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On an extension of the Jacobi-Hamilton method

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In a note (*) that was published in 1890, **Volterra** showed that there is some interest to the problem of extending the beautiful work of JACOBI and HAMILTON on the equations that express the idea that first variation of a simple integral is zero to the case of a multiple integral. He accomplished that generalization in the case of the integral:

$$\iint U\, dx_4\, dx_5 \,,$$

in which U is a function of x_1, x_2, x_3, x_4, x_5, and the functional determinants:

$$\frac{D(x_1, x_2)}{D(x_4, x_5)}\,, \qquad \frac{D(x_2, x_3)}{D(x_4, x_5)}\,, \qquad \frac{D(x_3, x_1)}{D(x_4, x_5)}\,.$$

(However, that function is not homogeneous with respect the last three quantities.) I propose to treat the more general case, i.e., the case of the integral:

$$I = \iiint \cdots \int f\left(x_{r+1}, \ldots, x_n, x_1, \ldots, x_r, \frac{\partial x_{r+1}}{\partial x_1}, \ldots, \frac{\partial x_n}{\partial x_r} \right) dx_1 \cdots dx_r \tag{1}$$

(in which $x_{r+1}, \ldots, x_n$ are certain functions of $x_1, \ldots, x_r$), which includes that of JACOBI as a special case for $r = 1$. I shall appeal to what **Volterra** said in a previous note (**) for those of the definitions and theorems that relate to n-dimensional space.

I. – First of all, let us put the integral I into a more convenient form. Set:

(*) "Sopra una estensione della teoria JACOBI-HAMILTON der calcolo della variazione," Accademia dei Lincei, Rendiconti, vol. VI, pp. 127.

(**) "Delle variabili complesse negli iperspazi," *loc. cit.*, vol. V, pp. 159.

$$F = f\,\frac{D(x_1,\ldots,x_r)}{D(\omega_1,\ldots,\omega_r)} \tag{2}$$

while supposing that $x_1, \ldots, x_r$ are expressed as functions of the r parameters $\omega_1, \ldots, \omega_r$. One sees that one can write:

$$I = \int\cdots\int F\,d\omega_1\,d\omega_2\ldots d\omega_r\ , \tag{3}$$

in which F denotes a homogeneous function of degree one in the functional determinants, $\dfrac{D(x_{i1},x_{i2},\ldots,x_{ir})}{D(\omega_1,\ldots,\omega_r)}$, where $i_1, \ldots, i_r$ is an arbitrary combination of the n whole numbers from 1 to r. That is because one has:

$$\frac{\partial x_{i+k}}{\partial x_i} = \frac{D(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)} : \frac{D(x_1,x_2,\ldots,x_r)}{D(\omega_1,\ldots,\omega_r)}\ .$$

Conversely, *the integral* (3) *can always be reduced to the form* (1), *no matter what form the function F of* $x_1, \ldots, x_n$ *takes, which is homogeneous of degree one with respect to the quantities:*

$$\lambda_{i_1,i_2,\ldots,i_r} = \frac{D(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)}\,,$$

because the equality (2) defines f as a function of $x_1, \ldots, x_n$ and the functional determinants:

$$\frac{D(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)}\,,$$

which are certain functions of the quantities:

$$\frac{\partial x_{r+1}}{\partial x_1}\,,\ \ldots\ ,\ \frac{\partial x_n}{\partial x_r}\,.$$

It suffices to choose from among the variables $x_1, \ldots, x_n$, the letters $x_1, \ldots, x_r$ with respect to which the equations of the multiplicity S_r considered are soluble.

