{r}=L_{2}-L_{1}=m\left(\cos \varphi_{1}-\sin \varphi_{2}\right), \tag{15}
\end{equation*}
$$

in which $\varphi_{1}$ and $\varphi_{2}$ are the angles that the curve defines with the $t$-axis at the point $1(2$, respectively). Equation (15) has a certain analogy with the law of energy.

## § 4. - Adapting the basic concepts of differential geometry to Newtonian mechanics.

We first consider the motion of a material point along a line that we choose to be the $x$-axis. The path of motion in the course of time $t$ can be represented by a time-path curve in the $t x$-plane. Along such a curve, we would like to adapt the basic concepts of differential geometry, as they were defined in the previous subsection, in such a way that we replace group of orthogonal transformations everywhere with that of the Galilei transformations, which is given by:

$$
\begin{equation*}
t^{\prime}=t, \quad x^{\prime}=x-a t \tag{1}
\end{equation*}
$$

The infinitesimal transformation that generates that group reads:

$$
\begin{equation*}
\delta t=0, \quad \delta x=-t \delta a . \tag{2}
\end{equation*}
$$

The only finite invariants of the group are then the functions of $t$ alone.
We now define the arc-length element, which we would like to call $d \sigma$, analogously to $\S \mathbf{3}$ by the following postulates:
A) Let the square of the arc-length element be a quadratic differential form with coefficients that can depend upon $x$ and $t$ :

$$
d \sigma^{2}=E(x, t) d t^{2}+2 F(x, t) d t d x+G(x, t) d x^{2} .
$$

$B$ ) The form shall be invariant under the group of Galilei transformations [equation (1)].
The integral of the arc-length element along a curve segment is called the proper time $\left({ }^{1}\right)$ of the process of motion that the curve segment represents.
C) Congruent processes of motion that take place only at two different absolute times $t$ shall require the same proper time for their evolution.
$D$ ) If the material point is at rest then the time in the ordinary sense shall equal the proper time during which it is at rest, so it will be equal to $\int d \sigma$ along the corresponding segment of the $t$-axis.

Since functions of $t$ are the only finite invariants, we deduce from $A$ ) and $B$ ) that $d \sigma^{2}$ must have the form:

$$
d \sigma^{2}=f(t) d t^{2}
$$

We deduce from $C$ ) that $f(t)$ reduces to a constant, and from $D$ ), that this constant has the value 1. Thus:

[^6]\[

$$
\begin{equation*}
d \sigma=d t \tag{3}
\end{equation*}
$$

\]

The "proper time" of a process of motion is equal to time in the ordinary sense. As above, we once more introduce:

$$
\begin{equation*}
u=\int d t=\int d t^{\prime}=u^{\prime}=t=t^{\prime} \tag{4}
\end{equation*}
$$

as the parameter and define the first and second extensions of the group (1):

$$
\begin{align*}
& \frac{d t^{\prime}}{d u^{\prime}}=\frac{d t}{d u}, \quad \frac{d x^{\prime}}{d u^{\prime}}=\frac{d x}{d u}-a  \tag{1.a}\\
& \frac{d^{2} t^{\prime}}{d u^{\prime 2}}=\frac{d^{2} t}{d u^{2}}, \quad \frac{d^{2} x^{\prime}}{d u^{\prime 2}}=\frac{d^{2} x}{d u^{2}} \tag{1.b}
\end{align*}
$$

When we employ the same abbreviations as in the previous subsection, the infinitesimal transformation that generates the two-fold extended group can be written:

$$
\begin{align*}
& \delta t=0, \quad \delta u=0, \quad \delta t_{u}=0, \quad \delta t_{u u}=0,  \tag{5}\\
& \delta x=-t \delta a, \quad \delta x_{u}=-\delta a, \quad \delta x_{u u}=0 \text {. }
\end{align*}
$$

We once more define the concept of curvature $1 / \rho$ by the following postulate:
A) The curvature $1 / \rho$ shall be a second-order differential invariant under the group of Galilei transformations [equation (1)].
B) The curvature shall be of first-order in $t_{u u}$ and $x_{u u}$, so in a certain sense, it is the simplest differential invariant.
C) Congruent processes of motion that take place at only different absolute times $t$ shall have the same value of curvature at homologous locations.
D) A uniformly-accelerated motion with acceleration 1 shall have at least one location that corresponds to a curvature of $1 / \rho=1$.

We then set:

$$
\frac{1}{\rho}=F\left(u, t, x, t_{u}, x_{u}, t_{u u}, x_{u u}\right)
$$

In that way, due to equation (5), $F$ must satisfy the following partial differential equation:

$$
\begin{equation*}
-t \frac{\partial F}{\partial x}-\frac{\partial F}{\partial x_{u}}=0 . \tag{6}
\end{equation*}
$$

That corresponds to the following system of ordinary differential equations:

$$
\begin{equation*}
\frac{d t}{0}=-\frac{d x}{t}=\frac{d t_{u}}{0}=-\frac{d x_{u}}{1}=\frac{d t_{u u}}{0}=\frac{d x_{u u}}{0}=\frac{d u}{0} . \tag{7}
\end{equation*}
$$

The integrals of that system are: $t, t_{u}, t_{u u}, x_{u u}, u, x-t x_{u}$. Now, due to the meaning of $u$ [equation (4)]:

$$
t_{u}=1, \quad t_{u u}=0, \quad x_{u u}=\frac{d^{2} x}{d t^{2}}, \quad u=t, \quad x-t x_{u}=x-t \frac{d x}{d t},
$$

so when we also observe $B$ ), we will have:

$$
\frac{1}{\rho}=f\left(t, x-t x_{u}\right) \frac{d^{2} x}{d t^{2}} .
$$

Due to $C$ ), the function $f$ reduces to a constant, and from $D$ ), that constant has the value 1 . Thus:

$$
\begin{equation*}
\frac{1}{\rho}=\frac{d^{2} x}{d t^{2}} \tag{8}
\end{equation*}
$$

We once more set the velocity equal to:

$$
\frac{d x}{d t}=w
$$

and define a quantity $L$ by the two postulates:
A) $\frac{d L}{d x}=\frac{m}{\rho}$, in which $w$ is a constant.
$B$ ) For $w=0$, one will also have $L=0$.

It follows from $A$ ) that:

$$
\frac{d L}{d t}=\frac{m}{\rho} \frac{d x}{d t}=m \frac{d^{2} x}{d t^{2}} \frac{d x}{d t}=\frac{d}{d t} \frac{1}{2} m w^{2},
$$

from which it will follow that:

$$
\begin{equation*}
L=\frac{1}{2} m w^{2} \tag{9}
\end{equation*}
$$

when one considers $B$ ).
We again refer to the quantity $L$ as the vis viva of the motion. Analogous to equation (15) in § 3, here one has the equation:

$$
\begin{equation*}
m \int_{x_{1}, t_{1}}^{x_{2}, t_{2}} \frac{d x}{\rho}=L_{2}-L_{1} \tag{10}
\end{equation*}
$$

which is, in fact, the energy equation here.
We would now like to move on to the concept of the vis viva of a material point that moves arbitrarily in space in such a way that we add the following postulate to $A$ ) and $B$ ), which refer to motions along the $x$-axis:
C) The vis viva $L$ of an arbitrarily-moving material point shall not depend upon the direction, but merely upon the absolute value of the velocity $w$.

Due to equation (9), it will follow from this that:

$$
\begin{equation*}
L=\frac{1}{2} m w^{2}=\frac{1}{2} m\left[\left(\frac{d x}{d t}\right)^{2}+\left(\frac{d y}{d t}\right)^{2}+\left(\frac{d z}{d t}\right)^{2}\right] . \tag{11}
\end{equation*}
$$

If we have $n$ different material points then we would like to denote the vis viva of the $h^{\text {th }}$ one of them by $L_{h}$. The constant $m$ that enters into postulate $A$ ) does not need to be the same for all points in it. One has ( ${ }^{1}$ ):

$$
\begin{equation*}
L_{h}=\frac{1}{2} m_{h}\left[\left(\frac{d x_{h}}{d t}\right)^{2}+\left(\frac{d y_{h}}{d t}\right)^{2}+\left(\frac{d z_{h}}{d t}\right)^{2}\right] . \tag{12}
\end{equation*}
$$

We would like to call the same expression, but constructed in the coordinates $x_{h}^{\prime}, y_{h}^{\prime}, z_{h}^{\prime}, t^{\prime}$, which refer to a uniformly-moving coordinate system, the vis viva relative to the moving system and denote it by $L_{h}^{\prime}$ :

$$
\begin{equation*}
L_{h}^{\prime}=\frac{1}{2} m_{h}\left[\left(\frac{d x_{h}^{\prime}}{d t}\right)^{2}+\left(\frac{d y_{h}^{\prime}}{d t}\right)^{2}+\left(\frac{d z_{h}^{\prime}}{d t}\right)^{2}\right] . \tag{13}
\end{equation*}
$$

Finally, we would like to define the concept of the vis viva of a system of $n$ points $L$ ( $L^{\prime}$, respectively), and indeed by the postulate:
D)

$$
\begin{equation*}
L=\sum_{h=1}^{n} L_{h}, \tag{14}
\end{equation*}
$$

[^7]\[

$$
\begin{equation*}
L^{\prime}=\sum_{h=1}^{n} L_{h}^{\prime} . \tag{15}
\end{equation*}
$$

\]

## § 5. - Systematic construction of Newtonian mechanics.

We are now in a position to be able to develop the system of fundamental equations of mechanics on the basis of the Galilei transformations in a logically rigorous form. We then have $n$ material points.
I. Let $X,-T$ be a series of variables that experiences the transformation that is contragredient to a linear transformation of $x$, $t$, i.e., one must have:

$$
\begin{equation*}
X x-T t=X^{\prime} x^{\prime}-T^{\prime} t^{\prime} \tag{1}
\end{equation*}
$$

identically.
We let $X$ denote the $x$-component of the total force that acts upon the system from the outside, and let $T$ denote the effect of that force. The pairs of variables $(Y,-T),(Z,-T)$ might have an analogous meaning.
II. The system moves in such a way that the following equation will be true:

$$
\begin{equation*}
\sum_{h=1}^{n} \frac{d L_{h}}{d t}=\frac{d L}{d t}=T \tag{2}
\end{equation*}
$$

in which $L_{h}$ means the expression that was defined in the last subsection (see pp. 25).
III. Equation (2) shall remain covariant under the group of Galilei transformations, i.e., it shall go to:

$$
\begin{equation*}
\sum_{h=1}^{n} \frac{d L_{h}^{\prime}}{d t^{\prime}}=\frac{d L^{\prime}}{d t^{\prime}}=T^{\prime} \tag{3}
\end{equation*}
$$

Covariance is understood to mean that when, say, the variable-pair $x_{h}$ and $t$ is transformed into $x_{h}^{\prime}$ and $t^{\prime}$ by a Galilei transformation, the variable-pair $X,-T$ will suffer the contragredient transformation, in the sense of I.
IV. There exist $n$ subsystems of our system of material points for which postulates of the same form as I, II, III should be valid in their own right, i.e., every such subsystem (say, the $k^{\text {th }}$ one) will also belong to a variable-pair $\bar{X}_{k},-\bar{T}_{k}$, that suffers the transformation contragredient to $x, t$.

Moreover, when $\bar{L}_{k}$ is the vis viva of that system, one has:

$$
\begin{equation*}
\frac{d \bar{L}_{k}}{d t}=\bar{T}_{k}, \quad \frac{d \bar{L}_{k}^{\prime}}{d t}=\bar{T}_{k}^{\prime} \tag{4}
\end{equation*}
$$

$\bar{X}_{k}$ is what we call the $x$-components of the force that acts upon the $k^{\text {th }}$ subsystem, and $\bar{T}_{k}$ is the effect of that force.
V. Each force component that acts upon a subsystem is the sum of quantities, each of which originates in a well-defined material point that does not belong to the subsystem, and a quantity that does not at all originate in the material points that belong to the total system, and which is the sum of the force components in a system of forces components that act from the outside that act upon the individual subsystems from the outside.

We now infer the following conclusions from those assumptions: Due to equation (1), the transformation that $X$ and $-T$ suffer when $x, t$ experience the Galilei transformation is initially given by:

$$
\begin{align*}
T^{\prime} & =T-a X, \\
X^{\prime} & =X . \tag{5}
\end{align*}
$$

We conclude from II and III, as at the conclusion of § 3 , that the following equations must exist:

$$
\begin{equation*}
\sum_{h=1}^{n} m_{h} \frac{d^{2} x_{h}}{d t^{2}}=X, \quad \sum_{h=1}^{n} m_{h} \frac{d^{2} y_{h}}{d t^{2}}=Y, \quad \sum_{h=1}^{n} m_{h} \frac{d^{2} z_{h}}{d t^{2}}=Z, \tag{6}
\end{equation*}
$$

and when we set:

$$
\begin{equation*}
\sum_{h=1}^{n} m_{h}=M \tag{7}
\end{equation*}
$$

and once more understand $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ to mean the coordinates of the center of mass:

$$
\begin{equation*}
M \frac{d^{2} \mathfrak{x}}{d t^{2}}=X, \quad M \frac{d^{2} \mathfrak{y}}{d t^{2}}=Y, \quad M \frac{d^{2} \mathfrak{z}}{d t^{2}}=Z \tag{8}
\end{equation*}
$$

Those equations express the so-called center of mass theorem. In words, it says: The acceleration of the center of mass of a system is equal to the total force that acts upon the system, divided by the sum of the coefficients $m$ that occur in the expression for the vis viva. The quotient of the $x$-component of the force that acts upon a system and the $x$-component of the acceleration of the center of mass will also be called the mass of that system such that one can also say: The mass of the system is equal to the sum of the constants $m_{h}$ that occur in the expression for $L$.

