ELECTRODYNAMICS

BY

DR. J. FRENKEL

PROFESSOR OF THEORETICAL PHYSICS
AT THE POLYTECHNIC INSTITUTE
IN LENINGRAD

TRANSLATED BY

D. H. DELPHENICH

VOLUME ONE

GENERAL MECHANICS OF ELECTRICITY

WITH 32 FIGURES

BERLIN
PUBLISHED BY JULIUS SPRINGER
1926

TO THE MEMORY OF MY BROTHER

SERGIUS

(21 SEPTEMBER 1900 – 7 APRIL 1920)

WHO WAS MY FIRST STUDENT
AND ASSISTANT

FOREWORD

The present textbook on electrodynamics offers some peculiarities in regard to the arrangement of the topics, as well as the manner of presentation, that I would like to mention briefly here.

As far as the arrangement of the topics is concerned, I have followed the example of theoretical mechanics in presenting them. In theoretical mechanics, one first examines the general laws of the interaction and motion of material particles, or "points," which are later applied to certain specialized systems that idealize the various material bodies, namely, rigid bodies, elastic bodies, fluids, and gases. Accordingly, in this first volume, I will consider only the general laws of the interaction and motion of *electrically-charged* material particles, which are also often treated as points, and in that way all interactions between them will come down to their charges.

I shall call those particles (or "electric mass-points") *electrons* without having to go into the detailed consideration of the phenomena that arise from them in material bodies in so doing.

The electromagnetic (and optical) properties of material bodies shall first be investigated in the second volume, and indeed from a macroscopic viewpoint.

It should also be pointed out here that, in analogy with mechanics, the bodies in "electric fluids" can be classified as free-moving charges (conductors) and "elastically" bound charges, whether bound electrically or magnetically.

I hope to be able to treat the microscopic electrodynamics of the simplest material systems (atoms and molecules) once I have succeeded in carrying out the necessary quantum-theoretical recasting of classical electrodynamics.

In the present first volume, phenomena that are independent of time will be considered to begin with. In that way, I shall begin by presenting electrostatics, not in terms of isolated electric charges, but in terms of the simplest *neutral* systems, i.e., dipoles. Magnetostatics will not be constructed on the basis of fictitious magnetic poles, but on the basis of the somewhat-idealized stationary electric currents. In so doing, I shall employ the *energy principle* as a guideline in order to exhibit the basic properties of electric and magnetic fields and their effect on electric dipoles or currents, resp. (or their elements). Combining that principle with the *equivalence principle* between electric dipoles and currents (relative to their interaction with each other) will imply the general differential equations for time-constant electromagnetic fields, and especially the laws of *Coulomb* and *Biot-Savart*.

In Section Two, the laws obtained will be generalized to time-dependent phenomena, and indeed, by means of the *principle of the relativity of velocity* (in an entirely-narrow conception of it) and the *principle of the conservation of electricity*. The *Maxwell-Lorentz* equations thus-obtained will then be applied to the determination of the electromagnetic field in the most-important cases. The foundations of the electromagnetic theory of light will be laid by examining the electromagnetic field of an oscillator.

Moreover, the concepts of energy, momentum, etc., will be exhibited for arbitrary fields, and the classical theory of the force of inertia (i.e., the electromagnetic theory of mass) and radiation damping, as well as the theory of the translatory and rotational motion of a spatially-extended electron will be developed. In conclusion, that classical presentation will be considered critically.

Section Three is dedicated to laying the foundations of the special theory of relativity and its application to electromagnetic effects (i.e., fields) and the equations of motion of the electron. In it, the time-dependent processes will be regarded as static phenomena in a four-dimensional universe. We shall not go into the general theory of relativity in this book since it is concerned with gravitational effects more than with electrical ones.

By way of introduction, I have given a brief presentation of the foundations of vector and tensor calculus. The reader will find all of the mathematical apparatus that will be used in what follows there. In the treatment of tensors and the coordinate-wise representation of vector quantities and operations, I have restricted myself to rectangular coordinate systems, for simplicity. However, they will be the ones that are used almost always in Section Three.

This brief overview of the contents of the book shall now be completed by some mention of the things that are missing from it. *The historical development of the study of electricity is ignored completely here*. Namely, I believe that the historical and logical viewpoint should not, by any means, be confused with each other, at least, not when the branch of science in question has to be developed as a closed logical system, which is the case for classical mechanics and classical electrodynamics. I have tried to present the modern study of electricity in a way that is as simple and systematic as possible without having to recall the history of its development. I would prefer to leave the study of that topic to others.

In particular, I have ignored the theory of the ether in all of its forms completely. Undoubtedly, the ether has played a very important and fruitful role as the working hypothesis in the development of electrodynamics. First, since the time of *Huyghens*, it was the foundation for the theory of optical phenomena. Then, since the time of *Faraday* and *Maxwell*, it was the bridge between optics and electromagnetism. Much later, one sought to make the ether responsible for all physical phenomena. However, now is the time to recognize that the ether has played out its historical role and that it has the right to a place of honor in only the history of physics. Its introduction, even as an auxiliary concept, into the presentation of the modern theory of electricity cannot at all clarify, but rather only obfuscate, that theory and burden it with a number of illusory problems, such as all problems that relate to the motion of the Earth "in the ether."

Due to the fact that my treatment of the theory of electricity is "ahistorical" from the outset, I have refrained from citing any references to the literature, whether textbooks or treatises, and have perhaps left many names that are very important in the development of electrodynamics unmentioned.

In conclusion, I would like to express my heartfelt thanks to Herrn Prof. *M. Born* for reviewing the manuscript. Furthermore, I owe my deepest thanks to Herren Prof. *P. Ehrenfest*, *V. Bursian*, *G. Krutkow*, and especially Herrn cand. *W. Elsasser* for their technical assistance.

It is a pleasant duty for me to mention that it was a stipend from the International Education Board that made it possible for me to work in a foreign country.

In particular, I would also like to thank the publishing house of *Julius Springer* for their care and generosity in every sense of the word.

Göttingen, September 1926.

J. Frenkel

TABLE OF CONTENTS

PART TWO

Time-dependent electromagnetic effects

Chapter Five: The general laws of the electromagnetic fields. Page

Chapter Six: The electromagnetic fields of moving point-charges (electrons).

Chapter Seven: Energy and momentum for time-varying electromagnetic phenomena. Dynamics of electrons.

PART THREE

The theory of relativity

Chapter Eight: Foundations of the theory of relativity.

Chapter Nine: Applying the theory of relativity to electromagnetic phenomena.

INTRODUCTION

FUNDAMENTALS OF VECTOR AND TENSOR CALCULUS

A. – Addition, inner and outer multiplication of vectors.

§ 1.

The various physical quantities are ordinarily arranged into two classes, namely, *scalars* and *vectors*. The former are determined completely by giving their *numerical values*. In order to specify the other ones completely, one must give their *directions*, in addition to the numerical magnitudes. Some typical scalars that one considers are time, the mass of a body, etc.; some typical vectors are velocity, force, etc. Later, we will see that vectors are a special case of quantities of a more general type, namely, the so-called *tensors*.

Ordinarily, we will denote the vector quantities by German letters and their magnitudes by the corresponding Latin letters, or even by enclosing the vector symbol between two vertical lines. For example, $|\mathfrak{A}| = A$ shall mean the magnitude of the vector $\mathfrak{A}$.

The simplest vector quantity, which also serves as an intuitive representation of all other quantities, is a rectilinear segment that points from one point Q to another P and can be regarded as the displacement of a material point.

The displacement OP can be replaced with a series of other displacements OA, AB, BC, CP that define a broken line and call them the *components* of OP. When one is given two components OA and AP, one can consider the vector OP to be the diagonal of a parallelogram whose sides are equal in length and point in the same directions as OA and AP, resp.

The replacement of a vector with a number of components (which can obviously be accomplished in infinitely-many different ways) is called a *geometric decomposition*. The opposite operation to a decomposition consists of the replacement of a number of arbitrary segments (vectors) $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$ with a single segment $\mathfrak{F}$ for which they play the role of components and is called *geometric addition*, which will be denoted symbolically by the equation:

$$\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2 + \ldots + \mathfrak{F}_n = \sum_k \mathfrak{F}_k \,. \tag{1}$$

The vector $\mathfrak{F}$ in that is called the *geometric sum* of $\mathfrak{F}_1$, $\mathfrak{F}_2$, etc. It is easy to prove that the geometric sum is independent of the sequence of individual summands (i.e., the components) and that the addition of an arbitrary number of summands can be replaced with their geometric sum. The geometric addition then satisfies the *commutative* and *associative* laws, just like the ordinary (algebraic) kind.

A vector that is equal in length and oppositely-directed to another one $\mathfrak{B}$ will be denoted by $-\mathfrak{B}$. The geometric sum $\mathfrak{A} + (-\mathfrak{B})$ will be written in the simple form $\mathfrak{A} - \mathfrak{B}$ and is called the geometric *difference* of the vectors $\mathfrak{A}$ and $\mathfrak{B}$.

§ 2.

If one draws two planes that are perpendicular to a line MN through the endpoints to a segment OP then they will cut out a segment $O_1 P_1$ from that line that is called the *projection* of OP onto MN. Thus, that projection will be considered to be *positive* as an ordinary (scalar) quantity when the direction $O_1 P_1$ coincides with the positive direction of the line MN and negative in the opposite case.

If one denotes the projection of a vector $\mathfrak{A}$ onto another $\mathfrak{B}$ (or a line that points in the same direction as $\mathfrak{B}$) by A_B then from the definition above, one will have:

$$A_B = A \cos(\mathfrak{A}, \mathfrak{B}) .$$

The product $A_B \cdot B = AB \cos(\mathfrak{A}, \mathfrak{B}) = AB_A$ is called the *inner* (or "scalar") product of the vectors $\mathfrak{A}$ and $\mathfrak{B}$, and will be denoted by the symbol $\mathfrak{A} \cdot \mathfrak{B}$, or simply $\mathfrak{A}\mathfrak{B}$. It is easy to see that the projection of the geometric sum of two or more segments (e.g., OQ and QP in Fig. 1) onto any line will be equal to the algebraic sum of the projections of the individual summands, i.e., $(\mathfrak{A} + \mathfrak{B})_C = A_C + B_C$. If one multiplies that equation by C then one will get:

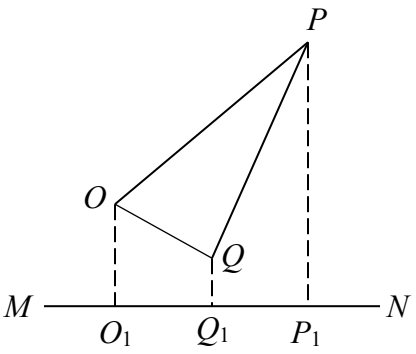


Figure 1.

$$(\mathfrak{A} + \mathfrak{B})\,\mathfrak{C} = \mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{C} \qquad (2)$$

from the definition of the inner product. That formula, which expresses the *distributive* law, can obviously be generalized to an arbitrary number of summands in each factor:

$$\left(\sum_p \mathfrak{A}_p\right)\left(\sum_q \mathfrak{B}_q\right) = \sum_p \sum_q \mathfrak{A}_p \mathfrak{B}_q . \qquad (2.a)$$

§ 3.

One can regard the (planar) surface that is bounded by a closed, planar line σ (to the extent that only its orientation and area S are under consideration) as a vector quantity and represent it by a segment $\mathfrak{S}$ that is perpendicular to its plane and has a length that is proportional to S. The double-valuedness in the direction of $\mathfrak{S}$ that then arises will be avoided by giving a certain *sense of traversal* along the bounding curve σ and assigning a *unique* direction to the representative segment for that traversal (or orbiting), which is a sense that is ordinarily that of a *right-handed* screw.

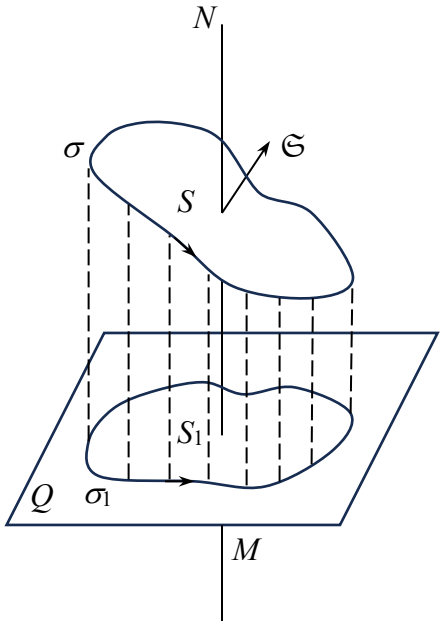


Figure 2.

If one draws lines through all points of the bounding curve that are perpendicular to a certain plane Q then one will get a curve σ_1 in that way that is referred to as the *projection* of σ onto Q. The surface S_1 that is bounded by σ_1 is correspondingly called the *projection* of S onto Q. That projection is considered to be an ordinary (i.e., scalar) quantity whose sign can be determined in such a way that one chooses a certain sense of traversal in the plane Q to be positive. In the event that this sense of traversal coincides with the sense of traversal along σ_1, S_1 will be positive; in the opposite case, it will be negative. As one easily sees, S_1 is numerically equal to the product of the projected area S with the cosine of the angle α between the plane that contains S and the plane Q. When representing the latter (or even better, the chosen sense of traversal along it) by the line MN that points perpendicular to it in the sense of the right-hand screw rule (Fig. 2), one can identify the projection of the surface S onto Q with the projection of the segment $\mathfrak{S}$ that represents S onto MN.

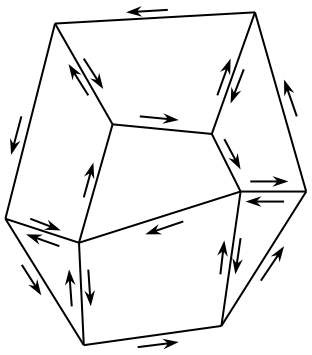


Figure 3.

If the "boundary curve" σ consists of nothing but rectilinear segments then one can replace the planar surface S in question with a number of other planar surfaces $S_1, S_2, \ldots$ that define a polyhedral surface that is bounded by σ. The sense of traversal along the "curves" (which are polygons, in our present case) $\sigma_1, \sigma_2, \ldots$ that bound the surfaces $S_1, S_2, \ldots$ shall be chosen in such a way that each edge that belongs to two different polygons will be traversed in opposite directions, while the sense of traversal on the edges of the original "outer" polygon σ remains unchanged (Fig. 3). Under those conditions, one can obviously assert that the algebraic sum of the projections of the "component surfaces" $S_1, S_2, \ldots$ onto any plane Q will always be equal to the corresponding projections of S. It will follow from this that for any type of "decomposition" of S into component surfaces (or of σ into component

curves), the geometric sum of the vectors $\mathfrak{S}_1$, $\mathfrak{S}_2$, … must have the same value $\mathfrak{S}$. In that sense, we will refer to the various polyhedral surfaces that are bounded by σ as *equivalent*.

Obviously, a polyhedral surface can also be bounded by a *non-planar* polygonal curve. However, as long the sense of traversal along that "curve" (σ) is given and the sense of traversal is established on the individual component polygons (σ_i) according to the aforementioned rule, the geometric sum of the component surfaces (or the vectors that represent them) $\mathfrak{S}_i$ will have the same value $\mathfrak{S}$ for all polyhedral surface that are bounded by the same polygonal curve σ ([1]). We would like to refer to that vector $\mathfrak{S}$, whose magnitude and direction are determined uniquely by the form and sense of traversal of the polygon σ, as the *geometric moment* of σ. That definition of the geometric moment can be easily generalized to arbitrary closed curves since such a curve can be regarded as a limiting case of a polygon with infinitely-small edges. For plane curves, the magnitude of the moment is equal to the area, and its direction is associated with the sense of traversal along the aforementioned right-hand screw rule.

§ 4.

The simplest plane figure is a triangle or a parallelogram (which can be decomposed into two equal triangles). The geometric moment of a parallelogram whose sides are defined by the vectors $\mathfrak{A}$ and $\mathfrak{B}$ is called the *outer product* (or "vector product") of those vectors and will be denoted by the symbol $\mathfrak{A} \times \mathfrak{B}$. It is then assumed that the segments $\mathfrak{A}$ and $\mathfrak{B}$ point away from the same point and that the sense of traversal of the parallelogram starts from that point in the direction of $\mathfrak{A}$ and ends up by traversing $\mathfrak{B}$ in the opposite direction. The inversion of the sense of traversal will then correspond to the inversion of the sequence of $\mathfrak{A}$ and $\mathfrak{B}$. That will then imply the equation:

$$\mathfrak{B} \times \mathfrak{A} = - \mathfrak{A} \times \mathfrak{B} \, .$$

The magnitude of the outer product is obviously independent of the sequence of the two vectors, and as is easy to see, it will be equal to $A\,B \sin(\mathfrak{A}, \mathfrak{B})$.

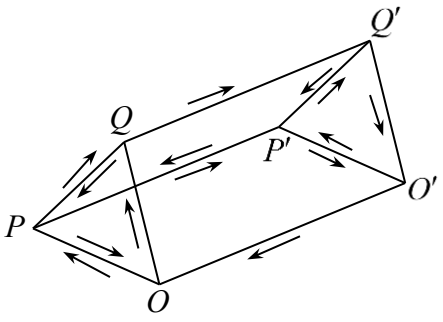


Figure 4.

Although the commutative law is not true for outer multiplication, as opposed to the inner kind, the distributive law proves to be valid in both cases. Namely, if we construct two parallelograms that are determined by the vectors $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{B}$, $\mathfrak{C}$ in such a way that the have a common side ($\mathfrak{C}$), and extend the figure that we get (Fig. 4, in which $OP = O'P' = \mathfrak{A}$, $PQ = P'Q' = \mathfrak{B}$, and $OO' = PP' = QQ' = \mathfrak{C}$) by the

([1]) That will follow immediately when one considers the projections of σ and σ_i (i = 1, 2, 3, …) onto any plane.

triangles OPQ and $O'P'Q'$ with opposite directions of traversal and moments then we will get the parallelogram $OQQ'O'$ whose moment must obviously be equal to the geometric sum of the moments $OPP'O'$ and $PQQ'P'$. On the other hand, since $OQ = OP + PQ = \mathfrak{A} + \mathfrak{B}$, it will then follow that:

$$\mathfrak{A} \times \mathfrak{C} + \mathfrak{B} \times \mathfrak{C} = (\mathfrak{A} + \mathfrak{B}) \times \mathfrak{C} . \tag{3}$$

That equation can be generalized to an arbitrary number of summands, like the corresponding equation for inner multiplication:

$$\left(\sum_p \mathfrak{A}_p \right) \times \left(\sum_q \mathfrak{B}_q \right) = \sum_p \sum_q \mathfrak{A}_p \times \mathfrak{B}_q . \tag{3.a}$$

§ 5.

If $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are three non-coplanar (i.e., they do not lie in the same plane) segments then the double product $\mathfrak{A}\ (\mathfrak{B} \times \mathfrak{C})$ means the volume of a parallelopiped whose sides are equal to those segments and point in the same direction (with the + or – sign). That follows from the fact that the magnitude of the outer product $\mathfrak{B} \times \mathfrak{C}$ is equal to the area of the parallelogram $(\mathfrak{B}, \mathfrak{C})$, which can be regarded as the base surface of the parallelepiped $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, while its direction is perpendicular to that surface. Since different surfaces can be chosen to be the base surface of the parallelepiped, the expressions $\mathfrak{B}\ (\mathfrak{C} \times \mathfrak{A})$ and $\mathfrak{C}\ (\mathfrak{A} \times \mathfrak{B})$ will give the same volume, and indeed with the same sign as $\mathfrak{A}\ (\mathfrak{B} \times \mathfrak{C})$, when the vectors $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are permuted *cyclically*. For arbitrary vectors, one will then have:

$$\mathfrak{A}\ (\mathfrak{B} \times \mathfrak{C}) = \mathfrak{B}\ (\mathfrak{C} \times \mathfrak{A}) = \mathfrak{C}\ (\mathfrak{A} \times \mathfrak{B}) . \tag{4}$$

The double outer product $\mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C})$ represents a vector that points perpendicular to $\mathfrak{A}$ in the plane of $\mathfrak{B}$, $\mathfrak{C}$. That is why it can be expressed in the form $\beta\,\mathfrak{B} + \gamma\,\mathfrak{C}$, in which β and γ are two scalar coefficients between which the relation $\beta\ (\mathfrak{A}\mathfrak{B}) + \gamma\ (\mathfrak{A}\mathfrak{C}) = 0$ must exist. If we correspondingly set $\beta = \alpha\ \mathfrak{A}\mathfrak{C}$ and $\gamma = -\alpha\ \mathfrak{A}\mathfrak{B}$, in which α means a new scalar coefficient, then we will have:

$$\mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C}) = \alpha\ \{(\mathfrak{A}\mathfrak{C})\ \mathfrak{B} - (\mathfrak{A}\mathfrak{B})\ \mathfrak{C}\} .$$

Now, it is easy to show that $\alpha = 1$, independently of the magnitude and direction of the vectors $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, such that:

$$\mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C}) = (\mathfrak{A}\mathfrak{C})\ \mathfrak{B} - (\mathfrak{A}\mathfrak{B})\ \mathfrak{C} . \tag{5}$$

We shall skip the proof of the statement above since the identity (5) will be exhibited below in a different way. It follows from (5) that:

$$\mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C}) + \mathfrak{B} \times (\mathfrak{C} \times \mathfrak{A}) + \mathfrak{C} \times (\mathfrak{A} \times \mathfrak{B}) = 0 , \tag{5.a}$$

in which the individual summands are obtained from each other by cyclically permuting the vectors $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$.

An application of (4) and (5) will then give:

$$(\mathfrak{A} \times \mathfrak{B})\,(\mathfrak{C} \times \mathfrak{D}) = \mathfrak{C}\,[\mathfrak{D} \times (\mathfrak{A} \times \mathfrak{B})] = \mathfrak{C}\,[\mathfrak{A}\,(\mathfrak{B}\,\mathfrak{D}) - \mathfrak{B}\,(\mathfrak{A}\,\mathfrak{D})] ,$$

i.e.:

$$(\mathfrak{A} \times \mathfrak{B})\,(\mathfrak{C} \times \mathfrak{D}) = (\mathfrak{C}\,\mathfrak{A})\,(\mathfrak{B}\,\mathfrak{D}) - (\mathfrak{C}\,\mathfrak{B})\,(\mathfrak{A}\,\mathfrak{D}) .$$

§ 6.

In conclusion, we would like to briefly consider the coordinate representation of vector quantities and operations.

We imagine a rectangular coordinate system whose axes OX_1, OX_2, OX_3 point in the directions of three unit vectors $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$, resp. Along with the orthogonality conditions:

$$\mathfrak{e}_i \cdot \mathfrak{e}_k = \begin{cases} 1 & \text{for} \quad i = k, \\ 0 & \text{for} \quad i \neq j, \end{cases} \tag{7}$$

those vectors shall satisfy the conditions:

$$\mathfrak{e}_2 \times \mathfrak{e}_3 = \mathfrak{e}_1 , \quad \mathfrak{e}_3 \times \mathfrak{e}_1 = \mathfrak{e}_2 , \quad \mathfrak{e}_1 \times \mathfrak{e}_2 = \mathfrak{e}_3 , \tag{7.a}$$

which express the “right-hand screw character” of the coordinate system.

Let $\mathfrak{r}$ be the radius vector OP of a point P relative to the coordinate origin O. Its *components* along the coordinate axes (i.e., the *coordinates* x_1, x_2, x_3 of the point P) are defined by the vector equation:

$$\mathfrak{r} = x_1\,\mathfrak{e}_1 + x_2\,\mathfrak{e}_2 + x_3\,\mathfrak{e}_3 . \tag{8}$$

On the other hand, if we construct the *projection* of $\mathfrak{r}$ onto any axis (X_i) then, from (2), and according to (7) and (8), we will have:

$$r_i = \mathfrak{r}\,\mathfrak{e}_i = x_i .$$

The components of the radius vector will then coincide with its projection onto the axes in the case considered. Obviously, the same thing must be true for all other vectors such that components of a vector $\mathfrak{A}$ can be defined by the equations:

$$A_i = \mathfrak{A}\ \mathfrak{e}_i\ , \tag{8.a}$$

instead of equation (8).

The components of a geometric sum $\mathfrak{A} + \mathfrak{B} + \ldots$ are obviously equal to the algebraic sum of the corresponding components of the individual summands.

If we set:

$$\mathfrak{A} = \sum_i A_i\ \mathfrak{e}_i \quad \text{and} \quad \mathfrak{B} = \sum_k B_k\ \mathfrak{e}_k$$

in the products $\mathfrak{A}\ \mathfrak{B}$ and $\mathfrak{A} \times \mathfrak{B}$ then, from (7), that will give:

$$\mathfrak{A}\ \mathfrak{B} = \sum_i A_i\ B_i\ = A_1\ B_1 + A_2\ B_2 + A_3\ B_3\ . \tag{9}$$

In particular, for $\mathfrak{B} = \mathfrak{A}$:

$$\mathfrak{A}\ \mathfrak{A} = A^2\ = A_1^2 + A_2^2 + A_3^2\ . \tag{9}$$

Furthermore:

$$\mathfrak{A} \times \mathfrak{B} = \sum_i \sum_k A_i\ B_k\ (\mathfrak{e}_i \times \mathfrak{e}_k)\ = \sum_{i<k} \sum (A_i\ B_k - A_k\ B_i)(\mathfrak{e}_i \times \mathfrak{e}_k)\ ,$$

i.e., from (7.a):

$$\mathfrak{A} \times \mathfrak{B} = (A_2\ B_3 - A_3\ B_2)\ \mathfrak{e}_1 + (A_3\ B_1 - A_1\ B_3)\ \mathfrak{e}_2 + (A_1\ B_2 - A_2\ B_1)\ \mathfrak{e}_3\ , \tag{10}$$

or

$$(\mathfrak{A} \times \mathfrak{B})_1 = A_2\ B_3 - A_3\ B_2\ , \quad (\mathfrak{A} \times \mathfrak{B})_2 = A_3\ B_1 - A_1\ B_3\ , \quad (\mathfrak{A} \times \mathfrak{B})_3 = A_1\ B_2 - A_2\ B_1\ . \tag{10.a}$$

One can easily exhibit the identities (4) and (5) by means of those formulas. In the case of (5), e.g., one will have, from (10.a):

$$\begin{aligned} [\mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C})]_1 &= A_2\ (\mathfrak{B} \times \mathfrak{C})_3 - A_3\ (\mathfrak{B} \times \mathfrak{C})_2 \\ &= A_2\ (B_1\ C_2 - B_2\ C_1) - A_3\ (B_3\ C_1 - B_1\ C_3) \\ &= B_1\ (A_1\ C_1 + A_2\ C_2 + A_3\ C_3) - C_1\ (A_1\ B_1 + A_2\ B_2 + A_3\ B_3) \\ &= B_1\ (\mathfrak{A}\ \mathfrak{C}) - C_1\ (\mathfrak{A}\ \mathfrak{B})\ , \end{aligned}$$

which agrees with (5).

B. – The differential operations of vector calculus.

§ 7.

One can regard vectors, just like scalars as *variable* quantities (in magnitude and direction), and indeed, as *independent* variables (i.e., arguments) or as *dependent* ones (i.e., functions). Therefore, the following four cases can appear:

1. A scalar function of a scalar argument $\alpha(t)$.
2. A vectorial function of a scalar argument $\mathfrak{A}(t)$.
3. A scalar function of a vectorial argument $\varphi(\mathfrak{r})$.
4. A vectorial function of a vectorial argument $\mathfrak{F}(\mathfrak{r})$.

For the sake of intuitiveness, we will consider t to be the time and $\mathfrak{r}$ to be the radius vector of various points in space (relative to a "fixed" point O) from the outset. Therefore, $\alpha(t)$, $\mathfrak{A}(t)$ will be functions of time, and $\varphi(\mathfrak{r})$, $\mathfrak{F}(\mathfrak{r})$ will be functions of positions, such that the "position" will be defined when we are given $\mathfrak{r}$.

One can illustrate a scalar function of position $\varphi(\mathfrak{r})$ by constructing a family of surfaces:

$$\varphi = c = \text{const.}$$

for equidistant values of c. The curves that are orthogonal to those surfaces at each point will then be given by the direction of the most-rapid increase in φ.

One ordinarily refers to the vectorial functions of position $\mathfrak{F}(\mathfrak{r})$ as "vector fields," and illustrates them by constructing a family of lines ("streamlines," when $\mathfrak{F}$ is a velocity, "lines of force," when $\mathfrak{F}$ means a force) that go through each point in the direction of the vector $\mathfrak{F}$ that belongs whose "density" (i.e., number of lines per unit area of the surface that it is perpendicular to) is proportional to the magnitude of $\mathfrak{F}$ ([2]).

§ 8.

In the case of a vector function of a scalar argument $\mathfrak{A}(t)$, the "vectorial derivative," $d\mathfrak{A}/dt$ corresponds to the ordinary *derivative* $d\alpha/dt$ of a scalar function of the same argument and is defined to be the limit of the vector:

([2]) When a surface S is not perpendicular to F, but skew to it, the number of those lines that cut the surface per unit area will be measured by the *projection* of F onto the normal to S.

$$\frac{1}{\Delta t}\left[\mathfrak{A}\,(t+\Delta t)-\mathfrak{A}\,(t)\right]$$

as $\Delta t \to 0$.

In the case of a scalar function of a vector argument $\mathfrak{r}$, the corresponding differential operation can be defined as follows:

Imagine a closed surface S that includes the point P in question ($OP = \mathfrak{r}$).

Draw a unit vector $\mathfrak{n}$ at each point S that points in the direction of the *exterior* normal. If we divide S into infinitely-many elements dS and define the product $\mathfrak{n}\,\varphi\,dS$, in which $\mathfrak{n}$ and φ refer to two arbitrary points of dS, then the geometric sum of the infinitely-small vectors $\mathfrak{n}\,\varphi\,dS$ in the limit as $dS \to 0$ will be a vector quantity that is independent of the aforementioned point, which one denotes by:

$$\oint \varphi\,\mathfrak{n}\,dS$$

and calls the *surface integral* of the function $\varphi\,(\mathfrak{r})$ ([3]). We will assume that this function is *continuous* over all of the volume that comes under consideration ([4]).

If one divides the surface integral of φ by the volume V that is enclosed by the surface S and contracts S to the (always interior) point P then that will give a vector quantity in the limit as $S \to 0$ that is independent of the form of S or its alterations under the passage to the limit ([5]). As we will see in the next section, that quantity corresponds to the ordinary "derivative" since it determines the magnitude and direction of the rate of change in $\varphi\,(\mathfrak{r})$ "at the point P." One calls it the *gradient* of φ and ordinarily denotes it by the symbol grad φ. One then has:

$$\operatorname{grad}\varphi = \lim_{S\to 0}\cdot\frac{1}{V}\oint \varphi\,\mathfrak{n}\,dS\,. \tag{11}$$

One can define two different differential quantities for a vectorial function $\mathfrak{F}\,(\mathfrak{r})$ that will correspond to the gradient of a scalar when the product $\varphi\,\mathfrak{n}$ in (11) is replaced with the inner or outer product of $\mathfrak{n}$ and $\mathfrak{F}$. In the first case, we will get a scalar differential quantity that is called the *divergence* of $\mathfrak{F}$:

$$\operatorname{div}\mathfrak{F} = \lim_{S\to 0}\cdot\frac{1}{V}\oint \mathfrak{n}\,\mathfrak{F}\,dS \tag{11.a}$$

and in the second case, we will get a vector quantity called the *rotation* (or "curl") of $\mathfrak{F}$:

([3]) The circle in the integral symbol means that the surface S is closed.

([4]) I.e., the ratio of the difference $\varphi\,(r_2) - \varphi\,(r_1)$ between two different points to the distance between those points $|\,r_2 - r_1\,|$ must also remain *finite*.

([5]) For the proof of that, see the Remark in **§ 11**.

$$\text{rot}\,\mathfrak{F} = \lim_{S\to 0}\cdot\frac{1}{V}\oint \mathfrak{n}\times\mathfrak{F}\,dS\,. \tag{11.b}$$

One can refer to the divergence as the *inner gradient* of $\mathfrak{F}$ and the rotation as the *outer* one. If one introduces the vectorial operator:

$$\nabla = \lim_{S\to 0}\cdot\frac{1}{V}\oint \mathfrak{n}\ldots dS \tag{12}$$

and regards it as symbolic vector then the three differential operations (11), (11.a), (11.b) can be expressed symbolically by the corresponding multiplication operations:

$$\text{grad}\;\varphi = \nabla\varphi\,, \qquad \text{div}\;\mathfrak{F} = \nabla\mathfrak{F}, \quad \text{rot}\;\mathfrak{F} = \nabla\times\mathfrak{F}\,. \tag{12.a}$$

§ 9.

That manner of expression is often very convenient and very enlightening in the context of the analytical properties of the various differential operations. On the other hand, their geometric interpretation is expressed more intuitively and directly by the usual terminology. In the case of the vectorial differential quantities grad φ and rot $\mathfrak{F}$, that interpretation can be recognized immediately by a simple specialization of the form of S. Namely, let S be the surface of an infinitely-small cylinder with the end surfaces S', S'', and a lateral surface Σ. We will denote the corresponding exterior normals by $\mathfrak{n}'$, $\mathfrak{n}''$, and ν, resp., and the height of the cylinder by h. We next define the projection of the vector $\oint \varphi\,\mathfrak{n}\,dS$ onto the cylinder axis, which points from S' to S'' (so it coincides with $\mathfrak{n}''$). That projection is equal to the inner product $\mathfrak{n}''\oint \varphi\,\mathfrak{n}\,dS$, or since $\mathfrak{n}''$ is constant, to $\oint \varphi(\mathfrak{n}''\mathfrak{n})\,dS$. Since $\nu\,\mathfrak{n}'' = 0$, the integral above will reduce to the sum of the parts that correspond to the two end surfaces, i.e., since $\mathfrak{n}''\,\mathfrak{n}' = -1$, to the difference $\int\varphi''dS'' - \int\varphi'dS'$. By definition, the ratio of that difference to the volume of the cylinder $V = S'h = S''h$ in the limit as $S''\to 0$ and $h\to 0$ must be equal to the projection of the vector grad φ onto $\mathfrak{n}''$. Since one will then have $\int\varphi''dS'' - \int\varphi'dS' = (\varphi''-\varphi')S''$, one will get:

$$\mathfrak{n}''\,\text{grad}\;\varphi = \text{grad}_{\mathfrak{n}''}\;\varphi = \lim_{h\to 0}\frac{\varphi''-\varphi'}{h} = \frac{\partial\varphi}{\partial h}. \tag{13}$$

Therefore: The projection of grad φ onto any direction is equal to the rate of increase of φ in that direction, or in other words, it is equal to the (partial)derivative of φ along the corresponding rectilinear axis. It follows from this that the vector grad φ determines the direction and magnitude

of *fastest* increase in the function φ ($\mathfrak{r}$). The lines that represent that vector, or rather, the corresponding vector field, are the curves that are orthogonal to the surfaces φ = const. The geometric meaning of the gradient will be explained completely by that connection.

In order to clarify the meaning of the vector rot $\mathfrak{F}$, we define the projection of the integral $\oint \mathfrak{n} \times \mathfrak{F}\, dS$ onto $\mathfrak{n}''$, in the same way as before. In that way, (4) will imply that:

$$\mathfrak{n}'' \text{ rot } \mathfrak{F} = \frac{1}{S''h} \oint \mathfrak{n}''(\mathfrak{n} \times \mathfrak{F})\, dS = \frac{1}{S''h} \oint \mathfrak{F}(\mathfrak{n}'' \times \mathfrak{n})\, dS = \frac{1}{S''h} \oint \mathfrak{F}(\mathfrak{n}'' \times \nu)\, d\Sigma ,$$

since the outer product $\mathfrak{n}'' \times \mathfrak{n}$ will vanish on the end surfaces of the cylinder. As far as the lateral surface Σ is concerned, that outer product will have a length equal to 1 and it will point in the same direction as the tangent to the intersecting curve or the generator to the cylinder σ towards the side that corresponds to the direction of the axis $\mathfrak{n}''$ in the sense of the right-hand screw rule. If we set $\mathfrak{n}'' \times \nu = \tau$ ("tangent vector") and $d\,\Sigma = h\, d\sigma$ then we will have:

$$\frac{1}{S''h} \oint \mathfrak{F}(\mathfrak{n}'' \times \nu)\, d\Sigma = \frac{1}{S''} \oint \mathfrak{F} \cdot \tau\, d\sigma ,$$

and as a result:

$$\text{rot}_{\mathfrak{n}''} \mathfrak{F} = \lim_{\sigma \to 0} \frac{1}{S''} \oint F_\tau\, d\sigma . \tag{14}$$

The integral that appears on the right-hand side is generally called the *line integral* of the vector $\mathfrak{F}$. In particular, for a *closed* curve (note the circle in the integral symbol!), one calls it the *circulation* of that vector. That "circulation" is non-zero when the lines that represent the vector field $\mathfrak{F}$ are closed or helical since the tangential projection of $\mathfrak{F}$ will keep the same sign at all points of a closed curve in that case. That will happen, e.g., when $\mathfrak{F}$ means the velocity of the various particles of a rotating solid body or a rotating fluid mass. Those places where rot $\mathfrak{F}$ is non-zero are called *vortices* of the vector field $\mathfrak{F}$ ($\mathfrak{r}$). The vortex points generally define continuous lines, namely, the so-called *vortex lines* (or vortex filaments), which can be regarded as curved rotational axes in the case of fluids.

One can easily show by means of the formula (14) that the vector rot $\mathfrak{F}$ will then point in the same direction as the rotational axis, and its magnitude will be equal to twice the angular velocity. That explains the choice of the term "rotation."

The expression (11.a) for the divergence cannot be converted by specializing the surface, due to the scalar character of the differential quantity. However, its geometric meaning will become immediately clear when one illustrates the vector field $\mathfrak{F}$ by the corresponding "$\mathfrak{F}$-lines." The product $F_n\, dS$ can then be interpreted as the number of lines that go through the surface element,

and indeed outwards when F_n is positive or inwards when F_n is negative. The integral $\oint F_n \, dS$, which is called the *flux* of $\mathfrak{F}$ through S in the context of that picture ([6]), will then be equal to the excess in the number of lines that enter S over the number that exist it. (Obviously, that excess can just as well prove to be positive as negative.) When the divergence of $\mathfrak{F}$ vanishes inside of S, from (11.a), the total flux of the vector $\mathfrak{F}$ through S must also be equal to zero. That means that the lines that represent the vector field $\mathfrak{F}$ ($\mathfrak{r}$) will go through the region that is enclosed by S without beginning or ending in it. However, if div $\mathfrak{F}$ is non-zero at a point or in a region then the flux of $\mathfrak{F}$ through a surface that encloses that point must also be non-zero. In particular, when div $\mathfrak{F} > 0$, we will have a *source* of the $\mathfrak{F}$-lines inside of S, i.e., a place where they begin and *diverge* in different directions. That explains the choice of terminology "divergence." Negative values of div $\mathfrak{F}$ correspond to a "sink" (i.e., a "negative source") of the $\mathfrak{F}$-lines to which they must converge in all directions. (That is why the quantity – div $\mathfrak{F}$ is often called the "convergence.")

§ 10.

We must still mention a differential operation that relates to the vector function $\mathfrak{F}$ ($\mathfrak{r}$) and has a close analogy with the gradient of a scalar function. However, one can define that operation only when one is given a second vector or a vector function $\mathfrak{A}$ that is not itself differentiated since it only determines the *direction* in which the differentiation of the function $\mathfrak{F}$ ($\mathfrak{r}$) must proceed at each point. We then recall formula (13) of the foregoing section. If we replace φ with $\mathfrak{F}$ in it then that will give an operation that we will denote by the symbol ($\mathfrak{n}''$ grad):

$$(\mathfrak{n}'' \text{ grad})\, \mathfrak{F} = \lim_{h \to 0} \frac{1}{h} (\mathfrak{F}'' - \mathfrak{F}') = \frac{\partial \mathfrak{F}}{\partial h}. \tag{15}$$

Upon inverting the argument by which we arrived at formula (13) from (11), we will get the following definition of the new operation, which corresponds to formulas (11), (11.a), (11.b):

$$(\mathfrak{A} \text{ grad})\, \mathfrak{F} = \lim_{S \to 0} \frac{1}{V} \oint (\mathfrak{A}\, \mathfrak{n})\, \mathfrak{F} \, dS \,,$$

([6]) Even when S is not closed. In that case, as usual, the normal $\mathfrak{n}$ must point in the direction that is associated with the sense of traversal along the boundary curve σ according to the right-hand screw rule.

in which $\mathfrak{A}$ should mean an initially-constant unit vector, just like $\mathfrak{n}''$. However, we immediately free ourselves of that restriction when we understand the values of $\mathfrak{A}$ under the integral sign to mean the values at the points in question (to which the surface S should contract), i.e., to set:

$$(\mathfrak{A}\ \text{grad})\ \mathfrak{F} = \lim_{S\to 0}\left\{\frac{1}{V}\oint (\mathfrak{A}\,\mathfrak{n})\,\mathfrak{F}\,dS\right\}_{\mathfrak{A}=\text{const.}}, \tag{15.a}$$

in full generality. The vector $(\mathfrak{A}\ \text{grad})\ \mathfrak{F}$ is then equal to the "partial derivative" of $\mathfrak{F}$ in the direction of $\mathfrak{A}$, multiplied by the magnitude of $\mathfrak{A}$ ([7]). If we consider, e.g., two infinitely-close points $\mathfrak{r}_1$ and $\mathfrak{r}_2$ and set $\mathfrak{A} = \mathfrak{r}_2 - \mathfrak{r}_1 = d\,\mathfrak{r}$ then the vector $(d\,\mathfrak{r}\ \text{grad})\ \mathfrak{F}$ will mean nothing but the vector difference $d\,\mathfrak{F} = \mathfrak{F}(\mathfrak{r}_2) - \mathfrak{F}(\mathfrak{r}_1)$, just as $d\,\mathfrak{r}\cdot\text{grad}\ \varphi = d\varphi = \varphi(\mathfrak{r}_2) - \varphi(\mathfrak{r}_1)$. Note that the scalar product $\mathfrak{A}\,\text{grad}\,\varphi$ can be reduced to the combined operation that is defined by (15.a) when one replaces the vectorial function $\mathfrak{F}$ with the scalar φ. Namely, one has:

$$\begin{aligned}(\mathfrak{A}\ \text{grad})\ \varphi &= \lim_{S\to 0}\left\{\frac{1}{V}\oint (\mathfrak{A}\,\mathfrak{n})\,\varphi\,dS\right\}_{\mathfrak{A}=\text{const.}} = \lim_{S\to 0}\left\{\frac{1}{V}\oint \mathfrak{A}\cdot(\mathfrak{n}\,\varphi)\,dS\right\}_{\mathfrak{A}=\text{const.}}\\ &= \left(\mathfrak{A}\cdot\lim_{S\to 0}\frac{1}{V}\oint \mathfrak{n}\,\varphi\,dS\right) = \mathfrak{A}\cdot\text{grad}\ \varphi\,.\end{aligned}$$

When one assumes that $\mathfrak{F}$ is constant in (15.a), not $\mathfrak{A}$, that will give:

$$\lim_{S\to 0}\left\{\frac{1}{V}\oint (\mathfrak{A}\,\mathfrak{n})\,\mathfrak{F}\,dS\right\}_{\mathfrak{F}=\text{const.}} = \mathfrak{F}\lim_{S\to 0}\frac{1}{V}\oint \mathfrak{n}\,\mathfrak{A}\,dS = \mathfrak{F}\ \text{div}\ \mathfrak{A}\,.$$

([7]) In connection with (15.a), one can define two other differential operations of the same type, namely:

$$(\mathfrak{A}\times\text{grad})\,\mathfrak{F} = \lim_{S\to 0}\left\{\frac{1}{V}\oint (\mathfrak{A}\times\mathfrak{n})\,\mathfrak{F}\,dS\right\}_{\mathfrak{A}=\text{const.}},$$

$$(\mathfrak{A}\times\text{grad})\times\mathfrak{F} = \lim_{S\to 0}\left\{\frac{1}{V}\oint (\mathfrak{A}\times\mathfrak{n})\times\mathfrak{F}\,dS\right\}_{\mathfrak{A}=\text{const.}}.$$

However, those operations are inessential for practical applications of the vector calculus and can then be reduced to the previous ones. Indeed, the following identities will be true:

$$(\mathfrak{A}\times\text{grad})\,\mathfrak{F} = (\mathfrak{A}\ \text{rot}\ \mathfrak{F}) \quad \text{and} \quad (\mathfrak{A}\times\text{grad})\times\mathfrak{F} = \mathfrak{A}\ \text{rot}\ \mathfrak{F} + (\mathfrak{A}\ \text{grad})\ \mathfrak{F} - \mathfrak{A}\ \text{div}\ \mathfrak{F}\,.$$

Now, it is easy to show that in the general case when the two quantities $\mathfrak{A}$ and $\mathfrak{F}$ are considered to be variable, the integral $\frac{1}{V}\oint(\mathfrak{A}\,\mathfrak{n})\,\mathfrak{F}\,dS$ must be equal to simply the sum of the expressions above in the limit. In fact, if one sets $\mathfrak{A} = \mathfrak{A}_0 + \Delta\mathfrak{A}$ and $\mathfrak{F} = \mathfrak{F}_0 + \Delta\mathfrak{F}$, in which $\mathfrak{A}_0$ and $\mathfrak{F}_0$ mean the values of $\mathfrak{A}$ and $\mathfrak{F}$ at the point in question, then one will have:

$$(\mathfrak{A}\,\mathfrak{n})\,\mathfrak{F} = (\mathfrak{A}_0\,\mathfrak{n})\,\mathfrak{F}_0 + (\mathfrak{A}_0\,\mathfrak{n})\,\Delta\mathfrak{F} + (\Delta\mathfrak{A}\,\mathfrak{n})\,\mathfrak{F}_0 + (\Delta\mathfrak{A}\,\mathfrak{n})\,\Delta\mathfrak{F}\ ,$$

and furthermore:

$$\lim_{S\to 0}\frac{1}{V}\oint(\mathfrak{A}_0\,\mathfrak{n})\,\mathfrak{F}_0\,dS = 0\ ,$$

$$\lim_{S\to 0}\frac{1}{V}\oint(\mathfrak{A}_0\,\mathfrak{n})\,\Delta\mathfrak{F}\,dS = \lim_{S\to 0}\frac{1}{V}\oint(\mathfrak{A}_0\,\mathfrak{n})\,\mathfrak{F}\,dS\ = (\mathfrak{A}\ \mathrm{grad})\,\mathfrak{F}\ ,$$

$$\lim_{S\to 0}\frac{1}{V}\oint(\Delta\mathfrak{A}\cdot\mathfrak{n})\,\mathfrak{F}_0\,dS\ = \lim_{S\to 0}\frac{1}{V}\oint(\mathfrak{A}\,\mathfrak{n})\,\mathfrak{F}_0\,dS\ = \mathfrak{F}\ \mathrm{div}\ \mathfrak{A}\ ,$$

and since $\Delta\mathfrak{F}$ and $\Delta\mathfrak{A}$ are infinitely-small quantities (due to the assumption that the functions $\mathfrak{F}$ and $\mathfrak{A}$ are continuous):

$$\lim_{S\to 0}\frac{1}{V}\oint(\Delta\mathfrak{A}\,\mathfrak{n})\,\Delta\mathfrak{F}\,dS = 0\ .$$

As a result:

$$\lim_{S\to 0}\frac{1}{V}\oint(\mathfrak{A}\,\mathfrak{n})\,\mathfrak{F}\,dS = (\mathfrak{A}\ \mathrm{grad})\,\mathfrak{F} + \mathfrak{F}\ \mathrm{div}\ \mathfrak{A}\ . \qquad (15.b)$$

§ 11.

