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The electrodynamics of rotating electrons.

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The Uhlenbeck-Goudsmit conception of the rotating electron will be employed, following Thomas, for the presentation of the equations of motion in a given electromagnetic field by means of special relativity. The electron will thus be treated simply as a point whose magnetic properties are coupled with a well-defined six-vector ("moment tensor"). In this way, one arrives at the explanation that Thomas already gave for the origin of the anomalous Zeeman effect in a more thorough and rigorous way. In conclusion, the electromagnetic field that is generated by a "rotating" electron will be determined, and this will suggest the possibility that the structure of the atomic nucleus is induced mainly by the magnetostatic interaction between electrons and protons.

§ 1. Introduction. Uhlenbeck and Goudsmit [2]) have recently applied, to great effect, the concept of rotating quantized electrons that was already proposed by H. A. Compton to the problem of the multiplet structure of the spectral term in the domain of optics and Röntgen rays. They started from the fact that in a coordinate system S', in which a electron that circles a nucleus is at rest, an additional magnetic field strength arises:

$$\mathfrak{H}' = -\frac{1}{c}[\mathfrak{v}\ \mathfrak{E}]. \tag{1}$$

In this, $\mathfrak{v}$ means the translational velocity of the electron relative to the coordinate system S that is fixed in the nucleus, and $\mathfrak{E}$ is the electric field strength that prevails relative to this system and is generated by the nucleus.

If one now ascribes a proper magnetic moment $\mathfrak{m}'$ to the electron then the magnetic field strength (1) must correspond to an additional magnetic energy:

$$U' = -\mathfrak{m}'\ \mathfrak{H}' = \mathfrak{m}'\left[\frac{\mathfrak{v}}{c}\mathfrak{E}\right]. \tag{1a}$$

Uhlenbeck and Goudsmit have now shown that the structure of the optical and Röntgen multiplet terms are explained immediately when one ascribes the values $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc., to the azimuthal quantum number (k), in agreement with the Landé normalization,

[1]) International Education Board Fellow for 1926.

[2]) Nature **117**, 264, Feb. 20, 1926.

and adds the resulting "relativistic correction) for the thermal energy to the mean value of the extra magnetic energy (1a) under the assumption that $\mathfrak{m}'$ equals one-half the Bohr magneton and that the ratio of the magnetic moment $\mathfrak{m}'$ to the corresponding mechanical impulse moment has the same value $\frac{e}{2cm_0}$ ($e < 0$ is the charge of the electron, m_0 is its mass, c is the velocity of light) as it does for the orbital motion. In other words, the impulse moment of the "proper rotation" must therefore also be set equal to one-half the ordinary elementary Bohr value $h/2\pi$.

The concept of rotating electrons makes it further possible to give a complete explanation for the anomalous Zeeman effect (in which, e.g., the noteworthy *Paschen-Back effect* of the hydrogen lines is made understandable) when the "atomic hull" of the previous Sommerfeld-Landé schema is replaced with the proper rotation of the electron. However, one must, while preserving the previous value $\frac{1}{2}\frac{h}{2\pi}$ for the impulse moment of the electron, ascribe a magnetic moment that is twice as large $\mathfrak{m} = 2\mathfrak{m}'$, so it equals a complete Bohr magneton. The ratio of the two moments – viz., the magnetic and the mechanical ones – must then equal:

$$\kappa = \frac{e}{cm_0}, \tag{2}$$

by assumption, in contradiction to the previous assumption that necessarily arises for the explanation of the multiplet structure.

§ 2. The Thomas theory. Thomas [1]) sought to give a resolution of the contradiction on the grounds of the following relativistic argument:

One considers the electron at two successive points $t' = t$ and $t'' = t + dt$. Let the corresponding "rest systems" that result from S by a Lorentz transformation without rotation be S' and S''. It may now be easily shown that S' can be obtained from S'' directly by an infinitesimal Lorentz transformation that corresponds to an infinitesimal relative velocity $d\mathfrak{v} = \dot{\mathfrak{v}}\,dt$ ($\dot{\mathfrak{v}}$ = acceleration), and likewise an infinitesimal rotation of the coordinate axes that is given (approximately) by the vector:

$$d\mathfrak{w} = \frac{1}{2c^2}[\dot{\mathfrak{v}}\mathfrak{v}]\,dt\,. \tag{3}$$

Now, according to Thomas, the temporal change in the impulse moment of the electron $\mathfrak{m} / k$ must be determined by the usual differential equation:

[1]) Nature, April 10, 1926, pp. 514. The manuscript of this paper was cordially made available to me by Dr. W. Pauli at the end of February, and this gave rise to my own paper.

$$\frac{d'}{dt}\left(\frac{\mathfrak{m}}{\kappa}\right) = [\mathfrak{m}\ \mathfrak{H}'], \tag{4a}$$

where:

$$\frac{d'}{dt}\left(\frac{\mathfrak{m}}{\kappa}\right) = \frac{d}{dt}\left(\frac{\mathfrak{m}}{\kappa}\right) - \left[\frac{d\mathfrak{w}}{dt}\frac{\mathfrak{m}}{\kappa}\right] \tag{4b}$$

means the rate of change of the vector $\mathfrak{m} / \kappa$ relative to a coordinate system that goes from S' to S'' in a time dt by means of a translational acceleration $\dot{\mathfrak{v}}$ and a rotational velocity $d\mathfrak{w} / dt$.

If one replaces κ with the value (2) and observe that (in the first approximation) $\frac{m_0}{e}\dot{\mathfrak{v}} = \mathfrak{E}$ then, from (1), one has:

$$\left[\frac{d\mathfrak{w}}{dt}\frac{\mathfrak{m}}{\kappa}\right] = + \frac{1}{2}\left[\left[\frac{\mathfrak{v}}{c}E\right]\mathfrak{m}\right] = \frac{1}{2}[\mathfrak{m}\ \mathfrak{H}'],$$

and it results from (4a) and (4b) that:

$$\frac{d}{dt}\left(\frac{\mathfrak{m}}{\kappa}\right) = \frac{1}{2}[\mathfrak{m}\ \mathfrak{H}']. \tag{5}$$

In this way, we obtain an equation in the usual form (4) when we introduce the apparent moment:

$$\mathfrak{m}' = \frac{1}{2}\mathfrak{m}$$

in place of the actual magnetic moment $\mathfrak{m}$, and then replace the ratio κ with:

$$\kappa' = \frac{\kappa}{2}.$$

Thus, (5) becomes:

$$\frac{d}{dt}\left(\frac{\mathfrak{m}'}{\kappa'}\right) = [\mathfrak{m}'\ \mathfrak{H}'], \tag{5a}$$

in agreement with (4).

We then see that from the standpoint of the usual theory the temporal change in the impulse moment of the electron corresponds to a rotational force whose moment $\mathfrak{f}' = [\mathfrak{m}'\ \mathfrak{H}']$ equals one-half the actual rotational moment $\mathfrak{f} = [\mathfrak{m}\ \mathfrak{H}']$. Correspondingly, for the consideration of the change in energy that is induced by this rotational work, one must compute with an "apparent" magnetic energy $U' = -\frac{1}{2}(\mathfrak{m}\ \mathfrak{H}') = -(\mathfrak{m}'\ \mathfrak{H}')$.