Now, in order to apply assumption IV, we select $n$ subsystems for which it should be valid. We choose the $k^{\text {th }}$ subsystem to consist of the points $m_{1}, m_{2}, \ldots, m_{k}$. Thus, for this system, IV will take the form of the equations:

$$
\begin{align*}
& \frac{d \bar{L}_{k}}{d t}=\sum_{h=1}^{k} \frac{d L_{h}}{d t}=\bar{T}_{k},  \tag{9}\\
& \frac{d \bar{L}_{k}^{\prime}}{d t}=\sum_{h=1}^{k} \frac{d L_{h}^{\prime}}{d t}=\bar{T}_{k}^{\prime} . \tag{10}
\end{align*}
$$

Just as equation (5) was deduced from equations (2) and (3), one can conclude the existence of the following equations from equations (9) and (10), together with IV:

$$
\begin{equation*}
\sum_{h=1}^{k} m_{h} \frac{d^{2} x_{h}}{d t^{2}}=\bar{X}_{k}, \quad \sum_{h=1}^{k} m_{h} \frac{d^{2} y_{h}}{d t^{2}}=\bar{Y}_{h}, \quad \sum_{h=1}^{k} m_{h} \frac{d^{2} z_{h}}{d t^{2}}=\bar{Z}_{k} . \tag{11}
\end{equation*}
$$

If we denote the coordinates of the center of mass of $k^{\text {th }}$ subsystem by $\mathfrak{x}_{k}, \mathfrak{y}_{k}, \mathfrak{z}_{k}$ and set:

$$
\begin{equation*}
\sum_{h=1}^{k} m_{h}=M_{k} \tag{12}
\end{equation*}
$$

then we will get:

$$
\begin{gathered}
M_{k} \frac{d^{2} \mathfrak{x}_{k}}{d t^{2}}=\bar{X}_{k}, \quad M_{k} \frac{d^{2} \mathfrak{y}_{k}}{d t^{2}}=\bar{Y}_{k}, \quad M_{k} \frac{d^{2} \mathfrak{z}_{k}}{d t^{2}}=\bar{Z}_{k} \\
(k=1,2,3, \ldots, n) .
\end{gathered}
$$

From our definition above, $M_{k}$ is the mass of the $k^{\text {th }}$ subsystem.
We now introduce quantities $X_{1}, X_{2}, \ldots, X_{n}, Y_{1}, \ldots$ by the following definition:

$$
\left.\begin{array}{c}
\bar{X}_{k+1}-\bar{X}_{k}=X_{k+1}, \quad \bar{Y}_{k+1}-\bar{Y}_{k}=Y_{k+1}, \quad \bar{Z}_{k+1}-\bar{Z}_{k}=Z_{k+1},  \tag{14}\\
(k=1,2, \ldots, n-1) \\
\bar{X}_{1}=X_{1}, \quad \bar{Y}_{1}=Y_{1}, \quad \bar{Z}_{1}=Z_{1} .
\end{array}\right\}
$$

Due to equation (11), we will then have:

$$
\begin{equation*}
m_{h} \frac{d^{2} x_{h}}{d t^{2}}=X_{h}, \quad m_{h} \frac{d^{2} y_{h}}{d t^{2}}=Y_{h}, \quad m_{h} \frac{d^{2} z_{h}}{d t^{2}}=Z_{h} \quad(h=1,2, \ldots, n) \tag{15}
\end{equation*}
$$

If we regard the point with the coefficient $m_{h}$ as a subsystem then from IV and (15), $X_{h}$ will be $x$-component of the force that acts upon it, etc. Moreover, from our definition, $m_{h}$ is the mass of that point. The coefficients $m_{h}$ in the expression for the vis viva are then nothing but the masses of the individual points, when regarded as subsystems.

Now, let $X_{1}^{2}$ be the part of the force that acts upon $m_{1}$ that originates in $m_{2}$, and let $X_{2}^{1}$ be the part of the force that acts upon $m_{2}$ that originates in $m_{1}$. From V, one then has:

$$
\begin{align*}
& X_{1}=X_{1}^{2}+X_{1}^{*},  \tag{16}\\
& X_{2}=X_{2}^{1}+X_{2}^{*} .
\end{align*}
$$

The system in question might consist of only the points $m_{1}$ and $m_{2}$. From V, one then has that $X_{1}^{*}$ is the force that is exerted upon $m_{1}$ from the outside, and $X_{2}^{*}$ is the force that is exerted upon $m_{2}$. It follows, in general, from equation (6) and equation (15) that:

$$
\begin{equation*}
\sum_{h=1}^{n} X_{h}=X, \quad \sum_{h=1}^{n} Y_{h}=Y, \quad \sum_{h=1}^{n} Z_{h}=Z . \tag{17}
\end{equation*}
$$

In our case, in particular, we have:

$$
X_{1}+X_{2}=X .
$$

From the last part of assumption V, one further has:

$$
X_{1}^{*}+X_{2}^{*}=X
$$

so

$$
\begin{equation*}
X_{1}^{2}+X_{2}^{1}=0 . \tag{18}
\end{equation*}
$$

That equation expresses the theorem of the equality of action and reaction.
One calls a system for which the force $(X, Y, Z)$ that acts from the outside equals zero a free system. Due to equation (5), such a thing can also be characterized then by the fact that:

$$
T^{\prime}=T,
$$

or from equations (2) and (3), that $\frac{d L}{d t}=\frac{d L^{\prime}}{d t^{\prime}}$. For such a thing, the center of mass is known to move along a straight line. If one performs the differentiation in equation (2) then one will find that:

$$
\sum_{h=1}^{n} m_{h}\left(\frac{d^{2} x_{h}}{d t^{2}} \frac{d x_{h}}{d t}+\frac{d^{2} y_{h}}{d t^{2}} \frac{d y_{h}}{d t}+\frac{d^{2} z_{h}}{d t^{2}} \frac{d z_{h}}{d t}\right)=T,
$$

from which, equation (15) will imply the following relation for $T$ :

$$
\begin{equation*}
T=\sum_{h=1}^{n}\left(X_{h} \frac{d x_{h}}{d t}+Y_{h} \frac{d y_{h}}{d t}+Z_{h} \frac{d z_{h}}{d t}\right) . \tag{19}
\end{equation*}
$$

$T$ is therefore, in fact, what one ordinarily calls the effect.
The same equation is true for $T^{\prime}$ in the primed coordinates.

A mechanical system that is defined by Postulates I to V (pp. 26,27) has the property that each of its parts behaves like the whole. We call such a system an incomplete system.

One can now attempt to base the law of energy, not in the form of II (pp. 26), but by starting from the form that expressed at the beginning of § 2 [equation (1)], so by defining a mechanical system by the following assumptions:
I. The $n$ materials points move in such a way that the equation:

$$
\begin{equation*}
L+V\left(x_{h}, y_{h}, z_{h}\right)=\text { const. } \tag{20}
\end{equation*}
$$

is fulfilled, in which $V$ depends upon only the mutual distances between the material points.
II. Equation (20) remains invariant under the group of Galilei transformations, but the constant on the right-hand side can possibly change. One will then have:

$$
\begin{equation*}
L^{\prime}+V\left(x_{h}^{\prime}, y_{h}^{\prime}, z_{h}^{\prime}\right)=\text { const. } \tag{21}
\end{equation*}
$$

III. Equations of the form (20) and (21) might also exist for $n$ selected subsystems:

$$
\begin{align*}
L+V\left(x_{h}, y_{h}, z_{h}\right) & =\text { const. }, \\
L^{\prime}+V\left(x_{h}^{\prime}, y_{h}^{\prime}, z_{h}^{\prime}\right) & =\text { const. }, \tag{22}
\end{align*}
$$

in which $V_{k}$ depends upon only the mutual distances between the material points that belong to the $k^{\text {th }}$ subsystem.

This system also has the property that each of its subsystems obeys the same laws as the total system.

We infer from I and II, precisely as in § 2 [equation (9)], the equations:

$$
\begin{equation*}
\sum_{h=1}^{n} m_{h} \frac{d^{2} x_{h}}{d t^{2}}=0, \quad \sum_{h=1}^{n} m_{h} \frac{d^{2} y_{h}}{d t^{2}}=0, \quad \sum_{h=1}^{n} m_{h} \frac{d^{2} z_{h}}{d t^{2}}=0 . \tag{23}
\end{equation*}
$$

We infer from III the existence of the equation:

$$
\begin{equation*}
\sum_{h=1}^{k} m_{c_{h}} \frac{d^{2} x_{c_{h}}}{d t^{2}}=0 \tag{24}
\end{equation*}
$$

in precisely the same way, if $m_{c_{1}}, m_{c_{2}}, \ldots, m_{c_{k}}$ are the points that belong to the subsystem in question.

However, we can infer from (23) and (24) that:

$$
\frac{d^{2} x_{h}}{d t^{2}}=\frac{d^{2} y_{h}}{d t^{2}}=\frac{d^{2} z_{h}}{d t^{2}}=0 \quad(h=1,2, \ldots, n)
$$

Therefore, only uniform rectilinear motions of all points can take place in this system. The systems I, II, III on pp. 30 are therefore not suitable for the derivation of the general fundamental laws of mechanics. One likewise sees that a system in which the energy principle is expressed in the form of equation (20) cannot be constructed as an incomplete system, in the sense that was defined above.

However, we would still like to exhibit a system of assumptions in which the energy principle takes the form (20), but we must then drop the requirement that every subsystem must behave like the total system. We then pose the following assumptions:
I. The equation:

$$
L+V=\text { const. }
$$

is always fulfilled during the entire motion.
II. That equation remains invariant under the group of Galilei transformations (as before).
III. We select $n-1$ subsystems such that $k^{\text {th }}$ one consists of the points $m_{1}, m_{2}, \ldots, m_{k}$. The following equation is true for the $k^{\text {th }}$ one:

$$
\begin{equation*}
\sum_{h=1}^{k} m_{k}\left(\frac{d x_{h}}{d t} \frac{d^{2} x_{h}}{d t^{2}}+\frac{d y_{h}}{d t} \frac{d^{2} y_{h}}{d t^{2}}+\frac{d z_{h}}{d t} \frac{d^{2} z_{h}}{d t^{2}}\right)+\sum_{h=1}^{k}\left(\frac{\partial V}{\partial x_{h}} \frac{d x_{h}}{d t}+\frac{\partial V}{\partial y_{h}} \frac{d y_{h}}{d t}+\frac{\partial V}{\partial z_{h}} \frac{d z_{h}}{d t}\right)=0 . \tag{25}
\end{equation*}
$$

The $V$ in it is the same function that depends upon all $n$ points as in I.
IV. Equation (25) shall be invariant under the group of Galilei transformations:

$$
\begin{equation*}
\sum_{h=1}^{k} m_{k}\left(\frac{d x_{h}^{\prime}}{d t^{\prime}} \frac{d^{2} x_{h}^{\prime}}{d t^{\prime 2}}+\frac{d y_{h}^{\prime}}{d t^{\prime}} \frac{d^{2} y_{h}^{\prime}}{d t^{\prime 2}}+\frac{d z_{h}^{\prime}}{d t^{\prime}} \frac{d^{2} z_{h}^{\prime}}{d t^{\prime 2}}\right)+\sum_{h=1}^{k}\left(\frac{\partial V}{\partial x_{h}^{\prime}} \frac{d x_{h}^{\prime}}{d t}+\frac{\partial V}{\partial y_{h}^{\prime}} \frac{d y_{h}^{\prime}}{d t}+\frac{\partial V}{\partial z_{h}^{\prime}} \frac{d z_{h}^{\prime}}{d t}\right)=0 . \tag{26}
\end{equation*}
$$

We would like to call a mechanical system that is defined by those assumptions a complete system. The individual parts of it no longer behave like the whole since the function $V$ in equations (25) and (26) will indeed depend upon other quantities besides the mutual distances of the material points of the subsystem. We can once more deduce equation (23) from I and II. As in § 2, equation (9), the equations will follow from III and IV:

$$
\sum_{h=1}^{k} m_{h} \frac{d^{2} x_{h}}{d t^{2}}+\sum_{h=1}^{k} \frac{\partial V}{\partial x_{h}}=0
$$

$$
\begin{gather*}
\sum_{h=1}^{k} m_{h} \frac{d^{2} y_{h}}{d t^{2}}+\sum_{h=1}^{k} \frac{\partial V}{\partial y_{h}}=0  \tag{27}\\
\sum_{h=1}^{k} m_{h} \frac{d^{2} z_{h}}{d t^{2}}+\sum_{h=1}^{k} \frac{\partial V}{\partial z_{h}}=0 \\
(k=1,2, \ldots, n)
\end{gather*}
$$