We divide the volume V, which is bounded by the (*not* infinitely small) surface S, into two sub-volumes V_1 and V_2 and denote the surfaces that bound those sub-volumes by S_1 (S_2, resp.). The exterior normals $\mathfrak{n}_1$ and $\mathfrak{n}_2$ coincide with $\mathfrak{n}$ on the parts of S_1 and S_2 that are common to S. They are opposed to each other on the surface $S_{1,2}$ that separates V_1 from V_2 such that one will have either $\mathfrak{n}_2 = -\mathfrak{n}_1$ or $\mathfrak{n}_1 + \mathfrak{n}_2 = 0$ on it.

It will then follow that the sum of the two integrals $\oint\varphi\,\mathfrak{n}_1\,dS_1$ and $\oint\varphi\,\mathfrak{n}_2\,dS_2$ must be equal to the original surface integral $\oint\varphi\,\mathfrak{n}\,dS$ independently of the form of the separation surface $S_{1,2}$ (since the integrals $\oint\varphi\,\mathfrak{n}_1\,dS_{1,2}$ and $\oint\varphi\,\mathfrak{n}_2\,dS_{1,2}$ will mutually cancel).

By continuing that process, one can subdivide V into infinitely-many volume elements V_i that are bounded by infinitely-small, closed surfaces S_i such that the sum $\sum_i \oint \varphi\, \mathfrak{n}_i\, dS_i$ will always be equal to $\oint \varphi\, \mathfrak{n}\, dS$ ([8]). If we pass to the limit and observe that $\lim_{S_i \to 0} \frac{1}{V_i} \oint \varphi_i\, \mathfrak{n}_i\, dS_i = -$ grad φ_i then we will get:

$$\oint \varphi\, \mathfrak{n}\, dS = \lim_{V_i \to 0} \sum_i \operatorname{grad} \varphi_i\, V_i ,$$

or when the sum is replaced with a volume integral:

$$\oint \varphi\, \mathfrak{n}\, dS = \int \operatorname{grad} \varphi\, dV . \tag{16}$$

In the same way, one will get the transformation formulas from (8), (9), and (15.b):

$$\oint F_n\, dS = \oint \mathfrak{n}\, \mathfrak{F}\, dS = \int \operatorname{div} \mathfrak{F}\, dV , \tag{16.a}$$

$$\oint \mathfrak{n} \times \mathfrak{F}\, dS = \int \operatorname{rot} \mathfrak{F}\, dV , \tag{16.b}$$

$$\oint (\mathfrak{n}\, \mathfrak{A}) \mathfrak{F}\, dS = \int (\mathfrak{A} \operatorname{grad}) \mathfrak{F}\, dV + \int \mathfrak{F} \operatorname{div} \mathfrak{A}\, dV . \tag{16.c}$$

§ 12.

One will get similar transformation formulas for line integrals that are taken along a closed curve σ. Namely, if one replaces σ with a net of infinitely-small planar curves σ_i that bound the surface elements S_i of any surface S that is bounded (so it is not closed) by σ and observe that the sum of the integrals $\oint \tau_i\, \mathfrak{F}\, d\sigma_i$ must always remain equal to the original integral $\oint \tau\, \mathfrak{F}\, d\sigma$ (since

([8]) On the grounds of that theorem, one can easily prove the validity of the statement that we expressed in **§ 8** in regard to the independence of $\lim_{S \to 0} \frac{1}{V} \oint \varphi\, \mathfrak{n}\, dS$ on the form of the outer surface S. We imagine that the elements V_i (S_i) have *the same form and size*, except for the boundary elements whose number and magnitude *relative* to the integral $\oint \varphi\, \mathfrak{n}\, dS$ will go to zero as the number N of *all* elements increases. If the surface S itself is infinitely small and N is infinitely large then due to the assumed continuity of the function φ, one can consider all sub-integrals $\oint \varphi_i\, \mathfrak{n}_i\, dS_i$ to be equal to each other, up to higher-order infinitesimals, and as a result one can set $\oint \varphi_i\, \mathfrak{n}_i\, dS_i \approx \frac{1}{V} \oint \varphi\, \mathfrak{n}\, dS$. Since one has $V_i = V / N$, to the same degree of approximation, it will follow that the limiting value of $\frac{1}{V} \oint \varphi\, \mathfrak{n}\, dS$ for a vanishingly-small surface S of *any* form must coincide with the value of $\frac{1}{V_i} \oint \varphi_i\, \mathfrak{n}_i\, dS_i$ for an infinitely-small surface of a well-defined form (e.g., a cube), so it will be independent of the form of S.

the tangent vectors τ_i point in opposite direction on any separating line between two surfaces elements S_i) then, from (14), one will get:

$$\oint \tau\, \mathfrak{F}\, d\sigma = \sum_i \oint \tau_i\, \mathfrak{F}\, d\sigma_i = \sum_i \mathfrak{n}_i\, \mathrm{rot}\, \mathfrak{F}_i\, S_i$$

in the limit as S_i, $\sigma_i \to 0$, or:

$$\oint F_\tau\, d\sigma = \int \mathrm{rot}_n\, \mathfrak{F}\, dS\,, \tag{17}$$

in which $\mathrm{rot}_n\, \mathfrak{F} = \mathfrak{n} \cdot \mathrm{rot}\, \mathfrak{F}$, and $\mathfrak{n}$ means the normal to dS. Therefore, that normal will point in the direction that corresponds to the sense of traversal along the boundary curve σ that is defined by the tangent vector τ and the right-hand screw rule.

Formula (16.a) is ordinarily referred to as *Gauss*'s theorem and (17) is referred to as *Stokes*'s theorem. One can add a similar formula to formula (17) that is obtained by replacing the vector function $\mathfrak{F}$ ($\mathfrak{r}$) with a scalar function φ (r). As above, that will give the identity:

$$\oint \tau\, \varphi\, d\sigma = \sum_i \oint \tau_i\, \varphi_i\, d\sigma_i\,.$$

The conversion of a line integral $\oint \tau_i\, \varphi_i\, d\sigma_i$ that extends along an infinitely-small curve σ_i, takes place most simply in the following way: Imagine that σ_i is the generator of the cylindrical surface that we used while deriving formula (13) above, and instead of the inner product $\mathfrak{n}'' \oint \varphi\, \mathfrak{n}\, dS$, we form the outer product $\mathfrak{n}'' \times \oint \varphi\, \mathfrak{n}\, dS$ (the index i will be dropped later). In that way, we will get:

$$\mathfrak{n}'' \times \oint \mathfrak{n}\, \varphi\, dS = \oint \mathfrak{n}'' \times \mathfrak{n}\, \varphi\, dS = \int \mathfrak{n}'' \times \nu\, \varphi\, d\Sigma = h \oint \tau\, \varphi\, d\sigma\,,$$

just as we did when deriving formula (14), and as a result, from (11):

$$\mathfrak{n}'' \times \mathrm{grad}\, \varphi\, S' = \oint \tau\, \varphi\, d\sigma\,,$$

or when we reintroduce the index i :

$$\oint \tau_i\, \varphi_i\, d\sigma_i = \mathfrak{n}_i \times \mathrm{grad}\ \varphi_i\, S_i\,.$$

Upon summing and passing to the limit ($S_i \to 0$), we will then get the following formula:

$$\oint \tau\, \varphi\, d\sigma = \int \mathfrak{n} \times \mathrm{grad}\, \varphi\, dS\,. \tag{17.a}$$

§ 13.

Upon applying the differential operations that were considered above to the functions:

$$\nabla\varphi = \text{grad}\ \varphi, \quad \nabla\,\mathfrak{F} = \text{div}\,\mathfrak{F}\,, \quad \nabla\times\mathfrak{F} = \text{rot}\,\mathfrak{F}\,,$$

which correspond to the usual first-order derivatives, we will get the following five second-order derivatives:

$$(\nabla\nabla)\,\varphi = \text{div grad}\ \varphi\,, \qquad \nabla\,(\nabla\,\mathfrak{F}) = \text{grad div}\,\mathfrak{F}\,, \qquad \nabla\times\nabla\varphi = \text{rot grad}\ \varphi\,,$$

$$\nabla\,(\nabla\times\mathfrak{F}) = \text{div rot}\,\mathfrak{F}\,, \qquad \nabla\times(\nabla\times\mathfrak{F}) = \text{rot rot}\,\mathfrak{F}\,.$$

If the differential operator were not a symbolic vector, but a true one, then the double "products" $\nabla\times\nabla\varphi = (\nabla\times\nabla)\varphi$ and $\nabla\,(\nabla\times\mathfrak{F})$ would vanish identically (in φ or $\mathfrak{F}$, resp.), i.e., the following identities would exist:

$$\text{rot grad}\ \varphi = 0\,, \tag{18}$$

$$\text{div rot}\,\mathfrak{F} = 0\,. \tag{18.a}$$

Now, it is easy to see that this is actually the case, based upon the transformation formulas above.

If we imagine that the surface S is *closed* in (17) and (17.a) then the bounding curve σ will reduce to a point, and the corresponding line integral will vanish. In that way, we will get the identities:

$$\oint \mathfrak{n}\ \text{rot}\,\mathfrak{F}\ dS\ = 0\,, \qquad \oint \mathfrak{n}\times\text{grad}\,\varphi\ dS\ = 0\,,$$

or upon transforming that according to formulas (16.a) and (16.b):

$$\int \text{div rot}\,\mathfrak{F}\ dV\ = 0\,, \qquad \int \text{rot grad}\,\varphi\ dV\ = 0\,.$$

Since the volume that is enclosed by S is entirely arbitrary the integrands in those volume integrals must vanish identically, which will then imply formulas (18) and (18.a).

One ordinarily writes the operation $\nabla\nabla\varphi$ = div grad φ in the form $\nabla^2\varphi$, where $\nabla^2 = \nabla\nabla$ is called the *Laplace* operator (it is often denoted by Δ, as well).

From the definition of div, one can obviously write:

$$\nabla^2\varphi = \lim_{S\to 0}\frac{1}{V}\oint \mathfrak{n}\cdot\text{grad}\,\varphi\ dS\,,$$

or since $\mathfrak{n}$ grad $\varphi = (\mathfrak{n}$ grad$)\ \varphi$:

$$\nabla^2 \varphi = \lim_{S \to 0} \frac{1}{V} \oint (\mathfrak{n}\,\mathrm{grad})\, \varphi \, dS \,. \tag{19}$$

That formula shows that the operation ∇^2 can be applied to not only scalar functions, but also vectorial ones, and indeed, when we replace φ with $\mathfrak{F}$ in (19):

$$\nabla^2 \mathfrak{F} = \lim_{S \to 0} \frac{1}{V} \oint (\mathfrak{n}\,\mathrm{grad})\, \mathfrak{F} \, dS \,. \tag{19.a}$$

If one reintroduces the symbolic vector ∇ in place of the symbol grad and treats it as an ordinary factor (which must, however, always appear *in front of* the function $\mathfrak{F}$ to be differentiated) then it will follow from the algebraic identity (5) that when one sets $\mathfrak{A} = \mathfrak{n}$, $\mathfrak{B} = \nabla$ (or conversely) and $\mathfrak{C} = \mathfrak{F}$, one will have:

$$(\mathfrak{n}\,\nabla)\,\mathfrak{F} = \mathfrak{n}\,(\nabla\,\mathfrak{F}) - \nabla\times\,(\mathfrak{n}\times\mathfrak{F}) = \nabla\,(\mathfrak{n}\,\mathfrak{F}) - \mathfrak{n}\times(\nabla\times\mathfrak{F})\,.$$

Since:

$$\lim_{S \to 0} \frac{1}{V} \oint \mathfrak{n}\,(\nabla\,\mathfrak{F})\, dS \;= \mathrm{grad}\,(\nabla\,\mathfrak{F}) = \nabla \lim_{S \to 0} \frac{1}{V} \oint \mathfrak{n}\,\mathfrak{F}\, dS$$

and

$$\lim_{S \to 0} \frac{1}{V} \oint \mathfrak{n}\times(\nabla\times\mathfrak{F})\, dS \;= \mathrm{rot}\,\nabla\times\mathfrak{F} = \nabla\times\lim_{S \to 0} \frac{1}{V} \oint \mathfrak{n}\times\mathfrak{F}\, dS\,,$$

from the equations above, that will yield:

$$\nabla^2 \mathfrak{F} = \mathrm{grad}\,\mathrm{div}\,\mathfrak{F} - \mathrm{rot}\,\mathrm{rot}\,\mathfrak{F}\,. \tag{19.b}$$

We will prove that identity more rigorously in the next section by means of the coordinate representation for the vectorial differential operations.

§ 14.

The simplest way for one to get such a representation is as follows:

We consider the function $\varphi(\mathfrak{r})$ to be an ordinary scalar function of the three components x_1, x_2, x_3 of the vector argument $\mathfrak{r}$. From (13), when the vector $\mathfrak{n}''$ is replaced with any of the unit vectors $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$ that determines the coordinate axes, that will give:

$$\mathfrak{e}_i\,\mathrm{grad}\;\varphi = \mathrm{grad}_i\;\varphi = \frac{\partial \varphi}{\partial x_i} \qquad (i = 1, 2, 3)\,. \tag{20}$$

The components of the vector grad $\varphi = \nabla\varphi$ are then equal to the partial derivatives of the function $\varphi(x_1, x_2, x_3)$ with respect to the corresponding coordinates.

It will then follow that the operator ∇ can be defined quite independently of the nature of the function φ as a (symbolic) vector with the components $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$. When written in the form:

$$\nabla = \mathfrak{e}_1 \frac{\partial}{\partial x_1} + \mathfrak{e}_2 \frac{\partial}{\partial x_2} + \mathfrak{e}_3 \frac{\partial}{\partial x_3}, \tag{20.a}$$

it is called the *Hamiltonian* operator, and can be applied to scalar, as well as vectorial functions when the vector argument $\mathfrak{r}$ of those functions is replaced with the three scalar arguments x_1, x_2, x_3. In that way, one will get the following expressions for div $\mathfrak{F}$, rot $\mathfrak{F}$, and ($\mathfrak{A}$ grad) $\mathfrak{F}$:

$$\operatorname{div} \mathfrak{F} = \nabla\mathfrak{F} = \frac{\partial}{\partial x_1} F_1 + \frac{\partial}{\partial x_2} F_2 + \frac{\partial}{\partial x_3} F_3 , \tag{21}$$

$$\operatorname{rot} \mathfrak{F} = \nabla \times \mathfrak{F} = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \right) \mathfrak{e}_1 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \right) \mathfrak{e}_2 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \mathfrak{e}_3 , \tag{21.a}$$

i.e.:

$$\operatorname{rot}_1 \mathfrak{F} = \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \quad \operatorname{rot}_2 \mathfrak{F} = \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \quad \operatorname{rot}_3 \mathfrak{F} = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}, \tag{21.b}$$

and

$$(\mathfrak{A} \text{ grad } \mathfrak{F})_i = \mathfrak{A} \text{ grad } F_i = A_1 \frac{\partial F_i}{\partial x_1} + A_2 \frac{\partial F_i}{\partial x_2} + A_3 \frac{\partial F_i}{\partial x_3} \quad (i = 1, 2, 3). \tag{21.c}$$

One can also derive those formulas immediately from the corresponding defining equations (11.a), (11.b) [(14) and (15), resp.] by choosing S to be the surface of an infinitely-small parallelepiped whose sides are parallel to the coordinate axes. For example, in the case of div $\mathfrak{F}$, when those sides are denoted by Δx_i and the surfaces that are perpendicular to them are denoted by S_i', S_i'' ($i = 1, 2, 3$) (such that S_i' refers to the point x_i, and S_i'' refers to the point $x_i + \Delta x_i$), one will have:

$$\oint \mathfrak{n}\mathfrak{F}\, dS = \sum_{i=1}^{3} \left\{ \int (\mathfrak{n}'\mathfrak{F}')_i \, dS_i' + \int (\mathfrak{n}''\mathfrak{F}'')_i \, dS_i'' \right\},$$

or, since $\mathfrak{n}_i'' = \mathfrak{e}_i$ and $\mathfrak{n}_i' = -\mathfrak{e}_i$, i.e., $(\mathfrak{n}''\ \mathfrak{F}'')_i = +\ \mathfrak{F}_i''$ and $(\mathfrak{n}'\ \mathfrak{F}')_i = -\ \mathfrak{F}_i'$:

$$\oint \mathfrak{n}\mathfrak{F}\, dS = \sum_i \int (F_i'' - F_i')\, dS_i = \sum_i \frac{\partial F_i}{\partial x_i} \Delta_i \, dS_i = \left(\sum_i \frac{\partial F_i}{\partial x_i} \right) \cdot V ,$$

in which $V = \Delta x_i \cdot S_i = \Delta x_1 \cdot \Delta x_2 \cdot \Delta x_3$ means the volume of the parallelepiped. Obviously, those equations are fulfilled only approximately up to quantities of the same order as the products $V \cdot \Delta x_i$. However, if we pass to the limit $\Delta x_i \to 0$ then we will get the exact equation:

$$\lim_{\Delta x_i \to 0} \frac{1}{V} \oint \mathfrak{n}\, \mathfrak{F}\, dS = \sum_i \frac{\partial F_i}{\partial x_i} ,$$

which is equivalent to (19).

Formulas (20) and (21) will give:

$$\text{div grad } \varphi = \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 \varphi}{\partial x_3^2} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \varphi . \tag{22}$$

One can then, in fact, define the *Laplace* operator ∇^2 to be the square of the *Hamiltonian* operator (18.a):

$$\nabla^2 = (\nabla\nabla) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} . \tag{22.a}$$

Now, it is easy to convince oneself that the same expression will be obtained for ∇^2 on the basis of equation (19.b). Namely, if one constructs the projection of the right-hand side of (19.b) onto any axis [e.g., the first one (X_1)] then, from (20), (21), and (21.b), one will have:

$$\frac{\partial}{\partial x_1} \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) - \frac{\partial}{\partial x_2} \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \right) = \frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_2^2} + \frac{\partial^2 F_1}{\partial x_3^2} ,$$

and that is nothing but the first component of the vector $\nabla^2 \mathfrak{F}$, when ∇^2 is regarded as a *scalar* factor in that way. Note that the projection of that vector onto any direction $\mathfrak{n}$ is equal to $\text{div grad } F_n$, i.e.:

$$| \nabla^2 \mathfrak{F} |_n = \nabla^2 F_n . \tag{22.b}$$

§ 15.

Now that we have defined and explained the differential operations of vector calculus, we must (as in the usual vector calculus) establish the rules that govern the application of those operations to sums, products, and compositions of functions.

The “derivative” of a geometric sum of several vector functions of a scalar or vectorial argument is obviously equal to the geometric sum of the corresponding derivatives, just like with ordinary differentiation. For instance:

$$\frac{d}{dt}\{\mathfrak{A}(t)+\mathfrak{B}(t)\}=\frac{d\mathfrak{A}}{dt}+\frac{d\mathfrak{B}}{dt},\quad \operatorname{grad}(\varphi+\psi)=\operatorname{grad}\varphi+\operatorname{grad}\psi,\quad \operatorname{div}(\mathfrak{E}+\mathfrak{F})=\operatorname{div}\mathfrak{E}+\operatorname{div}\mathfrak{F},$$

etc.

Moreover, one can easily prove that in the case of a product of two functions of a *scalar* argument, the same formulas that are true for scalar functions will still be valid, and indeed one has:

$$\frac{d}{dt}(\varphi\,\mathfrak{A})=\frac{d\varphi}{dt}\mathfrak{A}+\varphi\frac{d\mathfrak{A}}{dt}, \tag{23}$$

$$\frac{d}{dt}(\mathfrak{A}\;\mathfrak{B})=\frac{d\mathfrak{A}}{dt}\mathfrak{B}+\mathfrak{A}\frac{d\mathfrak{B}}{dt}, \tag{23.a}$$

$$\frac{d}{dt}(\mathfrak{A}\times\mathfrak{B})=\frac{d\mathfrak{A}}{dt}\times\mathfrak{B}+\mathfrak{A}\times\frac{d\mathfrak{B}}{dt}. \tag{23.b}$$

For example, the last formula is obtained as follows:

$$\frac{d}{dt}(\mathfrak{A}\times\mathfrak{B})=\lim_{\Delta t\to 0}\frac{1}{\Delta t}[(\mathfrak{A}+\Delta\mathfrak{A})\times(\mathfrak{B}+\Delta\mathfrak{B})-\mathfrak{A}\times\mathfrak{B}]$$

$$=\lim_{\Delta t\to 0}\left\{\frac{\Delta\mathfrak{A}}{\Delta t}\times\mathfrak{B}+\mathfrak{A}\times\frac{\Delta\mathfrak{B}}{\Delta t}+\frac{\Delta\mathfrak{A}}{\Delta t}\times\frac{\Delta\mathfrak{B}}{\Delta t}\Delta t\right\}=\frac{d\mathfrak{A}}{dt}\times\mathfrak{B}+\mathfrak{A}\times\frac{d\mathfrak{B}}{dt}.$$

The derivative of a product of two factors with respect to a scalar argument is therefore equal to the sum of the derivatives that are obtained when each of variables is regarded as variable, while the other one is regarded as constant. That rule can be easily generalized to double, and even more complicated products, and will *also remain valid when one differentiates functions of a vector argument* ($\mathfrak{r}$). That assertion can be proved in the same way for each type of differentiation, as we did in **§ 10** for the two different operations that apply to the expression $\lim_{S\to 0}\frac{1}{V}\oint(\mathfrak{A}\,\mathfrak{n})\,\mathfrak{F}\,dS$. In the context of the coordinate representation of the vectorial differential operations, one can also consider them to be a direct consequence of the corresponding theorem for scalar arguments.

In the simplest case of two factors (which might be scalar or vectorial functions of the vector argument $\mathfrak{r}$), one must distinguish between four types of products:

$$\varphi\,\psi,\quad \varphi\,\mathfrak{F},\quad \mathfrak{E}\,\mathfrak{F},\quad \mathfrak{E}\times\mathfrak{F},$$

and six corresponding types of derivatives, namely:

$$\nabla(\varphi\,\psi)=\operatorname{grad}(\varphi\,\psi),\qquad \nabla(\varphi\,\mathfrak{F})=\operatorname{div}(\varphi\,\mathfrak{F}),\quad \nabla\times(\varphi\,\mathfrak{F})=\operatorname{rot}(\varphi\,\mathfrak{F}),$$

$$\nabla\cdot(\mathfrak{E}\times\mathfrak{F})=\operatorname{div}(\mathfrak{E}\times\mathfrak{F}),\quad \nabla\times(\mathfrak{E}\times\mathfrak{F})=\operatorname{rot}(\mathfrak{E}\times\mathfrak{F}),$$

and

$$\nabla\,(\mathfrak{E}\;\mathfrak{F}) = \operatorname{grad}\,(\mathfrak{E}\;\mathfrak{F})\,.$$

One can carry out those differentiations completely by the detour of the coordinate representation. However, it is simpler and more convenient to perform the corresponding calculations with the help of the vectorial differential operation ∇ directly. In so doing, as was stated many times before, the latter can be considered to be an ordinary factor, but it must appear *immediately before* the function to be differentiated. When the last requirement is not fulfilled from the outset, the sequence of factors must be first switched by applying the algebraic identities (4) and (5). In that way, we will get the following formulas:

$$\nabla\,(\varphi\,\psi) = \varphi\,\nabla\psi + \psi\,\nabla\varphi\,,$$

$$\nabla\,(\varphi\,\mathfrak{F}) = \varphi\,\nabla\mathfrak{F} + \nabla\varphi\cdot\mathfrak{F}\,, \qquad \nabla\times(\varphi\,\mathfrak{F}) = \varphi\,\nabla\times\mathfrak{F} + \nabla\varphi\times\mathfrak{F}\,,$$

with no further analysis, or with the usual notations:

$$\operatorname{grad}\,(\varphi\,\psi) = \varphi\,\operatorname{grad}\,\psi + \psi\,\nabla\,(\varphi\,\psi) = \varphi\,\nabla\psi + \psi\,\nabla\varphi\,, \tag{24}$$

$$\operatorname{div}\,(\varphi\,\mathfrak{F}) = \varphi\,\operatorname{div}\,\mathfrak{F} + \operatorname{grad}\,\varphi\cdot\mathfrak{F}\,, \tag{24.a}$$

$$\operatorname{rot}\,(\varphi\,\mathfrak{F}) = \varphi\,\operatorname{rot}\,\mathfrak{F} + \operatorname{grad}\,\varphi\times\mathfrak{F}\,. \tag{24.b}$$

In the last formula, the sequence of the two factors in the second term on the right-hand side might seem initially doubtful. In order to determine the correct sequence, one must revert to the original definition of the corresponding operation in this and analogous cases. If one considers $\mathfrak{F}$ to be constant then, from (11.b), one will have:

$$\operatorname{rot}\,(\varphi\,\mathfrak{F}) = \lim_{S\to 0}\frac{1}{V}\oint \mathfrak{n}\times(\varphi\mathfrak{F})\,dS = \lim_{S\to 0}\left[\frac{1}{V}\oint \mathfrak{n}\,\varphi\,dS\right]\times\mathfrak{F} = \operatorname{grad}\,\varphi\times\mathfrak{F}\,,$$

which agrees with (24.b).

Moreover, from (4):

$$\{\nabla\,(\mathfrak{E}\times\mathfrak{F})\}_{\mathfrak{F}\,=\,\text{const.}} = \mathfrak{F}\,(\nabla\times\mathfrak{E})\,, \qquad \{\nabla\,(\mathfrak{E}\times\mathfrak{F})\}_{\mathfrak{E}\,=\,\text{const.}} = -\,\mathfrak{E}\,(\nabla\times\mathfrak{F})\,,$$

and as a result:

$$\operatorname{div}\,(\mathfrak{E}\times\mathfrak{F}) = \mathfrak{F}\,\operatorname{rot}\,\mathfrak{E} - \mathfrak{E}\,\operatorname{rot}\,\mathfrak{F}\,. \tag{25}$$

By means of the identity (5), when one sets $\mathfrak{A} = \nabla$, $\mathfrak{B} = \mathfrak{E}$, and $\mathfrak{C} = \mathfrak{F}$, that will give:

$$\{\nabla\times(\mathfrak{E}\times\mathfrak{F})\}_{\mathfrak{F}\,=\,\text{const.}} = (\mathfrak{F}\,\nabla)\,\mathfrak{E} - \mathfrak{F}\,(\nabla\,\mathfrak{E})\,,$$

$$\{\nabla \times (\mathfrak{E} \times \mathfrak{F})\}_{\mathfrak{E}\,=\,\text{const.}} = \mathfrak{E}\,(\nabla\,\mathfrak{F}) - (\mathfrak{E}\,\nabla)\,\mathfrak{F}\ ,$$

so

$$\text{rot}\,(\mathfrak{E} \times \mathfrak{F}) = (\mathfrak{F}\ \text{grad})\ \mathfrak{E} - (\mathfrak{E}\ \text{grad})\ \mathfrak{F} + \mathfrak{E}\ \text{div}\ \mathfrak{F} - \mathfrak{F}\ \text{div}\ \mathfrak{E}\ , \tag{26}$$

and in the same way, with $\mathfrak{A} = \mathfrak{F}$, $\mathfrak{B} = \nabla$, $\mathfrak{C} = \mathfrak{E}$, ($\mathfrak{A} = \mathfrak{E}$, $\mathfrak{B} = \nabla$, $\mathfrak{C} = \mathfrak{F}$, resp.):

$$\{\nabla\,(\mathfrak{E}\ \mathfrak{F})\}_{\mathfrak{F}\,=\,\text{const.}} = (\mathfrak{F}\ \nabla)\ \mathfrak{E} + \mathfrak{F} \times (\nabla \times \mathfrak{E})\ ,$$
$$\{\nabla\,(\mathfrak{E}\ \mathfrak{F})\}_{\mathfrak{E}\,=\,\text{const.}} = (\mathfrak{E}\ \nabla)\ \mathfrak{F} + \mathfrak{E} \times (\nabla \times \mathfrak{F})\ ,$$

i.e.:

$$\text{grad}\,(\mathfrak{E}\ \mathfrak{F}) = (\mathfrak{E}\ \text{grad})\ \mathfrak{F} + (\mathfrak{F}\ \text{grad})\ \mathfrak{E} + \mathfrak{E} \times \text{rot}\ \mathfrak{F} + \mathfrak{F} \times \text{rot}\ \mathfrak{E}\ . \tag{27}$$

If the functions φ or $\mathfrak{F}$ do not depend upon $\mathfrak{r}$ *directly*, but only by way of another *scalar* function of that argument $f(\mathfrak{r})$ then their vectorial derivatives grad, div, rot, ($\mathfrak{A}$ grad) with respect to $\mathfrak{r}$ (just like in the usual differential calculus) will reduce to the *corresponding* (ordinary, inner, outer, resp.) products of the vectorial derivatives, e.g., the gradient of f times the derivative of φ or $\mathfrak{F}$ with respect to the scalar argument f. The following formulas will then be true:

$$\nabla\varphi\,(f) = (\nabla f)\,\frac{d\varphi}{df}\,, \qquad \nabla\mathfrak{F}\,(f) = (\nabla f)\cdot\frac{d\mathfrak{F}}{df}\,, \quad \nabla \times \mathfrak{F}\,(f) = \nabla f \times \frac{d\mathfrak{F}}{df}\,,$$
$$(\mathfrak{A}\nabla)\ \mathfrak{F}\,(f) = (\mathfrak{A}\,\nabla f)\,\frac{d\mathfrak{F}}{df}\,,$$

or in the usual notation:

$$\text{grad}\ \varphi\,(f) = (\text{grad}\ f)\,\frac{d\varphi}{df}\,, \tag{28}$$

$$\text{div}\ \mathfrak{F}\,(f) = (\text{grad}\ f)\cdot\frac{d\mathfrak{F}}{df}\,, \tag{28.a}$$

$$\text{rot}\ \mathfrak{F}\,(f) = (\text{grad}\ f)\times\frac{d\mathfrak{F}}{df}\,, \tag{28.b}$$

$$(\mathfrak{A}\ \text{grad})\ \mathfrak{F}\,(f) = (\mathfrak{A}\,\text{grad}\ f)\,\frac{d\mathfrak{F}}{df}\,. \tag{28.c}$$

In order to explain those rules of differentiation, we would like to derive formula (28.b) more thoroughly. If we set:

$$\mathfrak{F} = \mathfrak{F}_0 + \Delta\mathfrak{F} = \mathfrak{F}_0 + \left(\frac{d\mathfrak{F}}{df}\right)_0 \Delta f = \mathfrak{F}_0 + \left(\frac{d\mathfrak{F}}{df}\right)_0 (f - f_0)$$

in

$$\text{rot}\ \mathfrak{F} = \lim_{S\to 0}\frac{1}{V}\oint \mathfrak{n}\times\mathfrak{F}\,dS\ ,$$

in which the index 0 refers to the point considered, then when we recall that:

$$\oint \mathfrak{n} \times \mathfrak{F}_0 \, dS = 0 \qquad \text{and} \qquad \oint f_0 \, \mathfrak{n} \times \left(\frac{d\mathfrak{F}}{df} \right)_0 dS = 0 \, ,$$

we will have:

$$\oint \mathfrak{n} \times \mathfrak{F} \, dS = \oint f \, \mathfrak{n} \times \left(\frac{d\mathfrak{F}}{df} \right)_0 dS = \left[\oint f \, \mathfrak{n} \, dS \right] \times \left(\frac{d\mathfrak{F}}{df} \right)_0 ,$$

and as a result:

$$\operatorname{rot} \mathfrak{F} = \lim_{S \to 0} \frac{1}{V} \left[\oint f \, \mathfrak{n} \, dS \right] \times \left(\frac{d\mathfrak{F}}{df} \right)_0 = \left[\lim_{S \to 0} \frac{1}{V} \oint f \, \mathfrak{n} \, dS \right] \times \left(\frac{d\mathfrak{F}}{df} \right)_0 = (\operatorname{grad} f) \times \frac{d\mathfrak{F}}{df} \, .$$

§ 16.

The differentiation of more complicated scalar or vectorial functions of a vector function ($\mathfrak{r}$) can be reduced to the differentiation of the *simplest* functions of that type by means of the formulas that were cited above. The simplest functions are initially the radius vector $\mathfrak{r}$ itself, along with its magnitude r and its square $(\mathfrak{r}\mathfrak{r}) = r^2$, and the linear functions $\mathfrak{k} \; \mathfrak{r}$ and $\mathfrak{k} \times \mathfrak{r}$, where $\mathfrak{k}$ means a constant vector. Indeed, one can calculate the various derivatives of those functions by combining the general formulas above. However, a simpler way to achieve that goal is when one starts directly from the geometric meanings of those functions and the corresponding operations or employs their coordinate representations.

For example, one can easily see that the integral $\oint \mathfrak{n} \, \mathfrak{r} \, dS$ is equal to three times the volume that is bounded by the surface S (since dS is the base surface of a skew cone with height n r), from which, it will follow that $\lim_{S \to 0} \oint \mathfrak{n} \, \mathfrak{r} \, dS = 3$, i.e.:

$$\operatorname{div} \mathfrak{r} = 3 \, . \tag{29}$$

One will arrive at the same result much more simply by means of the coordinate representation. Namely, since $r_i = x_i$, from (21), one will have:

$$\operatorname{div} \mathfrak{r} = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} = 3 \, .$$

One will get the formula:

$$\operatorname{rot} \mathfrak{r} = 0 \tag{29.a}$$

in the same way, and since $r^2 = x_1^2 + x_2^2 + x_3^2$:

$$\text{grad } r^2 = 2\,\mathfrak{r}\,, \tag{29.b}$$

or since, from (28), grad $r^2 = 2r$ grad r :

$$\text{grad } r = \frac{\mathfrak{r}}{r}\,. \tag{29.c}$$

The gradient of r is then equal to a unit vector that points in the same direction as $\mathfrak{r}$. That result follows from the defining equation (13) with no calculation.

One easily obtains the following formulas in the same way:

$$\text{grad } (\mathfrak{k}\ \mathfrak{r}) = (\mathfrak{k}\ \text{grad})\ \mathfrak{r} = \mathfrak{k}\,, \tag{30}$$

$$\text{div } (\mathfrak{k} \times \mathfrak{r}) = 0\,, \tag{30.a}$$

$$\text{rot } (\mathfrak{k} \times \mathfrak{r}) = 2\,\mathfrak{k}\,, \tag{30.b}$$

$$(\mathfrak{A}\ \text{grad})\ (\mathfrak{k} \times \mathfrak{r}) = (\mathfrak{k} \times \mathfrak{A})\,. \tag{30.c}$$

For example, from (21.a), one has:

$$\text{rot}_1\ (\mathfrak{k} \times \mathfrak{r}) = \frac{\partial}{\partial x_2}(\mathfrak{k}\times\mathfrak{r})_3 - \frac{\partial}{\partial x_3}(\mathfrak{k}\times\mathfrak{r})_2 = \frac{\partial}{\partial x_2}(k_1 x_2 - k_2 x_1) - \frac{\partial}{\partial x_3}(k_3 x_1 - k_1 x_3) = 2k_1\,.$$

Otherwise, from (26) and (27):

$$\text{rot } (\mathfrak{k} \times \mathfrak{r}) = -\ (\mathfrak{k}\ \text{grad})\ \mathfrak{r} + \mathfrak{k}\ \text{div}\ \mathfrak{r}\,,$$
$$\text{grad } (\mathfrak{k}\ \mathfrak{r}) = (\mathfrak{k}\ \text{grad})\ \mathfrak{r} + \mathfrak{k}\ \text{rot}\ \mathfrak{r} = (\mathfrak{f}\ \text{grad})\ \mathfrak{r}\,,$$

such that one of the formulas (30) will suffice in order for one to derive formula (30.b) and the second of (30) by means of (29) and (29.a).

In order to explain the rules of differentiation that were presented, we would like to consider some even more complicated functions.

1. $\varphi = r^n$. From (28) and (29.c), we will get:

$$\text{grad } r^n = n\,r^{n-1}\,\text{grad}\,r = n\,r^{n-2}\,\mathfrak{r}\,, \tag{31}$$

in which n is an entirely-arbitrary number.

2. $\mathfrak{F} = r^n\,\mathfrak{r}$. It will follow from (24) and (24.a) that:

$$\text{div } r^n\,\mathfrak{r} = n\,r^{n-2}\,(\mathfrak{r}\,\mathfrak{r}) + 3r^n\,,$$

i.e.:

$$\text{div } r^n \mathfrak{r} = (n+3)\ r^n , \tag{31.a}$$

and furthermore, due to (29.a):

$$\text{rot } r^n \mathfrak{r} = 0 . \tag{31.b}$$

That equation is a special case of the identity (18), since from (31), one will have $r^n \mathfrak{r} = \text{grad} \frac{r^{n+2}}{n+2}$.Note that for $n = -2$, that formula must be replaced with $\mathfrak{r} / r^2 = \text{grad ln } r$.

3. $\mathfrak{F} = (\mathfrak{k}\ \mathfrak{r})\ \mathfrak{r}$ (viz., the component of $\mathfrak{r}$ that is parallel to $\mathfrak{k}$). From (24) and (30), one will have:

$$\text{div } (\mathfrak{k}\ \mathfrak{r})\ \mathfrak{r} = 3\ \mathfrak{k}\ \mathfrak{r} + \mathfrak{r}\ \mathfrak{k} = 4\ \mathfrak{k}\ \mathfrak{r} ,$$
$$\text{rot } (\mathfrak{k}\ \mathfrak{r})\ \mathfrak{r} = \mathfrak{k} \times \mathfrak{r} .$$

4. $\mathfrak{F} = (\mathfrak{k} \times \mathfrak{r}) \times \mathfrak{r}$ (viz., the component of $\mathfrak{r}$ that is perpendicular to $\mathfrak{k}$). From (26), one will have:

$$\text{rot } [(\mathfrak{k} \times \mathfrak{r}) \times \mathfrak{r}] = (\mathfrak{r}\ \text{grad})\ (\mathfrak{k} \times \mathfrak{r}) - [(\mathfrak{k} \times \mathfrak{r})\ \text{grad}]\ \mathfrak{r} + (\mathfrak{k} \times \mathfrak{r})\ \text{div } \mathfrak{r} - \mathfrak{r}\ \text{div } (\mathfrak{k} \times \mathfrak{r}) ,$$

and furthermore, from (30) – (30.c):

$$\text{rot } [(\mathfrak{k} \times \mathfrak{r}) \times \mathfrak{r}] = \mathfrak{k} \times \mathfrak{r} - \mathfrak{k} \times \mathfrak{r} + 3\ \mathfrak{k} \times \mathfrak{r} = 3\ \mathfrak{k} \times \mathfrak{r} .$$

Obviously, one will arrive at the same result when one replaces $(\mathfrak{k} \times \mathfrak{r}) \times \mathfrak{r}$ with $(\mathfrak{k}\ \mathfrak{r})\ \mathfrak{r} - r^2\ \mathfrak{k}$ using (5).

C. – Coordinate transformations and tensors.

§ 17.

If we go from the original coordinate system (X_1, X_2, X_3) to another one (X_1', X_2', X_3') that is also rectangular and has the same origin O, but with different directions for the axes then we will get new values for the coordinates of a point P, i.e., the components of its radius vector $OP = \mathfrak{r}$, namely:

$$x_k' = \mathfrak{r}\ \mathfrak{e}_k' \qquad\qquad (k = 1, 2, 3),$$

in which $\mathfrak{e}_1'$, $\mathfrak{e}_2'$, $\mathfrak{e}_3'$ are the new "coordinate vectors," i.e., unit vectors that determine the direction of the new axes. If the new coordinate system (X') is a right-hand screw system, just like the original one (X) (which will always be assumed in what follows) then one can think of it as arising from the latter by a rotation.

The relationship between the "original" and "new" components of the radius vector, or any other vector $\mathfrak{A}$, are determined by the quantities:

$$\cos(X_i, X_{i'}) = \mathfrak{e}_i \mathfrak{e}'_{i'} = \alpha_{ii'} \qquad (i,\ i' = 1, 2, 3), \tag{32}$$

which one can consider to be the new components of the original coordinate vectors.

If one sets:

$$\mathfrak{A} = \sum_i A_i \mathfrak{e}_i = \sum_{i'} A'_{i'} \mathfrak{e}'_{i'}$$

then one will have:

$$A'_{i'} = \mathfrak{A}\, \mathfrak{e}'_{i'} = \left(\sum_i A_i \mathfrak{e}_i \right) \mathfrak{e}'_{i'} = \sum_i A_i (\mathfrak{e}_i \mathfrak{e}'_{i'}),$$

i.e.:

$$A'_{i'} = \sum_{i=1}^{3} \alpha_{ii'} A_i\ , \tag{32.a}$$

and likewise:

$$A_i = \sum_{i'=1}^{3} \alpha_{ii'} A'_{i'}\ . \tag{32.b}$$

Since one has:

$$\mathfrak{e}'_{i'} = \sum_i \alpha_{ii'} \mathfrak{e}_i \quad \text{and} \qquad \mathfrak{e}_i = \sum_{i'} \alpha_{ii'} \mathfrak{e}'_{i'}$$

from (32), it will follow that the components of an arbitrary vector will transform in the same way as the coordinate vectors under a rotation, or *covariantly* ([9]).

