PART THREE

THE THEORY OF RELATIVITY

CHAPTER EIGHT

FOUNDATIONS OF THE THEORY OF RELATIVITY

§ 1. – Space-time symmetry of the equations of electromagnetism.

In Chap. VI, **§ 4**, we saw that the differential equations for the electromagnetic potential can be written in a form that is completely symmetric with respect to the four indices when we combine time multiplied by $\sqrt{-1}c = i\,c$ as the fourth coordinate x_4 with the three spatial coordinates x_1, x_2, x_3 and treat the scalar potential multiplied by i as the corresponding component of a four-dimensional vector potential. In so doing, we have restricted ourselves to the case of a point charge. We would now like to examine the case of a continuous distribution of electric charge and current with volume densities ρ (j, resp.). The quantities ρ and j are known to be coupled with each other by the relation:

$$\operatorname{div} \mathrm{j} + \frac{1}{c}\frac{\partial \rho}{\partial t} = 0\,,$$

which expresses the law of conservation of electricity. If we introduce an arbitrary rectangular coordinate system X_1, X_2, X_3, then that equation will take the form:

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3} + \frac{1}{c}\frac{\partial \rho}{\partial t} = 0\,,$$

or with the notations:

$$i\,c\,t = x_4, \tag{1}$$

$$i\,\rho = j_1\,, \tag{2}$$

we can write:

$$\sum_{k=1}^{4} \frac{\partial j_k}{\partial x_k} = 0\,. \tag{2.a}$$

That equation is entirely analogous to equation (19.a), Chap. VI, which we will write out again here for the sake of completeness. With the notation:

$$i\,\varphi = A_4\,, \tag{3}$$

we will then have:

$$\sum_{k=1}^{4} \frac{\partial A_k}{\partial x_k} = 0\,. \tag{3.a}$$

The differential equations for the electromagnetic potentials:

$$\nabla^2\mathfrak{A} - \frac{1}{c^2}\frac{\partial^2\mathfrak{A}}{\partial t^2} = -4\pi\,\mathrm{j}\,, \qquad \nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\,\varphi\,,$$

can now be combined into the system of equations:

$$\sum_{k=1}^{4}\frac{\partial^2 A_k}{\partial x_k^2} = -4\pi j_k \qquad (k = 1, 2, 3, 4). \tag{4}$$

Moreover, from the formulas:

$$\mathfrak{E} = -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t}, \qquad \mathfrak{H} = \operatorname{rot}\mathfrak{A}\,,$$

we will get the components of the electric and magnetic field strengths:

$$E_k = -\frac{\partial\varphi}{\partial x_k} - \frac{1}{c}\frac{\partial A_k}{\partial t} = i^2\frac{\partial\varphi}{\partial x_k} - i\frac{\partial A_k}{\partial x_4} = i\left(\frac{\partial A_4}{\partial x_k} - \frac{\partial A_k}{\partial x_4}\right) \qquad (k = 1, 2, 3)$$

and

$$H_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}, \qquad H_2 = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}, \qquad H_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}.$$

It will then be possible to regard the six quantities:

$$\left.\begin{array}{lll} H_1 = H_{23}, & H_2 = H_{31}, & H_3 = H_{12}, \\ -i\,E_1 = H_{14}, & -i\,E_2 = H_{24}, & -i\,E_3 = H_{34} \end{array}\right\} \tag{5}$$

as the components of a *four-dimensional skew-symmetric* tensor that are given by differentiating the potential components with respect to the coordinates according to the formulas:

$$H_{kl} = \frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l} = -H_{lk} \qquad (k, l = 1, 2, 3, 4). \tag{5.a}$$

We would now like to replace the components of the electric and magnetic field strengths in *Maxwell*'s fundamental equations I and II, **§ 3**, Chap. V, with the tensor components H_{kl} .

That will give the first group, namely:

$$\operatorname{rot}\mathfrak{E} + \frac{1}{c}\frac{\partial\mathfrak{H}}{\partial t} = 0, \qquad \operatorname{div}\mathfrak{H} = 0\,,$$

in the form of:

$$\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} + \frac{1}{c}\frac{\partial H_1}{\partial t} = i\left(\frac{\partial H_{34}}{\partial x_2} + \frac{\partial H_{32}}{\partial x_3} + \frac{\partial H_{23}}{\partial x_4}\right) = 0\,,$$

$$\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} + \frac{1}{c}\frac{\partial H_2}{\partial t} = i\left(\frac{\partial H_{14}}{\partial x_3} + \frac{\partial H_{43}}{\partial x_1} + \frac{\partial H_{31}}{\partial x_2}\right) = 0\,,$$
$$\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} + \frac{1}{c}\frac{\partial H_3}{\partial t} = i\left(\frac{\partial H_{24}}{\partial x_1} + \frac{\partial H_{41}}{\partial x_2} + \frac{\partial H_{12}}{\partial x_4}\right) = 0\,,$$
$$\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = \frac{\partial H_{23}}{\partial x_1} + \frac{\partial H_{31}}{\partial x_2} + \frac{\partial H_{12}}{\partial x_3} = 0\,,$$

in coordinate notation, i.e., as a unified system of equations:

$$\frac{\partial H_{ik}}{\partial x_l} + \frac{\partial H_{kl}}{\partial x_i} + \frac{\partial H_{li}}{\partial x_k} = 0\,, \tag{6}$$

in which i, k, l are three different numbers from the sequence 1, 2, 3, 4. Note that equations (6) will also remain valid when two or all three of those numbers are equal to each other. For example, for $k = l$, one will have $\frac{\partial H_{ik}}{\partial x_k} + \frac{\partial H_{ki}}{\partial x_k} = 0$ and $\frac{\partial H_{kk}}{\partial x_i} = 0$ since $H_{ik} = -H_{ki}$ and $H_{kk} = 0$.

In the same way, we will get the second group of *Maxwell* equations, namely:

$$\operatorname{rot}\mathfrak{H} - \frac{1}{c}\frac{\partial\mathfrak{E}}{\partial t} = 4\pi j\,, \qquad \operatorname{div}\mathfrak{E} = 4\pi\rho\,,$$

in the form:

$$\frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3} - \frac{1}{c}\frac{\partial E_1}{\partial t} = \frac{\partial H_{12}}{\partial x_2} + \frac{\partial H_{13}}{\partial x_3} + \frac{\partial H_{14}}{\partial x_4} = 4\pi j_1\,,$$
$$\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} = -\,i\left(\frac{\partial H_{41}}{\partial x_1} + \frac{\partial H_{42}}{\partial x_2} + \frac{\partial H_{41}}{\partial x_3}\right) = 4\pi\rho\,,$$

i.e.:

$$\sum_{k=1}^{4}\frac{\partial H_{kl}}{\partial x_l} = 4\pi j_k \qquad (k = 1, 2, 3, 4). \tag{7}$$

The formulas:

$$\mathrm{j} = \operatorname{rot}\mathfrak{M} + \frac{1}{c}\frac{\partial\mathfrak{P}}{\partial t}, \qquad \rho = -\operatorname{div}\mathfrak{P}$$

can be represented in a completely-analogous way [cf., (21) and (19.a), Chap. V]. In that way, the magnetic and electric polarization play the same roles as $\mathfrak{H} / 4\pi$ and $-\,\mathfrak{E} / 4\pi$, resp. If one sets:

$$\left.\begin{aligned} M_1 &= P_{23}, \quad M_2 = P_{31}, \quad M_3 = P_{12}, \\ i\,P_1 &= P_{14}, \quad i\,P_2 = P_{24}, \quad i\,P_3 = P_{34} \end{aligned}\right\} \tag{8}$$

then one will have:

$$j_k = \sum_{k=1}^{4} \frac{\partial P_{kl}}{\partial x_l} . \tag{8.a}$$

Just like $H_{kl} = - H_{lk}$, one can regard the quantities $P_{kl} = - P_{lk}$ as the components of a four-dimensional skew-symmetric tensor. Recall that despite their formula analogy and equal dimension, they are generally completely different since the quantities P_{kl} cannot satisfy an equation of the form (6).