For $k \neq n, \sum_{h=1}^{k} \frac{\partial V}{\partial x_{h}}$ will not vanish identically, in general. The equations of motion for the individual material points can be derived by subtraction:

$$
\begin{equation*}
m_{h} \frac{d^{2} x_{h}}{d t^{2}}=-\frac{\partial V}{\partial x_{h}}, \quad m_{h} \frac{d^{2} y_{h}}{d t^{2}}=-\frac{\partial V}{\partial y_{h}}, \quad m_{h} \frac{d^{2} z_{h}}{d t^{2}}=-\frac{\partial V}{\partial y_{h}} \quad(h=1,2, \ldots, n) . \tag{28}
\end{equation*}
$$

Each part of the complete system can be regarded as incomplete. If we consider, e.g., the $r^{\text {th }}$ subsystem and set:

$$
\begin{aligned}
& -\frac{\partial V}{\partial x_{i}}=X_{i}, \quad-\frac{\partial V}{\partial y_{i}}=Y_{i}, \quad-\frac{\partial V}{\partial z_{i}}=Z_{i}, \\
& -\sum_{h=1}^{r} \frac{\partial V}{\partial x_{h}}=X, \text { etc., } \sum_{i=1}^{k} X_{i}=\bar{X}_{k}, \text { etc. }, \\
& -\sum_{h=1}^{r}\left(\frac{\partial V}{\partial x_{h}} \frac{d x_{h}}{d t}+\frac{\partial V}{\partial y_{h}} \frac{d y_{h}}{d t}+\frac{\partial V}{\partial z_{h}} \frac{d z_{h}}{d t}\right)=T, \\
& -\sum_{h=1}^{k}\left(\frac{\partial V}{\partial x_{h}} \frac{d x_{h}}{d t}+\frac{\partial V}{\partial y_{h}} \frac{d y_{h}}{d t}+\frac{\partial V}{\partial z_{h}} \frac{d z_{h}}{d t}\right)=\bar{T}_{k}
\end{aligned}
$$

then the quantities $\bar{X}_{k}$, etc., $\bar{T}_{k}, X$, etc., $T$, thus-defined, will fulfill all of assumptions I to V on pps. 26,27 by which we have defined an incomplete system. Each part of a complete system is then, in fact, an incomplete system, which justifies the terminology.

All of the considerations in $\S \S \mathbf{2 , 4 , 5}$ related to motions in geometric space. The meaning for motions relative to substantial coordinate systems can be inferred from the considerations in § $\mathbf{1}$. In particular, we see from the last part of § $\mathbf{5}$ that a substantial system $S$ can already be regarded as being in a state of absolute rest, in the sense of § 1, when the law of energy in the form of equation (20) is fulfilled relative to $S$, as well as in a substantial system $S^{\prime}$ that moves uniformly and rectilinearly relative to $S$.

## § 6. - The relativity principle of Newtonian mechanics and electromagnetic processes.

The fundamental equations of Newtonian mechanics describe the motions of bodies relative to a body that can be regarded as being in a state of absolute rest. The entire study of electromagnetism is based upon the so-called Maxwell field equations for bodies "at rest," and we must first clarify the meaning that the word "rest" would have here. We once more operate in geometric space to begin with and consider a body in it that is in a state of absolute rest. Inside of it, let the vectors $\mathfrak{E}, \mathfrak{H}, \mathfrak{D}, \mathfrak{B}, \mathfrak{i}$, and the scalars $\rho, \varepsilon, \mu$, $\sigma$ be given as functions of the coordinates $\xi, \eta, \zeta$, and time $\tau$ in absolute space that are connected by the following system of equations:

$$
\begin{align*}
\operatorname{curl} \mathfrak{H} & =4 \pi\left(\mathfrak{i}+\frac{\partial \mathfrak{D}}{\partial \tau}\right), & & \mathrm{I}  \tag{I}\\
\operatorname{div} \mathfrak{D} & =\rho, & & \text { II }  \tag{II}\\
\operatorname{curl} \mathfrak{E} & =-\frac{\partial \mathfrak{B}}{\partial \tau}, & & \text { III } \\
\operatorname{div} \mathfrak{B} & =0, & & \text { IV } \\
\mathfrak{D} & =\frac{\varepsilon}{4 \pi} \mathfrak{E}, \quad \mathfrak{B}=\mu \mathfrak{H}, \quad \mathfrak{i}=\sigma \mathfrak{E} . & & \mathrm{V}
\end{align*}
$$

We restrict ourselves in that to regarding $\varepsilon, \mu, \sigma$ as constants. Now, the statement of Maxwell's theory is that there are empirically-observable bodies in which the electromagnetic processes proceed according to equations I to IV, but the coordinates must now be defined relative to a reference system that is rigidly coupled with the body (so a substantial one), instead of $\xi, \eta, \zeta$, and we must understand the vectors and scalars $\mathfrak{E}, \mathfrak{H}, \mathfrak{D}, \mathfrak{B}, \mathfrak{i}, \rho, \varepsilon, \mu, \sigma$ to mean the measurable quantities, namely, the electric and magnetic field strengths, the electric and magnetic excitations, the intensity of the conduction current, the charge density, the dielectric constant, the magnetic permeability, and Ohm's conductivity. We would like to say of such bodies that they can be regarded as being in a state of absolute rest in the electromagnetic sense. Whether a body that is in a state of rest in the mechanical sense (with the definition in § 1) is also at rest in the electromagnetic sense in a certain case can only be decided by experiments.

If $x, y, z$ are the coordinates, and $t$ is the time in the substantial coordinate system that is coupled with our body, and we map the empirical phenomena into absolute space by the equations:

$$
\begin{equation*}
x=\xi, y=\eta, z=\zeta, t=\tau \tag{1}
\end{equation*}
$$

then the picture of our body in geometric space will be at rest, and the changes in its electromagnetic properties will coincide with the changes in the vectors and scalars $\mathfrak{E}, \ldots, \sigma$ in geometric space.

The law that Maxwell's equations do not remain invariant under the group of Galilei transformations is usually expressed. That is still correct when one understands the word
"invariant" in the narrower sense. However, in a certain sense, one can also assert the covariance of the system of Maxwell's equations under the aforementioned group. We introduce new variables into equations I to V :

$$
\begin{equation*}
\xi^{\prime}=\xi-a \tau, \quad \eta^{\prime}=\eta, \quad \zeta^{\prime}=z, \quad \tau^{\prime}=\tau . \tag{2}
\end{equation*}
$$

We will then get:

$$
\begin{align*}
\operatorname{curl}^{\prime} \mathfrak{H} & =4 \pi\left(\mathfrak{i}+\frac{\partial \mathfrak{D}}{\partial \tau}-a \frac{\partial \mathfrak{D}}{\partial \xi^{\prime}}\right) \\
\operatorname{div}^{\prime} \mathfrak{D} & =\rho \\
\operatorname{curl}^{\prime} \mathfrak{E} & =-\left(\frac{\partial \mathfrak{B}}{\partial \tau^{\prime}}-a \frac{\partial \mathfrak{B}}{\partial \xi^{\prime}}\right), \\
\operatorname{div}^{\prime} \mathfrak{B} & =0, \\
\mathfrak{D} & =\frac{\varepsilon}{4 \pi} \mathfrak{E}, \quad \mathfrak{B}=\mu \mathfrak{H}, \quad \mathfrak{i}=\sigma \mathfrak{E} .
\end{align*}
$$

In it, one has:

$$
\operatorname{div}^{\prime}=\frac{\partial}{\partial \xi^{\prime}}+\frac{\partial}{\partial \eta^{\prime}}+\frac{\partial}{\partial \zeta^{\prime}}
$$

and one understands curl' analogously. Quantities that are not transformed by the group (2) also enter into equations I to V, e.g., the components of $\mathfrak{E}, \mathfrak{H}, \ldots$, etc. We would now like to subject those quantities to the identity transformation:

$$
\begin{equation*}
\mathfrak{E}^{\prime}=\mathfrak{E}, \quad \mathfrak{H}^{\prime}=\mathfrak{H}, \quad \mathfrak{D}^{\prime}=\mathfrak{D}, \quad \mathfrak{B}^{\prime}=\mathfrak{B}, \quad \rho^{\prime}=\rho . \tag{3}
\end{equation*}
$$

In addition, the quantities $\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \tau}$ will occur, which we would like to treat as actual variables, as one frequently does in vector calculus. We let that series of variables suffer the transformation that is contragredient $\left({ }^{1}\right)$ to (2) by setting:

$$
\begin{equation*}
\frac{\partial^{\prime}}{\partial \xi}=\frac{\partial}{\partial \xi}, \quad \frac{\partial^{\prime}}{\partial \eta}=\frac{\partial}{\partial \eta}, \quad \frac{\partial^{\prime}}{\partial \varphi}=\frac{\partial}{\partial \varphi}, \quad \frac{\partial^{\prime}}{\partial \tau}=\frac{\partial}{\partial \tau}-a \frac{\partial}{\partial \xi} . \tag{4}
\end{equation*}
$$

Maxwell's equations can then be written in the following form relative to a coordinate system that moves with velocity $a$, which we obtain by substituting (3) and (4) in $\mathrm{I}^{\prime}$ to $\mathrm{V}^{\prime}$ :

$$
\begin{align*}
\operatorname{curl}^{\prime} \mathfrak{H}^{\prime} & =4 \pi\left(\mathfrak{i}^{\prime}+\frac{\partial^{\prime} \mathfrak{D}^{\prime}}{\partial \tau^{\prime}}\right) \\
\operatorname{div}^{\prime} \mathfrak{D}^{\prime} & =\rho^{\prime}
\end{align*}
$$

[^8]\[

$$
\begin{array}{rlrl}
\operatorname{curl}^{\prime} \mathfrak{E}^{\prime} & =-\frac{\partial^{\prime} \mathfrak{B}^{\prime}}{\partial \tau^{\prime}}, & & \mathrm{III}^{\prime \prime} \\
\operatorname{div}^{\prime} \mathfrak{B}^{\prime} & =0, & \mathrm{IV}^{\prime \prime} \\
\mathfrak{D}^{\prime} & =\frac{\varepsilon}{4 \pi} \mathfrak{E}^{\prime}, \quad \mathfrak{B}^{\prime}=\mu \mathfrak{H}^{\prime}, \quad \mathfrak{i}^{\prime}=\sigma \mathfrak{E}^{\prime} . & & \mathrm{V}^{\prime \prime}
\end{array}
$$
\]

Those equations have precisely the same form as equations I to V. One can then assert the covariance of Maxwell's field equations under the group (2), in a certain sense.

Let $K$ be a body that can be regarded as being at rest, and let $K^{\prime}$ be a body that moves relative to $K$ with a velocity of $a$ in the $x$-direction. The electromagnetic processes inside of $K$ will then be described in coordinates relative to $K$ by the system of equations I to V , while the processes inside of $K$, but in coordinates that relate to $K^{\prime}$, will be described by equations $\mathrm{I}^{\prime}$ to $\mathrm{V}^{\prime}$ (when we replace $\xi^{\prime}, \eta^{\prime}, \zeta^{\prime}$ with the coordinates $x^{\prime}, y^{\prime}, z^{\prime}$ in the substantial system $K^{\prime}$ ). The processes inside of the moving body $K^{\prime}$ cannot be derived in any way in the absence of a special hypothesis.

If we would like to preserve the analogy to mechanics then that would suggest that such a hypothesis might be to postulate the validity of Newton's principle of relativity for the electromagnetic processes that take place in moving bodies, as well. I have implemented those ideas in calculations in a previous paper $\left({ }^{1}\right)$, and here I would like to only recapitulate the connection to that train of thought.

The velocity $a$ will occur in the equations for the processes inside of $K^{\prime}$ in any case. Our first assumption is that the equations should go to the ones for bodies at rest for $a=0$.

Secondly, the equations should be covariant under the group of Galilei transformations in the same sense as they are for bodies at rest.

When we understand $d \mathfrak{E} / d t$ and analogous expression to mean the change in $\mathfrak{E}$ at a point of the body $K^{\prime}$ that moves with it, that will imply the following equations:

$$
\begin{array}{rlrl}
\operatorname{curl} \mathfrak{H} & =4 \pi\left(\mathfrak{i}+\frac{\partial \mathfrak{D}}{\partial \tau}\right), & & \text { I. } a \\
\operatorname{div} \mathfrak{D} & =\rho, & & \text { II. } a \\
\operatorname{curl} \mathfrak{E} & =-\frac{\partial \mathfrak{B}}{\partial \tau}, & & \text { III. } a \\
\operatorname{div} \mathfrak{B} & =0, & \text { IV. } a \\
\mathfrak{D} & =\frac{\varepsilon}{4 \pi} \mathfrak{E}, \quad \mathfrak{B}=\mu \mathfrak{H}, & \mathfrak{i}=\sigma \mathfrak{E} . & \text { V. } a
\end{array}
$$

The coordinates in them are taken in the substantial system $K^{\prime}$. They are the well-known Hertzian equations for moving bodies, which are generally obtained only for uniformly-moving bodies from our derivation.

