On the methodology of the Dirac equation

By. W. Franz in Munich

Submitted by A. Sommerfeld at the session on 2 November 1935.

Translated by D. H. Delphenich

———

Overview

Part III
Applications

Appendix
Algebra of Clifford numbers

————

Introduction

    Sauter gave the impetus for solving the Dirac equation without representing the Dirac operators and corresponding eigenfunctions by matrices in two papers ([1]). Moreover, the eigenfunctions can be represented as hypercomplex numbers that belong to the number field that is generated by the Dirac operators. Obviously, that method offers a mnemonic advantage in that it makes the introduction of the somewhat-abstruse relationships between the matrices (in particular, their signs) superfluous, and in place of them, it bases all calculations on the simpler rules of calculation for the Dirac operators. However, it seemed to emerge from the subsequent investigations, especially that of **Bechert** ([2]), that the advantage of **Sauter**'s method was already exhausted. Of course, it offers a great advantage in the derivation of general relations from the Dirac equation (such as, say, the treatment of the WKB process by **Bechert**, *ibid.*, § 2) to not write down the γ matrices explicitly, but to employ only the simple relations that exist between them. However, that way of proceeding is by no means characteristic of **Sauter**'s method. [A subsequent treatment of the Dirac equation with no Ansatz for the γ matrices was given by **Temple** ([3]) before Sauter's first paper.] Moreover, what is typical of Sauter's method

   ([1]) **F. Sauter**, Zeit. Phys. **63** (1930), 803 and **64** (1930), 295.
   ([2]) **K. Bechert**, Zeit. Phys. **79** (1932), 26 and Helv. Phys. Acta **6** (1933), 82.
   ([3]) **G. Temple**, "The operational wave equation…," Proc. Roy. Soc. **127** (1933), 349.

consists of the fact that one represents the *solutions* (not just the equation itself) as aggregates of γ's.

The following examination will show that the method not only possesses a mnemonic value, but that it is also suitable as an autonomous starting point for general arguments and special calculations. Indeed, treating the wave function as hypercomplex numbers will offer no essential advantage as long as one restricts oneself to imitating those processes that have proved to be expedient in the matrix treatment of the Dirac equation. However, when one overlooks that close historical connection and gives due consideration to the spirit of hypercomplex numbers in the choice of the process applied, that will imply an advantage in the theory of the equation, as well as its practical application. The most obvious advantage for the theoretical treatment lies in the conceptual simplicity and transparency of the method, which admits, e.g., an especially simple proof of the Lorentz invariance of the equation (cf., § 6). In order to apply Sauter's method to the practical calculation of wave functions to any advantage, one will require a systematic process of solution that does not let the association of the γ's with the coordinates get lost at any point.

The first part of the following study is dedicated entirely to the theory of the Dirac equation, the second part deals with question of practical calculation, and the third part should explain the foregoing with some simple and important examples. In order to let the method of hypercomplex numbers emerge clearly, we will ignore the existence of all theories that start from the matrix representation (treat the connections with them only parenthetically, resp.)

Part I. Theory of the Dirac equation

§ 1. Theory of the Pauli equation

In order to make the later discussions of the Dirac equation more intuitive and easier to understand, the most important points shall be explained by using the simpler example of the Pauli equation for an electron. We shall then briefly introduce the derivation of the Pauli equation, as well as the most important points in its interpretation. Naturally, we must simply copy the Pauli paper ([1]) at many points, while we shall ignore those points in the original paper that are connected with the matrix representation of the Pauli wave functions as having been agreed upon.

a) *Derivation of the equation.* – The starting point is the classical Hamilton function for an electron with a magnetic moment:

(1) $$H = H_0 + H_1 ,$$

in which H_0 is the Hamilton function of an electron without spin (including pertinent relativistic corrections), while:

([1]) **W. Pauli**, Zeit. Phys. **43** (1927), 601.

(2)
$$H_1 = -\frac{e\hbar}{4m^2c^2}(\mathfrak{s}\,[\mathfrak{E}\,\mathfrak{p}]) - \frac{e\hbar}{2mc}\,(\mathfrak{s}\,\mathfrak{H})$$

implements the spectroscopically-known fact that the electron has a magnetic moment with a magnitude of one Bohr magneton. $\mathfrak{s}$ is the unit vector in the direction of the moment.

In the quantization of equation (1), the problem arises of how to replace the unit vector $\mathfrak{s}$ with a suitable vector operator σ. When the commutation relations for moments:

(3)
$$[\mathfrak{M}, \mathfrak{M}] = -\frac{\hbar}{i}\,\mathfrak{M}$$

are applied to the spin moment $M = \hbar/2\,\sigma$, that will yield:

(4)
$$[\sigma, \sigma] = 2i\sigma.$$

If one observes that σ_x, σ_y, σ_z can assume only the values ± 1 – i.e., that:

(5)
$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1,$$

then after a brief calculation, that will imply that:

(6)
$$\sigma_y\sigma_z = -\sigma_z\sigma_y = i\sigma_x,$$

and two corresponding equations that arise from (6) by cyclic permutation of the coordinates. In that way, the three components of the operator σ prove to be quaternions (except for a factor i).

In order for the operator σ to actually define a meaningful replacement for the classical vector $\mathfrak{s}$, one must also be able to ascribe a necessary vector character to it. That means that $(\sigma\,\mathfrak{e})$ – viz., the scalar product of σ with any unit vector – must possess the square $(\sigma\,\mathfrak{e})^2 = 1$, corresponding to equation (5), and that an equation like (6) [e.g., $(\mathfrak{e}_y\,\sigma)$ $(\mathfrak{e}_z\,\sigma) = i\,([\mathfrak{e}_y\,\mathfrak{e}_z]\,\sigma)$] must be true for any two perpendicular unit vectors, moreover. One easily convinces oneself that this is, in fact, the case, and that one can replace (5) and (6) with the following known vector relation:

(7)
$$(\mathfrak{a}\,\sigma)\,(\mathfrak{b}\,\sigma) = (\mathfrak{a}\,\mathfrak{b}) + i\,([\mathfrak{a}\,\mathfrak{b}]\,\sigma),$$

in which $\mathfrak{a}$ and $\mathfrak{b}$ are arbitrary vectors.

b) *The numbers of the field of quaternions* all have the form $A = a_0 + a_1\,\sigma_1 + a_2\,\sigma_2 + a_3\,\sigma_3$, or $a_0 + (\mathfrak{a}\,\sigma)$. The a_i are ordinary complex numbers. The product:

$$\{a_0 + (\mathfrak{a}\ \sigma)\}\{a_0 - (\mathfrak{a}\ \sigma)\} = a_0^2 - \mathfrak{a}^2$$

is a quaternion-free quantity. If it is non-zero then the number A will possess a reciprocal $\left(\text{viz., } A^{-1} = \dfrac{a_0 - (\mathfrak{a}\sigma)}{a_0^2 - \mathfrak{a}^2}\right)$. By contrast, if $a_0^2 - \mathfrak{a}^2 = 0$ then A will be a divisor of zero and not possess a reciprocal. As one easily convinces oneself, the divisors of zero all have the form $(1 + (\mathfrak{e}\ \sigma))$, as a right or left factor. ($\mathfrak{e}$ is a unit vector.)

The property of zero divisors that is most important to us is their capacity to "reduce." If we multiply, say, the general four-parameter quaternion number A on the right by the zero divisor $(1 + \sigma_3)$ then the product $(a_0 + a_1\ \sigma_1 + a_2 \cdot i\sigma_2 + a_3)(1 + \sigma_3)$ will no longer depend upon the two parameters $(a_0 + a_3)$ and $(a_1 + i\ a_2)$. If we also multiply on the right by $(1 + \sigma_3)$ then that will yield the one-parameter quantity $(a_0 + a_3) \cdot (1 + \sigma_3)^2$. Each multiplication by $(1 + \sigma_3)$ has reduced the number of parameters by one-half. The same reduction will come about when one employs any two zero divisors to multiply on the left and right, instead of $(1 + \sigma_3)$.

c) *Solutions to the equation.* – The solutions to the Pauli equation, like its coefficients, must belong to the field of quaternions. However, a system of solutions of the equations that is complete in the field of quaternions is not physical useful, with no further assumptions. The physically-meaningful quantities have the form $\bar{\psi}_n \Pi \psi_m$, in which ψ_m is a solution of the Pauli equation, $\bar{\psi}_m$ is a solution of the adjoint equation, and Π is a quantity that belongs to the field of quaternions. Naturally, in order for a system of solutions to be physically-useful, all of the expressions that are formed in that way (in particular, the quantities that characterize the probability of presence) must have a quaternion-free relationship to each other; i.e., one must have:

$$(8) \qquad\qquad \bar{\psi}_n \Pi \psi_m = P_{nm} \cdot N,$$

in which the P_{nm} contain no quaternions, while N is a quaternionic quantity that is a universal constant. Only the ratios of the P_{nm} enter into the physical interpretation.

The possibility of reduction by zero divisors that was discussed in *b*) will assist one in fulfilling the requirement (8). If we possess, say, a system of solutions that is complete in the total field of quaternions then we multiply the ψ_m on the right and $\bar{\psi}_n$ on the left with any constant zero divisor. The property of the ψ_m and the $\bar{\psi}_n$ that they are solutions of the Pauli equation and its adjoint, resp., will not be perturbed by that. However, the totality of quadratic quantities $\bar{\psi}_n \Pi \psi_m$ will be a one-parameter family, as required by (8). One can see very easily that the physical results are independent of the special choice of the two zero divisors. We will provide the general proof of that later in our treatment of the Dirac equation (cf., § **4**).

Obviously, the Ansatz that ψ is a column matrix is a special case of the given general prescription; hence:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 & \psi_3 \\ \psi_2 & \psi_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a zero divisor, while the matrix $\begin{pmatrix} \psi_1 & \psi_3 \\ \psi_2 & \psi_4 \end{pmatrix}$ is the most general aggregate of the unit matrix and the three quaternion matrices. The greater generality of our prescription above can be employed to our advantage in the construction of the theory, as long as we base it upon the hypercomplex number domain, not just matrices. Namely, whereas when we are working with matrices, the demand that we should use the simplest matrix representation of the σ, as well as the ψ, compels us to ascribe the transformation properties of spinors to the ψ's and to accept the difficulties that are linked with them, while leaving the σ invariant (i.e., represent them by the same simplest matrices in any reference system), the natural treatment of σ as a vector and correspondingly ψ as an invariant on the grounds of **Sauter**'s method also proves to be the most suitable. The basis for that is the fact that in the field of quaternions, any component of σ [= (σ $\mathfrak{e}$)], like any zero divisor [= $(1 + (\sigma \mathfrak{e}))$], exhibits basically the same simple structure.

    d) *Reality properties.* – According to (1), the Pauli equation reads:

$$(9) \qquad\qquad\qquad H\ \psi = -\ \frac{\hbar}{i}\frac{\partial}{\partial t}\psi\ .$$

The adjoint equation to that is:

$$(9a) \qquad\qquad\qquad \bar{\psi}\cdot H^{*} = \frac{\hbar}{i}\frac{\partial}{\partial t}\bar{\psi}\ .$$

(The asterisk shall suggest a change of sign in the imaginary unit, while the differentiations in H^{*} shall act on the left.) It will then follow from this directly that one can get a solution of (9a) from a solution ψ of equation (9) replacing i with $-i$, as well as inverting the sequence of all factors. (The latter prescription will become meaningless when one has written out the solution in a form that contains the non-commuting elements – viz., the quaternions – only linearly.)

    For any ψ, the given prescription will yield the solution $\bar{\psi}$ of the adjoint equation that is associated with the same state as ψ, in such a way that the density will be defined as it must be by that prescription (and such that it is essentially the only prescription that will lead to a well-defined density). We can assume, perhaps, that ψ has the form:

$$(10) \qquad\qquad\qquad \psi = (\psi_0 + \psi_1\ (\mathfrak{e}'\ \sigma))(1 + (\mathfrak{e}\ \sigma)),$$

in which $\mathfrak{e}' \perp \mathfrak{e}$. Any function with a factor of $(1 + (\mathfrak{e}\ \sigma))$ on the right, which will generally have the form $(\varphi_0 + \varphi_1\ (\mathfrak{e}'\ \sigma) + \varphi_2\ (\mathfrak{e}\ \sigma) + \varphi_3\ ([\mathfrak{e}\ \mathfrak{e}']\ \sigma)(1 + (\mathfrak{e}\ \sigma))$, can be put into that form, since:

$$(\mathfrak{e}\ \sigma)\ (1 + (\mathfrak{e}\ \sigma)) = 1 \cdot (1 + (\mathfrak{e}\ \sigma))$$

and

$$([\mathfrak{e}\ \mathfrak{e}']\ \sigma)(1 + (\mathfrak{e}\ \sigma)) = (\mathfrak{e}'\ \sigma)(1 + (\mathfrak{e}\ \sigma)).$$

    The function that is adjoint to (10) reads:

(11) $$\bar{\psi} = (1 + (\mathfrak{e}\,\sigma))\,(\psi_0^* + \psi_1^* \cdot (\mathfrak{e}\,\sigma))\,;$$

it will follow from this that:

$$\bar{\psi}\psi = (\psi_0\,\psi_0^* + \psi_1\,\psi_1^*)\,(1 + (\mathfrak{e}\,\sigma)^2),$$

which is, in fact, definite.

The Hermiticity of the quadratic form $\bar{\psi}_m\,\Pi\,\psi_n$ that belongs to the self-adjoint operator Π [or more precisely, the quaternion-free expression that corresponds to equation (8) that is associated with that operator] is a trivial consequence of our definition of the adjoint, as well as its compatibility with the rules of calculation (7) for the σ. The quantities N that appear in (8) can obviously be assumed to be self-adjoint [e.g., $= (1 + (\mathfrak{e}\,\sigma)^2)$], so it will follow from:

$$[\bar{\psi}_n\,\Pi\,\psi_m]_{\text{adj.}} = \bar{\psi}_m\,\Pi\,\psi_n$$

that:

$$\{P_{nm}\}_{\text{adj.}} = P_{mn}\,.$$

However, "adjoint" and "complex-conjugate" mean the same thing for quaternion-free quantities,, so we can write this as:

$$P_{mn}^* = P_{nm}\,,$$

with which the asserted Hermiticity is proved.

§ 2. The Dirac equation and its adjoint.　Physical interpretation.

The Dirac equation for the electron reads:

(12) $$D\psi \equiv \left\{\sum_{k=1}^{4} \gamma_k\left(\frac{\hbar}{i}\frac{\vec{\partial}}{\partial x_k} - \frac{e}{c}\Phi_k\right) - i\frac{E_0}{c}\right\}\psi = 0.$$

γ_1, γ_2, γ_3, γ_4, are the four Dirac operators, which are coupled by the relations:

(13) $$\gamma_i\,\gamma_k + \gamma_k\,\gamma_i = 2\delta_{ik}\,.$$

The adjoint equation is defined [1] such that it implies a continuity equation in the following way:

(14) $$0 = \bar{\psi}(D\psi) - (\bar{\psi}\bar{D})\psi \equiv \sum_{k=1}^{4}\frac{\partial}{\partial x_k}j_k\,.$$

It follows from this that the adjoint equation to (12) reads:

---

[1]　See **A. Sommerfeld**, *Wellenmechanischer Ergänzungsband*, (1929), pp. 124.

(15)
$$\bar{\psi}\bar{D} \equiv \bar{\psi}\left\{\sum_{k=1}^{4}\gamma_k\left(-\frac{\hbar}{i}\frac{\bar{\partial}}{\partial x_k}-\frac{e}{c}\Phi_k\right)-i\frac{E_0}{c}\right\}=0.$$

That implies:

(16)
$$\sum_{k=1}^{4}\frac{\partial}{\partial x_k}\bar{\psi}\,\gamma_k\psi=0$$

as a continuity equation, or:

(16a)
$$\sum_{k=1}^{4}\frac{\partial}{\partial x_k}\,j_k=0,$$

in which j_k is the four-current:

(17)
$$j_k=\bar{\psi}\,\gamma_k\psi\;.$$

However, it should be remarked that j_k is determined by the continuity equation (16) only up to an additive divergence-less quantity. (See the example of the iterated Dirac equation § **16a**.) The precise form of the current expression must be deduced from other conditions.