Quantities such as the inner product of two vectors or especially the square of a vector $A^2 = \mathfrak{A}\mathfrak{A}$ will obviously remain unchanged, or *invariant*, such that one will have:

$$A_1^2 + A_2^2 + A_3^2 = A_1'^2 + A_2'^2 + A_3'^2\ .$$

If one expresses the new components in terms of the original ones here by means of formulas (32.a) then one will get the identity:

$$\sum_{i'} A_{i'}'^2 = \sum_{i'} \left(\sum_k \alpha_{ki'} A_k \right) \left(\sum_l \alpha_{li'} A_l \right) = \sum_k \sum_l A_k A_l \left(\sum_{i'} \alpha_{ki'} \alpha_{li'} \right),$$

from which it will follow that:

([9]) That is true for only rectangular coordinate systems. In the general case of skew coordinate axes, when the components of a vector are different from the corresponding projections, only the latter will transform covariantly, while the former will transform *contravariantly*, i.e., the *new* components will have the same relationship to the original ones that the *original* coordinate vectors have to the new ones.

$$\sum_{i'} \alpha_{ki'}\, \alpha_{li'} = \begin{cases} 1 & \text{for} \quad k = l, \\ 0 & \text{for} \quad k \neq l. \end{cases} \tag{33}$$

Due to the known formula:

$$\sum_{i'} \alpha_{ki'}\, \alpha_{li'} = \sum_{i'} \cos(X_k\, X'_{i'})\, \cos(X_l\, X'_{i'}) = \cos(X_k\, X_l)\,,$$

those relations express the fact that the coordinates considered are rectangular, and that is why they are called "orthogonality conditions." One will get the reciprocal relations to (33) in the same way:

$$\sum_{i} \alpha_{ik'}\, \alpha_{il'} = \begin{cases} 1 & \text{for} \quad k' = l', \\ 0 & \text{for} \quad k' \neq l'. \end{cases} \tag{33.a}$$

Therefore, the relations (33) and (33.a) are not independent but follow immediately from each other.

§ 18.

In contrast to ordinary scalar quantities, which are independent of the orientation of the coordinate system, the components of a vector have no well-defined values that are independent of that orientation, even though they are also scalar quantities. We must then distinguish two types of scalars in the coordinate representation of vectors: The ordinary *invariant* scalars, on the one hand, and the *variant* ones, i.e., the scalars that transform covariantly with the coordinate vectors, on the other. The introduction of variant scalars that represent a vector when combined into a triple allows us to go further in that direction and construct quantities that relate to the vectors in the same way that the latter relate to the ordinary (i.e., invariant) scalars.

For example, we can define the products of each pair of components of the vectors $\mathfrak{A}$ and $\mathfrak{B}$. In that way, we will get nine quantities in the original coordinate system (X):

$$\begin{gathered} A_1\, B_1\,,\ A_1\, B_2\,,\ A_1\, B_3\,, \\ A_2\, B_1\,,\ A_2\, B_2\,,\ A_2\, B_3\,, \\ A_3\, B_1\,,\ A_3\, B_2\,,\ A_3\, B_3\,, \end{gathered}$$

and nine *completely-different* quantities in the new coordinate system (X'):

$$\begin{gathered} A'_1\, B'_1,\ \ A'_1\, B'_2,\ \ A'_1\, B'_3, \\ A'_2\, B'_1,\ \ A'_2\, B'_2,\ \ A'_2\, B'_3, \\ A'_3\, B'_1,\ \ A'_3\, B'_2,\ \ A'_3\, B'_3, \end{gathered}$$

but which *correspond* to the original ones in the same way that the new components of a vector correspond to the original components of *the same* vector.

On those grounds, one can regard the quantities $A_i\, B_k$ and $A'_{i'}\, B'_{k'}$ as the (scalar) components of one and the same *composite quantity* relative to the coordinate systems (X) and (X'). That composite quantity, which initially possesses no geometrically intuitive meaning, is the simplest representative of the so-called *tensor quantities*, or more precisely, *second-rank tensors.*

One defines a second-rank tensor in the general case to be a quantity ${}^2\mathfrak{T}$ that can be represented by $3^2 = 9$ *divariant* scalars, i.e., nine scalar quantities whose values depend upon the choice (i.e., orientation) of the coordinate system in the same way that the values of the product of two *monovariant* (i.e., representing an ordinary vector) scalars ([10]). Those divariant scalars are called the *components* of the tensor. If they are known for a coordinate system (X), and indeed equal to T_{ik} $(i,\ k = 1,\ 2,\ 3)$, then one can calculate them for any other coordinate system by using the transformation formulas [cf., (32.a)]:

$$T'_{i'k'} = \sum_i \sum_k \alpha_{ii'}\, \alpha_{kk'}\, T_{ik}\ . \tag{34}$$

Conversely, the original components of ${}^2\mathfrak{T}$ can be expressed in terms of the new ones by means of the formulas:

$$T_{ik} = \sum_{i'} \sum_{k'} \alpha_{ii'}\, \alpha_{kk'}\, T'_{i'k'}\ , \tag{34.a}$$

which correspond to the formulas (32.b). [Obviously, one can also obtain it by solving equations (34) for the T_{ik} directly.]

One defines tensors of rank three, four, and higher in a completely-similar way. A tensor of rank n ${}^n\mathfrak{T}$ is then represented in coordinate notation by 3^n components that are n-fold variant scalars, i.e., scalars that transform like the products of n vector components under coordinate transformations (rotations of the original coordinate systems). In that way, one can consider the vector components to be monovariant scalars and correspondingly consider vector quantities to be first-rank tensors. The ordinary invariant scalars are added to that sequence as the tensors of rank "zero."

§ 19.

We would initially like to treat only the "ordinary" tensors, i.e., the second-rank tensors. Due to the linear character of the transformation equations (34) or (34.a), we can conclude immediately that the sums (or differences) of the corresponding components of two different tensors define a new tensor. That new tensor, which corresponds to the geometric sum (or difference) of two vectors

([10]) In particular, the *coordinates* of a point.

is called the tensor sum (difference, resp.) of the other two ${}^2\mathfrak{P}$, ${}^2\mathfrak{Q}$, and will be denoted by ${}^2\mathfrak{P} + {}^2\mathfrak{Q}$ (${}^2\mathfrak{P} - {}^2\mathfrak{Q}$, resp.). In general, the equation:

$$ {}^2\mathfrak{T} = {}^2\mathfrak{P} + {}^2\mathfrak{Q} + {}^2\mathfrak{R} + \ldots \tag{35} $$

means that $T_{ik} = P_{ik} + Q_{ik} + R_{ik} + \ldots$ ($i, k = 1, 2, 3$).

One can associate any tensor ${}^2\mathfrak{T}$ with the components T_{ik} with another one ${}^2\tilde{\mathfrak{T}}$ whose components are determined by the condition:

$$ \tilde{T}_{ik} = T_{ki} . \tag{36} $$

That condition means that one will get the components of ${}^2\tilde{\mathfrak{T}}$ when one transposes rows with columns in the component matrix of ${}^2\mathfrak{T}$:

$$ \begin{array}{lll} T_{11} , & T_{12} , & T_{13} , \\ T_{21} , & T_{22} , & T_{23} , \\ T_{31} , & T_{32} , & T_{33} . \end{array} $$

It is easy to see that the condition (36) is invariant under coordinate transformations, i.e., that the new components of the corresponding tensors that are calculated from T_{ik} and $\tilde{T}_{ik}$ using the formula (34) will satisfy the same condition:

$$ \tilde{T}'_{i'k'} = T'_{k'i'} . $$

That shows that the quantities $\tilde{T}_{ik}$ actually define a tensor. That tensor is called the *transpose* of ${}^2\mathfrak{T}$. In particular, the following special cases should be noted:

1. $T_{ki} = T_{ik}$. The tensor ${}^2\mathfrak{T}$ is said to be *symmetric* and is identical to the transposed tensor ${}^2\tilde{\mathfrak{T}}$. The number of *distinct* components of ${}^2\mathfrak{T}$ is equal to six.

2. $T_{ki} = - T_{ik}$. The tensor ${}^2\mathfrak{T}$ is said to be *skew-symmetric* or *antisymmetric* and is equal and opposite to its transpose (${}^2\mathfrak{T} + {}^2\tilde{\mathfrak{T}} = 0$). Since the "diagonal components" of ${}^2\mathfrak{T}$, i.e., T_{11}, T_{22}, T_{33}, will vanish in that case, the number of mutually-independent scalar quantities that determine ${}^2\mathfrak{T}$ will reduce to three, namely:

$$ T_{23} = - T_{32} = T_1 , \qquad T_{31} = - T_{13} = T_2 , \qquad T_{12} = - T_{21} = T_3 . \tag{36.a} $$

It will then follow that a skew-symmetric tensor ${}^2\mathfrak{T}$ must be completely-equivalent to $\mathfrak{T}$ with the components T_1, T_2, T_3 .

In fact, it is easy to prove that the quantities (36.a) can be just as well considered to be divariant scalars as monovariant ones. Obviously, a skew-symmetric tensor can always be constructed by means of *two* different vectors $\mathfrak{A}$ and $\mathfrak{B}$ whose components will satisfy the equations:

$$A_i\, B_k - A_k\, B_i = T_{ik} = -\, T_{ki}$$

in the original coordinate system. However, from formulas (10.a), the left-hand side of those equations are nothing but the components of the vector $\pm\, \mathfrak{A} \times \mathfrak{B}$, i.e., the outer products of the vectors $\mathfrak{A}$ and $\mathfrak{B}$. The scalars T_1, T_2, T_3 are then, in fact, the components of a vector:

$$\mathfrak{T} = \mathfrak{A} \times \mathfrak{B} \ .$$

An *asymmetric* tensor ${}^2\mathfrak{T}$, i.e., a tensor that is neither symmetric not skew-symmetric, can always be decomposed into a symmetric and a skew-symmetric part. Namely, if one sets:

$${}^2\mathfrak{P} = \tfrac{1}{2}\left({}^2\mathfrak{T} + {}^2\tilde{\mathfrak{T}}\right), \qquad \text{i.e.,} \qquad P_{ik} = \tfrac{1}{2}\left(T_{ik} + T_{ki}\right) = P_{ki}$$

and

$${}^2\mathfrak{Q} = \tfrac{1}{2}\left({}^2\mathfrak{T} - {}^2\tilde{\mathfrak{T}}\right), \qquad \text{i.e.,} \qquad Q_{ik} = \tfrac{1}{2}\left(T_{ik} - T_{ki}\right) = -\, Q_{ki}$$

then one will have:

$${}^2\mathfrak{T} = {}^2\mathfrak{P} + {}^2\mathfrak{Q} \ .$$

Since the statements above suggest that the skew-symmetric tensor ${}^2\mathfrak{Q}$ is equivalent to a vector $\mathfrak{Q}$, one will see that in the general case, a tensor (second rank) can be reduced to a symmetric tensor and a vector.

§ 20.

The meaning of such a reduction initially emerges in the *multiplication* of tensors with vectors or other tensors.

If one defines the following expressions that correspond to the inner product of two vectors:

$$\sum_k T_{ik}\, F_k \qquad \text{and} \qquad \sum_i T_{ik}\, F_i \ ,$$

in which F_1, F_2, F_3 are the components of a vector $\mathfrak{F}$ and observes that those expressions are completely equivalent, in regard to their type of transformation (“variance”) with ones that that are obtained from them by the Ansatz $T_{ik} = A_i\, B_k$, i.e.:

$$\sum_k A_i\, B_k\, F_k = A_i \sum_k B_k\, F_k = A_i\, \mathfrak{B}\, \mathfrak{F}$$

and

$$\sum_i A_i\, B_k\, F_i = B_k \sum_i A_i\, F_i = B_i\, \mathfrak{A}\, \mathfrak{F}\,,$$

then one will see that $\sum_k T_{ik}\, F_k$ must be equal to the i^{th} component of a vector, and $\sum_i T_{ik}\, F_i$ must be equal to the k^{th} component of another vector. We would like to denote the first of those vectors by ${}^2\mathfrak{T}\ \mathfrak{F} = \mathfrak{F}\ {}^2\mathfrak{T}$ and call it the *product* of ${}^2\mathfrak{T}$ and $\mathfrak{F}$. The second vector will then seem to be the product of ${}^2\tilde{\mathfrak{T}}$ and $\mathfrak{F}$, such that:

$$\left.\begin{aligned} ({}^2\mathfrak{T}\,\mathfrak{F})_i &= \sum_k T_{ik}\, F_k, \\ ({}^2\tilde{\mathfrak{T}}\,\mathfrak{F})_k &= \sum_i T_{ik}\, F_i\,. \end{aligned}\right\} \tag{37}$$

If the tensor ${}^2\mathfrak{T}$ is symmetric then the two products (37) will coincide. If one sets ${}^2\mathfrak{T} = {}^2\mathfrak{P} + {}^2\mathfrak{Q}$ in the general case, where ${}^2\mathfrak{P}$ is symmetric $= \frac{1}{2}({}^2\mathfrak{T} + {}^2\tilde{\mathfrak{T}})$, and ${}^2\mathfrak{Q}$ is skew-symmetric then with:

$$Q_{23} = -\,Q_{32} = Q_1\,, \qquad Q_{31} = -\,Q_{13} = Q_2\,, \qquad Q_{12} = -\,Q_{21} = Q_3\,,$$

that will give:

$$({}^2\mathfrak{T}\ \mathfrak{F})_i = ({}^2\mathfrak{P}\ \mathfrak{F})_i + (\mathfrak{F} \times \mathfrak{Q})_i\,,$$

i.e.:

$$\mathfrak{F} \cdot {}^2\mathfrak{T} = \mathfrak{F}\ {}^2\mathfrak{P} + \mathfrak{F} \times \mathfrak{Q}\,, \tag{37.a}$$

and likewise:

$$\mathfrak{F} \cdot {}^2\tilde{\mathfrak{T}} = \mathfrak{F}\ {}^2\mathfrak{P} - \mathfrak{F} \times \mathfrak{Q}\,.$$

The multiplication of a tensor by a vector then reduces to the corresponding multiplication of the symmetric part of that tensor and the outer multiplication of the vector that is equivalent to the skew-symmetric part by the vector considered ([11]).

Corresponding statements are true for the *multiplication of two tensors* ${}^2\mathfrak{T}$ and ${}^2\mathfrak{S}$. Namely, one can define two *invariant* scalar quantities from the components of ${}^2\mathfrak{T}$ and ${}^2\mathfrak{S}$:

$$\sum_i \sum_k T_{ik}\, S_{ki} \quad \text{and} \quad \sum_i \sum_k T_{ik}\, S_{ik}\,.$$

([11]) Along with the cited "inner" multiplication that was cited, one can also introduce an operation that corresponds to the outer multiplication of two vectors. However, it would not give a vector, but a tensor; see below.

One sees the invariance of those quantities under coordinate transformation directly by means of the Ansätze $T_{ik} = A_i\, B_k$, $S_{ik} = C_i\, D_k$, since the first one will reduce to $(\mathfrak{AD})\,(\mathfrak{BC})$ in that way, and the second one, to $(\mathfrak{AC})\,(\mathfrak{BD})$. The first of the aforementioned quantities will be denoted by ${}^2\mathfrak{T}\cdot{}^2\mathfrak{S}$ and will be called the scalar product of the tensors ${}^2\mathfrak{T}$ and ${}^2\mathfrak{S}$.

One will then have:

$$\sum_i \sum_k T_{ik}\, S_{ki} = {}^2\mathfrak{T}\,{}^2\mathfrak{S} = {}^2\mathfrak{S}\,{}^2\mathfrak{T}\,, \tag{38}$$

and correspondingly:

$$\sum_i \sum_k T_{ik}\, S_{ik} = {}^2\mathfrak{T}\,{}^2\tilde{\mathfrak{S}} = {}^2\tilde{\mathfrak{T}}\,{}^2\mathfrak{S}.$$

As long as *one* of the tensors ${}^2\mathfrak{T}$ and ${}^2\mathfrak{S}$ is symmetric, the latter product must coincide with (38). In the general case (viz., ${}^2\mathfrak{T}$ and ${}^2\mathfrak{S}$ are both asymmetric), when ${}^2\mathfrak{M}$ means the symmetric part of ${}^2\mathfrak{S}$, and ${}^2\mathfrak{N}$ means the skew-symmetric part, one will have:

$${}^2\mathfrak{T}\,{}^2\mathfrak{S} = {}^2\mathfrak{P}\,{}^2\mathfrak{M} - 2\,\mathfrak{Q}\,\mathfrak{N}\,, \tag{38.a}$$

and

$${}^2\mathfrak{T}\,{}^2\tilde{\mathfrak{S}} = {}^2\mathfrak{P}\,{}^2\mathfrak{M} + \mathfrak{Q}\,\mathfrak{N}\,.$$

One cannot get invariant scalars, but *divariant* ones, from the components of two tensors by forming the product and *simply* summing (relative to *one* pair of indices), i.e., the components of new tensors of the same (i.e., second) rank. One must then distinguish between the following four types of products:

$$\sum_l T_{il}\, S_{lk}\,, \qquad \sum_l T_{il}\, S_{kl}\,, \qquad \sum_l T_{li}\, S_{kl}\,, \qquad \sum_l T_{li}\, S_{lk}\,,$$

which we will refer to as the (i, k)-components of the *tensor products:*

$${}^2\mathfrak{T}\times{}^2\mathfrak{S}\,, \qquad {}^2\mathfrak{T}\times{}^2\tilde{\mathfrak{S}}\,, \qquad {}^2\tilde{\mathfrak{T}}\times{}^2\tilde{\mathfrak{S}}, \qquad {}^2\tilde{\mathfrak{T}}\times{}^2\mathfrak{S}\,,$$

resp. We will then have:

$$({}^2\mathfrak{T}\times{}^2\mathfrak{S})_{ik} = \sum_l T_{il}\, S_{lk}\,. \tag{39}$$

It will follow from that definition ([12]) that tensor multiplication is not commutative, in general. That operation will be commutative only when the two factors are symmetric.

Upon decomposing the latter tensors into their symmetric and skew-symmetric parts, from (39), that will give:

([12]) Which corresponds to the usual definition of the products of determinants and matrices.

$$^2\mathfrak{T} \times {}^2\mathfrak{S} = {}^2\mathfrak{P} \times {}^2\mathfrak{M} + {}^2\mathfrak{P} \times {}^2\mathfrak{N} + {}^2\mathfrak{Q} \times {}^2\mathfrak{M} + {}^2\mathfrak{Q} \times {}^2\mathfrak{N} .$$

Therefore, $^2\mathfrak{P} \times {}^2\mathfrak{M} = {}^2\mathfrak{M} \times {}^2\mathfrak{P}$. As far as the other three summands are concerned, they cannot be immediately reduced to the corresponding expressions in terms of the vectors $\mathfrak{Q}$ and $\mathfrak{N}$. One initially gets the components of $^2\mathfrak{Q} \times {}^2\mathfrak{N}$ in the form of:

$$(^2\mathfrak{Q} \times {}^2\mathfrak{N})_{11} = Q_{11}\, N_{11} + Q_{12}\, N_{21} + Q_{13}\, N_{31} = -\, Q_3\, N_3 - Q_2\, N_2 = -\, (\mathfrak{Q}\mathfrak{N}) + Q_1\, N_1 .$$

$$(^2\mathfrak{Q} \times {}^2\mathfrak{N})_{12} = Q_{11}\, N_{12} + Q_{12}\, N_{22} + Q_{13}\, N_{32} = -\, Q_2\, N_1 ,$$

i.e.:

$$(^2\mathfrak{Q} \times {}^2\mathfrak{N})_{ik} = \begin{cases} Q_k N_i - (\mathfrak{Q}\mathfrak{N}) & \text{for} \quad k = i , \\ -\, Q_k N_i & \text{for} \quad k \neq i . \end{cases} \tag{39.a}$$

The tensor $^2\mathfrak{Q} \times {}^2\mathfrak{N}$ is then composed from the two vectors $\mathfrak{Q}$ and $\mathfrak{N}$ by multiplying their components ([13]).

Moreover, one will have:

$$(^2\mathfrak{P}\ {}^2\mathfrak{N})_{11} = P_{11}\, N_{11} + P_{12}\, N_{21} + P_{13}\, N_{31} = -\, P_{12}\, N_3 + P_{12}\, N_2 ,$$
$$(^2\mathfrak{P}\ {}^2\mathfrak{N})_{12} = P_{11}\, N_{12} + P_{12}\, N_{22} + P_{13}\, N_{32} = \quad P_{11}\, N_3 - P_{13}\, N_1 .$$

We would like to call those expressions the components of the *outer* product of the vector $\mathfrak{N}$ and the tensor $^2\mathfrak{P}$ and denote them by:

$$(^2\mathfrak{P} \times {}^2\mathfrak{N})_{ik} = (\mathfrak{N} \times {}^2\mathfrak{P})_{ik} = -\, (\mathfrak{P}_i \times \mathfrak{N})_k . \tag{39.b}$$

Therefore, $\mathfrak{P}_i$ means a vector with the components P_{i1}, P_{i2}, P_{i3} .

To conclude this section, let us point out that the multiplication of tensors with vectors or other tensors, as it was defined above, obeys the *distributive law*, just like the inner and outer multiplication of vectors. We will then have:

$$^2\mathfrak{T}\ (\mathfrak{A} + \mathfrak{B}) = {}^2\mathfrak{T}\ \mathfrak{A} + {}^2\mathfrak{T}\ \mathfrak{B} ,$$
$$(^2\mathfrak{T} + {}^2\mathfrak{S})\ \mathfrak{A} = {}^2\mathfrak{S}\ \mathfrak{A} + {}^2\mathfrak{T}\ \mathfrak{A} ,$$

etc.

([13]) If one replaces the radius vector of a point by the corresponding skew-symmetric tensor $^2\mathfrak{r}$ whose components are $r_{23} = -\, r_{32} = x_1$, etc., then the sum over all particles of a rigid body $^2\mathfrak{M} = -\, \sum m\, (^2\mathfrak{r} \times {}^2\mathfrak{r})$ (m = the mass of a particle) will determine the *moment of inertia* of that body relative to an arbitrary axis, and indeed, the moment of inertia about an axis in the direction of the unit vector $\mathfrak{n}$ will be equal to the inner product $^2\mathfrak{M}\ \mathfrak{n}$.

§ 21.

From (37.a), the inner multiplication of the vector (${}^2\mathfrak{T}\ \mathfrak{A}$) with another vector $\mathfrak{B}$ will give an invariant scalar quantity that can be written as follows in the general case of an asymmetric tensor ${}^2\mathfrak{T} = {}^2\mathfrak{P} + {}^2\mathfrak{Q}$:

$$({}^2\mathfrak{T}\ \mathfrak{A})\ \mathfrak{B} = ({}^2\mathfrak{P}\ \mathfrak{A})\ \mathfrak{B} + \mathfrak{Q}\ (\mathfrak{A} \times \mathfrak{B})\ .$$

In what follows, we will always set $\mathfrak{Q} = 0$, i.e., we will consider only symmetric tensors. The product of ${}^2\mathfrak{T}$, $\mathfrak{A}$, and $\mathfrak{B}$ will then be independent of the sequence of factors, which is why we can denote it by simply ${}^2\mathfrak{T}\ \mathfrak{A}\ \mathfrak{B}$.

In particular, if we set $\mathfrak{A} = \mathfrak{B} = \mathfrak{r}$ then we will have:

$${}^2\mathfrak{T}\ \mathfrak{r}\ \mathfrak{r} = T_{11}\,x_1^2 + T_{22}\,x_2^2 + T_{33}\,x_3^2 + 2T_{23}\,x_2\,x_3 + 2T_{31}\,x_3\,x_1 + 2T_{12}\,x_1\,x_2\,. \tag{40}$$

We will then get a quadratic form in the coordinates x_1, x_2, x_3, in which the components of the tensor ${}^2\mathfrak{T}$ play the role of coefficients.

If one considers that tensor to be a constant quantity and the vector $\mathfrak{r}$ (i.e., the coordinates x_1, x_2, x_3) then the equation:

$${}^2\mathfrak{T}\ \mathfrak{r}\ \mathfrak{r} = \text{const.} \tag{40.a}$$

will determine a second-order surface, and indeed one that is entirely independent of the orientation of the coordinate system. That is why one can regard that surface (ellipsoid, hyperboloid) as the *coordinate-free geometric representation of the tensor in question*.

One can also introduce a corresponding representation for *vectors* since they can be considered to be first-rank tensors. Namely, instead of representing a vector $\mathfrak{A}$ by a *segment* (i.e., by a certain value of the radius vector $\mathfrak{r}$) that is proportional to it and points in the same direction, one can just as well use the *plane* that is perpendicular to it, i.e., the first-order surface that is defined by the equation:

$$\mathfrak{A}\ \mathfrak{r} \equiv A_1\,x_1 + A_2\,x_2 + A_3\,x_3 = \text{const.}$$

If one sets the constant above equal to 1 then the distance from the plane to the coordinate origin will be equal to $\dfrac{1}{\sqrt{A_1^2 + A_2^2 + A_3^2}} = \dfrac{1}{A}$, i.e., *inversely* proportional to the magnitude of the "first-rank tensor" that represents that plane. One will get a similar result for the geometric representation of the (symmetric) second-rank tensor according to (40.a). The "magnitude" of the tensor can then be defined by $|\,{}^2\mathfrak{T}\,| = \sqrt{{}^2\mathfrak{T}\ {}^2\mathfrak{T}}$.

As is known, one can transform the surface (40.a) to its "principal" or "symmetry axes" by a suitable rotation of the coordinate system, i.e., one can put its coordinate equation into the form:

$$ {}^2\mathfrak{T}\ \mathfrak{r}\,\mathfrak{r} \equiv \mathfrak{T}'_{11}\,x_1'^2 + \mathfrak{T}'_{22}\,x_2'^2 + \mathfrak{T}'_{33}\,x_3'^2 = \text{const.} = 1\ . \tag{40.b} $$

The corresponding coordinate axes are called the *principal axes* of the tensor that determines that surface (or is represented by it). They are then characterized by the vanishing of the components $\mathfrak{T}'_{ik}$ when $i \neq k$.

Note that such a "principal axis transformation" is impossible for asymmetric tensor, just like the geometric illustration of it that was described above.

§ 22.

The principal axes of a tensor can be defined by saying that the vector ${}^2\mathfrak{T}\mathfrak{r}$, which generally has a different direction from $\mathfrak{r}$, has the same direction as $\mathfrak{r}$ for the principal axis directions.

In fact, the vector ${}^2\mathfrak{T}\mathfrak{r}$ is equal to the gradient of the scalar quantity $\frac{1}{2}{}^2\mathfrak{T}\mathfrak{r}\mathfrak{r}$, from which it will follow that it has the direction of the *outer normal* to the surface (40.a) at the point in question that is determined by $\mathfrak{r}$.

If one introduces an undetermined scalar λ then the principal axes of the tensor ${}^2\mathfrak{T}$ will be determined by the vector equation:

$$ {}^2\mathfrak{T}\ \mathfrak{r} = \lambda\,\mathfrak{r}, \tag{41} $$

or the corresponding scalar equations:

$$ \left.\begin{aligned} (T_{11}-\lambda)\,x_1 &+ T_{12}\,x_2 &+ T_{13}\,x_3 &= 0,\\ T_{21}\,x_1 &+ (T_{22}-\lambda)\,x_2 &+ T_{23}\,x_3 &= 0,\\ T_{31}\,x_1 &+ T_{32}\,x_2 &+ (T_{33}-\lambda)\,x_3 &= 0. \end{aligned}\right\} \tag{41.a} $$

That will give the cubic equation for the scalar λ :

$$ \begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13}\\ T_{21} & T_{22}-\lambda & T_{23}\\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0 \tag{41.b} $$

whose roots are equal to precisely the principal components T'_{11}, T'_{22}, T'_{33} of $\mathfrak{T}$ ([14]). If we develop that equation in powers of λ then it will assume the following form:

$$\lambda^3 - T^{(1)}\lambda^2 - T^{(2)}\lambda - T^{(3)} = 0\,, \tag{41.c}$$

with:

$$T^{(1)} = T_{11} + T_{22} + T_{33}\,, \tag{42}$$

$$T^{(2)} = T_{23}^2 + T_{31}^2 + T_{12}^2 - T_{22}T_{33} - T_{33}T_{11} - T_{11}T_{22}\,, \tag{42.a}$$

and

$$T^{(3)} = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix}\,. \tag{42.b}$$

Since the roots of (41.c) are determined completely by quantities that are independent of the choice of coordinates, they must also be true for the coefficients $T^{(1)}$, $T^{(2)}$, $T^{(3)}$. That is why they are called *invariants*, and indeed, linear, quadratic, and cubic, resp., of the tensor ${}^2\mathfrak{T}$. The invariance of $T^{(1)}$ will follow immediately from the fact that for $T_{ik} = A_i B_k$, one will have $T^{(1)} = \mathfrak{A}\mathfrak{B}$. Similarly, one can recognize the invariance of $T^{(2)}$ since:

$$2T^{(2)} = {}^2\mathfrak{T}\,{}^2\mathfrak{T} - \left(T^{(1)}\right)^2\,.$$

By contrast, directly establishing the invariant character of (42.b) would require a special examination.

§ 23.

Along with constant tensors, one can also consider *variable* tensors that are functions of time or position, i.e., the radius vector (tensor functions, tensor fields). One can define the differential quantity that corresponds to the divergence of a vector by the following Ansatz, which is analogous to formula (11.a):

$$\operatorname{div} {}^2\mathfrak{T} = \lim_{S\to 0}\frac{1}{V}\oint {}^2\mathfrak{T}\,\mathfrak{n}\,dS\,. \tag{43}$$

([14]) We would like to skip the proof of this here since the question of the principal axis transformation of second-degree surfaces is discussed in textbooks on analytic geometry. We shall only remark that the most general condition for the reality of all three roots to (41.b) is the so-called *Hermitian* condition: T_{ki} = complex conjugate of T_{ik} . That means that when the tensor ${}^2\mathfrak{T}$ is complex, and indeed equal to ${}^2\mathfrak{A} + \sqrt{-1}\;{}^2\mathfrak{B}$, its real part ${}^2\mathfrak{A}$ will be symmetric, while the imaginary part ${}^2\mathfrak{B}$ must be skew-symmetric.

Since the product ${}^2\mathfrak{T}\mathfrak{n}$ represents a vector ([15]), the divergence of a tensor must also be a vector quantity. By a suitable specialization of the surface S (viz., a parallelepiped whose edges are parallel to the coordinate axes), (43) will imply the following coordinate-wise definition of that quantity [cf., (21)]:

$$(\text{div } {}^2\mathfrak{T})_i = \sum_k \frac{\partial T_{ik}}{\partial x_k}. \tag{43.a}$$

One can also define it symbolically as the product of the tensor ${}^2\mathfrak{T}$ with the *Hamiltonian* operator ∇. Note that, from (37.a), the divergence of an asymmetric tensor can be decomposed into the divergence of its symmetric part ${}^2\mathfrak{P}$ and the rotation of the vector $\mathfrak{Q}$ that corresponds to its skew-symmetric part:

$$\text{div } {}^2\mathfrak{T} = \text{div } {}^2\mathfrak{P} + \text{div } \mathfrak{Q} . \tag{43.b}$$

It follows from (43) that there is a transformation formula that corresponds to *Gauss*'s theorem (16.a):

$$\oint \mathfrak{n}\, {}^2\mathfrak{T}\, dS = \int \text{div } {}^2\mathfrak{T}\, dV . \tag{44}$$

One can also define two symbolic tensors by means of the symbolic vector ∇, and indeed a symmetric one with the components $\nabla_i \nabla_k = \dfrac{\partial^2}{\partial x_i \partial x_k}$, and a skew-symmetric one ${}^2\nabla$ with the component matrix:

$${}^2\nabla = \begin{pmatrix} 0 & \dfrac{\partial}{\partial x_3} & -\dfrac{\partial}{\partial x_2} \\ -\dfrac{\partial}{\partial x_3} & 0 & \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} & -\dfrac{\partial}{\partial x_1} & 0 \end{pmatrix} .$$

The first of those "tensors" is a second-order differential operator, while the second one is a first-order operator that is completely equivalent to ∇, which is only a different notation for the operations that are determined by ∇. Upon applying that operator to a vector function, according to the general multiplication formula (37.a) (in which ${}^2\mathfrak{P} = 0$ and $\mathfrak{Q} = \nabla$), we will get:

$${}^2\nabla \cdot \mathfrak{F} = \nabla \times \mathfrak{F} = \text{rot } \mathfrak{F} . \tag{45}$$

([15]) Indeed, a vector whose direction coincides will the direction of the outer normal of the surface that represents the tensor ${}^2\mathfrak{T}$, so $\mathfrak{n}$ will play the role of the radius vector.

Similarly, the "scalar product" of ${}^2\nabla$ with a tensor function will be:

$$ {}^2\nabla \cdot {}^2\mathfrak{T} = 2\,\nabla\mathfrak{Q} = 2\,\mathrm{div}\,\mathfrak{Q}\,, \tag{45.a} $$

in which:

$$ \mathrm{div}\,\mathfrak{Q} = \frac{\partial Q_{23}}{\partial x_1} + \frac{\partial Q_{31}}{\partial x_2} + \frac{\partial Q_{12}}{\partial x_3}\,. $$

The right-hand side of that equation is often called the "rotation" of the skew-symmetric tensor ${}^2\mathfrak{Q}$ and is denoted by rot ${}^2\mathfrak{Q}$ (rot ${}^2\mathfrak{Q}$ = div $\mathfrak{Q}$).

An operation that would correspond to the rotation of a vector is not generally applied to tensor functions.

When applying the operator ∇ to products of tensors with vectors or other tensors, one must consider rules of differentiation that are similar to the ones for the corresponding operations of vector calculus. We would not like to go into the details of that question here.

§ 24.

In conclusion, we must add a few words about tensors of higher rank.

We have already given the general definition of a tensor of rank n, ${}^n\mathfrak{T}$, at the end of **§ 18**. We can construct such tensors by multiplying or differentiating the components of tensors of lower rank (including vectors) (viz., "extension"). Conversely, we can also lower the rank of a tensor, and indeed by summing over the components with two equal indices (viz., "contraction"). A third-rank tensor ${}^3\mathfrak{T}$ with the components T_{ikl} will imply, e.g., in that way, three vectors with the components $\sum_{k=1}^{3} T_{ikk}$, $\sum_{k=1}^{3} T_{kik}$, $\sum_{k=1}^{3} T_{kki}$ (i = 1, 2, 3). The invariant scalar $T^{(1)} = \sum_{i=1}^{3} T_{ii}$ should be mentioned as an example of such a contraction that we know of already.

In contrast to ordinary tensors, the asymmetric tensors of higher rank *cannot* be decomposed into completely symmetric and completely skew-symmetric parts. One must halt that decomposition when one gets to summands that are symmetric in some pairs of indices and skew-symmetric with respect to all other ones.

A completely-symmetric tensor of rank n, ${}^n\mathfrak{T}$, can be illustrated or "represented" by an algebraic surface of order n :

$$ {}^n\mathfrak{T}\mathfrak{r}\mathfrak{r}\cdots\mathfrak{r} = \sum_{i_1\cdots i_n} T_{i_1 i_2 \cdots i_n}\, x_{i_1}\, x_{i_2} \cdots x_{i_n} = \mathrm{const.} $$

Therefore, the product ${}^n\mathfrak{T}\ \mathfrak{F}$ will mean a tensor of rank (n – 1) with the components:

$$\sum_{i_1=1}^{3} T_{i_1 i_2 i_3 \cdots i_n}\, \mathfrak{F}_{i_1}\,.$$

The tensors:

$$R_{i_1 i_2 \cdots i_n} = x_{i_1}\, x_{i_2} \cdots x_{i_n} \tag{46}$$

and

$$S_{i_1 i_2 \cdots i_n} = \frac{\partial^n \varphi}{\partial x_{i_1}\, \partial x_{i_2} \cdots \partial x_{i_n}} \tag{46.a}$$

might serve as the simplest examples of *symmetric* tensors of rank n (φ = any scalar function of $\mathfrak{r}$).

One can construct more complicated symmetric tensors from them by multiplying by invariants scalars and adding. (Obviously, only tensors of the same rank can be added together.)

Note that the n-variant scalar (46) can be written in the form:

$$R\,(n_1, n_2, n_3) = x_1^{n_1}\, x_2^{n_2}\, x_3^{n_3} \qquad (n_1 + n_2 + n_3 = n), \tag{46.b}$$

and the number of components of ${}^{n}\mathfrak{R}$ that are all equal to $R\,(n_1, n_2, n_3)$ is equal to:

$$\frac{n!}{n_1!\, n_2!\, n_3!}\,.$$

Corresponding statements are true for the tensor (46.a) when $R\,(n_1, n_2, n_3)$ is replaced with:

$$S\,(n_1, n_2, n_3) = \frac{\partial^n \varphi}{\partial x_1^{n_1}\, \partial x_2^{n_2}\, \partial x_3^{n_3}}\,.$$

PART ONE

TIME-INDEPENDENT ELECTROMAGNETIC EFFECTS

CHAPTER ONE

ELECTROSTATIC EFFECTS AND THE ENERGY PRINCIPLE.

§ 1. – Electric dipole.

In the presentation of the foundations of electrostatics, one ordinarily begins with the consideration of *electrified bodies*, which are contrasted from the ordinary *neutral* bodies that one regards as completely "free of electricity." That juxtaposition is known to be unjustified since any neutral body actually *contains hidden electric charges of two opposite types in equivalent – i.e., mutually-neutralizing – amounts*. Under ordinary circumstances, those charges are distributed uniformly over the volume of the body such that not only the body as a whole, but also every volume element in it *that is not too small*, will appear to be "neutral." However, that uniform distribution can be perturbed by certain influences, and indeed in such a way that an excess of electric charges of the one type will appear in one part of the body, while an equivalent excess of charges of the opposite type will appear in the other. Such a *spatial separation* of opposite electric charges in a neutral body expresses itself in the appearance of *forces of attraction* between the body and other neutral bodies in its vicinity, in such a way that a similar spatial separation of the hidden electric charges would be "induced" in the latter.

A body or sub-body that contains electric charges of one or the other type in excessive amounts is called *electrified*. Since all bodies are neutral in the normal state, the electrification of one of them must always be coupled with the opposite electrification of other bodies that were either in contact with it to begin with or defined a unified body with it. Conversely: The "neutralization" of the body in question, i.e., the vanishing of the excess electric charges of a certain type that were included in it, can take place only simultaneously with the neutralization of one or more other opposite electrified bodies.

It follows from this that the appearance or disappearance of *electric forces* (i.e., forces of electric origin) would always require the spatial separation or unification, resp. (convergence) of electric charges of opposite type, but not the "creation" or "annihilation" of those charges.

One can then assert that electricity does not represent a random and variable attribute of material bodies, but consists of elements – viz., "elementary charges" – that are just as indestructible and invariable as the matter itself. As a result of their permanent coupling with material bodies, those elementary charges must be considered to be an invariable and inseparable *property* of the elementary particles that the material body is constructed from, in the same way that the *masses* of those particles can be regarded as such a property. From that standpoint, it is necessary from the outset to represent the idea that the smallest *neutral* material particles – viz., atoms – are composite structures that consists of even smaller *electrified* particles. One calls those *smallest* of all material particles that are characterized by not only their mass, but also their electric charge, *electrons*. Obviously, we must assume that there are two types of electrons in any atom that have opposite charges of equivalent magnitudes.

Without going into the deeper foundations and further development of the theory of electrons, we would like to make the picture that was sketched out above the foundation upon which we will build the general principles of electrodynamics.

In that way, we will not start with "isolated" electric charges, but with the simplest *neutral* systems, which consist of two opposite charges (i.e., two oppositely-charge particles). Such systems are called *electric dipoles*. We will initially consider the forces that electric dipoles exert upon each other. In that way, one can treat an "isolated" charge as one end of an electric dipole, the other (opposite) end of which is found at a very large (i.e., practically infinite) distance.

The main advantage that way of looking at things has over the usual one (in which it is not dipoles, but individual charges, that are treated as the sources or points of application, resp., of electric forces) is not only the fact that electricity is always composed of equal and opposite charges, i.e., dipoles, but also the methodological advantage that will come to light in the following discussions ([16]).

§ 2. – The moment of an electric dipole.

For the sake of visualization, we will imagine an electric dipole as a small, fixed rod whose endpoints are endowed with opposite charges of equivalent magnitudes. The equivalence of the charges, i.e., the neutrality of the system that they define, will then be defined by the fact that when the length of the dipole vanishes, the forces and torques that it exerts upon other dipoles, just like the forces and torques that it experiences as a result of the other dipoles, should also vanish.

When the length of the dipole is very small compared to its distance from the other dipoles, it will be called *elementary*. [The mechanics (viz., the "active" and "passive" effects) of an elementary dipole must obviously remain unchanged in practice when one displaces it parallel to itself, i.e., without changing its orientation, in a spatial region whose linear dimension has the same order of magnitude as its length.] It can be easily proved from that principle that:

1. The mechanical effect of such an *elementary* dipole is *proportional to its length*, and

2. The proportionality factor, which will serve as a measure of the magnitude of the corresponding charges, is an *additive* quantity.

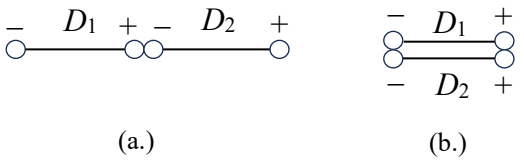
Let D_1 and D_2 be two *identical* and equally-oriented elementary dipoles that are found in a very small spatial region V. We ignore the effect of those dipoles on each other and focus our attention upon only the interaction of each of them with the dipoles that are found externally to them (i.e., at a great distance from V) then those interactions must be approximately equal, and independently of the positions of the two dipoles inside of V. That means that the mechanical effect of the system that is defined by D_1 and D_2 will be *twice* as large as the effect of both parts when taken individually in any relative configuration.

In particular, we consider the following two configurations:

([16]) That advantage is derived from the fact that the mechanical effects of the dipole can be determined from the *vector quantities* that characterize that dipole, while isolated charges are characterized by scalar quantities.

1. The *opposite* ends of D_1 and D_2 coincide with each other (Fig. 5.a).

2. The *same* ends – i.e., the two dipoles – coincide with each other (Fig. 5.b).



(a.) (b.)

Figure 5.

In the first case, the opposite charges that lie next to each other define a dipole of vanishing length. We then get a single dipole D with the same charges as D_1 and D_2, but *twice the length*. With that, it is also proved that the mechanical effects of different dipoles are proportional to their lengths ([1]). If we introduce the proportionality factor as a measure for the magnitude of the corresponding charge then a consideration of the second case will show that this quantity is *additive*, i.e., that the effect of two charges that are found at the same point is equal to the effect of *one* charge whose quantity is equal to the sum of the quantities of the two individual charges.

That additivity, which was established here for only charges of the first type, can be adapted to opposite charges when one characterizes them by equivalent magnitudes with opposite signs (– or +). The assignment of positive signs on charges of one type or the other remains entirely arbitrary.

If that assignment has been established and the unit of charge has been chosen then one can state the following theorem: *The mechanical effect of an elementary dipole is determined by the product:*

$$p = e\, l\,, \tag{1}$$

in which l means its length and e (> 0) means the absolute value of its charges.

One can consider the length of the dipole l to be the magnitude of a vector $\mathfrak{l}$, namely, the radius vector of its positive "pole" (i.e., the positive charge) relative to the negative pole. Since the charge e is obviously a scalar quantity, that notion corresponds to the fact that we can characterize an elementary dipole by the vector:

$$\mathfrak{p} = e\, \mathfrak{l}\,. \tag{1.a}$$

That vector, whose magnitude is given by (1), is called the *moment*, or the *electric moment*, of the dipole. In conjunction with some other quantities that we will consider later on, it determines not only the magnitude, but also the *direction*, of the forces that act upon an elementary dipole and are exerted by it.

§ 3. – Systems of elementary dipoles.

Let D_1 and D_2 be two elementary dipoles with moments $\mathfrak{p}_1$ and $\mathfrak{p}_2$ that are *different* (in magnitude, as well as direction) and found in the same small spatial region V. Since the mechanical

([1]) Strictly speaking, that theorem can be proved for only the special case considered. However, its generalization to arbitrary length ratios is quite obvious.

effect of any of them when the orientation is fixed depends upon only the moment $\mathfrak{p}_i = e_i \mathfrak{l}_i$ ($i = 1, 2$), but not the charge e_i, one can set $e_i = 1$ (i.e., $e_1 = e_2 = 1$), and in that way, the difference between the dipoles is reduced to merely the difference between their lengths. We now move them together in such a way that the positive end of the one coincides with the negative end of the other (Fig. 6). The external effects of the corresponding charges must cancel reciprocally then, and the system in question will be equivalent to an elementary dipole D with a length $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$, i.e., with a moment $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2$.

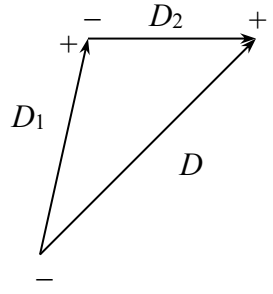


Figure 6.