Let it be remarked that the result above relates to the case in which no true magnetic field is present; i.e., when the magnetic field strength vanishes in the "nuclear coordinate system" *S*. If this field strength $\mathfrak{H}$ is non-zero then one must replace equation (5) with the following general equation:

$$\frac{d}{dt}\left(\frac{\mathfrak{m}}{\kappa}\right) = \frac{1}{2}[\mathfrak{m}\ \mathfrak{H}'] + [\mathfrak{m}\ \mathfrak{H}], \tag{5}$$

or:

$$\frac{d}{dt}\left(\frac{\mathfrak{m}'}{\kappa'}\right) = [\mathfrak{m}'\ \mathfrak{H}'] + 2\,[\mathfrak{m}'\ \mathfrak{H}].$$

The total magnetic energy is therefore expressed by the sum:

$$U = -\frac{1}{2}(\mathfrak{m}\ \mathfrak{H}') = (\mathfrak{m}\ \mathfrak{H}). \tag{6a}$$

The following objection can be raised against the Thomas argument.

First, one is dealing with the impulse moment and the magnetic moment of the electron as invariant quantities, which is certainly incorrect, since three-dimensional vectors must transform in a certain way under a Lorentz transformation.

Second, this theory relates exclusively to the "rotational motion" of the electrons. It should then follow that the complete magnetic moment, not one-half of it, is also appropriate to the translational motion in the case of $\mathfrak{H} = 0$, from the usual expression for the driving force ($\mathfrak{m}$ grad) $\mathfrak{H}'$. There is no proof that in the absence of an external magnetic field the precession velocity of the "electron axes" and that of the path plane are equal such that the resulting impulse moment would remain constant in quantity and direction.

In the sequel, we would like exhibit the precise equations of motion for the "rotating" electron by a consequently four-dimensional representation (in the sense of special relativity, just like Thomas) of the usual three-dimensional equations. Thus, one obtains a complete resolution of the contradiction that was suggested in § 1 between the explanation for the multiplet structure and the Zeeman effect.

In particular, it yields that the Thomas equation (5) determines, not the actual, but the mean, secular variation of the magnetic moment; i.e., it is only correct when one replaces $d / dt\, \mathfrak{m}$ and $\mathfrak{H}'$ with the corresponding mean values [1]).

§ 3. The moment tensor. We will ignore any sort of considerations regarding the structure of the electron from the outset and simply treat it as a point whose properties are characterized by certain scalar, vector, and tensor quantities.

We especially regard its magnetic properties in such a way that the given of the three-dimensional vector of the magnetic moment $\mathfrak{m}$ is fundamentally insufficient for its

[1]) From a written communication of Pauli that I received in connection with my paper, Thomas has developed the same theory as the one presented below, independently of myself. [Rem. by the editor]

complete characterization, since a three-dimensional vector must only be considered to be the spatial part (i.e., projection) of a four-dimensional vector (viz., a *four-vector*) or an anti-symmetric tensor (viz., a *six-vector*).

As is known, the magnetic field strength $\mathfrak{H}$ represents the spatial part of the electromagnetic field tensor $F_{\alpha\beta} = - F_{\beta\alpha}$ (α, β = 1, 2, 3, 4), whose temporal part determines the electric field strength $\mathfrak{E}$ according to the schema:

$$\begin{pmatrix} F_{23} & F_{31} & F_{12} & F_{14} & F_{24} & F_{34} \\ H_1 & H_2 & H_3 & -iE_1 & -iE_2 & -iE_3 \end{pmatrix}. \tag{I}$$

Correspondingly, we would like to define the magnetic moment of the electron $\mathfrak{m}$ as the spatial part of an anti-symmetric tensor $\mu_{\alpha\beta} = - \mu_{\beta\alpha}$ by means of the schema:

$$\begin{pmatrix} \mu_{23} & \mu_{31} & \mu_{12} & \mu_{14} & \mu_{24} & \mu_{34} \\ m_1 & m_2 & m_3 & +ip_1 & +ip_2 & +ip_3 \end{pmatrix}, \tag{II}$$

where p_1, p_2, p_3 are the spatial components of a three-dimensional vector $\mathfrak{p}$ that is analogous to the electric moment of a dipole [1]).

We would like to determine this vector from the condition that it should vanish ($\mathfrak{p}'$ = 0) in the coordinate system S' in which the electron is instantaneously at rest. It then follows, in an arbitrary coordinate system S relative to which the electron has the translational velocity $\mathfrak{v}$, from the known transformation formulas for the quantities (II) and (I), that:

$$\mathfrak{p} = \left[\frac{\mathfrak{v}}{c}\mathfrak{m}\right]. \tag{7}$$

One can also derive this result independently of the formulas above in the following way [2]): Let x_α be the coordinates of the electron and the time multiplied by ic ($ict = x_4$) relative to the system S. We construct the four-dimensional vector $\mu_{\alpha\beta}\dot{x}_\beta$ (the summation sign for equal index pairs will always omitted in what follows) from $\mu_{\alpha\beta}$ and $\dot{x}_\alpha = dx_\alpha / dt$, where $d\tau = dt\,\sqrt{1-v^2/c^2}$ means the proper time of the electron. The components of this vector $\mu'_{\alpha\beta}\dot{x}'_\beta$ vanish in the "rest system" S', so one has $\dot{x}'_1 = \dot{x}'_2 = \dot{x}'_3 = 0$, and from our assumption, $\mu'_{14} = \mu'_{24} = \mu'_{34} = 0$. However, it follows from this that for any other coordinate system S, the equations:

$$\mu_{\alpha\beta}\dot{x}_\beta = 0 \tag{7a}$$

1) This analogy will be clarified later on.

2) From a remark by W. Pauli.

are fulfilled, which express the vanishing of the vector above. If one substitutes the corresponding three-dimensional expressions for $\mu_{\alpha\beta}$ and $\dot{x}_\alpha$ then for all $\alpha = 1, 2, 3$ one gets the spatial components of the vector:

$$\frac{c}{\sqrt{1-v^2/c^2}}\left(\left[\frac{\mathfrak{v}}{c}\mathfrak{m}\right]-\mathfrak{p}\right),$$

while for $\alpha = 4$:

$$\mu_{\alpha\beta}\dot{x}_\beta = -\frac{i}{\sqrt{1-v^2/c^2}}(\mathfrak{v}\mathfrak{p}).$$

The vanishing of the second expression follows immediately from the vanishing of the first one – i.e., equation (7). As is known, one may define the following two invariant scalar quantities by means of the tensor components $\mu_{\alpha\beta} = -\mu_{\beta\alpha}$:

$$m^2 - p^2 = \tfrac{1}{2}\mu_{\alpha\beta}\,\mu_{\alpha\beta}$$

and

$$(\mathfrak{m}\ \mathfrak{p}) = i(\mu_{23}\,\mu_{14} + \mu_{31}\,\mu_{24} + \mu_{12}\,\mu_{34}).$$

Thus, due to (7) (i.e., since $\mathfrak{p}' = 0$), one has:

$$(\mathfrak{m}\ \mathfrak{p}) = \mathfrak{m}'\ \mathfrak{p}' = 0$$

and

$$m^2 - p^2 = m^2 - \left[\frac{\mathfrak{v}}{c}\mathfrak{m}\right]^2 = m'^2. \tag{8}$$

The last equation determines the independence of the magnetic moment of the electron from its translational velocity $\mathfrak{v}$. One can describe it in the form:

$$m = \frac{\mu}{\sqrt{1-v_\perp^2/c^2}},$$

where $v_\perp$ means the component of $\mathfrak{v}$ that is perpendicular to $\mathfrak{m}$; $m' = \mu$ is the magnitude of the magnetic moment in the "rest system."