Since the vectors $\mathfrak{Z}$ and $\mathfrak{Z}^*$ (electric and magnetic polarization potential) relate to the vector potential $\mathfrak{A}$ and the scalar potential φ in just the same way that $\mathfrak{P}$ and $\mathfrak{M}$ (= $\mathfrak{P}^*$) relate to j and ρ, resp., one can further set:

$$\left.\begin{array}{lll} Z_1^* = Z_{23}, & Z_2^* = Z_{31}, & Z_3^* = Z_{12}, \\ i\,Z_1 = Z_{14}, & i\,Z_2 = Z_{24}, & i\,Z_3 = Z_{34}, \end{array}\right\} \tag{9}$$

and

$$A_k = \sum_{k=1}^{4} \frac{\partial Z_{kl}}{\partial x_l} , \tag{9.a}$$

corresponding to (8) and (8.a).

From (20) and (22), Chap. V, the components of the *electromagnetic polarization potential* that arises from combining $\mathfrak{Z}$ and $\mathfrak{Z}^*$ will then satisfy the differential equations:

$$\sum_{k=1}^{4} \frac{\partial^2 Z_{kl}}{\partial x_k^2} = - 4\pi P_{kl} . \tag{9.b}$$

From (9.a) and (5.a), one can express the components of the electromagnetic field tensor H_{kl} in term of the components of the polarization potential as follows:

$$H_{kl} = \sum_{h=1}^{4} \frac{\partial}{\partial x_h} \left(\frac{\partial Z_{lh}}{\partial x_k} - \frac{\partial Z_{kh}}{\partial x_l} \right) . \tag{9.c}$$

Those expressions can coincide with (9.b) only when one has $\frac{\partial Z_{lh}}{\partial x_k} - \frac{\partial Z_{kh}}{\partial x_l} = \frac{\partial Z_{lk}}{\partial x_h}$. Those relations can be written in the form:

$$\frac{\partial Z_{lh}}{\partial x_k} + \frac{\partial Z_{hk}}{\partial x_l} + \frac{\partial Z_{lk}}{\partial x_h} = 0 ,$$

which is identical to equations (6). However, in reality, they are *not* fulfilled since from (9.b), corresponding equations for P_{kl} would then follow from them.

The known expression for the total force per unit volume (or the impulse per unit time $\mathfrak{f} = \rho\,\mathfrak{E} + \mathfrak{j} \times \mathfrak{H}$) and the corresponding work $l = c\,\mathfrak{j}\,\mathfrak{E}$ can be combined into the components of a "four-vector," and indeed one has:

$$f_1 = \rho\, E_1 + j_2\, H_3 - j_2\, H_2 = H_{12}\, j_2 + H_{13}\, j_3 + H_{14}\, j_4\,, \ldots$$

$$\frac{l}{c} = j_1\, E_1 + j_2\, H_2 + j_3\, H_2 = -\, i\,(H_{41}\, j_1 + H_{42}\, j_2 + H_{43}\, j_3)\,,$$

or with the notation:

$$\frac{i\,l}{c} = f_4\,, \tag{10}$$

$$f_k = \sum_{k=1}^{4} H_{kl}\; j_l \qquad (k = 1, 2, 3, 4). \tag{10.a}$$

If one substitutes $j_l = \frac{1}{4\pi}\sum_{n=1}^{4}\frac{\partial H_{kn}}{\partial x_n}$ in that, from (7), then one will have:

$$f_k = \frac{1}{4\pi}\sum_{l=1}^{4}\sum_{n=1}^{4} H_{kl}\,\frac{\partial H_{kl}}{\partial x_n} = \frac{1}{4\pi}\sum_{l}\sum_{n}\frac{\partial}{\partial x_n}(H_{kl}\, H_{ln}) - \frac{1}{4\pi}\sum_{l}\sum_{n} H_{ln}\,\frac{\partial H_{kl}}{\partial x_n}\,.$$

Moreover, switching the summation indices will give:

$$\sum_{l}\sum_{n} H_{ln}\,\frac{\partial H_{kl}}{\partial x_n} = \sum_{n}\sum_{l} H_{nl}\,\frac{\partial H_{kn}}{\partial x_l}\,,$$

or due to the fact that $H_{ln} = -\, H_{nl}$ and $H_{kn} = -\, H_{nk}$:

$$\sum_{l}\sum_{n} H_{ln}\,\frac{\partial H_{kl}}{\partial x_n} = \sum_{n}\sum_{l} H_{ln}\,\frac{\partial H_{nk}}{\partial x_l} = \tfrac{1}{2}\sum_{n}\sum_{l} H_{ln}\left(\frac{\partial H_{kl}}{\partial x_n} + \frac{\partial H_{nk}}{\partial x_l}\right),$$

and ultimately from (6):

$$\sum_{l}\sum_{n} H_{ln}\,\frac{\partial H_{kl}}{\partial x_n} = -\,\tfrac{1}{2}\sum_{l}\sum_{n} H_{ln}\,\frac{\partial H_{ln}}{\partial x_k} = -\,\tfrac{1}{4}\,\frac{\partial}{\partial x_k}\left(\sum_{l}\sum_{n} H_{ln}^2\right).$$

We then get:

$$f_k = \frac{1}{4\pi}\sum_{l}\sum_{n}\frac{\partial}{\partial x_n}(H_{kl} H_{ln}) + \frac{1}{16\pi}\,\frac{\partial}{\partial x_k}\left(\sum_{l}\sum_{n} H_{ln}^2\right).$$

If one introduces the relationships:

$$\Theta_{kn} = \delta_{kl}\, L + \frac{1}{4\pi}\sum_{l}\sum_{n} H_{kl}\, H_{ln}\,, \tag{11}$$

$$L = \frac{1}{16\pi}\sum_l \sum_n H_{ln}^2 = \frac{1}{8\pi}(H^2 - E^2), \tag{11.a}$$

$$\delta_{kl} = \begin{cases} 1 & \text{for} \quad k = n \\ 0 & \text{for} \quad k \neq n \end{cases} \tag{11.b}$$

then one will have:

$$f_k = \sum_{n=1}^{4} \frac{\partial \Theta_{kn}}{\partial x_n} \qquad (k = 1, 2, 3, 4). \tag{12}$$

One can easily convince oneself that the system of formulas that one just obtained is identical to the system of formulas:

$$\mathfrak{f} = -\frac{\partial \mathfrak{g}}{\partial t} - \text{div}\, {}^2\mathfrak{T}, \qquad l = -\frac{\partial \xi}{\partial t} - \text{div}\, \mathfrak{K},$$

which were derived in **§§ 3** and **4** of the previous chapter.