By definition, *the form* (1) *is equivalent to the form* (*):

(*) The form (3) is the analogue of the parametric form that was employed by WEIERSTRASS in the case of a simple integral. The extension to the case of a multiple integral was realized for the first time in a practical form by **HADAMARD** in his course at the Collège de France.

$$I = \int_{S_r} F(x_1, \ldots, x_n, \lambda_{1,2,\cdots,r}\,; \ldots; \lambda_{i_1,i_2,\cdots,i_r}\,; \ldots)\, d(\omega_1, \ldots, \omega_r) \tag{3}$$

or

$$I = \int_{S_r} F\left(x_1, \ldots, x_n, d(x_1, \ldots, x_r), \ldots, d(x_{i_1}, \ldots, x_{i_r})\right),$$

in which I have employed the notation of MÉRAY for multiple integrals and set:

$$d(x_{i_1}, \ldots, x_{i_r}) = \frac{D(x_{i_1}, \ldots, x_{i_r})}{D(\omega_1, \ldots, \omega_r)}\, d\omega_1\, d\omega_2 \ldots d\omega_r\ .$$

II. First variation. – Let us calculate the first variation of I. We will have:

$$\delta I = \int_{S_r} \left[\sum_i F_{x_i}\, \delta x_i + \sum_{i_1,\ldots,i_r} F_{\lambda_{i_1,\cdots,i_r}}\, \delta\lambda_{i_1,\ldots,i_r} \right] d(\omega_1, \ldots, \omega_r)\ ,$$

in which the F_{x_i} and $F_{\lambda_{i_1,\ldots,i_r}}$ are the partial derivatives of F with respect to x_i and $\lambda_{i_1,\ldots,i_r}$, resp., when they are considered to be independent variables, whereas $\partial F / \partial\omega_1$, for example, denotes the derivative that is taken while considering the x and the λ to be functions of $\omega_1, \ldots, \omega_r$.

Hence, upon integrating the terms in $\delta\lambda$ by parts:

$$\delta I = \int_{S_r} \left(\sum_{i=1}^{r} Q_i\, \delta x_i \right) d(\omega_1, \ldots, \omega_r) + \int_{S_{r-1}} \sum_{i_1,\ldots,i_r} F_{\lambda_{i_1,\cdots,i_r}}\, \delta x_{i_1}\, d(x_{i_2}, \ldots, x_{i_r})\ , \tag{4}$$

in which one sets:

$$Q_i \equiv F_{x_i} - \sum_{i_1,\ldots,i_{r+1}} \left[\frac{D(F_{\lambda_{i_1,\cdots,i_{r+1}}}, x_{i_1}, x_{i_2}, \ldots, x_{i_{r+1}}}{D(\omega_1, \omega_2, \ldots, \omega_r)} \right] \tag{5}$$

and lets S_{r-1} denote the at-most-$(r - 1)$-dimensional multiplicity that forms the boundary of S_r . One sees that if S_{r-1} remains fixed then one will have:

$$\delta I = \int_{S_r} \left(\sum_{i=1}^{r} Q_i\, \delta x_i \right) d(\omega_1, \ldots, \omega_r)\ .$$

Conforming to the terminology that is employed in the calculus of variations, we shall call any multiplicity that verifies the equation $\delta I = 0$ when the limits are fixed an *extremal*. For that to be the case, it is necessary and sufficient that S_r must be an integral multiplicity of the equations:

$$Q_1 = 0 , \qquad \ldots , \qquad Q_n = 0 . \tag{6}$$

III. Canonical equations of the extremals. – In order to arrive at the canonical form of equations (6), we set:

$$q_{i_1, i_2, \ldots, i_r} = F_{\lambda_{i_1,\ldots,i_r}} \tag{7}$$

and place ourselves in the general case, where we consider the λ to be independent variables (*), and the Hessian of F (which is null and of order C_n^r) to have at least one minor of order $C_n^r - 1$ that is non-zero. Under that condition, the relations (7) will show that upon further considering the λ to be independent variables, there will be one and only one relation between the q :

$$H(x_1, \ldots, x_n, q_{1,\ldots,r}; \cdots; q_{i_1,\ldots,i_r}; \cdots) = 0 . \tag{8}$$