§ 4. The temporal variation of the moment tensor. We now introduce the four-dimensional quantities that correspond to the magnetic energy – $(\mathfrak{m}\ \mathfrak{H}) = -m_\alpha H_\alpha$, and the magnetic rotational moment $[\mathfrak{m}\ \mathfrak{H}]$; i.e., the vector or anti-symmetric tensor with the components $m_\alpha H_\beta - m_\beta H_\alpha$. The four-dimensional "extension" of the energy function is obviously the scalar:

$$U = -\tfrac{1}{2}\mu_{\alpha\beta}F_{\alpha\beta} = -(\mathfrak{m}\ \mathfrak{H}) - (\mathfrak{p}\ \mathfrak{E}). \tag{9}$$

The corresponding "extension" for the rotational moment is given, as one easily recognizes, by the anti-symmetric four-dimensional tensor (i.e., the six-vector):

$$f_{\alpha\beta} = \mu_{\alpha\beta}F_{\beta\gamma} - \mu_{\beta\gamma}F_{\alpha\gamma} \tag{10}$$

with the spatial part:

$$(f_{23}, f_{31}, f_{12}) = [\mathfrak{m}\ \mathfrak{H}] + [\mathfrak{p}\ \mathfrak{E}] \tag{10a}$$

and the temporal part:

$$-i(f_{14}, f_{24}, f_{34}) = -[\mathfrak{m}\ \mathfrak{H}] - [\mathfrak{p}\ \mathfrak{E}]. \tag{10b}$$

We define the *impulse moment* of the electron to be the spatial part of the tensor:

$$\frac{1}{\kappa}\mu_{\alpha\beta}$$

with $\kappa = e / cm_0$.

The simplest four-dimensional "extension" of the differential equation (4) for the temporal variation of $\mu_{\alpha\beta}$ would then read:

$$\frac{\dot{\mu}_{\alpha\beta}}{\kappa} = f_{\alpha\beta}; \tag{11}$$

i.e.:

$$\frac{\dot{\mathfrak{m}}}{\kappa} = [\mathfrak{m}\ \mathfrak{H}] + [\mathfrak{p}\ \mathfrak{E}] \tag{11a}$$

and

$$\frac{\dot{\mathfrak{p}}}{\kappa} = [\mathfrak{p}\ \mathfrak{H}] - [\mathfrak{m}\ \mathfrak{E}], \tag{11b}$$

where the dot means differentiation with respect to proper time.

Equations (11a) and (11b) can be simultaneously satisfied only in the case where the vectors $\mathfrak{m}$ and $\mathfrak{v}$ are (*a priori*) independent of each other. However, the relation (7) must, in fact, exist between them, which means that equations (11a), (11b) are incompatible. Now, it is easy to modify the general equation (11) in such a way that the condition (7a) is fulfilled. To that end, we introduce an initially indeterminate four-dimensional vector a_α and define the invariant scalar:

$$-\mu_{\alpha\beta}a_\alpha\dot{x}_\beta = -\tfrac{1}{2}\mu_{\alpha\beta}(a_\alpha\dot{x}_\beta - a_\beta\dot{x}_\alpha), \tag{12}$$

which vanishes, from (7a). We add this scalar to the *energy function U*; i.e., replace the former with:

$$U' = -\tfrac{1}{2}\mu_{\alpha\beta}(F_{\alpha\beta} + a_\alpha\dot{x}_\beta - a_\beta\dot{x}_\alpha) = -\tfrac{1}{2}\mu_{\alpha\beta}F'_{\alpha\beta}. \tag{12a}$$

We correspondingly replace the tensor $f_{\alpha\beta}$ with:

$$f'_{\alpha\beta} = \mu_{\alpha\beta}F'_{\beta\gamma} - \mu_{\beta\gamma}F'_{\alpha\gamma}, \tag{12b}$$

i.e.:

$$f'_{\alpha\beta} = f_{\alpha\beta} + a_\gamma(\dot{x}_\alpha\mu_{\beta\gamma} - \dot{x}_\beta\mu_{\alpha\gamma}), \tag{12c}$$

and the "equation of motion" (11) with:

$$\frac{\dot{\mu}_{\alpha\beta}}{\kappa} = f'_{\alpha\beta}, \tag{13}$$

or, when written out in full:

$$\frac{\dot{\mu}_{\alpha\beta}}{\kappa} = \mu_{\alpha\beta}F_{\alpha\beta} - \mu_{\alpha\beta}F_{\alpha\beta} + a_\gamma(\dot{x}_\alpha\mu_{\beta\gamma} - \dot{x}_\beta\mu_{\alpha\gamma}). \tag{13a}$$

We now determine the vector a_α in such a way that this equation is in harmony with the relation (7a). Moreover, it indeed follows from (13a), with consideration for (7a) and the identity relation:

$$\dot{x}_\alpha\dot{x}_\alpha = -c^2,$$

that:

$$\frac{\dot{\mu}_{\alpha\beta}}{\kappa}\dot{x}_\beta = -\frac{1}{\kappa}\mu_{\alpha\beta}\ddot{x}_\beta = \mu_{\alpha\gamma}F_{\beta\gamma}\dot{x}_\beta - a_\gamma\mu_{\alpha\gamma}\dot{x}_\beta\dot{x}_\beta = \mu_{\alpha\beta}(F_{\beta\gamma}\dot{x}_\beta + a_\gamma c^2),$$

or

$$\mu_{\alpha\gamma}\left(\frac{\ddot{x}_\gamma}{\kappa} + F_{\beta\gamma}\dot{x}_\beta + a_\gamma c^2\right) = 0.$$

One thus infers that:

$$a_\gamma = \frac{1}{\kappa c^2}(\kappa F_{\gamma\beta}\dot{x}_\beta - \ddot{x}_\gamma). \tag{14}$$

Independently of this expression for a_γ, one gets from (13a), with consideration of (7a):

$$\frac{1}{\kappa}\dot{\mu}_{\alpha\beta}\dot{\mu}_{\alpha\beta} = \mu_{\alpha\beta}\mu_{\alpha\gamma}F_{\beta\gamma} - \mu_{\alpha\beta}\mu_{\beta\gamma}F_{\alpha\gamma} = 2\mu_{\alpha\beta}\mu_{\alpha\gamma}F_{\beta\gamma} = 0$$

(due to the anti-symmetric character of $F_{\beta\gamma}$); i.e.:

$$\frac{d}{d\tau}\mu_{\alpha\beta}^2 = 0,$$

or

$$\tfrac{1}{2}\mu_{\alpha\beta}^2 = m^2 - p^2 = \mu^2 = \text{const.} \tag{15}$$

This formula shows that the magnetic moment of the electron (as assessed in a "rest system") can, in fact, be quantized. If its magnitude were not constant then one could not speak of it being quantized.