Along with the four-current, one can introduce other physical quantities P when one applies the continuity equation to the result of a suitable perturbation calculation. Those quantities are given by the quadratic expression:

(18)
$$\Pi_{nm}=\bar{\psi}_n\,\Pi\,\psi_m\,,$$

in which Π is the operator that corresponds to the quantity P ([1]). Arbitrary (self-adjoint, see below) operators come into question for Π that belong to the number field that is generated by the coefficients of the Dirac equation. $\bar{\psi}_n$ and ψ_m are members of a "complete" system of solutions to the Dirac equation. (The concept of the completeness of the Dirac functions will be discussed in more detail in the next paragraph.)

The Π_{nm} can be endowed with physical meaning only when they have a γ-free relationship to each other; i.e.:

(19)
$$\Pi_{nm}=P_{nm}\,N,$$

in which is free of the γ's, and N is a universal constant quantity that belongs to the field of the Dirac operators.

§ 3. The number field of Dirac operators

The most important algebraic properties of the number field that is generated by the Dirac operators shall now be discussed by way of some examples. The proof be will carried out for the general case of n γ's in the Appendix.

([1]) Due to the definition of the density $\rho=\bar{\psi}\,\gamma_4\psi$, the eigenvalue equation that belongs to the operator Π will read $\gamma_4\,\Pi\,\psi=p\cdot\psi$. The quantity $\bar{\psi}\Pi\psi$ represents the P-density, while P corresponds to the operator $\gamma_4\Pi$, as an individual property of the electron.

The numbers that arise from γ_1, γ_2, γ_3, γ_4 by addition, subtraction, and multiplication all have the form:

$$(20) \qquad \begin{aligned} A &= a_0 + a_1\,\gamma_1 + a_2\,\gamma_2 + a_3\,\gamma_3 + a_4\,\gamma_4 \\ &+ a_{12}\,\gamma_{12} + a_{23}\,\gamma_{23} + \ldots + a_{34}\,\gamma_{34} \\ &+ a_{234}\,\gamma_{234} + a_{134}\,\gamma_{134} + a_{124}\,\gamma_{124} + a_{123}\,\gamma_{123} \\ &+ a_{1234}\,\gamma_{1234} + \ldots \;(\gamma_{12} = \gamma_1 \cdot \gamma_2, \text{ etc.}). \end{aligned}$$

Among those numbers, there are ones that possess a reciprocal – e.g., γ_4 . Numbers without reciprocals are zero divisors. The most important property of zero divisors for us is their ability to reduce. However, the relationships between quaternions are more complicated now, since not all zero divisors possess the same possibility of reduction.

Let us consider, say, the zero divisor $(1 + \gamma_4)$. It has the same form as our previous quaternionic zero divisor and also has the same algebraic properties. Its product with the general 16-component number (20) still includes eight independent parameters:

$$\begin{aligned} A \cdot (1 + \gamma_4) &= ((a_0 + a_4) + (a_1 + a_{14})\,\gamma_1 + (a_2 + a_{24})\,\gamma_2 + (a_3 + a_{34})\,\gamma_3 \\ &+ (a_{12} + a_{124})\,\gamma_{12} + (a_{23} + a_{234})\,\gamma_{23} + (a_{13} + a_{134}\,\gamma_{134})\,\gamma_{13} \\ &+ (a_{123} + a_{1234})\,\gamma_{123}\,)(1 + \gamma_4), \end{aligned}$$

while the two-sided product:

$$\begin{aligned} (1 + \gamma_4) &\cdot A \cdot (1 + \gamma_4) \\ &= ((a_0 + a_4) + (a_{12} + a_{124})\,\gamma_{12} + (a_{23} + a_{234})\,\gamma_{23} + (a_{13} + a_{341})\,\gamma_{13})\,(1 + \gamma_4)^2 \end{aligned}$$

includes only four. Under multiplication, $(1 + \gamma_4)$ will reduce the number of independent parameters of A by one-half. We can characterize that behavior of the number $(1 + \gamma_4)$ by the number $1/2$, which we would like to refer to as the *reduction factor r*.

The product of two commuting numbers with $r = 1/2$ is an even more powerful zero divisor. Let us consider, say, $(1 + \gamma_4)(1 + i\gamma_{12})$.

$$\begin{aligned} A\,(1 + \gamma_4)(1 + i\gamma_{12}) &= ((a_0 + a_4 - i\,a_{12} - i\,a_{124}) + (a_1 + a_{14} - i\,a_2 - i\,a_{24}) \cdot \gamma_1 \\ + (a_3 + a_{34} - i\,a_{34} - i\,a_{123}) \cdot \gamma_3 &+ (a_{13} + a_{134} - i\,a_{23} - i\,a_{234}) \cdot \gamma_{13})\,(1 + \gamma_4) \cdot (1 + i\gamma_{12}) \end{aligned}$$

includes $4 = 16 / 4$ independent parameters, while:

$$(1 + \gamma_4)(1 + i\gamma_{12})\,A\,(1 + \gamma_4)(1 + i\gamma_{12}) = (a_0 + a_4 - i\,a_{12} - i\,a_{124}) \cdot (1 + \gamma_4)^2\,(1 + i\gamma_{12})^2$$

has only $1 = 16 / 4 \cdot 4$. The reduction factor of our number is therefore $r = 1 / 4$. There is no stronger reduction in the field of Dirac operators (except for 0 with $r = 0$).

We shall mention only in passing that there is an especially weak zero divisor with $r = 3 / 4$. [E.g., $(1 + \gamma_4) + (1 + i\,\gamma_{12})$].

The field of Dirac operator possesses different kinds of *subfields*. We shall discuss the ones that have some practical meaning for us briefly.

A *quaternion field* will be generated by hypercomplex numbers γ_{21}, γ_{32}, γ_{13} that will go to the field that was discussed before (§ **1b**) by the substitution $\sigma_k = -\gamma_k\,\gamma_{123}$.

One constructs a *biquaternion field* from the eight basic quantities 1, γ_{12}, γ_{23}, γ_{31}, γ_{14}, γ_{24}, γ_{34}, γ_{1234}. It can be decomposed into two quaternion fields ([1]). One of those two fields includes all numbers with the factor $(1 + \gamma_{1234})$, while the second one includes all numbers with the factor $(1 - \gamma_{1234})$. The product of any number of the first kind with any number of the second kind will be zero. Any of the two quaternion fields can be constructed from one unit $e^{\pm} = \frac{1}{2}(1 \pm \gamma_{1234})$ and three quaternions $\gamma_{21} \cdot e^{\pm}$, $\gamma_{32} \cdot e^{\pm}$, $\gamma_{13} \cdot e^{\pm}$.

The three quantities γ_1, γ_2, γ_3 likewise generate a biquaternion field. It decomposes into two quaternion fields with the units $\frac{1}{2}(1 + i\gamma_{123})$ and $\frac{1}{2}(1 - i\gamma_{123})$.

The simplest subfield of the Dirac field will be generated by a single γ – say, γ_4. It can be decomposed into two fields of the same type as the usual complex numbers, namely, the γ-free multiples of $(1 + \gamma_4)$ and $(1 - \gamma_4)$.

§ 4. Reduction of the system of solutions.
Independence of the physical results of the special reduction.

The solutions of the Dirac equation are hypercomplex numbers from the Dirac number domain. Therefore, a complete system would refer to a system of functions from which one could develop any function from that number domain. However, such a system would not fulfill the physical condition (19), since the Π_{nm} would be a 16-parameter manifold. However, it is easy to reduce the 16-parameter manifold of Π_{nm} to a one-parameter one with the help of the reduction capability of the zero divisors that was discussed in § **3**.

Let ψ'_m be a complete system of solutions to the Dirac equation in the Dirac number fields, and let $\bar{\psi}'_n$ be a complete system of solutions to the adjoint equation. We right-multiply all ψ'_m by a zero divisor Γ_1 whose reduction factor is $r = 1/4$, and left-multiply all $\bar{\psi}'_n$ on the left by Γ_2 with $r = 1/4$. The system $\psi_m = \psi'_m \cdot \Gamma_1$ and $\bar{\psi}_n = \Gamma_2 \cdot \bar{\psi}'_n$ that arises is a system of solutions of the Dirac equation (their adjoints, resp.) that satisfies the requirement (19). Not all functions of the Dirac number field can be developed in the solutions ψ_m, but only the functions with the right factor Γ_1.

In that way, along with the concept of a complete system of solutions in the Dirac number field, we will also arrive at a concept of physical completeness, which we would like to refer to casually as "completeness" in what follows: A system of functions is *complete* when all functions with the right factor Γ_1 can be developed from it.

The physical results will be independent of the special choice of zero divisors along as they only have the reduction factor $r = 1/4$. In order to show that, we prove the:

([1]) Here, we employ the word "decompose" in place of the usual one of "reduction," since we have already used the word "reduce" in a different sense.

Theorem:

A system of functions that is useful from (19) *will experience no essential change when one multiplies all* ψ_m *on the right and all* $\bar{\psi}_n$ *on the left by a common constant factor* C_1 (C_2, resp.).

Proof: The ratios of the P_{nm} to each other will not be changed by such a multiplication; only one $N' = C_2 \, N \, C_1$ will come from N. (The only case that must be excluded is $C_2 \, N \, C_1 = 0$.) We now imagine that a complete system of functions has been generated by Γ_1, Γ_2 . The foregoing theorem says that a further reduction with two arbitrary factors Γ'_1, Γ'_2 will not change the physical results. The same thing will be true when one further reduces a system that was obtained from Γ_1, Γ_2 by way of Γ'_1, Γ'_2. One sees from this that any physical relation that follows from eigenfunctions of one system can also follow from eigenfunctions of the other system. One will then get a one-to-one correspondence between the two with the help of complete systems that are obtained from Γ_1, Γ_2 and Γ'_1, Γ'_2 with no change in the physical results; that implies independence of the special choice of the factors Γ_1 and Γ_2.

Above and beyond that, it follows from the cited theorem that any system of functions that fulfills (19) can be uniquely associated with a subsystem of ours by a complete system that is obtained by reduction (namely, when one reduces with Γ_1, Γ_2). Our reduction process then yields the essentially-unique solution to the following problem: Find a system of the largest-possible number of functions from the Dirac number field that satisfy the condition (19).

The condition that one must have $C_2 \, N \, C_1 \neq 0$ creates no difficulties. If we would like to further reduce, say, a system with norm N by factors Γ_1, Γ_2 then we would choose $C_1 = c_1 \, \Gamma_1$, $C_2 = c_2 \, \Gamma_2$. c_1 and c_2 can always be chosen such that $C_2 \, N \, C_1$ is non-zero.

§ 5. Forming the adjoints. Reality properties.

We write down the Dirac equation (12) and its adjoint (15), while distinguishing between real and imaginary quantities:

$$(21) \qquad \left\{ -ic\left(\frac{\hbar}{i}\overrightarrow{\nabla} - \frac{e}{c}\mathfrak{A}, \gamma \right) + \left(-\frac{\hbar}{i}\frac{\overrightarrow{\partial}}{\partial t} - V \right)\gamma_4 - E_0 \right\} \psi = 0,$$

$$(22) \qquad \bar{\psi}\left\{ -ic\left(-\frac{\hbar}{i}\overleftarrow{\nabla} - \frac{e}{c}\mathfrak{A}, \gamma \right) + \left(\frac{\hbar}{i}\frac{\overleftarrow{\partial}}{\partial t} - V \right)\gamma_4 - E_0 \right\} = 0.$$

We have introduced $x_4 = ict$, $\Phi_4 = (ic / e) V$ and $\Phi_k = \mathfrak{A}_k$ for $k = 1, 2, 3$ into this. ∇ is the "vector" $\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right)$, and γ is the vector $(\gamma_1, \gamma_2, \gamma_3)$.

The form of equations (21) and (22) shows that one will get a solution $\bar{\psi}$ to the adjoint equation (22) from a solution ψ to the Dirac equation (21) when one inverts the sequence of all factors and gives the quantities i, γ_1, γ_2, γ_3 the opposite signs. One must show that a function $\bar{\psi}$ that is produced in that way will also be, in fact, the adjoint function to ψ; i.e., that $\bar{\psi}$ and ψ must be associated with the same state. The proof follows from the fact that the definition that was given for the adjoint function is essentially the only one that will lead to a well-defined density $\bar{\psi} \gamma_4 \psi$. Let ψ and $\bar{\psi}$, say, be any solution of the Dirac equation and its adjoint, resp. As § **4** showed, we can assume, with no loss of generality, that ψ has a right-hand factor of $(1 + i\,\gamma_{12}) (1 + \gamma_4)$ split off, and $\bar{\psi}$ has a right-hand factor of $(1 + i\,\gamma_{12}) (1 + \gamma_4)$ split off. We have chosen the two basically arbitrary factors Γ_1 and Γ_2 in such a way that they are mutually adjoint. ψ and $\bar{\psi}$ will then have the following forms:

$$\psi = (\psi_0 + \psi_1\,\gamma_1 + \psi_3\,\gamma_3 + \psi_{13}\,\gamma_{13})(1 + i\,\gamma_{12}) (1 + \gamma_4),$$
$$\bar{\psi} = (1 + i\,\gamma_{12}) (1 + \gamma_4) (\bar{\psi}_0 - \bar{\psi}_1\,\gamma_1 - \bar{\psi}_3\,\gamma_3 - \bar{\psi}_{13}\,\gamma_{13}).$$

We have changed the signs in γ_1, γ_2, γ_3 and inverted the factors everywhere in the basic quantities γ_1, γ_3, γ_{13}. The complex quantities $\bar{\psi}_0$, $\bar{\psi}_1$, $\bar{\psi}_3$, $\bar{\psi}_{13}$ have still not been related to ψ_0, ψ_1, ψ_3, ψ_{13}. The density will be:

$$\bar{\psi}\,\gamma_4\,\psi = (\bar{\psi}_0\,\psi_0 + \bar{\psi}_1\,\psi_1 + \bar{\psi}_3\,\psi_3 + \bar{\psi}_{13}\,\psi_{13}) (1 + i\,\gamma_{12})^2 (1 + \gamma_4)^2.$$

It follows form this that the density will actually be defined when we set $\bar{\psi}_0 = \psi_0^*$, $\bar{\psi}_1 = \psi_1^*$, $\bar{\psi}_3 = \psi_3^*$, $\bar{\psi}_{13} = \psi_{13}^*$, corresponding to the adjoint definition above. One further sees that this definition is essentially the only one that will lead to a well-defined density. The solution $\bar{\psi}$ that is obtained by that prescription will therefore not only be a solution of the adjoint equation to begin with, but it will be, as such, *the* function that is adjoint to ψ. We can say somewhat more generally ([1]):

One will get the adjoint to a quantity when one inverts the sequence of all factors and gives the quantities i, γ_1, γ_2, γ_3 the opposite sign. For operators, one must add: One must change its direction – i.e., $\overset{\rightarrow}{\nabla}$ must become $\overset{\leftarrow}{\nabla}$, $\dfrac{\overset{\rightarrow}{\partial}}{\partial t}$ must become $\dfrac{\overset{\leftarrow}{\partial}}{\partial t}$, and conversely.

The proof of the Hermiticity of the quadratic expressions that belong to the self-adjoint operators is almost a word-for-word transcription of the proof for the Pauli equation (§ **1**d).

([1])   See **F. Sauter**, "Atomarer Photoeffekt…," Ann. Phys. (Leipzig) **235** (1931), pp. 9.

§ 6. Transformation properties

We must remark that the question of the transformation properties of the ψ's is initially a completely meaningless way of speaking. Only the demand that the physical quantities, namely, the ratios of the P_{nm} [formula (19)], should transform as tensors according to their physical interpretation has any meaning. All further statements about the ψ's are pure conventions, since they represent only mathematical tools.