[^9]Since we denote the coordinates in the substantial rest system $K$ by $x, y, z, t$, we must properly write equations I. $a$ to V. $a$ consistently in terms of $x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}$, where:

$$
\begin{equation*}
x^{\prime}=x-a t, \quad y^{\prime}=y, \quad z^{\prime}=z, \quad t^{\prime}=t . \tag{5}
\end{equation*}
$$

If we then map the processes inside of $K^{\prime}$ into geometric space by the mapping law:

$$
\begin{equation*}
x^{\prime}=\xi, \quad y^{\prime}=\eta, \quad z^{\prime}=\zeta, \quad t^{\prime}=\tau \tag{6}
\end{equation*}
$$

then the image of $K^{\prime}$ will be at rest, and the images of the processes that are given by I. $a$ to V.a will coincide with the processes in geometric space that are given by I to V, but from our definition, that means: $K^{\prime}$ can also be considered to be in a state of absolute rest. In complete analogy to Newtonian mechanics, we will then have the theorem:

If a body $K$ can be regarded as being in a state of absolute rest then that can also be said of any body $K^{\prime}$ that moves uniformly and rectilinearly with respect to $K$.

If we, like H. Hertz, postulate the validity of equations I. $a$ to V. $a$ for the processes in an arbitrarily-moving body then, from our definition, any body can be regarded as being in a state of absolute rest, in the electromagnetic sense.

In particular, it is known that it follows from equations I to V that plane waves propagate in geometric space in all directions with the same constant absolute velocity $1 / \sqrt{\varepsilon \mu}$. It follows from the relativity principle of Newtonian mechanics (which is expressed by equations I. $a$ to V.a here) that electric waves will also propagate in all directions with the same velocity $1 / \sqrt{\varepsilon \mu}$ relative to any body (whose constants have the values $\varepsilon$ and $\mu$ ) that can be regarded as being in a state of absolute rest. When the body consists of. e.g., air, and we let it become ever thinner and thinner, the values of $\varepsilon$ and $\mu$ will converge to 1 , so the value of the velocity will also converge to 1. If the body in which the wave motion takes place moves in the $x$-direction with the constant velocity $w$, contrary to my standpoint, then the waves will propagate in that direction with velocity $1+w$, versus my standpoint. That value will always be different from 1 by the value $w$, which is independent of the nature of the body. Now, when the body becomes ever thinner and thinner, so everything that moves with it will become ever less and less, and although the state of motion results with velocity $w$, I can succeed in making it approach the state in which nothing at all moves against me anymore by that thinning process. However, when I consider the propagation of light in the air at rest from the outset, contrary to my standpoint, it will result with velocity 1 , so it will differ by a finite amount. It must then be that when two actually-observed states converge to each other without limit, the wave velocities that correspond to them must always differ by a finite quantity. That is a conceptual difficulty in Hertz's equations, so in the postulation of Newton's principle of relativity in electrodynamics, as well. In general, that alone would not compel us to abandon Hertz's equations, and with it, Newton's principle of relativity, in the absence of any new experimental facts.

## Section Two

## § 7. - Electromagnetic processes in moving bodies and the relativity principle according to H . A. Lorentz.

We would now like to derive the equations for the electromagnetic processes in moving bodies from an entirely different system of assumptions. We shall then start from the same definition of a body "at rest" $K$ that was used in § 6.

A body for which $\varepsilon=\mu=1, \sigma=0$ might be called a refraction-free body. We assume that Maxwell's field equations are given for bodies at rest, so the following equations will be valid in the interiors of refraction-free bodies:

$$
\begin{array}{rlrl}
\operatorname{curl} \mathfrak{H} & =4 \pi \frac{\partial \mathfrak{E}}{\partial t}, & \text { I } \\
\operatorname{div} \mathfrak{E} & =\rho, & \text { II } \\
\operatorname{curl} \mathfrak{E} & =-\frac{\partial \mathfrak{B}}{\partial t}, & & \text { III }  \tag{III}\\
\operatorname{div} \mathfrak{H} & =0 . & \text { IV }
\end{array}
$$

The first assumption that we shall make is the following one:
A. If a refraction-free body $S^{\prime}$ moves rectilinearly with constant velocity $\mathfrak{w}$ relative to a substantial system $S$ that is at rest in the electromagnetic sense then the electromagnetic processes in the interior of $S^{\prime}$ shall still evolve according to equations I to IV, and indeed the coordinates will be referred to the system $S$ in that $\left({ }^{1}\right)$, except that equation I differs insofar as the convection current $\rho \mathfrak{w}$ appears in the displacement current $\partial \mathfrak{E} / \partial t$, so it will read:

$$
\operatorname{curl} \mathfrak{H}=4 \pi\left(\frac{\partial \mathfrak{E}}{\partial t}+\rho \mathfrak{w}\right)
$$

As a second assumption, the Michelson experiment suggests the following one:
B. Let $x, y, z, t$ be the coordinates relative to $S$, and let $x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}$ the ones relative to $S^{\prime}$. Plane waves shall then propagate inside of $S^{\prime}$ in all directions with the constant velocity 1 . Now since, from I, the waves that propagate in $S^{\prime}$ with constant velocity 1 in all directions, as well as

[^10]in relative to $S$, assumption $B$ defines a relation between the quantities $x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}$ and $x, y, z, t$. That relation can obviously be expressed as follows:

It must follow from:

$$
x^{2}+y^{2}+z^{2}=t^{2}
$$

that:

$$
x^{\prime 2}+y^{\prime 2}+z^{\prime 2}=t^{\prime 2} .
$$

C. The following properties of the Galilei transformations will remain true for the relations between the primed and unprimed coordinates, which will be mediated by the connection between the coordinates $S$ and $S^{\prime}$, instead of the Galilei transformations [§ 6, equation (5)], and we lay the $x$-axis in the direction of motion of $S^{\prime}$ with respect to $S$ and assume that the velocity is $a$ :
a) $x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}$ shall be linear homogeneous functions of $x, y, z, t$.
b) The coefficients depend upon only the velocity $a$.
c) The values of $t^{\prime}$ are symmetric around the direction of motion (i.e., the $x$-axis).
d) The transformations that correspond to different values of $a$ define a group.
$e)$ The coordinate planes correspond to each other at each point in time.
f) The length of every segment in $S^{\prime}$ shall be determined uniquely.

We next conclude from $a$ ), $b$ ), and $e$ ) that:

$$
\begin{align*}
& x^{\prime}=\varphi_{1}(a)(x-a t), \\
& y^{\prime}=\varphi_{1}(a) y  \tag{1}\\
& z^{\prime}=\varphi_{1}(a) y \\
& t^{\prime}=b_{1}(a) t+b_{2}(a) x+b_{3}(a) z+b_{4}(a) z
\end{align*}
$$

From $c$ ), $t^{\prime}$ can depend upon $y$ and $z$ only the combination $y^{2}+z^{2}$, from which it follows that:

$$
b_{3}=b_{4}=0 .
$$

$B$ ) implies the relation:

$$
\begin{equation*}
x^{\prime 2}+y^{\prime 2}+z^{\prime 2}-t^{\prime 2}=[\varphi(a)]^{2}\left(x^{2}+y^{2}+z^{2}-t^{2}\right) . \tag{2}
\end{equation*}
$$

That equation must be fulfilled identically by means (1), i.e., the coefficients of $x^{2}, y^{2}, z^{2}$, $t^{2}$ must vanish individually. Upon substituting the values of $y^{\prime}$ and $z^{\prime}$, we will find:

$$
\varphi_{2}(a)=\varphi_{3}(a)=\varphi(a),
$$

so:

$$
\begin{align*}
& y^{\prime}=\varphi(a) y,  \tag{3}\\
& z^{\prime}=\varphi(a) z,
\end{align*}
$$

and

$$
\begin{equation*}
x^{\prime 2}-t^{\prime 2}=[\varphi(a)]^{2}\left(x^{2}-t^{2}\right) . \tag{4}
\end{equation*}
$$

If we denote the hyperbolic sine (cosine, respectively) of a quantity $\psi$ by $\sinh \psi(\cosh \psi$, respectively), so:

$$
\begin{equation*}
2 \sinh \psi=e^{\psi}-e^{-\psi}, \quad 2 \cosh \psi=e^{\psi}+e^{-\psi} \tag{5}
\end{equation*}
$$

then every linear homogeneous transformation of $x$ and $t$ that fulfills the identity (4) can be written:

$$
\begin{align*}
x^{\prime} & =x \varphi(a) \cosh \psi+t \varphi(a) \sinh \psi,  \tag{6}\\
t^{\prime} & =-x \varphi(a) \sinh \psi+t \varphi(a) \cosh \psi,
\end{align*}
$$

in which $\psi$ is a variable parameter that can possibly be a function of $a$. Upon comparing (1) and (6), one will get the identities:

$$
\begin{align*}
x \varphi(a) \cosh \psi+t \varphi(a) \sinh \psi & =\varphi_{1}(a)(x-a t), \\
-x \varphi(a) \sinh \psi+t \varphi(a) \cosh \psi & =b_{1}(a) t+b_{2}(a) x . \tag{7}
\end{align*}
$$

A comparison of the coefficients of $x$ and $t$ will then give:

$$
\begin{array}{ll}
\varphi(a) \cosh \psi-\varphi_{1}(a)=0, & \varphi(a) \sinh \psi+b_{1}(a)=0,  \tag{8}\\
\varphi(a) \sinh \psi+a \varphi_{1}(a)=0, & \varphi(a) \cosh \psi-b_{1}(a)=0 .
\end{array}
$$

If we denote the hyperbolic tangent by tanh $\psi$ then it will follow from the first of equations (8) upon division that:

$$
\begin{equation*}
\tanh \psi=-a, \tag{9}
\end{equation*}
$$

from which, the relations between sine, cosine, and tangent will imply that:

$$
\begin{equation*}
\sinh \psi=\frac{a}{\sqrt{1-a^{2}}}, \quad \cosh \psi=\frac{1}{\sqrt{1-a^{2}}} . \tag{10}
\end{equation*}
$$

It then follows from (8) that:

$$
\varphi_{1}(a)=\frac{\varphi(a)}{\sqrt{1-a^{2}}}
$$

$$
\begin{align*}
& b_{1}(a)=\frac{\varphi(a)}{\sqrt{1-a^{2}}}  \tag{11}\\
& b_{2}(a)=\frac{-a \varphi(a)}{\sqrt{1-a^{2}}}
\end{align*}
$$

It will then follow from equations (1) and (3) that:

$$
\begin{equation*}
x^{\prime}=\varphi(a) \frac{x-a t}{\sqrt{1-a^{2}}}, \quad y^{\prime}=\varphi(a) y, \quad z^{\prime}=\varphi(a) z, \quad t^{\prime}=\varphi(a) \frac{-a t+t}{\sqrt{1-a^{2}}} \tag{12}
\end{equation*}
$$

We determine the function $\varphi(a)$ by the requirement $d$ ). To that end, of the transformations that are given by (12), we consider the ones that correspond to the parameter values $a_{1}$ and $a_{2}$ : We let the corresponding coordinate values be $x_{1}^{\prime}, \ldots, t_{1}^{\prime}\left(x_{2}^{\prime}, \ldots, t_{2}^{\prime}\right.$, respectively), and we express the $x_{2}^{\prime}$, etc., in terms of the $x_{1}^{\prime}$, etc. We will then get:

$$
\begin{equation*}
x_{2}^{\prime}=\frac{\varphi\left(a_{2}\right)}{\varphi\left(a_{1}\right)} \frac{x_{a}^{\prime}-b t_{1}^{\prime}}{\sqrt{1-b^{2}}}, \quad y_{2}^{\prime}=\frac{\varphi\left(a_{2}\right)}{\varphi\left(a_{1}\right)} y_{1}^{\prime}, \quad z_{2}^{\prime}=\frac{\varphi\left(a_{2}\right)}{\varphi\left(a_{1}\right)} z_{1}^{\prime}, \quad t_{2}^{\prime}=\frac{\varphi\left(a_{2}\right)}{\varphi\left(a_{1}\right)} \frac{-b x_{1}^{\prime}+t_{1}^{\prime}}{\sqrt{1-b^{\prime 2}}} . \tag{13}
\end{equation*}
$$

In that way, one has:

$$
\begin{equation*}
b=\frac{a_{2}-a_{1}}{1-a_{1} a_{2}} . \tag{14}
\end{equation*}
$$