As is known, the equations of motion of a non-magnetic electron read:

$$m_0 \ddot{x}_\alpha = \frac{e}{c} F_{\alpha\beta} \dot{x}_\beta ,$$

or, with $\frac{e}{m_0 c} = \kappa$:

$$\ddot{x}_\alpha = \kappa F_{\alpha\beta} \dot{x}_\beta . \tag{15a}$$

If one neglects the force that arises from the magnetic moment in comparison to the Lorentz force $e\left(\mathfrak{E} + \left[\frac{\mathfrak{v}}{c}\mathfrak{H}\right]\right)$, which corresponds to the four-vector $\frac{e}{c} F_{\alpha\beta} \dot{x}_\beta$, one will have, from (15) and (15a):

$$a_\gamma \approx 0. \tag{15b}$$

In this approximation – i.e., upon neglecting the perturbation to the translational motion of the electron that is implied by the magnetic force – one can thus determine its "rotational motion" – i.e., the temporal variation of the vector $\mathfrak{m}$ – by way of the simple equations (11) or (11a).

If one substitutes $\mathfrak{p} = \left[\frac{\mathfrak{v}}{c}\mathfrak{m}\right]$ in (11), using (7), then one has:

$$\frac{\dot{\mathfrak{m}}}{\kappa} \approx [\mathfrak{m}\ \mathfrak{H}] + \left[\left[\frac{\mathfrak{v}}{c}\mathfrak{m}\right]\mathfrak{E}\right]. \tag{16}$$

We now consider the case in which the electron moves around the nucleus in a weak external magnetic field $\mathfrak{H}$. In a still larger degree of approximation (viz., by neglecting the terms that are quadratic in $1/c$), one can then set:

$$\mathfrak{E} \sim \frac{m_0}{e}\frac{d\mathfrak{v}}{dt}. \tag{16a}$$

With that, the second term on the right-hand side of (16) assumes the form:

$$\frac{m_0}{ec}\left[[\mathfrak{v}\,\mathfrak{m}]\frac{d\mathfrak{v}}{dt}\right].$$

We would now like to compute the mean value of this expression for the unperturbed motion.

One has (for the unperturbed motion!):

$$\overline{\frac{d}{dx}[[\mathfrak{v}\,\mathfrak{m}]\mathfrak{v}]} = \overline{\left[[\mathfrak{v}\,\mathfrak{m}]\frac{d\mathfrak{v}}{dx}\right]} + \left[\left[\frac{d\mathfrak{v}}{dx}\mathfrak{m}\right]\mathfrak{v}\right] = 0.$$

One further has the identity:

$$\left[[\mathfrak{v}\,\mathfrak{m}]\frac{d\mathfrak{v}}{dx}\right]+\left[\left[\mathfrak{m}\frac{d\mathfrak{v}}{dx}\right]\mathfrak{v}\right]+\left[\left[\frac{d\mathfrak{v}}{dx}\mathfrak{v}\right]\mathfrak{m}\right] = 0.$$

From this, it follows that:

$$\overline{\left[[\mathfrak{v}\,\mathfrak{m}]\frac{d\mathfrak{v}}{dx}\right]} = \frac{1}{2}\left[\mathfrak{m}\overline{\left[\frac{d\mathfrak{v}}{dx}\mathfrak{v}\right]}\right],$$

or, from (16a) and (1):

$$\overline{\left[\left[\frac{\mathfrak{v}}{\mathfrak{c}}\mathfrak{m}\right]\mathfrak{E}\right]} \approx \frac{1}{2c}[\mathfrak{m}[\overline{\mathfrak{E}}\mathfrak{v}]] = \frac{1}{2}[\mathfrak{m}\overline{\mathfrak{H}'}]. \tag{16b}$$

The secular variation of the vector $\mathfrak{m}$ is then determined in the approximation above from the equation:

$$\frac{1}{\kappa}\overline{\frac{d\mathfrak{m}}{dt}} \approx [\mathfrak{m}\;\mathfrak{H}] + \tfrac{1}{2}[\mathfrak{m}\overline{\mathfrak{H}'}]. \tag{17}$$

This is the corrected Thomas equation (6).

§ 5. Derivation of the equations of motion from Hamilton's principle. We will now carry out a more rigorous derivation of the differential equation (13) for the "rotational motion" of the electron on the basis of Hamilton's principle. With that, we will, at the same time, obtain the precise differential equation for the translational motion.

We thus set, as usual:

$$\delta\int L\,d\tau = 0, \tag{18}$$

with the supplementary conditions:

$$\dot{x}_\alpha^2 = -c^2, \tag{18a}$$

$$\mu_{\alpha\beta}\,\dot{x}_\beta = 0. \tag{18b}$$

We then write the Lagrangian function in the form:

$$L = \frac{e}{c}\varphi_\alpha\dot{x}_\alpha + T^* + \tfrac{1}{2}\mu_{\alpha\beta}F_{\alpha\beta}, \tag{19}$$

where T^* means the *kinetic energy* of the rotational motion.

We consider this energy, in the context of the usual three-dimensional mechanics, as a function of the "angular velocity," which we will characterize by the anti-symmetric tensor $\omega_{\alpha\beta} = -\omega_{\alpha\beta}$. We then set, by definition:

$$\delta T^* = \frac{\mu_{\alpha\beta}}{2\kappa}\delta\omega_{\alpha\beta}\,. \tag{19a}$$

In order to determine the variation of $\mu_{\alpha\beta}$, we next observe the corresponding operation in ordinary mechanics. The work done by the magnetic torque $[\mathfrak{m}\ \mathfrak{H}]$ for a virtual infinitesimal rotation $\delta\mathfrak{w}$ is equal to the inner product $(\delta\mathfrak{w}\ [\mathfrak{m}\ \mathfrak{H}])$. On the other hand, it must be equal to the increase in magnetic energy $-\ \delta(-\ \mathfrak{m}\ \mathfrak{H}) = (\delta\mathfrak{m}, \mathfrak{H})$. One then has $(\delta\mathfrak{m}, \mathfrak{H}) = (\delta\mathfrak{w}, [\mathfrak{m}\ \mathfrak{H}])$ or $(\delta\mathfrak{m}, \mathfrak{H}) = ([\delta\mathfrak{w}, \mathfrak{m}]\ \mathfrak{H})$, and as a result:

$$\delta\mathfrak{m} = [\delta\mathfrak{w}, \mathfrak{m}].$$

The corresponding four-dimensional variational formula must be derived in the same way that formula (10) is derived from the three-dimensional expression for the rotational moment $[\mathfrak{m}\ \mathfrak{H}]$. If one then introduces the four-dimensional anti-symmetric "rotation tensor" $\delta\Omega_{\alpha\beta}$, whose spatial part is equal to the vector $\delta\mathfrak{w}$, then one has:

$$\delta\mu_{\alpha\beta} = \delta\Omega_{\alpha\gamma}\,\mu_{\beta\gamma} - \delta\Omega_{\beta\gamma}\,\mu_{\alpha\gamma}\,. \tag{13b}$$

