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On the equations of equilibrium of a solid, elastic body

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On knows the fundamental role that is played by the particular integral $1 / r$ in the equation $\Delta u = 0$ in potential theory. One likewise knows some particular integrals of the equations of equilibrium of an isotropic elastic body that play a role in that part of the theory of elasticity that is entirely analogous to the one that is played by the function $1 / r$ in potential theory. The aforementioned particular integrals have in common that they are homogeneous of degree – 1, and that they have just one real singular point at a finite distance.

It is natural to propose the question of whether there exist particular integrals of the equations of equilibrium for an arbitrary crystalline body that enjoy the same properties as the function $1 / r$.

I hope to give a satisfactory answer to that question with the following results.

In the first chapter, I shall give a formula that represents all analytic, homogeneous integrals of degree – 1 of a partial differential equation with constant coefficients. Upon giving convenient values to the arbitrary elements in that formula, one will find that the differential equations of the form $f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) u = 0$, where f is a definite form, always admit a certain number of integrals that are regular for any real system of variables, except for the system $x = y = z = 0$.

With the aid of these integrals, in the second chapter, I shall deduce a formula from the known theorem of BETTI that permits one to express the components of the deformation inside of a body when one is given those components on the surface, as well as the forces that act upon the surface.

The aforementioned formula is composed of three types of integrals that are perfectly analogous to the integrals in potential theory that represent the potentials of an extended, three-dimensional mass, a simple layer, and a double layer. One will find a study of these integrals in the third chapter.

In the same chapter, I shall show, moreover, that one can express any homogeneous integral of negative integer degree of the equations of equilibrium that is regular for real values as a linear function of the derivatives of the regular integrals of degree – 1. I then show, in turn, what the physical significance of the homogeneous integrals of degree – 1 is, and I solve the problem of equilibrium for an infinitely large elastic medium that is undeformed at infinity.

CHAPTER I

§ 1. Homogeneous solutions of degree – 1 of linear differential equations with constant coefficients.

Homogeneous functions of degree – 1 that satisfy a homogeneous linear differential equation with constant coefficients:

$$f\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) u = 0 \tag{1}$$

can be obtained from the particular integral:

$$u = \frac{1}{\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3}$$

upon forming the expression:

$$u = \int \frac{\psi(\xi, \eta)\, d\xi}{f_2(\xi, \eta)(\xi x_1 + \eta x_2 + x_3)}. \tag{2}$$

In that formula, the variables are coupled to each other by the relation:

$$f(\xi, \eta) = f(\xi, \eta, 1) = 0,$$

$f_2(\xi, \eta)$ denotes the derivative $\partial f / \partial \eta$, and $\psi(\xi, \eta)$ is an integer rational function of η of degree $n - 1$, where n is the degree of f, with respect to which ξ will be an analytic function.

Having said that, we make the following hypotheses on $f(\xi_1, \xi_2, \xi_3)$:

1. The coefficient of ξ_2^n is non-zero.
2. The factors of f (if it is reducible) are all unequal.

By virtue of hypothesis 1, we can write:

$$f(\xi, \eta) = f_0\, \eta^n + f_1\, \eta^{n-1} + \dots + f_n,$$

in which f_0 is certainly non-zero. By virtue of the second hypothesis, the roots $\eta_1, \ldots, \eta_n$ of the equation $f(\xi, \eta)$ = will be unequal, in general.

If we now substitute these roots for η successively in the expression (2) and take the sum of the results then we will have a symmetric function of the roots η, and one will obtain its expression as a function of ξ in just the following manner:

Decompose the rational function of η:

$$\frac{\psi(\xi,\eta)}{f_2(\xi,\eta)(\xi x_1+\eta x_2+x_3)}$$

into simple fractions.

Provided that the variable ξ has a value that makes the roots η_ν of the equation $f(\xi, \eta) = 0$ are all unequal, one will obtain:

$$\frac{\psi(\xi,\eta)}{f_2(\xi,\eta)(\xi x_1+\eta x_2+x_3)} = \sum_{\nu=1}^{n}\frac{\psi(\xi,\eta_\nu)}{f_2(\xi,\eta_\nu)(\xi x_1+\eta_\nu x_2+x_3)}\cdot\frac{1}{\eta-\eta_\nu}, \tag{3}$$

where η_0 is determined by the equation $\xi x_1 + \eta_0 x_2 + x_3 = 0$.

If we now develop the two sides of equation (3) in negative powers of η and write that the coefficients of $1 / \eta$ are equal then we will get the desired expression:

$$\sum_{\nu=1}^{n}\frac{\psi(\xi,\eta_\nu)}{f_2(\xi,\eta_\nu)(\xi x_1+\eta_\nu x_2+x_3)} = -\frac{\psi(\xi,\eta_0)}{x_2 f(\xi,\eta_0)}. \tag{4}$$

Now set:

$$\psi(\xi, \eta) = k_1\,\eta^{n-1} + k_2\,\eta^{n-2} + \ldots + k_n,$$

with the following values for the coefficients k:

$$\begin{aligned} k_1 &= f_0\psi_1, \\ k_2 &= f_1\psi_1 + f_0\psi_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ k_n &= f_{n-1}\psi_1 + f_{n-2}\psi_2 + \ldots + f_0\psi_n, \end{aligned} \tag{5}$$

in which the ψ denote indeterminate analytic functions.

Upon now forming the definite integral:

$$u = -\frac{1}{2\pi i}\int_C \frac{\psi(\xi,\eta_0)}{x_2 f(\xi,\eta_0)}\,d\xi, \tag{6}$$

where the closed contour C must contain no other singular points that the roots of the equation $f(\xi, \eta_0) = 0$, one will obtain a homogeneous integral of degree – 1 of equation (1), which would obviously follow from equation (4). We shall prove that one can choose the indeterminate functions ψ_ν in such a manner that the first n coefficients of the development of u in increasing powers of x_2 will be equal to the n corresponding coefficients in the development of a homogeneous function of degree – 1.

Upon developing the function $-\dfrac{\psi(\xi,\eta_0)}{x_2 f(\xi,\eta_0)}$ in decreasing powers of η, one will find:

(7)
$$\frac{\psi(\xi,\eta_0)}{x_2 f(\xi,\eta_0)} = -\frac{1}{x_2}\left(\frac{\psi_1}{\eta}+\frac{\psi_2}{\eta^2}+\cdots+\frac{\psi_n}{\eta^n}+\sum_{\nu=1}^{\infty}\frac{\psi_{n+\nu}}{\eta^{n+\nu}}\right).$$

This development converges, provided that one gives η a value that satisfies the inequality $|\eta| > R$, in which R denotes the absolute value of the largest root in η of the equation $f(\xi, \eta) = 0$.

Let $\xi = -x_2 / x_1$ be a regular point for the functions $\psi_1(x), \ldots, \psi_n(x)$, and choose the integration contour to be a circle C that has the point $-x_2 / x_1$ for its center and a radius ρ that is small enough that C contains no other singular point of the functions ψ. Suppose that this condition is verified if $\rho < \delta$, and set $\xi = -x_2 / x_1 + \rho\, e^{\theta i}$, from which, one will get $\eta_0 = -x_2 / x_1\, \rho\, e^{\theta i}$.

Because the roots of the equation $f(\xi, \eta_0) = 0$ all become equal to $-x_2 / x_1$ for $x_2 = 0$, one can choose x_2 to be small enough that the circle C contains all of the roots, and at the same time, that the modulus of η_0 is greater than R.

If the preceding conditions are verified then one can integrate both sides of equation (7), which will give the result:

(8)
$$\begin{aligned} u &= -\frac{1}{2\pi i}\int_C \frac{\psi(\xi,\eta_0)}{x_2 f(\xi,\eta_0)}\,d\xi \\ &= \frac{1}{x_1}\psi_1\left(-\frac{x_3}{x_1}\right)-\frac{x_2}{x_1^2}\psi_2'\left(-\frac{x_3}{x_1}\right)+\cdots+(-1)^{n-1}\frac{x_2^{n-1}}{\underline{|n-1}\,x_1^n}\psi_n^{(n-1)}\left(-\frac{x_3}{x_1}\right)+\ldots \end{aligned}$$

However, the coefficient $\dfrac{(-1)^\nu}{\underline{|\nu}}\dfrac{1}{x_1^{\nu+1}}\psi_{\nu+1}^{(\nu)}\left(-\dfrac{x_3}{x_1}\right)$ of x_2^ν in this is a homogeneous function of the variables x_3 and x_1 of degree $-(\nu + 1)$, that one can choose arbitrarily if $\nu \leq n - 1$.

Therefore, formula (6) will indeed give us all homogeneous integrals of degree – 1 of equation (1) that are developable in increasing powers of x_2 .

Since one can always perform a linear change of variables in such a manner that a function of the type considered here will be regular for $x_2 = 0$ and that hypothesis 1 will be verified at the same time, one can consider the problem of finding the analytic, homogeneous integrals of degree – 1 of equation (1) as having been solved. Nonetheless, one restriction still remains, namely, hypothesis 2. However, it is easy to see that this restriction has no importance, because the expression (6), which satisfies equation (1) identically, does not cease to satisfy that equation if the function $f(\xi_1, \xi_2, \xi_3)$ has, perchance, a multiple factor. Moreover, the development (8) will have the same form once more in that case, and the coefficients of the first n terms will be arbitrary functions.

One must then observe that the integrals, whose expressions we have given in the form of a definite integral, can be presented in a form that is devoid of any integration sign, because one can easily perform the integration in formula (8) by using equation (4), and one will find the following sum of residues as the expression for u:

$$u = \sum_{\nu=1}^{n} \frac{\psi(\xi_\nu, \eta_\nu)}{x_1 f_2(\xi_\nu, \eta_\nu) - x_2 f_1(\xi_\nu, \eta_\nu)}, \tag{9}$$

where ξ_ν, η_ν denote the coordinates of the point of intersection of the lines:

$$f(\xi, \eta) = 0, \qquad \xi x_1 + \eta x_2 + x_3 = 0,$$

and f_1 and f_2 are defined by the formulas:

$$f_1 = \frac{\partial f}{\partial \xi}, \qquad f_2 = \frac{\partial f}{\partial \eta}.$$

Upon employing homogeneous coordinates ξ_1, ξ_2, ξ_3 in place of ξ and η, one can give u a more symmetric form. Set $f_\nu = \partial f / \partial \xi_\nu$, so one has:

$$n\, f(\xi_1, \xi_2, \xi_3) = f_1\, \xi_1 + f_2\, \xi_2 + f_3\, \xi_3 = 0.$$

Since we have:

$$f_1\, \xi_1 + f_2\, \xi_2 + f_3\, \xi_3 = 0,$$

we deduce, upon introducing three constants k_1, k_2, k_3 :

$$\frac{\xi_1}{x_2 f_3 - x_3 f_2} = \frac{\xi_2}{x_3 f_1 - x_1 f_3} = \frac{\xi_3}{x_1 f_2 - x_2 f_1} = \frac{k_1\xi_1 + k_2\xi_2 + k_3\xi_3}{\begin{vmatrix} k_1 & k_2 & k_3 \\ x_1 & x_2 & x_3 \\ f_1 & f_2 & f_3 \end{vmatrix}}.$$

In order for the last expression to not be illusory, it is necessary that the k must satisfy the inequality:

$$k_1\, \xi_1 + k_2\, \xi_2 + k_3\, \xi_3 \neq 0.$$

By introducing the expressions $\xi = \xi_1 / \xi_3$ and $\eta = \xi_2 / \xi_3$, one will find that:

$$\frac{\psi(\xi, \eta)}{x_1 f_2 - x_2 f_1} = \frac{\xi_3\, \xi_3^{n-2}\, \psi\left(\dfrac{\xi_1}{\xi_3}, \dfrac{\xi_2}{\xi_3}\right)}{x_1 f_2(\xi_1, \xi_2, \xi_3) - x_2 f_1(\xi_1, \xi_2, \xi_3)}$$

$$= \frac{\psi(\xi_1, \xi_2, \xi_3)(k_1\xi_1 + k_2\xi_2 + k_3\xi_3)}{\begin{vmatrix} k_1 & k_2 & k_3 \\ x_1 & x_2 & x_3 \\ f_1 & f_2 & f_3 \end{vmatrix}},$$

where one lets $\psi(\xi_1, \xi_2, \xi_3)$ denote the homogeneous function $\xi_3^{n-2}\psi\left(\frac{\xi_1}{\xi_3}, \frac{\xi_2}{\xi_3}\right)$ of degree $n - 2$. The expression for u in terms of homogeneous coordinates thus becomes:

$$u = \sum_{\nu=1}^{n} \frac{\psi(\xi_1^\nu, \xi_2^\nu, \xi_3^\nu)(k_1\xi_1^\nu + k_2\xi_2^\nu + k_3\xi_3^\nu)}{\begin{vmatrix} k_1 & k_2 & k_3 \\ x_1 & x_2 & x_3 \\ f_1^\nu & f_2^\nu & f_3^\nu \end{vmatrix}}, \tag{10}$$

in which ξ_1^ν, ξ_2^ν, ξ_3^ν are the coordinates of the points of intersection of the lines:

$$f(\xi_1, \xi_2, \xi_3) = 0, \qquad x_1\,\xi_1 + x_2\,\xi_2 + x_3\,\xi_3 = 0,$$

and $f_\alpha^\nu = f_\alpha^\nu(\xi_1^\nu, \xi_2^\nu, \xi_3^\nu)$.