In order for a group to be given by (12), it will then be necessary and sufficient that:

$$
\frac{\varphi\left(a_{2}\right)}{\varphi\left(a_{1}\right)}=\varphi(b)
$$

i.e., the function $\varphi(a)$ must fulfill the functional equation:

$$
\begin{equation*}
\varphi(v)=\varphi(u) \varphi\left(\frac{v-u}{1-u v}\right) \tag{15}
\end{equation*}
$$

in which $u$ and $v$ are any two values of the argument. When we differentiate (15) with respect to $u$ and set $u=0$ in the result, we will get:

$$
\begin{equation*}
0=\varphi^{\prime}(0) \varphi(v)+\left(v^{2}-1\right) \varphi(0) \varphi^{\prime}(v), \tag{16}
\end{equation*}
$$

in which $\varphi^{\prime}(v)$ means the derivative of $\varphi$. However, one has $\varphi(0)=1$, as we will see from (15) when we set $u=0$ in it. Moreover, we set $\varphi^{\prime}(0)=m$. The differential equation (16) will then have the single solution:

$$
\begin{equation*}
\varphi(u)=\left(\frac{1+u}{1-u}\right)^{m} . \tag{17}
\end{equation*}
$$

As one convinces oneself upon substituting, that function fulfills the functional equation (15).
That value of $\varphi(a)$ is substituted in equation (12). We now imagine a substantial segment that is perpendicular to the $x$-axis and let $r$ denote its length in $S$, while $r^{\prime}$ is its length in $S^{\prime}$. One then has:

$$
r^{\prime 2}=y^{\prime 2}+z^{\prime 2}, \quad r^{2}=y^{2}+z^{2},
$$

so

$$
r^{\prime 2}=r^{2}\left(\frac{1+a}{1-a}\right)^{2 m}
$$

Now, one will have $r^{\prime 2}>r^{2}$ or $r^{\prime 2}<r^{2}$ according to whether $a$ is positive or negative. However, does not at all depend upon the motion itself, but only upon which side of the $x$-axis is counted as positive. Should the length of a segment in $S^{\prime}$ depend upon only the motion, as assumption $f$ ) would demand, then $r^{\prime 2} / r^{2}$ would have to be independent of the sign of $a$, so:

$$
\left(\frac{1+a}{1-a}\right)^{2 m}=\left(\frac{1-a}{1+a}\right)^{2 m}
$$

i.e., $m=0$, so $\varphi(a)=1$. The transformation equations will ultimate read:

$$
\begin{equation*}
x^{\prime}=\frac{x-a t}{\sqrt{1-a^{2}}}, \quad y^{\prime}=y, \quad z^{\prime}=z, \quad t^{\prime}=\frac{-a t+t}{\sqrt{1-a^{2}}} \tag{18}
\end{equation*}
$$

then.
In order to perform the entire calculation, it is essential to assume that:

$$
\begin{equation*}
|a|<1 \tag{19}
\end{equation*}
$$

because otherwise no real transformation of the type that is required by $B$ ) and $C$ ) could be given at all.

We would now like to transform equations I', II, III, IV, which are valid inside of the moving body $S^{\prime}$, but when referred to the coordinates in $S$, into the coordinates in the moving body $S^{\prime}$.

As we have seen that will happen by way of equations (18), according to the assumptions $A$ ), $B$ ), C). The result of the transformation reads $\left({ }^{1}\right)$ :

$$
\begin{aligned}
\operatorname{curl}^{\prime} \mathfrak{H}^{\prime} & =4 \pi\left(\frac{\partial \mathfrak{E}^{\prime}}{\partial t^{\prime}}+\rho^{\prime} \mathfrak{w}^{\prime}\right), & \mathrm{I}^{\prime \prime} \\
\operatorname{div}^{\prime} \mathfrak{E}^{\prime} & =\rho^{\prime}, & \mathrm{II}^{\prime \prime} \\
\operatorname{curl}^{\prime} \mathfrak{E}^{\prime} & =-\frac{\partial \mathfrak{H}^{\prime}}{\partial t^{\prime}}, & \mathrm{III}^{\prime \prime} \\
\operatorname{div}^{\prime} \mathfrak{H}^{\prime} & =0 . & \mathrm{IV}^{\prime \prime}
\end{aligned}
$$

In those equations, we have:

$$
\begin{gather*}
\mathfrak{E}_{x^{\prime}}^{\prime}=\mathfrak{E}_{x}, \quad \mathfrak{E}_{y^{\prime}}^{\prime}=\frac{\mathfrak{E}_{y}-w \mathfrak{H}_{z}}{\sqrt{1-w^{2}}}, \quad \mathfrak{E}_{z^{\prime}}^{\prime}=\frac{\mathfrak{E}_{z}+w \mathfrak{H}_{y}}{\sqrt{1-w^{2}}}, \\
\mathfrak{H}_{x^{\prime}}^{\prime}=\mathfrak{H}_{x}, \quad \mathfrak{H}_{y^{\prime}}^{\prime}=\frac{\mathfrak{H}_{y}+w \mathfrak{E}_{z}}{\sqrt{1-w^{2}}}, \quad \mathfrak{H}_{z^{\prime}}^{\prime}=\frac{\mathfrak{H}_{z}-w \mathfrak{E}_{y}}{\sqrt{1-w^{2}}},  \tag{20}\\
\rho^{\prime}=\rho \sqrt{1-w^{2}}, \quad \mathfrak{w}_{x}^{\prime}=\mathfrak{w}_{y}^{\prime}=\mathfrak{w}_{z}^{\prime}=0,
\end{gather*}
$$

when we understand $\mathfrak{E}_{x}$ to mean the $x$-component of $\mathfrak{E}$, and analogously for the quantities $\mathfrak{E}_{y}, \mathfrak{E}_{z}$, $\ldots$, and further set $\mathfrak{w}_{x}=w, \mathfrak{w}_{y}=\mathfrak{w}_{z}=0$.

We must also set $a=w$ in equations (18).
Since equations $\mathrm{I}^{\prime \prime}$ to $\mathrm{IV}^{\prime \prime}$ have precisely the same form as equations I to IV, we can say: Equations I to IV will remain covariant under the group (18) (which we refer to as the group of Lorentz transformations) when one transforms the quantities $\mathfrak{E}, \mathfrak{H}, \rho, \mathfrak{w}$ according to the rules that are given by (20). That fact is analogous to the covariance under Galilei transformations that we learned about in the previous subsection, except that here another transformation of those quantities appears that is not transformed by the group itself ( $\mathfrak{E}, \mathfrak{H}$, etc.).

We can infer some important conclusions about the mappings of the systems $S$ and $S^{\prime}$ into absolute (geometric) space. Once more, let $\xi, \eta, \zeta$ be the coordinates, and let $t$ be the time in geometric space. In the system $S^{\prime}$, we have two substantial points whose connecting line is parallel to the $x$-axis. We then call the quantity $x_{2}^{\prime}-x_{1}^{\prime}=l$, in which $x_{1}^{\prime}$ and $x_{2}^{\prime}$ are the abscissas of the two points in $S^{\prime}$, the empirical distance between the two points. In the system $S$, two substantial points might have the empirical distance $l$ when $x_{2}-x_{1}=l$.
$\left({ }^{1}\right)$ The recalculation was first performed by H. A. Lorentz. The most elegant version of that is found in Minkowski, Göttinger Nachrichten (1908). Minkowski found the result with no calculation when he regarded equations (18) as a rotation in four-dimensional space in the $x t$-plane through a complex angle (loc. cit., pp. 59, et seq.)

I can now perform the mapping into geometric space according to the law:

$$
\begin{equation*}
x=\xi, y=\eta, z=\zeta, t=\tau . \tag{21}
\end{equation*}
$$

The image will then be at rest in $S$, which I express as follows: I regarded $S$ as being in a state of absolute rest. We now consider two points in $S$ that have the empirical distance $x_{2}-x_{1}=l$. The abscissas of their image are then:

$$
\begin{equation*}
\xi_{1}=x_{1}, \quad \xi_{2}=x_{2} . \tag{22}
\end{equation*}
$$

We call the quantity $\xi_{2}-\xi_{1}$ the absolute distance between the two points. It will follow from (22) that when a system is regarded as being in a state of absolute rest, the empirical distance between two points will be equal to the absolute one. However, we now perform the mapping according to the law:

$$
\begin{equation*}
x^{\prime}=\xi, \quad y^{\prime}=\eta, \quad z^{\prime}=\zeta, \quad t^{\prime}=\tau \tag{23}
\end{equation*}
$$

The image of $S^{\prime}$ will now be at rest. We now regard $S^{\prime}$ as being in a state of absolute rest, and $S$ as moving with the velocity $w$. We will now have:

$$
x_{1}^{\prime}=\frac{x_{1}-w t}{\sqrt{1-w^{2}}}, \quad x_{2}^{\prime}=\frac{x_{2}-w t}{\sqrt{1-w^{2}}} .
$$

Due to (23), one has:

$$
x_{1}^{\prime}=\xi_{1}, \quad x_{2}^{\prime}=\xi_{2},
$$

so:

$$
\begin{equation*}
\xi_{2}-\xi_{1}=\frac{x_{2}-x_{1}}{\sqrt{1-w^{2}}}=\frac{l}{\sqrt{1-w^{2}}} . \tag{24}
\end{equation*}
$$

i.e., when a system $S$ is regarded as moving in the $x$-direction with a velocity $w$, the absolute distance between two substantial points will be increased in comparison to the empirical one by a ratio of $1: \sqrt{1-w^{2}}$. The ratio of the empirical distance to the absolute one then depends upon my own whims. One refers to the fact that the empirical distances in a system get smaller compared to the absolute ones when I assume that motion of the image of our system in geometric space moves faster as the Lorentz contraction, which is therefore not at all a contraction in proper sense of the word, but merely an expression for the type of map that we make into absolute space. When one deduces from Lorentz's theory that the motion of Earth makes it contract by a ratio of 1 : $\sqrt{1-w^{2}}, w$ must be understood to mean the speed of the Earth relative to the Sun, i.e., we map the processes from the world into absolute space in such a way that the image of the Sun is at rest, we must do that in order to make the electromagnetic and mechanical processes that take place at the same location also overlap in their image in geometric space.

Finally, we would like to express the fact that the same equations for $\mathfrak{E}, \mathfrak{H}$, etc., will arise in geometric space under the transformations (21) and (23) due to the covariance of the equations for
refraction-free bodies, and those two will coincide with equations I to IV in § 6, in the following form:

When a system $S$ can be considered to be in a state of absolute rest, so can every system $S^{\prime}$ that moves uniformly and rectilinearly with respect to $S$. That theorem is mathematically equivalent to the covariance of the equations for refraction-free bodies under the group of Lorentz transformations. We call that theorem: The theorem of the relativity of electrodynamics.

Now, we can employ that theorem as a postulate in order to exhibit new equations. Thus, Minkowski $\left({ }^{1}\right)$ has derived the equations for electromagnetic processes in moving bodies that are not refraction-free from three axioms that we would like to express in conjunction with what was said up to now:
A) Maxwell's field equations (§ 6, I to IV) are valid in bodies at rest.
$B$ ) The equations for a body that moves rectilinearly with a velocity $w$ shall reduce to the ones for bodies at rest when $w=0$.
C) Any velocity of a body shall be smaller than the speed of propagation of plane waves in a refraction-free body, so:

$$
|w|<1 .
$$

$D)$ The equations for arbitrary moving bodies shall remain covariant in the same sense as they are for refraction-free bodies.

Minkowski derived a system of equations for arbitrary ponderable bodies from those assumptions that I would not like to write out here. However, I shall refer to his own work.

In the next section, we would like to apply the postulate of covariance under the Lorentz transformation to some further cases.

## § 8. - The basic concepts of electromagnetic mechanics.

When we map an empirical system in which electromagnetic processes play out into geometric space, it can happen that the image of one and the same substantial segment can have different lengths according to whether we perform the map in the mechanical sense or in the electromagnetic picture. However, since theoretical physics must strive to achieve a unified picture of all physical processes in geometric space and the extension of the law of mapping from Newtonian mechanics to electrical processes proves to be unsuitable, that suggests that one might extend the law of mapping from Lorentzian electrodynamics to mechanics. A theory of mechanics that would be

[^11]suited to that has already been proposed by Einstein ( ${ }^{1}$ ) and Minkowski $\left({ }^{2}\right)$. The theories of mechanics of the two cited researchers differ somewhat. However, it is common to both of them that the fundamental equations are covariant under the group of Lorentz transformations. We would like to call such a theory of mechanics an electromagnetic theory of mechanics.

The structure of Newtonian mechanics, which we have in the first chapter, allows us to construct an electromagnetic theory of mechanics in a precisely parallel way, and indeed simply by replacing the group of Galilei transformations with that of the Lorentz transformations, or expressed more physically, the relativity postulate of Newtonian mechanics with that of electrodynamics.

The basic concepts of electromagnetic mechanics then define a second analogy to the basic concepts of differential geometry, and indeed that analogy is much closer and more transparent that in Newtonian mechanics, wherein lies the greater harmony that Minkowski boasted of in his own mechanics.