Indeed, the scalar ξ, the vectors $\mathfrak{g}$ and $\mathfrak{K}$, and the three-dimensional tensor ${}^2\mathfrak{T}$ are combined into a symmetric four-dimensional tensor whose components are given by (11). The following relations then exist (cf., Chap. VII, **§§ 3** and **4**):

$$\Theta_{11} = L - \frac{1}{4\pi}(H_{12}^2 + H_{13}^2 + H_{14}^2) = \frac{1}{8\pi}(2H^2 - E^2 - 2H_3^2 - 2H_2^2 + 2E_1^2)$$
$$= \frac{1}{8\pi}(H_1^2 - H_2^2 - H_3^2 + E_1^2 - E_2^2 - E_3^2) = -T_{11};$$
$$\Theta_{22} = -T_{22}; \qquad \Theta_{33} = -T_{33};$$

$$\Theta_{12} = -\frac{1}{4\pi}(H_{13}H_{23} + H_{14}H_{24}) = -\frac{1}{4\pi}(-H_2H_1 - E_1E_2) = -T_{12},$$
$$\Theta_{23} = -T_{23}; \qquad \Theta_{31} = -T_{31};$$

$$\Theta_{14} = -\frac{1}{4\pi}(H_{12}H_{42} + H_{13}H_{43}) = -\frac{1}{4\pi}(i\,E_2H_3 - i\,E_3E_2) = \frac{i}{4\pi}[\mathfrak{E} \times \mathfrak{H}]_1 = -\frac{i}{c}K_1 = i\,c\,g_1,$$
$$\Theta_{24} = -\frac{i}{c}K_2 = -i\,c\,g_2; \qquad \Theta_{34} = -\frac{i}{c}K_3 = -i\,c\,g_3;$$

$$\Theta_{44} = +\frac{1}{8\pi}(H^2 - E^2) - \frac{1}{4\pi}(H_{41}^2 + H_{42}^2 + H_{43}^2) = \frac{1}{8\pi}(H^2 + E^2) = \xi,$$

which can be represented by the matrix:

$$\left(\begin{array}{ccc|c} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ \Theta_{21} & \Theta_{22} & \Theta_{23} & \Theta_{24} \\ \Theta_{31} & \Theta_{32} & \Theta_{33} & \Theta_{34} \\ \hline \Theta_{41} & \Theta_{42} & \Theta_{43} & \Theta_{44} \end{array}\right) = \left(\begin{array}{c|c} -{}^2\mathfrak{T} & -i\,c\,\mathfrak{g} \\ \hline \dfrac{1}{i\,c}\,\mathfrak{K} & \xi \end{array}\right). \tag{12.a}$$

When one considers that matrix, one can write the first three equations in (12) in the form:

$$f_k = -\frac{\partial T_{k1}}{\partial x_1} - \frac{\partial T_{k2}}{\partial x_2} - \frac{\partial T_{k3}}{\partial x_3} - \frac{\partial g_k}{\partial t} \qquad (k = 1, 2, 3),$$

and the fourth one in the form:

$$l = -\,i\,c\,f_4 = -\frac{\partial K_1}{\partial x_1} - \frac{\partial K_2}{\partial x_2} - \frac{\partial K_3}{\partial x_3} - \frac{\partial \xi}{\partial t}.$$

The cited representation of the basic quantities and equations of the electromagnetic field, which goes back to *H. Minkowski*, can be interpreted *in a purely-formal mathematical way* as follows:

We imagine an (in reality, nonexistent) four-dimensional space that relates to three-dimensional space in the same way that the latter does to the plane. We further imagine a *rectangular coordinate system X* in that space, i.e., we draw four *mutually-perpendicular* axes X_1, X_2, X_3, X_4 . We will clarify the analytical meaning of the "orthogonality" of the four axes in the following sections. We shall initially remark that it corresponds to the assumed *orthogonality of the spatial axes* X_1, X_2, X_3, in conjunction with the *symmetry of the cited equations relative to all four indices*. In that way, we use the fourth axis as a "graphical representation" of time, multiplied by ic, so the four quantities $x_1, x_2, x_3, x_4 = ict$ are collectively the rectangular components of a four-vector" $\mathfrak{r}$ that characterized the position and time of any event. We accordingly introduce the following four-dimensional vectors: $\mathfrak{j}$ (four-current), $\mathfrak{A}$ (four-potential), $\mathfrak{f}$ (impulse per unit volume in four-dimensional space), and the four-dimensional tensors: ${}^2\mathfrak{H}$ (skew-symmetric field tensor) and ${}^2\Theta$ (symmetric impulse or stress-energy tensor). Finally, we define the four-dimensional operator $\Box$, which corresponds to the ordinary operator ∇, as a symbolic "four-vector" with the components $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}$. The basic equations of the electromagnetic field can then be represented in a *coordinate-free* notation as relations between the four-dimensional vectors and tensors above.

Equations (2.a), (3.a), and (4) can be written with no further analysis in the form:

$$\left.\begin{aligned} \Box\,\mathfrak{j} \;&= \mathbf{div}\,\mathfrak{j} = 0, \\ \Box\,\mathfrak{A} \;&= \mathbf{div}\,\mathfrak{A} = 0, \\ \Box^2\mathfrak{A} &= -4\pi\,\mathfrak{j}, \end{aligned}\right\} \tag{13}$$

which is completely-analogous to the corresponding equations for a time-constant magnetic field (and a stationary electrical current).

We would like to represent formulas (5.a), (6), and (7) as follows:

$$\left.\begin{array}{l} {}^2\mathfrak{H} = \square \times \mathfrak{A} = \mathbf{rot}\,\mathfrak{A}, \\ \mathbf{rot}\ {}^2\mathfrak{H} = 0, \\ \square^2\,\mathfrak{H} = \mathbf{div}\ {}^2\mathfrak{H} = 4\pi\,\mathbf{j}. \end{array}\right\} \tag{13.a}$$

Here, the analogy with the corresponding three-dimensional ("time-less") formulas will be spoiled by the fact that a skew-symmetric four-dimensional tensor (in contrast to the three-dimensional one) is *not* congruent to a vector. In fact, the tensor ${}^2\mathfrak{H}$ is determined by *six* independent quantities, not four, from which its twelve non-vanishing pair-wise equal and opposite components $H_{kl} = -H_{lk}$ are constructed. Following *Minkowski*, one cares to refer to it as a *six-vector*, i.e., as a vector that is defined by its projections onto the *six coordinate planes* ($X_2 X_3$, $X_3 X_1$, $X_1 X_2$, $X_1 X_4$, $X_2 X_4$, $X_3 X_4$), not onto the four coordinate axes ([1]).

The operation **div** ${}^2\mathfrak{H}$ that appears in (13.a) corresponds to the divergence of an ordinary three-dimensional tensor. In that way, one will have:

$$(\mathbf{div}\ {}^2\mathfrak{H})_k = \sum_{l=1}^{4} \frac{\partial H_{kl}}{\partial x_l}$$

(cf., Introduction, **§ 23**).

The operation **rot** ${}^2\mathfrak{H}$ corresponds to the **rot** of a four-vector in the sense that it will take a skew-symmetric tensor of rank two to a tensor of rank three whose non-vanishing components are on the left-hand side of equations (6). Those four quantities can be regarded as the components of a four-vector that is congruent to the aforementioned third-rank tensor.

If one introduces a skew-symmetric tensor ${}^2\mathfrak{H}^*$ that is "dual" to ${}^2\mathfrak{H}$ by way of the matrix:

$$\begin{pmatrix} 0 & H_{12} & H_{13} & H_{14} \\ H_{21} & 0 & H_{23} & H_{24} \\ H_{31} & H_{32} & 0 & H_{34} \\ H_{41} & H_{42} & H_{43} & 0 \end{pmatrix} = \begin{pmatrix} 0 & H_{43}^* & H_{24}^* & H_{32}^* \\ H_{34}^* & 0 & H_{41}^* & H_{13}^* \\ H_{42}^* & H_{14}^* & 0 & H_{21}^* \\ H_{23}^* & H_{31}^* & H_{12}^* & 0 \end{pmatrix}$$

then one will have:

$$\frac{\partial H_{23}}{\partial x_1} + \frac{\partial H_{31}}{\partial x_2} + \frac{\partial H_{12}}{\partial x_3} = \frac{\partial H_{41}^*}{\partial x_1} + \frac{\partial H_{42}^*}{\partial x_2} + \frac{\partial H_{43}^*}{\partial x_3} = (\mathbf{div}\ {}^2\mathfrak{H})_4\ ,$$

$$\frac{\partial H_{43}}{\partial x_2} + \frac{\partial H_{24}}{\partial x_3} + \frac{\partial H_{32}}{\partial x_4} = \frac{\partial H_{12}^*}{\partial x_2} + \frac{\partial H_{13}^*}{\partial x_3} + \frac{\partial H_{14}^*}{\partial x_4} = (\mathbf{div}\ {}^2\mathfrak{H})_1\ ,$$