It is self-explanatory that the quantities $\delta\Omega_{\alpha\beta}$ (like the $\delta\mathfrak{w}$) do not represent exact differentials – i.e., there is no "angle coordinate" $\Omega_{\alpha\beta}$ that corresponds to the coordinates x_α (so one has an anholonomic system). Nevertheless, along with the relations:

$$\delta\dot{x}_\alpha = \frac{d}{d\tau}\delta x_\alpha\,, \tag{20}$$

one must obviously also have the corresponding commutation relations for $\delta\Omega_{\alpha\beta}$ and $d\Omega_{\alpha\beta} = \omega_{\alpha\beta}\ d\tau$; i.e.:

$$\delta\omega_{\alpha\beta} = \frac{d}{d\tau}\delta\Omega_{\alpha\beta}\,. \tag{20a}$$

By means of the formulas above and the relations:

$$\delta\varphi_\alpha = \frac{\partial\varphi_\alpha}{\partial x_\gamma}\delta x_\gamma\,, \qquad \dot{\varphi}_\alpha = \frac{\partial\varphi_\alpha}{\partial x_\gamma}\dot{x}_\gamma\,,$$

$$\delta F_{\alpha\beta} = \frac{\partial F_{\alpha\beta}}{\partial x_\gamma}\delta x_\gamma\,, \qquad \dot{F}_{\alpha\beta} = \frac{\partial F_{\alpha\beta}}{\partial x_\gamma}\dot{x}_\gamma\,,$$

we get:

$$\delta L = \frac{e}{c}\frac{\partial\varphi_\alpha}{\partial x_\gamma}\dot{x}_\alpha\delta x_\gamma - \frac{e}{c}\frac{\partial\varphi_\alpha}{\partial x_\gamma}\dot{x}_\gamma\delta x_\alpha + \frac{d}{d\tau}\left(\frac{e}{c}\varphi_\alpha\delta x_\alpha\right) - \frac{\dot{\mu}_{\alpha\beta}}{2\kappa}\delta\Omega_{\alpha\beta}$$

$$+\frac{d}{d\tau}\left(\frac{\mu_{\alpha\beta}}{2\kappa}\delta\Omega_{\alpha\beta}\right)+\frac{1}{2}\mu_{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial x_\gamma}\delta x_\gamma+\frac{1}{2}F_{\alpha\beta}(\delta\Omega_{\alpha\gamma}\mu_{\beta\gamma}-\delta\Omega_{\beta\gamma}\mu_{\alpha\gamma}),$$

or since:

$$\frac{\partial\varphi_\alpha}{\partial x_\beta}-\frac{\partial\varphi_\beta}{\partial x_\alpha}=F_{\beta\alpha},$$

one has:

$$\delta L=\left(\frac{e}{c}F_{\alpha\beta}\dot{x}_\beta+\frac{1}{2}\mu_{\beta\gamma}\frac{\partial F_{\beta\gamma}}{\partial x_\alpha}\right)\delta x_\alpha$$

$$+\frac{1}{2}\left(-\frac{\dot{\mu}_{\alpha\beta}}{\kappa}+\mu_{\alpha\beta}F_{\beta\gamma}-\mu_{\beta\gamma}F_{\alpha\gamma}\right)\delta\Omega_{\alpha\beta}+\frac{d}{d\tau}\left(\frac{e}{c}\varphi_\alpha\delta x_\alpha+\frac{\mu_{\alpha\beta}}{2\kappa}\delta\Omega_{\alpha\beta}\right).$$

Likewise, from (18a) and (18b), with the addition of the undetermined Lagrange multipliers λ and a_α (α = 1, 2, 3, 4):

$$\lambda\dot{x}_\alpha\delta\dot{x}_\alpha=-\delta x_\alpha\frac{d}{d\tau}(\lambda\dot{x}_\alpha)+\frac{d}{d\tau}(\lambda\dot{x}_\alpha\delta x_\alpha)=0$$

and

$$a_\alpha\delta(\mu_{\alpha\beta}\dot{x}_\beta)=\frac{d}{d\tau}(a_\alpha\mu_{\alpha\beta}\delta x_\beta)-\delta x_\alpha\frac{d}{d\tau}(\mu_{\beta\alpha}a_\beta)+\frac{1}{2}\delta\Omega_{\alpha\beta}a_\gamma(\dot{x}_\alpha\mu_{\beta\gamma}-\dot{x}_\beta\mu_{\alpha\gamma})=0.$$

With the usual assumption that the variations δx_α, $\delta\Omega_{\alpha\beta}$ vanish at the boundaries of the integral (18), it then follows from (18), (18a), and (18b) (by adding the above expressions and setting the coefficients of δx_α and $\delta\Omega_{\alpha\beta}$ equal to zero) that:

$$\frac{d}{d\tau}(\lambda\dot{x}_\alpha+\mu_{\beta\alpha}a_\beta)=\frac{e}{c}F_{\alpha\beta}\dot{x}_\beta+\frac{1}{2}\mu_{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial x_\alpha}\tag{21}$$

and

$$\frac{1}{\kappa}\dot{\mu}_{\alpha\beta}=\mu_{\alpha\gamma}F_{\beta\gamma}-\mu_{\alpha\gamma}F_{\beta\gamma}+a_\gamma(\dot{x}_\alpha\mu_{\beta\gamma}-\dot{x}_\beta\mu_{\alpha\gamma}).$$

The latter equation agrees with (13a); the former one is the generalization of the usual equation of motion (15a) for a non-magnetic electron.

We correspondingly set:

$$\lambda=m_0+\lambda',\tag{21a}$$

where λ' means an additional term that is independent of the magnetic moment of the electron. After performing the differentiation on the left-hand side of (21), we get, from (15):

$$\lambda'\ddot{x}_\alpha+\dot{\lambda}'\dot{x}_\alpha+\mu_{\beta\alpha}\dot{a}_\alpha+\dot{\mu}_{\beta\alpha}a_\alpha=\kappa m_0c^2a_\alpha+\frac{1}{2}\mu_{\alpha\beta}\frac{\partial F_{\beta\gamma}}{\partial x_\alpha}.$$