In order to obtain a symmetric expression for u in the form of a definite integral, one agrees to express the variable ξ in formula (6) by an auxiliary variable s in the following manner:

Define ξ_1, ξ_2, and ξ_3 as functions of s by the equation:

$$x_1\,\xi_1 + x_2\,\xi_2 + x_3\,\xi_3 = \begin{vmatrix} k_1 & k_2 & k_3 \\ x_1 & x_2 & x_3 \\ a_1 s + b_1 & a_2 s + b_2 & a_3 s + b_3 \end{vmatrix} \tag{11}$$

that must verified for any values of the quantities k_ν.

We suppose that the a_ν and b_ν are arbitrary real constants that are, however, independent of the k_ν.

For $k_\nu = x_\nu$, formula (11) gives us:

$$x_1\,\xi_1 + x_2\,\xi_2 + x_3\,\xi_3 = 0.$$

The expression $\xi = \xi_1 / \xi_3$ gives $\eta_0 = \xi_2 / \xi_3$ and:

$$d\xi = \frac{\xi_3\,d\xi_1 - \xi_1\,d\xi_2}{\xi_3^2},$$

which is an expression that will take the form:

$$d\xi = -\frac{x_2}{\xi_3^2}\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} ds,$$

after an easy calculation, or, if we denote the determinant by (x, a, b):

$$d\xi = -\frac{x_2}{\xi_3^2}(x, a, b)\, ds.$$

Upon introducing these expressions for ξ, η_0, and $d\xi$ into formula (6), one will finally find that:

(12) $$u = \frac{1}{2\pi i}\int_C \frac{\psi(\xi_1, \xi_2, \xi_3)(a, b, x)}{f(\xi_1, \xi_2, \xi_3)} ds\,.$$

We make the following remarks in regard to this formula: The contour C must contain at least one of the roots of the equation $f(\xi_1, \xi_2, \xi_3) = 0$, since otherwise u would be zero.

Suppose that one has fixed an integration contour C. It is then clear that one can vary the arbitrary constants a_ν and b_ν in a continuous manner without that having any influence on the value of u, provided that none of the zeroes of $f(\xi_1, \xi_2, \xi_3)$ cross the contour C during that variation.

It is clear that one can also vary the x_ν under the same condition without the integral (12) ceasing to represent the same analytic function. In particular, suppose that the contour C is the axis of real s and that f is a definite form. Two systems of values of the constants a_ν, b_ν will then give the same value to u if one can pass from the one system to the other by a continuous variation without encountering the system for which there are real roots of the equation $f = 0$. However, if there is a real root of the equation $f = 0$ at s then the corresponding values of ξ_1, ξ_2, ξ_3 will be $\xi_1 = \xi_2 = \xi_3 = 0$; in this case, if we set $k_\nu = a_\nu$ then equation (11) will give us:

$$a_1\,\xi_1 + a_2\,\xi_2 + a_3\,\xi_3 = (a, b, x) = 0.$$

The condition $(a, b, x) = 0$ is then necessary in order for $f = 0$ to have a real root at s.

Upon supposing that $\psi(-x_1, -x_2, -x_3) = \psi(x_1, x_2, x_3)$, moreover, we will then seek what the value of $u(x_1, x_2, x_3)$ will be when one changes the signs of the variables x_1, x_2, x_3. Since one cannot pass from the point (x_1, x_2, x_3) to the point $(-x_1, -x_2, -x_3)$ without encountering values for which one has $(a\text{¸}\, b, x) = 0$, it is necessary to vary the quantities a_ν, b_ν at the same time as the x_ν. Suppose, for example, that the values of the quantities av, bv are such that (a, b, v) preserves the value 1 when (x_1, x_2, x_3) passes from the point (x_1, x_2, x_3) to the point $(-x_1, -x_2, -x_3)$. Since the functions ψ and f do not change signs then, one will have:

$$u(-x_1, -x_2, -x_3) = u(x_1, x_2, x_3).$$

§ 2. The case in which $f(\xi_1, \xi_2, \xi_3)$ is a definite form.

Suppose that the form $f(\xi_1, \xi_2, \xi_3)$ is a definite form; i.e., that the equation:

$$f(\xi_1, \xi_2, \xi_3) = 0$$

has no real solution other than the obvious solution:

$$\xi_1 = \xi_2 = \xi_3 = 0 .$$

We shall prove that in this case, a certain number of the homogeneous integrals of degree – 1 enjoy the property of being holomorphic in the neighborhood of any real point, except for only the origin.

The functions that are considered are homogeneous, so it will suffice to prove that our functions are regular for all real values that satisfy the condition:

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

Upon giving the values a_1, a_2, a_3 to the quantities k_1, k_2, k_3, one deduces from (11) that:

$$a_1\xi_1 + a_2\xi_2 + a_3\xi_3 = (a, b, x),$$

from which one concludes that the distance from the point ξ_1, ξ_2, ξ_3 to the origin is never less than:

$$r = \frac{|(a,b,x)|}{\sqrt{a_1^2 + a_2^2 + a_3^2}} .$$

Suppose (as is always possible) that the numbers a_ν, b_ν are chosen in such a manner that r is greater than a given non-zero quantity – say, d. If we let m denote the minimum of $f(\xi_1, \xi_2, \xi_3)$ for the real values that satisfy the equation:

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$$

then we can affirm that the minimum of $f(\xi_1, \xi_2, \xi_3)$ for the real values of s is not less than $\mu\, d^n$.

The equation $f(\xi_1, \xi_2, \xi_3) = 0$ then admits no real root at s. Moreover, since the coefficient of s^n in that equation is:

$$f\left(\begin{vmatrix} x_2 & x_3 \\ a_2 & a_3 \end{vmatrix}, \begin{vmatrix} x_3 & x_1 \\ a_3 & a_1 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ a_1 & a_2 \end{vmatrix}\right),$$

one can always choose a_1, a_2, a_3 in such a manner that this coefficient will have an absolute value that is greater than a given positive quantity – say, A. Since the other coefficients are always finite, it is clear that one can describe a circle C whose center is at the point $s = 0$ and whose radius r is independent of x_1, x_2, x_3 and large enough that all of the roots of the equation $f(\xi_1, \xi_2, \xi_3) = 0$ at s are interior to C.

Now take the integration contour in formula (12) to be a semi-circle C_1 with radius ρ_1 that is larger than ρ and has its diameter along the real s axis and its center at the origin. I then say that the function $u(x_1, x_2, x_3)$ that is defined by the equation:

$$u = \int_{C_1} \frac{\psi(\xi_1, \xi_2, \xi_3)(a, b, x)}{f(\xi_1, \xi_2, \xi_3)} ds,$$

where ψ denotes an integer rational function that is homogeneous of degree $n - 2$, has no real singularities.

One now sees that the absolute value of $f(\xi_1, \xi_2, \xi_3)$ will not go below the quantity $A(\rho_1 - \rho)^n$ when s traverses the curvilinear part of C_1. Let m be the smaller of the numbers $\mu\, d^n$ and $A(\rho_1 - \rho)^n$; m is the a lower limit to the absolute values that $f(\xi_1, \xi_2, \xi_3)$ takes when s describes the contour C_1 . Having said that, one can find two numbers η and $m_1 < m$ in such a manner that the inequality:

$$|\, f(x_1 + h_1, x_2 + h_2, x_3 + h_3, s) - f(x_1, x_2, x_3, s)\,| \leq m_1$$

is verified for all h_v that satisfy the inequalities:

$$(13) \qquad |\, h_v\,| < h \qquad (v = 1, 2, 3).$$

In the preceding inequality, one has denoted $f(\xi_1, \xi_2, \xi_3)$ by $f(\xi_1, \xi_2, \xi_3, s)$.

It is now easy to see that the development of $\dfrac{(a, b, x)\psi}{f}$ in increasing powers of h_v converges for all h_v that satisfy inequalities (13). Set:

$$(14) \qquad \frac{(a, b, x)\psi}{f} = \sum_{\lambda_1\lambda_2\lambda_3} \Phi_{\lambda_1\lambda_2\lambda_3} h_1^{\lambda_1} h_2^{\lambda_2} h_3^{\lambda_2},$$

and let G be an upper limit of the values of $(a, b, x)\ \psi$ for the values of the variables considered. We have shown that the absolute value of $f(x_1 + h_1, x_2 + h_2, x_3 + h_3, s)$ is not less than $m - m_1$, and in turn, we find that:

$$\left|\Phi_{\lambda_1\lambda_2\lambda_3}\right| < \frac{G}{m - m_1} \frac{1}{h^{\lambda_1+\lambda_2+\lambda_3}}.$$

Since the development (14) is then uniformly convergent, one has the right to write:

$$(15) \qquad u = \int_{C_1} \frac{(a, b, x)\psi}{f} ds = \sum_{\lambda_1\lambda_2\lambda_3} h_1^{\lambda_1} h_2^{\lambda_2} h_3^{\lambda_2} \int_{C_1} \Phi_{\lambda_1\lambda_2\lambda_3} ds.$$

It then follows that the function u will be developable in the neighborhood of an arbitrary real point x_1, x_2, x_3 that satisfies the equation $x_1^2 + x_2^2 + x_3^2 = 1$ and that this development will converge for all h_ν that are less than h. Let that development be:

$$u = \sum_{\lambda_1\lambda_2\lambda_3} u_{\lambda_1\lambda_2\lambda_3} h_1^{\lambda_1} h_2^{\lambda_2} h_3^{\lambda_2} ,$$

so the development of u around a point x_1, x_2, x_3 that satisfies the equation $x_1^2 + x_2^2 + x_3^2 = r^2$ will be written:

$$u = \sum_{\lambda_1\lambda_2\lambda_3} \frac{u_{\lambda_1\lambda_2\lambda_3}}{r^{\lambda_1+\lambda_2+\lambda_3}} h_1^{\lambda_1} h_2^{\lambda_2} h_3^{\lambda_2} , \tag{16}$$

and will converge, in turn, if the h_ν satisfy the inequality:

$$| h_\nu | < h \sqrt{x_1^2 + x_2^2 + x_3^2} , \tag{17}$$

and, *a fortiori*, if $\sqrt{h_1^2 + h_2^2 + h_3^2} < h \sqrt{x_1^2 + x_2^2 + x_3^2}$.

It is important to observe that the development (16) is also uniformly convergent if one considers it to be a function of the real variables x_1, x_2, x_3 that is subject to the condition:

$$\frac{1}{h}\sqrt{h_1^2 + h_2^2 + h_3^2} < \sqrt{x_1^2 + x_2^2 + x_3^2} ,$$

because in this case again each term in the development of u will be lower in absolute value than the corresponding term of a convergent series whose terms are independent of the terms in the development of u.

§ 3. Application to a system of differential equations.

In what follows, we will have a particular need for integrals that are homogeneous of degree – 1 of two systems of differential equations, namely:

$$\sum_{\mu=1}^{3} \Delta_{\mu\lambda} u_\mu = 0, \tag{18a}$$

$$\sum_{\lambda=1}^{3} \Delta_{\lambda\mu} v_\lambda = 0, \tag{18b}$$

in which the $\Delta_{\lambda\mu}$ denote the symbols of the operation of the form:

$$\Delta_{\lambda\mu} = \sum_{\alpha\beta} \binom{\lambda\mu}{\alpha\beta} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} \qquad (\alpha, \beta = 1, 2, 3),$$

in which the $\binom{\lambda\mu}{\alpha\beta}$ denote constant coefficients.