([1]) Note that in the case of two dimensions, a skew-symmetric tensor reduces to a scalar.

$$\frac{\partial H_{41}}{\partial x_3}+\frac{\partial H_{13}}{\partial x_4}+\frac{\partial H_{34}}{\partial x_1} = \frac{\partial H^*_{23}}{\partial x_3}+\frac{\partial H^*_{24}}{\partial x_4}+\frac{\partial H^*_{21}}{\partial x_1} = (\mathbf{div}\ {}^2\mathfrak{H})_2\ ,$$
$$\frac{\partial H_{21}}{\partial x_4}+\frac{\partial H_{42}}{\partial x_1}+\frac{\partial H_{14}}{\partial x_2} = \frac{\partial H^*_{34}}{\partial x_4}+\frac{\partial H^*_{31}}{\partial x_1}+\frac{\partial H^*_{32}}{\partial x_2} = (\mathbf{div}\ {}^2\mathfrak{H})_3\ ,$$

such that the equation **rot** ${}^2\mathfrak{H} = 0$ can be replaced with:

$$\mathbf{div}\ {}^2\mathfrak{H}^* = 0\ .$$

Entirely similar formulas are true for the skew-symmetric tensors ${}^2\mathfrak{P}$ and ${}^2\mathfrak{Z}$. We do not need to write them out again.

According to formulas (8.a), we further have that the components of the four-impulse are:

$$\mathfrak{f} = {}^2\mathfrak{H} \cdot \mathfrak{j}\ , \tag{14}$$

in which the inner product of the tensor ${}^2\mathfrak{H}$ and the vector $\mathfrak{j}$ is defined by analogy with the corresponding product of three-dimensional quantities [Introduction (37)]. Finally, equations (12) can be written in the form:

$$\mathfrak{f} = \mathbf{div}\ {}^2\Theta\ , \tag{14.a}$$

from which, (9) and (9.a) will imply that:

$${}^2\Theta = {}^2\boldsymbol{\delta}\,\frac{1}{8\pi}\,({}^2\mathfrak{H}\cdot{}^2\mathfrak{H}) + \frac{1}{4\pi}\,({}^2\mathfrak{H}\times{}^2\mathfrak{H}) \tag{14.b}$$

[cf., Introduction (39)].

Note that when one projects four-dimensional vectors onto the time axis, one will get ordinary scalar quantities, while projecting onto ordinary space will yield three-dimensional vectors that are very closely connected with those scalars physically. The components of four-dimensional tensors along the "spatial axes" correspondingly define ordinary three-dimensional tensors, while their components along one spatial axis and the time axis define ordinary vectors, and their component along the time axis is a scalar [cf., the matrix (12.a)].

In what follows, we will mainly employ the *coordinate-wise* representation of electromagnetic quantities and equations. We have formulated them in a coordinate-free notation only in order to highlight their analogy with the ordinary vectorial equations for time-constant magnetic and electric field. That formal analogy allows us to treat the electromagnetic phenomena in ordinary three-dimensional space that depend upon time as *static* (i.e., time-independent) phenomena in a fictitious four-dimensional space.

§ 2. – The Lorentz transformation.

Ordinary Euclidian space is *isotropic*, i.e., all directions are regarded as completely equivalent. One can also refer to that physical equivalence of the various directions as their *relativity* in the sense that any direction can be determined only *relative* to other directions that are assumed to be known. An *absolutely* determined direction that is distinguished by special physical properties cannot be given at all.

It is just that isotropy of space that guarantees the possibility of a coordinate-free formulation of the physical laws, and in particular, the laws of electromagnetic phenomena, as relationships between vectorial (as well as scalar and tensorial) quantities. If a direction in space were physically distinguished ([1]) then one could formulate the aforementioned laws only with respect to that direction. Obviously, it is possible, and in many cases useful, to replace that coordinate-free formulation of the physical laws with a coordinate-wise one. In that way, all (three-dimensional) vectors will be represented by their components relative to three arbitrarily-chosen axes or directions. We have referred to those components as *variant* scalars in the Introduction since their values are determined only relative to the chosen coordinate system and will vary under the transition from one coordinate system to another.

Since all coordinate systems are regarded as equivalent due to the relativity of directions, it would be absurd to ask which values of the components of a certain vector are the "correct" or "true" values. The triples of scalars (A_1, A_2, A_3) and (A_1', A_2', A_3'), which represent the components of one and the same vector $\mathfrak{A}$ relative to two different coordinate systems X and X', are completely equivalent. The only thing that can and must be of interest to us is the relationship between the two scalar triples. If the quantities (A_1, A_2, A_3) are assumed to be known and determine the directions of the X'-axes relative to X by the corresponding cosines of the angles $\alpha_{ii'}$ then the quantities (A_1', A_2', A_3') must be expressible in terms of the A_i and $\alpha_{ii'}$ in some way. In other words, one must establish only the formulas by which the components of a vector *transform* under any sort of changes in the coordinate directions.

Now, the following should be emphasized in particular: *The components of different vectors transform in the same way* (i.e., according to the same formulas), or on other words: The corresponding components of the different vectors are *covariant* scalar quantities. That remark initially seems to be entirely trivial and would follow directly from one's intuition. However, in reality, it has a much deeper sense and defines the analytical expression for the *principle of the invariance of physical laws*. By that, we mean: The physical laws can be expressed by *equations* that couple different vector quantities (as well as invariant scalars, tensors, etc.) with each other. In a coordinate representation of the vectors, those equations will have the following form:

$$A_i = B_i ,$$

in which $\mathfrak{A}$ and $\mathfrak{B}$ mean two *physically-different* vectors. We imagine, e.g., that $\mathfrak{A}$ is the product of the mass of a particle with its acceleration, and $\mathfrak{B}$ is the external force that acts on that particle.

([1]) For example, as one believed the "vertical" direction to be in the Middle Ages.

If the components of the acceleration and the force are not covariant quantities then the equation above cannot be true independently of the choice of the coordinate directions. Analytically, the *relativity* of those directions, namely, their *physical equivalence*, means nothing but the covariance of the corresponding vector components for all coordinate transformations. The coordinate directions are relative, and the vector components are variant. However, the physical laws that couple those vector components with each other *must be absolute and invariant*. That requirement will be fulfilled by the *covariance* of the corresponding vector components. Obviously, the same thing is true for not only vectors, but for tensor quantities of arbitrary rank.

In what follows, we will restrict ourselves to rectangular coordinate systems. As is known, in that way, the components of a vector will coincide with its projections, such that we will not need to distinguish between the two (as would happen in the general case of skew coordinate systems). We have exhibited the transformation formulas for vector and tensor components for this case in the Introduction, **§ 18**. Obviously, those formulas will also remain valid when we fix, e.g., the third axis and restrict ourselves to coordinate transformations in a plane (e.g., the X_1X_2-plane). In that way, we can treat the component that bears the index 3 as an invariant scalar and treat the ones with the indices 1, 2 as the components of two-dimensional vectors.

We would now like to take a step in the opposite direction and consider the transformation of the coordinate axes and the transformation of the vector and tensor components in the fictitious *four-dimensional space* of the previous section in an entirely analogous way in which the fourth axis (viz., the time axis) can no longer be regarded as fixed.