The question of the transformation properties of a solution is all the more pointless since it is not even possible to associate that solution with a well-defined coordinate system in a meaningful way. Suppose that the Dirac equation reads:

$$(23) \qquad \left\{ \sum_{k=1}^{4} \left(\gamma_k^{(1)}, p_k^{(1)} - \frac{e}{c} \Phi_k \right) - i \frac{E_0}{c} \right\} \psi = 0$$

in a coordinate system (1). We introduce new rectangular coordinates into (23):

$$(24) \qquad x_i^{(2)} = \sum_k a_{ik} x_k^{(1)} \qquad \text{with} \qquad \sum_i a_{ik} a_{il} = \delta_{kl} \ ;$$

the converse reads:

$$(24a) \qquad x_i^{(1)} = \sum_k a_{ik} x_k^{(2)} \ .$$

If we employ $\gamma_i^{(2)} = \sum_k a_{ik} \gamma_k^{(1)}$ as an abbreviation, with $\gamma_i^{(1)} = \sum_k a_{ik} \gamma_k^{(2)}$ as its inverse, then (23) will imply:

$$(23a) \qquad \left\{ \sum_{k=1}^{4} \left(\gamma_k^{(2)}, p_k^{(2)} - \frac{e}{c} \Phi_k^{(2)} \right) - i \frac{E_0}{c} \right\} \psi = 0.$$

However, as one can convince oneself (cf., Appendix f), the $\gamma_i^{(2)}$ satisfy the relations of the four Dirac operators, and (23a) is the Dirac equation in system (2). The function ψ will then satisfy the Dirac equation in system (2) in precisely the same way that it satisfies the Dirac equation in system (1). The $\gamma_i^{(1)}$ are no better or worse than the $\gamma_i^{(2)}$ as basic quantities, and the relations between them are completely symmetric, moreover. It would therefore be completely unmotivated to say: ψ is the solution to the Dirac equation in coordinate system (1). Moreover, if the ψ are to possess any meaning then that statement must be independent of the coordinate system.

It is likewise meaningless to say that (23) is the Dirac equation in system (1), but not in system (2); indeed, (23) and (23a) are identical. Any Dirac equation "in any system" will differ from (23) by at most a change of symbols – i.e., not at all.

In this way, **Sauter**'s method differs *quite essentially* from the treatment of matrices. If one represents the γ's and the ψ's by matrices (column matrix, resp.) then there probably will be some sense to speaking of a solution $\psi^{(1)}$ in system (1), as opposed to a solution $\psi^{(2)}$ in system (2). However, differences of that kind are always statements about the representation of the γ's by matrices. If we would like to solve, e.g.,

(23), on the basis of the coordinate system (1) then we would first represent $\gamma_1^{(1)}$, $\gamma_2^{(1)}$, $\gamma_3^{(1)}$, $\gamma_4^{(1)}$ by four matrices that are as simple as possible, and assume that $\psi^{(1)}$ is a column matrix (likewise on the grounds of simplicity). If we would now like to express the $x_k^{(1)}$ in terms of the $x_k^{(2)}$ by introduction of those matrices then we would obtain (23a), but the coefficients of $p_k^{(2)}$ – namely, the $\gamma_k^{(2)}$ – would no longer be matrices of the simplest form. If we were to attempt to correct this by subjecting the matrices to a suitable automorphism, in addition, then $\psi^{(1)}$ would lose its simple form as a column matrix. With the original matrix representation of the γs and the ψ in (23), we would then have, in fact, established the coordinate system (1), since the four Dirac equations would assume their simplest form in that system. However, all forms of the Dirac equations, such as (23) and (23a), are just as simple in regard to the Dirac number field.

The necessity of reducing the solution raises no difficulties, since the prescription: "all ψ's must have a common right-hand factor of reduction factor 1/4 split off" is invariant under Lorentz transformations of the γs.

There is essentially *only one* Dirac equation, and there is *only one* solution ψ that characterizes a certain state, up to an inessential constant factor. The latter statement follows from the independence of the results on the special choice of reduction that was proved in § **4**. The equation, like its solutions, has a meaning that is independent of the coordinate system; a solution can be characterized by the state that it belongs to in any coordinate system. – The crucial issue for the physical interpretation is the continuity equation and the definition of the current that is required by the Dirac equation uniquely. One will then be compelled to associate the current in the $x_k^{(1)}$-direction with the quantities $j_k^{(1)} = \bar{\psi}\,\gamma_k^{(1)}\,\psi$, and the current in the $x_k^{(2)}$-direction with the quantities $j_k^{(2)} = \bar{\psi}\,\gamma_k^{(2)}\,\psi \equiv \sum_k a_{ik}\bar{\psi}\,\gamma_k^{(1)}\,\psi$. That is, the Dirac equation compels us to ascribe the transformation properties of a four-vector to the four-current, as would also be physically necessary. In the same way, the Dirac equation also compels us to ascribe not only the physical meaning to other quantities, but also the transformation properties that the meaning implies. Those transformation properties will be recovered correctly when we treat ψ as an invariant and $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ as a four-vector.

It might transpire that we could establish some part of the transformation properties of the Dirac equation without having to speak of spinors. However, it just so happens that the introduction of spinors is completely unmotivated as long as one does not speak of matrices. When one works with matrices, one must choose between two evils: Either allow arbitrarily complicated matrix representations for the γs or ascribe the transformation properties of spinors to the γs. As a rule, one prefers the latter as the lesser evil. Since the question of a matrix representation does not exist at all in the application of **Sauter**'s method, one would have no reason to introduce spinors. Furthermore, it seems to me that the transformation properties of the currents would also be best examined by basing them on matrices without making use of the fact that the ψ's transform like spinors, since spinors are only unnecessary gimmicks that obscure the simple state of affairs in that context.

§ 7. Normalization condition. Completeness relation.

The charge density in the state n must be associated with the quantity $\bar{\psi}_n\,\gamma_4\,\psi_n$. For the true, real charge density ρ_n, one will have:

(25) $$\bar{\psi}_n \gamma_4 \psi_n = \rho_n \cdot N,$$

corresponding to (19), in which N is the common "norm" of all eigenfunctions. In order for all states n to be normalized correctly, one must have $\int \rho_n \, d\tau = 1$; i.e.:

(26) $$\int \bar{\psi}_n \gamma_4 \psi_n \, d\tau = N.$$

This norm is invariant under Lorentz transformations. The fact that the four-current $\bar{\psi}_n \gamma_4 \psi_n$ is divergence-free and the boundary condition at spatial infinity guarantees the invariance of $\int \bar{\psi}_n \gamma_4 \psi_n \, d\tau$. (26), together with the orthogonality condition:

(27) $$\int \bar{\psi}_n \gamma_4 \psi_m \, d\tau = 0 \qquad \text{for } n \neq m,$$

will imply the possibility of calculating the development coefficients of arbitrary functions in terms of the ψ_n. The completeness relation reads:

(28) $$\sum_n \int f \, \bar{\psi}_n \, d\tau \cdot \int \psi_n \, g \, d\tau = N \cdot \int f \, \gamma_4 \, g \, d\tau,$$

with arbitrary ([1]) functions f and g. (28) is true for continuous eigenfunctions, as well as discrete ones, as long as one understands $\sum_n$ to mean an integration over the continuous parameter in the former case.

Part II. Practical calculation issues

§ 8. Rules of calculation

The direct use of the relations (13) is appropriate for problems that are treated in tensor form while maintaining the four-dimensional symmetry. For practical calculations, the fourth coordinate will be distinguished, as a rule, and especially with the introduction of stationary states. The physically-given quantities will then have the form of scalars and three-vectors. It is correspondingly practical to distinguish the fourth component γ_4 of the γ_α, as well, and combine the first three components (γ_1, γ_2, γ_3) into a three-vector γ. When one further introduces the abbreviation τ for γ_{123}, one will get:

(29)

	Four scalars:	$1, \, \tau, \, \gamma_4, \, \tau\gamma_4,$
	Four three-vectors:	$\gamma, \, \gamma\tau, \, \gamma\gamma_4, \, \gamma\tau\gamma_4$

as the basic quantities of the Dirac number field, in which one has:

([1])   In the sense of the completeness definition in § **4**.

(30) $$\gamma = (\gamma_1, \gamma_2, \gamma_3), \qquad \tau = \gamma_{123}.$$

In order to be able to calculate with those quantities conveniently, the rules of calculation must be put into vectorial form. One will arrive at that in the most logical way when one thinks of the Dirac number field as being generated by the two scalars τ and γ_4 and the vector γ, which are linked by the relations:

(31)
$$(\mathfrak{v}\ \gamma)(\mathfrak{w}\ \gamma) = (\mathfrak{v}\mathfrak{w}) + ([\mathfrak{v}\mathfrak{w}]\ \gamma)\ \tau,$$
$$\tau^2 = -1, \quad \gamma_4^2 = 1,$$
$$\gamma\ \tau = \tau\ \gamma, \qquad \gamma\ \gamma_4 + \gamma_4\ \gamma = 0, \qquad \tau\ \gamma_4 + \gamma_4\ \tau = 0.$$

($\mathfrak{v}$, $\mathfrak{w}$ are arbitrary complex vectors.) [If one multiplies the first equation in (31) on the left by $1 = (-i\ \tau)^2$ and sets $\sigma = -i\ \gamma\ \tau$ then one will get the known vectorial relation for the Dirac spin matrices.]

The relations of ordinary vector analysis can be applied to hypercomplex vectors with no further assumptions, as long as one only carefully avoids the impermissible permutation of factors. We cite, say, the following rules:

(32a) $$([\mathfrak{a}\mathfrak{b}]\ \mathfrak{c}) = (\mathfrak{a}\ [\mathfrak{b}\mathfrak{c}]),$$

(32b) $$([\mathfrak{a}\mathfrak{b}]\ [\mathfrak{c}\ \mathfrak{d}]) = \overline{\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}} - \overline{\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}}$$

(32c) $$[[\mathfrak{a}\mathfrak{b}]\ \mathfrak{c}] = \underline{\mathfrak{a} \cdot \mathfrak{b} \cdot \mathfrak{c}} - \mathfrak{a} \cdot (\mathfrak{b}\ \mathfrak{c}).$$

Non-commutativity makes it necessary to denote scalar multiplication by lines above or below the symbols; however, the application of the relations will raise no difficulties, moreover.

The following rules, which can be derived from (31), are useful:

(33)
$$\begin{cases}
\gamma(\mathfrak{v}\gamma) = \mathfrak{v} + [\mathfrak{v}\gamma]\tau, \\
(\mathfrak{v}\gamma) \cdot \gamma = \mathfrak{v} - [\mathfrak{v}\gamma]\tau, \\
\quad (\gamma\gamma) = 3, \quad [\gamma\gamma] = 2\gamma\tau, \quad (\gamma[\gamma\gamma]) = 6\tau, \\
(\gamma[\gamma\mathfrak{v}]) = 2(\mathfrak{v}\gamma)\tau.
\end{cases}$$

It is convenient to present some formulas with more factors, as required; we cite the following especially simple and important one:

(34) $$([\overline{\gamma\mathfrak{e}}] \cdot \gamma \cdot [\gamma\ \mathfrak{e}]) = -2\mathfrak{e}\ (\gamma\mathfrak{e}).$$

Simplifications can be achieved in the calculation of self-adjoint expressions. We cite:

$$(35) \quad \begin{cases} \gamma(\mathfrak{v}\,\gamma) + \mathrm{adj.} = 2\mathfrak{v}, \\ \gamma(\mathfrak{v}\,\gamma)\,i\,\tau + \mathrm{adj.} = -2[\mathfrak{v}, i\,\tau], \\ i\gamma\,\gamma_4 + \mathrm{adj.} = 0. \end{cases}$$

§ 9. Change of coordinate system. Normalization issues.

When one would like to relate two states that are described by two differently-normalized eigenfunctions, the problem will arise of how to give them a common norm – i.e., of renormalizing at least one of them. How one is to go about doing that can be inferred from §§ **4** and **7**: Let ψ' be a function that has not yet been normalized in the desired way, and let $\bar{\psi}'$ be its adjoint. If one would like to normalize them by $N = \Gamma_1 \cdot \Gamma_2$, in which N, as well as Γ_1 and Γ_2, possesses the reduction factor $r = 1/4$ then one would form $\psi = \psi' \cdot c_2\,\Gamma_2$, $\bar{\psi} = \Gamma_1\,c_1 \cdot \psi'$, in which c_1 and c_2 are constants that prevent the vanishing of the product. One will then arrive at the desired normalizations when one includes a suitable complex normalization factor in c_1. If one lets, say, $N' = \int \bar{\psi}' \gamma_4 \psi' \, d\tau$ then the condition for the correct normalization will read:

$$\Gamma_1\,c_1\,N'\,c_2\,\Gamma_2 = N.$$

One no longer needs to deal with the functions themselves under renormalization then, but only with the norm N' (N, resp.).

One will get an important special case when one changes the coordinate system. Let a function be given – say, $\psi^{(1)}$ – that is normalized by $N^{(1)} = F(\gamma_1^{(1)}, \gamma_2^{(1)}, \gamma_3^{(1)}, \gamma_4^{(1)})$, in which F is a certain (simple) polynomial in the $\gamma_\alpha^{(1)}$ [viz., the components of the four-vector γ_α relative to the axes of a coordinate system (1)]. That function is renormalized by appending a right-hand factor in such a way that the normal will be $N^{(2)} = F(\gamma_1^{(2)}, \gamma_2^{(2)}, \gamma_3^{(2)}, \gamma_4^{(2)})$. $N^{(2)}$ might then emerge from $N^{(1)}$ in such a way that the index 1 is replaced with 2 everywhere. One must the form:

$$(36) \qquad \psi^{(2)} = \psi^{(1)} \cdot S, \qquad \bar{\psi}^{(2)} = S' \cdot \bar{\psi}^{(1)}.$$

Suitable S and S' can be exhibited in two different ways. First, we would like to carry out the general prescription in the beginning of this paragraph

$$(38) \qquad N = (1 + i\,\gamma_{12})(1 + \gamma_4).$$

We renormalize by multiplying $\psi^{(1)}$ on the right and $\psi^{(1)}$ on the left by $N^{(2)}$, and then divide by a suitable real factor $\sqrt{N}$. We then set:

(39)
$$S = S' = \frac{(1+i\gamma_{12}^{(2)})(1+\gamma_4^{(2)})}{\sqrt{N}}.$$

(37) will then imply that:

(37a) $\frac{1}{N}(1+i\gamma_{12}^{(2)})(1+\gamma_4^{(2)})(1+i\gamma_{12}^{(1)})(1+\gamma_4^{(1)})(1+i\gamma_{12}^{(2)})(1+\gamma_4^{(2)}) = (1+i\gamma_{12}^{(2)})(1+\gamma_4^{(2)})$.

The orthogonal transformation $\gamma_i^{(1)} = \sum_k a_{ki}\gamma_i^{(2)}$ exists between the $\gamma_i^{(1)}$ and the $\gamma_i^{(2)}$.