We shall get the desired integrals in the following manner: If we eliminate two of the unknowns from equations (18*a*) or (18*b*) then we will get a differential equation that can be written in the symbolic form:

$$\begin{vmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{vmatrix} v = 0.$$

That differential equation will be linear, homogeneous of sixth order, and have constant coefficients. Let f denote the function that one obtains by replacing the symbols of the operations $\partial / \partial x_1$, $\partial / \partial x_2$, $\partial / \partial x_3$ with the variables ξ_1, ξ_2, ξ_3 .

From the preceding, the desired integrals will be represented by the formulas:

$$(19) \qquad v_a = \frac{1}{\pi}\int_C \frac{(a,b,x)\psi_a}{f} ds = \frac{1}{\pi}\sum_{\nu=1}^{3} \int_C \frac{\psi_a(\xi,\eta_\nu)\,d\xi}{f_2(\xi,\eta_\nu)(\xi x_1 + \eta_\nu x_2 + x_3)},$$

which then comes down to the determination of the function ψ.

Upon introducing the expressions (19) into equations (18*b*), one will find that:

$$\sum_{\nu=1}^{3} \int_C \frac{\Delta_{1\mu}^{\nu}\psi_1 + \Delta_{2\mu}^{\nu}\psi_2 + \Delta_{3\mu}^{\nu}\psi_3}{f_2(\xi,\eta_\nu)(\xi x_1 + \eta_\nu x_2 + x_3)} d\xi = 0 \qquad (\mu = 1, 2, 3),$$

in which the $\Delta_{\lambda\mu}^{\nu}$ denote the functions that one obtains by replacing $\partial / \partial x_1$, $\partial / \partial x_2$, $\partial / \partial x_3$ with ξ, η_ν, 1, respectively, in the $\Delta_{\lambda\mu}$.

One sees that one can satisfy the preceding equations by taking the ψ to be functions of degree four that depend upon three arbitrary constants and are defined by the formulas:

$$\psi_1 = \begin{vmatrix} k_1 & \Delta_{21} & \Delta_{31} \\ k_2 & \Delta_{22} & \Delta_{32} \\ k_3 & \Delta_{23} & \Delta_{33} \end{vmatrix}, \qquad \psi_2 = \begin{vmatrix} \Delta_{11} & k_1 & \Delta_{31} \\ \Delta_{12} & k_2 & \Delta_{32} \\ \Delta_{13} & k_3 & \Delta_{33} \end{vmatrix}, \qquad \psi_3 = \begin{vmatrix} \Delta_{11} & \Delta_{21} & k_1 \\ \Delta_{12} & \Delta_{22} & k_2 \\ \Delta_{13} & \Delta_{23} & k_3 \end{vmatrix}.$$

Finally, take the integration contour C to be the semi-circle that was defined in the preceding number, so formulas (19) will represent integrals of the system (18*b*) whose only real singular points at a finite distance are the point $x_1 = x_2 = x_3 = 0$. Let β_1, β_2, β_3 denote the integrals thus obtained. One will now get the analogous integrals of the system (18*a*) immediately by exchanging the indices of the Δ in the expressions for the ψ between them. Call these integrals α_1, α_2, α_3 . It results from formula (10), § 1 that these integrals α and β will be algebraic functions.

CHAPTER II.

Green's method.

§ 1. Proof of a fundamental theorem.

Consider an arbitrary elastic body. Let u_1, u_2, u_3 be the components of the displacement of a point of a body, and let x_1, x_2, x_3 be the rectangular coordinates of the same point in the natural state. One then knows that the potential for the interior forces can be expressed with the aid of a definite quadratic form in the six variables:

$$\delta_{\nu\nu} = \frac{\partial u_\nu}{\partial x_\nu}, \qquad \delta_{\lambda\mu} = \frac{\partial u_\lambda}{\partial x_\mu} + \frac{\partial u_\mu}{\partial x_\lambda}, \qquad\qquad (\lambda, \mu, \nu = 1, 2, 3).$$

Let f be that form, and let dS be the volume element S of the body considered, so the aforementioned potential is equal to the integral $\int f\, dS$ when it is extended over the volume S.

Upon applying the principle of virtual velocities, one imagines another deformation whose components are v_1, v_2, v_3, and one considers the integral $\int \Delta\, dS$, where Δ is the bilinear form:

$$\Delta = \frac{\partial f}{\partial \delta_{11}}\frac{\partial v_1}{\partial x_1} + \frac{\partial f}{\partial \delta_{22}}\frac{\partial v_2}{\partial x_2} + \frac{\partial f}{\partial \delta_{33}}\frac{\partial v_3}{\partial x_3}$$
$$+ \frac{\partial f}{\partial \delta_{23}}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right) + \frac{\partial f}{\partial \delta_{31}}\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right) + \frac{\partial f}{\partial \delta_{12}}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right).$$

A consideration of the integral $\int \Delta\, dS$ will lead us to the fundamental theorem whose proof defines the objective of this paragraph.

I first observe that the form Δ is far from being the most general bilinear form that one can define with the first derivatives of the functions u and v. Meanwhile, since the theorems that I intend to prove still persist in the general case, I suppose that Δ is an arbitrary bilinear form of the first derivatives of the functions u_1, u_2, u_3, v_1, v_2, v_3 . To abbreviate, set:

$$u_{\mu\beta} = \frac{\partial u_\mu}{\partial x_\beta}, \qquad v_{\lambda\alpha} = \frac{\partial v_\lambda}{\partial x_\alpha}.$$

We express the form Δ by the formula:

(2) $$\Delta = \sum_{\lambda\mu\alpha\beta} \binom{\lambda\mu}{\alpha\beta} v_{\lambda\alpha} u_{\mu\beta} \qquad (\lambda, \mu, \alpha, \beta = 1, 2, 3),$$

in which the symbols $\binom{\lambda\mu}{\alpha\beta}$ denote constants, and each of the indices takes on the values 1, 2, 3, independently of the other ones.

We suppose that the six functions u_1, u_2, u_3, v_1, v_2, v_3 and their derivatives of the first two orders are continuous functions in a certain real domain D.

Introduce the quantities $T_{\lambda\alpha}$ and $\mathfrak{T}_{\mu\beta}$ that are defined by the formulas:

$$T_{\lambda\alpha} = \frac{\partial\Delta}{\partial v_{\lambda\alpha}}, \qquad \mathfrak{T}_{\mu\beta} = \frac{\partial\Delta}{\partial u_{\mu\beta}}.$$

Take a volume S inside the domain D that is bounded by a surface ω that possesses a well-defined tangent plane at every point. Let $d\omega$ be the element of ω.

One now deduces, in a well-known manner, two expressions for the integral $\int_S \Delta\, dS$ from the identities:

$$\Delta = \sum_{\lambda\alpha} T_{\lambda\alpha} v_{\lambda\alpha} = \sum_{\mu\beta} \mathfrak{T}_{\mu\beta} u_{\mu\beta},$$

namely:

(3) $$\int_S \Delta\, dS = \int_\omega \sum_\lambda v_\lambda \sum_\alpha T_{\lambda\alpha} \cos(nx_\alpha)\, d\omega - \int_S \sum_\lambda v_\lambda \sum_\alpha \frac{\partial T_{\lambda\alpha}}{\partial x_\alpha} dS$$

and

$$\int_S \Delta\, dS = \int_\omega \sum_\mu u_\mu \sum_\beta \mathfrak{T}_{\lambda\alpha} \cos(nx_\beta)\, d\omega - \int_S \sum_\mu u_\mu \sum_\beta \frac{\partial \mathfrak{T}_{\mu\beta}}{\partial x_\beta} dS,$$

in which n denotes the external normal of the surface ω and $\cos(n\, x_\alpha)$ ($\alpha = 1, 2, 3$) denote its direction cosines.

One derives the following expressions for $T_{\lambda\alpha}$ and $\mathfrak{T}_{\mu\beta}$ from equation (1):

(4) $$\begin{aligned} T_{\lambda\alpha} &= \sum_{\mu\beta} \binom{\lambda\mu}{\alpha\beta} u_{\mu\beta} = \sum_{\mu=1}^{3} \left[\binom{\lambda\mu}{\alpha 1} \frac{\partial}{\partial x_1} + \binom{\lambda\mu}{\alpha 2} \frac{\partial}{\partial x_2} + \binom{\lambda\mu}{\alpha 3} \frac{\partial}{\partial x_3} \right] u_\mu, \\ \mathfrak{T}_{\mu\beta} &= \sum_{\lambda\alpha} \binom{\lambda\mu}{\alpha\beta} v_{\lambda\alpha} = \sum_{\mu=1}^{3} \left[\binom{\lambda\mu}{1\beta} \frac{\partial}{\partial x_1} + \binom{\lambda\mu}{2\beta} \frac{\partial}{\partial x_2} + \binom{\lambda\mu}{3\beta} \frac{\partial}{\partial x_3} \right] v_\lambda, \end{aligned}$$

so one has:

(5) $$\sum_{\alpha=1}^{3} \frac{\partial T_{\lambda\alpha}}{\partial x_\alpha} = \sum_{\alpha\beta\mu} \binom{\lambda\mu}{\alpha\beta} \frac{\partial^2 u_\mu}{\partial x_\alpha\, \partial x_\beta},$$

$$\sum_{\beta=1}^{3}\frac{\partial \mathfrak{T}_{\mu\beta}}{\partial x_\beta} = \sum_{\alpha\beta\lambda}\binom{\lambda\mu}{\alpha\beta}\frac{\partial^2 v_\lambda}{\partial x_\alpha\,\partial x_\beta}.$$

If one now introduces the symbols of the operations:

(6)
$$\Delta_{\lambda\mu} = \sum_{\alpha\beta}\binom{\lambda\mu}{\alpha\beta}\frac{\partial^2}{\partial x_\alpha\,\partial x_\beta},$$

(7)
$$\Delta_{\lambda\mu}^{\alpha} = \binom{\lambda\mu}{\alpha 1}\frac{\partial}{\partial x_1} + \binom{\lambda\mu}{\alpha 2}\frac{\partial}{\partial x_2} + \binom{\lambda\mu}{\alpha 3}\frac{\partial}{\partial x_3},$$
$$\nabla_{\lambda\mu}^{\beta} = \binom{\lambda\mu}{1\beta}\frac{\partial}{\partial x_1} + \binom{\lambda\mu}{2\beta}\frac{\partial}{\partial x_2} + \binom{\lambda\mu}{3\beta}\frac{\partial}{\partial x_3}$$

then one can write formulas (4) and (5) as:

(8)
$$T_{\lambda\alpha} = \sum_{\mu}\Delta_{\lambda\mu}^{\alpha}u_\mu, \qquad \mathfrak{T}_{\mu\beta} = \sum_{\lambda}\nabla_{\lambda\mu}^{\beta}u_\lambda,$$

(9*a*)
$$\sum_{\alpha}\frac{\partial T_{\lambda\alpha}}{\partial x_\alpha} = \sum_{\mu}\Delta_{\lambda\mu}u_\mu,$$

(9*b*)
$$\sum_{\beta}\frac{\partial \mathfrak{T}_{\mu\beta}}{\partial x_\beta} = \sum_{\lambda}\Delta_{\lambda\mu}v_\lambda.$$

I now suppose that the functions u_λ and v_λ satisfy the systems of differential equations:

(10)
$$\sum_{\mu}\Delta_{\lambda\mu}u_\mu = U_\lambda, \qquad \sum_{\lambda}\Delta_{\lambda\mu}v_\lambda = V_\mu,$$

in which the symbols U_λ and V_μ denote continuous, uniform functions. Upon now forming the difference of the expressions (3), while recalling equations (10), I obtain the equation:

(11)
$$\int_S \sum_{\rho}(U_\rho v_\rho - V_\rho u_\rho)\,dS$$
$$= \int_\omega \left\{\sum_{\lambda} v_\lambda \sum_{\alpha} T_{\lambda\alpha}\cos(nx_\alpha) - \sum_{\lambda} u_\lambda \sum_{\alpha}\mathfrak{T}_{\lambda\alpha}\cos(nx_\alpha)\right\} d\omega.$$

Upon setting:

$$T_\lambda = \sum_{\alpha} T_{\lambda\alpha}\cos(nx_\alpha), \qquad \mathfrak{T}_\lambda = \sum_{\alpha}\mathfrak{T}_{\lambda\alpha}\cos(nx_\alpha),$$

to abbreviate, one can write formula (11) in the form:

$$\int_S \sum_\rho (U_\rho v_\rho - V_\rho u_\rho)\, dS = \int_\omega \sum_\lambda (v_\lambda T_\lambda - u_\lambda \mathfrak{T}_\lambda)\, d\omega . \tag{12}$$

This formula is the expression of the fundamental theorem that plays the same role for the systems (10) that GREEN's theorem does for the LAPLACE equation. In the particular case in which Δ is the variation of the internal potential of an elastic body, the fundamental theorem is identical to the known BETTI theorem.