We introduce a new coordinate system $X'(X'_1, X'_2, X'_3, X'_4)$ in place of the original coordinate system X (X_1, X_2, X_3, X_4) according to the transformation formulas [cf., Introduction (32.a) and (32.b)]:

$$x'_{i'} = \sum_{i=1}^{4} \alpha_{ii'}\, x_i \,, \tag{15}$$

$$x_i = \sum_{i'=1}^{4} \alpha'_{ii'}\, x'_{i'} \,, \tag{15.a}$$

which must satisfy the *orthogonality condition:*

$$x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \tag{16}$$

identically. That condition will imply the following relations between the transformation coefficients:

$$\alpha'_{ii'} = \alpha_{ii'} \qquad (i,\ i' = 1, 2, 3, 4), \tag{16.a}$$

$$\sum_{i=1}^{4} \alpha_{ii'}\, \alpha_{i'k} = \delta_{ii'} \,, \qquad \sum_{i=1}^{4} \alpha_{ki}\, \alpha_{ki'} = \delta_{ii'} \,, \tag{16.b}$$

in which one has:

$$\delta_{ii'} = \begin{cases} 1 & \text{for} \quad i' = i, \\ 0 & \text{for} \quad i' \neq i. \end{cases} \tag{16.c}$$

We define those coefficients in accordance with their intuitive geometric meaning in terms of ordinary three-dimensional transformations as the cosines of the angles between the old and new axes:

$$\alpha_{ii'} = \cos(X_i, X'_{i'}) = \alpha'_{ii'} .$$

We can imagine that the new system X' arises from a "rotation" of the original axis-cross in four-dimensional space that preserves orthogonality.

Note that those terms, which are connected with the corresponding terms in ordinary analytic geometry, are nothing more than words in the case considered that actually carry no geometric meaning. However, in that way, we can imagine the corresponding three-dimensional, or even two-dimensional, structures, and in that way not only better conceptualize the analytical connections, but at the same time, better predict them by reasoning by analogy. The quantities x_1, x_2, x_3, x_4 and x'_1, x'_2, x'_3, x'_4, which represent the original (new, resp.) coordinates of the various points in X-space, need to assume not only real or imaginary values, but entirely-arbitrary complex numbers (cf., Chap. VI, **§ 4**).

We would like to fix those coordinates as the rectangular components of the four-dimensional radius vector $\mathfrak{r}$. In that way, we would like to establish that the components of the various four-dimensional vectors ($\mathfrak{j}$, $\mathfrak{A}$, $\mathfrak{f}$) that we introduced in **§ 1** transform in accordance with the same formulas (15), (15.a) as the components of $\mathfrak{r}$ (so they are covariant). As far as the components of the differential operator $\Box$ is concerned, its covariance is a direct consequence of the orthogonality of the transformation that is represented by those formulas (just like in the three-dimensional case). The tensor components H_{kl}, Θ_{kl} shall transform like the product of the corresponding coordinates, so they shall be doubly covariant.

One can easily show that under those assumptions *the fundamental equations of the electromagnetic field* (2.a)-(10) *remain invariant under the transformations that are determined by formulas* (15)-(15.c). That means: When one expresses the original components in the old equations as functions of the new ones, one will get the equations of the same form in those new components from that process.

We can easily prove that assertion directly. For example, we consider equations (2.a) and (4). Since the symbol $\Box_k = \partial / \partial x_k$ and the quantities A_k transform like the x_k, we will get from (16) that:

$$\sum_{k=1}^{4} \Box_k A_k = \sum_{k'=1}^{4} \Box'_{k'} A'_{k'} ,$$

and as a result, due to (2.a):

$$\sum_{k'=1}^{4} \Box'_{k'} A'_{k'} = 0 .$$

Moreover, the square of the operator $\Box$, i.e., the sum $\sum_{k=1}^{4} \Box_k^2$, obviously represent an invariant operator. If we then substitute:

$$j_k = \sum_{k'=1}^{4} \alpha_{kk'} \, j'_{k'} \quad \text{and} \qquad A_k = \sum_{k'=1}^{4} \alpha_{kk'} \, A'_{k'}$$

in (4) according to (15.a) then we will have:

$$\sum_{k=1}^{4} \square_k^2 A_k = \sum_{l=1}^{4} \square_l'^2 A_k = \sum_{k'=1}^{4} \alpha_{kk'} \sum_{l=1}^{4} \square_l'^2 A'_{k'} = - \, 4\pi \sum_{k'=1}^{4} \alpha_{kk'} \, j'_{k'} \; .$$

Those four equations (k = 1, 2, 3, 4) can be written in the form:

$$\sum_{k'=1}^{4} \alpha_{kk'} \, y_{k'} = 0 \; ,$$

and the quantities $y_{k'} = \sum_{l=1}^{4} \square_l'^2 A'_{k'} + 4\pi \, j'_{k'}$ can be treated as four unknowns. Since the determinant of the coefficients $\alpha_{kk'}$ is equal to 1 due to the conditions (16.b), those quantities must vanish identically, and we will get:

$$\sum_{l=1}^{4} \square_l'^2 A'_{k'} = - \, 4\pi \, j'_{k'} \, .$$

However, the stated invariance of the fundamental equations can also be recognized in a very instructive indirect way, and indeed from *their symmetry relative to all four indices* 1, 2, 3, 4, in conjunction with their invariance for those orthogonal transformations for which the fourth axis remains fixed, so ones that correspond to an ordinary rotation of the spatial system of axes X_1, X_2, X_3. That “spatial” invariance is a direct consequence, and at the same time, the condition for the isotropy of the ordinary three-dimensional space of the relativity of all spatial directions; that is why it can be assumed. Due to the aforementioned symmetry property of equations (2.a) to (10), one can assert that they will also remain invariant when one performs an orthogonal three-dimensional transformation of the coordinates x_2, x_3, x_4 (instead of x_1, x_2, x_3) and transform the corresponding components of all four-dimensional vectors (and tensors) covariantly. However, two successive orthogonal transformations of each of the three coordinates will give a transformation of all four coordinates that is also orthogonal, i.e., it fulfills the condition (16).

With the use of geometric terminology, we can then say that *the fundamental equations of the electromagnetic field are invariant under all rotations of the four-dimensional system of axes X*. However, with that, the property of *isotropy*, namely, the relativity of directions in ordinary space, is adapted to the demands of four-dimensional space. In particular, we can consider the “fourth” direction, which carries the time coordinates, to be relative (i.e., undetermined) and treat the fourth components of the vectors $\mathfrak{r}$, $\mathfrak{j}$, $\mathfrak{A}$, etc., (i.e., time, charge density, scalar potential, which are usually regarded as invariant scalar quantities) as *variant*, just like the spatial components of ordinary vectors. The absolute magnitude, or square, of those vectors, e.g., $x_1^2 + x_2^2 + x_3^2$ (viz., the square of the spatial distance) will likewise be variant quantities, and only the sum of the squares of all four components, which we can define to be the square of the corresponding four-dimensional vector, is actually regarded as invariant.

Now, one might ask whether that four-dimensional relativity of direction and the four-dimensional orthogonal transformations that express it analytically have a well-defined *physical* sense, or are they entirely meaningless from a physical standpoint?

In order for them to possess an actual physical meaning, it is obviously necessary (but still not initially sufficient) that the transformed quantities $t' = x_4' / ic$, $\rho' = j_4' / c$, $\varphi' = A_4' / i$, etc., should be *real*, just like the original ones.

One cares to refer to the four-dimensional orthogonal transformations that satisfy that relativity condition as the *Lorentz transformations* ([1]).