Therefore:

(38a) $$N^{(1)} = 1 + \sum_k a_{k4}\gamma_k^{(2)} + i\sum_{k,l}a_{k1}a_{l2}\gamma_{lk}^{(2)} + i\sum_{k,l,m}a_{k1}a_{l2}a_{m4}\gamma_{klm}^{(2)}.$$

Only terms that include nothing but distinct $\gamma_k^{(2)}$ will be found in the two-fold and three-fold products (cf., orthogonality relations, tensor character of the γ_α products). From (37a), only those terms that are endowed with 1, $\gamma_4^{(2)}$, $i\gamma_{12}^{(2)}$, $i\gamma_{124}^{(2)}$ will make a contribution to $N^{(1)}$; all of those quantities can be replaced with 1. That will yield the following value for N:

(40)
$$N = 1 + a_{44} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{42} & a_{44} \end{vmatrix}.$$

For the case of a purely spatial rotation, ($a_{44} = 1$), one will have:

(40a)
$$N = 2\left(1 + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right).$$

The renormalization will become even simpler when employs:

(41)
$$N = (1 + i\,\gamma_{12})(1 + \gamma_{1234})$$

as the norm, since γ_{1234} is invariant under orthogonal transformations. [Of course, this norm has the drawback that it not self-adjoint. However, that is not a serious disadvantage. If one determines ψ in such a way that it splits off a right-hand factor $(1+i\gamma_{12}) \cdot (1 + \gamma_{1234})$ then a left-hand factor of $(1+i\gamma_{12}) \cdot (1 - \gamma_{1234})$ will split off from $\bar\psi$. If one multiplies all $\bar\psi$ on the left by γ_3 then one will get N as the norm.] Only the factor must be renormalized; we set:

(42)
$$S = S' = \frac{1+i\gamma_{12}^{(2)}}{\sqrt{N}}$$

and get from (37):

$$\frac{1}{N} \cdot (1 + i\gamma_{12}^{(2)})(1 + i\gamma_{12}^{(1)})(1 + i\gamma_{12}^{(2)}) = (1 + i\gamma_{12}^{(2)}),$$

from which, it will follow that:

$$(43) \qquad\qquad N = 1 + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

The extraordinary simplicity of the renormalization is connected essentially with the fact that we have employed the zero divisor properties of N. Without using that, one can fulfill (37) when one chooses S' and S in such a way that $S'S = 1$, $S'\gamma_i^{(1)}S = \gamma_i^{(2)}$ for $i = 1$, 2, 3, 4. One can then resolve the problem of the renormalization by a spinor transformation. However, it is clear that the spinor transformation will generally be much more laborious than the previous conversion (41), (42), (43). A single quantity with especially simple properties like our norm N can naturally be transformed with much less effort than the totality of all numbers in the Dirac field. Generally, one would like to mean that the spinor transformation performs an especially good service when one would like to replace all $\gamma_i^{(1)}$ in ψ with $\gamma_i^{(2)}$. However, that advantage is only apparent. Namely, it is impractical to get back to the $\gamma_i^{(1)}$ with the help of the spinor transformation and to the $x_i^{(1)}$ by the direct introduction of the Lorentz transformation from the $\gamma_i^{(2)}$ and $x_i^{(2)}$. In fact, the γ_1 and x_1 appear in ψ in an organically-coupled manner, so it would be reasonable to subject both of them to the Lorentz transformation directly at the same time.

This is most obvious in the case of the free electron or the *s*-state of the electron in a central field (cf., § **14**, end). Nevertheless, in many practical cases, the use of the spinor transformation for renormalization can be appropriate. The simple cases of practical importance, such as simple rotations, also correspond to very simple transformation operators *S*, and in addition one can employ the spinor transformation advantageously to replace the $\gamma_i^{(1)}$ with the $\gamma_i^{(2)}$, if not to the entire eigenfunction then at least for the purpose of reducing the zero divisors Γ that were introduced; indeed, one has $\Gamma^{(1)} \cdot S = S \cdot \Gamma^{(2)}$.

It must be stressed that the spinor transformation that was introduced for renormalization differs essentially from the usual spinor transformation of the eigenfunctions. Namely, ψ is usually multiplied by the transformation-operator on the left – i.e., on the inner, open, sensitive side. In that way, the properties of the function will be changed quite essentially; they will no longer satisfy the same equation as before. Things are different for us: We multiply ψ by the transformation-operator on the right, hence, on the outer, insensitive side, which is still armored by the factor Γ. ψ will remain a solution of the same equation as before, and its properties will not be altered in any way.

§ 10. Relations between the physical quantities.

For the applications of the Dirac equation, it is probably always useful to avoid calculating with the explicit eigenfunctions (to the extent that there are any) and to employ relations between the physical quantities that follow immediately from the Dirac equation instead of those eigenfunctions. The ideal goal would be to express all quantities that appear free of the γ_α in terms of the norm $\int \bar{\psi}\, \gamma_4\, \psi\, d\tau$. Since that ideal goal cannot always be attained, one can mostly implement essential simplifications with the help of the Dirac equation. One can obtain relations that one can use for that purpose from the Dirac equation:

(44)
$$\left\{ -c\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\mathfrak{A}, i\gamma \right) + \left(-\frac{\hbar}{i}\frac{\vec{\partial}}{\partial t} - V \right)\gamma_4 - E_0 \right\}\psi_n = 0$$

and its adjoint:

(44a)
$$\bar{\psi}_m\left\{ -c\left(\frac{\hbar}{i}\overleftarrow{\nabla} - \frac{e}{c}\mathfrak{A}, i\gamma \right) + \left(\frac{\hbar}{i}\frac{\overleftarrow{\partial}}{\partial t} - V \right)\gamma_4 - E_0 \right\}\psi_n = 0$$

when one multiplies (44) by $(\bar{\psi}_{m*}$ on the left and (44a) on the right by $(_*\psi_n)$ and adds or subtracts the two equations that then arise. In place of the star *, one can use the quantities 1, γ, τ, $\gamma\tau$, γ_4, $\gamma\gamma_4$, $\tau\gamma_4$, $\gamma\tau\gamma_4$, in sequence. We will get 16 relations (8 scalar and 8 vectorial) in that way that we would like to summarize briefly. The quantities that appear all contain one right-hand factor ψ_n and one left-hand factor $\bar{\psi}_m$; let those factors be omitted for the sake of brevity. In addition, we introduce the following abbreviations:

$$\mathfrak{p} = \frac{1}{2}\frac{\hbar}{i}\left(\vec{\nabla} - \overleftarrow{\nabla} \right),$$

$$\Delta\mathfrak{p} = \frac{\hbar}{i}\left(\vec{\nabla} + \overleftarrow{\nabla} \right),$$

$$E = -\frac{1}{2}\frac{\hbar}{i}\left(\frac{\vec{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial t} \right),$$

$$\Delta E = -\frac{1}{2}\frac{\hbar}{i}\left(\frac{\vec{\partial}}{\partial t} + \frac{\overleftarrow{\partial}}{\partial t} \right).$$

The 16 relations then read:

(a) $c\,(\Delta\mathfrak{p}, i\gamma) - (\Delta E, \gamma_4) = 0$ (continuity equation),

(b) $\qquad c\,(\mathfrak{p} - \dfrac{e}{c}\mathfrak{A})\,\gamma_4 + [\Delta\mathfrak{p},\,\gamma]\,\tau\,\gamma_4 - (E - V)\,i\,\gamma = 0,$

(45)   (c) $\qquad c\,(\mathfrak{p} - \dfrac{e}{c}\mathfrak{A}) - \dfrac{i}{2}\,\Delta E\,\gamma_4\,\gamma - E_0\,i\,\gamma = 0,$

(d) $\qquad \dfrac{c}{2}\,(\Delta\mathfrak{p},\,i\,\gamma)\,\gamma_4 + (E - V) - E_0\,\gamma_4 = 0,$

(e) $\qquad -\dfrac{i}{2}\,c\,\Delta\mathfrak{p}\,\tau + c\,[\mathfrak{p} - \dfrac{e}{c}\mathfrak{A},\,i\,\gamma] + \Delta E\,\gamma\,\tau\,\gamma_4 = 0,$

(f) $\qquad -\,c\,(\mathfrak{p} - \dfrac{e}{c}\mathfrak{A},\,i\,\gamma) + (E - V)\,\gamma_4 - E_0 = 0,$

(g) $\qquad +\tfrac{1}{2}\,(\Delta\mathfrak{p},\,i\,\gamma)\,\tau\,\gamma_4 + \tfrac{1}{2}\cdot\Delta E\cdot\tau + E_0\,\tau\,\gamma_4 = 0,$

(i) $\qquad ic\,(\mathfrak{p} - \dfrac{e}{c}\mathfrak{A})\,\tau\,\gamma_4 - \dfrac{c}{2}\,[\Delta\mathfrak{p},\,i\,\gamma]\,\gamma_4 + \dfrac{1}{2}\,\Delta E\,\gamma\,\tau = 0,$

(k) $\qquad -\,i\,\dfrac{c}{2}\,\Delta\mathfrak{p} - c\,[\mathfrak{p} - \dfrac{e}{c}\mathfrak{A},\,i\,\gamma]\,\tau + (E - V)\,\gamma\,\gamma_4 = 0,$

(45)   (l) $\qquad [\mathfrak{p} - \dfrac{e}{c}\mathfrak{A},\,i\,\gamma]\,\tau\,\gamma_4 + \dfrac{i}{2}\,\Delta\mathfrak{p}\,\gamma_4 + \dfrac{1}{2}\,\Delta E\,\gamma - E_0\,\gamma\,\gamma_4 = 0,$

(m) $\qquad -\,c\,(\mathfrak{p} - \dfrac{e}{c}\mathfrak{A},\,\gamma_4)\,\gamma + \dfrac{i}{2}\,\Delta E = 0,$

(n) $\qquad c\,[\mathfrak{p} - \dfrac{e}{c}\mathfrak{A},\,i\,\gamma]\,\gamma_4 - \dfrac{i}{2}\,\Delta\mathfrak{p}\,\tau\,\gamma_4 + (E - V)\,i\,\gamma\,\tau - E_0\,i\,\gamma\,\tau\,\gamma_4 = 0,$

(o) $\qquad -\,c\,(\mathfrak{p} - \dfrac{e}{c}\mathfrak{A},\,i\gamma_4)\,\tau - \dfrac{\Delta E}{2}\,\tau\,\gamma_4 - E_0\,\tau = 0,$

(p) $\qquad c\,(\mathfrak{p} - \dfrac{e}{c}\mathfrak{A},\,i\gamma)\,\tau\,\gamma_4 + (E - V)\,\tau = 0,$

(q) $\qquad c\,(\mathfrak{p} - \dfrac{e}{c}\mathfrak{A})\,\tau - \dfrac{c}{2}\,[\Delta\mathfrak{p},\,\tau] + (E - V)\,i\,\gamma\,\tau\,\gamma_4 - i\,\gamma\,\tau\,E_0 = 0.$

§ 11. Relations between mean values in a stationary state.

If one integrates equations (45) over space then one will get relations between mean values. In that way, a simplification of (45) will come about, since the integral of $\Delta \mathfrak{p}$ will vanish. We would like to summarize the formulas that one gets when ψ_n and $\bar{\psi}_m$ belong to the same stationary state, and thus $m = n$; ΔE will also vanish then. That will yield (we do not indicate the spatial mean explicitly):

$$(b) \quad c\,(\mathfrak{p}-\frac{e}{c}\mathfrak{A})\,\gamma_4 - (E-V)\,i\,\gamma = 0,$$

$$(c) \quad c\,(\mathfrak{p}-\frac{e}{c}\mathfrak{A}) - E_0\,i\,\gamma = 0,$$

$$(d) \quad (E-V) - E_0\,\gamma_4 = 0, \qquad\qquad 1,\, i\,\gamma,\, \gamma_4$$

$$(e) \quad c\,[\mathfrak{p}-\frac{e}{c}\mathfrak{A},i\gamma] = 0,$$

$$(f) \quad -c\,(\mathfrak{p}-\frac{e}{c}\mathfrak{A},i\gamma) + (E-V)\gamma_4 - E_0 = 0,$$

$$(46)$$

$$(g) \quad \tau\,\gamma_4 = 0,$$

$$(h) \quad (E-V)\,\tau\,\gamma_4 = 0, \qquad\qquad \tau\,\gamma_4$$

$$(i) \quad (\mathfrak{p}-\frac{e}{c}\mathfrak{A})\cdot\tau\,\gamma_4 = 0,$$

$$(k) \quad -c\,[\mathfrak{p}-\frac{e}{c}\mathfrak{A},i\,\gamma\,\tau_4] + (E-V)\gamma\,\gamma_4 = 0,$$

$$(l) \quad c\,[\mathfrak{p}-\frac{e}{c}\mathfrak{A},i\,\gamma]\,\gamma\,\tau_4 + E_0\gamma\,\gamma_4 = 0,$$

$$(m) \quad (\mathfrak{p}-\frac{e}{c}\mathfrak{A},\gamma\,\gamma_4) = 0,$$

$$(n) \quad c\,[\mathfrak{p}-\frac{e}{c}\mathfrak{A},\gamma\,\gamma_4] + (E-V)\,i\,\gamma\,\tau - E_0\,i\,\gamma\,\tau\,\gamma_4 = 0, \qquad i\,\gamma\,\tau,\, \gamma\,\gamma_4,\, i\,\gamma\,\tau\,\gamma_4,\, \tau$$

$$(o) \quad c\,(\mathfrak{p}-\frac{e}{c}\mathfrak{A},i\,\gamma\,\tau) - E_0\tau = 0,$$

$$(p) \quad c\,(\mathfrak{p}-\frac{e}{c}\mathfrak{A},i\,\gamma)\,\tau\,\gamma_4 + (E-V)\,\tau = 0,$$

$$(q) \quad c\,(\mathfrak{p}-\frac{e}{c}\mathfrak{A})\,\tau + (E-V)\,i\,\gamma\,\tau\,\gamma_4 - i\,\gamma\,\tau E_0 = 0.$$

The equations split into three groups, and the first group of quantities is associated with $1,\ i\gamma,\ \gamma_4$. The second group contains only $\tau\,\gamma_4$, for which the mean value will vanish in any stationary state. The remaining equations yield relations between quantities that

are endowed with two-fold and three-fold products of the γ_α . No relations can be derived between quantities that belong to distinct groups of those three.

Equations (46) are valid for only those solutions of the wave equation whose squares are integrable over all space. However, they are also valid in the case of plane waves without spatial integration, since $\Delta\mathfrak{p} = 0$ at any location in space for plane wave. Plane waves can occur when $\mathfrak{A} = 0$ and the potential V is constant [when its changes can be neglected, resp., and thus, in particular, for free electrons or for the electrons that are scattered by the nucleus or released photo-electrically and at a large distance from the nucleus (pure spherical waves)]. The relations (46) can then be replaced completely by the following ones, which we would now like to write out completely [$\boldsymbol{\beta} = \dfrac{c\,\mathfrak{p}}{E-V}$, $(E - V)^2 = (c\,\mathfrak{p})^2\, E_0^2$]:

$$(b) \qquad \bar\psi\, i\,\gamma\,\psi \;=\; \boldsymbol{\beta}\,\bar\psi\,\gamma_4\,\psi\,,$$

$$(d) \qquad \bar\psi\,\psi = \sqrt{1-\beta^2}\cdot\bar\psi\,\gamma_4\,\psi\,,$$

$$(g) \qquad \bar\psi\,\tau\,\gamma_4\,\psi = 0,$$

(47)

$$(k) \qquad \bar\psi\,\gamma\,\gamma_4\,\psi \;=\; [\boldsymbol{\beta},\bar\psi\, i\gamma\tau\,\psi]\,,$$

$$(o) \qquad \bar\psi\,\tau\,\psi \;=\; -\,(\boldsymbol{\beta},\bar\psi\, i\gamma\tau\,\psi)\,,$$

$$(n) \qquad \bar\psi\, i\,\gamma\,\tau\,\gamma_4\,\psi = \sqrt{1-\beta^2}\cdot\bar\psi\, i\gamma\tau\,\psi - \frac{\boldsymbol{\beta}}{\sqrt{1-\beta^2}}(\boldsymbol{\beta},\bar\psi\, i\gamma\tau\,\psi)\,.$$

(b) and (d) express the current, $\bar\psi\,\psi$ expresses the density, and (k), (o), and (n) relate expressions with two-fold and three-fold products of the γ_α to $\bar\psi\, i\,\gamma\,\tau\,\psi$.

Equations (b) and (d) clarify a difficulty that **Sauter** ([1]) mentioned. Namely, **Sauter** found that, from Dirac (46c), one has S_r (= $ec\bar\psi\, i\,\gamma\,\psi$) = $\dfrac{e\,v}{\sqrt{1-\beta^2}}\bar\psi\psi$ for the photo current. He concluded from this that one has $S_r = e\,\mathfrak{v}\cdot\rho$ [that is, $S_r = e\mathfrak{v}\bar\psi\,\gamma_4\,\psi$], as opposed to the non-relativistic case. Now, (47) is, however, applicable to the spherical waves of the photo current (since $V = 0$ at great distances from the nucleus, and in addition terms $\sim 1 / r^3$ can be neglected, which practically makes a plane wave out of the spherical wave). (47d) shows that the equations $S_r = \dfrac{e\,v}{\sqrt{1-\beta^2}}\bar\psi\psi$ and $S_r = e\mathfrak{v}\cdot\bar\psi\,\gamma_4\,\psi$ are not mutually exclusive, but are identical.