§ 2. Application of Green's method to the systems: $\sum_\mu \Delta_{\lambda\mu}\, u_\lambda = U_\mu$, $\sum_\lambda \Delta_{\lambda\mu}\, v_\lambda = V_\mu$.

Let u_1, u_2, u_3 be three functions that satisfy the equations:

$$\sum_{\mu=1}^{3} \Delta_{\lambda\mu} u_\mu = U_\lambda ,$$

and suppose that the continuity conditions of § 1 are verified. Upon taking the functions v to be the functions $\beta_\mu(x_1 - x_1^0, x_2 - x_2^0, x_3 - x_3^0)$, we can apply formula (12) of § 1 to the condition of excluding part of the domain of integration S that is internal to a close surface ω'. We suppose that ω' is a sphere with the arbitrarily small radius r and the point $A(x_1^0, x_2^0, x_3^0)$ as its center. Let $\sigma_{\lambda\alpha}$ and σ_λ be the functions that are deduced from the function β in the same manner that the quantities $\mathfrak{T}_{\lambda\alpha}$ and $\mathfrak{T}_\lambda$ are deduced from the functions v. Let S' denote the domain S minus the sphere ω'. An application of formula (12) then gives us:

$$\int_\omega \sum_\rho (\beta_\rho T_\rho - u_\rho \sigma_\rho)\, d\omega + \int_{\omega'} \sum_\rho (\beta_\rho T_\rho - u_\rho \sigma_\rho)\, d\omega' = \int_{S'} \sum_\rho U_\rho \beta_\rho dS' . \tag{13}$$

Now, make the radius r decrease, so the integral:

$$\int_{\omega'} \sum_\rho \beta_\rho T_\rho d\omega'$$

obviously converges to zero, since the integrals β_ρ are homogeneous functions of degree – 1, and T_ρ remains finite for $r = 0$.

We now see what the other part of the integral that belongs to the sphere ω' becomes. It suffices to consider the integral:

$$J_1 = \int u_1 \left[\sigma_{11} \cos(nx_1) + \sigma_{12} \cos(nx_2) + \sigma_{13} \cos(nx_3)\right] d\omega' ,$$

where n denotes the external normal to the volume S' – i.e., the interior normal to the sphere ω'. Let $\sigma'_{\alpha\beta}$ denote the value of the homogeneous function $\sigma_{\alpha\beta}$ of degree – 2 on the surface of a sphere of radius 1 that is concentric with ω', and denote the surface element of that sphere by $d\omega_1$. The values of the functions $\sigma_{\alpha\beta}$ at two points along the same radius – one of which is on ω', and the other of which is on ω – are then related by the formula:

$$\sigma_{\alpha\beta} = \frac{1}{r^2}\sigma'_{\alpha\beta},$$

and one will have:

$$d\omega' = r^2\, d\omega_1,$$

moreover. One can then write:

$$J_1 = \int u_1 [\sigma'_{11}\cos(nx_1) + \sigma'_{12}\cos(nx_2) + \sigma'_{13}\cos(nx_3)]\, d\omega_1 .$$

In that formula, u_1 depends upon only r. Set $u_1(x_1^0, x_2^0, x_3^0) = u_1^0$; due to the continuity of u_1, we can choose r to be small enough that $\left| u_1 - u_1^0 \right|$ is less than some arbitrarily small quantity δ. Moreover, if we let g be the largest absolute value of the quantity in brackets in the expression for J_1 then we will have:

$$\begin{aligned}
&\left| J_1 - u_1^0 \int_{\omega_1} [\sigma'_{11}\cos(nx_1) + \sigma'_{12}\cos(nx_2) + \sigma'_{13}\cos(nx_3)]\, d\omega_1 \right| \\
&= \int_{\omega_1} (u_1 - u_1^0)[\sigma'_{11}\cos(nx_1) + \sigma'_{12}\cos(nx_2) + \sigma'_{13}\cos(nx_3)]\, d\omega_1 \\
&< \delta g \int d\sigma_1 \\
&< 4\pi\,\delta g,
\end{aligned}$$

i.e.:

$$\lim_{r=0} J_1 = u_1^0 \int_{\omega'} \sigma_1\, d\omega' .$$

If we set:

(14) $$L_\rho = \int_{\omega'} \sigma_\rho\, d\omega',$$

to abbreviate, then we will have:

$$\lim_{r=0} \int_{\omega'} \sum_\rho u_\rho \sigma_\rho\, d\omega' = L_1 u_1^0 + L_2 u_2^0 + L_3 u_3^0 .$$

The integral in the right-hand side of equation (13) obviously preserves a finite value when we make r tend to zero, because if we denote the upper limits of $|U_\rho|$ and $|r\,\beta_\rho|$ by G and g, resp., then we will have:

$$\left| \int_{S'} \sum_\rho U_\rho \beta_\rho\, dS' \right| < G\, g \int_{S'} \frac{dS'}{r} .$$

However, one knows that the integral in the right-hand side preserves a finite value no matter how small r is. Consequently, it is legitimate to write:

$$\lim_{r=0} \int_{S'} \sum_\rho U_\rho \beta_\rho dS' = \int_S \sum_\rho U_\rho \beta_\rho dS .$$

The result of all of these passages to the limit is expressed by the formula:

(15) $$L_1 u_1^0 + L_2 u_2^0 + L_3 u_3^0 = \int_\omega \sum_\rho (\beta_\rho T_\rho - u_\rho \sigma_\rho)\, d\omega - \int_S \sum_\rho U_\rho \beta_\rho dS .$$

One likewise obtains an analogous formula for the functions v_μ , which satisfy the adjoint system:

$$\sum_\mu \Delta_{\mu\lambda} v_\mu = V_\lambda .$$

Upon letting τ_ρ denote the quantity that is analogous to the σ_ρ that is deduced from the integral α, and letting M_ρ denote the integral:

$$M_\rho = \int_{\omega'} \tau_\rho d\omega' \qquad (\rho = 1, 2, 3),$$

the formula can be written:

(16) $$M_1 v_1^0 + M_2 v_2^0 + M_3 v_3^0 = \int_\omega \sum_\rho (\alpha_\rho \mathfrak{T}_\rho - v_\rho \tau_\rho)\, d\omega - \int_S \sum_\rho V_\rho \alpha_\rho dS .$$

Furthermore, imagine the case in which the point $A(x_1^0, x_2^0, x_3^0)$ is situated on the surface ω, and suppose that the surface ω has a well-defined tangent plane at A. To that effect, describe the sphere ω' with its center at A and let v denote the part of ω is external to the sphere ω'. If we apply the reciprocity theorem to the value S' then if we let ω' denote the part of the spherical surface ω' that is internal to S then we will have:

$$\int_{\omega'} \sum_\rho (\beta_\rho T_\rho - u_\rho \sigma_\rho)\, d\omega' + \int_v \sum_\rho (\beta_\rho V_\rho - u_\rho \sigma_\rho)\, dv = \int_{S'} \sum_\rho u_\rho \beta_\rho dS' .$$

One proves, as in the preceding, that the limit of the integral on the right-hand side is a finite quantity for $r = 0$. Similarly, one finds the limit of the first integral to be:

$$\lim_{r=0} \int_{\omega'} \sum_\rho (\beta_\rho T_\rho - u_\rho \sigma_\rho)\, d\omega' = - u_1^0 \int_{\omega'} \sigma_1 d\omega' - u_2^0 \int_{\omega'} \sigma_2 d\omega' - u_3^0 \int_{\omega'} \sigma_3 d\omega' ,$$

in which the integrals in the right-hand side must be taken over the part of the spherical surface ω' that is on the internal side of the tangent plane to ω at the point A. (The internal side of the tangent plane is the one with the interior normal.) Since the σ_ρ are even functions, it follows that the value of $\int \sigma_\rho d\omega'$ must be equal to $\frac{1}{2} L_\rho$. One also proves that the limit of the integral:

$$\int_v \sum_\rho (\beta_\rho V_\rho - u_\rho \sigma_\rho)\, dv$$

is a finite quantity for $r = 0$. That limit can then be expressed by the integral:

$$\int_\omega \sum_\rho (\beta_\rho T_\rho - u_\rho \sigma_\rho)\, d\omega.$$

We are thus led to the formula:

$$L_1 u_1^0 + L_2 u_2^0 + L_3 u_3^0 = 2\int_\omega \sum_\rho (\beta_\rho T_\rho - u_\rho \sigma_\rho)\, d\omega - 2\int_S \sum_\rho U_\rho \beta_\rho dS,$$

and similarly, to the analogous formula:

$$M_1 v_1^0 + M_2 v_2^0 + M_3 v_3^0 = 2\int_\omega \sum_\rho (\alpha_\rho \mathfrak{T}_\rho - v_\rho \tau_\rho)\, d\omega - 2\int_S \sum_\rho V_\rho \alpha_\rho dS.$$

Finally, if A is a point that is external to the volume S then I will recall that one has:

$$\int_\omega \sum_\rho (\beta_\rho T_\rho - u_\rho \sigma_\rho)\, d\omega - \int_S \sum_\rho U_\rho \beta_\rho dS = 0,$$
$$\int_\omega \sum_\rho (\alpha_\rho \mathfrak{T}_\rho - v_\rho \tau_\rho)\, d\omega - \int_S \sum_\rho V_\rho \alpha_\rho dS = 0,$$

because in that case, no singular point will be found inside of ω.

We shall calculate the values of the coefficients L and M in the following paragraph. Since it will then result that these coefficients are non-zero, formulas (15) and (16) will permit one to calculate the values of the functions u and v inside of a volume S if we know the values of these functions and certain linear functions of the first derivatives for the points (x_1, x_2, x_3) that belong to the surface ω.

In particular, these formulas will apply to the theory of equilibrium of an arbitrary crystalline elastic body. In that case, the quantities T_ρ will denote the components of the pressure on the surface of the body considered. We shall return to that application in the last chapter.

§ 3. Calculation of the coefficients L and M.

We have defined the coefficient L_ρ by the formula:

$$L_\rho = \int_\omega \sigma_\rho d\omega,$$

in which the integral must be taken over a certain spherical surface ω. Meanwhile, if we take the functions u_λ to have constant values then an application of formula (15), § 2 will show us that L_ρ is independent of the special form of the surface of integration, in such a way that ω can be an arbitrary closed surface that contains the origin, provided that it can be converted into the sphere ω by a continuous deformation. In particular, take ω to be a cylinder C that is parallel to the x_1-axis, and whose bases have the equations $x_1 = a$ and $x_1 = -a$. Upon letting ds denote the linear element of the intersection of the cylinder C with the x_2, x_3-plane, one can write the expression for L_ρ as:

$$L_\rho = \int_s \int_{-a}^{+a} \sigma_\rho \, ds \, dx_1 + B,$$

in which B is the sum of the two integrals that are taken over the bases of C. However, if one calls the area of the base A and lets σ_0 denote the largest absolute value of σ_ρ when x_1 is equal to unity then one will have:

$$|B| < \frac{2A\sigma_0}{a^2}.$$

It will then follow that $\lim B = 0$ for a infinite. As a result, one can write:

$$L_\rho = \lim_{a=\infty} \int_s \int_{-a}^{+a} \sigma_\rho \, ds \, dx_1 = L_\rho = \int_{-\infty}^{+\infty} \int_s \sigma_\rho \, ds \, dx_1 .$$

However, it is clear that one can choose a positive quantity a_0 such that the two integrals:

$$\int_a^{\infty} \sigma_\rho \, dx_1 , \qquad \int_{-a}^{-\infty} \sigma_\rho \, dx_1$$

have values that are less than an arbitrarily small quantity if a is positive and larger than a_0 . It then follows that one has the right to invert the order of integrations in the formula for L_ρ . First, calculate the integral:

$$J(a) = \int_{-a}^{-a} \sigma_\rho \, dx_1 .$$

If we suppose, for the moment, that x_2 has a positive value then we can employ the expression for β_ν that we gave in Chapter I, § 3:

$$\beta_\nu = \frac{1}{\pi} \int_C \sum_{\alpha=1}^{6} \frac{\psi_\nu(\xi, \eta_\alpha) \, d\xi}{f_2(\xi, \eta_\alpha)(\xi x_1 + \eta_\alpha x_2 + x_3)},$$

in which the contour C must contain only the zeroes of $\xi x_1 + \eta_a x_2 + x_3$ whose imaginary parts are positive. One deduces the following expression for $\sigma_{\rho\nu}$ [see § 1, form. (8)] from that expression for β_ν :

$$(1) \qquad \sigma_{\rho\nu} = -\frac{1}{\pi}\int_C \sum_{\alpha=1}^{6} \frac{\Delta_{1\rho}^{\nu}\psi_1 + \Delta_{2\rho}^{\nu}\psi_2 + \Delta_{3\rho}^{\nu}\psi_2}{f_2(\xi,\eta_\alpha)(\xi x_1 + \eta_\alpha x_2 + x_3)^2}\,d\xi\,,$$

in which the expressions $\Delta_{\mu\rho}^{\nu}$ denote the linear functions in ξ and η that one obtains by replacing the symbols $\partial / \partial x_1$, $\partial / \partial x_2$, $\partial / \partial x_3$ with ξ, η_a, and 1, respectively, in the expressions (7), § 1.