That result can easily be generalized to an arbitrary number of elementary dipoles. If they are found in a sufficiently-small spatial region ([1]) then *that system will be equivalent in regard to its mechanical interaction with external* (i.e., very distant) *dipoles to a single dipole whose moment is equal to the geometric sum of its electric moments.*

Conversely, one can replace a given elementary dipole with a moment of $\mathfrak{p}$ will a system of arbitrarily-many elementary dipoles (coupled to each other rigidly) whose moments $\mathfrak{p}_1$, $\mathfrak{p}_2$, … satisfy the condition:

$$\mathfrak{p}_1 + \mathfrak{p}_2 + \ldots = \mathfrak{p} \ .$$

Obviously, the theorems above still remain valid for *non-elementary* dipoles (i.e., ones whose lengths are not small compared to their distances from other dipoles that they interact with). However, in that way, the mechanical effects of such dipoles are still not determined by giving their moments. Nonetheless, one can decompose a non-elementary dipole into a *chain of elementary ones* that are characterized completely by their moments and *positions*. Namely, if one couples the endpoints A and B of the dipole in question with the moment $e \cdot AB$ by a broken line with infinitely-small rectilinear elements $A P_1$, P_1P_2, …, $P_n B$, and imagines that the charges $+ e$ and $- e$ are concentrated at the points P_1, P_2, … then one will get a chain of elementary dipoles with the moments $e\, AP_1$, $e\, P_1 P_2$, etc., (Fig. 7) whose geometric sum is equal to $e\, AB$. However, in so doing, one must observe that in this case, the individual elementary dipoles *cannot* be displaced independently of each other, at least not in a region whose linear dimensions are comparable to the length AB.

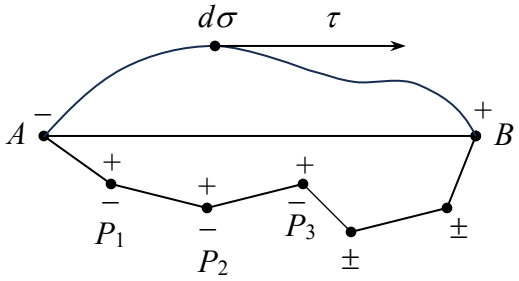


Figure 7.

The form of the broken line $A\, P_1\, P_2$, …, B remains completely arbitrary. In particular, one can pass to the limiting case of a *curve* with continuous curvature. Each element $d\sigma$ of that curve will then be an elementary dipole that is associated with a moment $d\, \mathfrak{p} = e\ \tau\, d\sigma$, in which τ means a unit vector in the direction of $d\sigma$ ("tangential vector").

([1]) With the same linear dimension as its length.

The problem of determining the interaction of non-elementary dipoles will be reduced to the corresponding problem for elementary dipoles by means of that decomposition. In that way, one can regard each elementary dipole as simply *a point* since it is not its length that should come under consideration, but only its electric moment. When the moment is fixed, one can always make the length arbitrarily small by increasing the charge. The kinematical description of an elementary dipole will then reduce to the data of its "position," i.e., the radius vector $\mathfrak{r}$ of the point that represents it and its "orientation," i.e., the direction of its electric moment (whose magnitude we shall regard as constant).

§ 4. – The statics of an elementary dipole. Electric field strength.

We shall now assume that all of the dipoles that act upon the elementary dipole in question are fixed, while the latter can displace and rotate arbitrarily. We now ask how the external force $\mathfrak{F}$ and moment (i.e., torque) $\mathfrak{M}$ depends upon its position ($\mathfrak{r}$) and orientation ($\mathfrak{p}$).

In order to answer that question, we would like to assume that *those force effects have a conservative character*, i.e., it can be derived a still-undetermined *energy function* U ($\mathfrak{r}$, $\mathfrak{p}$). We will refer to that assumption, which has fundamental significance for what follows, as the *energy principle* ([1]).

The position of the dipole in question D will initially be fixed. However, it might rotate arbitrarily around the point in question P. It will then follow from the energy principle that will be oriented in a certain direction that corresponds to a minimum of energy as a function of $\mathfrak{p}$ (for $\mathfrak{r}$ = const.). It is possible from the outset that there are *several* such stable equilibrium orientations ([2]).

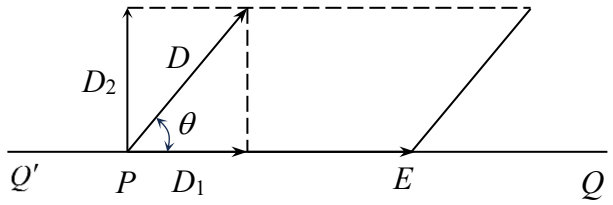


Figure 8.

Now let PQ be one of those directions. If the electric moment $\mathfrak{p}$ of the dipole falls in the direction PQ then the moment that acts upon it will be equal to zero. We now assume that the dipole rotates out of that direction through an angle θ (Fig. 8) and replace it with two dipoles D_1 and D_2 with moments $p_1 = p \cos\theta$ and $p_2 = p \sin\theta$ that are parallel (perpendicular, resp.) to PQ. Since D_1 experiences no torque, from what said above, the torque $\mathfrak{M}$ that is exerted on D must coincide with the torque $\mathfrak{M}_2$ that acts on D_2.

([1]) The energy principle means that the work that is done under a displacement or rotation of the dipole by the forces and torques that act upon it depend upon only its *initial* and *final* positions (orientations, resp.), but not upon the intermediate positions and orientations.

([2]) If *no* equilibrium orientation exists then the work done in returning the dipole to the original orientation would generally be non-zero.

Obviously, for sufficiently-small values of θ, D must have the ambition to point in the direction PQ, i.e., the moment $\mathfrak{M}$ must point perpendicular to the plane (D, PQ), and completely independent of whether there are or are not other equilibrium directions. However, one can say the same thing of the moment that acts upon D_2 due to the equality $\mathfrak{M}_2 = \mathfrak{M}$. However, since the angle between D_2 and the line $Q'PQ$ is a maximum, we can conclude immediately that *PQ is the only stable equilibrium direction*, i.e., the only direction that corresponds to the condition U ($\mathfrak{p}$) = minimum.

Obviously, the moment $\mathfrak{M}_2$ must be proportional to the electric moment of D_2, i.e., the quantity $p \sin \theta$. If we denote the proportionality factor (which is independent of p and θ) by E then since $\mathfrak{M}_2 = \mathfrak{M}$, we will have:

$$M = E\, p \sin \theta. \tag{2}$$

The magnitude of the vector $\mathfrak{M}$ is then equal to the area of a parallelogram that is spanned, on the one hand, by the vector $\mathfrak{p}$ and a line segment of length E that points from P to Q, on the other.

As a result, if one considers E to be the magnitude of a vector $\mathfrak{E}$ that is represented by that line segment then one can determine the vector $\mathfrak{M}$ as the outer product of the vectors $\mathfrak{p}$ and $\mathfrak{E}$ by using the equation:

$$\mathfrak{M} = \mathfrak{p} \times \mathfrak{E}\ . \tag{3}$$

The vector function $\mathfrak{M}$ ($\mathfrak{r}$, $\mathfrak{p}$) will decompose into two factors as a result of that formula. One of them depends upon *only* $\mathfrak{p}$ (and indeed linearly), while the other depends upon *only* $\mathfrak{r}$ (and in a way that is still completely unknown).

Now, it is easy to obtain the corresponding decomposition of the energy function. The work that must be done *against* $\mathfrak{M}$ in order to increase the angle θ by $d\theta$ is equal to the product $M\, d\theta$. On the other hand, it is equal to the corresponding increase in energy dU. From (2), we will then get $dU = E\, p \sin \theta\, d\theta$, and as a result:

$$U = -\, E\, p \cos \theta + C\ ,$$

since $\theta' = \theta + \pi$. The potential energy $U + U'$ of the system that is defined by the two dipoles (relative to the other "external" dipoles) will then reduce to 2 C. However, since the resultant electric moment of that system is equal to zero, it can experience *no forces whatsoever*. That is why the quantity C must also be independent of $\mathfrak{r}$. As a result, we can set $C = 0$. We accordingly get the following general expression for the energy as a function of $\mathfrak{p}$ *and* r:

$$U = -\, p\, E \cos \theta = -\, \mathfrak{p}\ \mathfrak{E}\ . \tag{4}$$

If one imagines that the form of the function $\mathfrak{E}(\mathfrak{r})$ is given then one can calculate the force $\mathfrak{F}$ that acts on the dipole from (4) using the energy principle. The work done by that force under a displacement of the dipole through the line segment $d\,\mathfrak{r}$ *with its orientation fixed* is, in fact, determined from the equation:

$$\mathfrak{F}\, d\,\mathfrak{r} = -\, dU = d\,(\mathfrak{p}\ \mathfrak{E}) = d\,\mathfrak{r} \cdot \operatorname{grad}(\mathfrak{p}\ \mathfrak{E})$$

(see the Introduction, **§ 10**), i.e., due to the arbitrariness in $d\,\mathfrak{r}$:

$$\mathfrak{F} = \operatorname{grad}(\mathfrak{p}\ \mathfrak{E}) = (\mathfrak{p}\ \operatorname{grad})\ \mathfrak{E} + \mathfrak{p} \times \operatorname{rot}\mathfrak{E} \tag{5}$$

[Introduction, formula (27)].

The vector, or better yet, the vector function $\mathfrak{E}(\mathfrak{r})$, which determines the effect of all "external" dipoles on the one in question (if one ignores the presence of the latter at the point in question), is called the (external) *electric field strength*. One can evaluate the direction and magnitude of that field strength experimentally using formula (2) by means of a dipole that can be displaced and rotated freely. Indeed, the direction of $\mathfrak{E}$ coincides with the one that the dipole seeks. The corresponding magnitude E is equal to the ratio of the largest moment that is exerted on the dipole in an orientation that is perpendicular to the latter to its electric moment.

It should be noted that the force $\mathfrak{F}$ vanishes for $\mathfrak{p} \perp \mathfrak{E}$ ($\theta = 90^{\circ}$) [according to (5)]. By contrast, the moment $\mathfrak{M}$ vanishes for $\mathfrak{p} \parallel \mathfrak{E}$, while $\mathfrak{F}$ attains its maximum magnitude for the position of the dipole in question. In particular, for $\theta = 0$, $\mathfrak{F}$ falls in the direction of the fastest increase of E, for $\theta = 180^{\circ}$, it falls in the opposite direction, i.e., the direction of steepest descent. The first of those orientations (for a fixed position) corresponds to the stable equilibrium, while the second one corresponds to the labile one [U = minimum or maximum, resp., according to (4)].

§ 5. – The vortex-free character of the electric field and its effect on isolated charges (poles).

The forces that act upon a non-elementary dipole in a given electric field can be determined by decomposing that dipole into a chain of elementary ones (**§ 3**). According to the energy principle, in order to do that, we need to calculate only the potential of the dipole in question. Obviously, that potential energy U is equal to sum of the energies that correspond to the individual elementary dipoles ([1]). If they define a curve σ (Fig. 7) then $dU = -\, d\,\mathfrak{p} \cdot \mathfrak{E} = -\,(\tau\, e\, d\sigma\, \mathfrak{E}) = -\, e\, E_{\tau}\, d\sigma$, in which $\mathfrak{E}$ is the electric field strength *at the location in question*, and E_{τ} means its projection onto $d\sigma$, and as a result:

([1]) In that way, it is obvious that one must deal with their energies relative to external electrical systems, but not relative to each other.

$$U = -\, e \int_A^B E_\tau \, d\sigma \,. \tag{6}$$

The integration is performed from the negative to the positive end of the dipole in question.

The force effect that is exerted upon the dipole is obviously composed of two forces $\mathfrak{f}_A$ and $\mathfrak{f}_B$ that act on its ends. We now imagine that those ends, i.e., the corresponding *poles* (with charges $-e$ and $+e$), are displaced from the points A, B to the neighboring points A', B'. The work that is done by that will be $\mathfrak{f}_A \, \delta \mathfrak{f}_A + \mathfrak{f}_B \, \delta \mathfrak{f}_B$, where $\delta \mathfrak{f}_A$ and $\delta \mathfrak{f}_B$ mean infinitely-small displacements AA' (BB', resp.). On the other hand, from the energy principle, that work done must be equal to the reduction in the potential energy of the dipole, i.e., to the difference $-\delta U = -\,(U' - U)$. Due to the arbitrariness of the curves σ and σ', we can consider the latter to be the lengthening of the former that is defined by the line segments AA' and BB' and correspondingly set $\int_{A'}^{B'} = \int_{A'}^{A} + \int_{A}^{B} + \int_{B}^{B'} = \int_{A}^{B} + \int_{B}^{B} - \int_{A}^{A'}$ (the integrands have been omitted, for the sake of simplicity). We will then have:

$$-\delta U = e \int_B^{B'} E_{\tau'} \, d\sigma' - e \int_A^{A'} E_{\tau'} \, d\sigma' = e\,(\mathfrak{E}_B \, \delta \mathfrak{r}_B) - e\,(\mathfrak{E}_A \, \delta \mathfrak{r}_A)\,.$$

If one compares that expression with the original one:

$$-\delta U = \mathfrak{f}_B \, \delta \mathfrak{f}_B + \mathfrak{f}_A \, \delta \mathfrak{f}_A$$

then that will give:

$$\mathfrak{f}_B = e\, \mathfrak{E}_B\,, \qquad \mathfrak{f}_A = -\, e\, \mathfrak{E}_A\,.$$

As a result, the force $\mathfrak{f}$ that is exerted on an isolated pole with charge e will be equal to:

$$\mathfrak{f} = e\, \mathfrak{E}\,. \tag{7}$$

That formula corresponds to the usual definition of the electric field strength, namely, the force that acts on a unit (positive) charge at the point in question.

The energy of the dipole (AB) is a well-defined quantity that must remain independent of its decomposition into elementary dipoles, i.e., the form of the integration curve (6). If one connects two such curves σ_1 and σ_2 into a *closed curve* σ and integrates along σ_1 in the positive direction (from A to B) and along σ_2 in the negative one (from B to A) then that must always give zero. We then reach the conclusion that the circulation of the vector $\mathfrak{E}$ along any closed curve vanishes, i.e.:

$$\oint E_\tau \, d\sigma = 0\,.$$

It will then follow from *Stokes's* formula [(17), Introduction] that the electric field strength must satisfy the condition:

$$\text{rot } \mathfrak{E} = 0 \tag{8}$$

in all of space. That condition, which can be regarded as a direct consequence of the energy principle, expresses the basic property of the electric field, namely, it is *vortex-free*. Since (8) is fulfilled identically in $\mathfrak{r}$, according to [(18), Introduction], one can set:

$$\mathfrak{E} = -\text{ grad } \varphi\,, \tag{9}$$

in which φ means a still-undetermined function that is called the *scalar* or *electric potential*. One correspondingly refers to the field of vectors $\mathfrak{E}$ as a *vortex-free*, or *potential*, *field*.

The physical meaning of the potential φ is clarified by formula (6). Namely, from (9), one has:

$$E_\tau = -\text{ grad}_\tau \varphi = -\frac{d\varphi}{d\sigma}\,,$$

and as a result:

$$U = e\,(\varphi_B - \varphi_A) = e_B\,\varphi_B + e_A\,\varphi_A\,.$$

The potential energy of an isolated pole is therefore equal to:

$$U = e\,\varphi\,. \tag{10}$$

The electric potential φ at any point P is then equal to the potential energy of a unit (positive) charge that is found at that point relative to all charges that contribute to that potential.

Since the force $\mathfrak{f}$ that acts upon the pole in question must be equal to the negative gradient of its potential energy, we further get $\mathfrak{f} = -\text{ grad } U = -\,e\text{ grad }\varphi$, or from (9), $\mathfrak{f} = e\,\mathfrak{E}$, i.e., formula (7).

§ 6. – Reducing the effects of dipoles to those of isolated poles.

In the usual presentation of electrostatics, which starts from a consideration of the interaction of isolated poles, formula (7) is cited as a definition from the outset. The energy principle [in the form (9)] is derived from the assumption that the force between two electric poles has the direction of the line that connects them, i.e., it is a "central force." The force and torque that are exerted on a dipole can then be calculated from the two forces that act upon its individual poles.

In the case of an *elementary* dipole with an infinitely-small length $P_1\,P_2 = 1$, in the calculation of the moment $\mathfrak{M}$, one can consider the forces $\mathfrak{f}_1 = e_1\,\mathfrak{E}_1 = -\,e\,\mathfrak{E}_1$ and $\mathfrak{f}_2 = e_2\,\mathfrak{E}_2 = +\,e\,\mathfrak{E}_2$ to be equal in magnitude, but opposite in direction, and as a result set $\mathfrak{M} = \mathfrak{l} \times e\,\mathfrak{E}$, where $\mathfrak{E}$ means the electric field strength at any point P of the dipole (Fig. 9). In that way, one will come to formula

(3). If one considers the difference between the forces $\mathfrak{f}_2$ and $\mathfrak{f}_1$ then that will give the force that acts on the dipole as their geometric sum:

$$\mathfrak{F} = \mathfrak{f}_1 + \mathfrak{f}_2 = e\,(\mathfrak{E}_2 - \mathfrak{E}_1)\,.$$

Due to the smallness of l, one can then set $\mathfrak{E}_2 - \mathfrak{E}_1 = (\mathfrak{l}\ \mathrm{grad})\ \mathfrak{E}$ (Introduction, **§ 10**), and as a result:

$$\mathfrak{F} = (\mathfrak{p}\ \mathrm{grad})\ \mathfrak{E}\,. \tag{11}$$

That formula will coincide with (5) when one observes the condition that rot $\mathfrak{E} = 0$, which follows from the energy principle.

As far as the potential energy of an elementary dipole is concerned, it will reduce to the sum of the corresponding energies of its two ends, i.e.:

$$\begin{aligned} U &= e_1\,\varphi_1 + e_2\,\varphi_2 = e\,(\varphi_2 - \varphi_1) \\ &= e\,(\mathfrak{l},\ \mathrm{grad}\ \varphi) = -\,(\mathfrak{p}\ \mathfrak{E})\,, \end{aligned}$$

which agrees with (4).

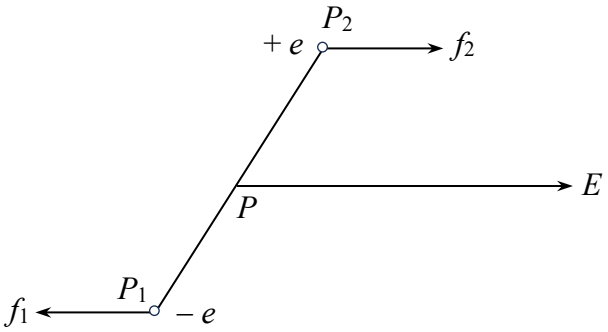


Figure 9.

Although that (usual) way of looking at things is somewhat simpler than the one that we have proposed in the previous sections, the latter has the advantage that it can be adapted immediately to the action between *electrical currents*, and in that way the analogy and differences between the two types of action will emerge more clearly.

CHAPTER TWO

ELECTROKINETIC (MAGNETIC) EFFECTS AND THE ENERGY PRINCIPLE

§ 1. – Electric currents.

Any orderly motion of electricity in material bodies under which the opposite charges *are displaced relative to each other* is called an *electric current*. An electrical current can then be created by the motion of opposite charges in opposite directions or by the motion of charges of a certain type, while the other ones are in a state of complete rest. The case in which both types of charge move in the same direction with different velocities can be regarded as the superposition of one of the two aforementioned cases with a collective motion of the material body in question, which is a collective motion that remains entirely ineffectual from the standpoint of electricity.

If one imagines that the opposite charges are coupled with each other pair-wise into elementary dipoles then the current process will reduce to *the time variation of the electric moment of those dipoles* at each moment.

Let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be the radius vectors of the ends P_1 and P_2 of one such dipole relative to a fixed point O. If we consider the charges $e_1 = -e$ and $e_2 = +e$ that define it to be constant then that will give:

$$\frac{d\mathfrak{p}}{dt} = e\,(\mathfrak{v}_2 - \mathfrak{v}_1) = e_1\,\mathfrak{v}_1 + e_2\,\mathfrak{v}_2 \tag{1}$$

for the time derivative of its electric moment $\mathfrak{p} = e\,P_1\,P_2 = e\,(\mathfrak{r}_2 - \mathfrak{r}_1)$, in which $\mathfrak{v}_1 = d\,\mathfrak{r}_1 / dt$ and $\mathfrak{v}_2 = d\,\mathfrak{r}_2 / dt$ mean the "absolute" velocities of the two charges, and $\mathfrak{v}_2 - \mathfrak{v}_1$ means their relative velocity.

In what follows, we would like to call the product of the charge of a particle with its velocity the *electric impulse* of that particle (by imitation of the usual *mechanical* impulse, which is equal to the product of *mass* and velocity). The content of formula (1) can be expressed as follows: The time derivative of the electric moment of a dipole is equal to the geometric sum of the electric impulses of the poles that define it.

The electric impulse per unit volume of the body in question is called the *current density* (at the point in question). One can also define the density of the electric current $\mathfrak{J}$ to be the time derivative of the electric moment $\mathfrak{P}$ per unit volume, i.e., the geometric sum of the moments of the dipoles that are found in a very small volume at the moment considered, divided by the magnitude of that volume element.

One will then have:

$$\mathfrak{J} = \frac{d\mathfrak{P}}{dt}\,. \tag{2}$$

The vector $\mathfrak{P}$ is ordinarily referred to as the *electric polarization* of the body at the point in question. Its introduction is especially convenient when the positive and negative charges that are found in any volume element always remain in the same element, i.e., one cannot separate them from each other. That is the case for the so-called *dielectric* bodies, whose molecules can be regarded as dipoles that are composed of inseparable charges.

By contrast, for the *electrical conductors* (such as electrolytes and metals), the individual charges (e.g., electrons, ions) drift about independently of each other throughout the entire volume of the body. In that case, in order to be able to preserve the concept of polarization and the relation (2), one must show that the opposite charges that are coupled with each other into elementary dipoles will switch with each other from time to time in order for their mutual distance to always remain small (compared to the dimensions of the body).

§ 2. – Stationary electric currents.

If the current density at each point of the body under consideration remains constant in time (which can be true for only conductors) then the electrical current is called *stationary* (or also "direct current").

In that case, one can show that the electric charges move in *closed curves* and that the charges that leave from one side of a volume element will be immediately replaced with other ones that enter from the opposite side.

Imagine that there are N particles with charge e in a unit volume, and that they move with the same velocity $\mathfrak{v}$. The part of the current density that corresponds to those charges is obviously equal to $N\,e\,\mathfrak{v}$. Let dS be a surface element whose normal $\mathfrak{n}$ defines the angle θ with the direction of motion (i.e., with $\mathfrak{v}$) (Fig. 10). The number $d\nu$ of charges of the type in question that go through dS during the time interval dt is equal to the product of N with the volume of a cylinder with the base surface dS times the height $v\,dt \cdot \cos\theta = (\mathfrak{v}\,\mathfrak{n}) \cdot dt = v_n\,dt$. If one counts that number as positive when $\theta < 90^\circ$ and negative when $\theta > 90^\circ$ then one will have:

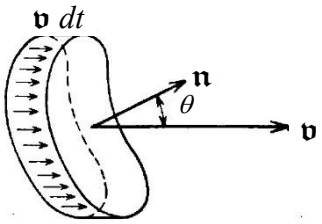


Figure 10.

$$d\nu = N\,(\mathfrak{v}\,\mathfrak{n})\,dS\,dt\,.$$

Upon multiplying that expression by e and summing over all types of charges (with different e or $\mathfrak{v}$) that are found in the body in question (or better yet, they *move* in it), we will get the total *amount of electricity* that flows through dS in time dt:

$$dQ = \sum N\,e\,(\mathfrak{v}\,\mathfrak{n})\,dS\,dt = (\sum N\,e\,\mathfrak{v})\,\mathfrak{n}\,dS\,dt\,.$$

Note that the part of the electrical current through dS that consists of positive charges will be positive or negative according to whether it moves in the direction of the normal $\mathfrak{n}$ ($\theta < 90^{o}$) or the opposite direction ($\theta > 90^{o}$), resp. Conversely, the corresponding part that consists of negative charges will be negative in the former case and positive in the latter.

The sum $\sum N\, e\, \mathfrak{v}$ is obviously nothing but the *current density* $\mathfrak{J}$ that was defined above. The amount of electricity dQ / dt that flows through dS per unit time is called the *strength* of the corresponding current. The electrical current strength for an arbitrary surface is expressed by the integral:

$$I = \int J_n \, dS \, . \tag{3}$$

In the case of a closed surface, that current strength must be equal to the decrease per unit time in the amount of electricity in the volume V that is bounded by S (as usual, $\mathfrak{n}$ means the exterior normal). That theorem is an immediate consequence of the principle of the indestructibility of electric charges (Chap. I, **§ 1**). If we denote the *electric charge density*, i.e., the sum $\sum e$ of the charges in a unit volume, by ρ, then we can express the aforementioned "conservation law" in the form:

$$\oint J_n \, dS \; = - \frac{d}{dt} \int \rho \, dV \, .$$

Moreover, by means of *Gauss*'s formula, we will get:

$$\int \operatorname{div} \mathfrak{J} \, dV \; = - \frac{d}{dt} \int \rho \, dV \, ,$$

or since $\dfrac{d}{dt} \int \rho \, dV \; = \int \dfrac{\partial \rho}{\partial t} dV$:

$$\int \left(\operatorname{div} \mathfrak{J} + \frac{\partial \rho}{\partial t} \right) dV \; = 0 \, .$$

Due to the arbitrariness in the volume V, the integrand must vanish identically, such that the law of conservation of electric charge can be expressed as the following differential equation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathfrak{J} = 0 \, . \tag{4}$$

In the case of a *stationary* electric current that is of interest to us, we will have $\partial \rho / \partial t = 0$, and as a result:

$$\operatorname{div} \mathfrak{J} = 0 \, . \tag{4.a}$$

That equation shows that the "current lines," i.e., the curves that represent the vector field $\mathfrak{J}$, are *source-free*, i.e., they have no starting or ending points. As long as the electric current takes place

in a *bounded* space, they must then be *closed curves*. The set of all such curves that go through an open surface S (or the line σ that bounds it) can be regarded as a current thread or "tube." The current strengths $I = \int J_n \, dS$ keeps the same value for arbitrary cross-sections of that "tube."

One can then think of any body in which a stationary electric current is found as a set of such "current tubes" with infinitely-small cross-sections that play the same role in regard to the *current distribution* that, e.g., the ordinary volume elements (with infinitely-small sides) do in the distribution of electric charge (or polarization). We will refer to each infinitely-thin tubular current element as a *linear* electric current.

Clearly, there are no linear currents, in the mathematical sense of the word. However, one can treat an electric current that flows in a very thin metal wire (as long as its interaction with *other* such currents is considered) as linear in practice, just as we have treated the two poles of a dipole as point charges in the previous chapter.

§ 3. – The magnetic moment of a linear current.

The electrical currents, or rather, the bodies in which they circulate, exert certain actions upon each other that one refers to as *electrokinetic* or *magnetic*. Those actions can be studied most simply with *linear* electric currents, which play the same role in this context that the electric dipoles play in the study of electrostatic actions. The elementary dipoles then correspond to *elementary linear currents*, which are characterized by the fact that the corresponding current lines (which we will think of as infinitely-thin rigid wires) are *planar* and *very small* compared to their mutual distances.

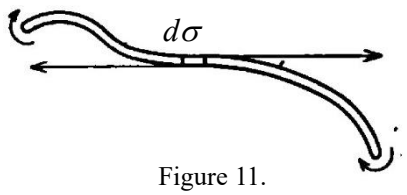


Figure 11.

We next observe the following completely-obvious fact: If the current line contracts to a point then its mechanical (i.e., magnetic) effect must vanish. It must likewise vanish when the current line contracts to a "double line" of finite length, i.e., to a line that consists of two practically-overlapping halves (Fig. 11), such that the sum of the electric impulses in each double element of that line ($d\sigma$) will be equal to zero.

One can conclude from this that the geometric quantity that is decisive for the mechanical action of an elementary current line *is not its length*, but the *area of the surface that it bounds*.

We next exhibit two completely-identical elementary current lines of rectangular form that have the same orientation and are found next to each other in a region V of roughly the same dimensions as their own lengths. The mechanical effect of the system that they define on external (distant) electric currents must obviously be twice as large as the corresponding effect of each of the two current lines considered, and it must be entirely *independent of their relative positions* ([1]). In particular, one can place them in such a way that two of their sides coincide (Figs. 12.a, b).

([1]) As long as they remain in the region V.

Those coincident sides will then define a double current line whose mechanical effect is equal to zero. The other six sides define a new rectangular current line whose form is different in the cases a) and b), but have the same area 2 *S*, where *S* means the area of the two isolated current lines. That will then show that the mechanical effect of an elementary current line is *independent of its form and proportional to its area* ([1]).

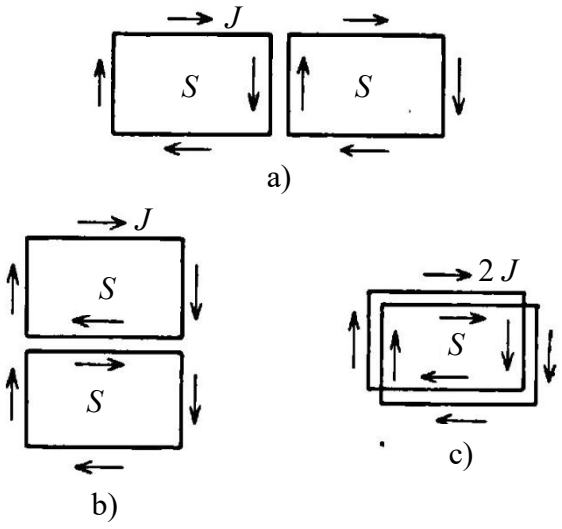


Figure 12.

Instead of placing the current lines that were considered above next to each other, one can overlap them, as is suggested in Fig. 12.c. In that way, one will, in fact, get an isolated current line with the same form and circumference as the original one, *but with a current strength* 2 *I that is twice as large*.

We then see that the mechanical effect that an elementary current line exerts on an external (distant) current or experiences from such currents *is proportional to the area S that it encloses and the current strength I*, i.e., it is determined from the product *I S*. That product, which corresponds to the electric moment of an (elementary) dipole, is called the *magnetic moment* of the current under consideration (or the current line).

When one treats the interaction between electric currents, it is convenient to introduce a new unit for current strength that we will first determine later. Let the ratio of that new "electrokinetic" unit to the original "electrostatic" one be *c*. That means that the "electrostatic" strength of a current *I* corresponds to the electrokinetic strength:

$$i = \frac{I}{c}. \tag{5}$$

The magnetic moment of a current line is then determined by the formula:

$$m = i\,S\,. \tag{5.a}$$

Just as in the case of a dipole, one can consider *m* to be the magnitude of a *vector quantity*. We define that vector quantity by:

$$\mathfrak{m} = i\,S\,\mathfrak{n}\,, \tag{5.b}$$

in which $\mathfrak{n}$ means the normal to the current plane. Its direction shall be associated with the sense of traversal on the current line σ that is determined by the direction of the current using the right-

([1]) That proof is carried out for only current lines of rectangular form here. However, its generalization to current lines of entirely-arbitrary form raises no difficulties. Namely, one can decompose any such line into rectangles and rectangular triangles (on the boundary). In that way, a triangle will be equivalent to one-half of the corresponding rectangle (in regard to its effect).

hand screw rule. The *magnetic moment of a planar current line is then equal to its geometric moment* $S\,\mathfrak{n}$ (cf., Introduction, **§ 4**), *multiplied by the current strength* i. That definition can be adapted directly to arbitrary non-planar current lines (see below, **§ 4**).

Note that the opposite direction of current for a given orientation of σ must obviously have the same consequence as the opposite orientation, namely, the opposite of the direction of the forces and torques that act on the current line or by it. From (5.b), that would correspond to the opposite sign for $\mathfrak{m}$. One then sees from this that the magnetic moment of an elementary current line must determine both the *magnitude* and *direction* of its mechanical effect ("passive" and "active") completely.

§ 4. – Systems of elementary currents. Non-elementary currents.

Let σ_1 and σ_2 be two elementary current lines with magnetic moments $\mathfrak{m}_1$ and $\mathfrak{m}_2$ that are found in a very small spatial region V. Since the forms of those lines (insofar as it is independent of the area) and their current strengths are irrelevant to their mechanical effect, just like their relative positions in V, one can replace them with two parallelograms with the same current strengths $i_1 = i_2 = 1$ that lie next to each other. In that way, we will get the picture that is suggested by Fig. 4 (Introduction, **§ 4**). By adding the two "triangular currents" $PQOP$ and $P'Q'O'P'$ with equal and opposite moments to the "parallelogram currents" $PQQ'P'P$ and $OPP'O'O$ being considered, we will get a single resultant elementary current that is represented by the parallelogram $OQQ'O'O$. The magnetic moment of that current $\mathfrak{m}$ is obviously equal to the geometric sum of the moments of the other two parallelogram currents, i.e.:

$$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 .$$

The result can be easily generalized to an arbitrary number of elementary linear currents, as long as the current lines in question are found to be sufficiently close to each other and at a large distance from the other currents that they interact with.

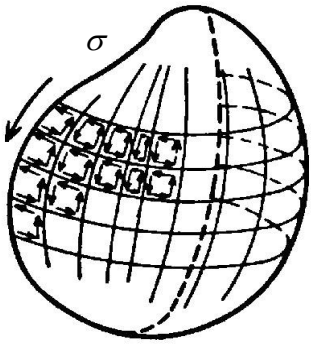


Figure 13.

Just as we had decomposed a non-elementary dipole into a chain of elementary ones in the previous chapter (**§ 3**), we can replace a non-elementary linear current, i.e., a current line of arbitrary form and magnitude, *with a network of elementary current lines*. That replacement or "decomposition" corresponds completely to the decomposition into closed curves or the surfaces that are bounded by them that was considered in the Introduction (**§ 3**). We must only imagine that those curves are the carriers of electrical currents of *the same strength* and then determine the sense of traversal on each curve from the direction of current.

In that way, the question of the mechanical effect of an arbitrary non-elementary current line σ will be reduced to the calculation of the corresponding effect of the network of current lines that replaces it. One can imagine that this network of current lines spans an *entirely-arbitrary* surface S that is bounded by σ (Fig. **13**). Therefore, every network of lines, *with the exception of* σ, can be considered to be a doubled current line, such that the mechanical effect of the entire network of currents must always be equal to the effect of the current line σ.

According to (5.b), one can express the magnetic moment of an elementary current line that bounds the surface element dS with the normal $\mathfrak{n}$ by:

$$d\,\mathfrak{m} = i\,\mathfrak{n}\,dS\,.$$

The geometric sum of all those moments, i.e., the integral:

$$\mathfrak{m} = i\int \mathfrak{n}\,dS\,, \tag{6}$$

is called the *magnetic moment of the current line* σ *in question.* That vector is obviously *equal to the product of the current strength i* with the *geometric moment of the curve* σ that was defined before (Introduction, **§ 3**). For sufficiently-small dimensions of that curve, the mechanical effect of the corresponding current will be characterized completely by the vector $\mathfrak{m}$, even when σ *is not a plane curve* (so when the current is not "elementary," properly speaking). Otherwise, in order to evaluate the total effect of the current in question, one must calculate the corresponding effects of the elementary currents that replace it individually.

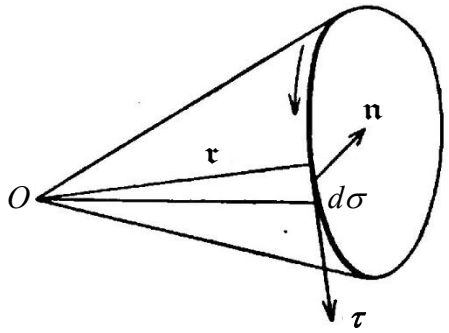


Figure 14.

Since the vector $\mathfrak{m}$ or $\int \mathfrak{n}\,dS$ depends upon only σ, but not upon the form of the surface S, it would seem reasonable to express it in the form of a line integral that is taken along σ. The conversion is given by simply specializing the surface S. Indeed, we would like to consider S to be the outer surface of a *cone* whose vertex might lie at an arbitrary O (Fig. 14).

Since the normal $\mathfrak{n}$ keeps the same direction at all points of a triangle through O and is determined by $d\sigma$, one can take the area of that triangle to be the surface element dS. If we further observe that $\mathfrak{n}$ falls in the direction of the outer product $\mathfrak{r} \times \tau$, where $\mathfrak{r}$ means the radius vector to a point of σ ($d\sigma$, resp.), and τ means the corresponding tangent vector, then we will have:

$$\mathfrak{n}\,dS = \tfrac{1}{2}\mathfrak{r} \times \tau\,d\sigma\,,$$

and as a result, from (6):

$$\mathfrak{m} = \tfrac{1}{2}\oint \mathfrak{r} \times \boldsymbol{\tau}\, i\, d\sigma\,. \tag{6.a}$$

That expression has a very illuminating physical interpretation. If we imagine, for the moment, that the current line σ is replaced with a very thin current thread (e.g., a wire) of cross-section q then the current strength can be represented as the product of q with the *current density* $j = (1 / c)\, J$. Moreover, since the direction of $\mathfrak{j}$ coincides with $\boldsymbol{\tau}$, and the product $q\, d\sigma$ means the volume dV of the corresponding element of the current thread, we will get:

$$\boldsymbol{\tau}\, i\, d\sigma = \mathfrak{j}\, dV = \sum \frac{e}{c}\mathfrak{v}\,, \tag{6.b}$$

i.e., the "current element" $\boldsymbol{\tau}\, i\, d\sigma$ is equal to the electrical impulse of the charges that are present in that element ($d\sigma$), when expressed in the electrokinetic units, just like j. One can correspondingly define the integral $\oint \mathfrak{r} \times \boldsymbol{\tau}\, i\, d\sigma$ to be the *electric impulse moment* or the *electric angular momentum* of the current considered, which corresponds to the usual definition of the *mechanical* angular momentum when one replaces the mass of the material particle with the charge. As one can see from the argument that was posed, that electric angular momentum, which must be defined relative to any point (O), is independent of the choice of that point.

The magnetic moment of a linear electric current is then equal to one-half of the resulting electric angular momentum of the moving charges that define that current.

§ 5. – The statics of electric currents. The magnetic field.

Since an elementary electrical current is determined completely by its magnetic moment in the context of its interaction with other currents, we can treat that interaction in the same way that we treated the elementary electric dipole. Just as we did with the latter, we will ignore the dimensions of the current line completely, so we will actually treat it as a point that is characterized kinematically by its position ($\mathfrak{r}$) and the orientation of the vector $\mathfrak{m}$ that is coupled with it, and statically by the force $\mathfrak{F}$ and torque $\mathfrak{M}$ that acts on it. We will further assume that even in that case, the two force effects can be derived from an energy function U ($\mathfrak{r}$, $\mathfrak{m}$) that we will call the *magnetic energy* of the elementary current considered relative to the other currents that it is found to interact with.

Under those circumstances, one must obviously once more find the same expressions for U, $\mathfrak{M}$, $\mathfrak{F}$ that were exhibited in the previous chapter for an elementary dipole when one introduces the magnetic moment $\mathfrak{m}$ in place of $\mathfrak{p}$ (viz., the electric moment), and replaces $\mathfrak{E}$, the electric field strength, with the corresponding magnetic vector quantity, namely, the so-called *magnetic field strength* $\mathfrak{H}$. We then get the following formulas, which are completely analogous to equations (4), (3), and (5) in Chap. I:

$$U = - \mathfrak{m}\ \mathfrak{H}\ , \tag{7}$$

$$\mathfrak{M} = - \mathfrak{m} \times \mathfrak{H}\ , \tag{8}$$

$$\mathfrak{F} = \text{grad}\ (\mathfrak{m}\ \mathfrak{H})\ . \tag{9}$$

One observes that in so doing, one says *absolutely nothing* about the relationship between magnetic and electric field strengths as functions of $\mathfrak{r}$. The vector fields $\mathfrak{E}\ (\mathfrak{r})$ and $\mathfrak{H}\ (\mathfrak{r})$ can have an entirely-different structure from the outset. It is actually easy to derive the fact that a certain contrast exists between the two fields in that regard from the energy principle.

Namely, one now considers the magnetic energy of a *non-elementary* current relative to the currents in whose field $\mathfrak{H}$ one finds it. One can obviously determine that energy U from the sum of the infinitely-small contributions:

$$dU = - \mathfrak{H}\ d\ \mathfrak{m} = - i\ \mathfrak{H}\ \mathfrak{n}\ dS = - i\ H_n\ dS\ ,$$

which correspond to the individual elementary currents that replace the current in question (**§ 4**). One will then have:

$$U = - i \int H_n\ dS\ . \tag{10}$$

The surface integral:

$$\Phi = \int H_n\ dS \tag{10.a}$$

means the "magnetic flux" through any surface that is bounded by the current line σ. It plays the same role as the line integral of the electric field strength for the energy of a non-elementary dipole [Chap. I, formula (6)].

Since the magnetic energy of a given current of the type that one decomposes into elementary currents is independent of the choice of the outer surface S that one imagines is spanning the network of those elementary currents, in particular, the integral (10.a) must have the same value for two different surfaces S' and S''. By inverting the normal direction on one of those surfaces, one will then get $\int H_{n'}\ dS' + \int H_{n''}\ dS'' = 0$, i.e.:

$$\oint H_n\ dS = 0\ ,$$

in which S means the closed surface that is defined by S' and S'', and $\mathfrak{n}$ means the corresponding exterior (or interior) normal to it.

One will get the same result when one contracts the current line σ to a point. In that way, the surface S in (10) will be closed, and the energy U will obviously be equal to zero.

It follows from the fact that the magnetic flux through any closed surface vanishes and *Gauss*'s formula [$\oint H_n\ dS = \int \text{div}\ \mathfrak{H}\ dV$] that the magnetic field must fulfill the condition:

$$\operatorname{div} \mathfrak{H} = 0 \tag{11}$$

in all of space. That condition is a direct consequence of the energy principle and corresponds to the condition rot $\mathfrak{E} = 0$ for the electric field. It means that the *magnetic field is source-free*, i.e., that the "magnetic lines of force" are closed curves in general.

Due to the character of equation (11) as an identity with respect to $\mathfrak{r}$, one can appeal to the identity div rot $\mathfrak{F} = 0$ [Introduction, formula (18.a)] and set:

$$\mathfrak{H} = \operatorname{rot} \mathfrak{A}, \tag{12}$$

in which $\mathfrak{A}$ means an as-yet-undetermined vector function. We will call that vector function, which corresponds to the scalar potential φ, the *vector potential* or also the *electrokinetic potential*.

Note that the vector potential is *not determined uniquely* by the magnetic field strength. That is because one can add the gradient of an arbitrary scalar function to $\mathfrak{A}$. Since its rotation vanishes identically, $\mathfrak{H}$ will remain unchanged in that way. By a suitable choice of that scalar function, one can determine the vector potential in such a way that it will satisfy a given scalar condition, e.g., the condition that it is source-free:

$$\operatorname{div} \mathfrak{A} = 0 \,. \tag{12.a}$$

If one substitutes (12) in (10) then, from *Stokes*'s formula [Introduction, (17)], that will give:

$$U = -\, i \oint A_\tau \, d\sigma = -\oint \mathfrak{A} \boldsymbol{\tau} \, i \, d\sigma \,. \tag{13}$$

By means of that formula, the magnetic energy of the current line in question will be expressed as a sum of parts that are associated with the *individual elementary of that line* [and not the elementary *currents* that replace them, as in the original definition by formula (10)]. However, we will see immediately that such a decomposition (in contrast to the corresponding decomposition in the case of electric dipoles) is not always permissible, with no further conditions.

§ 6. – The effect of the magnetic field on isolated current elements and moving charges.