However, if, conversely, one knows the function H (which is chosen to be arbitrary, but not homogeneous) then a well-known theory will show that if the q are coupled by just the relation (8) then one can fund a function F that is homogeneous of degree one with respect to the independent variables λ, such that the relations (7) are verified. It suffices to take:

$$\frac{F}{\lambda_{1,2,\ldots,r}} = \frac{\sum\limits_{i_1,\ldots,i_r} q_{i_1,\ldots,i_r} H_{q_{i_1,\ldots,i_r}}}{H_{q_{1,\ldots,r}}} , \tag{10}$$

if $H_{q_{1,2,\ldots,r}}$, for example, is not identically zero, and one will then know that one has:

$$\frac{\lambda_{1,2,\ldots,r}}{-H_{q_{1,2,\ldots,r}}} = \ldots = \frac{\lambda_{i_1,\ldots,i_r}}{-H_{q_{1,2,\ldots,r}}} = \ldots = \frac{F_{x_1}}{H_{x_1}} = \ldots = \frac{F_{x_n}}{H_{x_n}} . \tag{11}$$

That results from a change of variables, without one having to be preoccupied with equations (6). In particular, one will see that one will have:

$$I = \int_{S_r} \left[\sum_{i_1,\ldots,i_r} q_{i_1,\ldots,i_r} H_{q_{i_1,\ldots,i_r}} \right] d(\omega_1, \ldots, \omega_r) \tag{12}$$

when one sets:

(*) Which will not be true when one takes into account the equalities $\lambda_{i_1,\ldots,i_r} = \dfrac{D(x_{i_1}, \ldots, x_{i_r})}{D(\omega_1, \ldots, \omega_r)}$, because one will then have:

$$\sum_{s=1}^{r+1} (-1)^s \lambda_{i_1,\ldots,i_{s-1}, i_{s+1},\ldots,i_{r+1}} \times \lambda_{i_s, i_2,\ldots,i_r} = 0 . \tag{9}$$

$$q_{i_1,\ldots,i_r} = F_{\frac{D(x_{i_1},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)}}, \tag{13}$$

and in this case the quantities $q_{i_1,\ldots,i_r}$ are not only coupled by the relation (8), but also by the equations:

$$\sum_{s=1}^{r+1} (-1)^s H_{q_{i_1,\ldots,i_{s-1},i_{s+1},\ldots,i_{r+1}}} \times H_{q_{i_s,i_2,\ldots,i_r}} = 0, \tag{14}$$

which result from the identities (9), (11).

If we now introduce the new variables into equations (6) *then we will see that we can regard the extremals as being defined by the following canonical system, which is analogous to that of* HAMILTON (to which it will reduce for $r = 1$):

$$\left.\begin{aligned} &H(x_1,\ldots,x_n,q_{1,2,\ldots,r};\cdots;q_{i_1,\ldots,i_r};\cdots) = 0, && (8)\\ &\cdots = \frac{\sum_{i_2,\ldots,i_r} d(q_{i_1,i_2,\ldots,i_r};x_{i_2},\ldots,x_{i_r})}{H_{x_i}} = \cdots = \frac{d(x_{i_1},x_{i_2},\ldots,x_{i_r})}{H_{q_{i_1,i_2,\ldots,i_r}}} = \cdots && (15)\end{aligned}\right\}$$

One can further write that in the form:

$$\left.\begin{aligned} &\sum_{i_2,\ldots,i_r} \frac{D(q_{i_2,\ldots,i_r},x_{i_2},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)} = \rho H_{x_{i_1}}, \qquad (i_1 = 1,\ldots,n), && (16)\\ &\frac{D(q_{i_2,\ldots,i_r},x_{i_2},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)} = -\rho H_{q_{i_1,\ldots,i_r}}, \quad (i_1,i_2,\ldots,i_r = 1,\ldots,n), && (17)\\ &H(x_1,\ldots,x_n,q_{1,\ldots,r},\ldots,q_{i_1,\ldots,i_r},\ldots) = 0. && (8)\end{aligned}\right\} \quad (E)$$

Furthermore, observe that one can replace the condition (8) with another one that is simpler.