We will have, in turn, an expression of the following form for the integral J:

$$(2) \qquad J(a) = -\frac{1}{\pi}\int_{-a}^{+a} \int_C \sum_{\alpha=1}^{6} \frac{A_2^a \cos(nx_2) + A_3^a \cos(nx_3)}{f_2(\xi,\eta_\alpha)(\xi x_1 + \eta_\alpha x_2 + x_3)^2}\,d\xi\,dx_1\,.$$

If we denote $A_2^a \cos(n, x_2) + A_3^a \cos(n, x_3)$ by $\Phi(\xi, \eta_a)$, to abbreviate, then upon performing the integration over x_1 (which is obviously legitimate), we will have:

$$J(a) = \frac{1}{\pi}\int_C \sum_{\alpha=1}^{6} \frac{2a\,\Phi(\xi,\eta_a)\,d\xi}{f_2(\xi,\eta_\alpha)(-a\xi + \eta_\alpha x_2 + x_3)}\,.$$

However, it is easy to perform the integration in that formula, which will give us the result:

$$J(a) = i\sum_\nu \frac{4a\,\Phi(\xi_\nu,\eta_\nu)}{\left(a\dfrac{\partial f}{\partial \eta_\nu} - x_2\dfrac{\partial f}{\partial \xi_\nu}\right)(-a\xi_\nu + \eta_\nu x_2 + x_3)}$$

$$+\ i\sum_\nu \frac{4a\,\Phi(\xi'_\nu,\eta'_\nu)}{\left(-a\dfrac{\partial f}{\partial \eta'_\nu} - x_2\dfrac{\partial f}{\partial \xi'_\nu}\right)(-a\xi'_\nu + \eta'_\nu x_2 + x_3)}\,,$$

in which one gives values to the ξ_ν, η_ν that satisfy the equations:

$$f(\xi, \eta) = 0, \qquad a\xi + \eta\, x_2 + x_3 = 0,$$

and values ξ'_ν, η'_ν to the that satisfy the equations:

$$f(\xi', \eta') = 0, \qquad -a\xi' + \eta'\, x_2 + x_3 = 0;$$

ξ and ξ' must have positive imaginary parts in both cases. With the aid of these linear equations, one can write the expression for $J(a)$ as follows:

$$J(a) = 2i\sum_\nu \frac{a\,\Phi(\xi_\nu, \eta_\nu)}{\left(a\dfrac{\partial f}{\partial \eta_\nu} - x_2\dfrac{\partial f}{\partial \xi_\nu}\right)(\eta_\nu x_2 + x_3)} + 2i\sum_\nu \frac{a\,\Phi(\xi'_\nu, \eta'_\nu)}{\left(-a\dfrac{\partial f}{\partial \eta'_\nu} - x_2\dfrac{\partial f}{\partial \xi'_\nu}\right)(\eta'_\nu x_2 + x_3)},$$

and we will have to seek the limit of that function for $a = \infty$.

Since the straight lines that determine ξ, ξ', η, η' tend to the line $\xi = 0$ for $a = \infty$, the corresponding values of η will satisfy the equation:

$$f(0, \eta) = 0.$$

Since the imaginary part of ξ is positive, moreover, and we have supposed that x_2 is positive, it will then follow that the imaginary part of η must be negative, and that the imaginary part of η' must be positive. It is true that this line of reasoning supposes that the roots of $f(0, \eta) = 0$ are finite, but one can always arrange that the aforementioned roots are both finite and unequal by a change of coordinates. Having said that, one easily finds the desired limit:

$$\lim_{a=\infty} J(a) = 2i\sum_{\nu=1}^{3} \frac{\Phi(0, \eta_\nu)}{\dfrac{\partial f}{\partial \eta_\nu}(\eta_\nu x_2 + x_3)} - 2i\sum_{\nu=1}^{3} \frac{\Phi(0, \eta'_\nu)}{\dfrac{\partial f}{\partial \eta'_\nu}(\eta'_\nu x_2 + x_3)}.$$

However, by introducing the value of Φ, one will find:

$$\int_{-\infty}^{+\infty} \sigma_\rho\, dx_1 = 2i\sum_{\nu=1}^{3} \frac{A_2^\nu \cos(nx_2) + A_3^\nu \cos(nx_3)}{f_2(0, \eta_\nu)(\eta_\nu x_2 + x_3)} - 2i\sum_{\nu'=1}^{3} \frac{A_2^{\nu'} \cos(nx_2) + A_3^{\nu'} \cos(nx_3)}{f_2(0, \eta'_\nu)(\eta'_\nu x_2 + x_3)}.$$

In the deduction of this formula, we have supposed that x_2 is larger than zero, but one sees that the equality will still persist when x_2 is negative, because σ_ρ does not change sign when one changes the signs of x_2 and x_3, and the right-hand side has the same property. We now have to calculate:

$$\int_s J(\infty)\, ds,$$

but since the value of that integral does not depend upon the form of the contour s, one concludes that A_3^ν and $A_3^{\nu'}$ must satisfy the relations:

$$A_2^\nu \eta_\nu + A_3^\nu = 0, \qquad A_2^{\nu'} \eta_\nu + A_3^{\nu'} = 0.$$

Upon inferring A_3^ν and $A_3^{\nu'}$ from these formulas and recalling that n denotes the internal normal to s, one will arrive at the formula:

$$L_\rho = -\,2i\int_S \sum_{\nu=1}^{3} \frac{A_2^\nu (n_\nu dx_2 + dx_3)}{f_2(0,\eta_\nu)(\eta_\nu x_2 + x_3)} + 2i\int_S \sum_{\nu'=1}^{3} \frac{A_2^{\nu'} (\eta' dx_2 + dx_3)}{f_2(0,\eta'_\nu)(\eta'_\nu x_2 + x_3)},$$

but it is easy to perform the integrations here; one finds that:

$$\int \frac{n_\nu dx_2 + dx_3}{\eta_\nu x_2 + x_3} = 2\pi i, \qquad \int \frac{n'_\nu dx_2 + dx_3}{\eta'_\nu x_2 + x_3} = -\,2\pi i,$$

and as a result:

$$L_\rho = 4\pi \sum_{a=1}^{6} \frac{A_2^a}{f_2(0,\eta_a)},$$

in which one must extend the summation over all roots of the equation $f(0,\ \eta) = 0$. If one observes that the sum in the right-hand side is nothing but the coefficient of $1\ /\ \eta$ in the development of $\dfrac{A_2}{f(0,\eta)}$ in decreasing powers of η then it will be easy to simplify the expression for L_ρ. Indeed, we have [see forms. (1) and (2) of this paragraph]:

$$A_2 = \nabla^2_{1\rho}\psi_1 + \nabla^2_{2\rho}\psi_2 + \nabla^2_{3\rho}\psi_3$$

$$= \nabla^2_{1\rho}\begin{vmatrix} k_1 & \Delta_{21} & \Delta_{31} \\ k_2 & \Delta_{22} & \Delta_{32} \\ k_3 & \Delta_{23} & \Delta_{33} \end{vmatrix} + \nabla^2_{2\rho}\begin{vmatrix} \Delta_{11} & k_1 & \Delta_{31} \\ \Delta_{12} & k_2 & \Delta_{32} \\ \Delta_{13} & k_3 & \Delta_{33} \end{vmatrix} + \nabla^2_{3\rho}\begin{vmatrix} \Delta_{11} & \Delta_{21} & k_1 \\ \Delta_{12} & \Delta_{22} & k_2 \\ \Delta_{13} & \Delta_{23} & k_3 \end{vmatrix},$$

from which one infers the following value for the coefficient η^5 in A_2 by recalling formulas (6) and (7), § 1:

$$\binom{1\rho}{22}\begin{vmatrix} k_1 & \binom{21}{22} & \binom{31}{22} \\ k_2 & \binom{22}{22} & \binom{32}{22} \\ k_3 & \binom{23}{22} & \binom{33}{22} \end{vmatrix} + \binom{2\rho}{22}\begin{vmatrix} \binom{11}{22} & k_1 & \binom{31}{22} \\ \binom{12}{22} & k_2 & \binom{32}{22} \\ \binom{13}{22} & k_3 & \binom{33}{22} \end{vmatrix} + \binom{3\rho}{22}\begin{vmatrix} \binom{11}{22} & \binom{21}{22} & k_1 \\ \binom{12}{22} & \binom{22}{22} & k_2 \\ \binom{13}{22} & \binom{23}{22} & k_3 \end{vmatrix},$$

or, more simply:

$$= k_\rho \begin{vmatrix} \binom{11}{22} & \binom{21}{22} & \binom{31}{22} \\ \binom{12}{22} & \binom{22}{22} & \binom{32}{22} \\ \binom{13}{22} & \binom{23}{22} & \binom{33}{22} \end{vmatrix}.$$

However, the determinant that multiplies k_ρ here is equal to the coefficient of η^6 in $f(0, \eta)$; one will consequently have:

$$L_\rho = 4\pi k_\rho .$$

Since L_ρ does not depend upon any of the coefficients $\binom{\alpha\beta}{\lambda\mu}$, and since one will have obtained the value of M_ρ by the same calculation, by taking one's point of departure to be the formulas in which one changes $\binom{\alpha\beta}{\lambda\mu}$ into $\binom{\beta\alpha}{\lambda\mu}$, one will conclude that the value of M_ρ is:

$$M_\rho = 4\pi k_\rho .$$

As a result, formulas (15) and (16) of the preceding paragraph will take the forms:

$$(15a) \qquad k_1 u_1^0 + k_2 u_2^0 + k_3 u_3^0 = \frac{1}{4\pi}\int_\omega \sum_\rho (T_\rho \beta_\rho - u_\rho \sigma_\rho)\, d\omega - \frac{1}{4\pi}\int_S \sum_\rho U_\rho \beta_\rho \, dS$$

and

$$(16a) \qquad k_1 v_1^0 + k_2 v_2^0 + k_3 v_3^0 = \frac{1}{4\pi}\int_\omega \sum_\rho (\mathfrak{T}_\rho \alpha_\rho - v_\rho \tau_\rho)\, d\omega - \frac{1}{4\pi}\int_S \sum_\rho V_\rho \alpha_\rho \, dS ,$$

respectively. In the case where the point (x_1^0, x_2^0, x_3^0) is a point of the surface ω with a well-defined tangent plane, one will have the formulas:

$$(17a) \qquad k_1 u_1^0 + k_2 u_2^0 + k_3 u_3^0 = \frac{1}{2\pi}\int_\omega \sum_\rho (T_\rho \beta_\rho - u_\rho \sigma_\rho)\, d\omega - \frac{1}{2\pi}\int_S \sum_\rho U_\rho \beta_\rho \, dS ,$$

$$(18a) \qquad k_1 v_1^0 + k_2 v_2^0 + k_3 v_3^0 = \frac{1}{2\pi}\int_\omega \sum_\rho (\mathfrak{T}_\rho \alpha_\rho - v_\rho \tau_\rho)\, d\omega - \frac{1}{2\pi}\int_S \sum_\rho V_\rho \alpha_\rho \, dS .$$

CHAPTER III.

Applications

§ 1. Application to the theorem of equilibrium of an elastic solid body.