However, we would next like to attempt to treat the quantity:

$$dU = -\, \mathfrak{A} \, \boldsymbol{\tau} \, i \, d\sigma \tag{13.a}$$

as the magnetic energy of the current element $d\sigma$ (recall that the vector $i\,\boldsymbol{\tau}\,d\sigma$ represents the electric impulse of the charges that are found in $d\sigma$). The force $d\,\mathfrak{F}$ that acts upon that current element must then be calculated from the formula $d\,\mathfrak{F} = -\,\mathrm{grad}\,(dU)$, i.e.:

$$d\,\mathfrak{F} = \mathrm{grad}\,(i\,\boldsymbol{\tau}\,d\sigma\,\mathfrak{A}) = (i\,\boldsymbol{\tau}\,d\sigma\,\mathrm{grad})\,\mathfrak{A} + i\,\boldsymbol{\tau}\,d\sigma\,\times\,\mathrm{rot}\,\mathfrak{A}\,.$$

If one sets $\mathrm{rot}\,\mathfrak{A} = \mathfrak{H}$ and observes that the vector $(i\,\boldsymbol{\tau}\,d\sigma\,\mathrm{grad})\,\mathfrak{A}$ is equal to the difference $\mathfrak{A}_2 - \mathfrak{A}_1 = d\,\mathfrak{A}$ for the endpoints of the line segment $d\sigma$ then one can write the formula above as follows:

$$d\,\mathfrak{F} = i\,d\,\mathfrak{A} + i\,\boldsymbol{\tau}\,\times\,\mathfrak{H}\,d\sigma\,. \tag{13.b}$$

Upon integrating along the current line σ, the first term on the right-hand side of (13.b) will drop out since one obviously has $\oint d\mathfrak{A} = 0$. The total force that acts on that line is then equal to:

$$\mathfrak{F} = i\oint \boldsymbol{\tau}\,d\sigma\times\mathfrak{A}\,.$$

As long as one considers that total force, it is therefore entirely irrelevant whether the corresponding elementary force is defined by (13.b) or by the simpler formula:

$$d\,\mathfrak{F} = i\,\boldsymbol{\tau}\,d\sigma\times\,\mathfrak{H}\,. \tag{14}$$

We shall now attempt to derive that elementary force from the change in total energy U that results from a very small displacement of the current line considered in the magnetic field. In so doing, it is not necessary to regard that line as rigid (as we did for the elementary currents). Rather, we can treat it as a flexible and extensible string.

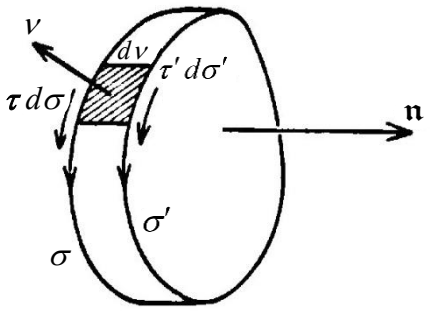


Figure 15.

Let the "original" and "displaced" current lines be denoted by σ and σ' (Fig. 15), and the corresponding energies by U and U'. Moreover, let S' be an arbitrary surface that is bounded by σ'. Since the corresponding surface for σ can be chosen to be entirely arbitrary, one can represent it by adding the surface Σ that is described by σ under displacement (i.e., spanned by σ and σ') (viz., $S = \Sigma + S'$). From (10), one will then have:

$$U' - U = \delta U = \int i\,H_\nu\,d\Sigma\,,$$

in which ν means the normal to $d\Sigma$. We would like to take the surface $d\Sigma$ to be the surface that is described by the line element $d\sigma$, i.e., the area of the parallelogram with the sides $\boldsymbol{\tau}\, d\sigma$ and $\delta\mathfrak{r}$. (We can consider the infinitesimal displacements of the various points of *ds* to be equal and point in the same direction in the first approximation.) Since $\nu\, d\Sigma = \boldsymbol{\tau}\, d\sigma \times \delta\mathfrak{r}$, we will get $H_\nu\, d\Sigma = \mathfrak{H}\nu\,\delta\Sigma = \mathfrak{H}\cdot(\boldsymbol{\tau}\, d\sigma \times \delta\mathfrak{r}) = -\,\delta\mathfrak{r}\cdot(\boldsymbol{\tau}\, d\sigma \times \mathfrak{H})$, and as a result:

$$-\,\delta U = \oint (i\,\boldsymbol{\tau}\, d\sigma \times \mathfrak{H})\,\delta\mathfrak{r}\,.$$

That expression, which represents the loss of energy under the displacement in question, must obviously be equal to the total work due to the electric forces that act upon the individual current elements. That will give:

$$\oint i\,\boldsymbol{\tau}\, d\sigma \times \mathfrak{H}\cdot\delta\mathfrak{r} = \oint d\mathfrak{F}\cdot\delta\mathfrak{r}\,.$$

In that equation, $\delta\mathfrak{r}$ is regarded as a very small vector quantity that varies *continuously* along σ. However, it is otherwise entirely arbitrary since the displacements of the various (non-sequential) points of σ are independent of each other, by assumption. That is why the individual elements of the integral above, or rather, the corresponding factors of $\delta\mathfrak{r}$, are equal to each other. In that way, we will get:

$$d\mathfrak{F} = i\;\boldsymbol{\tau}\, d\sigma \times \mathfrak{H}\,,$$

i.e., the formula (14). The previous formula (13.b) then proves to be false ([1]). As a result, we must recognize that the assumption of a magnetic energy in the individual elements of a current line is also false.

That "undecomposability" of the magnetic energy of a closed current line is explained by the fact that the motion of the individual electric charges that defines the current *is not a stationary* (i.e., time-independent) process, even in the case of stationary currents. However, the electric and magnetic energy, as they were defined above, refer to only those mechanical effects that do not depend upon time explicitly.

For the individual current elements, we must then drop the concept of energy and consider only the corresponding elementary force that is determined by the formula (14). We can again represent that elementary force as a sum of individual forces that correspond to the individual charges (viz., electrons) that define the current element in question. Indeed, according to (6.b), for a particle with charge *e* that moves in the magnetic field with a velocity $\mathfrak{v}$, we will get the following "magnetic" or "electromagnetic" force:

$$\mathfrak{f} = \frac{e}{c}\,\mathfrak{v} \times \mathfrak{H}\,. \qquad (14.a)$$

([1]) It is easy to see that the integral $\oint d\mathfrak{A}\cdot\delta\mathfrak{r}$ is non-zero, in general.

Note that this force can be treated as an electric force that is required by a *fictitious electric field* of strength:

$$\mathfrak{E} = \frac{1}{c}\,\mathfrak{v} \times \mathfrak{H}\ . \tag{14.b}$$

Although the form (14.a) cannot be derived from an energy function, when calculating the total energy of the *stationary* currents (linear and nonlinear) of the individual moving charges (viz., electrons), one can however attribute that total energy to a fictitious magnetic energy [cf., (13.a)]:

$$u = -\,\frac{e}{c}\,\mathfrak{v}\mathfrak{A}\,. \tag{15}$$

That fictitious energy corresponds to a force:

$$\mathfrak{f} = -\ \text{grad}\ u = \frac{e}{c}\,(\mathfrak{v}\ \text{grad})\ \mathfrak{A} + \frac{e}{c}\ \mathfrak{v} \times \text{rot}\ \mathfrak{A}\ ,$$

i.e.:

$$\mathfrak{f} = \frac{e}{c}\,(\mathfrak{v}\ \text{grad})\ \mathfrak{A} + \frac{e}{c}\ \mathfrak{v} \times \mathfrak{H}\ , \tag{15.a}$$

whose first (fictitious) term must vanish upon summing over all charges (or also any ones that define closed current lines).

———————

CHAPTER THREE

THE STRUCTURE OF ELECTRIC AND MAGNETIC FIELDS AS IT RELATES TO THE EQUIVALENCE PRINCIPLE

§ 1. – The equivalence of elementary dipoles and currents.

In the foregoing chapters, we have studied the mechanical effects of dipoles and currents only in regard to their *passive* aspects, while the "active" dipoles and currents from which the effect in question originated were considered only *indirectly* by way of the electric (magnetic, resp.) field that they generated.

In order to solve our main problem – viz., determining the interaction of dipoles or currents, i.e., of charges at rest or in motion – we must now examine the question of the structure of the fields that they create. Indeed, the assumption that the aforementioned interaction can be derived from a mutual potential energy (i.e., the "energy principle") allowed us to exhibit the general properties of those fields – viz., the vortex-free character of the electric field and the absence of sources for the magnetic one. However, their determination was still not complete with that, but only reduced to a simpler problem, namely, determining the scalar potential φ and the vector potential $\mathfrak{A}$.

The energy principle did not allow us to go further. That is why we must complete it by means of a different principle. However, before we formulate that principle, we would like to make some remarks by way of introduction that will be inessential for our presentation, but important in a historical context. The effect that a given magnetic field $\mathfrak{H}$ has on an elementary electric current with the moment $\mathfrak{m}$ is identical to the effect that an elementary electric dipole with a moment of $\mathfrak{p} = \mathfrak{m}$ would experience in a fictitious electric field $\mathfrak{E} = \mathfrak{H}$. In place of the fictitious electric field, one can imagine a fictitious *magnetic dipole* that consists of two opposite "magnetic poles" $\pm \mu$ at a very small distance l from each other, in which:

$$\mu \mathfrak{l} = \mathfrak{m} ,$$

corresponding to the relation $e \mathfrak{l} = \mathfrak{p}$ for real electric dipoles. Those "elementary magnetic dipoles" or "elementary magnets" (M) shall then react to the real magnetic field $\mathfrak{H}$ in just the same way that the corresponding electric currents (S) should.

According to the energy principle, the action on S must be equal and opposite to the reaction of the "external" current S' that creates the field $\mathfrak{H}$ (since both effects will be a result of the same "mutual" potential energy). If one would also like to adapt the energy principle to the magnet M that "passively" replaces the current S then one must assert that this replacement must also take place actively, i.e., that M creates the same magnetic field ($\mathfrak{H}'$) as S.

If one imagines S' to be a second elementary current, in particular, then one can express the assertion above as follows: The interaction of an elementary magnet and an elementary current is identical to the interaction of two elementary currents or two elementary magnets with equal corresponding moments.

Recall that the action of magnetic forces was first discovered in so-called "naturally magnetic" bodies or "natural magnets." Therefore, those magnets were first regarded as real magnetic dipoles whose properties were assumed to be identical to the properties of electric dipoles (*Coulomb*). One explains the inseparability of opposite "magnetic charges" in the same way as the inseparability of opposite electric charges in dielectric bodies, namely, by the assumptions that the opposite magnetic poles should always remain in the same molecules of the body under consideration. According to that theory, which goes back to *Weber*, a natural magnet is regarded as a system of elementary "molecular" magnets.

Once the magnetic effects of electric currents had been discovered and studied, those *Weberian* "molecular magnets" were later replaced with equivalent *molecular currents* by *Ampére*, which one now considers to be orbiting electrons and calls *magnetons*.

Thus, the magnetic "poles" and "dipoles" were recognized as fictitious mathematical structures. Nonetheless, even today, they have been preserved as a very useful, if not indispensable, tool for representing electrokinetic interactions.

With those historical remarks, which have no direct relationship to our line of reasoning, we would like to again give no attention to the fictitious magnetic dipole and express the following *equivalence principle* between the elementary electric dipoles and electric currents:

For a suitable choice of the ratio c between the electrostatic and electrokinetic units, the interaction of elementary electric dipoles will be identical to the interaction of the elementary electric currents that are found to have the same relative configuration and whose magnetic moments have the same (relative) orientation and numerical magnitudes as the electric moments of the corresponding dipoles.

Since the "passive" equivalence of elementary dipoles and currents with respect to their behavior in given external fields was already established above, the equivalence principle means that the magnetic field of an elementary current *coincides completely* ($\mathfrak{H} = \mathfrak{E}$) with the electric field of an elementary dipole with the same position, orientation, and a numerically-equal moment ($\mathfrak{m} = \mathfrak{p}$). In particular, it must be emphasized that this agreement has an *asymptotic character*, in the sense that it is true for only sufficiently-distant spatial points. Obviously, it can no longer exist in the immediate neighborhood of the two structures. However, if one considers those structures to be infinitely-small, e.g., point-like, then the domain of validity of the equivalence principle must extend to all of space, with the exclusion of such "singular" points.

§ 2. – The fundamental equations of electric and magnetic fields in empty space.

The potential energy U of an elementary dipole D relative to several other dipoles D', D'', etc., must obviously be equal to the sum of the energies U', U'', etc., that characterize the interaction of D with D', D'', … *individually*. If one denotes the electric moment of D by $\mathfrak{p}$ and the field

strengths at the point P where D is found that originate in D', D'', … by $\mathfrak{E}'$, $\mathfrak{E}''$, …, resp., then one will have:

$$U' = -(\mathfrak{p}\,\mathfrak{E}')\,,\quad U'' = -(\mathfrak{p}\,\mathfrak{E}'')\,,\qquad \ldots,$$

and as a result:

$$U = U' + U'' + \ldots = -\mathfrak{p}\,(\mathfrak{E}' + \mathfrak{E}'' + \ldots)\,.$$

On the other hand, that resultant energy must be equal to (minus) the inner product of $\mathfrak{p}$ with the resultant field strength $\mathfrak{E}$. That will imply the (almost-trivial) result that:

$$\mathfrak{E} = \mathfrak{E}' + \mathfrak{E}'' + \ldots,$$

i.e., the resultant field strength is composed vectorially from the elementary field strengths. Clearly, that result is also true for the magnetic field strengths.

One can replace any arbitrary system of electric dipoles or currents with a system of elementary (i.e., their dimensions are infinitely small) dipoles or currents as long as the point in question, for which the resultant (total) field strength is to be determined, is an *external* one, i.e., it is not charged, and no current flows through it. However, its distance from such "critical" or "singular" points can otherwise be arbitrarily small since one can think of the elementary dipole and currents as infinitely-small compared to that distance.

Outside of the "singular" points, it is therefore possible to consider the total electric or magnetic field strengths to be a geometric sum of infinitely-many components that *all originate in elementary dipoles or currents.*

From the energy principle, the electric and magnetic field strengths must satisfy the conditions:

$$\operatorname{rot}\mathfrak{E} = 0\,,\qquad \operatorname{div}\mathfrak{H} = 0$$

in all of space, including the singular points.

Since:

$$\operatorname{rot}(\mathfrak{F}' + \mathfrak{F}'' + \ldots) = \operatorname{rot}\mathfrak{F}' + \operatorname{rot}\mathfrak{F}'' + \ldots$$

and

$$\operatorname{div}\mathfrak{F} = \operatorname{div}\mathfrak{F}' + \operatorname{div}\mathfrak{F}'' + \ldots,$$

it will follow from the equivalence principle that the equations:

$$\operatorname{div}\mathfrak{E} = 0\,, \tag{1}$$

$$\operatorname{rot}\mathfrak{H} = 0 \tag{2}$$

must also be true *outside of the singular points*, and they are obtained from the energy equations above by switching the vectors $\mathfrak{E}$ and $\mathfrak{H}$.

Therefore, the electric, as well as magnetic, field strengths are free of both vortices and sources outside of the singular point. However, it can be easily shown that the equations above can no longer be fulfilled at such singular points, i.e., *that singular points are sources of the electric fields and vortices are the sources of the magnetic fields*.

In order to prove that, we next observe that from the formulas $\mathfrak{E} = -$ grad φ and $\mathfrak{H} =$ rot $\mathfrak{A}$, equations (1) and (2) are equivalent to the following equations for the potentials φ and $\mathfrak{A}$:

$$\nabla^2 \varphi = 0 \tag{3}$$

(*Laplace* equation), and:

$$\text{rot rot } \mathfrak{A} = \text{grad div } \mathfrak{A} - \nabla^2 \mathfrak{A} = 0 \, .$$

One can replace the last equation by an equation of the same type as (3):

$$\nabla^2 \mathfrak{A} = 0 \, , \tag{4}$$

with the condition [cf., Chap. II, formula (12.a)] that:

$$\text{div } \mathfrak{A} = 0 \, . \tag{4.a}$$

We now consider the vector $\varphi \, \mathfrak{E} = - \varphi$ grad φ. Since:

$$\text{div } \varphi \, \mathfrak{E} = \varphi \, \text{div } \mathfrak{E} + \mathfrak{E} \, \text{grad } \varphi = - \varphi \, \nabla^2 \varphi - \mathfrak{E}^2 ,$$

upon integrating over an entirely-arbitrary volume V and applying *Gauss's* theorem, we will get:

$$- \oint \varphi \, E_n \, dS \; = \int \varphi \nabla^2 \varphi \, dV + \int E^2 \, dV \, , \tag{5}$$

in which S means the surface that encloses the volume in question. If we shift that surface to infinity, i.e., extend the volume integral over all of space, then that will give:

$$- \lim_{S \to \infty} \oint \varphi \, E_n \, dS \; = \int E^2 \, dV \, ,$$

under the assumption that φ remains continuous everywhere and it satisfies equation (3).

We now assume that φ, and as a result E, vanish at infinitely-distant points, and indeed in such a way that the surface integral $\oint \varphi \, E_n \, dS$ will tend to zero independently of the form of the surface S. Since S drops off in proportion to the square of the distance (R), that assumption must be fulfilled when φ is inversely proportional to R (and as a result, E is inversely proportional to R^2) or decrease even faster with the distance (see below, **§ 3**). The volume integral $\int E^2 \, dV$ must also vanish then,

and since E^2 is a non-negative quantity, it must be equal to zero *identically*. The existence of a non-vanishing electric field under the aforementioned assumptions about the potential φ and its behavior at points at infinity is then incompatible with the identical existence of equation (3), and as a result, equation (1). However, since those equations must be fulfilled outside of the singular points, they must break down *at* those points.

The corresponding proof for the magnetic field can be performed on the basis of equation (4), just like in the case above, when one considers the components of $\mathfrak{A}$ individually. However, one can complete that proof without decomposing $\mathfrak{A}$ into its components on the basis of the identity:

$$\operatorname{div}(\mathfrak{A} \times \mathfrak{H}) = \mathfrak{H} \operatorname{rot} \mathfrak{A} - \mathfrak{A} \operatorname{rot} \mathfrak{H} .$$

Namely, if one replaces rot $\mathfrak{A}$ with $\mathfrak{H}$ and rot $\mathfrak{H}$ with $-\nabla^2\mathfrak{A}$ then that will give the following formula:

$$\oint (\mathfrak{A} \times \mathfrak{H})_n \, dS = \int \mathfrak{A} (\nabla^2 \mathfrak{A}) \, dV + \int H^2 \, dV , \tag{6}$$

which is entirely analogous to (5), and from it, one can infer the same conclusion, and in fact under the same assumptions about the vector potential $\mathfrak{A}$ as in the case of the scalar potential φ.

§ 3. – The fundamental equations of electromagnetic fields for singular points.

We now suppose that the electric charge e is concentrated in a small volume v that can contract to a point Q. Let $\mathfrak{E}$ be the electric field strength that *this* charge creates. Moreover, let V be an arbitrary volume that includes the volume v (the point Q, resp.), and let S be the surface that bounds it.

From *Gauss's* formula, the following relation will be true:

$$\oint E_n \, dS = \int \operatorname{div} \mathfrak{E} \, dV ,$$

or since $\mathfrak{E} = 0$ outside of v :

$$\oint E_n \, dS = \int \operatorname{div} \mathfrak{E} \, dv .$$

It will then follow from this that the electric flux through a closed surface that includes the electric charge in question will not depend upon the size, form, and position of that surface (relative to v or Q). In the limit as v goes to zero, the electric field strength at each point of S will have a well-defined direction and a magnitude that is proportional to e for a given position of Q, such that one can set:

$$\oint E_n \, dS = C_1 \, e , \tag{7}$$

in which C_1 means a coefficient of proportionality. From the foregoing, that coefficient must be independent of the position of the point Q, as long as it remains inside of S (S is now considered to be a fixed outer surface).

We now imagine that there are several point-like charges inside of S. If one denotes the magnitudes of those charges by e', e'', …, and the corresponding field strengths $\mathfrak{E}'$, $\mathfrak{E}''$, etc., then one will have:

$$\oint E'_n \, dS = C_1 e' , \qquad \oint E''_n \, dS = C_1 e'' , \qquad \ldots ,$$

where C_1 always means the same coefficient as in (7).

On the other hand, we obviously have that the total electric charge inside of S is $e = e' + e'' + \ldots$, and that the corresponding resulting field strength is $\mathfrak{E} = \mathfrak{E}' + \mathfrak{E}'' + \ldots$ Moreover, since one has $E'_n + E''_n + \cdots = E_n$, it will follow upon adding the equations above that:

$$\oint E_n \, dS = C_1 \, e ,$$

i.e., a formula with the same form as (7), but with a more-general meaning since now the charge that is found inside of S can be distributed throughout the volume V in a completely-arbitrary way.

We now imagine, in particular, a *continuous volume distribution* under which an infinitely-small volume element dV will contain a likewise infinitely-small charge $de = \rho \, dV$ (ρ = spatial charge density). We will then have $e = \int \rho \, dV$, and as a result:

$$\oint E_n \, dS = \int \operatorname{div} \mathfrak{E} \, dV = C_1 \int \rho \, dV .$$

Due to the arbitrariness in V, that will give the following differential equation for the electric field of singular points:

$$\operatorname{div} \mathfrak{E} = - \nabla^2 \varphi = C_1 \rho . \tag{8}$$

One can consider that equation to the generalization of (1) [(3), resp.]. Therefore, it will also remain valid when ρ becomes infinite, i.e., in the case of a non-continuous distribution of electricity in space. However, in the latter case, it is more advantageous to operate with the corresponding integral equation (7), and not with (8).

We now move on to consider magnetic fields, and we would first like to imagine that the field $\mathfrak{H}$ is created by a very thin current wire that can contract down to a (closed) line σ. Upon applying the *Stokes* formula to a closed curve σ' that is different from σ and *encircles* σ *once*, we will get the equation:

$$\oint H_{\tau'} \, d\sigma' = \int \operatorname{rot}_n \mathfrak{H} \, ds ,$$

in which s means the part of an arbitrary surface S that is bounded by σ' and is cut out from that surface by the current wires (since rot $\mathfrak{H} = 0$ outside of the current wire).

Since one can lay a set of surfaces S through the same cross-section of the wire s that are bounded by various lines σ', and on the other hand the surfaces that are bounded by the same line σ' can cut the current wire in question at different locations, it would follow from the formula above that: *First of all*, the circulation of the vectors $\mathfrak{H}$ along a line that encircles one of the current wires once will be independent of its quantity, form, and position (relative to σ), and *secondly*, that the flux of the vector rot $\mathfrak{H}$ through various cross-sections of the current wire will have the same value.

An argument that is completely analogous to the one that led us to exhibit formula (7) for an arbitrary charge distribution inside of the closed surface S will give the following formula, which is analogous to (7):

$$\oint H_{\tau'}\, d\sigma' = C_2\, i\,, \tag{9}$$

in which C_2 means a new coefficient of proportionality, and i means the total strength of the current that goes through σ', i.e., the algebraic sum of the current strengths for the various current wires that encircle σ' with + or – signs according to the direction of those currents relative to the sense of traversal on the curve σ'.

If one imagines, in particular, a continuous distribution of the current strength with the finite spatial density $\mathfrak{j}$ then one will have:

$$i = \int j_n\, dS\,,$$

i.e.:

$$\oint H_{\tau'}\, d\sigma' = \int \mathrm{rot}_n\, \mathfrak{H}\, ds = C_2 \int j_n\, dS\,,$$

and as a result, due to the arbitrariness in the surface S:

$$\mathrm{rot}\, \mathfrak{H} = -\nabla^2\, \mathfrak{A} = C_2\, \mathfrak{j}\,. \tag{9.a}$$

One can consider that formula to be the generalization of formula (3) [(4), resp.] to singular points of the magnetic field.

It should be pointed out that equation (9) is compatible with the identity div rot $\mathfrak{H} = 0$, due to (4.a), Chap. II, *as long as the currents in question are stationary*, which was always assumed up to now. The aforementioned second consequence of the formula $\oint H_{\tau'}\, d\sigma' = \int \mathrm{rot}_n\, \mathfrak{H}\, ds$ (viz., the independence of $\int \mathrm{rot}_n\, \mathfrak{H}\, ds$ on the choice of current wire cross-section s) corresponds to the equality of the current strength $i = \int j_n\, dS$ for the various cross-sections of those current wires.

§ 4. – Relationship between the constants C_1 and C_2 . Non-elementary currents and double layers. Non-elementary dipoles and solenoids.

Formulas (7) and (9) are coupled with each other directly by the equivalence principle. For that reason, one of them must be derivable from the other. Such a derivation would simultaneously allow us to determine the relationship between the two constants C_1 and C_2 .

Up to now, we have represented an elementary dipole as a rectilinear *line segment* with oppositely-charged ends, and an elementary current as a *closed planar line*. The two representations, which are quite different geometrically, can be combined into the same geometric structure to some extent, namely, a right *cylinder* whose lateral surface can play the role of the current lines, while the end surfaces play the role of the ends of the dipoles. By reducing the lateral dimension of the cylinder in comparison to its length, we will get an infinitely-thin rod that approaches the original linear picture of a dipole. On the other hand, upon reducing the height of the cylinder in comparison to its lateral dimension, we will get an infinitely-thin disc whose band-like lateral surface approaches the original linear picture of a current. Upon replacing the ends of the dipoles with two end surfaces S' and S'' for a cylinder, we can think of the corresponding charges $-e$ and $+e$ as being *uniformly* distributed over those surfaces, so we can represent the dipole as a *double layer* with the electric *surface density* $\pm \eta = \pm e / S$ ($S = S' = S''$).

We likewise imagine replacing the current line with the lateral surface of a cylinder that has the current distributed uniformly over that surface, i.e., a current of strength $k = i / l$ flows through the generating line in a unit of time, in which l means the length of the generator, i.e., the height of the cylinder. k is called the *surface density of the electric current.*

The electric moment of such a "cylindrical" dipole $\mathfrak{p} = e\,\mathfrak{l}$ can be correspondingly put into the following form:

$$\mathfrak{p} = \eta\, S\, l\, \mathfrak{n} = i_p\, S\, \mathfrak{n}\ , \tag{10}$$

in which $\mathfrak{n} = (1 / l)\,\mathfrak{l}$ is the exterior normal to the positive-charged cylinder surface S'', and one has:

$$i_p = \eta\, l\ . \tag{10.a}$$

The expression (10) for $\mathfrak{p}$ is obviously identical to the usual expression for the *magnetic* moment of an elementary current with strength i_p that circulates around the lateral surface of the cylinder (or the corresponding baseline).

On the other hand, we consider an actual "cylindrical current" of strength i, such that its electric moment $\mathfrak{m} = i\, S\, \mathfrak{n}$ can be expressed in the form:

$$\mathfrak{m} = k\, l\, S\, \mathfrak{n} = e_m\, \mathfrak{l}\ , \tag{11}$$

which corresponds to the usual expression for the electric moment of a dipole with length l and charge:

$$e_m = k\, S\ . \tag{11.a}$$

From the equivalence principle, one can identify the magnetic field of an elementary current with the electrostatic field of an elementary dipole. We will think of that "replacement dipole" as a cylindrical disc that is bounded by the current line σ and whose (infinitely-small) thickness l is coupled with the surface density $\pm \eta_m$, of the electric charge that covers its two sides by the formula (10.a) or (Fig. 16):

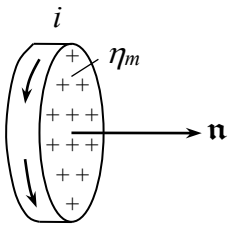


Figure 16.

$$\eta_m \, l = i \, . \qquad (10.b)$$

Conversely, if we consider the electric field $\mathfrak{E}$ of an elementary dipole to be the magnetic field of an elementary current then it would be preferable to represent that "replacement current" in the form of an infinitely-thin cylindrical whose length coincides with the dipole length l and whose (infinitely-thin) cross-section S, with the surface density k_p is coupled with the replacement current by the relation:

$$k_p \, S = e \, , \qquad (11.b)$$

which corresponds to the formula (11.a) ($\pm e$ are the "true" electric charges of the dipole in question). We will refer to a current of that form (Fig. 17) as a *solenoidal* current or *solenoid*.

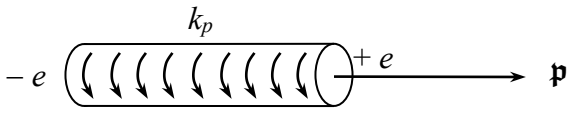


Figure 17.

We now imagine a *non-elementary* current line σ and initially replace it with a network of elementary currents with the magnetic moment $d\,\mathfrak{m} = i\,\mathfrak{n}\,dS$, in which S means an arbitrary surface that is bounded by σ. Moreover, we can replace that elementary current with a "disc-shaped" electric dipole with the base surfaces dS' and dS'' that that arise from dS by displacing it in the direction of the negative (positive, resp.) normal $\mathfrak{n}$ through a distance of l / 2. In that way, the surface density of the corresponding replacement charges will be determined by the relation (10.b).

We would like to assume that the length l is the same for all surface elements dS, for the sake of simplicity. The set of all "disc dipoles" that replace the current in question will then define an electric *double layer* of infinitely-small thickness l and infinitely-large charge density $\eta_m = i / l$ that is bounded by σ. *Outside* of that double layer, the electric field $\mathfrak{E}$ that it creates will coincide with the magnetic field $\mathfrak{H}$ that is created by the current i. However, it should be clear that such a coincidence does not exist inside of the double layer. In order to determine the electric field inside of the double layer, i.e., between the surfaces S' and S'', we next apply the formula (7) to a closed surface S that includes a small piece of *one* of the two layers (e.g., the negative layer) (Fig. 18). If we denote the part of S that lies between S' and S'' and is parallel to those surfaces by s and the other (viz., external) part by s' then from (7), we will have:

$$\int E_n \, ds + \int E'_{n'} \, ds' = - \, C_1 \, \eta_m \, s \, .$$

Since η_m (due to our assumption concerning l) is infinitely large, while the electric field strength $\mathfrak{E}'$ on s' is equal to the magnetic field strength $\mathfrak{H}$, and as a result, it must remain *finite*, we can keep the first term in the left-hand side of the equation above and set simply:

$$\int E_n \, ds = - C_1 \, \eta_m \, s \, .$$

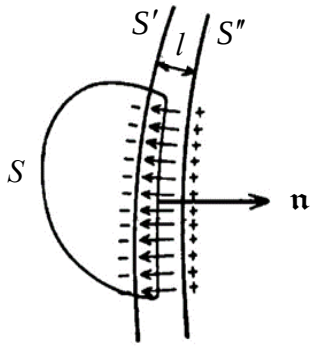


Figure 18.

Due to the arbitrariness of s, it will then follow from this that:

$$E_n = - C_1 \, \eta_m \, .$$

Moreover, it is easy to see that the electric field strength inside of the double layer must be parallel to the normal $\mathfrak{n}$. That is because in the opposite case, the line integral $\oint E_{\tau'} \, d\sigma'$ would need to have a non-zero value for a closed curve σ' that lies partly inside and partly outside of the double layer, which would contradict the energy principle.

We will then reach the conclusion that inside of the double layer, the electric field strength $\mathfrak{E}$ will be parallel to the normal $\mathfrak{n}$ and equal to:

$$E = - C_1 \, \eta_m \, . \tag{12}$$

The negative sign in that formula means that for $C_1 > 0$, the vector $\mathfrak{E}$ points in the opposite direction to the normal. By contrast, for $C_1 < 0$, it must point in the same direction as that normal.

Now let σ' be a closed line that encircles the current line σ under consideration once, and as a result, it will go through the double layer that replaces it once. We shall denote the part of σ' that is included in that layer by σ_i' and the external part by σ_a'. From the energy principle, we will then have:

$$\oint E_{\tau'} \, d\sigma' = \int \mathfrak{E} \tau_a' \, d\sigma_a' + \int \mathfrak{E} \tau_i' \, d\sigma_i' = 0 \, . \tag{12.a}$$

Moreover, from (12), we have:

$$\int \mathfrak{E} \tau_i' \, d\sigma_i' = - \eta_m \int \mathfrak{n} \tau_i' \, d\sigma_i' = \mp C_1 \eta_m \, l \, ,$$

in which the upper sign corresponds to the positive sense of traversal on σ' relative to σ, i.e., $\mathfrak{n} \tau_i' > 0$ (Fig. 19), and the upper one corresponds to the negative sense. If we choose the positive sense of traversal on σ' then, from (12.a), we will have:

$$\int \mathfrak{E}\,\tau'_a\, d\sigma'_a = -\,C_1\,\eta_m\, l\,.$$

Now, one can replace $\eta_m\, l$ with i on the right-hand side of that equation and replace $\mathfrak{E}$ with $\mathfrak{H}$ on the left. Moreover, since the integral $\int \mathfrak{H}\tau'_i\, d\sigma'_i$ must vanish when the thickness l of the double layer is infinitely small, one can extend the integral over the entire line σ'. In that way, we will get the equation:

$$\oint H_{\tau'}\, d\sigma' = C_1\, i\,,$$

which will coincide with equation (9) that was exhibited above when $C_1 = C_2$.

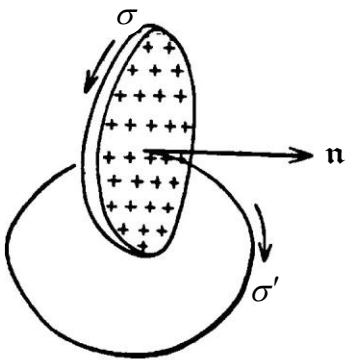


Figure 19.

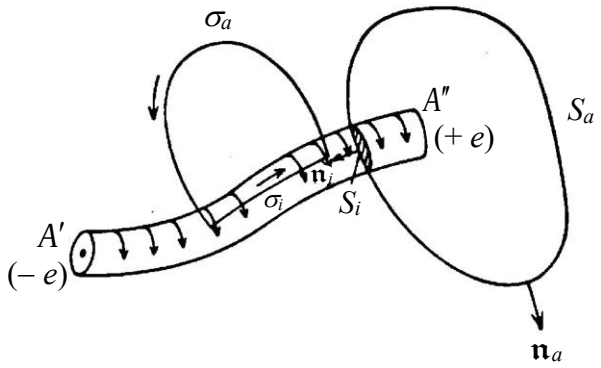


Figure 20.

One can also prove the equality of the coefficients C_1 and C_2 in the opposite way, i.e., one can replace a non-elementary electric dipole with a solenoidal current that is equivalent to it.

Namely, let σ be any curve that goes from the negative end of the dipole (A') to the positive one (A'') and along which we have a chain of elementary dipoles with the moments $d\,\mathfrak{p} = e\,\tau\, d\sigma$. If we regard that dipole as a cylindrical rod whose axis is defined by the corresponding line element $d\sigma$, and we replace it with a solenoidal current that flows through the lateral surface of that cylinder, then that will give a non-elementary current solenoid in the form of an infinitely-thin tube that connects the ends of the dipole with each other (Fig. 20). For the sake of simplicity, we would like to consider the cross-section of that tube S to be constant. If we then determine the (likewise constant) surface density of that replacement current (k_p) by way of the relation (11.b) then the magnetic field $\mathfrak{H}$ that it creates outside of the tube would coincide with the electric field of the dipole in question ([1]).

As far as the direction and magnitude of the magnetic field inside of the solenoid is concerned, one can determine it as follows:

([1]) It is assumed in all of this that the direction of the current corresponds to the positive direction along the axis curve σ (from A' to A'') in the sense of the right-hand screw rule.

Let $\sigma_i + \sigma_a$ be a closed line that runs partly inside of the tube parallel to its axis (σ_i) and partly outside of it in an entirely-arbitrary way. From formula (9), which we will now consider to be the starting formula, we will get in our present case (when the direction of integration along σ_i coincides with the positive direction of the curve σ):

$$\int H_{\tau_i}\, d\sigma_i + \int H_{\tau_a}\, d\sigma_a = C_2\, \kappa_p\, \sigma_i\,.$$

Since $\mathfrak{H}$ is equal to $\mathfrak{E}$ outside of the solenoid, and as a result, it must remain finite (except in the close vicinity of the ends of the dipole), and since the current density $k_p = e / S$ is infinitely-large, moreover, the equation will reduce to $\int H_{\tau_i}\, d\sigma_i = C_2\, \kappa_p\, \sigma_i$, or:

$$H_{\tau_i} = C_2\, \kappa_p\,,$$

due to the arbitrariness in σ_i. Now, if the vector $\mathfrak{H}$ has components that are perpendicular to its axis (σ) inside of the solenoid then the integral $\oint H_n\, dS$ will need to have a non-zero value for a surface (whose projection onto the plane of Fig. 20 will be suggested by the line $\sigma_i + \sigma_a$) that cuts the solenoid and runs parallel to its axis in its interior, but that would contradict the energy principle.

It would then follow from this that inside of the solenoid, the vector $\mathfrak{H}$ points parallel to its axis for $C_2 > 0$ or anti-parallel for $C_2 < 0$ and has the constant magnitude:

$$H = C_2\, k_p\,. \tag{13}$$

We now consider a closed surface S'' that includes one of the two ends of the solenoid, e.g., the one that corresponds to the positive end of the dipole. If we denote the external part of that surface by S_a and the internal one (that is cut out by the solenoid) by S_i then according to the energy principle, we will have:

$$\int H_{n_i}\, dS_i + \int H_{n_a}\, dS_a = 0\,,$$

or since $\int H_{n_i}\, dS_i = -\, C_2\, \kappa_p\, S = -\, C_2\, e$ (observe that the exterior normal to S_i points in the opposite direction to the axis curve σ) and $\int H_{n_a}\, dS_a = \int E_{n_a}\, dS_a = \oint E_{n''}\, dS''$, one will have:

$$\oint E_{n''}\, dS'' = C_2\, e\,.$$

That formula will coincide with the original formula (7) when one sets $C_2 = C_1$.

§5. – Determining the electric field from the charge distribution.

Let the charge e' be concentrated at a point P'. *On the grounds of symmetry*, it will follow that the field $\mathfrak{E}$ that is created by that charge must possess *radial symmetry* about the point P' ([1]). In other words, on the surface of a sphere *S* whose center coincides with P', $\mathfrak{E}$ must have a constant magnitude and point in the same direction as the external or internal normal. If we apply the general formula $\int E_n \, dS = C_1 \, e$ to our present case then that will give:

$$E = \frac{C_1 \, e'}{4\pi R^2},$$

in which *R* means the radius of the sphere, i.e., the distance from the point in question *P* to P'. For $C_1 e' > 0$, the vector $\mathfrak{E}$ points in the same direction as the radius vector $P'P = \mathfrak{R}$, whereas for $C_1 e' < 0$, it points from the point *P* to P'.

By definition, the electric field strength $\mathfrak{E}$ is nothing but the *force* that is exerted upon a unit *positive* charge at the point in question (or *would be* exerted if that unit charge were actually found there). In agreement with experimental facts, we would like to assume that the direction of the field strength that is created by a *positive* charge corresponds to a *force of repulsion* ([2]). That means that the coefficient C_1 is *positive*. However, its absolute value can be chosen quite arbitrarily. The magnitude of the electrical unit charge will be fixed by that quantity. Ordinarily, one sets:

$$C_1 = 4\pi,$$

and as a result:

$$\mathfrak{E} = \frac{e'}{R^3}\mathfrak{R} = \frac{e'}{R^2}\mathfrak{R}_0, \tag{14}$$

in which $\mathfrak{R}_0 = (1 / R)\, \mathfrak{R}$ means the unit vector that points in the direction $P'P$.

The force of interaction between two point charges e' and *e* that are found at a distance *R* from each other is correspondingly expressed by the well-known *Coulomb* formula:

$$f = \frac{e' e}{R^2}, \tag{14.a}$$

in which the case $f > 0$ corresponds to a mutual repulsion and the case *f* corresponds to an attraction.

([1]) That *symmetry principle* means the equal justification of *relativity* of the various directions in space. If the electric field of e' were not symmetric relative to P' then one could not consider the various directions to be equivalent.

([2]) I. e, that charges of the same type mutually repel each other.

In theoretical investigations, one often sets $C_1 = C_2 = 1$, following *H. A. Lorentz*. In that way, other so-called "rational" units are introduced in place of the usual electrostatic and electrokinetic units. In what follows, we shall employ the usual units ($C_1 = C_2 = 4\pi$) exclusively.

Let O be an arbitrary fixed point, and let $OP' = \mathfrak{r}'$, $OP = \mathfrak{r}$ be the radius vectors from the points P' and P relative to O. The radius vectors $P'P = \mathfrak{R}$, can obviously be represented as the geometric difference between $\mathfrak{r}$ and $\mathfrak{r}'$:

$$\mathfrak{R} = \mathfrak{r} - \mathfrak{r}' \,. \tag{15}$$

When differentiating $\mathfrak{R}$ or any function of $\mathfrak{R}$, either of the vectors $\mathfrak{r}$ and $\mathfrak{r}'$ can play the role of the argument, while the other one can be regarded as a constant parameter. If we consider the "source point" P' (i.e., the vector $\mathfrak{r}'$) to be fixed and the "reference point" P, i.e., the vector $\mathfrak{r}$, to be variable then we will get "reference-point derivatives," which we will generally denote by the symbol ∇ (grad, div, rot). In order to notate the corresponds "source-point derivatives" ($\mathfrak{r}'$ variable, $\mathfrak{r}$ = const.), we will use primed symbols ∇' (grad′, div′, rot′). Obviously, we will have the relation:

$$\nabla' = -\nabla \tag{15.a}$$

for any function of $\mathfrak{R}$ and each type of differentiation. The vector $(1/R^3)\mathfrak{R}$ is equal to the "reference-point gradient" of the function $\frac{1}{R}\left(\frac{1}{R^3}\mathfrak{R} = -\operatorname{grad}\frac{1}{R}\right)$, as is easy to see. From (14), it will then follow that $\mathfrak{E} = -$ grad φ, where:

$$\varphi = \frac{e'}{R} \,. \tag{16}$$

That will be the usual expression for the *electric potential* of a point charge when one adds the "boundary condition" that *the potential should vanish at an infinite distance* ($R = \infty$).

The product of φ and the quantity e at a point P where charge is concentrated is equal to the *mutual potential energy* of the two charges:

$$U = \frac{e'e}{R} \,. \tag{16.a}$$

The negative gradient of that quantity relative to $\mathfrak{r}$ or $\mathfrak{r}'$ then represents the force that acts on e as a result of e' (on e' as a result of e, resp.). From (15.a), those forces are opposite in direction and have a magnitude of $e'e/R^2$.

When several charges e_1', e_2', … are present at the points P_1', P_2', …, the resultant electric field strength $\mathfrak{E}$ at the reference point P in question is equal to the geometric sum of the vectors

$\mathfrak{E}_k = \frac{e'_k}{R_k^3}\mathfrak{R}_k$ ($\mathfrak{R}_k = P'_k P$), and the resultant potential of the algebraic sum of the corresponding potentials $\varphi_k = e'_k / R_k$. If one then imagines replacing the isolated points charges with a continuous spatial distribution of electricity and denotes the charge de' that the volume element dV' is endowed with then that will give the following integral expressions for $\mathfrak{E}$ and φ:

$$\mathfrak{E} = \int \frac{\rho'}{R^3}\mathfrak{R}\, dV', \tag{17}$$

$$\varphi = \int \frac{\rho'}{R} dV'. \tag{17.a}$$

While integrating, $\mathfrak{r}$ is considered to be a constant parameter, and ρ' is regarded as a given function of the vector argument $\mathfrak{r}'$. The integration shall extend over all of space. Clearly, the locations where $\rho' = 0$ will then remain irrelevant to that process. One can easily verify that the integral (17) is equal to minus the gradient of the integral (17.a) with respect to $\mathfrak{r}$. That follows from the fact that differentiating with respect to the vector $\mathfrak{r}$ can be performed under the integral sign, just like the usual differentiation of an integral with respect to a scalar parameter.

The formula (17.a) obviously represents the solution of the differential equation (8), namely, $\nabla^2\varphi = -4\pi\rho$. In fact, one can get it by the *direct integration* of that equation while considering the *boundary conditions.*

We next suppose that the charge density ρ vanishes everywhere except for a certain point P'. The potential φ at a point P can obviously depend upon only the distance $\varphi = R$, i.e., it must be a function of the magnitude of the vector $\mathfrak{R}$, but not its direction. Since grad $\varphi(R) = \frac{d\varphi}{dR}\frac{1}{R}\mathfrak{R}$ [Introduction, (28)], we will have:

$$\nabla^2\varphi = \text{div grad } \varphi = \frac{d\varphi}{dR}\frac{1}{R}\,\text{div}\,\mathfrak{R} + \mathfrak{R}\,\text{grad}\left(\frac{1}{R}\frac{d\varphi}{dR}\right) = \frac{3}{R}\frac{d\varphi}{dR} + R\frac{d}{dR}\left(\frac{1}{R}\frac{d\varphi}{dR}\right),$$

or:

$$\nabla^2\varphi = \frac{1}{R}\frac{d^2(R\varphi)}{dR^2}. \tag{18}$$

If one sets:

$$\frac{d^2(R\varphi)}{dR^2} = 0 \tag{18.a}$$

at all points, with the possible exception of P' ($R = 0$) then it will follow from integrating twice that $R\varphi = AR + B$, i.e.:

$$\varphi = A + \frac{B}{R}. \tag{18.b}$$

The first of the two integration constants is determined from the boundary condition $\varphi = 0$ for $R = \infty$ ($A = 0$), while the second one is determined from a condition of the form $\oint \mathrm{grad}_n\, \varphi\, dS = 4\pi e'$, where e' means the charge that is concentrated at P', and S is an arbitrary (i.e., infinitely small) surface that encloses that charge. Clearly, one will have $B = e'$ in that way.