([1]) **F. Sauter**, "Atomarer Photoeffekt…" Ann. Phys. (Leipzig) **9** (1931), pp. 246.

§ 12. Spin averaging.

For $\mathfrak{A} = 0$, the Dirac equation of a stationary state of energy E will reads:

(48)
$$\left\{ -(\gamma \nabla) + \frac{E-V}{\hbar c} \gamma_4 - \frac{E_0}{\hbar c} \right\} \psi = 0.$$

This equation contains only real quantities, in addition to the four γ_α; it is free of the imaginary unit i. It then follows that for any solution ψ, the quantities that are endowed with i and the ones that are not endowed with i will satisfy the equation by themselves.

However, it would be false to believe that one could obtain a physically-useful solution without the assistance of the imaginary unit. The solution must indeed include a right zero divisor with $r = 1/4$. However, there are no numbers in the real Dirac number field with $r = 1/4$. Hence, for every solution, there will be a "complex conjugate" solution that is linearly independent of it and emerges when one changes the sign of the imaginary unit in it. One has the following theorem:

Two complex-conjugate solutions:

1. *are orthogonal,*
2. *have everywhere-equal densities, and*
3. *have everywhere-opposite currents and opposite magnetic moments.*

In order to prove 1., one sets, with no loss of generality: $\psi = \varphi \cdot (1 + \gamma_4)(1 + i\,\gamma_{12})$, in which φ is free of γ_4 and i. One will then have $\bar{\psi}^* = (1 - i\,\gamma_{12})(1 + \gamma_4)\,\bar{\varphi}$ (the asterisk shall suggest the change of sign in the imaginary unit), and the "transition density":

$$\bar{\psi}^* \gamma_4 \psi = (1 - i\,\gamma_{12})(1 + \gamma_4)\,\bar{\varphi}\,\gamma_4\,\varphi\,(1 + \gamma_4)(1 + i\,\gamma_{12}) = 0,$$

hence, above all, $\int \bar{\psi}^* \gamma_4 \psi \, d\tau = 0$ (viz., orthogonality). [In order to see that the transition density vanishes, one must only observe that $\bar{\varphi}\,\gamma_4\,\varphi$ is self-adjoint, and can therefore contain only terms with γ_4 and $\gamma\,\gamma_4$ (indeed, they will be terms endowed with precisely one factor of γ_4, since φ and $\bar{\varphi}$ are free of γ_4), and that $(1 - i\,\gamma_{12})(1 + i\,\gamma_{12}) = 0$ and $(1 + \gamma_4)(1 + \gamma_4) = 0$.] One can prove the assertions 2. and 3. likewise. (On the grounds of normalizing ψ^*, one must only append a right-hand factor of $i\,\gamma_{12}$, which will also be done in what follows.) However, one can understand these theorems directly with the help of the reality properties of the physical quantities: $\bar{\psi}\,\gamma_4\,\psi$, $\bar{\psi}\,i\gamma\psi$, $\bar{\psi}\,i\gamma\tau\psi$, and $\bar{\psi}^*\,\gamma_4\,\psi^*$, $\bar{\psi}^*\,\gamma\psi^*$, $\bar{\psi}^*\,i\gamma\tau\psi^*$ all have real ratios. One will then have, up to a common normalization factor:

$$\bar{\psi}\,\gamma_4\,\psi = (\bar{\psi}\,\gamma_4\,\psi)^* = \bar{\psi}^*\,\gamma_4\,\psi^*,$$

$$\bar{\psi}\,i\,\gamma\psi = (\bar{\psi}\,i\,\gamma\psi)^* = -\,\bar{\psi}^*\,i\gamma\psi^*,$$

and likewise:

$$\bar{\psi}\, i\,\gamma\,\tau\,\psi = -\,\bar{\psi}^*\, i\,\gamma\,\tau\,\psi^*\,.$$

2. and 3. are proved with that.

We would like to call the degeneracy that is given by the existence of the two orthogonal eigenfunctions ψ and ψ^* briefly *spin degeneracy*, although that is not entirely correct.

In many cases, one starts with physical conditions under which the two states ψ and ψ^* will not differ, such that one can speak of a true degeneracy. The physical states themselves are of no interest then, but only the means over the two states, which we would like to call briefly *spin averages*, and they will be denoted by a double overbar in what follows. We then set:

$$\overline{\overline{\bar{\psi}\cdots\psi}} = \tfrac{1}{2}(\bar{\psi}\cdots\psi + \bar{\psi}^*\cdots\psi^*)\,.$$

The following relations are true for spin averages:

$$\overline{\overline{\bar{\psi}\, i\,\gamma\,\psi}} = \overline{\overline{\bar{\psi}\, i\,\gamma\,\tau\,\psi}} = \overline{\overline{\bar{\psi}\, i\,\gamma\,\tau\gamma_4\,\psi}} = 0,$$

and from (45), that will imply:

$$(c) \qquad \overline{\overline{\bar{\psi}\, c\,\mathfrak{p}\,\psi}} = 0,$$

$$(49) \qquad\qquad (g) \qquad \overline{\overline{\bar{\psi}\,\tau\,\gamma_4\,\psi}} = 0,$$

$$(n) \qquad \overline{\overline{\bar{\psi}[\mathfrak{p},\gamma\,\gamma_4]\psi}} = 0,$$

$$(q) \qquad \overline{\overline{\bar{\psi}\,\mathfrak{p}\,\tau\,\psi}} = 0.$$

(One must observe from this that $\overline{\overline{\bar{\psi}\,\Delta\mathfrak{p}\,i\,\gamma\,\psi}} = 0$ will probably follow from $\overline{\overline{\bar{\psi}\,i\,\gamma\,\psi}} = 0,$ but not $\overline{\overline{\bar{\psi}\,\mathfrak{p}\,i\,\gamma\,\psi}} = 0.$)

The two states that we have assimilated into the *spin average* exhibit a difference of sign in not only their magnetic moments, but also in various other properties, such as current, orbital momentum, etc. The direction of the magnetic moment by itself does not represent as trivial a case of degeneracy as our change of sign in the imaginary unit.

A more immediate kind of spin average than the one that was introduced above can be obtained for plane waves ($V = 0$, free electrons) $\psi = a \cdot e^{i(\mathfrak{p}\tau)/\hbar}$ with constant a. This does not fit into the previous schema, since it is, in fact, complex. a can differ from $\{- i\, c\,(\mathfrak{p}\,\gamma) + E\,\gamma_4 + E_0\}$ only by a right-hand constant factor. There are two independent reduced solutions that possess opposite magnetic moments ($\approx i\,\gamma\,\tau$). If we average over those two states then, from (47), all quantities will vanish, except for:

$$\bar{\psi} i \gamma \psi = \beta \bar{\psi} \gamma_4 \psi, \quad \bar{\psi} \psi = \frac{1}{\sqrt{1-\beta^2}} \bar{\psi} \gamma_4 \psi.$$

Part III. Applications.

§ 13. Transition from the Dirac equation to the Pauli equation.

The transition from the Dirac equation:

(50) $$\left\{ -i c \left(\mathfrak{p} - \frac{e}{c}\mathfrak{A}, \gamma\right) + (E - V)\gamma_4 - E_0 \right\} \psi = 0$$

to the Pauli equation can result from the following splitting of ψ:

(51) $$\psi = (1 + \gamma_4) \psi^+ + (1 - \gamma_4) \psi^-.$$

ψ^+ and ψ^- include only the γ_1, γ_2, γ_3, but not γ_4; (51) implies no loss in generality. We substitute (51) into (50) and get:

$$(1 + \gamma_4) \left\{ -i c \left(\mathfrak{p} - \frac{e}{c}\mathfrak{A}, \gamma\right)\psi^- + (E - V - E_0)\psi^+ \right\}$$

(52)

$$+ (1 - \gamma_4) \left\{ -i c \left(\mathfrak{p} - \frac{e}{c}\mathfrak{A}, \gamma\right)\psi^+ - (E - V + E_0)\psi^- \right\} = 0.$$

If one multiplies (52) on the left by $\frac{1}{2}(1 \pm \gamma_4)$ then the first (second, resp.) line will remain unchanged, while the other one will vanish. Since the curly brackets are free of γ_4, it will follow that they must vanish individually. That will imply the two equations:

(53a) $$-i c \left(\mathfrak{p} - \frac{e}{c}\mathfrak{A}, \gamma\right) \psi^- + (E - V - E_0) \psi^+ = 0,$$

(53b) $$-i c \left(\mathfrak{p} - \frac{e}{c}\mathfrak{A}, \gamma\right) \psi^+ - (E - V - E_0) \psi^- = 0.$$

The equations show that $\psi^+ \gg \psi^-$ in the non-relativistic approximation ($E - E_0 = W \ll E_0$). One then eliminates ψ^- and represents an equation for ψ^+ alone. With $\psi^+ = \varphi$, one will get:

(54) $$\left\{ (W - V)(2E_0 + W - V) - c^2 \left(\mathfrak{p} - \frac{e}{c}\mathfrak{A}, \gamma\right)^2 - \frac{\hbar c}{2E_0 + W - V} \cdot (\mathfrak{E}, \gamma) i c \left(\mathfrak{p} - \frac{e}{c}\mathfrak{A}, \gamma\right) \right\} \varphi = 0.$$

One can calculate the products $\left(\mathfrak{p}-\dfrac{e}{c}\mathfrak{A},\gamma\right)^2$ and $\left(\mathfrak{E},\ \gamma\right)\left(\mathfrak{p}-\dfrac{e}{c}\mathfrak{A},\gamma\right)$ with the help of the first rule of calculation (31). (One must observe that $[\mathfrak{p}\ \mathfrak{A}] + [\mathfrak{A}\ \mathfrak{p}] \neq 0$, etc., in those calculations.) With the abbreviation $\sigma = -\,i\,\gamma\,\tau$, that will yield:

$$
(55) \qquad \left\{ (W-V)(2E_0 + W - V) - c^2\left(\mathfrak{p}-\frac{e}{c}\mathfrak{A}\right)^2 + \hbar c \cdot e\,(\mathfrak{H}\sigma) \right.
$$

$$
\left. + \frac{\hbar c e}{2E_0 + W - V}\left(\left(\sigma[\mathfrak{E},\mathfrak{p}-\frac{e}{c}\mathfrak{A}]\right) - ic\left(\mathfrak{E},\mathfrak{p}-\frac{e}{c}\mathfrak{A}\right)\right) \right\} = 0.
$$

The vector σ, which was initially introduced as an abbreviation, satisfies the relation (7) for the Pauli spin operators. By making some associated omissions, (55) will go to the Pauli equation, corrected by the term $\sim (\mathfrak{E},\mathfrak{p})$.

The fourth coordinate was singled out by the Ansatz (51), which will then destroy the four-dimensional symmetry. Nevertheless, an exact treatment of equation (55) would necessarily yield a Lorentz-invariant physics. However, one would be compelled to work with a very complicated and unattractive equation of continuity, as well as the definition of the current-density. However, the transition from (50) to (55) will obviously make sense only when one makes some non-relativistic omissions that will actually destroy the Lorentz invariance of the physical properties. Nevertheless, the invariance under spatial rotation does remain preserved under the transition to the Pauli equation, not only for the physical properties, but also for the equation itself. Since we transform $(\gamma_1,\ \gamma_2,\ \gamma_3)$ for fixed γ_4 like a spatial polar three-vector, $\sigma = -\,i\,(\gamma_{23},\ \gamma_{31},\ \gamma_{12})$ will transform like an axial vector, just like $\mathfrak{H}$ and $[\mathfrak{E},\mathfrak{p}]$; the operator in equation (55) is therefore an invariant. The consideration of the transformation properties of the Dirac equation (§ **6**) will remain valid for the Pauli equation when they are adapted to three dimensions.

§ 14. Electron in a central field.

An electron in a central field satisfies the Dirac equation:

$$
(56) \qquad \left\{ -(\gamma\nabla) + \frac{E - V(r)}{\hbar c}\gamma_4 - \frac{E_0}{\hbar c} \right\}\psi = 0,
$$

which does not contain the imaginary unit (a special case of § **12**). We would like to exhibit a complete system of equations of that equation in a systematic way. It is best for one to initially look for those solutions of (56) that are free of the imaginary unit, just like the operator in the equation. That will have two advantages: First, it is precisely those solutions that contain the γ_α in a meaningful relationship with the associated coordinates that make the evolution of the solution simple and intuitive. On the other hand, one can arrive at two orthogonal reduced solutions at a single blow by multiplying by two

complex-conjugate reduction factors. (For the time being, the terms "real" and "complex" are related to the explicit appearance of the imaginary unit in the γ_α aggregates here.)

The way to integrate without introducing matrices was mapped out by **Temple** ([1]). (**Temple** generally used his method only to exhibit the radial equation and its solutions.) ([2]). One first determines a complete law of operators that commute with the Hamilton function and with each other. The following two will serve to achieve that goal:

(57) $$\mathfrak{N}_3 \equiv [\mathfrak{r} \, \nabla]_3 + \tfrac{1}{2} \, \gamma_{12} \equiv \frac{\partial}{\partial \varphi} + \tfrac{1}{2} \gamma_{12} \, ,$$

$$P \equiv \{1 - ([\mathfrak{r} \, \nabla] \, \gamma) \, \tau\} \, \gamma_4 \, .$$

φ is the azimuth around the x_3-axis, which will make it the axis of a polar coordinate system r, ϑ, φ. $\mathfrak{N}_3$ is, up to a constant factor, the x_3-component of the orbital momentum + spin momentum, P is a kind of square root of the square of that momentum sum $\mathfrak{N} = [\mathfrak{r} \, \nabla] + \tfrac{1}{2} \gamma \, \tau$. ($P^2 = - \, \mathfrak{N}^2 + \tfrac{1}{4}$).

Since we seek solutions of the Dirac equation that do not include the imaginary unit, we can employ only those operators that are free of the imaginary unit and possess *real* eigenvalues, in addition. That demand is reasonable for P; by contrast, $\mathfrak{N}_3$ possesses pure imaginary eigenvalues. For that reason, we will not employ $\mathfrak{N}_3$ itself, but $\mathfrak{N}_3^2$.

We bring both the Hamilton function and the two operators P and $\mathfrak{N}_3^2$ into diagonal form simultaneously; i.e., we look for those ψ for which the Dirac equation (56) is fulfilled and one has, at the same time:

(58) $$P \cdot \psi = p \, \psi,$$

(59) $$\mathfrak{N}_3^2 \, \psi = - \, m^2 \, \psi,$$

in which p and m^2 are real eigenvalues. We shall now first integrate equations (58) and (59), which are simpler to deal with than (56).

Eigenfunctions of $\mathfrak{N}_3^2$.

When the eigenvalue equation is written out, it will read:

(59a) $$0 = \left(\frac{\partial}{\partial \varphi} + \frac{1}{2} \gamma_{11} \right)^2 + m^2 \equiv \left(\frac{\partial}{\partial \varphi} + \left(m + \frac{1}{2} \right) \gamma_{12} \right) \cdot \left(\frac{\partial}{\partial \varphi} - \left(m - \frac{1}{2} \right) \gamma_{12} \right).$$

([1]) **G. Temple**, Proc. Roy. Soc. London **127** (1930), pp. 349.
([2]) **Sauter** has proved the possibility of obtaining all eigenfunctions in that way (*loc. cit.*, end of paper 2).

The number domain that the coefficients of this equation belong to is isomorphic to the field of complex numbers. $(\gamma_{12})^2 = -1$, so γ_{12} is the "imaginary unit" of the number domain. (59a) possesses the solutions:

(60)
$$\psi = e^{\gamma_{12}\left(m-\frac{1}{2}\right)\varphi} \cdot a_+ + e^{-\gamma_{12}\left(m-\frac{1}{2}\right)\varphi} \cdot a_- ,$$

with half-integer m. $a_\pm$ are integration constants, which might depend upon r and ϑ, but not upon φ. The exponential function is determined by the exponential series, or (what amounts to the same thing) by $e^{\alpha\gamma_{12}} = \cos\alpha + \gamma_{12}\sin\alpha$. (The exponential function can be employed, not only in the number field that is generated by γ_{12}, but also in non-commutative number domains; however, one must observe that the rule $e^a \cdot e^b = e^{a+b}$ is only true when $ab = ba$.)