We once more start with motions of a material point along the $x$-axis, which we shall represent as time-parameterized trajectories in the $x t$-plane. The transition to a system in the $x t$-plane that moves with velocity $a$ means a homogeneous linear coordinate transformation that is given by:

$$
\begin{equation*}
t^{\prime}=\frac{a x+t}{\sqrt{1-a^{2}}}, \quad x^{\prime}=\frac{x-a t}{\sqrt{1-a^{2}}}, \quad|a|<1 \tag{1}
\end{equation*}
$$

[see § 7, equation (18)]. If $x, t$ are rectangular coordinates then $x^{\prime}, t^{\prime}$ will be skew. Each line that goes through the origin, and whose direction tangent is less than 1 , belongs to another line as the $x^{\prime}$-axis. That pair defined a hyperbolic involution, namely, that of the conjugate diameter to the hyperbola:

$$
x^{2}-t^{2}=1
$$

The infinitesimal transformation that generates the group (1) reads:

$$
\begin{equation*}
\delta t=-x \delta a, \quad \delta x=-t \delta a \tag{2}
\end{equation*}
$$

We now define the arc-length element precisely as in $\S 4$ (pp.22). Postulates $A$ ), $C$ ), $D$ ) remain the same verbatim. However, we now have:
B) The form shall remain invariant under the group of Lorentz transformations [equation (1)].

The proper time is defined as it was in $\S 4$.
We shall next exhibit the finite invariants of the group. Due to equation (2), an invariant $F(x$, $t$ ) must satisfy the partial differential equation:

[^12]\[

$$
\begin{equation*}
t \frac{\partial F}{\partial x}+x \frac{\partial F}{\partial t}=0 \tag{3}
\end{equation*}
$$

\]

It will then follow that $F$ can only be a function of $t^{2}-x^{2}$; thus:

$$
d \sigma^{2}=f\left(t^{2}-x^{2}\right)\left(d t^{2}-d x^{2}\right)
$$

As in $\S 4$, one concludes from $C)$ and $D$ ) that $f\left(t^{2}-x^{2}\right)=1$, so:

$$
\begin{equation*}
d s=\sqrt{d t^{2}-d x^{2}} \tag{4}
\end{equation*}
$$

If we have a time-parameterized trajectory for which $|d x / d t|<1$ then the proper time, as measured from any point, will be quantity that always increases. As in § 4, we introduce it as a parameter:

$$
\begin{equation*}
u=\int d \sigma=\int \sqrt{d t^{2}-d x^{2}}=\int \sqrt{d t^{\prime 2}-d x^{\prime 2}}=\int d \sigma^{\prime}=u^{\prime} \tag{5}
\end{equation*}
$$

The "proper time" here is different from the time in the ordinary sense. As postulate $D$ ) demands, they will agree only in the case of rest. We would like to assume that $|d x / d t|<1$ for any time-parameterized trajectory. We now define the first and second extensions of the group (1):

$$
\begin{array}{ll}
\frac{d t^{\prime}}{d u^{\prime}}=\frac{-a \frac{d x}{d u}+\frac{d t}{d u}}{\sqrt{1-a^{2}}}, & \frac{d x^{\prime}}{d u^{\prime}}=\frac{\frac{d x}{d u}-a \frac{d t}{d u}}{\sqrt{1-a^{2}}} \\
\frac{d^{2} t^{\prime}}{d u^{\prime 2}}=\frac{-a \frac{d^{2} x}{d u^{2}}+\frac{d^{2} t}{d u^{2}}}{\sqrt{1-a^{2}}}, & \frac{d^{2} x^{\prime}}{d u^{\prime 2}}=\frac{\frac{d^{2} x}{d u^{2}}-a \frac{d^{2} t}{d u^{2}}}{\sqrt{1-a^{2}}} \tag{1.b}
\end{array}
$$

The infinitesimal transformation that generates the two-fold extended group reads:

We now define the concept of the curvature $1 / r$ by the following four postulates, which correspond to the one for classical mechanics ( $\S 4, \mathrm{pp} .23$ ).
A) The curvature shall be a second-order differential invariant under the group of Lorentz transformations.

We adopt postulates $B$ ) and $C$ ) from § 4 verbatim.
The unit of measurement for curvature was established by $D$ ) in $\S 4$. We would like to do that here as follows:
D) There shall be at least one location where motion with constant proper acceleration $\frac{d}{d t} \frac{d x}{d \sigma}$ $=1$ corresponds to the curvature $1 / \rho=1$.

We then set:

$$
\frac{1}{\rho}=F\left(u, t, x, t_{u}, x_{u}, t_{u u}, x_{u u}\right)
$$

According to $A$ ), $F$ must be a finite invariant of the group that is generated by (6). It must then satisfy the following partial differential equation:

$$
\begin{equation*}
t \frac{\partial F}{\partial x}+x \frac{\partial F}{\partial t}+t_{u} \frac{\partial F}{\partial x_{u}}+x_{u} \frac{\partial F}{\partial t_{u}}+t_{u u} \frac{\partial F}{\partial x_{u u}}+x_{u} \frac{\partial F}{\partial t_{u u}}=0 . \tag{7}
\end{equation*}
$$

That corresponds to the following system of ordinary differential equations:

$$
\begin{equation*}
\frac{d u}{0}=\frac{d t}{x}=\frac{d x}{t}=\frac{d t_{u}}{x_{u}}=\frac{d x_{u}}{t_{u}}=\frac{d t_{u u}}{x_{u u}}=\frac{d x_{u u}}{t_{u u}} \tag{8}
\end{equation*}
$$

The integrals of that system are: $u, t^{2}-x^{2}, t t_{u}-x x_{u}, t_{u}^{2}-x_{u}^{2}, t_{u} t_{u u}-x_{u} x_{u u}, t_{u u}^{2}-x_{u u}^{2}$.
Now, due to equation (5), one has:

$$
\begin{aligned}
t_{u}^{2}-x_{u}^{2} & =1, \\
x_{u} x_{u u}-t_{u} t_{u u} & =0 .
\end{aligned}
$$

It will then follow from $A$ ) and $B$ ) that $1 / \rho$ must have the form:

$$
\frac{1}{\rho}=f\left(u, t^{2}-x^{2}, t t_{u}-x x_{u}\right) \sqrt{ \pm\left(x_{u u}^{2}-t_{u u}^{2}\right)} .
$$

Due to $C$ ), the function $f$ will reduce to a constant, so:

$$
\frac{1}{\rho}=c \sqrt{x_{u u}^{2}-t_{u u}^{2}},
$$

in which $c$ is possibly an imaginary constant. In order to apply the postulate $D$ ), we next set:

$$
\begin{equation*}
\frac{d x}{d t}=w \tag{9}
\end{equation*}
$$

and remark that it will follow from equation (5) that:

$$
\left.\begin{array}{rl}
\frac{d t}{d u}=\frac{d t}{d \sigma}= & \frac{1}{\sqrt{1-w^{2}}}, \quad \frac{d x}{d u}=\frac{d x}{d \sigma}=\frac{w}{\sqrt{1-w^{2}}} \\
\frac{d^{2} t}{d u^{2}} & =\frac{d^{2} t}{d \sigma^{2}}=\frac{w}{\left(1-w^{2}\right)^{2}} \frac{d w}{d t} \\
\frac{d^{2} x}{d u^{2}}=\frac{d^{2} x}{d \sigma^{2}}=\frac{1}{\left(1-w^{2}\right)^{2}} \frac{d w}{d t} . \tag{11}
\end{array}\right\}
$$

It will follow from (10) and (11) that:

$$
\left(\frac{d^{2} x}{d u^{2}}\right)^{2}-\left(\frac{d^{2} t}{d u^{2}}\right)^{2}=\left(\frac{d w}{d t}\right)^{2} \frac{1}{\left(1-w^{2}\right)^{3}}
$$

It will then follow that:

$$
\begin{equation*}
+\sqrt{x_{u u}^{2}-t_{u u}^{2}}=\frac{1}{\sqrt{\left(1-w^{2}\right)^{3}}} \frac{d w}{d t}=\frac{d}{d t} \cdot \frac{d x}{d u} . \tag{12}
\end{equation*}
$$

Now, since $D$ ) says that it follows from $\frac{d}{d t} \cdot \frac{d x}{d u}=1$ that there must also be at least one place where 1 / $\rho=1$, we must have:

$$
\begin{equation*}
\frac{1}{\rho}=\sqrt{x_{u u}^{2}-t_{u u}^{2}}=\sqrt{\left(\frac{d^{2} x}{d u^{2}}\right)^{2}-\left(\frac{d^{2} t}{d u^{2}}\right)^{2}} \tag{13}
\end{equation*}
$$

We define the concept of vis viva $L$ by the same postulates $A$ ) and $B$ ) as in $\S 4$ (pp. 24), word-forword.

The fact that $\frac{d L}{d x}=\frac{m}{\rho}$ implies that:

$$
\frac{d L}{d t}=\frac{m}{\rho} w=\frac{m w}{\sqrt{\left(1-w^{2}\right)^{3}}} \frac{d w}{d t}=\frac{d}{d t} \frac{m}{\sqrt{1-w^{2}}},
$$

and due to $B$ ) and equation (10):

$$
\begin{equation*}
L=m\left(\frac{1}{\sqrt{1-w^{2}}}-1\right)=m\left(\frac{d t}{d \sigma}-1\right) . \tag{14}
\end{equation*}
$$

The concept of the transformations contragredient to the Galilei transformations was also important to the structure of classical mechanics. Here as well, we would like to define two variables $X$ and $-T$ ( $X^{\prime}$ and $-T^{\prime}$, resp.) that suffer the contragredient transformation when $x, t$ suffer a Lorentz transformation, i.e., they fulfill the identity:

$$
\begin{equation*}
X x-T t=X^{\prime} x^{\prime}-T^{\prime} t^{\prime} \tag{15}
\end{equation*}
$$

[see § 5, equation (1)]. That will imply the relations:

$$
\begin{equation*}
X^{\prime}=\frac{X-a T}{\sqrt{1-a^{2}}}, \quad T^{\prime}=\frac{-a X+T}{\sqrt{1-a^{2}}} . \tag{16}
\end{equation*}
$$

Here, the contragredient transformation is identical to the original one.
For the motion of a material point in space, we use the demand $C$ ) on pp. 25 to postulate the validity of formula (14), but we now set:

$$
w^{2}=\left(\frac{d x}{d t}\right)^{2}+\left(\frac{d y}{d t}\right)^{2}+\left(\frac{d z}{d t}\right)^{2}
$$

so

$$
\begin{equation*}
L=m\left(\frac{d t}{\sqrt{d t^{2}-d x^{2}-d y^{2}-d z^{2}}}-1\right)=m\left(\frac{d t}{d \sigma}-1\right), \tag{17}
\end{equation*}
$$

in which one then sets $d \sigma^{2}=d t^{2}-d x^{2}-d y^{2}-d z^{2}$, and one again understands $\int d \sigma$ to mean the "proper time" of a motion. If we have several material points then we would like to understand $L_{h}$ to mean the vis viva of the $h^{\text {th }}$ point and set:

$$
\begin{equation*}
L_{h}=m_{h}\left(\frac{1}{\sqrt{1-w_{h}^{2}}}-1\right)=m_{h}\left(\frac{d t}{d \sigma_{h}}-1\right) \tag{18}
\end{equation*}
$$

in which $w_{h}$ is the velocity of the $h^{\text {th }}$ point, and:

$$
\begin{equation*}
d \sigma_{h}=d t \sqrt{1-w_{h}^{2}} . \tag{19}
\end{equation*}
$$

## § 9. - Einsteinian and Minkowskian mechanics.

We would now like to exhibit an electromagnetic theory of mechanics that precisely parallels the structure of classical mechanics in § 5, and in which we replace the Galilei transformations with the Lorentz transformations everywhere.

We next remark that due to the relations that were true in the context of Galilei transformations, viz.:

$$
\begin{equation*}
d t=d t^{\prime}=d \sigma \tag{1}
\end{equation*}
$$

the postulates in § $\mathbf{5}$ can be written very different ways. Each of those ways of writing things, which were equivalent in the context of Galilei transformation, have very different meanings for Lorentz transformations. Thus, in place of the quantities $X$ and - $T$ in Postulate I (pp. 26), we can also set $X d t$, $-T d t\left(X^{\prime} d t^{\prime},-T^{\prime} d t^{\prime}\right.$, respectively) without equation (1) in $\S \mathbf{5}$ changing its meaning. We can also set $d L / d \sigma$ in place of $d L / d t$ in Postulate II again without making any difference. We must consider those various notations while adapting this postulate to electromagnetic mechanics.

We would next like to exhibit the equations of motion for a system that consists of a single material point, and indeed using the following postulates:

We express Postulate I as it was stated in § $\mathbf{5}$ verbatim.
II. The point moves in such a way that the following equation will be true:

$$
\begin{equation*}
\frac{d L}{d \sigma}=T \tag{2}
\end{equation*}
$$

in which $L$ is given by equation (17) in $\S 8$.
III. Equation (2) shall be covariant under the group Lorentz transformations, i.e., it shall go to:

$$
\begin{equation*}
\frac{d L^{\prime}}{d \sigma^{\prime}}=T^{\prime} \tag{3}
\end{equation*}
$$

The covariance is understood to mean that $X,-T$ suffer the transformation that is contragredient to $x, t$.