If the point P is found at a sufficient distance from P' then one can replace the latter with an infinitely-small volume with the finite charge density ρ'. Due to the linear character of the equation $\nabla^2\varphi = -4\pi\rho$, that will give its complete solution in the form of the sum of elementary solutions (18.b) that originate in the individual elementary charges $B = de' = \rho'\, dV'$, i.e., in the form of the integral (17.a). In that way, it is assumed that the point P is found "in empty space," i.e., outside of the charged volume. However, one can easily show that *this assumption is inessential for electric charges with finite volume densities.* Namely, let v be an infinitely-small volume that includes the "reference point" P in question. If the electric charge density ρ is *finite* in v then the contribution of all of the charge ρv that is contained in v to the potential φ at P must have order of magnitude $v / \sqrt[3]{v} = v^{2/3}$, so it will vanish in the limit as $v \to 0$.

One will get the same result in the case where the electric charge *is distributed with finite surface density* η *on a surface* that goes through P ([1]) since in that way the contribution to the potential φ at a point of a surface element s that originates in that element will have order of magnitude $s / \sqrt{s} = \sqrt{s} \to 0$. However, for a distribution of electricity with finite *line density*, the assumption above (viz., that P is in empty space) will be essential. The potential of a charged line at the points of the line does not have a well-defined finite value, just like the potential of a point charge at the source point.

If the charge distribution in question is restricted to a *spatial region of finite extent* then one can treat the total charge $e' = \int \rho'\, dV'$ as a point charge at infinitely-distant reference points, and approximate its potential by the formula e'/R. Therefore, the condition that was posed above (**§ 2**) for the vanishing of a "charge-free" electric field will be fulfilled. At the same time, that will provide the proof that equation (17.a) represents *the only* solution to the equation $\nabla^2\varphi = -4\pi\rho$ *for all of space* that vanishes like e'/R at infinity because one could get any other solution from it by merely adding a "charge-free" field.

§ 6. – Determining the magnetic field from the current distribution.

The magnetic field that is created by a stationary electrical current of finite spatial density $\mathbf{j}$ can be determined from the equations:

([1]) The potential is expressed by the surface integral $\varphi = \int (\eta'/R)\, dS'$ in that case.

$$\mathfrak{H} = \text{rot}\,\mathfrak{A}\,, \qquad \nabla^2\,\mathfrak{A} = -\,4\pi\,\mathfrak{j}\,, \qquad \text{div}\,\mathfrak{A} = 0\,.$$

We first consider the second of those equations. Due to its complete analogy with the equation $\nabla^2\,\varphi = -\,4\pi\,\rho$ for the scalar potential, one can immediately write down its solution in the form (17.a), in which φ must be replaced with $\mathfrak{A}$, and ρ' (viz., the electric charge density) must be replaced by the electric current density $\mathfrak{j}'$ (at the point $\mathfrak{r}'$).

However, in order for the expression that is obtained by that:

$$\mathfrak{A} = \int \frac{1}{R}\,\mathfrak{j}'\,dV' \tag{19}$$

to represent the desired vector potential of the electrical current considered, it must satisfy the condition div $\mathfrak{A}$ = 0. Now, it is easy to see that this condition is actually fulfilled. Since "div" means a differentiation relative to $\mathfrak{r}$, and $\mathfrak{j}'$ is a function of the vector $\mathfrak{r}'$, we will initially have:

$$\text{div}\,\mathfrak{A} = \int \text{div}\left(\frac{1}{R}\,\mathfrak{j}'\right) dV' \;=\; \int \mathfrak{j}' \cdot \text{grad}\,\frac{1}{R}\;dV'\,.$$

Moreover, from (15.a), one will have:

$$\mathfrak{j}'\;\text{grad}\;\frac{1}{R} = -\,\mathfrak{j}'\;\text{grad}'\;\frac{1}{R} = -\,\text{div}'\left(\frac{1}{R}\,\mathfrak{j}'\right) + \text{div}'\,\frac{1}{R}\,\text{div}\,\mathfrak{j}',$$

and as a result:

$$\text{div}\,\mathfrak{A} = -\int \text{div}'\left(\frac{1}{R}\,\mathfrak{j}'\right) dV' + \int \frac{1}{R}\,\text{div}'\,\mathfrak{j}'\;dV'\,.$$

The divergence of $\mathfrak{j}'$ must vanish in the case considered of a *stationary* electrical current. If the spatial region that it streams through is bounded by a surface S' then the vector $\mathfrak{j}'$, or at least, its normal component $\mathfrak{j}'_{n'}$, must also vanish on that surface. One will then have:

$$\text{div}\,\mathfrak{A} = -\int \text{div}'\left(\frac{1}{R}\,\mathfrak{j}'\right) dV' \;= -\int \frac{1}{R}\,\mathfrak{j}'_{n'}\;dS' = 0\,.$$

We can now calculate the magnetic field strength from (19). Namely, we have:

$$\mathfrak{H} = \text{rot}\,\mathfrak{A} = \int \text{rot}\left(\frac{1}{R}\,\mathfrak{j}'\right) dV' \;= -\int \mathfrak{j}' \times \text{grad}\left(\frac{1}{R}\right) dV'\,,$$

i.e.:

$$\mathfrak{H} = \int \frac{1}{R^3} \mathfrak{j}' \times \mathfrak{R} \; dV' = \int \frac{1}{R^2} \mathfrak{j}' \times \mathfrak{R}_0 \; dV' , \tag{19.a}$$

which is entirely-analogous to the formula (17).

The product $\mathfrak{j}' dV'$ represents the electrical impulse of the charges that are found in the volume element dV'. In the case of a *linear* current, as is known, one must replace $\mathfrak{j}' dV'$ with $i' \boldsymbol{\tau}' d\sigma'$ (where i' means the current strength, while $d\sigma'$ means the element of the current line in question). In that way, formulas (19) and (19.a) will assume the following form:

$$\mathfrak{A} = i' \oint \frac{1}{R} \boldsymbol{\tau}' \, d\sigma' , \tag{20}$$

$$\mathfrak{H} = i' \oint \frac{1}{R^2} \boldsymbol{\tau}' \times \mathfrak{R}_0 \, d\sigma' . \tag{20.a}$$

It should be noted that the formula (19) [and as a result, (19.a), as well], or the corresponding formula for the surface distribution of electric current, will also remain valid for reference points that are found inside of the volume or surface that it flows through (assuming that the spatial or surface density, resp., of the current has a finite value), just like in the case of the scalar potential. By contrast, formulas (20) and (20.a) are valid for only "external" points, i.e., ones that lie outside of the current line.

From formula (13) in the previous chapter, one can represent the mutual potential energy of two linear currents in the form of a double integral:

$$U_m = - \, i' i \oint \oint \frac{1}{R} (\boldsymbol{\tau}' \cdot \boldsymbol{\tau}) \, d\sigma' \, d\sigma = - \, i' i \oint \oint \frac{\cos\theta}{R} d\sigma' \, d\sigma , \tag{20.b}$$

in which θ means the angle between the line elements $d\sigma'$ and $d\sigma$. The symmetry in that formula relative to the primed and unprimed quantities corresponds to the fact that the potential energy of σ relative to σ' (viz., U_m) is identical to the energy U'_m of σ' relative to σ. One will then have:

$$U_m = - \, i \int H_n \, dS = - \, i' \int H'_{n'} \, dS' = U'_m . \tag{20.c}$$

The elements in the integrals (20) and (20.a), i.e., the vectors $\frac{i'}{R} \boldsymbol{\tau}' \, d\sigma'$ and $\frac{i'}{R^2} \boldsymbol{\tau}' \times \mathfrak{R}_0 \, d\sigma'$, resp., are obviously to be regarded as the infinitely-small potential ($d\,\mathfrak{A}$) and the infinitely-small magnetic field strength ($d\,\mathfrak{H}$) that originate in the current element $i' \boldsymbol{\tau}' d\sigma'$. In that way, we will get the well-known *Biot-Savart* law:

$$d\,\mathfrak{H} = \frac{i'}{R^2} \boldsymbol{\tau}' \times \mathfrak{R}_0 \, d\sigma' . \tag{21}$$

One ordinarily considers that law, which we have derived theoretically on the basis of the energy and equivalence principle at the same time as its electrostatic analogue, namely, *Coulomb's* law, to be a basic experimentally-established fact.

If one considers a current element with a charge e' that moves with a velocity of $\mathfrak{v}'$, so it possesses an electric impulse $(e'/c)\mathfrak{v}'$, then that will give the following expressions for the vector potential and strength of the magnetic field:

$$\mathfrak{A} = \frac{e'}{cR}\mathfrak{v}' = \frac{\varphi}{c}\mathfrak{v}', \tag{22}$$

$$\mathfrak{H} = \frac{e'}{cR^2}\mathfrak{v}' \times \mathfrak{R}_0 = \frac{1}{c}\mathfrak{v}' \times \mathfrak{E}, \tag{22.a}$$

in which φ means the scalar (electrostatic) potential, and $\mathfrak{E}$ means the electric field strength at the reference point in question that originates in the same charge.

The expression (20.b) for the mutual potential energy of two current lines is obviously equal to the sum of the corresponding expressions for the various charge-pairs:

$$u_m = -\frac{1}{R}\left(\frac{e'}{c}\mathfrak{v}' \cdot \frac{e}{c}\mathfrak{v}\right) = -\frac{1}{c^2}(\mathfrak{v}' \cdot \mathfrak{v})u, \tag{23}$$

where $u = e'e/R$ means the mutual energy of the charges in question. One can correspondingly consider the quantity u_m to be their *mutual magnetic energy*.

However, that interpretation proves to be incorrect, as was already discussed above (Chap. II, **§ 6**). Namely, the motion of an isolated charge creates a magnetic field that changes in time at any *fixed* reference point, while all of our arguments that were concerned with the energy principle, referred to *time-constant* fields. We can already see the breakdown of the energy principle for isolated *moving* charges from the non-vanishing of the divergence of (22), and indeed we will have:

$$\operatorname{div}\left(\frac{e'}{cR}\mathfrak{v}'\right) = \frac{e'}{c}\mathfrak{v}' \operatorname{grad}\frac{1}{R} = -\frac{e}{cR^2}\mathfrak{v}'\mathfrak{R}_0 .$$

However, that breakdown will emerge clearly when we calculate the forces that the two charges exert upon each other from (23). From the formulas:

$$\mathfrak{f} = -\operatorname{grad} u_m = +\operatorname{grad}' u_m = -\mathfrak{f}',$$

we will get:

$$\mathfrak{f} = -\mathfrak{f}' = -\frac{e e'}{R^2}\frac{\mathfrak{v}\mathfrak{v}'}{c^2}\mathfrak{R}_0 . \tag{23.a}$$

In reality, the force that acts on e as a result of e' is equal to:

$$\mathfrak{f} = \frac{e}{c}\,\mathfrak{v} \times \mathfrak{H} = \frac{e}{c}\,\mathfrak{v} \times \left(\frac{e'}{c}\,\mathfrak{v}' \times \frac{1}{R^2}\,\mathfrak{R}_0 \right) ,$$

i.e.:

$$\mathfrak{f} = \frac{e e'}{R^2}\frac{1}{c^2}[(\mathfrak{v}\mathfrak{R}_0)\mathfrak{v}' - (\mathfrak{v}\mathfrak{v}')\mathfrak{R}_0] , \tag{23.b}$$

and the force on e' as a result of e will be:

$$\mathfrak{f}' = - \frac{e e'}{R^2}\frac{1}{c^2}[(\mathfrak{v}'\mathfrak{R}_0)\mathfrak{v} - (\mathfrak{v}\mathfrak{v}')\mathfrak{R}_0] , \tag{23.c}$$

correspondingly.

Those forces do not satisfy the principle of the equality of action and reaction then (which can be regarded as an immediate consequence of the energy principle. They will be equal and opposite only the case where the velocities $\mathfrak{v}$ and $\mathfrak{v}'$ are equal to each other or when they are perpendicular to the connecting line $\mathfrak{R}$.

When those two conditions are fulfilled at the same time, formulas (23.b) and (23.c) will reduce to (23.a), and we will get:

$$\mathfrak{f} = - \mathfrak{f}' = - \frac{e e'}{R^2}\frac{v^2}{c^2}\mathfrak{R}_0 .$$

In this case, the electromagnetic (or "electrokinetic") forces that are generated by the motion of the charges have the opposite direction to the corresponding electrostatic ones and are equal to the product of the latter with the ratio v^2 / c^2. We then see that two charges (e.g., electrons) that move with the same velocity $\mathfrak{v}$ perpendicular to their connecting line will exert a total force on each other of:

$$\frac{e e'}{R^2}\left(1 - \frac{v^2}{c^2}\right)$$

that one can interpret as a "weakened" electrostatic force. That total force must vanish for $v = c$, i.e., the electrostatic and electrokinetic forces must mutually compensate each other.

That result shows that the quantity c, which was initially introduced in connection with the equivalence principle as the ratio of the electrostatic unit to the electromagnetic one, has the meaning of a certain *"critical" velocity*. The actual determination of the ratio above shows that $c = 3 \times 10^{10}$ cm / s = 300,000 km per second. *The "critical velocity" then agrees precisely with the speed of light*. Of course, that agreement is no coincidence. Rather, it relates to the electromagnetic nature of light phenomena, on the one hand, and the finite speed of propagation of electromagnetic effects, on the other. We will deal with that question in the next part of this book. Here, we shall mention only the fact that the formulas above can be regarded as only approximate formulas for isolated charges that overlook the time variation of the fields that they create and have meaning for only small velocities ($v / c \ll 1$).

§ 7. – The graphical representation of electric and magnetic fields.

The lines that represent the direction and density of the electric field graphically will be called "lines of electric force." Correspondingly, the lines that represent the magnetic field are called "lines of magnetic force."

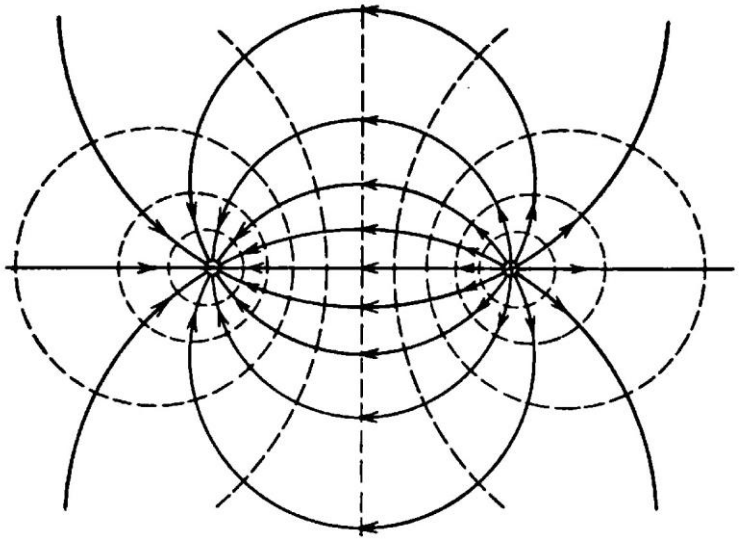


Figure 21.

It will follow from the vortex-free character of the electric field, i.e., from the fact that the condition rot $\mathfrak{E} = 0$ is fulfilled *in all of space*, that the lines of electric force are *not closed lines*. (For a closed line, the integral $\oint E_\tau \, d\sigma = \int \mathrm{rot}_n \, \mathfrak{E} \, dS$ would have a non-zero value.) For that reason, they must begin and end at some sort of point. The number of electric force lines that emanate from a volume V or converge inside of it is proportional to the electric flux through the outer surface S that bounds that volume, i.e., the integral $\oint E_\tau \, d\sigma$. (The lines of force that go through V will not contribute to that integral.) It follows from the equation $\oint E_\tau \, d\sigma = 4\pi e$ or its equivalent equation div $\mathfrak{E} = 4\pi\rho$ that the source points of the electric lines of force will coincide with positive charges and the sinks will coincide with the negative charges, and that the number of lines that "diverge" from a charge ("converge" to it, resp.) will be proportional to the magnitude e of the charge. In "empty space," the electric lines of force can neither begin nor end.

In practice, the electric field in the immediate neighborhood of a point charge is determined by *that* charge exclusively and is independent of other distant charges. We must then find the same line-like picture of the force lines here that we had in the case of an "isolated" point charge. The force lines will define a "line bundle" that is uniform in all directions. However, at some distance from e, they will become curved according to the configuration of the other charges. Opposite charges with the same absolute value will correspond to the same number of diverging (converging, resp.) lines. Therefore, in the case of a dipole, the force lines that emanate from the positive end must all come together at the negative end (Fig. 21). In that case (just like in the

general case of an arbitrary neutral system of electric charges), the electric lines of force will remain "inside" of that system, so to speak; none of them will go to infinity. When the total charge of the system is non-zero, a number of lines of force that correspond to that resultant charge must go to infinity (or come in from it), and indeed in the same radially-symmetric way as if the system considered were converging to a point charge. In fact, such cases cannot occur since matter is neutral on the whole.

Along with the lines of force, in order to make the electric field more intuitive, one can consider the "equipotential" or "level surfaces" that are orthogonal to those lines, i.e., the surfaces $\varphi = \text{const}$. The distance between two such surfaces that correspond to two slightly-different values of the potential φ is obviously inversely proportional to the electric field strengths at the points in question (due to the relation $E = -\partial\varphi / \partial n$, where ∂n means the length of the normal that is included between the two surfaces). In the vicinity of the individual point-charges, the level surfaces will be spherical. They can deform arbitrarily as the distance increases according to the configuration of the other charges.

In contrast to the electric lines of force, the *magnetic lines of force must always be closed.* That follows immediately from the source-free character of the magnetic field, i.e., the validity of the equation $\operatorname{div} \mathfrak{H} = 0$ for all of space. The magnetic lines of force can possibly go from infinity to infinity, i.e., they can close at a "point at infinity." One does not need to consider that case in detail then. Moreover, it follows from the relations $\oint H_\tau \, d\sigma = 4\pi i$ or $\operatorname{rot} \mathfrak{H} = 4\pi \mathbf{j}$ that in the case of a stationary electric current the magnetic lines of force will always *enclose* the current lines, which are also closed curves, due to the fact that $\operatorname{div} \mathbf{j} = 0$. That is because otherwise the integral would have a non-zero value along a closed line of force that does not enclose any current line, while the strength of the current i that flows through those lines would be equal to zero. In other words: One can consider the current lines to be the "vortex axes" of the magnetic lines of force, i.e., as annular axes that will be enclosed by the lines of force, say, like the way that a link in a chain is enclosed by the neighboring links.

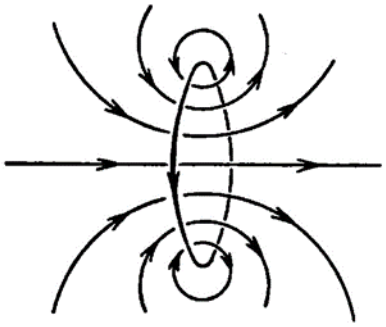


Figure 22.

In the immediate neighborhood of an element $\Delta\sigma'$ of a current line σ', i.e., at a distance from $\Delta\sigma'$ that is very small in comparison to the length of the element, and at the same time, to its radius of curvature, one can treat the element as a "rectilinear vortex axis." It follows on grounds of symmetry that the magnetic lines of force must be *circular* here. We will then get a family of *coaxial circles* as the graphic representation of the magnetic field. That *cylindrical symmetry* of the magnetic field in the neighborhood of a current element corresponds to the spherical symmetry of the electric field in the neighborhood of a point charge.

The integral $\oint H_\tau \, d\sigma$ is obviously equal to $2\pi r H$ for such a circular line of force with a radius of r. (The direction of integral must then correspond to the direction of the current in the sense of the right-hand screw rule.) It follows from this that $2\pi r H = 4\pi i'$, in which i' means the current strength, i.e.:

$$H = \frac{2i'}{r} . \tag{24}$$

The magnetic field strength in the immediate neighborhood of a current line is therefore inversely proportional to the first power of the distance. One can consider that rule to be the analogue of the *Coulomb* law for *current lines* (into which the stationary currents can be decomposed). The motion of *isolated* charges (i.e., electrons) does not represent a stationary process, and the law of *Biot-Savart*, which determines the magnetic field of such charges (or current elements) with no concern for its time variation, can therefore be regarded as only an approximate law. However, insofar as those current elements will be combined into a stationary linear current, the geometric sum of their *Biot-Savart* field strengths will be equal to *precisely* the resultant (time-constant) field strengths.

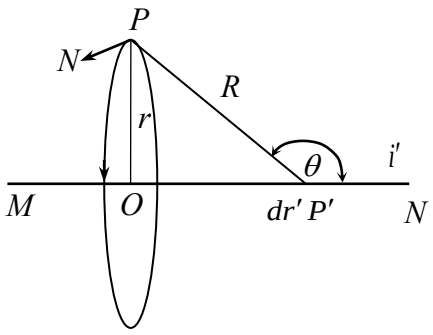


Figure 23.

The formula (24) can be derived from the *Biot-Savart* formula (21) for an *infinitely-long* (compared to *r*) *rectilinear* current (Fig. 23). Let *PO* be the altitude from the reference point *P* to the current line *MN* such that we will have $OP = \mathfrak{r}$, $OP' = \mathfrak{r}'$, $P'P = \mathfrak{R}$, with our usual notations. From (21), the field strength that is created by the current element $d\sigma'$ is:

$$dH = i' \frac{\sin\theta}{R^2} d\sigma' ,$$

in which θ means the angle $P P' N$. Since the direction of $d\mathfrak{H}$ is the same for all current elements (at the point in question, it is perpendicular to the plane of the page that the reader is viewing), the geometric integration reduces to the usual one: $H = \int dH$. Since $R = r / \sin\theta$ and $r' = -r \cot\theta$, we will have:

$$d\sigma' = dr' = -\frac{r}{\sin^2\theta} d\theta , \quad dH = \frac{i'}{r} \sin\theta \, d\theta ,$$

and as a result:

$$H = \frac{i'}{r} \int_0^{\pi} \sin\theta \, d\theta = \frac{2i'}{r} ,$$

which agrees with (24).

The magnetic field considered corresponds to a vector potential of the following form:

$$\mathfrak{A} = -2 \ln \frac{r}{a} \mathfrak{i} = 2 \ln \frac{a}{r} \mathfrak{i} , \tag{24.a}$$

in which $\mathfrak{i}$ means a vector whose magnitude and direction agree with the current strength, r means the distance from the reference point to the current line (but not from a well-defined point on it), and a means an arbitrary constant with the dimension of length. In fact, according to the general formula $\mathfrak{H} = \text{rot}\,\mathfrak{A}$, (24.a) will imply that:

$$\mathfrak{H} = -2\,\mathfrak{i} \times \text{grad} \ln \frac{a}{r} = 2\,\mathfrak{i} \times \frac{1}{r^2}\mathfrak{r} = \frac{2}{r}\mathfrak{i} \times \mathfrak{r}_0 \ .$$

That vector obviously represents the "cylindrical" field that was considered above. As far as the lines that represent the vector potential are concerned, they are the lines parallel to the cylinder axis. One sees from this that in the immediate neighborhood of a current line of entirely-arbitrary form, the "vector potential lines" define a "thread" that is parallel to it. However, in general, they are not suitable for an intuitive representation of the magnetic field.

§ 8. – The fields and interactions of elementary dipoles and currents.

The electrostatic ("scalar") potential φ of an arbitrary dipole $P_1'P_2'$ at a reference point P is obviously equal to the sum of the potentials of its two ends. If one denotes the corresponding charges by $e_1 = -\,e'$ and $e_2 = +\,e'$ and further sets $P_1'P = \mathfrak{R}_1$, $P_2'P = \mathfrak{R}_2$ then one will have:

$$\varphi = \varphi_1 + \varphi_2 = e'\left(\frac{1}{R_2} - \frac{1}{R_1}\right) .$$

In the case of an *elementary* dipole whose length $P_1'P_2' = \mathfrak{l}'$ is very small compared to the distance R_1 and R_2 one will then have $\frac{1}{R_2} - \frac{1}{R_1} = \mathfrak{l}'\,\text{grad}'\,\frac{1}{R}$, approximately, and as a result:

$$\varphi = \mathfrak{p}'\,\text{grad}'\,\frac{1}{R} = -\,\mathfrak{p}'\,\text{grad}\,\frac{1}{R}, \tag{25}$$

in which $\mathfrak{p}' = e'\,\mathfrak{l}'$ means the electric moment of the dipole, and R means its distance to the reference point. Since we have introduced that distance, we must treat the two ends of the dipole as a "double point" P' $(= P_1'P_2')$. One also calls such a double point a "double source" of the electric field.

The differentiations that are suggested by the symbols grad′ and grad refer to the vectors $\mathfrak{r}' = OP'$ and $\mathfrak{r} = OP$, in the usual way, where O is an arbitrary fixed point in space ($\mathfrak{R} = \mathfrak{r} - \mathfrak{r}'$). If one performs those differentiations then that will give:

$$\varphi = \frac{\mathfrak{p}'\mathfrak{R}}{R^3} = \frac{\mathfrak{p}'\mathfrak{R}_0}{R^2} = \frac{p'\cos\theta}{R^2}, \tag{25.a}$$

in which θ means the angle between $\mathfrak{p}'$ and $\mathfrak{R}$ ($\mathfrak{R}_0 = \mathfrak{R} / R$).

From formulas (30) and (31) (Introduction), one further finds that:

$$\mathfrak{E} = -\operatorname{grad}\varphi = -\frac{1}{R^3}\mathfrak{p}' + \frac{3}{R^5}(\mathfrak{p}'\mathfrak{R})\mathfrak{R},$$

i.e.:

$$\mathfrak{E} = \frac{1}{R^3}\{3(\mathfrak{R}_0\,\mathfrak{p}')\mathfrak{R}_0 - \mathfrak{p}'\} = \frac{1}{R^3}\left\{\frac{3}{R^2}(\mathfrak{R}\,\mathfrak{p}')\mathfrak{R} - \mathfrak{p}'\right\}. \tag{26}$$

One can see from that formula that the electric field strength lies in the plane $\mathfrak{p}'$, $\mathfrak{R}$ at the point P. Upon inner and outer multiplication of (26) by $\mathfrak{R}_0$, one will get the radial and azimuthal components of the vector E (the latter is in the direction of increase of the angle θ):

$$E_R = \frac{2p'}{R^3}\cos\theta, \tag{26.a}$$

$$E_\theta = \frac{p'}{R^3}\sin\theta. \tag{26.b}$$

The ratio $E_\theta / E_R = \frac{1}{2}\tan\theta$ is obviously equal to the tangent of the angle between $\mathfrak{E}$ and $\mathfrak{R}$.

If one replaces $\mathfrak{E}$ with $\mathfrak{H}$ and $\mathfrak{p}'$ with $\mathfrak{m}'$ in formulas (26), (26.a), and (26.b) then according to the equivalence principle, they will determine the magnetic field that is created by *an elementary current* with a magnetic moment of $\mathfrak{m}'$. One can then consider the corresponding field strength $\mathfrak{H}$ to be the (negative) gradient of a *scalar magnetic* potential:

$$\varphi_m = \frac{\mathfrak{m}'\mathfrak{R}_0}{R^2} = -\,\mathfrak{m}'\operatorname{grad}\frac{1}{R}.$$

On the other hand, that field strength can be represented in the general form $\mathfrak{H} = \operatorname{rot}\mathfrak{A}$, where $\mathfrak{A}$ means the vector potential of the elementary current considered. In order to determine that vector potential, one can start from the fact that the differential equation:

$$\operatorname{rot}\mathfrak{A} = -\operatorname{grad}\varphi_m = \operatorname{grad}\left(\mathfrak{m}'\operatorname{grad}\frac{1}{R}\right)$$

must be satisfied in all of space, *with the exclusion of the point* P' (where the current is thought to be concentrated). When one recalls the formulas (26) and (27) (Introduction), due to the constancy of $\mathfrak{m}'$, one will have:

$$\text{rot}\left(\mathfrak{m}'\times\text{grad}\frac{1}{R}\right) = -\,(\mathfrak{m}'\ \text{grad})\ \text{grad}\ \frac{1}{R} + \mathfrak{m}'\ \text{div grad}\ \frac{1}{R},$$

$$\text{grad}\left(\mathfrak{m}'\,\text{grad}\frac{1}{R}\right) = (\mathfrak{m}'\ \text{grad})\ \text{grad}\ \frac{1}{R} + \mathfrak{m}'\times\text{rot grad}\ \frac{1}{R}.$$

Now, since the expressions rot grad $(1/R)$ and div grad $(1/R) = \nabla^2\,(1/R)$ both vanish (the first one identically, and the second one, everywhere outside of the point P'), one will have:

$$\text{rot}\left(\mathfrak{m}'\times\text{grad}\frac{1}{R}\right) = -\ \text{grad}\left(\mathfrak{m}'\,\text{grad}\frac{1}{R}\right).$$

A comparison of that equation with the differential equation for $\mathfrak{A}$ above will give:

$$\mathfrak{A} = -\,\mathfrak{m}'\times\text{grad}\ \frac{1}{R} = \frac{1}{R^3}\,\mathfrak{m}'\times\mathfrak{R} = \frac{1}{R^2}\,\mathfrak{m}'\times\mathfrak{R}_0\,. \tag{27}$$

We next observe that $\tau'\,d\sigma' = d\,\mathfrak{r}'$. Moreover, since the vector grad $(1/r)$ remains constant $(=\mathfrak{k})$ during the integration, we will have:

$$\oint d\,\mathfrak{r}'(\mathfrak{r}'\mathfrak{k}) + \oint\mathfrak{r}'(d\mathfrak{r}'\mathfrak{k}) = \oint d\,\{\mathfrak{r}'(\mathfrak{r}'\mathfrak{k})\} = 0\,.$$

That will give:

$$\oint d\,\mathfrak{r}'(\mathfrak{r}'\mathfrak{k}) = -\oint\mathfrak{r}'(d\mathfrak{r}'\mathfrak{k}) = \oint\{d\,\mathfrak{r}'(\mathfrak{r}'\mathfrak{k}) - \mathfrak{r}'(d\mathfrak{r}'\mathfrak{k})\} = \tfrac{1}{2}\oint\mathfrak{k}\times(d\mathfrak{r}'\times\mathfrak{r}') = \tfrac{1}{2}\,\mathfrak{k}\times\oint d\mathfrak{r}'\times\mathfrak{r}',$$

i.e., from (6.a), Chap. II:

$$\mathfrak{A} = -\,i\oint d\,\mathfrak{r}'(\mathfrak{r}'\mathfrak{k}) = \text{grad}\,\frac{1}{r}\times\frac{1}{2}\oint\mathfrak{r}'\times i\,\tau'd\sigma' = \text{grad}\ \frac{1}{r}\times\mathfrak{m}',$$

in agreement with (27) ([1]).

([1]) One can arrive at this formula even more simply by means of the identity $\oint\tau'\psi\,d\sigma' = \int\mathfrak{n}'\times\text{grad}'\psi\,dS'$ [see Introduction, formula (17.a)]. Namely, if one sets $\psi = 1/R$ then if σ' is very small compared to R, that will give:

We now propose that a second elementary dipole (or current) with a moment of $\mathfrak{p}$ ($\mathfrak{m}$, resp.) is found at the reference point *P*. From (26), its potential energy relative to the first dipole $U = -\mathfrak{p}\mathfrak{E}$ can be written in the form:

$$U = \frac{1}{R^3}\{(\mathfrak{p}\ \mathfrak{p}') - 3\,(\mathfrak{R}_0\,\mathfrak{p})\,(\mathfrak{R}_0\,\mathfrak{p}')\} = \frac{1}{R^3}\left\{\mathfrak{p}\mathfrak{p}' - \frac{3}{R^2}(\mathfrak{R}\,\mathfrak{p})(\mathfrak{R}\,\mathfrak{p}')\right\}, \qquad (28)$$

whose symmetry in regard to $\mathfrak{p}$ and $\mathfrak{p}'$ corresponds to the equality of the energy of each dipole relative to the other one. We will then have $U = -\mathfrak{p}\mathfrak{E} = -\mathfrak{p}'\mathfrak{E}'$, in which $\mathfrak{E}'$ means the field strength that is created by the second dipole at the point P'. For that reason, we can refer to the quantity *U* as simply the *mutual potential energy* of the dipoles considered.

The *moment* that acts upon the second dipole can be calculated immediately by using the formula $\mathfrak{M} = \mathfrak{p} \times \mathfrak{E}$, and indeed, one will have:

$$\mathfrak{M} = \frac{1}{R^3}\{3\,(\mathfrak{p}'\ \mathfrak{R}_0)\,(\mathfrak{p} \times \mathfrak{R}_0) + \mathfrak{p} \times \mathfrak{p}'\}\ . \qquad (28.a)$$

The corresponding *force* is expressed by $\mathfrak{F} = (\mathfrak{p}\ \mathrm{grad})\ \mathfrak{E}$ or $\mathfrak{F} = -\ \mathrm{grad}\ U$. If we substitute the last formula in the expression (28) then that will give, after a simple calculation:

$$\mathfrak{F} = \frac{3}{R^5}(\mathfrak{p}\ \mathfrak{p}')\,\mathfrak{R} + \frac{3}{R^5}(\mathfrak{R}\ \mathfrak{p}')\,\mathfrak{p} + \frac{3}{R^5}(\mathfrak{R}\ \mathfrak{p}')\,\mathfrak{p} - \frac{15}{R^7}(\mathfrak{R}\ \mathfrak{p})\,(\mathfrak{R}\ \mathfrak{p}')\,\mathfrak{R}$$

or

$$\mathfrak{F} = \frac{3}{R^4}\{[\mathfrak{p}\ \mathfrak{p}' - 5\,(\mathfrak{R}_0\ \mathfrak{p})\,(\mathfrak{R}_0\ \mathfrak{p}')]\,\mathfrak{R}_0 + (\mathfrak{R}_0\ \mathfrak{p}')\,\mathfrak{p} + (\mathfrak{R}_0\ \mathfrak{p})\,\mathfrak{p}'\}. \qquad (28.b)$$

If one replaces the vector $\mathfrak{R}$ in (28.a) with the opposite vector $\mathfrak{R}' = PP' = \mathfrak{r}' - \mathfrak{r} = -\mathfrak{R}$ ($\mathfrak{R}'_0 = -\mathfrak{R}_0$) then one will get the moment $\mathfrak{M}'$ and the force $\mathfrak{F}'$ that the second dipole exerts on the first one. In that way, one will have $\mathfrak{M}' = -\mathfrak{M}$ and $\mathfrak{F}' = -\mathfrak{F}$, in agreement with the principle of the equality of "action and reaction," which is a direct consequence of the energy principle ([1]).

The same result will be obtained for the interaction of the elementary currents. We would need only to replace the electric moments $\mathfrak{p}$ and $\mathfrak{p}'$ with the magnetic ones $\mathfrak{m}$ and $\mathfrak{m}'$ in the formulas that were just derived.

$$i\oint \frac{\tau'}{R}\,d\sigma' = i\int \mathfrak{n}' \times \mathrm{grad}'\,\frac{1}{R}\,dS' \approx \mathrm{grad}\,\frac{1}{R} \times\ i\int \mathfrak{n}'\,dS' = \mathrm{grad}\,\frac{1}{R} \times \mathfrak{m}.$$

([1]) It should be noted that according to (28.a) and (28.b), the torque is inversely proportional to the *third* power of the mutual distance, and the force is inversely proportional to the *fourth* power.

Those formulas will also remain valid, in a certain sense, when one considers isolated *magnetons* instead of the elementary currents, i.e., isolated electrons that orbit around a certain center. From (6.a) and (6.b), Chap. II, the magnetic moment of such a magneton is equal to:

$$\mathfrak{m} = \frac{1}{2}\frac{e}{c}\,\mathfrak{r} \times \mathfrak{v}, \tag{29}$$

in which $\mathfrak{r}$ means the radius vector of the path, and $\mathfrak{v} = d\,\mathfrak{r} / dt$ means its velocity. The magnitude of the vector $\frac{1}{2}\mathfrak{r} \times \mathfrak{v}$ is equal to the *areal velocity* of the electron, i.e., the area that is described by the radius vector per unit time. If the motion of the electron takes place under the action of a force of attraction that points to the center of the atom *O* then that areal velocity must remain constant, just like the orbital plane. In that case ($\mathfrak{m}$ = const.), the "magneton" will be completely equivalent to a *stationary* elementary current. One must only understand $\mathfrak{A}$, $\mathfrak{H}$, etc., to mean the *temporal means* of the corresponding quantities for one or more orbits of the electron in question and leave the fluctuations that appear during an orbit out of consideration (cf., Chap. VII, **§ 9**).

§ 9. – The scalar potential of a non-elementary linear current.

The concept of a scalar magnetic potential that we applied to elementary currents above can be adapted to non-elementary linear currents. In order to do that, we must first replace the current considered (i', σ') with a network of elementary currents. From (25.a), the scalar potential of an elementary current with the magnetic moment $d\,\mathfrak{m} = i'\,\mathfrak{n}'\,dS'$ is equal to:

$$d\varphi_m = i'\frac{\cos\theta}{R^2}\,dS'\,.$$

The θ in that means the angle between the normal $\mathfrak{n}$ and the radius $\mathfrak{R}$ that is drawn from dS' to the reference point *P*. The product $\cos\theta\; dS'$ is then equal to the projection of the surface element dS' onto a spherical surface with center *P* that cuts that surface element. The ratio $\frac{\cos\theta}{R^2}\,dS'$ is then nothing but the *solid angle* $d\,\Omega'$ that dS' subtends at *P*. As a result, upon integrating, we will get:

$$\varphi_m = i'\,\Omega'\,. \tag{30}$$

It should be remarked that the sign of $d\,\Omega'$ will be positive (negative, resp.) according to whether the corresponding surface element is seen from the positive or negative side, resp.

One can define Ω' independently of the form of the surface S' to be the solid angle that is defined by a cone of rays that go from *P through the current line* σ'. However, one must observe that such a cone actually bounds *two* complementary solid angles, namely, Ω' and $4\pi - \Omega'$. In order to determine the potential (30), both angles must be used, according to whether the surface S' lies on one side of the reference point P or the other, but with *opposite signs*. If one then replaces the surface S' with another one S'' (Fig. 24) then one must replace the angle $\Omega' > 2\pi$ with $\Omega'' = -(4\pi - \Omega') = \Omega' - 4\pi$. That rule will also remain valid when one shifts the reference point P from one side of a given surface S' to the other. Under a shift from the negative to the positive, Ω' will jump by an amount 4π. That corresponds to jump in the potential φ_m by $4\pi i'$. If one would like to treat that potential as a *continuous* quantity then one would have to imagine that the transition through the surface S' is *obstructed*. One would then go from the positive side to the negative one only by the detour of an "almost"-closed line σ that actually surrounds the current line once. Since one would then have:

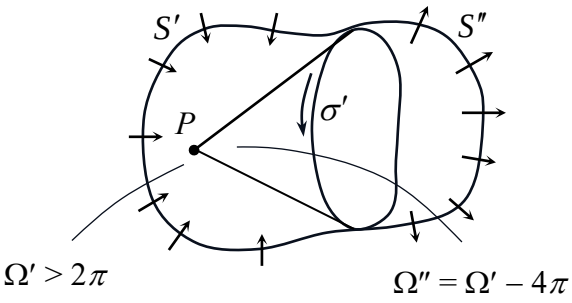


Figure 24.

$$H_\tau \, d\sigma = -\operatorname{grad}_\tau \varphi_m \, d\sigma = -\frac{\partial \varphi}{\partial \sigma} d\sigma = -d\varphi ,$$

it would follow that $\int H_\tau \, d\sigma = \int d\varphi = 4\pi i'$. We have already obtained that result in **§ 3**, and especially in **§ 4**, when we considered the electric field of the corresponding electric double layer instead of the magnetic field of the current in question. The "discontinuity surface" that was introduced above will then be nothing but the "double surface" of that layer.

From the formula (12), the electric field strength inside of such a double layer is equal to $4\pi\eta$ (where η means the surface density of the electric charge) and points from the positive to the negative side. From (30), the electric field outside of the double layer is obtained from the scalar potential:

$$\varphi = i'_e \Omega' , \tag{30.a}$$

in which $i'_e = \eta l$ is the electric moment per unit area of the layer.

An electric double layer does not need to be bounded by a closed curve σ'. It can also be closed, i.e., it can be defined by two closed surfaces with opposite charges, one of which is inside the other. Obviously, one can consider that case to be a limiting case of a vanishing boundary curve for a non-vanishing surface S'. It follows immediately from this that the electric field strength must vanish outside of the closed double layer. As far as the potential φ is concerned, it will be equal to zero externally and equal to the constant quantity $\pm 4\pi i$ internally. The upper sign is valid

for the case in which the internal surface is positive charged, while the negative sign is valid in the opposite case.

§ 10. – Electric and magnetic polarization and polarization potentials.

One can imagine that a *neutral* system of electric charges that is found in a bounded volume is replaced (and indeed in an infinitude of ways, in general) with an equivalent system of electric dipoles that also fill up that volume. For the sake of simplicity, we would like to assume that the electric charges are distributed continuously in the volume considered V and on the outer surface S that bounds it with a density of ρ (η, resp.). The *electric moment* must be correspondingly distributed in the equivalent system of dipoles. The volume density of that moment is called the *electric polarization*. If one denotes it by $\mathfrak{P}$ then the product $\mathfrak{P}\, dV$ will mean the resultant moment of the elementary dipoles that are found in the volume element dV.

We would now like to exhibit the relationship between ρ and $\mathfrak{P}$. We can do that in two ways, namely, first of all, by comparing the effect that a given external field ($\mathfrak{E}'$) exerts on the two systems, and secondly by comparing the fields ($\mathfrak{E}$) that they create outside of V. We will denote the actual system by C and the replacement (i.e., dipole) system by D.

The potential energy of C relative to the external field is expressed by the formula:

$$U = \int \varphi' \rho \, dV + \oint \varphi' \eta \, dS ,$$

in which φ' is the potential of that field. On the other hand, we will have:

$$U = - \int \mathfrak{E}' \mathfrak{P} \, dV$$

for the potential of D. If we substitute $\mathfrak{E}' = -$ grad φ' in that then from the identity:

$$\text{div } \varphi' \, \mathfrak{P} = \varphi' \text{ div } \mathfrak{P} + \mathfrak{P} \text{ grad } \varphi' ,$$

we will have:

$$U = \int \text{div}\,(\varphi' \mathfrak{P})\, dV - \int \varphi' \,\text{div}\, \mathfrak{P}\, dV = \oint \varphi' P_n \, dS - \int \varphi' \,\text{div}\, \mathfrak{P}\, dV .$$

In order for C and D to actually be equivalent, the following relations must then exist:

$$\rho = - \text{div } \mathfrak{P} , \tag{31}$$

$$\eta = P_n . \tag{31.a}$$

We would now like to verify whether the fields that are created by C and D will also become equivalent in that way. The volume element dV' (with radius vector $\mathfrak{r}'$), when considered to be a dipole with a moment of $\mathfrak{P}'\,dV'$, will create the potential $\mathfrak{P}'\,dV' \cdot \operatorname{grad}'(1/R)$ at a reference point with the radius vector $\mathfrak{r}$, where $\mathfrak{R} = \mathfrak{r} - \mathfrak{r}'$. The complete potential of D is expressed by the integral:

$$\varphi = \int \mathfrak{P}' \operatorname{grad}' \frac{1}{R}\, dV' .$$

By means of the identity:

$$\operatorname{div}'\left(\frac{\mathfrak{P}'}{R}\right) = \frac{1}{R}\operatorname{div}'\mathfrak{P}' + \mathfrak{P}' \operatorname{grad}' \frac{1}{R},$$

that integral can be represented in the form:

$$\varphi = \oint \frac{P_n'}{R}\, dS' - \int \frac{\operatorname{div}\mathfrak{P}'}{R}\, dV .$$

However, from (31) and (31.a), that is nothing but the potential of C:

$$\varphi = \oint \frac{\eta\, dS'}{R} + \int \frac{\rho'\, dV'}{R}.$$

If one further replaces $\operatorname{grad}'(1/R)$ with $-\operatorname{grad}(1/R)$ then since $\mathfrak{P}'$ depends upon only $\mathfrak{r}'$, but not upon $\mathfrak{r}$, one will have:

$$\mathfrak{p}' \operatorname{grad}' \frac{1}{R} = -\,\mathfrak{p}' \operatorname{grad} \frac{1}{R} = -\operatorname{div} \frac{\mathfrak{P}'}{R},$$

and as a result:

$$\varphi = -\int \operatorname{div}\frac{\mathfrak{P}'}{R}\, dV' = -\operatorname{div}\int \frac{\mathfrak{P}'}{R}\, dV' .$$

We would like to call the vector:

$$\mathfrak{Z} = \int \frac{\mathfrak{P}'}{R}\, dV' \tag{32}$$

the *electric polarization potential* (it is usually called the *Hertz* vector). The scalar potential φ is expressed in terms of $\mathfrak{Z}$ in the same way that ρ is expressed in terms of $\mathfrak{P}$, namely:

$$\varphi = -\operatorname{div}\mathfrak{Z} . \tag{32.a}$$

The polarization potential of an elementary dipole with the moment $\mathfrak{p}$ is obviously equal to:

$$\mathfrak{Z} = \frac{\mathfrak{p}}{R}. \tag{32.b}$$

Entirely analogous considerations and formulas can be applied to a system of *stationary currents*. If one replaces ρ with $\mathfrak{j}$ (viz., the current density on the surface *S*) in the foregoing argument and replaces η and $\mathfrak{k}$ (i.e., the current density on the surface *S*) then that will give:

$$U = -\int \mathfrak{A}' \cdot \mathfrak{j}\, dV - \oint \mathfrak{A}' \cdot \mathfrak{k}\, dS$$

for the potential energy of the system in question in an external magnetic field $\mathfrak{H}'$ with the vector potential $\mathfrak{A}'$. We now introduce a replacement system *D* that consists of elementary *magnetic dipoles* that are distributed continuously over the volume *V*. We denote the "magnetic polarization," i.e., the magnetic moment per unit volume, by $\mathfrak{M}$. The potential energy of *D* must then be expressed by the formula:

$$U = -\int (\mathfrak{H}' \mathfrak{M})\, dV\ .$$

By means of the identity:

$$\text{div}\ (\mathfrak{A}' \times \mathfrak{M}) = \mathfrak{M}\ \text{rot}\ \mathfrak{A}' - \mathfrak{A}'\ \text{rot}\ \mathfrak{M}$$

[Introduction, formula (25)], and since rot $\mathfrak{A}' = \mathfrak{H}'$, we will get:

$$-\int (\mathfrak{H}' \cdot \mathfrak{M})\, dV = -\int \text{div}(\mathfrak{A}' \times \mathfrak{M})\, dV - \int \mathfrak{A}'\ \text{rot}\ \mathfrak{M}\, dV\ ,$$

i.e.:

$$U = -\int \mathfrak{A}'\ \text{rot}\ \mathfrak{M}\, dV - \oint (\mathfrak{A}' \times \mathfrak{M})_n\, dS\ .$$

The equivalence of *C* and *D* is then guaranteed by the conditions:

$$\mathfrak{j} = \text{rot}\ \mathfrak{M} \tag{33}$$

and

$$\mathfrak{k} = \mathfrak{M} \times \mathfrak{n}. \tag{33.a}$$

[The last formula is obtained when one recalls the identity $(\mathfrak{A}' \times \mathfrak{M}) \cdot \mathfrak{n} = \mathfrak{A}' \cdot (\mathfrak{M} \times \mathfrak{n})$.]