Common eigensolutions of P.

The coefficients of equation (58) belong to the number domain that is generated by $\gamma_{12}, \gamma_{23}, \gamma_{31}$, and γ_4, which can be decomposed with the help of the units $\frac{1}{2}(1 \pm \gamma_4)$. One obtains two independent equations for two independent types of solutions ψ^+ and ψ^-, which contain the factors $1 + \gamma_4$ and $1 - \gamma_4$, resp. $[\psi^\pm = (1 \pm \gamma_4) \cdot \varphi^\pm$, where $\varphi^\pm$ is free of γ_4.]:

(61)
$$\pm \{1 - ([\mathfrak{r} \nabla] \gamma) \tau\} \varphi^\pm = p \cdot \varphi^\pm.$$

If one moves the term $\pm 1\ \varphi^\pm$ to the right-hand side and then iterates then that will give:

(61a)
$$\{-[\mathfrak{r}\nabla]^2 + ([\mathfrak{r}\nabla]\gamma)\tau\}\ \varphi^\pm = (p\mp1)^2 \cdot \varphi^\pm.$$

When one then substitutes $[\mathfrak{r}\nabla]\gamma\ \tau = (\mp p + 1)$ in (61), one will get:

(61b)
$$\{[\mathfrak{r}\nabla]^2 + (\mp p)(\mp p + 1)\}\ \varphi^\pm = 0.$$

However, (61b) is precisely the equation of the spherical functions of index $\mp p$.

Common eigenfunctions of $\mathfrak{N}_3^2$ and P.

The two functions $\varphi^\pm$ must be eigenfunctions of $\mathfrak{N}^2$, and thus possess the form (60). Hence, one must clearly have:

$$a_+ = P_{\mp p}^{m-1/2}(\cos\vartheta)\, a'_\pm , \qquad a_- = P_{\mp p}^{m+1/2}(\cos\vartheta) \cdot a'_\pm .$$

($a'_\pm$ is independent of ϑ.) Substituting this into (61) will give:

(62) $\{([\mathfrak{r}\,\nabla]\,\gamma)\,\tau + (\pm p - 1)\}\}\left(P_{\mp p}^{m-1/2}\,e^{\gamma_{12}\,(m-1/2)\varphi}\,a'_{+} + P_{\mp p}^{m+1/2}\,e^{-\gamma_{12}\,(m+1/2)\varphi}a'_{-}\right) = 0.$

The differentiations $[\mathfrak{r}\,\nabla]$ can be easily performed on the spherical functions explicitly. The third component $[\mathfrak{r}\,\nabla]_3 = \partial\,/\,\partial\varphi$ can be obtained by differentiating the factors $\pm\gamma_{12}(m\mp\tfrac{1}{2})$. One can apply the formula:

(63) $(\gamma_{12}\,[\mathfrak{r}\,\nabla]_1 - [\mathfrak{r}\,\nabla]_2)\cdot P^{\mu}\cdot e^{\gamma_{12}\,\mu\varphi} = P^{\mu+1}\cdot e^{\gamma_{12}\,(\mu+1)\varphi}$

to the other components. [One can infer this formula from, perhaps, **A. Sommerfeld**, *Wellenmech,*, Erg. Bd. pp. 297, (41a), when one combines the two equations there and replaces i with γ_{12} .] If one multiplies (63) by γ_{12} on the left and then adds it to the equation $\gamma_{12}\,[\mathfrak{r}\,\nabla]_3 = -\mu$ then that will yield:

(64) $([\mathfrak{r}\,\nabla]\,\gamma)\,\tau\cdot P^{\mu}\cdot e^{\gamma_{12}\,\mu\varphi} = -\mu\cdot P^{\mu}\cdot e^{\gamma_{12}\,\mu\varphi} + \gamma_{13}\cdot P^{\mu+1}\cdot e^{\gamma_{12}\,(\mu+1)\varphi}.$

Equation (62) can be split into two terms whose dependency upon φ is due to the factors $e^{-\gamma_{12}\,(m+1/2)\varphi}$ $[e^{\gamma_{12}\,(m-1/2)\varphi}$, resp.]. The two coefficients of these quantities in (62) must vanish individually. We write down the term with $e^{-\gamma_{12}\,(m+1/2)\varphi}$ (the coefficient of $e^{\gamma_{12}\,(m-1/2)\varphi}$ is the same as that of $e^{-\gamma_{12}\,(m+1/2)\varphi}$, up to a finite factor)

(65) $P_{\mp p}^{m+1/2}\cdot e^{-\gamma_{12}\,(m+1/2)\varphi}\left\{\gamma_{13}\cdot a'_{+} + \left(m+\tfrac{1}{2}\pm p - 1\right)\cdot a'_{-}\right\} = 0.$

That is, one then has: $\gamma_{13}\cdot a'_{+} = (\mp p - m + \tfrac{1}{2})a'_{-}$. The two independent common solutions of $\mathfrak{N}_3^2$ and P then yield:

(66) $\psi^{\pm} = \chi^{\pm}\cdot b^{\pm},$

with

$\chi^{\pm} = \left\{P_{\mp p}^{-1/2}e^{\gamma_{12}\,(m-1/2)\varphi}\left(\mp p - m + \tfrac{1}{2}\right) + \gamma_{13}\cdot P_{\mp p}^{m+1/2}e^{\gamma_{12}\,(m+1/2)\varphi}\right\}(1\pm\gamma_4).$

Common eigensolutions of $\mathfrak{N}_3^2$, P, and the Hamilton function.

The desired solution of the Dirac equation is a linear combination of the two solutions (66) with radially-dependent coefficients $b_{\pm}$. One will get the defining equation for those coefficients by substituting in the Dirac equation (56). We first form:

(67) $(\gamma\nabla)\cdot\chi^{\pm}$

$= (1\mp\gamma_4)\left\{\begin{array}{l}\gamma_1\left[(\nabla_1 + \gamma_{12}\nabla_2)P_{\mp p}^{m-1/2}e^{\gamma_{12}\,(m+1/2)\varphi}\left(\mp p - m + \tfrac{1}{2}\right) - \nabla_3 P_{\mp p}^{m+1/2}e^{\gamma_{12}\,(m-1/2)\varphi}\right] \\ +\gamma_3\left[(\nabla_1 - \gamma_{12}\nabla_2)P_{\mp p}^{m+1/2}e^{\gamma_{12}\,(m+1/2)\varphi} - \nabla_3\cdot P_{\mp p}^{m-1/2}e^{\gamma_{12}\,(m-1/2)\varphi}\left(\mp p - m + \tfrac{1}{2}\right)\right]\end{array}\right\}.$

The differentiations with respect to the polar angles can be carried out on the basis of two relations between spherical functions ([1]):

$$(\nabla_1 + \gamma_{12}\nabla_2)P_k^\mu e^{\gamma_{12}\mu\varphi}(k-\mu) - \nabla_3 P_k^{\mu+1}e^{\gamma_{12}(\mu-1)\varphi} = P_k^\mu \cdot e^{\gamma_{12}(\mu+1)\varphi}\left(\frac{\partial}{\partial r} + \frac{1+k}{r}\right),$$

(68)

$$(\nabla_1 - \gamma_{12}\nabla_2)P_k^{\mu+1}e^{\gamma_{12}(\mu+1)\varphi} + \nabla_3 \cdot P_k^\mu e^{\gamma_{12}\mu\varphi}(k-\mu) = P_{-k}^\mu \cdot e^{\gamma_{12}\mu\varphi}(k+\mu)\left(\frac{\partial}{\partial r} + \frac{1+k}{r}\right).$$

Therefore, (67) will give:

(69) $(\gamma\nabla)\chi^\pm$

$$= -(1\mp\gamma_4)\left\{\gamma_1 \cdot P_{\pm p}^{m+1/2}e^{\gamma_{12}(m+1/2)\varphi} - \gamma_3 \cdot P_{\pm p}^{m-1/2}e^{\gamma_{12}(m-1/2)\varphi}\left(\mp p + m - \tfrac{1}{2}\right)\right\}\left(\frac{\partial}{\partial r} + \frac{1\mp p}{r}\right)$$

$$= -\chi^\mp\left(\frac{\partial}{\partial r} + \frac{1\mp p}{r}\right)\gamma_3.$$

We now apply the operator of the Dirac equation (56) to $\psi^\pm$:

(70) $$\left\{-(\gamma\nabla) + \frac{E-V}{\hbar c}\gamma_4 - \frac{E_0}{\hbar c}\right\}\psi^\pm = \chi^\mp\left(\frac{\partial}{\partial r} + \frac{1\mp p}{r}\right)\gamma_3 \cdot b_\pm + \left(\pm\frac{E-V}{\hbar c} - \frac{E_0}{\hbar c}\right)\chi^\pm b_\pm.$$

The b must be determined in such a way that the sum of the two expressions on the right-hand side of (70) will vanish; that will imply the two equations:

$$\left(\frac{\partial}{\partial r} + \frac{1\mp p}{r}\right)\gamma_3 \cdot b_\pm - \left(\pm\frac{E-V}{\hbar c} - \frac{E_0}{\hbar c}\right)b_\mp = 0.$$

If one sets:

(71) $U_+ = -b_+, \qquad U_- = \gamma_3 \cdot b_-$

then that will yield radial equations that are free of the γ_α:

(72) $$\left(\frac{\partial}{\partial r} + \frac{1\mp p}{r}\right)U_\pm + \left(\pm\frac{E-V}{\hbar c} + \frac{E_0}{\hbar c}\right)U_\mp = 0.$$

If we understand the U_+ to mean solutions of (72) that are free of the γ_α then, from (66) and (71), the ultimate expression for the desired eigenfunctions will read:

([1]) See **F. Sauter**, Zeit. Phys. **63** (1930), pp. 807, (10). We shall use the second and fourth of those formulas, in which we have replaced i with $-\gamma_{12}$.

(73)
$$\psi = \left| \begin{array}{c} \left\{ -\gamma_{13} \cdot P_{-p}^{m+1/2} e^{\gamma_{12}(m+1/2)\varphi} + P_{-p}^{m-1/2} e^{\gamma_{12}(m-1/2)\varphi} \left(p + m - \tfrac{1}{2} \right) \right\} U_+ \\ + \left\{ \gamma_{13} \cdot P_{p}^{m+1/2} e^{\gamma_{12}(m+1/2)\varphi} + P_{p}^{m-1/2} e^{-\gamma_{12}(m-1/2)\varphi} \left(p - m + \tfrac{1}{2} \right) \right\} U_- \end{array} \right| (1 + \gamma_4)\, \Gamma.$$

Γ is an arbitrary constant factor.

One can get two orthogonal reduced solutions from (73) when one sets $\Gamma = (1 + i(\gamma\mathfrak{s})\tau)$ for one solution and $\Gamma = (1 - i(\gamma\mathfrak{s})\tau)\, i\,(\gamma\mathfrak{n})$ for the other one, in which $\mathfrak{s}$ and $\mathfrak{n} \perp \mathfrak{s}$ are unit vectors. The derivability of these two solutions from a single one by multiplying by a constant factor is what we referred to in § **12** as spin degeneracy. If one chooses $\mathfrak{s}$ to be the unit vector in the x_3-direction in particular then the resulting ψ will possess the eigenvalues $\mathfrak{N}_3\,\psi = \pm\, im\,\psi$, while in fact (73) generally possesses only the eigenvalue $-m^2$ for $\mathfrak{N}_3^2$.

The γ_α appear in (73) in an entirely logical manner. The appearance of $e^{\gamma_{12}\varphi}$ is then connected with the fact that φ means the rotation $1 \to 2$, while the appearance of γ_3 in isolation goes back to the fact that we have singled out the x_3-axis as the polar axis ([1]).

The coordinate symmetry in (73) is best expressed in the s-states ($p = 1$, $m = \tfrac{1}{2}$):

(74)
$$\psi_s = \left\{ U_+ + U_- \left(\gamma, \frac{\mathfrak{r}}{r} \right) \right\} (1 + \gamma_4).$$

The connection between the parameters m, p and the quantum numbers should be mentioned. If one chooses $\Gamma = (1 - i\,\gamma_1\,\gamma_2)$ then one-half the number m is precisely the magnetic quantum number. We infer the connection between p and the spin quantum number j from $P^2 = -\,\mathfrak{N}^2 + 1/4$, i.e.:

(75)
$$\left(p - \tfrac{1}{2} \right)\left(p + \tfrac{1}{2} \right) = j\,(j + 1), \qquad j = |\,p\,| - \tfrac{1}{2}.$$

§ 15. Remarks on the wave functions of free electrons.

The Dirac equation for the free electron reads:

(76)
$$\left\{ \sum_\alpha \left(\gamma_\alpha, \frac{\partial}{\partial x_\alpha} \right) + \frac{mc}{\hbar} \right\} \psi = 0.$$

One solves it with the Ansatz $\psi = a \cdot e^{\frac{i}{\hbar}\sum_\alpha (p_\alpha x_\alpha)}$. It follows that a must satisfy:

([1]) **F. Sauter** has recently found an especially beautiful solution to the Kepler problem for which no axis is distinguished, and the γ appear only in scalar products. As he cordially communicated to me, it will be published soon in Zeit. Phys.

(77) $$\{ i\sum_{\alpha}(\gamma_{\alpha}, p_{\alpha}) + mc\}a = 0,$$

and that p_{α} must satisfy $\sum p_{\alpha}^2 = (mc)^2$.

We would like to show, briefly, how one can exhibit solutions of (76) in a more direct way. We seek, say, a solution whose time dependency is $e^{-iEt/\hbar}$ and is independent of x_2 and x_3. The Dirac equation for that function reads:

(78) $$\left\{ \gamma_1 \cdot \frac{\partial}{\partial x_1} + \frac{E\gamma_4 + E_0}{\hbar c} \right\} \psi = 0$$

or also:

(78a) $$\frac{\partial \psi}{\partial x_1} = -\gamma_1 \frac{E\gamma_4 + E_0}{\hbar c} \psi,$$

with the solution:

(79) $$\psi = \exp\left(-\gamma_1 \frac{E\gamma_4 + E_0}{\hbar c} \cdot x_1 \right)$$

$$\left(= \cos\left(\frac{\sqrt{E^2 - E_0^2}}{\hbar c} \cdot x_1 \right) - \gamma_1 \frac{E\gamma_4 + E_0}{\sqrt{E^2 - E_0^2}} \sin\left(\frac{\sqrt{E^2 - E_0^2}}{\hbar c} \cdot x_1 \right) \right).$$

One can get two solutions of opposite impulse, spin, and energy from this when one multiplies by two "complex-conjugate" (in the sense of § **12**) reduction factors. If one chooses the factors to be $\left(1 \pm i\gamma_1 \frac{E\gamma_4 + E_0}{\sqrt{E^2 - E_0^2}} \right)$, in particular, then that will give the plane waves:

$$\exp\left(\mp i \frac{\sqrt{E^2 - E_0^2}}{\hbar c} x_1 \right) \left(1 \pm i\gamma_1 \frac{E\gamma_4 + E_0}{\sqrt{E^2 - E_0^2}} \right).$$

Equation (76) can be solved symmetrically in the four coordinates by means of the Ansatz $\psi =$ function of $\sum_{\alpha} \pi_{\alpha} x_{\alpha}$, in which π_{α} is a unit four-vector ($\sum_{\alpha} \pi_{\alpha}^2 = 1$). (76) will then yield:

(80) $$\frac{\partial \psi}{\partial(\sum \pi_{\alpha} x_{\alpha})} = -\frac{mc}{\hbar} \sum (\pi_{\alpha}\gamma_{\alpha}) \cdot \psi,$$

with the solution:

(81) $$\psi = \exp\left[-\frac{mc}{\hbar} \sum_{\alpha}(\pi_{\alpha}\gamma_{\alpha}) \cdot \sum_{\beta}(\pi_{\beta}\gamma_{\beta}) \right]$$

$$\left[= \mathfrak{Cos}\left(\frac{mc}{\hbar} \sum_\alpha (\pi_\alpha x_\alpha) \right) - \sum_\alpha (\pi_\alpha x_\alpha) \, \mathfrak{Sin}\left(\frac{mc}{\hbar} \sum_\alpha (\pi_\alpha x_\alpha) \right) \right].$$

Due to the boundary conditions at spatial infinity, π_1, π_2, π_3 must be pure imaginary, and since $\sum \pi_\alpha^2 = 1$, one must then have $\pi_4^2 > 1$.