Equations (2) and (3) can also be written:

$$
\begin{equation*}
m \frac{d}{d \sigma} \frac{d t}{d \sigma}=T, \quad m \frac{d}{d \sigma^{\prime}} \frac{d t^{\prime}}{d \sigma^{\prime}}=T^{\prime} \tag{4}
\end{equation*}
$$

from which it will follow that:

$$
m \frac{d}{d \sigma}\left(\frac{d t}{d \sigma}-a \frac{d x}{d \sigma}\right)=T-a X
$$

when one uses equations (1), (5), and (16) of § 8. When one subtracts the first of equations (4), it will follow that:

$$
\begin{equation*}
m \frac{d^{2} x}{d \sigma^{2}}=X \tag{5}
\end{equation*}
$$

One can derive the equations:

$$
\begin{equation*}
m \frac{d^{2} y}{d \sigma^{2}}=Y, \quad m \frac{d^{2} x}{d \sigma^{2}}=Z \tag{5.a}
\end{equation*}
$$

analogously. However, those are the basic equations for Minkowskian mechanics. The equations that relate to a system that moves with a velocity a relative to the system $x, y, z$ will follow from equations (5) and (5.a) by an application of the Lorentz transformation:

$$
\begin{equation*}
m \frac{d^{2} x^{\prime}}{d \sigma^{\prime 2}}=X^{\prime}, \quad m \frac{d^{2} y^{\prime}}{d \sigma^{\prime 2}}=Y^{\prime}, \quad m \frac{d^{2} z^{\prime}}{d \sigma^{\prime 2}}=Z^{\prime} \tag{6}
\end{equation*}
$$

If we set:

$$
\frac{d x}{d t}=w_{x}, \quad \frac{d y}{d t}=w_{y}, \quad \frac{d z}{d t}=w_{z}
$$

then it will follow from equation (2) that:

$$
T=m \frac{d}{d \sigma} \frac{d t}{d \sigma}=m \frac{1}{\sqrt{1-w^{2}}} \frac{d}{d t} \frac{1}{\sqrt{1-w_{x}^{2}-w_{y}^{2}-w_{z}^{2}}}=m \frac{1}{\left(1-w^{2}\right)^{2}}\left(w_{x} \frac{d w_{x}}{d t}+w_{y} \frac{d w_{y}}{d t}+w_{z} \frac{d w_{z}}{d t}\right) .
$$

Moreover, one has the analogue to equation (11) in § 8:

$$
\frac{d^{2} x}{d \sigma^{2}}=\frac{1}{\left(1-w^{2}\right)} \frac{d w_{x}}{d t}+\frac{w_{x}}{\left(1-w^{2}\right)^{2}}\left(w_{x} \frac{d w_{x}}{d t}+w_{y} \frac{d w_{y}}{d t}+w_{z} \frac{d w_{z}}{d t}\right), \text { etc., }
$$

so when one recalls equations (5) and (5.a):

$$
\begin{equation*}
T=X w_{x}+Y w_{y}+Z w_{z}=X \frac{d x}{d t}+Y \frac{d y}{d t}+Z \frac{d z}{d t} . \tag{7}
\end{equation*}
$$

However, as was remarked before at the beginning of this section, we can adapt Postulates I, II, and III of § $\mathbf{5}$ to electromagnetic mechanics in yet another way, and indeed in the following way:
I. Let $X^{*},-T^{*}$ be a pair of variables that suffer a linear transformation such that:

$$
\begin{equation*}
\left(X^{*} x-T^{*} t\right) d t=\left(X^{* \prime} x^{\prime}-T^{* \prime} t^{\prime}\right) d t^{\prime} \tag{8}
\end{equation*}
$$

when $x, t$ suffer a linear transformation.

The pairs $Y^{*},-T^{*}$ and $Z^{*},-T^{*}$ might have analogous meanings.
$X^{*}$ is called the $x$-component of the force that acts upon the point, and $T^{*}$ is the effect of that force component.
II. The point moves in such a way that:

$$
\begin{equation*}
\frac{d L}{d t}=T^{*} \tag{9}
\end{equation*}
$$

III. Equation (9) remains covariant under the group of Lorentz transformations, i.e., it will go to:

$$
\begin{equation*}
\frac{d L^{\prime}}{d t^{\prime}}=T^{* \prime} \tag{10}
\end{equation*}
$$

Equations (9) and (10) can also be written in the form:

$$
\begin{equation*}
m \frac{d}{d t} \frac{d t}{d \sigma}=T^{*}, \quad m \frac{d}{d t^{\prime}} \frac{d t^{\prime}}{d \sigma^{\prime}}=T^{* \prime} \tag{11}
\end{equation*}
$$

Due to (8), one has the equations:

$$
\begin{equation*}
X^{*} \frac{d t^{\prime}}{d t}=\frac{X^{*}-a T^{*}}{\sqrt{1-a^{2}}}, \quad T^{*} \frac{d t^{\prime}}{d t}=\frac{-a X^{*}+T^{*}}{\sqrt{1-a^{2}}} . \tag{12}
\end{equation*}
$$

It follows from (11) and (12) that:

$$
m \frac{d}{d t}\left(\frac{d t}{d \sigma}-a \frac{d x}{d \sigma}\right)=-a X^{*}+T^{*}
$$

and from that:

$$
\begin{equation*}
m \frac{d}{d t} \frac{d x}{d \sigma}=X^{*} \tag{13}
\end{equation*}
$$

and analogously:

$$
\begin{equation*}
m \frac{d}{d t} \frac{d y}{d \sigma}=Y^{*}, \quad m \frac{d}{d t} \frac{d z}{d \sigma}=Z^{*} \tag{13.a}
\end{equation*}
$$

and when referred to a moving system:

$$
\begin{equation*}
m \frac{d}{d t^{\prime}} \frac{d x^{\prime}}{d \sigma^{\prime}}=X^{* \prime}, \quad m \frac{d}{d t^{\prime}} \frac{d y^{\prime}}{d \sigma^{\prime}}=Y^{* \prime}, \quad m \frac{d}{d t^{\prime}} \frac{d z^{\prime}}{d \sigma^{\prime}}=Z^{* \prime} . \tag{14}
\end{equation*}
$$

Moreover:

$$
T^{*}=m \frac{d}{d t} \frac{d t}{d \sigma}=m \frac{d}{d t} \frac{1}{\sqrt{1-w^{2}}}=\frac{m}{\sqrt{\left(1-w^{2}\right)^{3}}}\left(w_{x} \frac{d w_{x}}{d t}+w_{y} \frac{d w_{y}}{d t}+w_{z} \frac{d w_{z}}{d t}\right) .
$$

Now, since:

$$
\frac{d}{d t} \frac{d x}{d \sigma}=\frac{1}{\sqrt{\left(1-w^{2}\right)}} \frac{d w_{x}}{d t}+\frac{w_{x}}{\sqrt{\left(1-w^{2}\right)^{3}}}\left(w_{x} \frac{d w_{x}}{d t}+w_{y} \frac{d w_{y}}{d t}+w_{z} \frac{d w_{z}}{d t}\right)
$$

it will follow that:

$$
\begin{equation*}
T^{*}=X^{*} \frac{d x}{d t}+Y^{*} \frac{d y}{d t}+Z^{*} \frac{d z}{d t}, \tag{15}
\end{equation*}
$$

due to equations (13) and (13.a).
Equations (13) and (13.a) are the basic equations of Einstein's mechanics.
We would now like to calculate the mass of the material point. In Newtonian mechanics, the mass was equal to the coefficient $m$ in the expression for the vis viva.

Since the definition of mass that was given on pp. 27 is independent of velocity, we can also express it in the following two forms:
I. One understands the longitudinal mass of a material point $\mu^{(l)}$ to mean the quotient of a force component and the component of acceleration in the same direction when the velocity is parallel to the acceleration. Thus:

$$
\mu^{(l)}=\frac{X}{\frac{d^{2} x}{d t^{2}}},
$$

under the condition that:

$$
\frac{d y}{d t}=\frac{d z}{d t}=0, \quad \frac{d^{2} y}{d t^{2}}=\frac{d^{2} z}{d t^{2}}=0 .
$$

II. One understands the transverse mass of a material point $\mu^{(t)}$ to mean the same quotient when the velocity is perpendicular to the velocity:

$$
\mu^{(t)}=\frac{X}{\frac{d^{2} x}{d t^{2}}}
$$

under the condition that:

$$
\frac{d x}{d t}=0, \quad \frac{d^{2} y}{d t^{2}}=\frac{d^{2} z}{d t^{2}}=0
$$

The two definitions are equivalent in classical mechanics, so one would then have:

$$
\mu^{(l)}=\mu^{(t)}=m .
$$

We would now like to calculate the masses for Minkowskian and Einsteinian mechanics. We start from equations (5) and (5.a) and define:

$$
\begin{array}{ll}
\mu^{(l)}=\frac{X}{\frac{d w_{x}}{d t}} & \left(w_{y}=w_{z}=0, \quad \frac{d w_{y}}{d t}=\frac{d w_{z}}{d t}=0\right), \\
\mu^{(t)}=\frac{X}{\frac{d w_{x}}{d t}} & \left(w_{x}=0, \quad \frac{d w_{y}}{d t}=\frac{d w_{z}}{d t}=0\right) .
\end{array}
$$

Now, one has:

$$
X=\frac{m}{\sqrt{1-w^{2}}} \frac{d}{d t} \frac{w_{x}}{\sqrt{1-w^{2}}}=m\left[\frac{w_{x}}{\left(1-w^{2}\right)^{2}}\left(w_{x} \frac{d w_{x}}{d t}+w_{y} \frac{d w_{y}}{d t}+w_{z} \frac{d w_{z}}{d t}\right)+\frac{1}{1-w^{2}} \frac{d w_{x}}{d t}\right],
$$

so it will follow from this that for $w_{y}=w_{z}=0, w=w_{x}$, one has:

$$
X=\frac{m}{\left(1-w_{x}^{2}\right)^{2}} \frac{d w_{x}}{d t}=\frac{m}{\left(1-w^{2}\right)^{2}} \frac{d w_{x}}{d t},
$$

and for $w_{x}=0, \frac{d w_{y}}{d t}=\frac{d w_{z}}{d t}=0$ :

$$
X=\frac{m}{1-w^{2}} \frac{d w_{x}}{d t},
$$

so:

$$
\begin{equation*}
\mu^{(l)}=\frac{m}{\left(1-w^{2}\right)^{2}}, \quad \mu^{(t)}=\frac{m}{1-w^{2}} . \tag{16}
\end{equation*}
$$

Analogously, we find from equations (13) and (13.a) of Einsteinian mechanics that:

$$
\begin{equation*}
\mu^{*(l)}=\frac{m}{\sqrt{\left(1-w^{2}\right)^{3}}}, \quad \mu^{*(t)}=\frac{m}{\sqrt{1-w^{2}}} . \tag{17}
\end{equation*}
$$

In electromagnetic mechanics, the mass of a material point does not coincide with the coefficient $m$ in the expression for the vis viva them which was introduced by the definition (pp. 24):

$$
\frac{d L}{d x}=\frac{m}{\rho} .
$$

The values of the longitudinal and transverse masses do not coincide in Einsteinian and Minkowskian mechanics, but their ratio does:

$$
\begin{equation*}
\frac{\mu^{(t)}}{\mu^{(l)}}=\frac{\mu^{*(t)}}{\mu^{*(l)}}=1-w^{2} . \tag{18}
\end{equation*}
$$

The Einsteinian values of the masses agree with the ones that Lorentz derived in his theory of the electron. It is interesting that those values, which Lorentz derived under a special hypothesis regarding the form of the electron, are implied directly by the basic equations for Einsteinian mechanics, which are derived in precisely the same as those of Newtonian mechanics in their own right when one only replaces the relativity principle of Newtonian mechanics with that of Lorentzian electrodynamics.