From this, it follows by multiplication by $\dot{x}_\alpha$, due to the relations $\dot{x}_\alpha^2 = -c^2$, $\dot{x}_\alpha \ddot{x}_\alpha = 0$ and $a_\alpha \dot{x}_\alpha = 0$, that:

$$-c^2 \dot{\lambda}' + \dot{\mu}_{\beta\alpha} a_\beta \dot{x}_\alpha = \frac{1}{2} \mu_{\beta\gamma} \frac{\partial F_{\beta\gamma}}{\partial x_\alpha} \dot{x}_\alpha = \frac{1}{2} \mu_{\beta\gamma} \frac{dF_{\beta\gamma}}{d\tau},$$

or:

$$-c^2 \dot{\lambda}' = \frac{d}{d\tau}\left(\frac{1}{2} \mu_{\beta\gamma} F_{\beta\gamma}\right) - \frac{1}{2} \dot{\mu}_{\alpha\beta} (F_{\alpha\beta} + a_\alpha \dot{x}_\beta - a_\beta \dot{x}_\alpha).$$

From (12a), (12b), and (13), we have:

$$\frac{1}{2} \dot{\mu}_{\alpha\beta} (F_{\alpha\beta} + a_\alpha \dot{x}_\beta - a_\beta \dot{x}_\alpha) = \frac{1}{2} \dot{\mu}_{\alpha\beta} F'_{\alpha\beta} = \frac{\kappa}{2} (\mu_{\alpha\gamma} F'_{\beta\gamma} - \mu_{\beta\gamma} F'_{\alpha\gamma}) F'_{\alpha\beta}$$
$$= \frac{\kappa}{2} (\mu_{\alpha\beta} F'_{\gamma\beta} F'_{\alpha\gamma} - \mu_{\beta\alpha} F'_{\gamma\alpha} F'_{\gamma\beta}) = \kappa \mu_{\alpha\beta} F'_{\alpha\gamma} F'_{\beta\gamma} = 0,$$

due to the anti-symmetric character of the tensor $\mu_{\alpha\beta}$. As a consequence, one has:

$$\lambda' = -\frac{1}{2c^2} \mu_{\alpha\beta} F_{\alpha\beta}. \tag{21b}$$

The increase in the mass m_0 is then equal to the *relativistic magnetic energy* of the electrons (relative to the nucleus and other particles that generate the field $F_{\alpha\beta}$), divided by the square of the velocity of light.

One can interpret the expression $\mu_{\alpha\beta}\, a_\alpha$ in (21) as the α-component of the additional impulse that originates in the absolute energy of the electron; i.e., the kinetic energy of its rotation.

By substituting (21a) in (21), this yields, due to (15):

$$\frac{d}{d\tau}(\lambda' \dot{x}_\alpha + \mu_{\alpha\beta} a_\alpha) = c^2 m_0 \kappa a_\alpha + \frac{1}{2} \mu_{\beta\gamma} \frac{\partial F_{\beta\gamma}}{\partial x_\alpha}. \tag{22}$$

One can use this equation only for the approximate determination of a_α. Moreover, when one neglects the left-hand side of (22) (since $c^2 m_0 \kappa = e\, c$), one indeed has:

$$a_\alpha = -\frac{1}{2ec} \mu_{\beta\gamma} \frac{\partial F_{\beta\gamma}}{\partial x_\alpha}. \tag{22a}$$

§ 6. The translational motion of the "rotating" electron in an atom. From equation (21), it follows that:

$$x_\alpha \frac{d}{d\tau}(\lambda \dot{x}_\beta + \mu_{\gamma\beta} a_\gamma) - x_\beta \frac{d}{d\tau}(\lambda \dot{x}_\alpha + \mu_{\gamma\alpha} a_\gamma) = \frac{1}{2}\mu_{\rho\sigma}\left(x_\alpha \frac{\partial F_{\rho\sigma}}{\partial x_\beta} - x_\beta \frac{\partial F_{\rho\sigma}}{\partial x_\alpha} \right),$$

or

$$\frac{d}{d\tau}\left\{ \lambda (x_\alpha \dot{x}_\beta - x_\beta \dot{x}_\alpha) + a_\gamma (x_\alpha \mu_{\gamma\beta} - x_\beta \mu_{\gamma\alpha}) \right\}$$
$$= \frac{1}{2}\mu_{\rho\sigma}\left(x_\alpha \frac{\partial F_{\rho\sigma}}{\partial x_\beta} - x_\beta \frac{\partial F_{\rho\sigma}}{\partial x_\alpha} \right) - a_\gamma (\dot{x}_\alpha \mu_{\beta\gamma} - \dot{x}_\beta \mu_{\alpha\gamma}) . \tag{23}$$

This equation can be regarded as the generalization of the "law of areas"; i.e., the usual formula for the rate of change of the ordinary impulse moment of the translational motion $m_0\left[\mathfrak{r} \frac{d\mathfrak{r}}{d\tau} \right]$. This impulse moment will then be replaced by the anti-symmetric tensor:

$$I_{\alpha\beta} = l\ (x_\alpha \dot{x}_\beta - x_\alpha \dot{x}_\alpha) + a_\gamma (x_\alpha \mu_{\gamma\beta} - x_\beta \mu_{\gamma\alpha}) , \tag{23a}$$

whose spatial part agrees with $m_0[\mathfrak{r}\,\dot{\mathfrak{r}}]$ in the first approximation. Let it be further remarked that the second term on the right-hand side of (23) is equal and opposite to the corresponding additional term in formula (13a) for the rate of change of the impulse moment of the rotational motion. If one sets:

$$\frac{\mu_{\alpha\beta}}{\kappa} = i_{\alpha\beta}$$

then, from (13a) and (23), the sum of both moments becomes:

$$\frac{d}{d\tau}(i_{\alpha\beta} + I_{\alpha\beta}) = \mu_{\alpha\gamma} F_{\beta\gamma} - \mu_{\beta\gamma} F_{\alpha\gamma} + x_\beta \frac{\partial U}{\partial x_\alpha} - x_\alpha \frac{\partial U}{\partial x_\beta} , \tag{23b}$$

where U means the *relative energy:*

$$U = -\tfrac{1}{2}\ \mu_{\alpha\gamma} F_{\beta\gamma} .$$

We now consider the case in which the electron moves in a radially-symmetric electric field $\mathfrak{E} = \psi(r)\ \mathfrak{r}$ in the absence of an (external) magnetic field. In this case, one has $U = -(\mathfrak{p}\ \mathfrak{E}) = -\psi(\mathfrak{p}\ \mathfrak{r})$, and as a result, for a, b = 1, 2, 3:

$$x_\beta \frac{\partial U}{\partial x_\alpha} - x_\alpha \frac{\partial U}{\partial x_\beta} = \psi(x_\alpha p_\beta - x_\beta p_\alpha) = E_\alpha p_\beta - E_\beta p_\alpha .$$

The resulting angular momentum, which corresponds to the spatial part of the tensor on the right-hand side of (23b), will then be equal to zero ($[p\ E] + [E\ p]) = 0$).

It follows from this that the resulting impulse moment of the electron in the case considered must remain constant in magnitude and direction.