In order to clarify the question that was formulated above, we would now like to consider a simple Lorentz transformation under which the second and third axis will remain fixed, so it corresponds to a "rotation" in the X_1X_4-plane. In that case, the general transformation formulas (15) and (15.a) will reduce to:

$$x_1' = \alpha_{11}\, x_1 + \alpha_{41}\, x_4 ,$$
$$x_4' = \alpha_{41}\, x_1 + \alpha_{44}\, x_4 ,$$

or

$$x_1 = \alpha_{11}\, x_1' + \alpha_{14}\, x_4' ,$$
$$x_4 = \alpha_{41}\, x_1' + \alpha_{44}\, x_4' ,$$

and $x_2 = x_2'$, $x_3 = x_3'$.

We imagine, for the moment, that X_4 is an ordinary axis that is perpendicular to X_1 and imagine that the transformation in question is a rotation through an angle of φ in the direction from X_4 to X_1. We will then have $\alpha_{11} = \cos(X_1, X_1') = \cos\varphi$, and:

$$\alpha_{14} = \cos(X_1, X_4') = \sin\varphi, \qquad \alpha_{41} = \cos(X_4, X_1') = -\sin\varphi,$$
$$\alpha_{44} = \cos(X_4, X_4') = \cos\varphi,$$

and as a result:

$$\left.\begin{aligned} x_1' &= x_1\cos\varphi - x_4\sin\varphi, \\ x_4' &= x_1\sin\varphi + x_4\cos\varphi \end{aligned}\right\} \tag{17}$$

or

$$\left.\begin{aligned} x_1 &= x_1'\cos\varphi + x_4'\sin\varphi, \\ x_4 &= -x_1'\sin\varphi + x_4'\cos\varphi. \end{aligned}\right\} \tag{17.a}$$

Therefore, if $x_4 = i\,c\,t$, x_1' is be real, and x_4' is imaginary ($= ict'$, where t' means a real quantity) then we must define the angle φ to be a pure imaginary quantity. However, we do not need to consider that angle itself, but only its tangent. Namely, we set:

([1]) From *H. A. Lorentz*, who introduced them for the first time, and indeed in a three-dimensional notation.

$$\tan \varphi = -i\,\beta = \frac{v}{i\,c}, \tag{17.b)}$$

in which $\beta = v / c$ is a *real* quantity (that corresponds to $\varphi = i\,\psi$, $\tan\psi = -\beta$). From (17), we will then have:

$$x_1' = \cos\varphi\,(x_1 - v\,t), \qquad i\,c\,t' = \cos\varphi\left(i\,c\,t - i\frac{v}{c}x_1\right),$$

i.e., due to:

$$\cos\varphi = \frac{1}{\sqrt{1+\tan^2\varphi}} = \frac{1}{\sqrt{1-\beta^2}},$$

we will have:

$$x_1' = \frac{x_1 - v\,t}{\sqrt{1-\beta^2}}, \qquad t' = \frac{t - v\,x_1 / c^2}{\sqrt{1-\beta^2}}, \tag{18}$$

and likewise, from (17.a) [or by solving (18) for x_1 and t]:

$$x_1 = \frac{x_1' + v\,t'}{\sqrt{1-\beta^2}}, \qquad t = \frac{t' + v\,x_1' / c^2}{\sqrt{1-\beta^2}}. \tag{18.a}$$

In order to ensure the reality of x_1' and t', it still remains to subject the parameter β to the condition that $\beta < 1$, i.e., to set:

$$v < c\,. \tag{18.b}$$

The quantity v, just like c, obviously means a *speed*. In the limiting case that is it is very small compared to c (= speed of light), formulas (18) will reduce to:

$$x_1' = x_1 - v\,t, \quad t' = t, \tag{19}$$

i.e., to the usual formulas for the transition from the original spatial coordinate system (X_1, X_2, X_3) to another one (X_1', X_2, X_3) that has the same orientation and moves with a constant velocity v in the positive X_1-direction relative to (X_1, X_2, X_3). The origins in the two systems O and O' must then coincide at the "initial moment" $t = t' = 0$.

We accordingly get the (1, 4)-components of the "four-current" $\mathbf{j}$ and the four-potential $\mathfrak{A}$ from:

$$j_1' = j_1 - \frac{v}{c}\rho, \qquad \rho' = \rho, \tag{19.a}$$

$$A_1' = A_1 - \frac{v}{c}\varphi, \qquad \varphi' = \varphi. \tag{19.a}$$

The (approximate) equations that follow from that covariance principle will likewise coincide with the usual formulas for the transformation that is determined by (19).

We then see that in the approximation in question, the rotation of the "time axis" X_4 (with one of the "spatial" axes) through an imaginary angle in four-dimensional space means nothing physically besides the transition from the rest state of the spatial system of axes to a *uniform, rectilinear motion of them*.

Now since the two axes X_4 and X_4' have the same right to be the time axis, one must obviously have the same right to regard the spatial coordinate system (X_1, X_2, X_3) as being "at rest" and (X_1', X_2, X_3) as "moving," and indeed in the opposite direction. *The relativity of directions, and in particular, the direction of the time axis in four-dimensional space, thus means nothing beyond the relativity of velocity in ordinary three-dimensional space.*

Recall that we postulated this kinematical relativity principle before in Chap. V as a generalization of the "energy equation" rot $\mathfrak{E} = 0$ for time-alternating fields, but in a somewhat-narrower sense. Indeed, there we were dealing with the motion of two current lines *relative to each other*, while the relativity of velocity that we now assert means that the electromagnetic phenomena play out in an entirely identical way relative to two *coordinate systems* that move relative to each other uniformly and rectilinearly, such that there is no possibility of inferring which system is "actually" moving from those phenomena. In other words, the *magnitude of the velocity is just as relative as its direction*.

Now, it seems that this relativity of velocity that would correspond to the isotropy of four-dimensional space exists only approximately, namely, in the limiting case of speeds that are very small compared to the "critical" speed c. If one would like to regard it as a completely *rigorous* and generally-valid physical law then one would meet up with the following difficulties:

One must restrict the speeds from the outset by the condition that $v < c$, i.e., speeds that are greater than c are regarded as not only *physically impossible*, but also impossible, *in principle*.

Moreover, in order for electromagnetic phenomena to play out according to the same laws (i.e., according to the same equations) from the standpoint of two observers that are comoving with the coordinate systems (X_1, X_2, X_3) and (X_1', X_2, X_3), the determination of lengths in the direction of motion and of times must be carried out according to the *Lorentz* formulas (18), not the usual so-called *Galilean* transformation formulas (19). Therefore, *different times* must be valid for the two observers X and X'. Events that appear to be simultaneous to one of them must be considered to be non-simultaneous by the other. In general, the *time duration* between two well-defined events shall be an undetermined *variant* quantity, just like the distance between the spatial points where those events took place. In that way, one must observe the following: The indeterminacy or variance of such distances for *non-simultaneous* events is a fact that has been known for some time and seems quite natural. For example, we imagine a stone that is thrown on a moving ship from aft to fore. The distance between the initial and final points on its path will be judged by the comoving observer in a completely-different way from the way that it is judged by the observer that is at rest on the land, who must still consider the displacement of the ship during the flight of the stone. That fact is expressed quite clearly by the equations of the *Galilean* transformation x_1' $= x - v\,t$, $t' = t$. Namely, if one denotes the spatial and temporal distances between the aforementioned events (viz., throwing and dropping of the stone) from the standpoints of the observe at rest and the comoving one by Δx_1, Δt ($\Delta x_1'$, $\Delta t'$, resp.) then according to the formulas above, one will have:

$$\Delta t' = \Delta t , \qquad \Delta x_1' = \Delta x_1 - v\,\Delta t .$$

However, if one is dealing with two *simultaneous* events at different spatial points then the spatial distance will seem to be a well-defined *invariant* quantity that is independent of the relative velocity ($\Delta t = 0$, $\Delta x_1' = \Delta x_1$).