Let f be a definite quadratic form of the six variables:

$$\delta_{\nu\nu} = \frac{\partial u_\nu}{\partial x_\nu}, \qquad \delta_{\lambda\mu} = \delta_{\mu\lambda} = \frac{\partial u_\lambda}{\partial x_\mu} + \frac{\partial u_\mu}{\partial x_\lambda}.$$

We have already recalled that $\int f\, dS$ can represent the potential of the internal forces on an elastic body. Take another system of variables:

$$\varepsilon_{\nu\nu} = \frac{\partial \upsilon_\nu}{\partial x_\nu}, \qquad \varepsilon_{\lambda\mu} = \varepsilon_{\mu\lambda} = \frac{\partial \upsilon_\lambda}{\partial x'_\mu} + \frac{\partial \upsilon_\mu}{\partial x'_\lambda}.$$

I recall that we have (§ 1, Chapter II):

$$\begin{aligned}\Delta &= \frac{\partial f}{\partial \delta_{11}}\varepsilon_{11} + \frac{\partial f}{\partial \delta_{22}}\varepsilon_{22} + \frac{\partial f}{\partial \delta_{33}}\varepsilon_{33} + \frac{\partial f}{\partial \delta_{23}}\varepsilon_{23} + \frac{\partial f}{\partial \delta_{31}}\varepsilon_{31} + \frac{\partial f}{\partial \delta_{12}}\varepsilon_{12} \\ &= \frac{\partial f}{\partial \varepsilon_{11}}\delta_{11} + \frac{\partial f}{\partial \varepsilon_{22}}\delta_{22} + \frac{\partial f}{\partial \varepsilon_{33}}\delta_{33} + \frac{\partial f}{\partial \varepsilon_{23}}\delta_{23} + \frac{\partial f}{\partial \varepsilon_{31}}\delta_{31} + \frac{\partial f}{\partial \varepsilon_{12}}\delta_{12}.\end{aligned}$$

It then follows that the quantities $T_{\lambda\alpha}$ are identical to the components of the stress that one ordinarily denotes by $t_{\lambda\alpha}$ (see, e.g., CLEBSCH, *Theorie d. Elasticität*).

Let X_1, X_2, X_3 be the rectangular components of the force that acts upon a volume element, so the components of the deformation satisfy the equations:

$$\sum_{\lambda=1}^{3} \frac{\partial t_{\lambda\alpha}}{\partial x_\alpha} = -X_\alpha \qquad (\alpha = 1, 2, 3).$$

However, these equations are identical to the differential equations (10), § 1, Chapter II. In order to see this, it will suffice to see the relationship between the problems that were treated above and the problem of the equilibrium of a solid, elastic body.

It remains to prove that the determinant of the functions $\Delta_{\lambda\mu}$ cannot be zero for any system of real values of the variables. Set $u_\lambda = v_\lambda$ in the expression for Δ, so:

$$2f = \Delta(u, u).$$

If one substitutes this in the equation $u_{\mu\beta} = x_\mu\,\xi_\beta$ then one will obtain:

$$2f = \sum_{\lambda,\mu,\alpha,\beta} \binom{\lambda\mu}{\alpha\beta} x_\lambda x_\mu \xi_\alpha \xi_\beta = \sum_{\lambda,\mu} x_\lambda x_\mu \sum_{\alpha,\beta} \binom{\lambda\mu}{\alpha\beta} \xi_\alpha \xi_\beta = \sum_{\lambda,\mu} \Delta_{\lambda\mu}(\xi_1, \xi_2, \xi_3)\, x_\lambda x_\mu \;.$$

That is, if one makes the following substitutions in $2f$:

$$\delta_{\nu\nu} = x_\nu\,\xi_\nu\,, \qquad\qquad \delta_{\lambda\mu} = x_\lambda\,\xi_\mu + x_\mu\,\xi_\lambda$$

then one will obtain a quadratic form in x_1, x_2, x_3 whose determinant is precisely the determinant of the functions $\Delta_{\lambda\mu}$.

Suppose that this determinant becomes zero for a system of real values of the variables – say, $\xi_\nu = \alpha_\nu$ – in which one of the quantities α_ν must be non-zero. One can then find a system of real values $x_\nu = a_\nu$ for which one of the quantities a_ν must be non-zero, and f will be equal to zero. However, in order for f to vanish, it is necessary that $\delta_{\nu\nu} = 0$, $\delta_{\lambda\mu} = 0$, or that the following equations must be verified:

$$a_\nu\,\alpha_\nu = 0, \qquad a_\lambda\,\alpha_\mu + a_\mu\,\alpha_\lambda = 0.$$

We can suppose that α_1 is non-zero; it will then follow that $a_1 = 0$. Upon substituting that value into the other equations, one will find that $a_2\alpha_1 = 0$, $a_3\alpha_1 = 0$, so one concludes that $a_2 = a_3 = 0$, which is contrary to the hypothesis. Thus, the determinant of the functions $\Delta_{\lambda\mu}$ is non-zero for any system of real values for the variables ξ_ν, except for the system $\xi_1 = \xi_2 = \xi_3 = 0$.

Q. E. D.

§ 2. Development into a series.

We have seen that the functions a and b enjoy the property of being developable into power series in the neighborhood of an arbitrary real point a_1, a_2, a_3, with the exception of $a_1 = a_2 = a_3 = 0$. Moreover, the development of $\alpha(a_1 + h_1, a_2 + h_2, a_3 + h_3)$ in powers of the variables h_1, h_2, h_3 converges for all values of these variables h that satisfy the inequality:

$$\sqrt{h_1^2 + h_2^2 + h_3^2} \le \mu\,\sqrt{a_1^2 + a_2^2 + a_3^2}\;,$$

in which m is a positive quantity that is independent of the quantities a_ν and h_ν and less than unity.

Having recalled that, suppose that u_1, u_2, u_3 form a system of integrals of the differential equations (7), Chapter II, that verify the necessary conditions for the application of BETTI's theorem in a domain that is bounded by two spherical surfaces 1 and 2 that have the origin for their centers and ρ_1 and ρ_2 for radii, resp. Suppose that the inequality $\rho_2 / \rho_1 < \mu^2$ is satisfied, so it will follow that:

$$\frac{\rho_2}{\mu} < \rho_1\,\mu\,.$$

Take a point x_1, x_2, x_3 that is situated between the spheres of radius ρ_2 / μ and $\rho_1\,\mu$ and let ξ_1, ξ_2, ξ_3 be a point on the sphere 2, so the development of:

$$\alpha\,(\xi_1 - x_1,\ \xi_2 - x_2,\ \xi_3 - x_3)$$

in powers of the variables ξ_v will converge if either:

$$\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} < \mu\,\sqrt{x_1^2 + x_2^2 + x_3^2}$$

or, upon considering the series to be a function of x_1, x_2, x_3, if:

$$\sqrt{x_1^2 + x_2^2 + x_3^2} > \frac{\rho_2}{\mu}\,.$$

Let η_1, η_2, η_3 be a point on the sphere 1, so the development of:

$$\alpha\,(\eta_1 - x_1,\ \eta_2 - x_2,\ \eta_3 - x_3)$$

in powers of the variables x_1, x_2, x_3 will converge if:

$$\sqrt{x_1^2 + x_2^2 + x_3^2} < \mu\,\rho_1\,.$$

Thus, the two developments of the functions:

$$\alpha\,(\eta_1 - x_1,\ \eta_2 - x_2,\ \eta_3 - x_3) \quad \text{and} \quad \alpha\,(\xi_1 - x_1,\ \xi_2 - x_2,\ \xi_3 - x_3)$$

have the space between the two spherical surfaces of radii ρ_2 / μ and $\rho_1\mu$ for their common domain of convergence.

These developments still converge in the same domain if the point ξ_1, ξ_2, ξ_3 is interior to the sphere 2 and the point η_1, η_2, η_3 is exterior to the sphere 1.

We now apply the formula (4), Chapter II, to the functions u_1, u_2, u_3. If one takes the domain S to be the space between the two spheres 1 and 2, and if we let ω_1 and ω_2 denote the surfaces of the two spheres 1 and 2 then we will get:

$$k_1\,u_1 + k_2\,u_2 + k_3\,u_3$$

$$= \frac{1}{4\pi}\int_{\omega_1} \sum_{\rho=1}^{3} T_\rho(\eta_1,\eta_2,\eta_3) \sum_{\lambda_1,\lambda_2,\lambda_3} \frac{x_1^{\lambda_1}\cdot x_2^{\lambda_2}\cdot x_3^{\lambda_3}}{\underline{|\lambda_1}\,\underline{|\lambda_2}\,\underline{|\lambda_3}} \frac{\partial^{\lambda_1+\lambda_2+\lambda_3}\beta_\rho(-\eta_1,-\eta_2,-\eta_3)}{\partial\eta_1^{\lambda_1}\partial\eta_2^{\lambda_2}\partial\eta_3^{\lambda_3}}\,d\omega_1$$

$$-\frac{1}{4\pi}\int_{\omega_1}\sum_{\rho=1}^{3}u_\rho(\eta_1,\eta_2,\eta_3)\sum_{\lambda_1,\lambda_2,\lambda_3}\frac{x_1^{\lambda_1}\cdot x_2^{\lambda_2}\cdot x_3^{\lambda_3}}{\underline{|\lambda_1}\underline{|\lambda_2}\underline{|\lambda_3}}\frac{\partial^{\lambda_1+\lambda_2+\lambda_3}\beta_\rho(-\eta_1,-\eta_2,-\eta_3)}{\partial\eta_1^{\lambda_1}\partial\eta_2^{\lambda_2}\partial\eta_3^{\lambda_3}}d\omega_1$$

$$+\frac{1}{4\pi}\int_{\omega_2}\sum_{\rho=1}^{3}T_\rho(\xi_1,\xi_2,\xi_3)\sum_{\lambda_1,\lambda_2,\lambda_3}(-1)^{\lambda_1+\lambda_2+\lambda_3}\frac{\xi_1^{\lambda_1}\cdot\xi_2^{\lambda_2}\cdot\xi_3^{\lambda_3}}{\underline{|\lambda_1}\underline{|\lambda_2}\underline{|\lambda_3}}\frac{\partial^{\lambda_1+\lambda_2+\lambda_3}\beta_\rho(x_1,x_2,x_3)}{\partial x_1^{\lambda_1}\,\partial x_2^{\lambda_2}\,\partial x_3^{\lambda_3}}d\omega_2$$

$$-\frac{1}{4\pi}\int_{\omega_2}\sum_{\rho=1}^{3}u_\rho(\xi_1,\xi_2,\xi_3)\sum_{\lambda_1,\lambda_2,\lambda_3}(-1)^{\lambda_1+\lambda_2+\lambda_3}\frac{\xi_1^{\lambda_1}\cdot\xi_2^{\lambda_2}\cdot\xi_3^{\lambda_3}}{\underline{|\lambda_1}\underline{|\lambda_2}\underline{|\lambda_3}}\frac{\partial^{\lambda_1+\lambda_2+\lambda_3}\sigma_\rho(x_1,x_2,x_3)}{\partial x_1^{\lambda_1}\,\partial x_2^{\lambda_2}\,\partial x_3^{\lambda_3}}d\omega_2\,.$$

However, it follows from the uniform convergence of the series that one can perform the integration by integrating each term; one will then obtain a development of the expression $k_1\,u_1 + k_2\,u_2 + k_3\,u_3$ that is valid in the space between the two spheres of radius $\rho_2\,/\,\mu$ and $\rho_1\,\mu$. That development will consist of two parts, one of which is a power series and the other of which is a series whose terms are the partial derivatives of the functions β_ν. The first of these parts converges in the interior of the sphere of radius $\mu\,\rho_1$, and the second one converges in the exterior of the sphere of radius $\rho_2\,/\,\mu$.