Indeed, H is a constant such that the x and q verify equations (16) and (17). That is because one will have:

$$\rho\frac{\partial H}{\partial\omega_1} \equiv \sum_i \rho H_{x_i}\frac{\partial x_i}{\partial\omega_1} + \sum_{q_{i_1,\ldots,i_r}} \rho H_{q_{i_1,\ldots,i_r}}\frac{\partial q_{i_1,\ldots,i_r}}{\partial\omega_1}$$

$$= \sum_i \frac{\partial x_i}{\partial\omega_1}\sum_{i_2,\ldots,i_r}\frac{D(q_{i_1,i_2,\ldots,i_r},x_{i_2},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)} - \sum_{i_1,\ldots,i_r}\frac{\partial q_{i_1,\ldots,i_r}}{\partial\omega_1}\frac{D(x_{i_1},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)}$$

$$\equiv \sum_{i_1,\ldots,i_r}\frac{\partial q_{i_1,\ldots,i_r}}{\partial\omega_1}\left[\left(\sum_s (-1)^s\frac{\partial x_{i_s}}{\partial\omega_1}\frac{D(x_{i_1},\ldots,x_{i_{s-1}},x_{i_{s+1}},\ldots,x_{i_r})}{D(\omega_2,\ldots,\omega_r)}\right) - \frac{D(x_{i_1},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)}\right]$$

$$+ \sum_{i_1,\dots,i_r} \sum_{s=1} (-1)^s \frac{\partial q_{i_1,\dots,i_r}}{\partial \omega_s} \left[\sum_{t=1} (-1)^t \frac{\partial x_{i_t}}{\partial \omega_1} \frac{D(x_{i_1}, \dots, x_{i_{t-1}}, x_{i_{t+1}}, \dots, x_{i_r})}{D(\omega_1, \dots, \omega_{s-1}, \omega_{s+1}, \dots, \omega_r)} \right] \equiv 0$$

in this case because the brackets are identically zero.

As a result, $\partial H / \partial \omega_1$ will be zero, and similarly for the other derivatives. Moreover, H is a constant, and in place of equation (8), one can be content to write that the *initial* value of H is zero. If one does not impose that condition then, after the change of variables:

$$\lambda_{i_1,\dots,i_r} = H_{q_{i_1,\dots,i_r}} , \tag{18}$$

equations (16) and (17) will represent the equations of the extremals of the integral:

$$\int_{S_r} f \cdot d(\omega_1, \dots, \omega_r) , \tag{19}$$

in which f is a function that is *not generally homogeneous* with respect to the λ that is defined by the equation:

$$f = \left(\sum_{i_1,\dots,i_r} q_{i_1,\dots,i_r} H_{q_{i_1,\dots,i_r}} \right) - H , \tag{20}$$

in which the q are determined by the system (18) as a function of the λ . One will no longer have quite the same problem anymore, because the value of the integral will not only depend upon the multiplicity S_r , but also upon its parametric representation as a function of $\omega_1, \dots, \omega_r$.

IV. Functions of hyperspaces. – Let us return to the expression (4) for δI . When S_r is an extremal, δI will reduce to the integral that is taken around the contour S_{r-1} of S_r . That integral can be written in the following way:

$$\delta I = \int_{\delta S_r} \left[\sum_{i_1,\dots,i_r} F_{\lambda_{i_1,\dots,i_r}} \frac{D(x_{i_1}, \dots, x_{i_r})}{D(\theta_1, \dots, \theta_r)} \right] d(\theta_1, \dots, \theta_r) ,$$

in which δS_r is the r-dimensional multiplicity that is generated by the infinitely-small displacement of S_{r-1}, which is a multiplicity whose coordinates can be represented as functions of the r parameters $\theta_1, \dots, \theta_r$. One sees that δI will be zero if one has:

$$\sum_{i_1,\dots,i_r} F_{\lambda_{i_1,\dots,i_r}} \frac{D(x_{i_1}, \dots, x_{i_r})}{D(\theta_1, \dots, \theta_r)} = 0$$

at every point of S_{r-1} .

In that case, we say that S_r is an extremal that is *transverse* to δS_r along S_{r-1} .