The vector potential of an elementary current (or magnetic dipole) with a moment of $\mathfrak{M}'\, dV'$ is equal to $\mathfrak{M}'\, dV' \times \text{grad}' \frac{1}{R}$. As a result, the magnetic field of our replacement system (outside *S*) is determined by the formula:

$$\mathfrak{A} = \int \mathfrak{M}' \times \operatorname{grad}' \frac{1}{R}\, dV' . \tag{34}$$

That formula can be easily transformed into the form:

$$\mathfrak{A} = \oint \frac{\mathfrak{M}' \times \mathfrak{n}}{R}\, dS' + \int \frac{\operatorname{rot} \mathfrak{M}'}{R}\, dV , \tag{34.a}$$

which coincides with the corresponding formula for the system of currents C, due to (33) and (33.a). One can write the expression (34) for $\mathfrak{A}$ as follows:

$$\mathfrak{A} = -\int \mathfrak{M}' \times \operatorname{grad}' \frac{1}{R}\, dV' = \int \frac{\operatorname{rot} \mathfrak{M}'}{R}\, dV' = \operatorname{rot} \int \frac{\mathfrak{M}'}{R}\, dV' .$$

If one defines the vector:

$$\mathfrak{Z}^* = \int \frac{\mathfrak{M}'}{R}\, dV' \tag{35}$$

to be the *magnetic polarization potential* of the system of currents in question then one can determine the ordinary vector potential from that by a formula that is completely analogous to (33):

$$\mathfrak{A} = \operatorname{rot} \mathfrak{Z}^* . \tag{35.a}$$

The polarization potentials $\mathfrak{Z}$ and $\mathfrak{Z}^*$ obviously satisfy the differential equations:

$$\nabla^2 \mathfrak{Z} = -4\pi \mathfrak{P} , \tag{36}$$

$$\nabla^2 \mathfrak{Z}^* = -4\pi \mathfrak{M} . \tag{36.a}$$

It should be pointed out that despite the similarity of the formulas:

$$\rho = -\operatorname{div} \mathfrak{P} \quad \text{and} \quad \mathfrak{j} = \operatorname{rot} \mathfrak{M}$$

with the corresponding field equations:

$$\operatorname{div} \mathfrak{E} = 4\pi\rho \qquad (\operatorname{rot} \mathfrak{H} = 4\pi \mathfrak{j}, \text{ resp.}),$$

the field strengths $\mathfrak{E}$ and $\mathfrak{H}$ *are generally completely different* from the vectors $-4\pi\mathfrak{P}$ and $4\pi\mathfrak{M}$. Namely, one must observe that the latter vanishes outside of the surface S, while the former does not. Moreover, $\mathfrak{E}$ and $\mathfrak{H}$ satisfy the "energy equations" $\operatorname{rot} \mathfrak{E} = 0$ and $\operatorname{div} \mathfrak{H} = 0$, while the quantities

rot $\mathfrak{P}$ and div $\mathfrak{M}$ remain indeterminate. In fact, the vectors $\mathfrak{P}$ and $\mathfrak{M}$ are defined by *only* the equations div $\mathfrak{P} = -\rho$, rot $\mathfrak{M} = \mathfrak{j}$, and the boundary conditions $\mathfrak{P}_n = \eta$, $\mathfrak{M} \times \mathfrak{n} = \mathfrak{k}$. That is why one can add an arbitrary vector of the form rot $\mathfrak{P}$ to $\mathfrak{M}$ and an arbitrary vortex-free vector to $\mathfrak{M}$ without the boundary conditions breaking down in that way. It is only in the distinguished case where the electric and magnetic field strengths themselves satisfy the following conditions on the surface *S*:

$$(\mathfrak{E} \cdot \mathfrak{n}) = -4\pi\eta \qquad (\mathfrak{H} \times \mathfrak{n} = 4\pi\mathfrak{k}, \text{ resp.}) \tag{37}$$

that one can identify $\mathfrak{P}$ and $\mathfrak{M}$ with the corresponding values of $-\frac{1}{4\pi}\mathfrak{E}$ and $\frac{1}{4\pi}\mathfrak{H}$, *inside of S*, i.e., one can set:

$$4\pi\mathfrak{P} = -\mathfrak{E}\,, \qquad 4\pi\mathfrak{M} = \mathfrak{H}\,. \tag{37.a}$$

CHAPTER FOUR

REPRESENTING ARBITRARY SYSTEMS BY MULTIPOLES. POTENTIAL THEORY.

§ 1. – Definition of a multipole.

Up to now, we have used the concept of the elementary dipole in a somewhat imprecise sense since we have assumed that the length of such a dipole was "small" compared to its distance from other dipoles. We would now like to replace (or better yet, extend) that imprecise physical concept with the following one that is quite precise, but purely mathematical:

The length of the dipole (l) shall be *infinitely small*, and the charges ($\pm e$) shall be *infinitely large* in such a way that its moment ($p = e\ l$) should have a *finite value*. In order to distinguish them from real physical dipoles, which always have a finite length and consist of charges of finite magnitude, we would like to refer to such a dipole as a *mathematical dipole*.

One must then consider a mathematical dipole to actually be a *point-like* structure – viz., a *double point* – just like an isolated point charge. They differ from each other by simply the fact that in the latter case, the point in question is coupled with a scalar quantity (charge), while in the former case it is coupled with a vector quantity (moment).

One can, in turn, regard two (mathematical) dipoles with equal and opposite infinitely-large moments that are found at an infinitely-small distance from each other as a *point*, and indeed, as a *quadruple* point. Such a quadruple point is called a *quadrupole*. A mathematical quadrupole then arises from four charges of equal or equal and opposite magnitudes $\pm e$ that define the corners of a parallelogram with infinitely-small sides l_1, l_2 (Fig. 25). In that way, one assumes that the product $e\ l_1\ l_2$, which corresponds to the moment of a dipole, possesses a finite value.

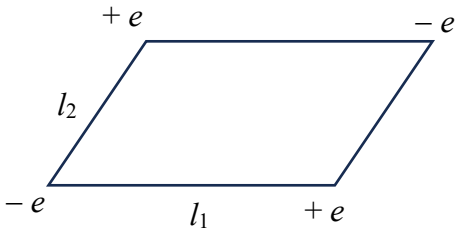


Figure 25.

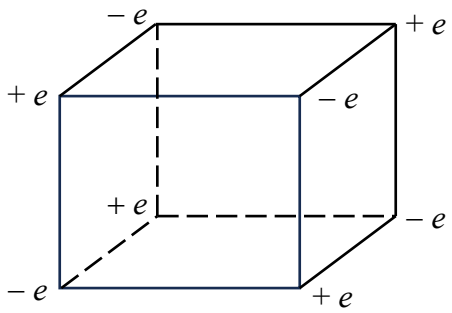


Figure 26.

When one switches the positive charges and the negative ones in a quadrupole, one will get the *opposite* quadrupole. Two opposite quadrupoles that are displaced relative to each other by an infinitely-small line segment l_3 in such a way that the product $e\ l_1\ l_2\ l_3$ remains finite define an *octuple* point or *octupole*.

One can then represent an octupole as an infinitely-small parallelepiped with the charges $\pm e$ at its corners (in the sequence $+ - + \ldots$) (Fig. 26).

By repeating the process that was described above, one will arrive at *multiple* points, or higher-order multipoles. One can define a 2^n-fold point (i.e., a pole of order n, in general) to be a quasi-point-like system that consists of two opposite poles of order $(n-1)$ at an infinitely-small distance from each other. If we resolve those poles of order $(n-1)$ into poles of lower order then we will ultimately get 2^n infinitely-large charges $\pm e$ that are displaced relative to each other though infinitely-small segments $l_1, l_2, \ldots, l_n$. The product $e\, l_1\, l_2 \ldots l_n$ must still have a finite value then.

The n vectors $\mathfrak{l}_1, \mathfrak{l}_2, \ldots, \mathfrak{l}_n$ can generally have arbitrary directions in space. However, they can all lie in the same plane or also on the same line. In the latter case, one calls the multipole in question *axial*.

One defines the *moment* of the pole of order n to be the quantity:

$$\frac{1}{n!}\, p^{(n)} = e\, l_1\, l_2 \ldots l_n , \tag{1}$$

which represents the *magnitude* of a *tensor of rank n*, as we will see. That tensor is the logical generalization of the vectorial moment of a "first-order pole," i.e., a dipole. It is determined by the magnitude $p^{(n)}$ and the n unit vectors $\mathfrak{l}_i / l_i = \mathfrak{a}_i$. The lines that point in the directions of those vectors are called the *axes* of the pole in question. One can define a point-charge as a "zero-order" pole.

§ 2. – The field and energy of a multipole.

Let P' be the (multiple) source-point of the electric field, and let P be the reference point for which the potential φ of that field is to be calculated.

In the case of a simple point, i.e., a point charge e', we have:

$$\varphi^{(0)} = \frac{e'}{R} .$$

We now imagine that this charge is displaced to the neighboring point P_1', while the opposite charge $-e'$ is found at P'. In the limit $P'P_1' = \mathfrak{l}_1' = l_1' \mathfrak{a}_1' \to 0$ and $e'\mathfrak{l}_1' = \mathfrak{p}'^{(1)}$ finite, we will get a double point, i.e., a mathematical dipole, whose potential is expressed by the known formula:

$$\varphi^{(1)} = (\mathfrak{l}_1' \operatorname{grad}')\varphi^{(0)} = (\mathfrak{p}'^{(1)} \nabla') \frac{1}{R} = p'^{(1)} (\mathfrak{a}_1' \nabla') \frac{1}{R} ,$$

and indeed not approximately, but quite exactly.

If one displaces that dipole from P' to the neighboring point P'_2 and at the same time attaches the opposite dipole to the point P' then in the limit $P'P'_2 = \mathfrak{l}'_2 = l'_2\,\mathfrak{a}'_2 \to 0$ while $p'^{(1)}\,l'_2 = \frac{1}{2}\,p'^{(2)}$ = finite, it will be a four-fold point, i.e., a quadrupole. Its potential at P is obtained from $\varphi^{(1)}$ in the same way that the latter was obtained from $\varphi^{(0)}$, i.e., by the formula:

$$\varphi^{(2)} = (\mathfrak{l}'_1\,\nabla')\,\varphi^{(1)} = \tfrac{1}{2}\,p'^{(2)}\cdot(\mathfrak{a}'_2\,\nabla')(\mathfrak{a}'_1\,\nabla')\frac{1}{R}\,.$$

If one generally denotes the potential of a well-defined pole $D^{(n'-1)}$ of order $(n'-1)$ by $\varphi^{(n'-1)}$ then one will get the following expression for the potential $\varphi^{(n')}$ of the pole $D^{(n')}$ of order n', which arises by displacing $D^{(n'-1)}$ through the infinitely-small segment $\mathfrak{l}'_{n'} = l'_{n'}\,\mathfrak{a}'_{n'}$ and adding the opposite pole at P':

$$\varphi^{(n')} = (\mathfrak{l}'_{n'}\,\nabla')\,\varphi^{(n'-1)}, \tag{2}$$

or, from (1):

$$\varphi^{(n')} = \frac{1}{n'!}\,p'^{(n')}(\mathfrak{a}'_{n'}\,\nabla')(\mathfrak{a}'_{n'-1}\,\nabla')\cdots(\mathfrak{a}'_1\,\nabla')\frac{1}{R}\,. \tag{2.a}$$

One can immediately conclude from that formula that the electric field of an n^{th}-order pole is independent of the sequence and the magnitudes of the individual infinitesimal displacements $\mathfrak{l}'_1$, $\mathfrak{l}'_2$, …, $\mathfrak{l}'_n$. It is determined completely by the scalar parameter $p'^{(n)}$ and the set of n unit vectors $\mathfrak{a}'_1$, $\mathfrak{a}'_2$, …, $\mathfrak{a}'_n$ (viz., "axes").

If a charge e is found at the point P then we will get the product:

$$U^{(0)} = e\,\varphi$$

for its potential energy relative to any system of electric charges that creates the potential φ at P. Upon displacing the charge through the infinitely-small segment $\mathfrak{l}_1$ and adding the opposite charge at P, when $e\,\mathfrak{l}_1 = \mathfrak{p}_1 = p^{(1)}\,\mathfrak{a}_1$ = finite, we will get a dipole whose energy is expressed by ([1]):

$$U^{(1)} = (\mathfrak{l}_1\,\nabla)U^{(0)} = (\mathfrak{p}_1\,\nabla)\,\varphi = p^{(1)}\,(\mathfrak{a}_1\,\nabla)\,\varphi\ .$$

Since $\nabla\varphi = -\,\mathfrak{E}$, one can write the formula above in the usual form $U^{(1)} = -\,\mathfrak{p}_1\,\mathfrak{E}$. If one replaces the charge e with a pole of order $(n-1)$ in this argument then one will get a pole $D^{(n)}$ of order n with the potential energy:

([1]) Recall that ∇ = grad means a differentiation with respect to $\mathfrak{r}$, and ∇' means a differentiation with respect to $\mathfrak{r}'$, where $\mathfrak{r} = OP$, $\mathfrak{r}' = OP'$, and $\mathfrak{R} = \mathfrak{r} - \mathfrak{r}'$.

$$U^{(n)} = (\mathfrak{l}_n \nabla) U^{(n-1)}, \tag{3}$$

instead of the dipole, or:

$$U^{(n)} = \frac{1}{n!} p^{(n)} (\mathfrak{a}_n \nabla)(\mathfrak{a}_{n-1} \nabla) \cdots (\mathfrak{a}_1 \nabla) \varphi_0, \tag{3.a}$$

where $p^{(n)}$ means the moment of the pole, and $\mathfrak{a}_1$, …, $\mathfrak{a}_n$ are the corresponding "axis vectors."

In particular, from (2.a) and (3.a), that will give the following formula for the mutual potential energy of the poles $D^{(n')}$ and $D^{(n)}$ when $\varphi = \varphi^{(n')}$:

$$U^{(n',n)} = \frac{1}{n!n'!} p'^{(n')} p^{(n)} (\mathfrak{a}'_1 \nabla') \cdots (\mathfrak{a}'_{n'} \nabla') (\mathfrak{a}_1 \nabla) \cdots (\mathfrak{a}_n \nabla) \frac{1}{R}. \tag{4}$$

Thus, one will have:

$$\nabla' = -\nabla, \tag{4.a}$$

such that the differential operations $(\mathfrak{a}'_1 \nabla')$, etc., can be replaced with $-(\mathfrak{a}'_1 \nabla)$, etc. [or one can replace $(\mathfrak{a}_1 \nabla)$ with $-(\mathfrak{a}_1 \nabla')$] in the actual calculation of $U^{(n',n)}$.

For $n' = n = 1$, we will get the expression (28), Chap. III, for the potential energy of two dipoles that we know already from (4). It differs only by the sign of the potential energy of a point-charge compared to a quadrupole. (The product $p_1 p'_1$ must then be replaced with $-p^{(2)} e'$ or $-p'^{(2)} e$.)

The expressions (2.a) and (4) can be calculated for arbitrary n' and n with no difficulty by means of the formulas:

$$\nabla (\mathfrak{a}\, \mathfrak{R}) = \mathfrak{a}\ (= \text{const.}), \qquad \nabla R^{-n} = -\, n R^{-(n+2)} \mathfrak{R},$$

and the general rule for the differentiation of the product of several factors. In that way, one will get a formula of the form:

$$\varphi^{(n)} = (-1)^n p^{(n)} \frac{Y^{(n)}}{R^{n+1}}, \tag{5}$$

for $\varphi^{(n)}$, where $Y^{(n)}$ means a function that depends upon only the direction (but not the magnitude) of the vector $\mathfrak{R}$, i.e., only the *unit vector* $\mathfrak{R}_0 = \mathfrak{R} : R$. That function consists of a number of summands that are composed of nothing but factors of the type $\mathfrak{a}_i\, \mathfrak{R}_0 = \cos(\mathfrak{a}_i, \mathfrak{R}_0)$ or $\mathfrak{a}_i\, \mathfrak{a}_k = \cos(\mathfrak{a}_i, \mathfrak{a}_k)$. $Y^{(n)}$ is a *symmetric* homogeneous function of degree n relative to the quantities $\mathfrak{a}_1$, …, $\mathfrak{a}_n$. For example, with the abbreviations $(\mathfrak{a}_i\, \mathfrak{R}_0) = \lambda_I$, $(\mathfrak{a}_i\, \mathfrak{a}_k) = \lambda_{ik}$, one will have:

$$Y^{(1)} = \lambda_1 ,$$
$$Y^{(2)} = \tfrac{1}{2}(3\lambda_1\,\lambda_2 - \lambda_{12}) ,$$
$$Y^{(3)} = \tfrac{1}{6}(15\lambda_1\,\lambda_2\,\lambda_3 - 3\lambda_1\,\lambda_{23} - 3\lambda_2\,\lambda_{31} - 3\lambda_3\,\lambda_{12}) ,$$
$$Y^{(4)} = \tfrac{1}{24}(105\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4 - 15\,(\lambda_2\,\lambda_3\,\lambda_{14} + \lambda_3\,\lambda_1\,\lambda_{24} + \lambda_1\,\lambda_2\,\lambda_{34} + \lambda_1\,\lambda_4\,\lambda_{23} + \lambda_2\,\lambda_4\,\lambda_{13} + \lambda_3\,\lambda_4\,\lambda_{12}) + 3\,(\lambda_{14}\,\lambda_{23} + \lambda_{24}\,\lambda_{31} + \lambda_{34}\,\lambda_{12}) ,$$

and in general:

$$Y^{(n)} = \frac{1}{n!}\{1\cdot 3\cdots(2n-1)\,\lambda_1\,\lambda_2\cdots\lambda_n - 1\cdot 3\cdots(2n-3)\sum\lambda_{12}\,\lambda_3\cdots\lambda_n + 1\cdot 3\cdots(2n-5)\sum\lambda_{12}\,\lambda_{34}\,\lambda_5\cdots\lambda_n\cdots\} , \tag{5.a}$$

where $\sum\lambda_{12}\,\lambda_3\cdots\lambda_n$, etc., means the sum of all the terms that were written out over all permutations of the foregoing indices. The proof of (5.a) can be achieved immediately by going from $Y^{(n)}$ to $Y^{(n+1)}$, while considering the aforementioned symmetry property.

The number of all terms that arise from $\lambda_{12}\,\lambda_{34}\,\ldots\,\lambda_{2k-1,\,2k}\;\;\lambda_{2k+1}\,\lambda_{2k+2}\,\ldots\,\lambda_n$ by various permutations of the numbers 1, 2, …, 2k is obviously equal to $\dfrac{(2k)!}{2^k\cdot k!}$. The number of terms in the sum $\sum\lambda_{12}\,\ldots\,\lambda_{2k-1,\,2k}\;\;\lambda_{2k+1}\,\ldots\,\lambda_n$ is therefore equal to:

$$\frac{(2k)!}{2^k\cdot k!}\cdot\frac{n!}{(2k)!(n-2k)!} = \frac{n(n-1)\cdots(n-2k+1)}{2\cdot 4\cdots 2k} .$$

In the case of a multipole with n coincident axes ($\lambda_1 = \lambda_2 = \ldots = \lambda_n$; $\lambda_{ik} = 1$), we will then get the following expression for $Y^{(n)}$:

$$Y^{(n)} = \frac{1\cdot 3\cdots(2n-1)}{1\cdot 2\cdots n}\sum_{k=0}^{2k\le n}(-1)^k\,\frac{n(n-1)\cdots(n-2k+1)}{2\cdot 4\cdots 2k\cdot(2n-1)\cdots(2n-2k+1)}\,\lambda^{n-2k} . \tag{5.b}$$

The functions (5.b) are called *Legendre polynomials*. One usually calls the more-general functions (5.a) *spherical functions* or also *harmonic functions* of order n. However, one does not consider the arguments of those functions to be the cosines $\lambda_1, \lambda_2, \ldots, \lambda_n$, but the angles θ, ψ, which establish the direction of the radius vector $\mathfrak{R}$ relative to any spherical coordinate system. (θ is the angle between $\mathfrak{R}$ and the polar axis, while ψ is the azimuth of $\mathfrak{R}$ in the meridian plane.) If one denotes the corresponding angles for the axes of the multipole in question by θ_i, ψ_i (i = 1, 2, …, n) then from a known formula in spherical trigonometry, one will have:

$$\lambda_i = \cos\,\theta_i\,\cos\,\theta + \sin\,\theta_i\,\sin\,\theta\cos\,(\psi - \psi_i)$$

and

$$\lambda_{ik} = \cos\,\theta_i \cos\,\theta_k + \sin\,\theta_i \sin\,\theta_k \cos\,(\psi_i - \psi_k)\,.$$

§ 3. – Representing arbitrary electric systems by multipoles.

Let S' be a system of electric charges that are found inside of a ball K' of radius a' with its center at P'. The electric potential φ of S' at a point P that lies outside of the ball can then be expressed by the sum:

$$S'\frac{\varepsilon'}{Q'P},$$

in which $Q'P$ means the distance from the charge ε' to the reference point P. (In that expression, we are using S' for the summation sign over all charges in the system considered.) If we denote the coordinates of ε' relative to P', i.e., the components of the vector $P'Q'$ in a completely-arbitrary (rectangular) coordinate system, by ξ_1', ξ_2', ξ_3', and the corresponding coordinates of P' and P by x_1', x_2', x_3' (x_1, x_2, x_3, resp.) then we will have:

$$(Q'P)^2 = \sum_{i=1}^{3}(x_i' + \xi_i' - x_i)^2\,,$$

and as a result, from *Taylor's* theorem:

$$\frac{1}{Q'P} = \frac{1}{R} + \sum_i \frac{\partial\,(1/R)}{\partial x_i'}\xi_i' + \frac{1}{2!}\sum_{i,k}\frac{\partial^2\,(1/R)}{\partial x_i'\,\partial x_k'}\xi_i'\,\xi_k' + \frac{1}{3!}\sum_{i,k,l}\frac{\partial^3\,(1/R)}{\partial x_i'\,\partial x_k'\,\partial x_l'}\xi_i'\,\xi_k'\,\xi_l' + \cdots,$$

in which:

$$R = \sqrt{\sum_i (x_i' - x_i)^2}$$

means the distance, as before.

Upon substituting that expression in $\varphi = S'\frac{\varepsilon'}{Q'P}$, we will get the following series for φ:

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \cdots + \varphi^{(n)} + \cdots, \tag{6}$$

with

$$\varphi^{(0)} = \frac{e'}{R}, \qquad \varphi^{(1)} = \sum_i \frac{\partial\,(1/R)}{\partial x_i'}e_i', \qquad \varphi^{(2)} = \frac{1}{2!}\sum_{i,k}\frac{\partial^2\,(1/R)}{\partial x_i'\,\partial x_k'}e_{ik}', \ \ldots, \tag{6.a}$$

in which the coefficients e', e_i', e_{ik}', … are defined by the formulas:

$$e' = S'\varepsilon',\ e_i' = S'\varepsilon'\,\xi_i',\ e_{ik}' = S'\varepsilon'\,\xi_i'\,\xi_k',\ \ldots \tag{6.b}$$

The quantity $\varphi^{(n)}$ will be called the n^{th}-order potential of the system in question. We will write it in the form:

$$\varphi^{(n)} = \sum \frac{1}{n_1!n_2!n_3!} \frac{\partial^n (1/R)}{\partial x_1'^{n_1} \partial x_2'^{n_2} \partial x_3'^{n_3}} e'(n_1, n_2, n_3) \qquad (n_1 + n_2 + n_3 = n), \tag{7}$$

with:

$$e'(n_1, n_2, n_3) = S'\varepsilon' \xi_1'^{n_1} \xi_2'^{n_2} \xi_3'^{n_3}. \tag{7.a}$$

For large values of *n*, that notation is much more convenient than the foregoing one in (6.a) and (6.b).

Since the point *P* lies *outside* the ball that encloses S' (i.e., the distance $P'Q'$ is smaller than $P'P = R$), the series (6) must converge absolutely and uniformly. One can regard that series as the development of φ in negative powers of the distance *R*. Indeed, $\varphi^{(n)}$ is proportional to the $(n + 1)^{\text{th}}$ power of *R*. We correspondingly set:

$$\varphi^{(n)} = \frac{H_n}{R^{n+1}}, \tag{8}$$

in which H_n no longer depends upon the magnitude of the vector $\mathfrak{R} = P'P$, but only on its direction. That dependency is expressed by the formula:

$$H_n = \sum_{n_1+n_2+n_3=n} \frac{e'(n_1, n_2, n_3)}{n_1!n_2!n_3!} [n_1, n_2, n_3], \tag{8.a}$$

in which:

$$[n_1, n_2, n_3] = R^{n+1} \frac{\partial^n (1/R)}{\partial x_1'^{n_1} \partial x_2'^{n_2} \partial x_3'^{n_3}} \tag{8.b}$$

are the simplest functions of the type considered. They can be determined directly in terms of the cosines $\lambda_i' = (x_i - x_i')/R$.

The parameters (6.b) or (7.a), which determined the electric properties of the system in question, might be referred to as the "electric moments" of order $n = n_1 + n_2 + n_3$ (relative to the given system of axes). The zero-order moment is nothing but the resultant electric charge of the system (e'). The first-order moments are the components of a vector that corresponds to the moment of an elementary (mathematical) dipole at the point P'. The second-order moments are components of a tensor that determines an elementary quadrupole. In the same way, the n^{th}-order moments mean the components of a *symmetric tensor of rank n* that corresponds to an n^{th}-order multipole.

According to (2.a), each summand in the expression (7) represents the potential of an n^{th}-order multipole whose axes all lie along three mutually-perpendicular directions, and whose moment is

equal to $\frac{n!}{n_1!n_2!n_3!}e'(n_1,n_2,n_3)$. The number of those summands obviously amounts to $(n+2)(n+1)/2$. However, due to the identity:

$$\frac{\partial^2(1/R)}{\partial x_1'^2}+\frac{\partial^2(1/R)}{\partial x_2'^2}+\frac{\partial^2(1/R)}{\partial x_3'^2}=0\,,$$

$n\,(n-1)/2$ identities of the form $\frac{\partial^{n-2}\nabla'^2(1/R)}{\partial x_1'^{n_1}\,\partial x_2'^{n_2}\,\partial x_3'^{n_3}}=0$ must exist between the functions $\frac{\partial^n(1/R)}{\partial x_1'^{n_1}\,\partial x_2'^{n_2}\,\partial x_3'^{n_3}}$ or $[n_1, n_2, n_3]$, i.e.:

$$[k_1+2,k_2,k_3]+[k_1,k_2+2,k_3]+[k_1,k_2,k_3+2]=0 \qquad (k_1+k_2+k_3=n-2).$$

It is possible to use those relations in order to express the $(n+1)(n+2)/2$ parameters $e'(n_1,n_2,n_3)$ that determine $\varphi^{(n)}$ or H_n in terms of:

$$\tfrac{1}{2}(n+2)(n+1)-\tfrac{1}{2}n(n-1)=2n+1$$

of them, and indeed in such a way that (7) will assume the form (5) (for $n'=n$), i.e., such the H_n will be represented in the form of a spherical function of order n (5.a), multiplied by a suitably-chosen coefficient $(e'^{(n)})$. That is because the number of independent parameters that determine the potential of an n^{th}-order multipole is equal to precisely $2n+1$ (viz., two parameters for each axis $\mathfrak{a}_k'$, e.g., the angles θ_k, ψ_k, and the resultant moment $e'^{(n)}$).

If one then sets:

$$(\mathfrak{a}_k'\nabla')=\alpha_{k_1}'\frac{\partial}{\partial x_1'}+\alpha_{k_2}'\frac{\partial}{\partial x_2'}+\alpha_{k_3}'\frac{\partial}{\partial x_3'},$$

in which α_{k_1}', α_{k_2}', α_{k_3}' mean the direction cosines of the unit vector $\mathfrak{a}_k'$ (so $\alpha_{k_1}'^2+\alpha_{k_2}'^2+\alpha_{k_3}'^2=1$), then it will always be possible to determine those direction cosines and the parameter $e'^{(n)}$ in such a way that the following identity will exist:

$$\sum_{n_1+n_2+n_3}\frac{e'(n_1,n_2,n_3)}{n_1!n_2!n_3!}\frac{\partial^n(1/R)}{\partial x_1'^{n_1}\,\partial x_2'^{n_2}\,\partial x_3'^{n_3}}=\frac{1}{n!}p'^{(n)}(\mathfrak{a}_1'\nabla')(\mathfrak{a}_2'\nabla')\cdots(\mathfrak{a}_n'\nabla')\frac{1}{R},$$

i.e., such that one can introduce other moments of the form:

$$p\,(n_1, n_2, n_3) = p'^{(n)} \sum \prod^{n_1} \alpha_{k_1,1} \prod^{n_2} \alpha_{k_2,2} \prod^{n_3} \alpha_{k_3,3}$$

that depend upon only $2n + 1$ independent parameters instead of the actual electric moments $e'(n_1, n_2, n_3)$.

One can also express that fact as follows: *The electric field of an arbitrary system of electric charges outside of a sphere that contains it is completely identical to the field of a system of multipoles of different orders that are concentrated at the center of the sphere*.

The actual determination of those multipoles in terms of the parameters $e'(n_1, n_2, n_3)$ is generally a very hard problem for $n > 1$, and we cannot go into the details of its solution here. It is simply soluble for only $n = 0$ and $n = 1$, and indeed, one will have $e = S'\varepsilon'$, and furthermore:

$$\varphi^{(1)} = \sum_i \frac{\partial (1/R)}{\partial x'_i} e'_i = \sum e'_i \nabla' \frac{1}{R} = (\mathfrak{p}' \nabla') \frac{1}{R},$$

in which $\mathfrak{p}'$ means the vector with the components $e'_i = S'\varepsilon'\xi'_i$, from which it will follow that $p'^{(1)} = |\,\mathfrak{p}'\,|$ and $\mathfrak{a}' = \mathfrak{p}' / |\,\mathfrak{p}'\,|$.

Note that from the theorem that was proved above, a number of arbitrary multipoles *of the same order* that are found *at the same point* can be replaced with a single "resultant" multipole. That theorem is the generalization of the corresponding theorem for elementary dipoles (Chap. I, **§ 3**). However, it is not possible to give a simple prescription for determining the axes and moments of a resultant multipole of order n (for $n > 1$) from the axes and moments of the individual "summands."

Moreover, one should note that the electric moments of a given system are *not determined uniquely*, in general, but will depend upon the choice of the center of the sphere (P'). The moment of order n will be independent of that choice only when the moments of all lower orders (i.e., the parameters $p'^{(n-1)}$, $p'^{(n-2)}$, …, $p'^{(0)} = e'$) vanish.

Similar results in regard to the possibility of representing the field of an arbitrary electric system by multipoles of different orders can be derived for the *potential energy* of such a system in a given external field.

Let S be a system of electric charges (ε) that are included inside of a ball K. The system S' that creates the field in question shall then be found *outside of K*. The potential of the field inside of K must satisfy the *Laplace* equation $\nabla^2 \varphi = 0$.

The potential energy of S (relative to S') is obviously equal to the sum of the corresponding energies for the individual charges, i.e.:

$$U = S\,\varepsilon\,\varphi\,(Q),$$

in which $\varphi\,(Q)$ means the value of the potential at the point Q that includes the charge ε. If one denotes the coordinates of Q relative to P (i.e., the components of the vector PQ) by ξ_1, ξ_2, ξ_3, and develops $\varphi\,(Q)$ in powers ξ_1, ξ_2, ξ_3 then that will give the following series for U :

$$U = U^{(0)} + U^{(1)} + U^{(2)} + \cdots + U^{(n)}\,, \tag{9}$$

with:

$$U^{(0)} = e\,\varphi\,, \quad U^{(1)} = \sum_i \frac{\partial \varphi}{\partial x_i} e_i\,, \quad U^{(2)} = \tfrac{1}{2}\sum_{i,k} \frac{\partial^2 \varphi}{\partial x_i\,\partial x_k} e_{ik}\,, \quad \ldots, \tag{9.a}$$

and

$$e = S\,\varepsilon\,, \quad e_i = S\,\varepsilon\,\xi_i\,, \quad e_{ik} = S\,\varepsilon\,\xi_i\,\xi_k\,, \tag{9.b}$$

or in a different notation:

$$U^{(n)} = \sum_{n_1+n_2+n_3=n} \frac{e(n_1,n_2,n_3)}{n_1!\,n_2!\,n_3!} \frac{\partial^n \varphi}{\partial x_1^{n_1}\,\partial x_2^{n_2}\,\partial x_3^{n_3}}\,, \tag{10}$$

with:

$$e\,(n_1, n_2, n_3) = S\,\varepsilon\,\xi_1^{n_1}\,\xi_2^{n_2}\,\xi_3^{n_3}\,. \tag{10.a}$$

The parameters (9.b) or (10.a) are the same electric moments of the system in question that determine its own field outside of *K*. Due to the *Laplace* equation:

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 \varphi}{\partial x_3^2} = 0\,,$$

precisely the same identities must exist between the n^{th}-order derivatives of φ with respect to x_1, x_2, x_3 that exist between the corresponding derivatives of *R* with respect to x_1', x_2', x_3' (cf., *supra*). It will follow that the n^{th}-order energy $U^{(n)}$ can be represented in the form (3.a):

$$U^{(n)} = \frac{1}{n!}\,p^{(n)}\,(\mathfrak{a}_1\nabla)(\mathfrak{a}_2\nabla)\cdots(\mathfrak{a}_n\nabla)\frac{1}{R}\,,$$

i.e., it is identical to the potential energy of a multipole of order *n* at the center of the sphere *P*, and indeed *the same multipole* that creates a potential that is identical to the n^{th}-order potential of *S* outside of the sphere *K*. In summary, we can then state the following: An arbitrary system of electric charges that can be separated from the other systems by a spherical surface (that encloses the former and excludes the latter) is completely equivalent to a number of multipoles of various orders that are concentrated at the center of the sphere in regard to its interaction with that system.

However, for the actual determination of the interaction of two such systems, it is more convenient to not introduce those multipoles explicitly, but to operate with the expressions that determine the potential of a system (S') and the potential energy of the other (*S*) as a function of the components of their electric moments $e'(n_1', n_2', n_3')$ [$e\,(n_1, n_2, n_3)$, resp.] (since those moments can be regarded as *known* quantities).

If we substitute $\varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \cdots$ in (9.a) then, according to (6) and (7), we will get the double series for U:

$$U = \sum U^{(n,n')}, \tag{11}$$

in which $U^{(n,n')}$ corresponds to the form that $U^{(n)}$ will assume when the potential φ is equal to $\varphi^{(n')}$.

Due to the relation:

$$\frac{\partial(1/R)}{\partial x_i} = -\frac{\partial(1/R)}{\partial x_i'},$$

we can appeal to the symbols (8.b) and write:

$$U^{(n,n')} = \frac{(-1)^{n'}}{R^{n+n'+1}} \sum\sum_{\substack{n_1+n_2+n_3=n \\ n_1'+n_2'+n_3'=n'}} \frac{e(n_1,n_2,n_3)}{n_1!n_2!n_3!} \cdot \frac{e'(n_1',n_2',n_3')}{n_1'!n_2'!n_3'!} [n_1+n_1', n_2+n_2', n_3+n_3']. \tag{11.a}$$

If we denote the coefficients of $1/R^{n+n'+1}$ in the cited equation by $H_{n,n'}$ and substitute that in:

$$\sum_{n+n'=k} H_{n,n'} = I_k \tag{11.b}$$

then we will have:

$$U = \frac{I_0}{R} + \frac{I_1}{R^2} + \frac{I_2}{R^3} + \cdots \tag{11.c}$$

Recall that U is not merely the energy of S and S', but at the same time, it also represents the energy of S' relative to S. [That fact corresponds to the symmetry of (11.a) relative to the "primed" and "unprimed" parameters.] From (11.c), the *mutual energy* of the two systems is represented in the form of an advancing series in negative powers of their mutual distance (more precisely, their distances from the center of the spherical surface that enclosed them) whose coefficients depend upon the orientation of the two systems relative to each other and to their connecting line ($\mathfrak{R}$) (except for I_0, which is equal to simply the product ee' of the resultant charges of S and S').

The coefficients $H_{n,0}$, $H_{0,n'}$ are identical to the previously-introduced spherical functions H_n ($H_{n'}$, resp.) [cf., formula (5)].

§ 4. – Harmonically-conjugate systems. Electric potential inside of a ball.

Two points Q and Q' are called "harmonically-conjugate" relative to the spherical surface K when they lie on a line that goes through the center P of the sphere and lie on the same side of P in such a way that:

$$\rho\rho' = a^2, \tag{12}$$

where $\rho = PQ,\ \rho' = PQ'$, and a is the radius of the sphere.

As a result, one of the points (e.g., Q) must lie inside of K, while the other one (e.g., Q') must lie outside of it. If one denotes the point where the line PQQ' intersects the surface of the sphere by A_0 (Fig. 27) then (12) can be written in the form:

$$\frac{PA_0}{PQ} = \frac{PQ'}{PA_0}.$$

Upon subtracting 1 from each side of that equation, we will get:

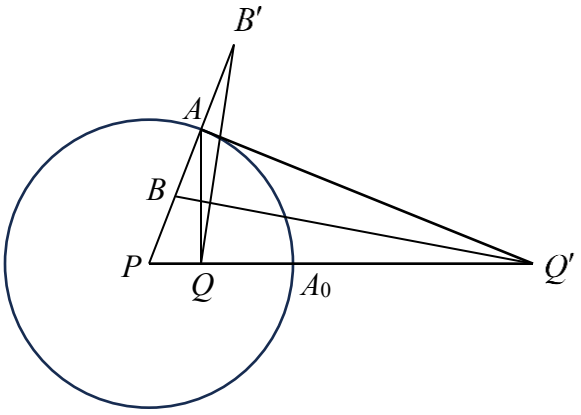


Figure 27

$$\frac{PA_0 - PQ}{PQ} = \frac{PQ' - PA_0}{PA_0},$$

i.e.:

$$\frac{QA_0}{PQ} = \frac{A_0Q'}{PA_0}$$

or

$$\frac{QA_0}{Q'A_0} = \frac{\rho}{a}.$$

Now, it is easy to see that this proportion is true for not only the point A_0, but for any other point A on the spherical surface. In order to prove that, we consider the triangles PQA and $PQ'A$. Since $PA = a$, from (12), we will have:

$$\frac{PQ}{PA} = \frac{PA}{PQ'}.$$

Since those triangles have a common angle (at P), they must be congruent. However, it would follow from this that:

$$\frac{QA}{PQ} = \frac{Q'A}{PA},$$

i.e.:

$$\frac{QA}{Q'A} = \frac{\rho}{a}. \qquad (12.a)$$

If we then imagine that two charges of the same sign ε and ε' are at Q and Q', resp., and have a ratio of:

$$\frac{\varepsilon}{\varepsilon'} = \frac{\rho}{a} = \frac{a}{\rho'} \qquad (12.b)$$

then their potentials $\frac{\varepsilon}{QA}$ ($\frac{\varepsilon'}{Q'A}$, resp.) *must be equal to each other for all points A* of the spherical surface.

Now, let B and B' be two harmonically-conjugate points on the line *PA*. The equations $PB \cdot PB' = a^2$ and $PQ \cdot PQ' = a^2$ imply that:

$$\frac{PB}{PQ'} = \frac{PQ}{PB'}.$$

As a result, since the triangles and PQB' and $PQ'B$ have a common angle at *P*, they must also be congruent. Therefore, one will have $QB' : PB' = Q'B : Q'P$ and $QB' : PQ = Q'B : BP$, i.e., with the notations $PB = R$, $PB' = R'$:

$$\frac{QB'}{Q'B} = \frac{R'}{\rho'} = \frac{\rho}{R}. \tag{13}$$

If one further denotes the electric potential of ε at B' and ε' at *B* by $\varphi' = \varepsilon / QB'$ ($\varphi = \varepsilon' / Q'B$, resp.) then from (13) and (12.b), one will have:

$$\frac{\varphi'}{\varphi} = \frac{R'}{a} = \frac{a}{R}. \tag{13.a}$$

That formula shows that the ratio $\varphi' : \varphi$ is independent of the direction of the line PA_0 on which the two "conjugate charges" are found and of the configuration of the latter on that line, as long as the relation (12) is fulfilled. If one then considers an arbitrary system *S* of charges ε inside of the sphere *K* and the corresponding "conjugate" system S' of external charges ε' then the potentials φ and φ' that are created by S' at *B* and *S* at B', resp., will have the same relationship (13.a) to each other that the potentials of the individual conjugate charges have to each other.

According to the results of the previous section, the potential φ' can be developed in a series outside of *K*:

$$\varphi' = \frac{H_0}{R'} + \frac{H_1}{R'^2} + \frac{H_1}{R'^3} + \cdots = \sum \frac{H_n}{R'^{n+1}},$$

in which H_0, H_1, H_2, … depends upon only the direction of the line *PA*. Correspondingly, we will get:

$$\varphi = \frac{H_0}{a} + H_1 \frac{R}{a^3} + H_2 \frac{R^2}{a^5} + \cdots = \sum H_n \frac{R^n}{a^{2n+1}}, \tag{14}$$

i.e., an advancing series in positive powers of *R* (for which $R < a$). For the limiting case of $R = R' = a$, the two series will obviously coincide.