## § 10. - Systematic construction of the electromagnetic mechanics for $n$ material points.

Now that we have studied the two ways of adapting Newtonian mechanics, we would now like to construct the electromagnetic mechanics of point systems precisely as we did in $\S \mathbf{5}$ for classical mechanics, and while preserving the same assumptions. We first start with the picture of Minkowskian mechanics and then express the assumptions in the following way:
I. We preserve the one in § $\mathbf{5}(\mathrm{pp} .26)$ verbatim.
II. The system moves in such a way that the equation is fulfilled:

$$
\begin{equation*}
\sum_{i=1}^{n} \frac{d L_{i}}{d \sigma_{i}}=T \tag{1}
\end{equation*}
$$

The $L_{i}$ and $d \sigma_{i}$ in it are the quantities that were defined by equations (18) and (19) in § 8.
III. Equation (1) shall remain covariant under the group of Lorentz transformations, i.e., it shall go to:

$$
\begin{equation*}
\sum_{i=1}^{n} \frac{d L_{i}^{\prime}}{d \sigma_{i}^{\prime}}=T^{\prime} . \tag{2}
\end{equation*}
$$

The Lorentz transformations are given here by the equations:

$$
\begin{equation*}
x_{h}^{\prime}=\frac{x_{h}-a t}{\sqrt{1-a^{2}}}, \quad t_{h}^{\prime}=\frac{-a x_{h}+t}{\sqrt{1-a^{2}}} \quad(h=1,2, \ldots, n) . \tag{3}
\end{equation*}
$$

We adopt assumptions IV and V from § 5 (pp. 26) verbatim.
We conclude from II and III, as in § 9, equation (5), that the following equations must be fulfilled during the motion of the system:

$$
\begin{equation*}
\sum_{h=1}^{n} m_{h} \frac{d^{2} x_{h}}{d \sigma_{h}^{2}}=X, \quad \sum_{h=1}^{n} m_{h} \frac{d^{2} y_{h}}{d \sigma_{h}^{2}}=Y, \quad \sum_{h=1}^{n} m_{h} \frac{d^{2} z_{h}}{d \sigma_{h}^{2}}=Z . \tag{4}
\end{equation*}
$$

A theorem that is analogous to the center of mass theorem cannot be expressed here. For that reason, the concept of "the mass of the system," as it was defined in § 5, has no meaning here. By contrast, one can conclude from IV that:

$$
\begin{equation*}
m_{h} \frac{d^{2} x_{h}}{d \sigma_{h}^{2}}=X_{h}, \quad m_{h} \frac{d^{2} y_{h}}{d \sigma_{h}^{2}}=Y_{h}, \quad m_{h} \frac{d^{2} z_{h}}{d \sigma_{h}^{2}}=Z_{h} \quad(h=1,2, \ldots, n), \tag{5}
\end{equation*}
$$

exactly as one does in Newtonian mechanics.
Moreover, the principle of the equality of action and reaction can be derived in exactly the same way as in § 5. Equations (4) and (5) have the same form relative to a moving system, but in terms of the primed quantities. The relativity principle in electrodynamics is fulfilled.

We would now like to adapt the picture that led us to Einsteinian mechanics in § 9 to a system of $n$ points. To that end, we express the assumptions in the following form:
I. Let $X_{h}^{*},-T_{h}^{*}(h=1,2, \ldots, n)$ be pairs of variables such that when $x_{h}, t_{h}$ suffer a linear transformation, they will suffer a linear transformation for which the identity:

$$
\begin{equation*}
\left(X_{h}^{*} x_{h}-T_{h}^{*} t\right) d t=\left(X_{h}^{* \prime} x_{h}^{\prime}-T_{h}^{* \prime} t_{h}^{\prime}\right) d t_{h}^{\prime} \tag{6}
\end{equation*}
$$

is valid.
$X_{h}^{*}$ is called the $x$-component of the force that acts upon the $h^{\text {th }}$ point, while $T_{h}^{*}$ is its effect. The concept of a total force that acts upon the system will not be introduced here as an autonomous concept, but the quantities $X^{*}$ and $T^{*}$ will be defined by:

$$
\begin{equation*}
X^{*}=\sum_{h=1}^{n} X_{h}^{*}, \tag{7}
\end{equation*}
$$

$$
\begin{equation*}
T^{*}=\sum_{h=1}^{n} T_{h}^{*} \tag{8}
\end{equation*}
$$

$Y^{*}, Z^{*}, Y_{h}^{*}$, etc., might have analogous meanings. $X^{*}$ is called the $x$-component of the total force that acts on the system.
II. Let the equation:

$$
\begin{equation*}
\sum_{h=1}^{n} \frac{d L_{h}}{d t}=T^{*} \tag{9}
\end{equation*}
$$

be fulfilled during the entire motion.
III. Equation (9) remains covariant under the group of Lorentz transformations, i.e., they go to:

$$
\begin{equation*}
\sum_{h=1}^{n} \frac{d L_{h}^{\prime}}{d t_{h}^{\prime}}=T^{* \prime} \tag{10}
\end{equation*}
$$

IV. This is the same as in § $\mathbf{5}$ word-for-word, except that all force components and effect quantities are endowed with an asterisk, and one adds the assumption that the quantities $X_{h}^{*}$ and $T_{h}^{*}$ are completely identical to the ones in I, II, and III.
V. drops out completely.

Equations (9) and (10) can then be written:

$$
\begin{equation*}
\sum_{h=1}^{n} m_{h} \frac{d}{d t} \frac{d t}{d \sigma_{h}}=\sum_{h=1}^{n} T_{h}^{*}, \quad \sum_{h=1}^{n} m_{h} \frac{d}{d t^{\prime}} \frac{d t^{\prime}}{d \sigma_{h}^{\prime}}=\sum_{h=1}^{n} T_{h}^{* \prime} . \tag{11}
\end{equation*}
$$

Due to equation (6), the quantities $T_{h}^{*}, X_{h}^{*}$ are connected with $X_{h}^{* \prime}, T_{h}^{* \prime}$ by relations that are analogous to equations (12) of § 9 :

$$
\begin{equation*}
X_{h}^{*} \frac{d t_{h}^{\prime}}{d t}=\frac{X_{h}^{*}-a T_{h}^{*}}{\sqrt{1-a^{2}}}, \quad T_{h}^{* \prime} \frac{d t_{h}^{\prime}}{d t}=\frac{-a X_{h}^{*}+T_{h}^{*}}{\sqrt{1-a^{2}}} \quad(h=1,2, \ldots, n) . \tag{12}
\end{equation*}
$$

We must apply the transformation that is given by (3) and (12) to equations (11). We easily see that this would be possible only if the right-hand sides of (9) and (10) were already in the form (8). The introduction of the concept of total force by a special definition, as in classical mechanics or also the electromagnetic mechanics that was proposed at the beginning of this section, in which the relation (7) could be proved, proves to be impossible here.

Upon applying the given transformation to equations (11), we conclude that:

$$
\sum_{h=1}^{n} m_{h} \frac{d}{d t}\left(\frac{d t}{d \sigma_{h}}-a \frac{d x_{h}}{d \sigma_{h}}\right)=\sum_{h=1}^{n}\left(T_{h}^{*}-a X_{h}^{*}\right),
$$

and upon subtracting the first equation (11), we will get:

$$
\begin{equation*}
\sum_{h=1}^{n} m_{h} \frac{d}{d t} \frac{d x_{h}}{d \sigma_{h}}=\sum_{h=1}^{n} X_{h}^{*}=X^{*} \tag{13}
\end{equation*}
$$

and analogously:

$$
\begin{equation*}
\sum_{h=1}^{n} m_{h} \frac{d}{d t} \frac{d y_{h}}{d \sigma_{h}}=Y^{*}, \quad \sum_{h=1}^{n} m_{h} \frac{d}{d t} \frac{d z_{h}}{d \sigma_{h}}=Z^{*} \tag{13.a}
\end{equation*}
$$

IV will then imply that:

$$
\begin{equation*}
m_{h} \frac{d}{d t} \frac{d x_{h}}{d \sigma_{h}}=X_{h}^{*}, \quad m_{h} \frac{d}{d t} \frac{d y_{h}}{d \sigma_{h}}=Y_{h}^{*}, \quad m_{h} \frac{d}{d t} \frac{d z_{h}}{d \sigma_{h}}=Z_{h}^{*} \quad(h=1,2, \ldots, n) \tag{14}
\end{equation*}
$$

We can also exhibit the principle of the equality of action and reaction here that we indeed inferred in § 5 (pp. 27) from a relation that is analogous to equation (7), it would be merely a definition of the concept of "total force," since equation (7) is merely a definition.

By contrast, a theorem that is analogous to the center of mass theorem will be proposed here, but generally for only a free system, i.e., a system for which:

$$
X^{*}=Y^{*}=Z^{*}=0 .
$$

It will then follow from equation (13) that:

$$
\frac{d}{d t} \sum_{h=1}^{n} m_{h} \frac{d x_{h}}{d \sigma_{h}}=0, \quad \text { so: } \quad \sum_{h=1}^{n} m_{h} \frac{d x_{h}}{d \sigma_{h}}=\text { const. }
$$

or when written differently:

$$
\sum_{h=1}^{n} m_{h} \frac{\frac{d x_{h}}{d \sigma_{h}}}{\sqrt{1-w_{h}^{2}}}=\text { const. }
$$

or when we infer the concept of transverse mass from equation (17) in $\S 9$ and denote the transverse mass of the $h^{\text {th }}$ point by $\mu_{h}^{*(t)}$ :

$$
\begin{equation*}
\sum_{h=1}^{n} \mu_{h}^{*(t)} \frac{d x_{h}}{d t}=\text { const. } \tag{15}
\end{equation*}
$$

We would not like to pursue electromagnetic mechanics in further detail since here we were only concerned with showing how one constructs it.


[^0]:    ${ }^{1}$ ) Göttinger Nachrichten, mathem.-phys, Kl. (1908).

[^1]:    ( ${ }^{1}$ ) Math. Ann. 66, pp. 354. "Without it (viz., the logical existence of absolute space), mechanics would lose all meaning as a logical science."
    $\left({ }^{2}\right)$ H. Hertz said: "As a peculiarity of a well-defined location in space, we shall appeal to its position relative to a coordinate system that is at rest with respect to the fixed distant stars in the heavens, but otherwise established by an arbitrary convention." (Prinzipien der Mechanik, Book II, § 299). However, that reference system can never be made into a universally-valid theory of mechanics, and it would also include the relationship between the basic laws and absolute space in Hertz's system more harmoniously.
    $\left(^{3}\right)$ Perhaps that emerges most clearly in L. Boltzmann, Prinzipe der Mechanik, Bd. I, § 2, pp. 7.
    $\left({ }^{4}\right)$ Philosophiae naturalis principia mathematica, Londini, 1726. "Axiomata leges motus corollarium V."
    ${ }^{\dagger}$ ) Translator: "Bodies enclosed in a given space have the same motions among themselves, whether that space is at rest or moves uniformly in a straight direction without circular motion."

[^2]:    $\left({ }^{1}\right)$ More precisely: The center of mass of the Sun-Earth system.
    $\left({ }^{2}\right)$ "Concipi autem animo solent huius spatii termini fixi, ad quos corpora referentur. Atque ista relation est id 'quod situs appelatur'." (L. Euler, Mechanica, Petropoli, 1736, § 6.) "But the fixed limits of this space, to which the bodies are referred, are usually conceived in the mind. And that relationship is 'what is called a location'."
    $\left(^{3}\right)$ Loc. cit., Chap. I, Propositio 19, § 77, Theorema.
    $\left({ }^{\dagger}\right)$ Translator: "If the space from which the relative motion is determined is either absolutely at rest or moving uniformly in a direction then the laws that are given in regard to motion and rest will also be relatively valid in regard to the state of the bodies."

[^3]:    ( ${ }^{1}$ ) See H. Poincaré (Livre jubilaire dédiè à H. A. Lorentz, pp. 270, et seq.).

[^4]:    ( ${ }^{1}$ ) J. R. Schütz (Göttinger Nachrichten, 1897) called those two demands collectively the "principle of the absolute conservation of energy," which he states by saying "the energy principle is valid independently of a constantlyadvancing motion of our material world relative to geometric space."

[^5]:    $\left(^{1}\right)$ Enzykl. d. math. Wiss., II, A. 6 ("Kontinuierliche Transformationsgruppen"), § 13.

[^6]:    $\left({ }^{1}\right)$ This term goes back to H. Minkowski (loc. cit., pp. 100).

[^7]:    ( ${ }^{1}$ ) Here, $L_{1}$ and $L_{2}$ have a different meaning from the one that they had in equation (10).

[^8]:    $\left.{ }^{1}\right)$ Which is analogous to the concept of the covariance of Newton's equations that was employed in §5.

[^9]:    $\left({ }^{1}\right)$ Ann. Phys. (Leipzig) 27 (108), pp. 897, et seq., under the title: "Das Relativitätsprinzip der Mechanik und die Gleichungen für die elektromagnetischen Vorgänge in bewegten Körpern."

[^10]:    $\left({ }^{1}\right)$ That assumption is already related to the fact of the aberration of the light from the fixed stars and was necessitated by the Fizeau experiment. That is because it implies that for the speed of light in a medium whose index of refraction is $n$ that moves with the speed $w$, the value of $\frac{1}{n}+w\left(1-\frac{1}{n^{2}}\right)$ will go to the constant value 1 that is true for refraction-free bodies $(n=1)$. The greater that $n$ is, the closer that Hertz's assumption (§6) will be fulfilled.

[^11]:    $\left({ }^{1}\right)$ Minkowski, loc. cit., pp. 72, et seq. I have implemented the same transition from refraction-free bodies to arbitrary ponderable ones with the help of the theory of electrons in a note in Ann. Phys. (Leipzig) 27 (1908), pp. 1059 , et seq., except that empty space entered in place of the refraction-free body, which is appropriate to the theory of electrons.

[^12]:    ( ${ }^{1}$ ) A. Einstein, Ann. Phys. (Leipzig) 17 (1905), pp. 917, et seq., and Jahrb. d. Radioaltivität u. Elektronik 4 (1907), pp. 431, et seq.
    $\left(^{2}\right)$ H. Minkowski, loc. cit., pp. 98, et seq. (Appendix: "Mechanik und Relativitätspostulate"). The foundations themselves are on pp. 107.