We shall now suppose, by extension of a known theory in the case of $r = 1$, that we can determine an extremal S_r by the following conditions:

1. It must pass through the arbitrarily-chosen $r - 1$-dimensional multiplicity Σ_{r-1} .

2. It must be transverse to a fixed multiplicity T_r .

From that, *one sees that every position of the multiplicity corresponds to a well-defined value of I,* $U_{\Sigma_{r-1}}$, whose variation will be given by the equation:

$$\delta U_{\Sigma_{r-1}} = \int_{\delta_1 S_r} \left[\sum_{i_1,\ldots,i_r} \frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})} \frac{D(x_{i_1},\ldots,x_{i_r})}{D(\theta_1,\ldots,\theta_r)} \right] d(\theta_1,\ldots,\theta_r)$$

when one sets:

$$\frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})} = F_{\lambda_{i_1,\ldots,i_r}} . \tag{20}$$

As a result, $U_{\Sigma_{r-1}}$ *is one of the functions that* VOLTERRA *studied under the name of functions of hyperspace,* and which I will call VOLTERRA functions or functions (V) of the multiplicity Σ_{r-1} . If one sets:

$$q_{i_1,\ldots,i_r} = \frac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})} \tag{21}$$

then one will see that this amounts to making the change of variables (7), and one can consequently state the following theorem:

If $x_1, \ldots, x_n$ *represent the coordinates of an* $r - 1$*-dimensional multiplicity and the quantities* $q_{i_1,\ldots,i_r}$ *represent the functional derivatives* $\dfrac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})}$ *of the function* $U_{\Sigma_{r-1}}$ *that was defined above then there will be a system of integrals of the equation* (E) *that coincides with the system that is composed of those quantities over* Σ_{r-1} .

The derivatives of the function U then verify a system of finite equations that is easy to form and that is the analogue of the JACOBI partial differential equation (which is obtained for $r = 1$). It is composed of equations (8) and (14), along with the equations that are obtained in the following manner: Let $\mu_{i_1,\ldots,i_r}$ denote the direction cosines of Σ_{r-1} relative to the axes $x_{i_1}, \ldots, x_{i_{r-1}}$. Equations (17), which verified on Σ_{r-1} , can be put into the form:

$$\ldots = \frac{\delta x_{i_1}\ \mu_{i_2,\ldots,i_r} - \delta x_{i_2}\ \mu_{i_1,i_2,\ldots,i_r} + \cdots}{H_{q_{i_1,\ldots,i_r}}} = \ldots$$

We have $C_n^r - 1$ equations in $n - 1$ unknowns $\frac{\delta x_2}{\delta x_1}, \ldots, \frac{\delta x_r}{\delta x_1}$, and upon eliminating those unknowns, they can yield equations of the form:

$$\varphi_i\,(x_1,\ldots,x_n,\ldots,\mu_{i_2,\ldots,i_r},\ldots,q_{i_1,\ldots,i_r},\ldots) = 0\ . \tag{22}$$

For example, in the case of $r = 2$, one will have:

$$H_{q_{ij}}\ \mu_k + H_{q_{jk}}\ \mu_i + H_{q_{ki}}\ \mu_j = 0 \qquad (i, j, k = 1, \ldots, n)\ .$$

By definition, *the derivatives* $\frac{\partial U}{\partial\,(x_{i_1},\ldots,x_{i_r})}$ *of* $U_{\Sigma_{r-1}}$ *are coupled with the coordinates* x_i *and the direction cosines* $\mu_{i_2,\ldots,i_r}$ *that relate to the point considered on* Σ_{r-1} *by equations* (8), (14), *and* (22). If one is content to note that the $\frac{\partial U}{\partial\,(x_{i_1},\ldots,x_{i_r})}$ satisfy equation (8) then one would not have reached an interesting result, because one knows that those derivatives are not determined uniquely (*) and that no matter what function (V) one considers, *one will always have to choose those derivatives in such a way that* (8) *is satisfied.*

V. Functions of degree one. – That result is no longer applicable in the case where the function (V) has degree one. Indeed, in that case, one agrees to take the derivatives of the function to be that one of the preceding systems that are functions of only the coordinates of the point at which one takes the derivatives. Moreover, one is *actually* stating a property of a function of degree one by saying that it satisfies the equation:

$$H\left(x_1,\ldots,x_r,\ldots,\frac{\partial U}{\partial\,(x_{i_1},\ldots,x_{i_r})},\ldots\right) = 0\ . \tag{8}$$

We then prove the following theorem:

If a function (V) *of degree one* $U_{\Sigma_{r-1}}$ *satisfies the partial functional differential equations* (8) *then it will verify the system* (E) *over the entire multiplicity* S_r , *such that one will have:*

(*) VOLTERRA, *loc. cit.*, pp. 160.

$$\left\{\begin{array}{l} \dfrac{D(x_{i_1},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)} = H_{q_{i_1,\ldots,i_r}}, \\ q_{i_1,\ldots,i_r} = \dfrac{\partial U}{\partial(x_{i_1},\ldots,x_{i_r})} \end{array}\right\} \qquad (i_1, \ldots, i_r = 1, \ldots, n)\,. \tag{23}$$

Indeed, by hypothesis, the q are functions of $x_1, \ldots, x_n$ such that one has (*):

$$A_{i_1,\ldots,i_{r+1}} \equiv \sum_{s=1}^{r+1}(-1)^{s+1}\frac{\partial}{\partial x_{i_s}} q_{i_1,\ldots,i_{s-1},i_{s+1},\ldots,i_{r+1}} = 0$$

identically.

Since equation (8) is verified for any $x_1, \ldots, x_n$, one will have:

$$\frac{\partial H}{\partial x_{i_1}} = 0 = H_{x_{i_1}} + \sum_{i_2,\ldots,i_{r+1}}\left[q_{i_2,\ldots,i_{r+1}}\frac{\partial q_{i_2,\ldots,i_{r+1}}}{\partial x_{i_1}} + \sum_s H_{q_{i_1,\ldots,i_{s-1},i_{s+1},\ldots,i_{r+1}}}\frac{\partial}{\partial x_{i_1}} q_{i_1,\ldots,i_{s-1},i_{s+1},\ldots,i_{r+1}}\right] \qquad (i_1 = 1, \ldots, n)$$

or

$$0 = H_{x_{i_1}} + \sum_{i_2,\ldots,i_{r+1}} H_{i_2,\ldots,i_{r+1}} A_{i_1,\ldots,i_{r+1}}$$
$$+ \sum_{i_2,\ldots,i_r}\left\{H_{q_{i_2,\ldots,i_{r+1}}}\frac{\partial}{\partial x_{i_1}} q_{i_1,\ldots,i_r} + (-1)^r\left[H_{q_{i_1,\ldots,i_{r-1},i_{r+1}}}\frac{\partial}{\partial x_{i_{r+1}}} q_{i_1,\ldots,i_r} + \cdots + H_{q_{i_1,\ldots,i_{r-1},i_n}}\frac{\partial}{\partial x_{i_n}} q_{i_1,\ldots,i_r}\right]\right\}.$$

By hypothesis, one can write these equations:

$$0 = \rho H_{x_{i_1}} + \sum_{i_2,\ldots,i_r}\left\{\frac{\partial}{\partial x_{i_1}} q_{i_1,\ldots,i_r}\frac{D(x_{i_1},x_{i_2},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)} + (-1)^r\left[\frac{\partial}{\partial x_{i_{r+1}}} q_{i_1,\ldots,i_r}\frac{D(x_{i_1},\ldots,x_{i_{r-1}},x_{i_{r+1}})}{D(\omega_1,\ldots,\omega_r)} + \cdots\right]\right\},$$

which are then equations (16):

$$-\rho H_{x_{i_1}} = \rho\sum_{i_2,\ldots,i_r}\frac{D(q_{i_1},\ldots,x_{i_2},\ldots,x_{i_r})}{D(\omega_1,\ldots,\omega_r)}\,.$$

Conversely, suppose that there exist functions $q_{i_1,\ldots,i_r}$ of $x_1, \ldots, x_r$ that verify the system (16), (17). Let us see if they can be considered to be the derivatives of a function of degree one, i.e., whether all of the quantities $A_{i_1,\ldots,i_{r+1}}$ are zero. We saw above that if the system (16), (17) is verified

(*) See the note that was cited before, vol. V, pp. 162.