Whereas a is a zero divisor, from (77), the solutions (79) and (81) are numbers that possess reciprocals. That is connected with the fact that we have replaced the eigenvalue condition $\dfrac{\hbar}{i} \dfrac{\partial}{\partial x_\alpha} \psi = p_\alpha \, \psi$ for the solution (77) with the weaker condition $-\left(\hbar \dfrac{\partial \psi}{\partial x_\alpha} \right)^2 = p_\alpha^2 \, \psi$. A similar phenomenon was observed in the Kepler problem. The solutions (73) of $r = 1/2$ that we presented in § **14** satisfy the eigenvalue equation $\mathfrak{N}_3^2 \, \psi = -m^2 \, \psi$; in order to satisfy the stronger condition $\mathfrak{N}_3 \, \psi = im \, \psi$, as well, solutions with $r = 1/4$ will be required. The means: The stronger the eigenvalue condition that is imposed upon the Dirac equation, the "stronger" (i.e., the more strongly reduced) that the zero divisor must be.

§ 16. Iterated Dirac equation.

The iterated Dirac equation will arise when one multiplies the two commuting operators:

$$(82) \qquad D^\pm \equiv \left(\sum_{k=1}^4 \gamma_k \left(\frac{\hbar}{i} \frac{\vec{\partial}}{\partial x_k} - \frac{e}{c} \Phi_k \right) \pm i \frac{E_0}{c} \right)$$

with each other; it will then read:

$$(83) \qquad 0 = D\psi \equiv D^+ \cdot D^- \psi.$$

a) *Adjoint equation and current expression.* – The definitions of the current and adjoint for the wave equations $D^\pm \psi = 0$ are obtained from the equation:

$$(84) \qquad \bar{\psi} \cdot D^\pm \psi - \bar{\psi} \cdot \bar{D}^\pm \psi \equiv \mathrm{Div} \, \bar{\psi} \, j^\pm \psi ,$$

which must be true for arbitrary ψ and $\bar{\psi}$.

$j^\pm$ is the operator of the four-current, which is generally defined by (84) only up to an additive divergence-free quantity, as we have remarked before. The continuity equation for $D^+ \cdot D^-$ can be inferred directly from (84):

$$\bar{\psi} \cdot D^+ (D^- \psi) - \bar{\psi} \cdot \bar{D}^+ (D^- \psi) \equiv \mathrm{Div} \, \bar{\psi} \, j^+ (D^- \psi),$$
$$(\bar{\psi} \cdot \bar{D}^+) D^- \psi - (\bar{\psi} \cdot \bar{D}^+) \bar{D}^- \psi \equiv \mathrm{Div} \, (\bar{\psi} \bar{D}^+) j^- \psi ,$$

Adding gives:

(85) $$\bar{\psi}(D^+D^-)\psi - \bar{\psi}(\bar{D}^+\bar{D}^-)\psi \equiv \text{Div } \bar{\psi}(j^+D^- + \bar{D}^+j^-)\psi .$$

Hence:

(86) $$\bar{D} \equiv \overline{D^+D^-} = \bar{D}^+ \cdot \bar{D}^-$$

and

(87) $$j = j^+ D^- + \bar{D}^+ j^- .$$

j is determined completely from j^+ and j^- when one stipulates that $j = j^+$ for $\bar{\psi}\bar{D}^+ = 0$, and $j = j^-$ for $D^-\psi = 0$.

If one substitutes $D^\pm$ from (82) and used $j_k^\pm = \gamma_k$ then that will yield:

(88) $$j_k = \frac{\hbar}{i}\left(\frac{\overrightarrow{\partial}}{\partial x_k} - \frac{\overleftarrow{\partial}}{\partial x_k}\right) - 2\frac{e}{c}\Phi_k + \frac{\hbar}{i}\sum_l\left(\frac{\overrightarrow{\partial}}{\partial x_l} + \frac{\overleftarrow{\partial}}{\partial x_l}\right)\gamma_k\gamma_l .$$

The associated expression for the current has the known Gordon form ([1]):

(89) $$J_k = \bar{\psi}j_k\psi = \frac{\hbar}{i}(\bar{\psi}\,\text{grad}\,\psi - \text{grad}\,\bar{\psi}\cdot\psi) - \frac{2e}{c}\bar{\psi}\Phi_k\psi + \frac{\hbar}{i}\sum_l\frac{\partial}{\partial x_l}(\bar{\psi}\,\gamma_k\gamma_l\,\psi) .$$

The continuity equation would also be satisfied, *per se*, if one dropped the last term – viz., the *polarization current* – since it is divergence-free. However, that term cannot be omitted if one does not wish to lose the agreement with the current definition $J_k = \bar{\psi}j_k\psi$ for the linear Dirac equation. One can infer that this current definition is precisely correct from the perturbation of eigenvalues by an electromagnetic field ([2]).

The expression $\bar{D}$ in (86) does not emerge immediately from the adjoint definition that applies to the linear Dirac equation; namely, by that prescription, the sequence of factors $\bar{D}^+$ and $\bar{D}^-$ must be inverted. However, $\bar{D}^+$ and $\bar{D}^-$, like D^+ and D^- (coincidentally), commute, such that in spite of everything, $\bar{D}$ is adjoint to D according to the old prescription. Therefore, the definition of adjoint that is valid for the linear Dirac equation can also be preserved for the *wave functions* of the iterated equation. That is in no way obvious; rather, it lies in our special form for the Dirac equation. If the equation were written in the asymmetric Dirac form:

$$D'^\pm = \sum_{k=1}^3 \alpha_k\left(p_k - \frac{e}{c}\Phi_k\right) \pm i\left(p_4 - \frac{e}{c}\Phi_4\right) - \alpha_0\cdot\frac{E_0}{c}$$

$$(\alpha_i\,\alpha_k + \alpha_k\,\alpha_i = 2\,\delta_{ik})$$

then the adjoint definition would have to be changed under the transition to the iterated equation, since D'^+ and D'^- do not commute.

([1]) **W. Gordon**, Zeit. Phys. **50** (1928), pp. 630.

([2]) I am grateful to Herrn Dr. **Scherzer** for the proof of this.

b) *Conversion to quaternions.* – When the iterated Dirac equation is written out, it will read:

$$(90) \qquad D\psi \equiv \left\{ -c\left(\mathfrak{p} - \frac{e}{c}\cdot\mathfrak{A}\right)^2 + (E-V)^2 - E_0^2 - \hbar e c[(\mathfrak{H}, i\gamma\tau) + (\mathfrak{E}, \gamma\gamma_4)]\right\}\psi = 0.$$

The coefficients belong to the domain of biquaternions, whose units are even-number products of the γ_1, γ_2, γ_3, γ_4. That number domain is decomposable (cf., § **3**). The equation $D\psi = 0$ decomposes into two independent equations:

$$D\frac{(1+\gamma_1\gamma_2\gamma_3\gamma_4)}{2}\cdot\psi\frac{(1+\gamma_1\gamma_2\gamma_3\gamma_4)}{2} = 0,$$

$$(91)$$

$$D\frac{(1-\gamma_1\gamma_2\gamma_3\gamma_4)}{2}\cdot\psi\frac{(1-\gamma_1\gamma_2\gamma_3\gamma_4)}{2} = 0.$$

We set:

$$(92) \qquad 1^\pm = \frac{(1-\gamma_1\gamma_2\gamma_3\gamma_4)}{2}, \qquad \sigma^\pm = -i\,\gamma\tau\,1^\pm,$$

to abbreviate. $\sigma^\pm$ are quaternion quantities, for which one has:

$$(93) \qquad (\mathfrak{a}\,\sigma^\pm)(\mathfrak{a}\,\sigma^\pm) = (\mathfrak{a}\,\mathfrak{b})\cdot 1^\pm + i\,([\mathfrak{a}\,\mathfrak{b}]\,\sigma^\pm) \qquad \text{and} \qquad (1^\pm)^2 = 1^\pm.$$

Since $\gamma\gamma_4\cdot 1^\pm = \pm i\,\sigma^\pm$, we will get two independent quaternion equations for $\psi^\pm = \psi\cdot 1^\pm$, instead of (90):

$$(94) \qquad \left\{ -c\left(\mathfrak{p} - \frac{e}{c}\cdot\mathfrak{A}\right)^2 + (E-V)^2 - E_0^2 + \hbar e c\,(\mathfrak{H}\pm i\mathfrak{E}, \sigma^\pm)\right\}\psi^\pm = 0.$$

The two equations (94) will be essentially different as long as one does not have $\mathfrak{E} = 0$. Namely, the sign of i is established by the Ansatz $\vec{\mathfrak{p}} = \frac{\hbar}{i}\vec{\nabla}$ and $E = -\frac{\hbar}{i}\frac{\partial}{\partial t}$. That essential difference will persist even in the case of $\mathfrak{H} = 0$, since the σ-relations (93) are not invariant under a change of sign of σ. The fact that the two equations coincide in the case of $\mathfrak{E} = 0$ should not be surprising since (90) is in fact already a quaternion equation then.

Kramers [1] derived equations (94) directly from classical-mechanical considerations.

c) *Reduction issues.* – The treatment of physical problems on the basis of the iterated Dirac equation is impossible in the case of non-vanishing electric fields in the number

---

[1] **H. A. Kramers**, *Zeeman-Festschrift*, s'Gravenhage, 1935, pp. 403.

field of biquaternions, since there is no unique reduction in that field. The uniqueness proof of § **4** breaks down in the field of biquaternions at the auxiliary condition $C_2 N C_1 \neq 0$, which cannot always be fulfilled by a choice of c_1 and c_2 (see the end of § **4**). There are two essentially different reductions that correspond to the two systems of solutions (94). That difficulty can be eliminated only extending the field to the complete Dirac field by the addition of an independent hypercomplex quantity; with that extension, the reduction will be unique. That will therefore mean that one needs the *entire* number field of the *linear* equation for the application of the iterated Dirac equation, even though the coefficients of the equation are all contained in a subfield.

Although one has to calculate only in the quaternion system $\overset{\pm}{\sigma}$ for the solution of the iterated Dirac equation (the necessity of basing things upon the entire Dirac field first comes to light in the *reduction* of solutions, while one can naturally get by with the biquaternions in the purely-mathematical solution of the iterated Dirac equation), I do not know whether the use of equations (94) for the exhibition of eigenfunctions of electrons has any advantage. For the calculation of eigenfunctions of the electrons in the central field, it is simplest for one to start with the linear equation directly in any case, as we did in § **14**.

d) *Conversion of the solutions to the solutions of two linear equations.* – The most general solution of the iterated Dirac equation is a linear combination of the solutions of the two linear equations. One sees that very easily when one integrates the equation:

$$(95) \qquad\qquad D^+ \cdot D^- \psi = 0$$

in two steps. We can initially regard (95) as a linear equation for the function $(D^- \cdot \psi)$ and then write:

$$(96) \qquad\qquad D^- \psi = \psi^+,$$

in which ψ^+ is any solution of the linear Dirac equation:

$$(97) \qquad\qquad D^+ \psi^+ = 0.$$

(96) is an inhomogeneous equation for ψ. Since $D^- \psi^+ = -2i \, (E_0 / c) \, \psi^+$ [cf., (82)], that will imply a particular solution of (96) $\psi = \dfrac{i\,c}{2E_0} \psi^+ \approx \psi^+$. The most general solution differs from the particular one by an integral of the homogeneous equation; i.e., i.e., $\psi = \psi^+ + \psi^-$. The statement is proved with that.

It follows from this, in particular, that the iterated Dirac equation will yield no other energy levels than the linear one. [The "neutron solution" of the Kepler that Temple gave ([1]) is not permissible, since it exhibits a singularity at the 0-point that will continue to exist when the singularity of the potential vanishes $(\alpha Z \to 0)$.]

([1]) **G. Temple**, Proc. Roy. Soc. London **145** (1934), pp. 344.

Appendix

Algebra of Clifford numbers.

n quantities γ_k that satisfy the equation:

(1) $$\gamma_i \, \gamma_k + \gamma_k \, \gamma_i = 2 \, \delta_{ik}$$

and are independent, moreover, generate a number field that we would like to denote with the abbreviation C_n , in honor of its discoverer **Clifford** ([1]). The numbers that are contained in C_n have the form:

(2) $$a_0 + \sum_i a_i \, \gamma_i + \sum_{i \neq k} a_{ik} \, \gamma_i \, \gamma_k + \ldots + a_{123\ldots n} \, \gamma_1 \, \gamma_2 \, \gamma_3 \ldots \gamma_n \, .$$

The a's are ordinary complex numbers, while the 0-fold, 1-fold, 2-fold, …, n-fold products of the various γ's define the basic quantities of the system. *The number of basic quantities is 2^n.* There are then $\binom{n}{v}$ products of $\sum_{v=0}^{n} \binom{n}{v} = (1 + 1)^n = 2^n$. One will see even more simply that the number of basic quantities is 2^n by induction; namely, when one adds the quantity γ_n to C_{n-1} , one will double the number of basic quantities.

Addition, subtraction, and multiplication of the numbers in C_n is defined uniquely; The operations that one uses in calculations will always lead back to numbers that belong to C_n . (That is the definition of the number field that is "generated" by the n γ's.) C_n differs from the field of ordinary complex numbers very essentially in two points: First, the commutative law of multiplication is not true, as (1) shows. Secondly, not every non-zero number possesses a reciprocal. The numbers without reciprocals are called zero divisors; there are different types of them. In what follows, we will show that the numbers in C_{2n} belong to $(2^n + 1)$ different types. Discovering and characterizing those types shall be the problem in the following three sections.

a) *Theorem of Clifford on the construction of the C_{2n} from quaternions. Isomorphism of C_{2n} with the 2n-rowed matrices.* – We shall next prove the theorem of Clifford that C_{2n} can be generated by n commuting quaternion fields. One can define a quaternion field to be a number field that is generated by two anticommmuting quantities with complex squares. ("Complex" shall mean: "not hypercomplex.")

We prove Clifford's theorem by induction: C_2 is obviously identical with the field of quaternions. C_{2n} arises from $C_{2(n-1)}$ by the addition of the quantities: $\gamma_1 \, \gamma_2 \ldots \gamma_{2(n-1)} \cdot \gamma_{2n-1}$ and $\gamma_1 \, \gamma_2 \ldots \gamma_{2(n-1)} \cdot \gamma_{2n}$; they both anticommute and have a real square, so they will generate a quaternion field. In addition, they will commute with $C_{2(n-1)}$. Clifford's theorem is then proved.

([1]) **Clifford**, "Applications of Grassmann's extensive algebra," Am. J. Math. **1** (1878), pp. 350. **Clifford** proves that C_{2n} would be generated by n commutating systems of quaternions.

The generalization follows immediately from Clifford's theorem, which is also proved directly by induction: A number of commuting Clifford fields $C_{n_i}^i$ generates a $C_{\sum_i n_i}$.