Now, consider the developments of some special functions. First, suppose that u_1, u_2, u_3 denote homogeneous functions of negative integer degree $-n$ that verify the necessary continuity conditions in all of space, with the exception of the origin. When the radius ρ_1 tends to infinity, the first two integrals will obviously tend to zero. In other words, the only things that remain will be homogeneous of degree $-n$, in such a way that one will get:

$$k_1\,u_1 + k_2\,u_2 + k_3\,u_3$$

$$=\sum_{\rho=1}^{3}\left(\sum_{\lambda_1\lambda_2\lambda_3}A^{\rho}_{\lambda_1\lambda_2\lambda_3}\frac{\partial^{\lambda_1+\lambda_2+\lambda_3}\beta_\rho(x_1,x_2,x_3)}{\partial x_1^{\lambda_1}\,\partial x_2^{\lambda_2}\,\partial x_3^{\lambda_3}}-\sum_{\mu_1\mu_2\mu_3}B^{\rho}_{\mu_1\mu_2\mu_3}\frac{\partial^{\mu_1+\mu_2+\mu_3}\sigma_\rho(x_1,x_2,x_3)}{\partial x_1^{\mu_1}\,\partial x_2^{\mu_2}\,\partial x_3^{\mu_3}}\right),$$

in which the values $A^{\rho}_{\lambda_1\lambda_2\lambda_3}$ and $B^{\rho}_{\mu_1\mu_2\mu_3}$ are constant coefficients and the indices must satisfy the conditions:

$$\begin{cases}\lambda_1+\lambda_2+\lambda_3 = n-1,\\ \mu_1+\mu_2+\mu_3 = n-2.\end{cases}$$

In particular, suppose that n is equal to unity, so:

$$k_1\,u_1 + k_2\,u_2 + k_3\,u_3 = A_1\,\beta_1 + A_2\,\beta_2 + A_3\,\beta_3\,.$$

Moreover, we know that the functions b depend linearly upon the constants k in such a way that one can write:

$$\beta_\rho = k_1\,\beta_{\rho 1} + k_2\,\beta_{\rho 2} + k_3\,\beta_{\rho 3}\,,$$

so the following expressions will result for the functions u:

$$u_\nu = A_1\,\beta_{1\nu} + A_2\,\beta_{2\nu} + A_3\,\beta_{3\nu} \qquad (\nu = 1, 2, 3).$$

If we set $u_\nu = a_\nu$ in these formulas then if we observe that $A_\rho = k_\rho$ in that case, we will obtain:

$$\alpha_\nu = k_1\,\beta_{1\nu} + k_2\,\beta_{2\nu} + k_3\,\beta_{3\nu} \qquad (\nu = 1, 2, 3).$$

Upon equating the coefficients of the constants k_ν in this formula and setting:

$$\alpha_\nu = k_1\,\alpha_{\nu 1} + k_2\,\alpha_{\nu 2} + k_3\,\alpha_{\nu 3}\,,$$

one will get the relation:

$$\alpha_{\lambda\mu} = \beta_{\nu\lambda}\,,$$

with the aid of which one can write the expression for the functions u in the following manner:

$$u_\nu = A_1\,\alpha_{\nu 1} + A_2\,\alpha_{\nu 2} + A_3\,\alpha_{\nu 3} \qquad (\nu = 1, 2, 3).$$

It results from this formula that the functions α_ν are the only integrals of equations (7), § 2 that are homogeneous of degree – 1 and have the property of being uniform and continuous, along with their derivatives of the first two orders, for all real values of the variables.

§ 3. Some properties of the integrals in the fundamental formulas.

In the fundamental formula (15), Chapter II, we let T_ρ denote certain linear functions of the first derivatives of the functions u_1, u_2, u_3 . We now leave aside that definition of the quantities T_ρ and suppose only that they are functions of the parameters that fix the position of a point on the surface ω that are finite and generally continuous.

Let x_1, x_2, x_3 be the coordinates of a real point A that is arbitrary moreover, and let ξ_1, ξ_2, ξ_3 be the coordinates of a point A' on the surface ω.

Consider the integral:

$$\varphi = k_1\,\varphi_1 + k_2\,\varphi_2 + k_3\,\varphi_3 = \frac{1}{4\pi}\int_\omega \sum_\rho T_\rho \beta_\beta \, d\omega; \tag{1}$$

we show that φ is a continuous function for any system of real values for x_1, x_2, x_3 . Since it results immediately from what we said in the preceding paragraph that φ is continuous for points exterior to ω, we shall take a point A on ω.

Describe a sphere s of radius ε with A as its center; let φ_0 be the part of the integral φ that is extended over the part ω_0 of ω that is interior to s and let φ denote the rest of it $\varphi - \varphi_0$. φ' is then a continuous function at A. However, we know that one can determine a finite and positive quantity g in such a manner that:

$$|\,\beta_\rho\,| < \frac{g}{r}\,,$$

in which r is the distance AA'. Furthermore, let G be an upper limit of the functions T_ρ, so one has:

$$|\varphi_0| < \frac{3}{4\pi} G \cdot g \int_{\omega_0} \frac{d\omega}{r}.$$

However, one knows from the elements of potential theory that the value of the integral in this inequality converges to zero with the radius ε. Therefore, one can determine ε to be small enough that the difference of the two values of φ_0 at points interior to s is less than an arbitrarily small quantity, so it will indeed result that $\varphi = \varphi_0 + \varphi'$ is a function that is continuous at A.

Now, pass on to the integral:

$$\vartheta = k_1\,\vartheta_1 + k_2\,\vartheta_2 + k_3\,\vartheta_3 = \frac{1}{4\pi}\int_S \sum_\rho U_\rho \beta_\rho dS,$$

in which U_ρ is a continuous function of the variables ξ_1, ξ_2, ξ_3.

It is clear that ϑ is a continuous function, along with its derivatives, for the points $A(x_1, x_2, x_3)$ that are exterior to the volume S.

In order to establish the continuity of ϑ for the points that belong to S, one only has to repeat the well-known proof of the continuity of the potential of an extended, three-dimensional mass.

One also establishes the continuity of the first derivatives of ϑ by the same method.

Now, return to formula (15*a*) of Chapter II. We have seen that the expression:

$$\frac{1}{4\pi}\int_\omega \sum_\rho (T_\rho \beta_\rho - U_\rho \sigma_\rho)\, d\omega - \frac{1}{4\pi}\int_S \sum_\rho U_\rho \beta_\rho\, dS \tag{2}$$

represents the function $k_1\,u_1 + k_2\,u_2 + k_3\,u_3$ for the points A that belong to the volume S, and if the point A is found to be on ω then the expression (2) will be equal to $\frac{1}{2}(k_1\,u_1 + k_2\,u_2 + k_3\,u_3)$, and finally, if A is outside of S then it will be equal to zero. However, from what we proved about the continuity of the integral φ and ϑ, it will follow that the brief changes in the expression (2) are due to the integral:

$$\varpi = k_1\varpi_1 + k_2\varpi_2 + k_3\varpi_3 = \frac{1}{4\pi}\int \sum_\rho u_\rho \sigma_\rho\, d\omega.$$

Let s be a point on the surface ω. Let ϖ_s denote the value of ϖ at the point s, let ϖ_{is} denote the limit of ϖ when the point $A(x_1, x_2, x_3)$ tends to s while staying inside of the surface, and let ϖ_{es} denote the limit of ϖ when the point A tends to s while staying outside of the surface. One then has the two relations:

(3)

$$\varpi_{is} = \varpi_s - \tfrac{1}{2}(k_1\,u_1 + k_2\,u_2 + k_3\,u_3),$$

$$\varpi_{es} = \varpi_s + \tfrac{1}{2}(k_1\, u_1 + k_2\, u_2 + k_3\, u_3).$$

Therefore, if u_1, u_2, u_3 are the values of three functions of the variables ξ_1, ξ_2, ξ_3 on the surface ω, and if the functions have continuous derivatives of the first two orders then formulas (3) will inform us of the discontinuity in the integral ϖ.

However, we shall prove that the formulas still persist under slightly more general conditions.

Let u_1, u_2, u_3 be functions of the parameters that fix the position of a point one ω. Suppose that these functions admit finite derivatives of first order, and take a point s in ω where the curvature is finite. Describe a sphere of radius ε whose center is at s and which cuts out a curve γ on the surface ω. If one lets ϖ' be the part of the integral that relates to the area ω_0 that is interior to γ then one will have:

$$\varpi' = \frac{1}{4\pi}\int_{\omega_0}\sum_\rho u_\rho \sigma_\rho d\omega = \frac{1}{4\pi}\int_{\omega_0}\sum_\rho u'_\rho \sigma_\rho d\omega + \frac{1}{4\pi}\int_{\omega_0}\sum_\rho (u_\rho - u'_\rho)\sigma_\rho d\omega.$$

However, since σ_ρ is a function that is homogeneous of degree – 2 and is regular outside of the point s, if one lets r denote the distance form s to another point of ω then one can set:

$$\sigma_\rho = \frac{\sigma_\rho^0}{r^2},$$

in which σ_ρ^0 is a function whose modulus has a finite upper limit, namely, g. One then has:

$$\frac{1}{4\pi}\int_{\omega_0}\sum_\rho (u_\rho - u'_\rho)\sigma_\rho d\omega = \int_{\omega_0}\sum_\rho \frac{u_\rho - u'_\rho}{r}\sigma_\rho^0 \cdot \frac{d\omega}{r}.$$

However, there exists a finite upper limit to $\frac{1}{r}(u_\rho - u'_\rho)$, no matter how small r is; let u be that limit, so one can write:

$$\left|\int_{\omega_0}\sum_\rho (u_\rho - u'_\rho)\sigma_\rho d\omega\right| < 3gu\int_{\omega_0}\frac{d\omega}{r}.$$

However, in potential theory, one shows that the integral has an absolute value that is less than:

$$\frac{2\pi\varepsilon}{\cos\gamma_0},$$

in which γ_0 denotes the largest angle between a normal to ω_0 and the normal at s.

The integral $\frac{1}{4\pi}\int_{\omega_0}\sum_\rho u'_\rho \sigma_\rho d\omega$ is no longer larger than $k_1u'_1 + k_2u'_2 + k_3u'_3$. We can then write the following inequality:

.

$$|\,\varpi'\,| < \left| k_1u'_1 + k_2u'_2 + k_3u'_3 \right| + \frac{3gu\varepsilon}{2\cos\gamma_0},$$

so it will then results that one can choose the radius ε to be small enough that the integral $\int_{\omega_0}\sum_\rho (u_\rho - u'_\rho)\sigma_\rho d\omega$ has an absolute value that is less than an arbitrarily small quantity δ.

Since the integral $\varpi - \varpi'$ is continuous in the neighborhood of *s*, one concludes that the integral:

$$\int_{\omega_0}\sum_\rho (u_\rho - u'_\rho)\sigma_\rho d\omega$$

is continuous at the point *s*. One deduces equations (3) from this immediately.

Since the right-hand sides of equations (3) are independent of the coefficients in the differential equations that define the functions β, one sees (see pp. 23) that formulas (3) also persist for the functions that are defined by the integral:

$$\frac{1}{4\pi}\int_\omega (v_1\tau_1 + v_2\tau_2 + v_3\tau_3)\, d\omega.$$

We have established that the functions φ are continuous; on the contrary, their first derivatives present discontinuities, whose nature we shall exhibit under the hypothesis that the T_ν are functions that have derivatives of first order.

Since τ_ρ denotes a linear function of arbitrary constants *k*, one can show, by writing:

$$\tau_\rho = k_1\tau_\rho^1 + k_2\tau_\rho^2 + k_3\tau_\rho^3,$$

in which the τ_ρ^ν are linear functions of the direction cosines of the normal to ω:

$$\tau_\rho^\nu = \sum_\alpha \tau_{\rho\alpha}^\nu \cos(nx_\alpha),$$

that one can deduce the following expression for $\tau_{\rho\alpha}^\nu$ (see formula (8), Chapter II, § 1 and Chapter III, § 2):

$$\tau_{\rho\alpha}^\nu = \sum_\mu \Delta_{\rho\mu}^\alpha \alpha_{\mu\nu}.$$

With the aid of these notations, we can substitute the following statement for formulas (3): $\int_\omega v\tau_\alpha^\lambda d\omega$ is a continuous function of (x_1, x_2, x_3) if λ is unequal to α; on the contrary, if $\lambda = \alpha$ then the integral will present a discontinuity that is defined by the formulas:

$$\left[\int_\omega v\tau_\alpha^\alpha d\omega\right]_s^{is} = -2\pi v_s\,, \qquad \left[\int_\omega v\tau_\alpha^\alpha d\omega\right]_s^{es} = 2\pi v_s\,.$$

Upon now differentiating the expression for φ_μ with the aid of the functions α:

$$\varphi_\mu = \frac{1}{4\pi}\int_\omega (T_1\alpha_{\mu 1} + T_2\alpha_{\mu 2} + T_3\alpha_{\mu 3})\,d\omega\,,$$

one will obtain:

$$\begin{aligned}\sum_\mu \Delta_{\lambda\mu}^\alpha \varphi_\mu &= -\frac{1}{4\pi}\int_\omega \sum_\mu (T_1\Delta_{\lambda\mu}^\alpha\alpha_{\mu 1} + T_2\Delta_{\lambda\mu}^\alpha\alpha_{\mu 2} + T_3\Delta_{\lambda\mu}^\alpha\alpha_{\mu 3})\,d\omega\,,\\ &= -\frac{1}{4\pi}\int_\omega (T_1\tau_{\lambda\alpha}^1 + T_2\tau_{\lambda\alpha}^2 + T_3\tau_{\lambda\alpha}^3)\,d\omega\,.\end{aligned}$$