As we already saw above [equation (15)], the magnitude of the tensor $\mu_{\alpha\beta}$, and consequently, also $i_{\alpha\beta}$, is constant in time. In the first approximation (viz., by neglecting the terms that are quadratic in $1/e$) one can, as a result, regard the magnitude of the impulse moment of the rotational motion $\mathfrak{i} = \frac{\mathfrak{m}}{\kappa}$ as constant in time. If one denotes the impulse moment of the translational motion (i.e., the spatial part of the tensor $I_{\alpha\beta}$) by $\mathfrak{J}$ then it follows, due to the condition $\mathfrak{i} + \mathfrak{J}$ = const., that the magnitude of $\mathfrak{J}$ also remains constant, and that both vectors $\mathfrak{i}$ and $\mathfrak{J}$ precess around their resultants with the same angular velocity. This result is very essential for atomic mechanics, since otherwise one could not quantize the impulse moment of an atom.

If one replaces the impulse moments $\mathfrak{i}$ and $\mathfrak{J}$ with the corresponding magnetic moments $\mathfrak{m} = \kappa\,\mathfrak{i}$ and $\mathfrak{M} = \frac{\kappa}{2}\mathfrak{J}$ then one sees that the sum $\mathfrak{m} + \mathfrak{M} = \frac{\kappa}{2}(\mathfrak{i} + \mathfrak{J}) + \frac{\kappa}{2}\mathfrak{i}$ does not represent a constant vector. Indeed, the magnitude of this vector remains constant, but its direction must precess around the atomic axis with the aforementioned angular velocity [1]). The angular velocity may not be simply determined; however, formula (17) shows that its mean value agrees with the ordinary Larmor velocity of the electron path in an external magnetic field $\bar{\mathfrak{H}}'$.

§ 7. The electromagnetic field of a "rotating" electron. If one considers the electron as a point charge and ignores its magnetic field then one can represent its electromagnetic field by the formula:

$$\varphi_\beta = \frac{k}{2\pi i}\oint \frac{dx'_\alpha}{S^2} = \frac{k}{2\pi i}\oint \frac{\left(\frac{dx'_\alpha}{d\tau'}\right)}{S^2}\,d\tau' \tag{24}$$

for the components of the four-potential. The integration is thus taken along a closed curve in the complex τ'-plane. [τ' is the proper time of the electron, $S^2 = \sum\,(x'_\alpha - x_\alpha)^2$ is its four-dimensional distance from the "origin" x_α; $k = 2e$] [2]).

If this curve encloses only one pole of the integrand, namely, the pole that corresponds to the real roots of the equation $R - c(t - t') = 0$:

$$[R^2 = \sum_{\alpha=1}^{3}(x'_\alpha - x_\alpha)^2\;],$$

then one gets the well-known Liénard-Wiechert formula for the retarded potential of a moving point charge by finding the residue:

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[1]) Such that no secular variation of $\mathfrak{m} + \mathfrak{M}$ appears.

[2]) Cf., my paper "Zur elektrodynamik punktförmiger Elektronen," ZS. f. Phys. **32**, 518, 1925.

$$\varphi_\alpha = k\left\{\frac{dx'_k}{d\tau'}\frac{1}{\dfrac{d(S^2)}{d\tau'}}\right\}_{t'=t-\frac{r}{c}} . \tag{24a}$$

We would now like to determine the additional electric field in a completely analogous way that is induced by the "rotation" of the electron; i.e., its magnetic moment.

The corresponding part of the four-potential ψ_α must obviously be representable in terms of the moment tensor $\mu'_{\alpha\beta}$ and the four-vector $(x'_\alpha - x_\alpha)$ by means of a complex integration of the same type as (24).

Furthermore, since ψ_α must be a linear vector function of $\mu'_{\alpha\beta}$, we come to the following Ansatz:

$$\psi_\alpha = \frac{Q}{2\pi i}\oint \mu'_{\alpha\beta}(x_\beta - x'_\beta)\, f(S)\, d\tau' , \tag{25}$$

where Q is a proportionality coefficient and $f(S)$ means an initially unknown function of S. For the determination of this function, we substitute (25) into the differential equation:

$$\sum_{\gamma=1}^{4}\frac{\partial^2\psi_\alpha}{\partial x_\gamma^2} = 0.$$

This yields:

$$\frac{\partial}{\partial x_\gamma} f(x_\beta - x'_\beta)\mu'_{\alpha\beta} = \mu'_{\alpha\beta} f + \mu'_{\alpha\beta}(x_\beta - x'_\beta)\frac{\partial f}{\partial x_\gamma},$$
$$\frac{\partial^2}{\partial x_\lambda^2} f(x_\beta - x'_\beta)\mu'_{\alpha\beta} = 2\mu'_{\alpha\beta}\frac{\partial f}{\partial x_\gamma} + \mu'_{\alpha\beta}(x_\beta - x'_\beta)\frac{\partial^2 f}{\partial x_\gamma^2},$$

and furthermore:

$$\frac{\partial f}{\partial x_\gamma} = \frac{df}{dS}\frac{x_\gamma - x'_\gamma}{S}, \qquad \frac{\partial^2 f}{\partial x_\gamma^2} = \frac{d^2 f}{dS^2}\frac{(x_\gamma - x'_\gamma)^2}{S^2} + \frac{df}{dS}\frac{S^2 - (x_\gamma - x'_\gamma)^2}{S^3},$$

and it follows that:

$$\sum_{\gamma=1}^{4}\frac{\partial^2\psi_\alpha}{\partial x_\gamma^2} = \frac{Q}{2\pi i}\oint \mu'_{\alpha\beta}(x_\beta - x'_\beta)\left\{\frac{d^2 f}{dS^2} + \frac{5}{S}\frac{df}{dS}\right\} d\tau' = 0;$$

i.e.:

$$\frac{d^2 f}{dS^2} + \frac{5}{S}\frac{df}{dS} = 0.$$

One will then have $f = 1 / S^4$, and from (25):

$$\psi_\alpha = \frac{Q}{2\pi i}\oint \frac{\mu'_{\alpha\beta}(x_\beta - x'_\beta)}{S^4}\,d\tau'. \tag{25a}$$

As one easily sees, the supplementary condition:

$$\sum_{\alpha=1}^{4}\frac{\partial\varphi_\alpha}{\partial x_\alpha} = 0$$

is fulfilled due to the fact that the integration path is closed.

For the determination of the coefficients Q, we consider the simplest case of an electron at rest with a moment $\mathfrak{m}'$ ($\mathfrak{p}' = 0$) that is constant in magnitude and direction. In this case, when one replaces the integration path with the imaginary axis [1]), one gets:

$$\psi_\alpha = +\frac{Q}{2\pi i}\mu'_{\alpha\beta}(x_\beta - x'_\beta)\int_{t'=t-i\infty}^{t'=t+i\infty}\frac{dt'}{[R^2 - c^2(t'-t)^2]^2}$$
$$= -\frac{Q}{2\pi c}\mu'_{\alpha\beta}(x_\beta - x'_\beta)\int_{-\infty}^{+\infty}\frac{d(x'_4 - x_4)}{[R^2 + (x'_4 - x_4)^2]^2};$$

i.e.:

$$\psi_\alpha = -\frac{Q\mu'_{\alpha\beta}(x_\beta - x'_\beta)}{4cR^3}.$$

If one thinks of the vector $\mathfrak{R}$ as pointing from the electron to the origin ($R_\alpha = x_\alpha - x'_\alpha$) and observes that φ_1, φ_2, φ_3 mean the components of the vector potential $\mathfrak{A}$ then one has:

$$\mathfrak{A} = \frac{Q}{4c}\frac{[\mathfrak{R}\,\mathfrak{m}]}{R^3}$$

and $\varphi_4 = i\,\varphi = 0$.