By contrast, according to the *Lorentz* formulas (18), the spatial distance between two events will remain a variant quantity even in the case of their "simultaneity." Obviously, that is connected with the fact that aforementioned "simultaneity" has no absolute meaning due to the *relativity of time*, which corresponds to the relativity of the direction of the time axis in four-dimensional space. That relativity of time (or more precisely, its *variance*) is closely connected by the Lorentz transformation with the variance of quantities such as charge density, scalar potential, etc., or the magnitude of three-dimensional vectors, i.e., quantities that would ordinarily be considered to be invariant scalars.

§ 3. – Einstein's relativity principle.

It is epistemologically clear from the outset that *motion* is a *relative* concept, i.e., that the motion of any body or the propagation of an action can be defined only relative to a coordinate system that one treats *quite arbitrarily* as being "at rest." All of the quantities that characterize the motion (viz., velocity, acceleration, etc.) must correspondingly be *variant* ones, i.e., they must depend upon the choice of that coordinate system. In the foregoing chapter, we deliberately overlooked that fact and treated the velocity as an "absolutely" determined quantity, i.e., we tacitly based it upon a coordinate system that was assumed to be "at rest" in an absolute sense of that word.

If *all of space* is filled with a continuous material medium (as was previously believed and is still asserted many times today when that medium is referred to as the "ether") whose parts are supposed to be in a state of eternal rest relative to each other then one can define the concept of rest uniquely from the physical standpoint as being at rest relative to that "ether." That would not be "absolute" rest in the epistemological sense of the word since one would still have to consider any motions of the "ether" as a solid entity in space relative to something else. However, such motions could be of no interest to the physicist, and rest relative to the "ether" would be equivalent to absolute rest for him.

However, in reality, there is no rational basis for filling up space with such a medium. The material world consists of nothing but electrons that act upon each other through empty space. That action-at-a-distance in the modern theory of the electron differs from the *actio in distans* of classical mechanics only by the fact that it is not "instantaneous," but *retarded*. The finite speed of propagation of electromagnetic actions has given rise to precisely the viewpoint that they are "local actions in the world-ether" of the same type as the propagation of sound in air or the elastic oscillations in solid bodies.

Now, remember how that "local action" was dealt with in classical mechanics by connecting it with the aforementioned "ether theory." One does not initially consider the body to be a continuous medium, but as a system of discrete mass-points (viz., atoms) that are found at *finite*, but very small, distances from each other. The interaction between those mass-points will then be treated as an *action-at-a-distance*, and indeed as an *instantaneous* action-at-a-distance since the force that is exerted upon any particle is determined the *simultaneous* configuration of the other ones, and mainly by the neighboring particles. If one assumes that this force varies in proportion to the relative displacement of the particles then one will get a finite speed of propagation for the "perturbations" of the normal equilibrium configuration of the particles in a known way. The passage to the limit of infinitely-small particles and infinitely-small distances, i.e., to the continuum theory, that one makes afterwards represents a mere mathematical fiction.

The "local action" of classical mechanics then means nothing but the usual "instantaneous action-at-a-distance." The fact that the latter takes place at very small distances ("small" compared to our ordinary macroscopic yardsticks) is basically inessential.

The retarded action-at-a-distance of the modern "electromechanics" can indeed be reduced to an instantaneous action-at-a-distance quite formally (cf., Chap., **§ 5**), but the corresponding series development has no immediate physical sense. That is why it is meaningless and useless to try to reduce the retarded electromagnetic action-at-a-distance to anything else. Rather, we can assert that all physical forces (to the extent that they represent an interaction between different electrons in the final analysis) *can be regarded as electromagnetic actions-at-a-distance that propagate in empty space with a well-defined speed of* $c = 3 \times 10^{10}$ cm/s.

One must recognize that fact as a fundamental principle that neither needs nor admits an "explanation." That is because an "explanation" would have to mean reducing it to something simpler and more fundamental. However, such a reduction of the fundamental facts is obviously impossible.

Since space is empty, except for electrons (which are point-like, in practice), one can speak of only *relative motion* and *relative rest*. Physical processes must satisfy the same laws when assessed in two coordinate systems that move relative to each other. In other words, the equations that express those laws and exist between the spatial coordinates and time, on the one hand, and different variant quantities that represent the components of certain vectors and tensors, on the other hand, must have an identical form in the two systems. We would now like to restrict ourselves to those coordinate systems that move relative to each other with a *uniform, rectilinear motion*, i.e., the so-called *inertial systems*. The general principle of the relativity of motion reduces to the principle of the relativity of *velocity* in this special case. That "special" principle of relativity was already recognized since the time of *Newton* in classical mechanics ([1]). Its adaptation to electrodynamics and the corresponding conversion of *Newtonian* mechanics was the contribution of *A. Einstein*. It is known that he also succeeded in generalizing the principle of relativity to arbitrary motions on the basis of the equivalence of inertial forces that are coupled with an accelerated motion and gravitational forces. However, we would not like to go into that general theory of relativity in this book since it mainly addresses gravitational effects.

The essential difference between the classical and *Einsteinian* theory of relativity originates in the fact in the latter considers the *finiteness* of the speed of propagation of electromagnetic action-

([1]) Although *Newton* himself spoke of a "space in a state of absolute rest."

at-a-distance, while in the former, that speed of propagation was regarded as infinitely large, so to speak.

Indeed, it follows from the principle of relativity that this speed of propagation *must have the same magnitude c* in two different inertial systems S (X_1, X_2, X_3) and $S'(X'_1, X'_2, X'_3)$, *and in particular, it must have the same magnitude for all directions in each system*.

For instance, if the last requirement were not fulfilled then one could not assert that the system (S) relative to which the light spreads out in all directions with the same speed c is "actually" at rest, while the other one relative to which that does not happen "actually" moves, namely, in the direction that corresponds to the smallest speed of light. However, since the systems S and S' are supposed to be completely equivalent, the process of light propagation must take place in the same way relative to both of them, i.e., with the same speed c in all directions.

We then see that the critical speed c, although it has only a *relative* sense, like any other velocity (i.e., it can be defined only relative to any reference system that can be regarded as being "at rest"), *is nonetheless regarded as an invariant quantity.*

That invariance of the critical speed is obviously incompatible with the usual picture of spacetime that one associates with classical mechanics. For example, if one replaces the thrown rock with a light or radio signal in the case of a moving ship that was considered above then it must propagate with the *same* speed c relative to the ship, whether forward, backward, to the right or to the left, that it has relative to the observer that stands on the shore. Obviously, that can happen only when the lengths and times are measured to be different in the corresponding reference systems, and in particular, when the concept of simultaneity can be assigned only a relative meaning. As long one imagines that the mechanical actions-at-a-distance are instantaneous and that they can all be combined with each other in time by the action of one body on the other ones, no matter how big their spatial dimensions might be, the "relativization" of simultaneity would seem to be physically excluded. However, as long as one considers the retarded character of the mechanical action-at-a-distance, that relativization will be not only possible, but indispensable. That is because there is no physical possibility of uniquely establishing the simultaneity of two spatially-separated events, e.g., on the planets Mars and Jupiter.

For some time now, we have been accustomed to ascribing no absolute sense to the *spatial* coincidence of two events that are not simultaneous. We must likewise reject any such absolute sense to the temporal coincidence of two spatially-distinct events. True absolute coincidence can only mean coincidence in *both* space *and* time.

In the foregoing sections, we saw that the fundamental equations of electrodynamics can be made consistent with the special principle of relativity only when we transform the spatial coordinates and time, not with the *Galilean* formulas (19), but with the *Lorentzian* ones, that reduce to (18) and (18.a) in the simplest case. The cited arguments shall convince us of the physical admissibility of those transformation formulas. Note that in that way *we do not need to change anything at all* in regard to our customary conception of space-time, as long as it refers to a *well-defined inertial system*. According to *Einstein*, the Lorentz transformation represents a new and unconventional *coupling* between the usual space-time quantities *in two different inertial systems*.