then H will be a constant. Therefore, the equations $\frac{\partial H}{\partial x_i} = 0$ will be verified, and if we then recall the preceding calculations then we will see that we have:

$$\sum_{i_2,\ldots,i_{r+1}} H_{q_{i_2,\ldots,i_{r+1}}} A_{i_1,\ldots,i_{r+1}} = 0 \qquad (i_1 = 1, \ldots, n) .$$

We will then have n linear homogeneous equations in the C_n^{r+1} quantities $A_{i_1,\ldots,i_{r+1}}$. Those equations are indeed verified when the $A_{i_1,\ldots,i_{r+1}}$ are zero. However, the argument does not prove that this must necessarily be true. Nonetheless, *we can confirm that this is true when* $r = n - 1$, because the preceding system can be written in the form:

$$H_{q_{1,\ldots,s-1,s+1,\ldots,n}} A_{1,\ldots,n} = 0 \qquad (s = 1, \ldots, n)$$

in that case.

Since we have previously supposed that it is possible to change the variables:

$$\frac{\lambda_{1,\ldots,r}}{H_{q_{1,\ldots,r}}} = \ldots = \frac{\lambda_{i_1,\ldots,i_r}}{H_{q_{i_1,\ldots,i_r}}} = \ldots,$$

such that the $H_{q_{i_1,\ldots,i_r}}$ will not be all zero, we will indeed have:

$$0 = A_{i_1,\ldots,i_{r+1}} \equiv \frac{\partial}{\partial x_1} q_{2,\ldots,n} - \frac{\partial}{\partial x_2} q_{1,3,\ldots,n} + \cdots + (-1)^{n+1} \frac{\partial}{\partial x_n} q_{1,\ldots,n-1} .$$

VI. Generalization of Jacobi's theorem. – *Suppose that one has found a function of degree one* $U_{\Sigma_{r-1}}$ *that depends upon one arbitrary parameter and verifies the "partial differential equation"* (9) *for any a. I say that the value of* $\partial U / \partial a$ *is constant along any extremal that verifies equations* (23). Indeed, if Σ_{r-1} and Σ'_{r-1} are two multiplicities that are situated along the extremal S_r then one will have:

$$\frac{\partial}{\partial a} U_{\Sigma'_{r-1}} - \frac{\partial}{\partial a} U_{\Sigma_{r-1}} = \int_{S_r} \sum \frac{\partial}{\partial a} \left[\frac{\partial U}{\partial (x_{i_1}, \ldots, x_{i_r})} \right] d\,(x_{i_1}, \ldots, x_{i_r}) \ ,$$

or upon setting:

$$V_{\Sigma_{r+1}} = \frac{\partial}{\partial a} U_{\Sigma_{r-1}} :$$

$$V_{\Sigma'_{r+1}} - V_{\Sigma_{r+1}} = \int_{S_r} \left[\frac{\partial}{\partial a} \Sigma\,(q_{i_1,\ldots,i_r}, H_{q_{i_1,\ldots,i_r}}) \right] d\,(\omega_1, \ldots, \omega_r) \ .$$

One will then indeed have:

$$V_{\Sigma'_{r+1}} = V_{\Sigma_{r+1}} = \int_{S_r} \frac{\partial H}{\partial a} d(\omega_1, \dots, \omega_r) = 0 .$$

Now consider a function $U_{\Sigma_{r-1}}$ of degree one that depends upon C_n^r arbitrary parameters $a_{i_1,\dots,i_r}$. We say that it is a complete integral of the equation $H = 0$ if the elimination of the parameters a from the equations:

$$q_{i_1,\dots,i_r} = \frac{\partial U}{\partial(x_{i_1}, \dots, x_{i_r})}$$

(in which the right-hand sides are certain functions of $x_1, \dots, x_n$, and the $a_{i_1,\dots,i_r}$) leads to the single equation:

$$H(x_1, \dots, x_n, \dots, q_{i_1,\dots,i_r}; \dots) = 0 .$$

We can then state the following theorem, which is the direct generalization of **JACOBI**'s theorem:

If a function of degree one $U_{\Sigma_{r-1}}$ *is a compete integral of equation* (8) *that depends upon* C_n^r *parameters* $a_{i_1,\dots,i_r}$ *then the equations:*

$$\left.\begin{aligned} \frac{\partial}{\partial a_{i_1,\dots,i_r}} U_{\Sigma_{r-1}} &= b_{i_1,\dots,i_r}, \\ q_{i_1,\dots,i_r} &= \frac{\partial U}{\partial(x_{i_1}, \dots, x_{i_r})} \end{aligned}\right\} \tag{24}$$

will define a system of integrals of the canonical equations (E) *for any constants a and b that leave those equations compatible.*

In other words, the multiplicities S_r that are generated by the multiplicities Σ_{r-1} that give constant values to the $\frac{\partial}{\partial a} U_{\Sigma_{r-1}}$ are extremal multiplicities. Indeed, if $\frac{\partial}{\partial a_{i_1,\dots,i_r}} U_{\Sigma_{r-1}}$ is constant then upon annulling its first variation, one must have:

$$\sum_{j_1,\dots,j_r} \frac{\partial}{\partial a_{i_1,\dots,i_r}} q_{j_1,\dots,j_r} \, d(x_{j_1}, \dots, x_{j_r}) = 0 . \tag{25}$$

Now, since equation (8) is verified for any a, one will have:

$$\sum_{j_1,\dots,j_r} \frac{\partial}{\partial a_{i_1,\dots,i_r}} q_{j_1,\dots,j_r} \, H_{q_{j_1,\dots,j_r}} = 0 \,. \tag{26}$$

However, from the definition of a complete integral, one easily sees that one of the functional determinants of order $C_n^r - 1$ that are formed from the:

$$\frac{\partial}{\partial a_{i_1,\dots,i_r}} q_{j_1,\dots,j_r}$$

must be non-zero, whereas the determinant of order C_n^r is zero. Consequently, equations (25) and (26) determine the same series of ratios for the $H_{q_{j_1,\dots,j_r}}$ and the $d(x_{j_1},\dots,x_{j_r})$. As a result, one can write:

$$\frac{D(x_{j_1},\dots,x_{j_r})}{D(\omega_1,\dots,\omega_r)} = \rho \, H_{q_{j_1,\dots,j_r}} \,,$$

and then from the theorem that was proved on pp. 9, one will also have:

$$\sum_{j_1,\dots,j_r} \frac{D(q_{j_1,\dots,j_r}, x_{j_2},\dots,x_{j_r})}{D(\omega_1,\dots,\omega_r)} = -\,\rho \frac{\partial H}{\partial x_{j_1}} \,,$$

which proves the theorem.

Let us give an example. The equations:

$$\frac{P^2}{y^2} + \frac{Q^2}{z^2} + \frac{R^2}{x^2} = 1 \,,$$

$$\frac{d(y,R) - d(z,Q)}{\frac{R^2}{x^3}} = \frac{d(z,P) - d(x,R)}{\frac{P^2}{y^3}} = \frac{d(x,Q) - d(y,P)}{\frac{Q^2}{z^3}} = \frac{d(y,z)}{\frac{P}{y^2}} = \frac{d(z,x)}{\frac{Q}{z^2}} = \frac{d(x,y)}{\frac{R}{x^2}}$$

are verified upon setting:

$$P = \frac{\partial U}{\partial(y,z)} \,, \quad Q = \frac{\partial U}{\partial(z,x)} \,, \quad R = \frac{\partial U}{\partial(x,y)} \,,$$

with

$$U = \int_L [z^2 \, a c - 2 \, x \, y \, b \, c] \, dx + y^2 \sqrt{1 - a^2 c^2 - b^2 c^2} \, dz$$

on any surface that is generated by the line L such that one has:

$$\frac{\partial U}{\partial a} = \text{const.}, \qquad \frac{\partial U}{\partial b} = \text{const.}, \qquad \frac{\partial U}{\partial c} = \text{const.}$$

We see that the difference between the case of $r = 1$ that JACOBI considered and the general case is manifested in the solution of the canonical equations (E). Indeed, the complete integral that JACOBI employed in order to solve them is no longer a function of n variables, *but a function of hyperspace*. That is one of the many situations in which the consideration of those functions is imposed by the nature of the problem.

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