The formula above for $\mathfrak{A}$ agrees with the usual expression for the vector potential of an elementary current with the moment $\mathfrak{m}$ when one sets:

$$Q = -4c. \tag{25b}$$

In the general case of an arbitrarily moving electron, one can calculate the integral (25a) by finding the residues. We therefore assume that we are dealing with the retarded potential; i.e., the residue relative to the real pole:

$$R_0 - c(t - t'_0) = 0.$$

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[1]) *loc. cit.*, pp. 523.

If one introduces the ordinary (complex) time t' as the independent variable, in place of the proper time τ' (where $d\tau' = dt'\sqrt{1-v'^2/c^2}$), then this yields:

$$S^4 = [R^2 - c^2 (t - t')^2]^2 = [R + c(t - t')]^2 \, [R - c\,(t - t')]^2,$$

and it follows that as $t' \to t'_0$:

$$R - c(t - t') = \left\{\frac{d}{dt'}[R - c(t - t')]\right\}(t' - t'_0) = c\left(1 - \frac{v'_R}{c}\right)(t' - t'_0),$$

i.e.;

$$S^4 = c^2 \, [R + c(t - t')]^2 \left(1 - \frac{v'_R}{c}\right)^2 (t' - t'_0)^2,$$

where v'_R means the projection of the velocity of the electron in the $\mathfrak{R}_0$-direction at the time point $t' = t'_0$.

From (25a), we then get:

$$\psi_\alpha = \frac{Q}{c^2\left(1 - \frac{v'_R}{c}\right)^2} \frac{1}{2\pi i} \oint \frac{F_\alpha(t')}{(t' - t'_0)^2} dt',$$

with the abbreviation:

$$F_a(t') = \frac{\mu'_{\alpha\beta}(x_\beta - x'_\beta)\sqrt{1 - v'^2/c^2}}{[R + c(t - t')]^2}.$$

Since the function $F(t')$ remains non-zero for $t' = t'_0$, it is known that one has:

$$\frac{1}{2\pi i} \oint \frac{F_\alpha(t')}{(t' - t'_0)^2} dt' = \left\{\frac{d}{dt'} F_\alpha(t')\right\}_{t' = t'_0}$$

It then results from (25b) that:

$$\psi_\alpha = -\frac{4}{c\left(1 - \frac{v'_R}{c}\right)^2} \left\{\frac{d}{dt'} \frac{\mu'_{\alpha\beta}(x_\beta - x'_\beta)\sqrt{1 - v'^2/c^2}}{[R + c(t - t')]^2}\right\}_{t' = t'_0}.$$

By performing the differentiation, this yields, due to the condition $\mu'_{\alpha\beta}\dfrac{dx'_\beta}{dt'} = 0$, by means of the relation $c\,(t' - t'_0) = R_0$:

$$\psi_\alpha = \frac{1}{c\left(1-\frac{v'_R}{c}\right)^2}\left\{\frac{x_\beta - x'_\beta}{R^2}\frac{d}{dt'}\mu^*_{\beta\alpha} + \left(1-\frac{v'_R}{c}\right)\frac{(x_\beta - x'_\beta)\mu^*_{\beta\alpha}}{R^3}\right\}, \tag{26}$$

where the index "0" is omitted, and we have set:

$$\mu^*_{\alpha\beta} = \mu'_{\alpha\beta}\sqrt{1-v^2/c^2}\,, \tag{26a}$$

to abbreviate.

In the case of an electron at rest with time-varying components of the moment tensor $\mu_{\alpha\beta}$, since $\mu_{\beta 4} = i\,p_\beta = 0$, (26) reduces to:

$$\left.\begin{aligned}\mathfrak{A} &= \frac{[\dot{\mathfrak{m}}\,\mathfrak{R}]}{cR^2} + \frac{[\mathfrak{m}\,\mathfrak{R}]}{R^3}\\ \varphi &= 0.\end{aligned}\right\} \tag{27}$$

Let it be remarked that, from (15), the magnitude of the magnetic moment $|\mathfrak{m}|$ must then remain constant [1]). Formula (27) above can be applied to the case of an electron that does not move to rapidly as the "zeroth-order approximation." We would not like to go into the calculation of the electric and magnetic field strengths, which follows with no difficulty from the usual formulas $\mathfrak{E} = -\frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t} - \text{grad}\,\varphi$, $\mathfrak{H} = \text{rot}\,\mathfrak{A}$.

In conclusion, we would now like to prove the following fact:

If the electrons are ascribed a magnetic moment with the magnitude of a Bohr magneton then for distances < 10^{-11} cm. their magnetic interaction, which is known to be inversely proportional to the fourth power of distance, should outweigh their electric (Coulomb) repulsion. This magnetic interaction can already make known the value of the screening constant for the inner electrons for heavy atoms. In the atomic nucleus, however, it must be a million times larger than the electrostatic forces. If one ascribes an impulse moment of the same magnitude as the electron's to the protons, and correspondingly a magnetic moment that is 2000 times smaller, then their magnetic interaction with each other and with the electrons should also strongly outweigh the electrostatic interaction. It thus seems justified to assert that the structure of the atomic nucleus is practically independent of the electric charges of the electrons and protons, and must be primarily induced by its magnetostatic interactions (in conjunction with the usual quantum conditions). This gives that, e.g., an electron and a proton at a distance of 5×10^{-13} cm can remain in static

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[1]) If the moment tensor were not subject to the condition $\mu_{\alpha\beta}x'_\beta = 0$ then the following two terms would enter into the expression above for $\mathfrak{A}$:

$$\frac{\dot{\mathfrak{p}}}{cR} + \frac{\mathfrak{p}}{R^2},$$

and would be, moreover:

$$\varphi = \frac{(\mathfrak{R}\dot{\mathfrak{p}})}{cR^2} + \frac{(\mathfrak{p}\mathfrak{R})}{R^3}.$$

equilibrium. However, this equilibrium would be unstable relative to the orientation of the magnetic axes of the two particles. If one thus assumes that the electron orbits around the nucleus then this would yield, in addition to the ordinary first-quantized path with a radius 0.55×10^{-8} cm, a second first-quantized path of radius 3×10^{-14} cm that is required by the magnetic attraction, where the electric attraction would seem to be a weak perturbing force. The quantities above work quite well for the measurements of the simplest nucleus. However, one may not conceal the complication here that the electron mass, due to its large velocity, increases to perhaps a thousand times the ordinary value, which will be, in part, compensated by the decrease in the mutual potential energy. I hope to treat this question more thoroughly in a later communication.

In conclusion, I would like to express my deepest thanks to Dr. W. Pauli for providing the impetus for this paper and many worthwhile consultations. I must further warmly thank Prof. P. Langevin and my friend G. Krutkow for some suggestions (and also the latter for reviewing the manuscript).

Hamburg-Nizza, April 1926.