From (8.a), the spherical functions H_n are determined by the electric moments $e'(n_1, n_2, n_3) = S\,\varepsilon\,\xi_1^{n_1}\,\xi_2^{n_2}\,\xi_3^{n_3}$ of the "internal system" *S*. That is because, from (12.b), when ξ_1, ξ_2, ξ_3 , and ξ_1', ξ_2', ξ_3' mean the components of the vectors ρ (ρ', resp.), that will imply that:

$$S\,\varepsilon\,\xi_1^{n_1}\,\xi_2^{n_2}\,\xi_3^{n_3} = S'\,\varepsilon'\,\frac{a}{\rho'}\left(\frac{a^2}{\xi_1'}\right)^{n_1}\left(\frac{a^2}{\xi_2'}\right)^{n_2}\left(\frac{a^2}{\xi_3'}\right)^{n_3},$$

i.e.:

$$e'(n_1, n_2, n_3) = a^{2n+1}\,S'\,\frac{\varepsilon'}{\rho'}\,\xi_1'^{-n_1}\,\xi_2'^{-n_2}\,\xi_3'^{-n_3}. \tag{14.a}$$

If one substitutes those expressions in the formula:

$$H_n = \sum_{n_1+n_2+n_3=n} \frac{e'(n_1, n_2, n_3)}{n_1!\,n_2!\,n_3!}\,[n_1, n_2, n_3] \tag{14.b}$$

then, from (14), one will get a representation of the "internal" potential φ that depends directly upon the nature of the external system that creates it. In that way, that nature (i.e., the configuration and magnitude of the charges ε') can be completely arbitrary since every external system of charges can always be associated with a conjugate internal system.

Upon replacing (14) in the formulas (9) and (10), where *S* might mean a completely-arbitrary internal system, one will get a new expression for the mutual potential energy of the two systems *S* and S'. That new representation of energy is *more general* than the previous one, which is given by formulas (11) to (11.c) since the series (11) will converge only in the case where *S* and S' can be separated from each other by *two* non-intersecting spherical surfaces. (The sum of the spherical radii $a + a'$ must be smaller than the distance *R* to its center.) By contrast, the new representation (with *one* spherical surface) will also be valid when the external charges lie arbitrarily-close to the spherical surface *K*, just like the internal ones.

If one introduces the parameter:

$$E'(n_1, n_2, n_3) = S'\,\frac{\varepsilon'}{\rho'}\,\xi_1'^{-n_1}\,\xi_2'^{-n_2}\,\xi_3'^{-n_3} \tag{15}$$

in place of $e'(n_1, n_2, n_3)$ and replaces the functions H_n with:

$$h_n = H_n\left(a^{2n+1}\right) = \sum_{n_1+n_2+n_3=n} \frac{E'(n_1, n_2, n_3)}{n_1!\,n_2!\,n_3!}\,[n_1, n_2, n_3] \tag{15.a}$$

then one will get the following expression for φ:

$$\varphi = h_0 + h_1 R + h_2 R^2 + \cdots = \sum h_n R^n . \tag{15.b}$$

However, in that way, the distance R cannot exceed the radius of the sphere: The series (15.b) will then converge only in the region $R < a$, in general.

The coefficients of that series h_n are spherical functions of order n, i.e., from (5.a), they are polynomials of degree n in the direction cosines λ of the vector $\mathfrak{R} = PB$. It will then follow that $h_n R^n$ is an entire homogeneous function of degree n in the components of that vectors, which we would like to denote by x_1, x_2, x_3 . (The fact can also be recognized directly when one observes that, by definition, one has:

$$[n_1, n_2, n_3] = R^{n+1} \frac{\partial^n (1/R)}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} .)$$

Since the derivatives $\dfrac{\partial^n \varphi}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}}$ in (10) refer to the center of the sphere ($x_1 = x_2 = x_3 = 0$), one can replace φ in it with the single term $h_n R^n$. The other terms (of lower and higher order) will give no contribution to the derivatives of order n. Therefore, from (10) and (15.a), that will imply:

$$U^{(n)} = \sum\sum_{\substack{n_1+n_2+n_3=n \\ n_1'+n_2'+n_3'=n}} \frac{e(n_1,n_2,n_3)}{n_1!\,n_2!\,n_3!} \frac{E'(n_1',n_2',n_3')}{n_1'!\,n_2'!\,n_3'!} [n_1,n_2,n_3;n_1',n_2',n_3'] , \tag{16}$$

with:

$$[n_1,n_2,n_3;n_1',n_2',n_3'] = \left\{ \frac{\partial^n}{\partial x_1^{n_1} \partial x_1^{n_1} \partial x_1^{n_1}} \left(R^{2n+1} \frac{\partial^n (1/R)}{\partial x_1^{n_1'} \partial x_1^{n_1'} \partial x_1^{n_1'}} \right) \right\}_{R=0} . \tag{16.a}$$

The cited representation for the energy ($U = U^{(0)} + U^{(1)} - \cdots$) initially has the advantage over the previous one that the dependency of the energy upon the *configuration* of the system S, i.e., the center of the sphere that encloses it, does not need to be expressed. In the event that the system in question can be displaced as a whole inside of that sphere (i.e., without colliding with the external system), that dependency can be determined as follows: One imagines that S is enclosed by a sphere K that is as small as possible. Now, let the center of that sphere be P, and let the center of the larger sphere that includes it be O. If one refers the coordinates of the external charges to O, as before, and those of the internal ones to P then the parameters $E'(n_1', n_2', n_3')$ will remain unchanged, while the quantities:

$$S\,\varepsilon (x_1 + \xi_1)^{n_1} (x_2 + \xi_2)^{n_2} (x_3 + \xi_3)^{n_3}$$

must be introduced in place of the constant parameters $e(n_1, n_2, n_3)$. Those functions are polynomials of degree n in x_1, x_2, x_3 (viz., the coordinates of P relative to O) whose coefficients are equal to the new (constant) values of the electric moments of S, except for numerical factors.

As we saw above, we can replace an "internal system" of electric charges with a number of multipoles at the center of the sphere. It is correspondingly possible to replace the conjugate external system S' (relative to the field that it generates inside of the sphere) with a number of *infinitely-distant charges* that are harmonically conjugate to the ordinary multipoles. In that way, a multipole of order *n* that creates the field:

$$\varphi' = p^{(n)} (\mathfrak{a}_1 \nabla)(\mathfrak{a}_2 \nabla)\ldots(\mathfrak{a}_n \nabla)\frac{1}{R'} = (-1)^n p^{(n)} \frac{Y_n}{R'^{n+1}}$$

outside of the sphere *K* will correspond to a system of infinitely-distant charges that create the field:

$$\varphi' = (-1)^n p^{(n)} \frac{Y_n R^n}{a^{2n+1}}$$

inside of K.

For example, a mathematical dipole corresponds to two opposite charges that lie along the same diameter (in opposite directions) and require the potential:

$$\varphi = \frac{\mathfrak{p} \cdot \mathfrak{R}}{a^3}$$

inside of the sphere. In that expression, $\mathfrak{p}$ means the electric moment of the dipole, as usual. Note that this potential will correspond to a *homogeneous* field of strength $\mathfrak{E} = -\mathfrak{p} / a^3$.

§ 5. – Equivalent surface charge.

From the results of the last two sections, one should now observe the following fact: A certain electric field can be created inside or outside of a closed surface by *different* charge distributions outside (inside, resp.) that surface.

That theorem is true in general for entirely-arbitrary closed (or infinite) surfaces ([1]) and is a direct consequence of *Coulomb*'s law or the *Laplace* equation, which is equivalent to it. One can further prove that for *any* (real) distribution of electric charges on one side of the surface *K* in question (i.e., it creates the same field on the other side), a distribution of electricity will exist on *that surface* itself. The mutual energy of two electric systems *S* and S' that are separated from each other by *K* can then be always calculated to be the mutual energy of two charged surfaces that coincide with *K*.

That theorem is quite simple to prove in the special case of a spherical surface. Let *S* and S' be two *harmonically-conjugate* systems (internal and external). The potentials that they create (φ' externally and φ internally) will then coincide on the spherical surface itself. One can then say that those potentials will go over to each other continuously when one crosses the surface. It is therefore

([1]) An infinite surface, e.g., a plane, can be treated as a surface that is closed at infinity.

possible to regard them as the external and internal values of the potential of one and the same system of electric charges Σ. Moreover, since one has the equation $\nabla^2\varphi' = 0$ outside of the sphere and $\nabla^2\varphi = 0$ inside of it, those charges can lie on only the boundary surface *K*. Their presence must express itself as a jump in the normal (i.e., radial) component of the electric field when one crosses *K*. By applying the general formula $\oint E_n\, dS = 4\pi\, e$ to an infinitely-small cylinder that includes the element *dS* of the spherical surface and whose lateral surface is perpendicular to it (Fig. 28), for a fixed *dS* and a vanishing height of the cylinder, we will get:

$$E'_n - E_n = 4\pi\,\eta\,, \tag{17}$$

in which $\eta = de\,/\,dS$ means the surface density of electricity on the element *dS*, $\mathfrak{E}'$ and $\mathfrak{E}$ are the electric field strengths on the external and internal surface of *dS*, and $\mathfrak{n}$ is the normal that points from inside to outside, i.e., in the direction of the radius vector $\mathfrak{R}$ ([1]). Since $\mathfrak{E}' = -\nabla'\varphi'$ and $\mathfrak{E} = -\nabla\varphi$, one can write formula (17) as follows:

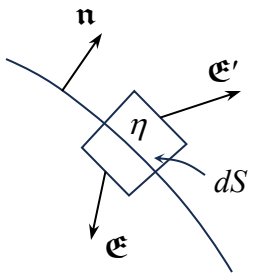


Figure 28.

$$\eta = \frac{1}{4\pi}\left(\frac{\partial\varphi}{\partial R} - \frac{\partial\varphi'}{\partial R'}\right)_{R=R'=a}.$$

Upon substituting the series (14) for φ and the corresponding series for φ', we will get:

$$\eta = \sum_n \frac{2n+1}{4\pi\, a^{n+2}}\, H_n\,. \tag{17.a}$$

That is the desired equivalent surface distribution of electric charge, and indeed for each of the two conjugate systems *S* and S' in the corresponding spatial regions. In particular, for $n = 0$, $H_0 = e$ (viz., a zero-order pole), that will give $\eta = e\,/\,(4\pi\, a^2)$, i.e., a uniform distribution of charge *e*. We then get the known result that a uniformly-charged spherical surface will exert the same action externally as if all of the charge were concentrated at its center ([2]). For $n = 1$ and $H_1 = p\cos\theta$ (viz., a dipole with the moment $\mathfrak{p}$, where θ is the angle between $\mathfrak{p}$ and $\mathfrak{R}$), one will likewise have:

$$\eta = \frac{3}{4\pi}\,\frac{p\cos\theta}{a^2}\,. \tag{17.b}$$

The total charge $\oint \eta\, dS$ is therefore equal to zero.

([1]) Cf., formula (12), Chap. III, for the electric field strengths inside of a *double layer*.

([2]) The external potential $\varphi' = e\,/\,R'$ will then correspond to the internal one $\varphi = e\,/\,a = \text{const}$.

We now assume that the systems S and S' are not conjugate, but completely independent of each other, so the internal one is characterized by the spherical functions H_n, and the external one is characterized by H'_n. One can represent their mutual energy in the form:

$$U = \oint \varphi \eta \, dS ,$$

in which the integral extends over the spherical surface.

Moreover, by means of the formulas (17.a) and $\varphi = \sum \frac{H'_{n'}}{a^{n'+1}}$ (for $R = a$), we will get:

$$U = \frac{1}{4\pi a^2} \sum_n \sum_{n'} \frac{2n+1}{a^{n+n'+1}} \oint H_n H'_{n'} \, dS . \tag{18}$$

That representation of energy must obviously be identical to the representation that was cited in the last section [see formula (16)]. However, it follows from this that the double series (18) can be reduced to the simple series:

$$U = \frac{1}{4\pi a^2} \sum_n \frac{2n+1}{a^{2n+1}} \oint H_n H'_n \, dS , \tag{18.a}$$

i.e., that one has:

$$\oint H_n H'_n \, dS = 0 \tag{18.b}$$

for $n' \neq n$. That relation, which can be easily verified directly, expresses the "orthogonality property" of the spherical functions of different order.

If one chooses an arbitrary sequence of spherical functions H_0, H_1, H_2, … that are normalized in such a way that:

$$\frac{1}{4\pi a^2} \int H_n^2 \, dS = 1 \tag{19}$$

then one can develop an arbitrary given distribution of electricity on the sphere η into those functions, i.e., one can represent it in the form:

$$\eta = \sum_{n=0}^{\infty} C_n H_n , \tag{19.a}$$

in which the coefficients C_n can be calculated from the formula:

$$\frac{1}{4\pi a^2} \int \eta H_n \, dS = \sum_{n=0}^{\infty} \frac{C_n}{4\pi a^2} \int H_n H'_n \, dS ,$$

i.e.:

$$C_n = \frac{1}{4\pi a^2}\int \eta H_n \, dS \, . \tag{19.b}$$

We shall now go on to the proof of the general theorem that was formulated at the beginning of this section.

Let S be a closed surface, and let φ' be the potential that an internal system of charges creates outside of S. Let the value of φ' on the surface itself be denoted by $\bar{\varphi}$.

In order for φ' to be regarded as the potential of a distribution of electricity on S with the surface density η, a function $\varphi(\mathfrak{r})$ must exist that represents the potential of that same charge distribution *inside of S*. Obviously, that function must satisfy the following conditions:

1. *Inside of S*, it will remain continuous, along with its first derivative ($\nabla\varphi$ = minus the electric field strength), while second derivative vanishes ($\nabla^2\varphi = 0$).

2. *On the surface S*, φ assumes the given value $\bar{\varphi}$ (i.e., the corresponding value of φ' on the surface).

Now, one can prove that such a function can always be found (for an arbitrary form of S and arbitrary values of $\bar{\varphi}$), and indeed *uniquely* (viz., *Dirichlet's principle*).

If one substitutes $\mathfrak{F} = \psi \, \nabla\psi$ in the general formula $\oint F_n \, dS = \int \operatorname{div} \mathfrak{F} \, dV$ then one will have:

$$\oint \psi \nabla_n \varphi \, dS = \int \psi \nabla^2\varphi \, dV + \int \nabla\psi \cdot \nabla\varphi \, dV \, . \tag{20}$$

In the special case of $\psi = \varphi$, we will get the formula (5), Chap. III, that we already used before:

$$\int (\nabla\varphi)^2 \, dV = + \int \varphi \nabla_n \varphi \, dS - \oint \varphi \nabla^2 \varphi \, dS \, , \tag{20.a}$$

which will imply that:

$$\varphi = 0$$

in the entire volume V, since $\nabla^2\varphi = 0$ and $\bar{\varphi} = 0$.

We shall initially overlook the condition that $\nabla^2\varphi = 0$ and consider the charge ΔJ that the integral $J = \int (\nabla\varphi)^2 \, dV$ will experience when the function φ is replaced with $\varphi + \delta\varphi$.

It will follow from the identity:

$$[\nabla(\varphi + \delta\varphi)]^2 = (\nabla\varphi)^2 + 2\,\nabla\varphi \cdot \nabla\delta\varphi + (\nabla\delta\varphi)^2$$

that

$$\Delta J = 2\int \nabla\varphi \cdot \nabla\delta\varphi \, dV + \oint (\nabla\delta\varphi)^2 \, dV \, ,$$

or from (20), with $\psi = \delta\varphi$:

$$\Delta J = \int (\nabla \delta\varphi)^2 \, dV - 2\int \delta\varphi \nabla^2 \varphi \, dV + 2 \oint \delta\varphi \nabla_n \varphi \, dS \, . \qquad (20.b)$$

If one now sets $\nabla^2 \varphi = 0$ inside of the surface S and compares φ with only those functions $\varphi + \delta\varphi$ that assume the same value $\bar{\varphi}$ as φ on S, i.e., ones for which one has $\delta\bar{\varphi} = 0$, then one will get an essentially-positive value for ΔJ. The condition $\nabla^2 \varphi = 0$, which defined the desired potential function above, is then completely equivalent to the following one: *The function φ shall make the integral J a minimum.* The fact that such a function will exist at all among all functions that assume the given boundary value $\bar{\varphi}$ on S can be regarded as a very illuminating fact. The fact that there is *only one* such function can be seen as follows: If $\varphi + \delta\varphi$ were another function of the same type then one would need to have $\delta\bar{\varphi} = 0$ and $\nabla^2 (\delta\varphi) = 0$. However, from (20.a), it would follow from this (when φ is replaced with $\delta\varphi$) that $\nabla(\delta\varphi)$ and $\delta\varphi$ would have to vanish in the entire volume V.

If the function φ has been found then one can define the desired surface density of the electricity on S by the formula (17), just as one did in the case of the spherical surface that was considered above. However, the actual determination of that function in general (i.e., for an arbitrary form of S and an arbitrary boundary value $\bar{\varphi}$) defines a very complicated problem analytically.

§ 6. – Green's function.

However, that problem can be reduced to a simpler one, namely, determining an auxiliary function that depends upon *only* the form of the surface S, but not on the boundary values $\bar{\varphi}$. The potential φ can be represented in the form of a surface integral over S by means of that function.

Let P^* be any point inside of S and let Σ be a very small surface that enclosed that point.

If one extends the volume integration that was suggested in formula (20), not over the total volume V, but only over the part V^* that lies between S and Σ then that will give the difference:

$$\oint \psi \nabla_n \varphi \, dS - \oint \psi \nabla_\nu \varphi \, d\Sigma$$

on the left-hand side of (20), where ν means the external normal to Σ.

If one switches the functions φ and ψ in (20) and further assumes that both of them remain finite inside of V^*, along with their gradients and second derivatives, then one will get:

$$\oint (\psi \nabla_\nu \varphi - \varphi \nabla_\nu \psi) \, d\Sigma = \oint (\psi \nabla_n \varphi - \varphi \nabla_n \psi) \, dS - \int (\psi \nabla^2 \varphi - \varphi \nabla^2 \psi) \, dV^* \qquad (21)$$

upon subtracting them. In the special case in which $\nabla^2 \varphi$ and $\nabla^2 \psi$ vanish inside of V^*, that will then imply that:

$$\oint (\psi \nabla_\nu \varphi - \varphi \nabla_\nu \psi)\, d\Sigma = \oint (\psi \nabla_n \varphi - \varphi \nabla_n \psi)\, dS\,. \tag{21.a}$$

Obviously, that equation is also valid *when the auxiliary function* ψ *becomes infinite inside of the surface* Σ, *just like* $\nabla\psi$ *and* $\nabla^2\psi$. For example, we can set $\psi = 1/R$, where R means the distance from the point P^* to any point P of the volume V^*. In fact, if we consider ψ to be a function of the radius vector $\mathfrak{r}$ of P then it will remain finite inside of V^*, along with its gradient $\nabla\psi$, while $\nabla^2\psi = 0$. We shall now go on to the case in which the surface Σ contracts to the point P^*. We will then have:

$$\lim \oint \psi \nabla_\nu \varphi\, d\Sigma = 0$$

since Σ will then decrease quadratically in R, and furthermore:

$$\lim \oint \varphi \nabla_\nu \psi\, d\Sigma = \varphi^* \cdot \lim \oint \nabla_\nu \frac{1}{R}\, d\Sigma = -\,4\pi\,\varphi^*\,,$$

since $1/R$ can be considered to be the potential of a unit charge at P^* ([1]). From (21), we will then get:

$$\varphi^* = \oint \frac{1}{R}\frac{\nabla_n \varphi}{4\pi}\, dS - \oint \frac{\varphi}{4\pi} \nabla_n \frac{1}{R}\, dS\,. \tag{21.b}$$

The first term on the right-hand side of that formula represents the potential of a surface charge of density $\frac{\nabla_n \varphi}{4\pi}$, and the second one represents the potential of a double layer whose electric moment per unit area amounts to $\varphi / 4\pi$ (see Chap. III, **§ 9**). It is easy to see that such a system would create a *zero* potential in the external space (i.e., no electric field). That is because if one replaces P^* with a point P' that lies externally to S then the function $\psi = 1/R$ (R = distance $P'P$) must remain finite and continuous in all of the volume V that S encloses. However, in that case, the left-hand and right-hand sides of (21) would vanish since the integrals $\oint \psi \nabla_n \varphi\, dS$ and $\oint \varphi \nabla_n \psi\, dS$ would be two equivalent transformations of the volume integral $\int (\nabla\varphi \cdot \nabla\psi)\, dV$. One can also interpret that fact physically by saying: When one crosses the double layer with the moment κ (per unit area), the potential will jump by an amount $4\pi\kappa$. Now, if:

$$\varphi = \bar{\varphi} \qquad \text{and} \qquad \kappa = \frac{\bar{\varphi}}{4\pi}$$

on the inner side of S then one must have $\varphi' = 0$ on the outer side.

([1]) φ^* means the value of φ at the point P^*.

The formula (21.a) is not entirely suitable for representing the potential φ inside of *S*, on the grounds that it must include not only the boundary values of φ (which must be equal to the corresponding boundary values of the external potential φ'), but also the boundary values of the gradient $\nabla\varphi$ *inside of S*, while those boundary values can be regarded as known only *outside of S*. In order to free oneself from those unknown boundary values, one replaces 1 / *R* in (21) with another function ψ of the two arguments $\mathfrak{r}$ (radius vector of the point *P*) and $\mathfrak{r}^*$, which will become identical to 1 / *R* when the distance PP^* becomes infinitely small (up to terms that will remain finite, and are thus irrelevant), satisfies the *Laplace* equation $\nabla^2\psi = 0$ for all $\mathfrak{r}$, just like 1 / *R*, and vanishes for those $\mathfrak{r}$ that belong to the points on the surface *S*. In place of (21.a), one will then get the formula:

$$\varphi(\mathfrak{r}^*) = -\frac{1}{4\pi}\oint \bar{\varphi}\,\nabla_n \psi(\mathfrak{r},\mathfrak{r}^*)\,dS\,, \tag{21.c}$$

which can serve to calculate the potential inside of *S* from its boundary values ([1]). The function $\psi(\mathfrak{r},\mathfrak{r}^*)$ is called the *Green* function of the surface *S*. One can easily prove that it is *symmetric* in both arguments $\mathfrak{r}$ and $\mathfrak{r}^*$, just like 1 / *R*. In order to prove that we consider the integral:

$$\int \nabla\psi(\mathfrak{r},\mathfrak{r}_1^*)\cdot\nabla\psi(\mathfrak{r},\mathfrak{r}_2^*)\,dV^*\,,$$

in which $\mathfrak{r}_1^*$, $\mathfrak{r}_2^*$ mean two points that lie inside of *S*, and V^* is the volume that is bounded by *S*, on the one hand, and the small surfaces Σ_1 and Σ_2 that enclose that point, on the other. By means of the transformation (20), instead of (21), that will give the equation:

$$\oint(\psi_1\nabla_{\nu_1}\psi_2 - \psi_2\nabla_{\nu_1}\psi_1)\,d\Sigma_1 + \oint(\psi_1\nabla_{\nu_2}\psi_2 - \psi_2\nabla_{\nu_2}\psi_1)\,d\Sigma_2 = \oint(\psi_1\nabla_{\nu}\psi_2 - \psi_2\nabla_{\nu}\psi_1)\,dS\,,$$

([1]) In the simplest case where the external potential φ' is replaced with a point-charge *e* that is found at the point *P* ($\mathfrak{r}$) inside of *S*, we will have $\bar{\varphi} = \bar{\varphi}' = e / R$ ($R = PP^*$) such that the formula (21) will reduce to:

$$\varphi(\mathfrak{r}) = -\frac{e}{4\pi}\oint\frac{1}{R}\nabla_n\psi(\mathfrak{r},\mathfrak{r}^*)\,dS\,.$$

That corresponds to a surface distribution of electricity with a density of:

$$\eta = -\frac{e}{4\pi}\nabla_n\psi(\mathfrak{r},\mathfrak{r}^*)\,.$$

Therefore, the total value of the equivalent surface charge:

$$\oint\eta\,dS = -\frac{e}{4\pi}\oint\nabla_n\psi(\mathfrak{r},\mathfrak{r}^*)\,dS$$

will be equal to *e*, as is easy to see.

in which we have set $\psi_1 = \psi(\mathfrak{r}, \mathfrak{r}_1^*)$ and $\psi_2 = \psi(\mathfrak{r}, \mathfrak{r}_2^*)$, to abbreviate. The integral on the right-hand side vanishes due to the boundary condition $\bar{\psi}_1 = \bar{\psi}_2 = 0$, and the one on the left reduces to the difference:

$$4\pi\,\psi_{2,1} - 4\pi\,\psi_{1,2} = 4\pi[\psi(\mathfrak{r}_2^*, \mathfrak{r}_1^*) - \psi(\mathfrak{r}_1^*, \mathfrak{r}_2^*)]$$

in the limits of $\Sigma_1 \to 0$ and $\Sigma_2 \to 0$. One will then have:

$$\psi(\mathfrak{r}_2, \mathfrak{r}_1) = \psi(\mathfrak{r}_1, \mathfrak{r}_2)\,. \tag{21.d}$$

In the special case of the spherical surface, the *Green* function will have the following form:

$$\psi(\mathfrak{r}_1, \mathfrak{r}_2) = \frac{1}{R} - \frac{a}{r_1}\frac{1}{R'},$$

in which a means the radius of the sphere $\mathfrak{r}_1$ means the radius vector that points from the center of the sphere to the point P_1, R means the distance $P_1\,P_2$, and R' means the distance $P_1'P_2$. Therefore, P_1' denotes the point that is conjugate to P_1 and lies outside of the spherical surface. In fact, P_2 lies on S, so $\psi(r_1, r_2)$ must vanish, from (12.a) (since $QA = P_1\,P_2$, $Q'A = P_1'P_2$, and $\rho = r$). Moreover, ψ satisfies the *Laplace* equation ($\nabla^2\psi = 0$) as a function of $\mathfrak{r}_2$ and coincides with $1 / R$ as $\mathfrak{r}_2 \to \mathfrak{r}_1$, in practice. With the use of the relation $\mathfrak{r}_1' = \frac{a^2}{r_1^2}\mathfrak{r}_1$ [cf., (12.b)], one can easily convince oneself of the symmetry relative to $\mathfrak{r}_1$ and $\mathfrak{r}_2$.

In the general case of an arbitrary surface S, the determination of the *Green* function in the form of a closed, analytical expression is impossible. However, if it is known then one can calculate the effective charge density on S by means of the formula (**§ 17**):

$$\eta = \frac{1}{4\pi}(\nabla_n\varphi - \nabla_n\varphi') = -\frac{1}{(4\pi)^2}\oint \bar{\varphi}\,\nabla_n\nabla_n^*\psi(\mathfrak{r}, \mathfrak{r}^*)\,dS - \frac{1}{4\pi}\nabla_n\varphi'.$$

Therefore, the potentials φ' and φ can be represented by the same surface integral $\oint(\eta / R)\,dS$.

In the foregoing, we have assumed that the external potential was known and sought to replace internal system of charges that created it with a surface charge. In the opposite case, when the potential φ is given inside of S and the external charges that create it are to be replaced with an equivalent surface charge, we must initially look for a potential function φ' in the external space that satisfies the equation $\nabla^2\varphi' = 0$ and is identical to φ on S. In that way, that problem can be reduced to the one that was considered before, in which one imagines that S is enclosed within a

very large external surface S' that will later be shifted to infinity, while the surface Σ contracts to the (external) point P^* in question. However, we cannot go further into those questions here.

§ 7. – Equivalent surface currents.

In the previous sections, we spoke of electric charges and electric fields. However, the results obtained will also remain true for magnetic fields when we imagine that the latter are created by fictitious magnetic poles. However, it is easy to rid ourselves of that fiction and replace the magnetic poles with equivalent electric currents.

In the case of a multipole, it suffices to, e.g., exclude "zero-order" poles and treat "first-order" poles, i.e., mathematical dipoles, as the elementary currents. In that way, one can replace the scalar potential of an n^{th}-order multipole $\varphi^{(n)}$ with an equivalent vector potential $\mathfrak{A}^{(n)}$. If $\varphi^{(n)}$ can be represented in the form $\varphi^{(n)} = (\mathfrak{a}_n \nabla)\varphi^{(n-1)}$, where $\nabla^2 \varphi^{(n-1)} = 0$, then that will give simply:

$$\mathfrak{A}^{(n)} = \mathfrak{a}_n \times \nabla \varphi^{(n-1)}, \tag{22}$$

since one will then have rot $\mathfrak{A}^{(n)} = -\nabla \varphi^{(n-1)}$ (cf., **§ 8**, Chap. III).

However, the introduction of the vector potential in place of the scalar potential is usually inappropriate.

When considering the magnetic field outside (inside, resp.) a closed surface *S* that is created by an internal (or external) system of currents, one can introduce an equivalent *surface current* instead of the equivalent surface distributions of "magnetic charges." The surface density of that current can be determined in the same way as the density of surface current. If one applies the general formula:

$$\oint \mathfrak{n} \times \mathfrak{H}\, dS = \int \text{rot}\, \mathfrak{H}\, dV = 4\pi \int \mathfrak{j}\, dV$$

to an infinitely-small cylinder that includes the element *dS* of the surface in question and whose base surfaces are parallel and equal to that element then one will get:

$$4\pi \mathfrak{k} = \mathfrak{n} \times (\mathfrak{H}' - \mathfrak{H}) \tag{23}$$

in the limiting case of vanishing height of the cylinder. In that equation, $\mathfrak{k}$ means the surface density of the electric current (according to $\int \mathfrak{k}\, dS = \int \mathfrak{j}\, dV$). $\mathfrak{H}'$ and $\mathfrak{H}$ are the magnetic field strengths on the outer and inner sides of *dS*.

Note that relationship between $\mathfrak{H}'$ and $\mathfrak{H}$ that is at the basis for formula (23) is completely different from the corresponding relationship between the electric field strengths $\mathfrak{E}'$ and $\mathfrak{E}$. In fact,

since the magnetic field satisfies the equation div $\mathfrak{H} = 0$ in all of space, the inner product must satisfy:

$$\mathfrak{n} \cdot (\mathfrak{H}' - \mathfrak{H}) = 0 , \tag{23.a}$$

i.e., the normal components of the magnetic field strengths *cannot* make a jump upon crossing the surface *S*. By contrast, equation (17) will be valid for the electric field strength in that case, which we can also write in the form:

$$\mathfrak{n} \cdot (\mathfrak{E}' - \mathfrak{E}) = 4\pi\,\eta , \tag{23.b}$$

as well as the equation:

$$\mathfrak{n} \times (\mathfrak{E}' - \mathfrak{E}) = 4\pi\,\eta , \tag{23.c}$$

which corresponds to the vanishing of the rotation of the vector $\mathfrak{E}$ in all of space and expresses the fact that the tangential component of the electric field strength can possess no discontinuity on *S*.

If one would then like to interpret the given external field $\mathfrak{H}'$ that is created by an unknown internal system of currents as the field of a surface current that is distributed on *S* then the field inside of *S* must be determined in such a way that the condition (23.a) is fulfilled.

When the external field can be represented by means of a scalar potential φ' ([1]), one can also derive the corresponding "conjugate" internal field $\mathfrak{H}$ from a scalar potential φ. In that way, φ must satisfy the *Laplace* equation ($\nabla^2\varphi = 0$) inside of *S* and the condition that:

$$\nabla_n \varphi = \nabla_n \varphi' \tag{24}$$

on the surface itself, which now replaces the previous condition that $\bar{\varphi} = \bar{\varphi}'$. Since the magnetic field is source-free, moreover, the integral must be:

$$\oint \nabla_n \varphi \, dS = 0 . \tag{24.a}$$

As far as the condition $\bar{\varphi} = \bar{\varphi}'$ is concerned, it *cannot* exist in the case considered, i.e., the potential must suffer a jump (Chap. III, **§ 9**):

$$\bar{\varphi}' - \varphi = 4\pi\, i_m \tag{24.b}$$

on the surface *S*. That corresponds to a magnetic *double layer* with a density (i.e., moment per unit area) of i_m. If that quantity were known then one could represent the two potentials φ and φ' by the integral:

$$\oint i_m \nabla_n \left(\frac{1}{R}\right) dS . \tag{24.c}$$

([1]) That is always the case when the magnetic lines of force do not close outside of *S*.

Here, we would like to treat only the simplest case in which S is a spherical surface. We can then represent φ' in the form:

$$\varphi' = \sum_n \frac{H_n}{R'^{n+1}}, \tag{25}$$

in which the coefficients H_n are regarded as known. If we correspondingly set:

$$\varphi = \sum_n \frac{H'_n}{a^{2n+1}} R^n \tag{25.a}$$

then the initially-unknown coefficients H'_n can be obtained from the relation:

$$\sum \frac{n H'_n}{a^{2n+1}} R^{n-1} = -\sum \frac{(n+1) H_n}{R'^{(n+2)}}, \qquad R' = R = a,$$

which follows from (24), i.e.:

$$H'_n = -\frac{n+1}{n} H_n . \tag{25.b}$$

From (24.b), that corresponds to the formula $4\pi i_m = -\sum \frac{2n+1}{n a^{n+1}} H_n$. If (25) reduces to the first-order term (so $\varphi^{(0)}$ must always be equal to zero), i.e., if one has:

$$\varphi' = \frac{\mathfrak{m}\mathfrak{R}'}{R'^3}$$

then from (25.b) and (25.a), one will have:

$$\varphi = -\frac{2\,\mathfrak{m}\mathfrak{R}}{a^3}.$$

It will follow from this that a homogeneous magnetic field $\mathfrak{H} = 2\,\mathfrak{m} / a^3$ prevails inside of the ball, while one has:

$$\mathfrak{H}' = \frac{1}{R'^3}[-\mathfrak{m} + 3\,\mathfrak{R}'(\mathfrak{m}\,\mathfrak{R}')]$$

outside of it. From (22), the scalar potentials φ' and φ correspond to the vector potentials:

$$\mathfrak{A}' = \frac{1}{R'^3}\mathfrak{m}\times\mathfrak{R}', \qquad \mathfrak{A} = \frac{1}{a^3}\mathfrak{m}\times\mathfrak{R},$$

resp. We get the surface density $\mathfrak{k}$ of the current at a point on the sphere with the radius vector $\mathfrak{a}$ from (23):

$$\mathfrak{k} = \frac{3}{4\pi a^4}\mathfrak{m}\times\mathfrak{a}\,.$$

That formula shows that the field in question can be generated by rotating the spherical surface, in the event that the latter is uniformly charged. In fact, if we denote the angular velocity by $\mathfrak{o}$ and the (constant) surface density of electric charge by η then the current density at the point $\mathfrak{a}$ must be expressed by the formula:

$$\mathfrak{k} = \frac{\eta}{c}\mathfrak{o}\times\mathfrak{a}$$

(since $\mathfrak{o}\times\mathfrak{a}$ is the linear velocity of the corresponding surface element). A comparison of that formula with the foregoing one will give the relation:

$$\mathfrak{m} = \frac{4\pi}{2}a^4\frac{\eta}{c}\mathfrak{o}\,,$$

which says nothing but the fact that $\mathfrak{m}$ is the resultant magnetic moment of the rotating charged sphere. Indeed, by definition, one will have:

$$\mathfrak{m} = \tfrac{1}{2}\oint \mathfrak{a}\times\mathfrak{k}\,dS\,,$$

which is identical to the cited formula since $\mathfrak{k} = \frac{\eta}{c}\mathfrak{o}\times\mathfrak{a}$ and $\mathfrak{a}\times\mathfrak{k} = \frac{\eta}{c}\{a^2\mathfrak{o}-(\mathfrak{a}\,\mathfrak{o})\mathfrak{a}\}$.

§ 8. – Induced electric and magnetic moments.

To conclude this chapter, we must briefly consider the question of what sort of changes the real electric (magnetic, resp.) system will suffer under the action of external fields. We understand a "real system" to be one that consists of any material bodies or bodies that are composed of a number of positive and negative electrons. As we have seen above, such a system can always be replaced with a set of multipoles of varying order. Up to now, we have treated those multipoles as having been "given," i.e., unchanging, or *fixed*. However, just as the real fixed body is not absolutely fixed in a mechanical context but will suffer small *deformations* under the influence of external forces, they are also not absolutely fixed in an electric (and magnetic) context. If one introduces any body into an external electric field then the opposite charges that are stuck in that body must experience oppositely-directed forces. Those forces must produce small displacements

of the charged particles up to the point that the *internal* electric forces that they would require (in analogy with elastic stresses) attain equilibrium with the external forces.

One cares to call the "electric deformation" of a body that was just suggested *polarization*. Phenomenologically, one can characterize it by the change in the electric moment of the body $e(n_1, n_2, n_3)$ or by the additional multipoles of various orders that must be added in order to produce the altered, "deformed" multipole.

Ordinarily, one considers only the change in the *first*-order moment (or multipole). In that way, one assumes that the additional or "induced" electric moment is proportional to the electric field strength $\mathfrak{E}$ (of the external field), in analogy with *Hooke*'s law for elastic deformations.

Since the first-order electric moments e_1, e_2, e_3 are nothing but the components of the dipole moment, which corresponds to a first-order multipole, one can consider the aforementioned additional moments, which we will denote by m_1, m_2, m_3, to be the components of the induced dipole moment $\mathfrak{p}$ that measures the electric polarization of the body in question. The simplest linear relation between $\mathfrak{p}$ and $\mathfrak{E}$ is expressed by the formula:

$$\mathfrak{p} = \alpha\, \mathfrak{E} \,. \tag{26}$$

That formula says that $\mathfrak{p}$ has the same direction as the vector $\mathfrak{E}$ and is proportional to it. Therefore, it will be assumed that the "polarization coefficient" α is an essentially-positive quantity. That means that the positive and negative charges will be shifted in the direction of the external force that acts upon them.

Formula (26) corresponds to *Hooke*'s law for ordinary *isotropic* bodies. A body for which that formula is valid is called "electrically isotropic." In the general case of an electrically-anisotropic body, one must replace (26) with the formula:

$$p_i = \sum_{k=1}^{3} \alpha_{ik} E_k \qquad (i, k = 1, 2, 3), \tag{26.a}$$

where the α_{ik} represent the components of the so-called "polarization tensor" ${}^2\alpha$ relative to the coordinate system in question. For a coordinate system that is fixed in the body, those components are regarded as constant parameters that characterize the body.

The polarization tensor can generally be treated as *symmetric*. Correspondingly, the coordinate system can always be chosen in such a way that the non-diagonal components of ${}^2\alpha$ will all vanish (cf., Introduction). In that way, (26.a) will reduce to:

$$p_1 = \alpha_{11} E_1 \,, \quad p_2 = \alpha_{22} E_2 \,, \quad p_3 = \alpha_{33} E_3 \,. \tag{26.b}$$

The potential energy of a fixed dipole with a moment of $\mathfrak{p}$ relative to the system that creates the field $\mathfrak{E}$ is known to be equal to $-\,\mathfrak{p}\,\mathfrak{E}$. If the moment $\mathfrak{p}$ is induced by the field itself then one must also consider the work that is done by the internal forces. Since those internal forces increase

linearly with $\mathfrak{p}$ and compensate for the field $\mathfrak{E}$, the aforementioned work is expressed by the formula ([1]):

$$U' = \tfrac{1}{2}\, \mathfrak{p}\ \mathfrak{E}\ .$$

One can interpret the quantity U' as the *internal* energy of the polarized body. It corresponds to the internal elastic energy of a mechanically-deformed body. If one would like to calculate the additional mechanical effect that originates in the polarization then the energy function *will not be* $-\ \mathfrak{p}\ \mathfrak{E}$, as with fixed dipoles, but the sum $-\ \mathfrak{p}\ \mathfrak{E} + U'$, i.e.:

$$U = -\tfrac{1}{2}\ \mathfrak{p}\ \mathfrak{E} = -\ U'\ . \tag{27}$$

That can also be seen directly when one starts from the expressions for the torque $\mathfrak{p} \times \mathfrak{E}$ or force $(\mathfrak{p}\,\mathrm{grad})\,\mathfrak{E}$, which are also valid here, and defines the corresponding energy function in the usual way while considering the proportionality between $\mathfrak{p}$ and $\mathfrak{E}$. If one sets $\mathfrak{p} = \alpha\,\mathfrak{E}$, for the sake of simplicity, then one will have:

$$(\mathfrak{p}\,\mathrm{grad})\,\mathfrak{E} = \alpha\,(\mathfrak{E}\ \mathrm{grad})\ \mathfrak{E} = \tfrac{1}{2}\alpha\,\mathrm{grad}\,E^2 = -\ \mathrm{grad}\ U\ ,$$

from which it will follow that:

$$U = -\ \tfrac{1}{2}\alpha\,E^2\ . \tag{27.a}$$

Note that the torque vanishes in this special case (viz., electric isotropy). In the general case, when the principal components of the polarization tensor α_{11}, α_{22}, α_{33} are different from each other, from (26.b), the components of the rotational moment can be expressed by:

$$(\mathfrak{p} \times \mathfrak{E})_1 = (\alpha_{22} - \alpha_{33})\, E_2\, E_3\ , \qquad \text{etc.}$$

As an example, we shall imagine that a small spherical particle whose normal electric moments are all equal to zero is found at a distance R from another charged particle. If the charge of the latter is equal to ε then one will have $E = \varepsilon / R^2$, and as a result:

$$U = -\ \frac{\alpha\,\varepsilon^2}{2R^4}\ .$$

That energy function corresponds to a force of attraction:

([1]) We imagine that $\mathfrak{p}$ is the length of a dipole that consists of a fixed negative charge – 1 and a displaceable positive charge + 1. The internal electric force $\mathfrak{E}$ that acts on the latter is proportional to the displacement $\mathfrak{p}$. We will then get $\int \mathfrak{E}\ d\mathfrak{p} = \tfrac{1}{2}\mathfrak{E}\mathfrak{p}$ for the work done by polarization.

$$f = -\frac{2\alpha\varepsilon^2}{R^5},$$

which is inversely proportional to the fifth power of distance. One has to deal with such forces in the elementary experiments regarding the attraction that a charged body exerts on a neighboring neutral body. If the field $\mathfrak{E}$ is created by a body that possesses no charge, but a natural dipole moment m, then from (26.a) and (26.b), Chap. III:

$$E^2 = \frac{m^2}{R^6}(1+3\cos^2\theta),$$

from which it will follow that:

$$U = -\frac{\alpha m^2}{2R^6}(1+3\cos^2\theta),$$

and since:

$$m^2\cos^2\theta = (\mathfrak{m}\mathfrak{R})^2 / R^2 = (\mathfrak{m}\,\mathfrak{R}_0)^2,$$

one will have:

$$\mathfrak{f} = -\frac{3\alpha}{R^7}[m^2\mathfrak{R}_0 + 4(\mathfrak{m}\mathfrak{R}_0)\mathfrak{R}_0 - (\mathfrak{R}_0\mathfrak{m})\mathfrak{m}],$$

or ultimately:

$$\mathfrak{f} = -\frac{3\alpha}{R^7}[\mathfrak{m}\times(\mathfrak{R}_0\times\mathfrak{m}) - 4(\mathfrak{m}\mathfrak{R}_0)^2\mathfrak{R}_0].$$

In this case, which can be realized, e.g., by the action of an elementary current (or magnet) on iron filings, we then have a force of attraction that is inversely proportional to the seventh power of distance.

Note that, no matter what the nature of the system that creates the field, the force $\mathfrak{f} = \operatorname{grad}\left(\frac{1}{2}\alpha E^2\right)$ must always correspond to an attraction, and indeed in the direction of the most-rapid increase in E^2 (assuming that $\alpha > 0$). That direction does not generally coincide with the direction of the radius vector $\mathfrak{R}$.

Formula (26) can be written in the form:

$$p_i = -\alpha\frac{\partial\varphi}{\partial x_i}.$$

It seems natural to generalize that relation to induced moments of higher order as follows:

$$p(n_1, n_2, n_3) = -\alpha^{(n)}\frac{\partial^n\varphi}{\partial x_1^{n_1}\,\partial x_2^{n_2}\,\partial x_3^{n_3}}. \qquad (28)$$

The derivative of the potential on the right-hand side is calculated for *the* point P of the body considered relative to which the moments $e\,(n_1, n_2, n_3)$ and $p\,(n_1, n_2, n_3)$ were determined.

However, that Ansatz is hardly permissible since the induced moments of various orders (just like the natural ones) must be coupled with each other by certain relations that depend upon the structure of the system in question. Those relations can be taken into consideration by the general Ansatz:

$$p\,(n_1, n_2, n_3) = -\sum_{n_1', n_2', n_3'} \alpha(n_1, n_2, n_3; n_1', n_2', n_3')\, \frac{\partial^{n'}\varphi}{\partial x_1^{n_1'}\, \partial x_2^{n_2'}\, \partial x_3^{n_3'}}\,, \tag{28.a}$$

in which the coefficients $\alpha(n_1, n_2, n_3; n_1', n_2', n_3')$ define a polarization tensor of rank $n + n' = n_1 + n_2 + n_3 + n_1' + n_2' + n_3'$. Both expressions (28) and (28.a) for the components of the "polarization of order n" must correspond to an energy:

$$U = \sum_{n_1+n_2+n_3=n} \tfrac{1}{2}\, p\,(n_1, n_2, n_{33})\, \frac{\partial^{n}\varphi}{\partial x_1^{n_1}\, \partial x_2^{n_2}\, \partial x_3^{n_3}}\,. \tag{28.b}$$

However, we cannot go further into that topic at this point.