If we multiply both sides of this by cos (n x_α) and take the sum with respect to the index α then we will have:

$$\sum_\alpha \cos(nx_\alpha)\sum_\mu \Delta_{\lambda\mu}^\alpha \varphi_\mu = -\frac{1}{4\pi}\int_\omega (T_1\tau_\lambda^1 + T_2\tau_\lambda^2 + T_3\tau_\lambda^3)\,d\omega\,. \tag{4}$$

Now, the integral in the right-hand side has the same form as the integral that defines the function ϖ, so we can account for the discontinuity in the expression (4) by means of the following formulas:

$$\left[\sum_{\alpha\mu}\cos(nx_\alpha)\Delta_{\lambda\mu}^\alpha\varphi_\mu\right]_s^{is} = \tfrac{1}{2}T_\lambda\,,$$

$$\left[\sum_{\alpha\mu}\cos(nx_\alpha)\Delta_{\lambda\mu}^\alpha\varphi_\mu\right]_s^{es} = -\tfrac{1}{2}T_\lambda\,.$$

Now, recall the study of the integral ϑ, while supposing that the functions U_ρ admit continuous first-order derivatives. By virtue of that hypothesis, one can write, upon applying a well-known formula of potential theory:

$$\frac{\partial\vartheta}{\partial x_\alpha} = \frac{1}{4\pi}\int_\omega \sum_\rho U_\rho\beta_\rho\cos(nx_\alpha)\,d\omega - \frac{1}{4\pi}\int_S \sum_\rho \frac{\partial U_\rho}{\partial x_\alpha}\beta_\rho\,dS\,.$$

Upon equating the coefficients of k_μ in the two sides of this equation and replacing the functions $\beta_{\rho\mu}$ with the equivalent expression $\alpha_{\mu\rho}$, one will find that:

$$\frac{\partial \vartheta_\mu}{\partial x_\alpha} = \frac{1}{4\pi}\int_\omega \sum_\rho U_\rho \alpha_{\mu\rho} \cos(nx_\alpha)\, d\omega - \frac{1}{4\pi}\int_S \sum_\rho \frac{\partial U_\rho}{\partial x_\alpha}\alpha_{\mu\rho}\, dS\,.$$

If one takes the derivatives of both sides with respect to x_β then one will get:

$$\frac{\partial^2 \vartheta_\mu}{\partial x_\alpha\, \partial x_\beta} = \frac{1}{4\pi}\int_\omega \sum_\rho U_\rho \frac{\partial \alpha_{\mu\rho}}{\partial x_\beta} \cos(nx_\alpha)\, d\omega - \frac{1}{4\pi}\int_S \sum_\rho \frac{\partial U_\rho}{\partial x_\alpha}\frac{\partial \alpha_{\mu\rho}}{\partial x_\beta}\, dS\,,$$

in which the last term is a continuous function.

Upon multiplying both sides of the equation by the coefficient $\begin{pmatrix}\lambda\mu\\ \alpha\beta\end{pmatrix}$ and summing over the indices α and β, one will find that:

$$\Delta_{\lambda\mu}\, \vartheta_\mu = \frac{1}{4\pi}\int_\omega \sum_\alpha \sum_\rho U_\rho \sum_\beta \begin{pmatrix}\lambda\mu\\ \alpha\beta\end{pmatrix} \frac{\partial \alpha_{\mu\rho}}{\partial x_\beta} \cos(nx_\alpha)\, d\omega \text{ + a continuous function.}$$

If one recalls the formula:

$$\sum_\beta \begin{pmatrix}\lambda\mu\\ \alpha\beta\end{pmatrix} \frac{\partial \alpha_{\mu\rho}}{\partial x_\beta} = -\,\Delta^\alpha_{\lambda\mu}\alpha_{\mu\rho}$$

then one can write the expression for $\Delta_{\lambda\mu}\, \vartheta_\mu$:

$$\Delta_{\lambda\mu}\, \vartheta_\mu = -\frac{1}{4\pi}\int_\omega \sum_\alpha \sum_\rho U_\rho \Delta^\alpha_{\lambda\mu}\alpha_{\mu\rho} \cos(nx_\alpha)\, d\omega \text{ + a continuous function.}$$

If we sum over μ then we will get:

$$\sum_\mu \Delta_{\lambda\mu}\vartheta_\mu = -\frac{1}{4\pi}\int_\omega \sum_\alpha \sum_\rho U_\rho \sum_\mu \Delta^\alpha_{\lambda\mu}\, \alpha_{\mu\rho} \cos(nx_\alpha)\, d\omega \text{ + a continuous function.}$$

However, one has the formula:

$$\sum_\alpha \sum_\mu \Delta^\alpha_{\lambda\mu}\, \alpha_{\mu\rho} \cos(nx_\alpha) = \tau^\rho_\lambda\,,$$

with the aid of which, one will get:

$$\sum_\mu \Delta_{\lambda\mu}\vartheta_\mu = -\frac{1}{4\pi}\int_\omega \sum_\rho U_\rho \tau^\rho_\lambda\, d\omega \text{ + a continuous function.}$$

We have proved that the integral that appears in this formula will experience a brief reduction that is equal to U_λ when the point $A(x_1, x_2, x_3)$ passes from the interior to the exterior of the surface ω. However, we know that $\sum_\mu \Delta_{\lambda\mu} \vartheta_\mu$ is equal to zero for the points that are exterior to S; as a result, one will have:

$$\sum_\mu \Delta_{\lambda\mu} \vartheta_\mu = U_\lambda \qquad (\lambda = 1, 2, 3).$$

It results from this that we have proved that the functions ϑ give us the solution to the following problem:

Determine the state of deformation of an unbounded elastic medium when that medium is subject to forces that act upon the volume elements that are interior to a certain surface and that one can suppose the deformation is zero at an infinite distance.

In particular, if one takes the volume S to be infinitely small then one will find that:

$$\vartheta = -\frac{1}{4\pi} \sum_\rho \beta_\rho \int_S U_\rho \, dS,$$

so, upon setting $\int_S U_\rho \, dS = -X_\rho$, one will deduce:

$$\vartheta_\nu = \frac{1}{4\pi}(X_1 \beta_{1\nu} + X_2 \beta_{2\nu} + X_3 \beta_{3\nu}).$$

It is then clear that these functions ϑ represent the components of the deformation in the bounded case, in which the medium is subjected to a force whose components are X_1, X_2, X_3 at just one point.

§ 4. Use of compensating functions.

By taking one's inspiration from GREEN's ideas, one can reduce the general problem of the equilibrium of an elastic body to some particular problems of the same nature.

First, envision the case in which one is given the components of the deformation on the surface ω of a body S, and the forces that act upon the volume elements of the body.

If one can resolve the problem of equilibrium in the particular case in which the components of the deformation on the surface ω are equal to the functions α_ν ($\xi_1 - x_1$, $\xi_2 - x_2$, $\xi_3 - x_3$), and the interior forces are equal to zero then one can also solve the general problem. Let γ_1, γ_2, γ_3 denote the components of the deformation in the particular problem and the components of the pressure on the surface by Γ_1, Γ_2, Γ_3. BETTI's theorem then gives us:

$$0 = \frac{1}{4\pi} \int_\omega \sum_\rho (\gamma_\rho T_\rho - \Gamma_\rho u_\rho) \, d\omega - \frac{1}{4\pi} \int_S \sum_\rho U_\rho \gamma_\rho \, dS.$$

One will find the solution to the general problem upon forming the difference between this expression and the right-hand side of equation (15*a*), § 3, Chap. II:

$$k_1\, u_1 + k_2\, u_2 + k_3\, u_3 = \frac{1}{4\pi}\int_\omega \sum_\rho (\Gamma_\rho - \sigma_\rho) u_\rho \, d\omega - \frac{1}{4\pi}\int_S \sum_\rho U_\rho (\alpha_\rho - \gamma_\rho)\, dS\,.$$

Consider the second problem, in which the forces that act upon the surface are known; let T_ρ denote the components of these forces; Let U_1, U_2, U_3 denote the components of the force that act upon a volume element of S. One then knows that the body S must be in equilibrium under the influence of these forces, which implies the six conditions:

$$\int_\omega T_\rho d\omega + \int_S U_\rho dS = 0,$$

$$\int_\omega (\xi_\lambda T_\rho - \xi_\rho T_\lambda)\, d\omega + \int_S (\xi_\lambda U_\rho - \xi_\rho U_\lambda)\, dS = 0.$$

The solution of this equilibrium problem is not unique. If one lets u_1, u_2, u_3 be functions that give a solution then one will obtain all of the other ones from the formulas:

$$u_\lambda + a_\lambda + p_\mu\, x_\nu - p_\nu\, x_\mu\,,$$

in which a_λ and p_μ denote constants.

The deformation that is defined by the functions α_1, α_2, α_3 corresponds to components of the pressure that are equal to τ_1, τ_2, τ_3. We have already found that $\int_\omega \tau_\rho d\omega = 4\pi\, k_\rho$. Upon substituting linear functions for the u in formula (15*a*), Chapter II, one will find that:

$$\int_\omega (\xi_\lambda \tau_\rho - \xi_\rho \tau_\lambda)\, d\omega = 4\pi\,(x_\lambda\, k_\rho - x_\rho\, k_\lambda).$$

Now, take the origin to be the center of gravity of the surface ω, and take the axes to be the principal axes of inertia for the surface ω.

Apply forces to the surface ω whose components are:

$$t_\lambda = b_\lambda + c_\mu\, x_\nu - c_\nu\, x_\mu \qquad (\lambda, \mu, \nu = 1, 2, 3).$$

If we write down the conditions for the forces – t and τ to bring about equilibrium then we will have:

$$\int_\omega t_\lambda d\omega = b_\lambda \cdot \int_\omega d\omega = 4\pi\, k_\lambda\,,$$

$$\int_\omega (\xi_\lambda t_\rho - \xi_\rho t_\lambda)\, d\omega = c_\mu \cdot \int_\omega (\xi_\lambda^2 + \xi_\rho^2)\, d\omega = 4\pi\,(x_\lambda\, k_\rho - x_\rho\, k_\lambda).$$

One sees that the values of the coefficients b and c that satisfy these equations are linear functions of the variables x_ν. Provided that the coefficients b and c are chosen in such a manner that they satisfy the equilibrium conditions, and one makes the hypothesis that the problem of equilibrium is possible, one can solve that problem in the particular case in which the forces that act upon ω are equal to:

$$\tau_\rho - t_\rho \,.$$

In that particular case, let δ_1, δ_2, δ_3 denote the components of the deformation, and let Λ_1, Λ_2, Λ_3 denote the components of the corresponding pressure. One has, in turn, the relations:

$$\tau_\rho - \Lambda_\rho = t_\rho$$

for the points on the surface ω.

Moreover, BETTI's theorem gives us:

$$0 = \frac{1}{4\pi}\int_\omega \sum (\delta_\rho T_\rho - \Lambda_\rho u_\rho)\, d\omega - \frac{1}{4\pi}\int_S \sum U_\rho \delta_\rho \, dS \,,$$

and formula (15), Chapter II gives us:

$$k_1 u_1 + k_2 u_2 + k_3 u_3 = \frac{1}{4\pi}\int_\omega \sum (T_\rho \alpha_\rho - u_\rho \tau_\rho)\, d\omega - \frac{1}{4\pi}\int_S \sum U_\rho \alpha_\rho \, dS \,,$$

so, upon taking the difference, one will get:

$$k_1 u_1 + k_2 u_2 + k_3 u_3 = \frac{1}{4\pi}\int_\omega \sum T_\rho (\delta_\rho - \alpha_\rho)\, d\omega - \frac{1}{4\pi}\int_\omega \sum U_\rho (\tau_\rho - \Lambda_\rho)\, d\omega$$
$$- \frac{1}{4\pi}\int_S \sum U_\rho (\alpha_\rho - \delta_\rho)\, dS \,.$$

However, the second integral here is equal to a linear function of the variables x_ν that takes the form:

$$k_1 u_1 + k_2 u_2 + k_3 u_3 + \begin{vmatrix} k_1 & k_2 & k_3 \\ p_1 & p_2 & p_3 \\ x_1 & x_2 & x_3 \end{vmatrix},$$

and which, in turn, represents a simple displacement of the body. Upon supposing that this function is equal to zero, we will get the solution:

$$k_1 u_1 + k_2 u_2 + k_3 u_3 = \frac{1}{4\pi}\int_\omega \sum T_\rho (\delta_\rho - \alpha_\rho)\, d\omega - \frac{1}{4\pi}\int_S \sum U_\rho (\alpha_\rho - \delta_\rho)\, dS \,.$$