It is now easy to prove the isomorphism ([1]) of C_{2n} with the field of $2n$-rowed matrices. Addition and multiplication of matrices is then defined in a known way by:

$$(3a) \qquad (A_{ik})(B_{ik}) = \left(\sum_l A_{il} B_{lk}\right),$$

$$(3b) \qquad (A_{ik}) + (B_{ik}) = (A_{ik} + B_{ik}).$$

We prove the isomorphism by induction. For $n = 1$, the isomorphism can be exhibited by the associations:

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma_1 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_2 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_1 \gamma_2 \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and its inverse:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \frac{1 + \gamma_1}{2}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to \frac{\gamma_2 + \gamma_1 \gamma_2}{2}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to \frac{\gamma_2 - \gamma_1 \gamma_2}{2}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \frac{1 - \gamma_1}{2}.$$

We now assume that we have represented C_{2n} by 2^{n-1} matrices. From Clifford's theorem, C_{2n} is a quaternion field with coefficients in $C_{2(n-1)}$, so the coefficients and the basic quantities will commute. Since the quaternions can be represented uniquely by two-rowed matrices, C_{2n} can be represented two-rowed matrices whose elements belong to $C_{2(n-1)}$, so the induction assumption will imply that they can be represented by 2^{n-1}-rowed matrices; the numbers in C_{2n} are then isomorphic to the 2^{n-1}-rowed matrices.

b) *Numbers types in C_{2n}. Rank. Possibility of reduction.* – The most fundamental property of numbers in the matrix field is their rank R. It is invariant under all automorphisms of the field. Now, from the previous section, C_{2n} is isomorphic to the field of $2n$-rowed matrices. It follows from this that the numbers of C_{2n} can be characterized uniquely by the rank of those matrices that are associated with them by an isomorphism of that sort. (The fact that rank is actually independent of the special representation of the C_{2n} is obvious. Namely, an automorphism would otherwise be possible as a way of getting around C_{2n}, and under that automorphism, two matrices of different rank would be associated with each other.) We can then classify the numbers of C_{2n} into $(2^n + 1)$ types whose rank is equal to 0, 1, 2, 3, …, 2^n. However, the type classification is not useful for us in the given form for two reasons: First, we would like to characterize the type directly by its algebraic behavior, and not indirectly by a complicated isomorphism. In addition, the rank R is not the quantity that is given to

---

([1]) We understand "isomorphism" to mean the possibility of establishing a *one-to-one* correspondence.

characterize the type, since that number would continue to be associated with different R when one considers it to belong to different C_{2n}. For example, the number 1 has the rank $R = 1$ in C_0 (= field of complex numbers), $R = 2$ in C_2 (= quaternions), and generally, rank $R = 2^n$ in C_{2n}.

We would now like to establish the algebraic meaning of the rank R of a number Γ. It will be true for matrices, and therefore for C_{2n}, as well. If:

$$(4) \qquad A = (X_{ik}) \equiv x_0 + \sum_i x_i\, \gamma_i + \sum_{i+k} x_i\, \gamma_i\, \gamma_k + \ldots + x_{12\ldots2n} \cdot \gamma_1\, \gamma_2 \ldots \gamma_{2n}$$

is the general number of the field, which depends upon 2^{2n} independent parameters X_{ik} (x_0, x_1, …, resp.), then:

$$(5) \qquad \Gamma_1 \cdot A \cdot \Gamma_2 \text{ will depend upon } R_1 \cdot R_2 \text{ parameters.}$$

If Γ_1 (Γ_2, resp.) possesses rank 2^n in this then $A\Gamma_1$ ($A\Gamma_2$, resp.) can be replaced with A, and it will follow that:

$$(6) \qquad \Gamma A \text{ and } A\Gamma \text{ will depend upon } 2^n \cdot R \text{ parameters.}$$

That shows that a right-hand (left-hand, resp.) factor of Γ will reduce the number of independent parameters, and indeed by the ratio of $R \cdot 2^n : 2^{2n}$. We shall call that ratio:

$$(7) \qquad \text{reduction factor } r = \frac{R}{2^n}.$$

r characterizes the type of any Clifford number Γ, so it will be independent of which special C_{2n} one embeds Γ in. The proof of the last assertion flows out of the fact that $C_{2(n+m)}$ can be generated from C_{2n} by adjoining quantities that commute with C_{2n}. (Clifford's theorem.)

The following theorems in regard to the isomorphism with matrices seem obvious:

1. The numbers with $r = 1$ possess reciprocals, and only those numbers.

2. Numbers with $r < 1$ are zero divisors and therefore do not possess reciprocals.

[One will make all theorems about matrix products, such as the ones cited above and (5), (6), easy to understand when one regards the 2^n-rowed matrices as homogeneous affine transformations of 2^n-dimensional space. A matrix of rank R will then mean a map to a subspace of dimension R. One proves Theorem 2, say, by the argument that a map to a subspace $R < 2^n$ (corresponding to $r < 1$) and a subsequent projection onto a line that is perpendicular to the subspace will yield a map of the entire space to the origin.]

$(2^n + 1)$ types of numbers are contained in C_{2n}, namely, the null ($r = 0$) type, then $(2^n - 1)$ types of zero divisors $\left(r = \dfrac{1}{2^n}, \dfrac{2}{2^n}, \dfrac{3}{2^n}, \cdots, \dfrac{2^2-1}{2^n} \right)$, and finally the *nondegenerate* numbers ($r = 1$), which possess reciprocals.

The numbers of C_{2n} with $r = 1 / 2^n$ all have essentially the form:

$$(8) \qquad \Gamma = (1 + i\,\gamma_1\,\gamma_2)(1 + i\,\gamma_3\,\gamma_4) \dots (1 + i\,\gamma_{2n-1}\,\gamma_{2n}) \qquad \left\{ r = \frac{1}{2^n} \right\}.$$

$(i\,\gamma_1\,\gamma_2)(i\,\gamma_3\,\gamma_4) \dots (i\,\gamma_{2n-1}\,\gamma_{2n})$ can also be replaced with any other independent commuting quantities with squares equal to 1.

One gets the numbers of the other types by adding different commuting zero divisors of type $r = 1/2^r$; e.g.:

$$(9) \qquad \Gamma = (1 + i\,\gamma_1\,\gamma_2) + (1 + i\,\gamma_3\,\gamma_4) + \dots + (1 + i\,\gamma_{2n-1}\,\gamma_{2n})$$

has the reduction factor $r = \dfrac{2^n - 1}{2^n}$.

c) Number types in C_0, C_2, C_4. – From the foregoing, the number types in C_0, C_2, C_4 are easy to establish.

C_0 *is the field of complex numbers;* other than zero ($r = 0$), C_0 contains nothing but nondegenerate numbers ($r = 1$). The general number A of the domain has one parameter.

C_2 *is the field of quaternions.* Along with zero ($r = 0$) and the numbers ($r = 1$), C_2 contains *one* type of zero divisor ($r = 1/2$). The general number A of the domain contains four parameters. The number of parameters will be reduced to two by a simple reduction with one zero divisor and to one by a two-sided reduction.

C_4 *is the Dirac number system.* Here, along with zero and the nondegenerate numbers, there are already three types of zero divisors $\left(r = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \right)$. $r = 1/4$ corresponds to numbers of the form $(1 + i\,\gamma_1\,\gamma_2)(1 + i\,\gamma_3\,\gamma_4) = (1 + i\,\gamma_1\,\gamma_2)(1 - \gamma_1\,\gamma_2\,\gamma_3\,\gamma_4)$. The sixteen components of a general number A will be reduced to four by those quantities in a one-sided way and to one in a two-sided way. Numbers of the type $r = 1/2$ are already contained in C_2. From (9), an example of $r = 3/4$ is $\Gamma = (1 + i\,\gamma_{12}) + (1 + i\,\gamma_{34})$. $A\Gamma$ and ΓA contain twelve parameters, while $\Gamma A\Gamma$ contain nine parameters.

d) Decomposability [1] *of C_{2n+1}.* We generally seek those C_r that are decomposable – i.e., they decompose into two sub-domains ($C_v = C_v^{(1)} + C_v^{(2)}$) – and indeed in such a way that each number in the first domain will give a product of zero when it is multiplied by any number in the second one. We then seek, in a known way [2], the possible units of the subsystem $e^{(1)}$, $e^{(2)}$, for which one then has: $e^{(i)} C_v^{(i)} = C_v^{(i)} \cdot e^{(i)} = C_v^{(i)}$, and in addition $(e^{(1)})^2 = e^{(1)}$, $(e^{(2)})^2 = e^{(2)}$, $e^{(1)} e^{(2)} = 0$. The $e^{(i)}$ must clearly commute with all numbers in C_v.

We first determine the numbers in C_v that commute with all numbers in C_v. The complex numbers belong to them in any event. For $v = 2n + 1$, $\gamma_1\,\gamma_2\,\gamma_3 \dots \gamma_v$ will also commute with all γ's, and therefore with all of C_v. Now, one easily sees that there are no other commuting quantities, in general. Namely, a product of γ's that does not contain all

---

[1] See rem. 1 on pp. 10.
[2] See, perhaps, *Enz. d. math. Wissensch.* I, 1, A, **4** (Study) (1898), pp. 165.

γ's will always anticommute with at least one γ that is either included in it or not (according to whether it contains an even or odd number of γ's as factors, resp.).

It follows immediately from the foregoing that the C_{2n} cannot be decomposed. Namely, only the numbers that are free of γ's will commute with C_{2n}, such that only they can be considered as possible units. However, since numbers that are free of γ's cannot be zero divisors, the condition $e^{(1)} \cdot e^{(2)} = 0$ cannot be fulfilled.

In C_{2n}, numbers of the form $a + b \cdot \gamma_1 \gamma_2 \gamma_3 \ldots \gamma_{2n+1}$ come into question as units. The zero divisors among them are the numbers $a (1 \pm i^n \cdot \gamma_1 \gamma_2 \gamma_3 \ldots \gamma_{2n+1})$. There are two units among them:

$$(10) \qquad e^{\pm} = \frac{1 \pm i^n \cdot \gamma_1 \cdot \gamma_2 \cdots \gamma_{2n+1}}{2}.$$

They fulfill the condition $e^{+} \cdot e^{-} = 0$ and are therefore suitable for the decomposition of C_{2n+1}. Obviously, one certainly has $e^{+} + e^{-} = 1$, so one will have:

$$(11) \qquad \Gamma = \Gamma e^{+} + \Gamma \cdot e^{-} = \Gamma^{+} + \Gamma^{-},$$

with $\Gamma^{\pm} = \Gamma \cdot e^{\pm}$, for any quantity Γ in C_{2n+1}.

The quantities Γ^{+} and Γ^{-} define a closed number field by themselves. Each of those fields is a C_{2n}. Namely, it can be generated by $2n$ anticommuting quantities with real squares (relative to the units $e^{\pm}$), say, by $\gamma_1 \cdot e^{\pm}$, $\gamma_2 \cdot e^{\pm}$, ..., $\gamma_{2n} \cdot e^{\pm}$. $\gamma_{2n+1} \cdot e^{\pm}$ can be expressed in terms of them, since $\gamma_1 \cdot e^{\pm} \cdot \gamma_2 \cdot e^{\pm} \ldots \gamma_{2n} \cdot e^{\pm} \cdot \gamma_{2n+1} \cdot e^{\pm}$ differs from $e^{\pm}$ only by a complex factor.

Due to decomposability, every equation in C_{2n+1} can be replaced with two independent equations in C_{2n}. Let, say, the equation:

$$(12) \qquad L \cdot \psi = 0$$

be given. We append a factor of $1 = e^{+} + e^{-}$ to L, as well as ψ, and get:

$$(12a) \qquad L\psi \equiv L (e^{+} + e^{-}) \cdot \psi (e^{+} + e^{-}) \equiv (L \cdot e^{+})(\psi e^{+}) + (L e^{-})(\psi e^{-}) \equiv L^{+} \psi^{+} + L^{-} \psi^{-}.$$

If we multiply the equation by e^{+} then the second term will drop out, while the first one will remain unchanged; an analogous situation will come about under multiplication by e^{-}. That will yield the equations:

$$(12b) \qquad L^{+} \psi^{+} = 0, \qquad L^{-} \psi^{-} = 0.$$

It would now be easy to classify the numbers in C_{2n+1} into types according to their capability of being reduced. However, aside from the fact that such a classification would have no practical interest due to the decomposability of C_{2n+1}, it would also not be "typical" in any way. For example, in C_3, one must ascribe different types to the numbers $(1 + \gamma_1)$ and $(1 + i \gamma_1 \gamma_2 \gamma_3)$ [namely, $(1 + \gamma_1) A (1 + \gamma_1)$ has two components and $(1 + i \gamma_1 \gamma_2 \gamma_3) A (1 + i \gamma_1 \gamma_2 \gamma_3)$ has four], although they are obviously algebraically equivalent.

The decomposability of C_{2n+1} is essentially linked with the fact that one allows complex coefficients, and not just real ones. If only real coefficients were allowed then the properties of the C_v would change very considerably. All C_v are *real-irreducible*. In addition, one easily sees that C_{4n+3}^{real} is isomorphic to $C_{4n+2}^{\text{complex}}$. ($\gamma_1\,\gamma_2\,...\,\gamma_{4n+3}$ is the imaginary unit of C_{4n+2}^{real}.) Likewise, C_{4n+2}^{real} is isomorphic to $C_{4n+1}^{\text{complex}}$. [For the meaning of Q, see e) below.] C_{2n}^{real} is an irreducible domain of a complicated structure. There are no numbers of rank $R = 1$ in C_{2n}^{real} for $n > 1$.

e) *Quaternions and biquaternions.* – One can derive complete sub-fields of C_n in various ways. The most important case is the "n-way geometric algebra" that **Clifford** gave, for which we would like to use the symbol Q_n. Q_n consists of those numbers in C_n that contain only even-number products of γ's:

$$(13) \qquad a_0 + \sum_{i\neq k} a_{ik}\,\gamma_i\,\gamma_k + \sum_{i,k,l,n} a_{iklm}\,\gamma_i\,\gamma_k\gamma_l\,\gamma_m\ + ...$$

Clearly, Q_n and C_{n-1} are isomorphic. One can in fact think of Q_n as being generated by the $n-1$ anticommuting, but otherwise independent, quantities $\gamma_1\,\gamma_2$, $\gamma_1\,\gamma_3$, $\gamma_1\,\gamma_4$, …, $\gamma_1\,\gamma_n$. Nevertheless, we would like to preserve both of the two symbols C_{n-1} and Q_n, since in practical applications the geometric meaning, and therefore the transformation properties of the two systems to be defined, are not at all different. One thinks of C_{n-1} as being generated by $n-1$ equivalent quantities, between which, no relations exist besides commutation relations. By contrast, one must think of Q_n as being generated by $n \cdot (n-1) / 2$ equivalent quantities (that are all two-fold products of n γ's), between which $(n-1)(n-2) / 2$ relations will exist in addition to the commutation relations. C_{n-1} is linked with the geometry of $(n-1)$-dimensional space, while Q_n is linked with the geometry of n-dimensional space.

Q_2 is the Gaussian complex numbers, as long as one allows only real coefficients in C_2.

Q_3 is the field of quaternions, not only in regard to the purely algebraic properties of C_2, but also in regard to the symmetry in the way that one imagines the basic quantities.

Clifford referred to Q_4 as the *biquaternions*. It corresponds to the rotations in four-dimensional space and defines the number field of the iterated Dirac equation. Since it is isomorphic to C_3, it can be decomposed in the same way as C_3.

f) *Isotropy of the Clifford number domain C_n.* – The quantities γ_i' that arise from the γ_k by an orthogonal transformation:

$$(14) \qquad \gamma_i' = a_{ik}\,\gamma_k \qquad \text{with} \quad \sum_i a_{ik}\,a_{il} = \delta_{kl}$$

will satisfy the same relations as the γ_k, namely:

$$(15) \qquad \gamma_i'\,\gamma_k' + \gamma_k'\,\gamma_i' = \delta_{ik}\,.$$

That invariance of the condition equations under orthogonal transformations is probably the most important property of Clifford numbers for the applications. It allows one to associate the n γ_k's with an n-dimensional coordinate space in an isotropic way. Namely, if one associates γ_k with the x_k-axis then the coordinate axes will be in no way distinguished, so any direction with the unit vector $\mathfrak{e}$ can be associated with $(\gamma \, \mathfrak{e}) = \sum_k \mathfrak{e}_k \, \gamma_k$ in a completely isotropic way. One can then think of the Clifford number domain C_n as being generated by a symbolic n-vector γ that possesses the components $(\gamma \, \mathfrak{e})$ in a direction $\mathfrak{e}$, and those components will then transform like vector components (while γ itself will take on a meaning that is independent of the coordinate system).

_____