The invariance of the critical speed is then, so to speak, guaranteed automatically since it plays the role of a constant parameter in the formulas that it enters into.

We would now like to show that the Lorentz transformation *is a necessary consequence of that invariance.* However, that transformation can be derived from that invariance and completely independent of the general laws of electrodynamics, but while considering the relativity of velocity (i.e., for uniform, rectilinear motion).

The force-free, uniform, rectilinear motion of any particle relative to the coordinate system S is expressed analytically by the linear equations:

$$\frac{x_1 - a_1}{\alpha_1} = \frac{x_2 - a_2}{\alpha_2} = \frac{x_3 - a_3}{\alpha_3} = \frac{t - t_0}{\alpha_0},$$

in which a, α, etc., mean constants.

When judged from a different coordinate system S' that moves relative to S with constant velocity, that force-free motion must also appear to be uniform and rectilinear, i.e., it must be likewise determined by the system of linear equations:

$$\frac{x_1' - a_1'}{\alpha_1'} = \frac{x_2' - a_2'}{\alpha_2'} = \frac{x_3' - a_3'}{\alpha_3'} = \frac{t' - t_0'}{\alpha_0'}.$$

It will follow from this that the quantities x_1, x_2, x_3, t, on the one hand, and x_1', x_2', x_3', t', on the other, must be expressed *linearly* in terms of each other.

We assume that the origins of S and S' (O, and O', resp.) coincide at a certain moment $t = t' = 0$, and we further imagine that a light or radio signal is sent out from O (O', resp.) at that moment. That signal must propagate in the form of a spherical wave with the same velocity speed c relative to S and S'. The center of that spherical wave must then remain at the point O for S, but at the point O' for S'.

We now imagine that the spherical wave in question meets any particle. Let the coordinates and time of that event be x_1, x_2, x_3, t in S and x_1', x_2', x_3', t' in S'. Therefore, the two equations must be fulfilled:

$$x_1^2 + x_2^2 + x_3^2 - c^2 t^2 = 0,$$

$$x_1'^2 + x_2'^2 + x_3'^2 - c^2 t'^2 = 0,$$

and as a result the equation:

$$x_1^2 + x_2^2 + x_3^2 - c^2 t^2 = x_1'^2 + x_2'^2 + x_3'^2 - c^2 t'^2,$$

as well, or with the notations $i\,c\,t = x_4$, $i\,c\,t' = x_4'$:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2. \tag{20}$$

We have then seen that the relations between the old and new coordinates (including the fourth one) must be linear, i.e., expressed by the formulas:

$$x'_{k'} = \sum_{k=1}^{4} \beta_{kk'}\, x_k \qquad (k' = 1, 2, 3, 4), \tag{20.a}$$

in which $\beta_{kk'}$ are constant coefficients that depend upon the magnitude and direction of the velocity v of S' relative to S, and also upon the relative orientations of the two systems.

It will follow from this that the two sums $\sum_{k=1}^{4} x'^2_k$ and $\sum_{k=1}^{4} x^2_k$ must also be identical when they are non-zero, i.e., when the event in question does not mean the coincidence of the wave that mentioned above and a particle, but something else, e.g., the coincidence of two different particles with each other.

In fact, by substituting the expressions (20.a) in $\sum_{k'=1}^{4} x'^2_{k'}$, we will get a quadratic form in the coordinates x_1, x_2, x_3, t that vanishes along with $\sum x'^2_{k'}$. However, from (20), the quadratic form $\sum x^2_k$ must also vanish as a result. For that reason, the two forms can differ from each other by only a constant proportionality factor. That factor A can depend upon the magnitude, but not the direction, of the relative velocity $\mathfrak{v}$ of S' relative to S. Now, the velocity of S relative to S' is obviously *equal and opposite* to the velocity of S' relative to S. As a result, when one has $\sum_{k'} x'^2_{k'}$ $= A(v)\sum_k x^2_k$, one must also have $\sum_k x^2_k = A(v)\sum_{k'} x'^2_{k'}$. One will then have $A(v) = 1$. That proves that equation (20) will remain true for *arbitrary* x_k and $x'_{k'}$. We have then once more discovered the orthogonality condition that is true for the general Lorentz transformation. If we then add the reality condition for x'_1, x'_2, x'_3, and $t' = x'_4 / ic$ then the equations (20.a) must produce a Lorentz transformation.

Instead of deriving the Lorentz transformation from the fundamental equations of the electromagnetic field, as we did in **§ 2** following *Minkowski*, we can obtain it directly from the relativity principle, in conjunction with the retarded character of the electromagnetic action-at-a-distance (or, as one usually says, with the constancy of the speed of light). That is how *Einstein* presented it in 1905. The possibility then arises of reversing *Minkowski*'s process, i.e., of deriving the electrodynamical fundamental equations *from the relativity principle* (while recalling the covariance property), or in any event, to make it plausible without having to appeal any other principles.

We can also employ the relativity principle in order to generalize those physical laws that relate to the special case of static phenomena to phenomena that depend upon time arbitrarily, as we already did in Chap. V, but in a much simpler and more systematic way. For example, if we consider the equation for stationary electrical currents $\nabla^2\mathfrak{A} = -4\pi\mathfrak{j}$ as having been proved then it will follow immediately from the relativity principle that the general equation must have the form $\sum_{l=1}^{4} \frac{\partial^2 A_k}{\partial x_l^2} = -4\pi j_k$ $(k = 1, 2, 3, 4)$. Finally, the relativity principle or the Lorentz transformation can always be applied when the influence of the velocity on any phenomenon that is known in the rest state of the electrical system considered is supposed to be exhibited.

In what follows, we will learn about some such methodological applications of the principle of relativity. Note that its fundamental significance to physics lies in precisely that direction and not in any epistemological criticism of the concepts of space and time. The principle of relativity (combined with the invariance of the speed of light) begins with the "relativization" of quantities that were regarded as invariant scalars up to now (e.g., time, charge density, the magnitude of an ordinary vector, etc.). However, we will then incorporate those variant quantities as the components (or projections) of four-dimensional vectors and tensors, and that will finally show us how we can obtain quantities and equations from them that are truly invariant and express the laws of nature for time-varying phenomena.

The center of mass of the theory of relativity lies precisely in that "constructive," methodological aspect of it, and not in its "destructive," critical aspect. Its main result is not the variance of the usual scalars and spatial vectors, but in the possibility of treating them as "temporal" ("spatial," resp.) projections of invariant four-dimensional "spacetime vectors." In that sense, one would have every right to refer to the theory of relativity as an "absolute theory."

§ 4. – Graphical representation of motion and a new derivation of the Lorentz transformation.

Before we go on to discuss the transformation formulas (18), we would like to derive them once more from *Einstein*'s relativity principle in an intuitive, geometric way that is connected with the usual graphical representation of motion.

As is known, the motion of a particle (or the propagation of an action) in a plane E (X_1, X_2) can be illustrated geometrically by means of a spatial coordinate system X, Y, Z, such that the plane E is represented by the XY coordinate plane, and time t is represented by the third axis Z. Therefore, the latter *does not* necessarily need to be directed perpendicular to the XY-plane (which we will refer to as "horizontal"). For the sake of simplicity, we will restrict ourselves to rectilinear motions that are parallel to the X-axis (or more precisely, to the line X_1 that is represented by that axis) and overlook the Y-axis. In order for the two coordinates x and z to have the same dimensions, we will set $z = k\,t$, where k means a coefficient with the dimension of a velocity that is initially completely arbitrary.

We denote the angle XOZ by ω (Fig. 35). The motion of a particle parallel to the line X_1 is represented graphically by a line in the XZ-plane. A *uniform* motion with a velocity of v then corresponds to a *straight* line:

$$\frac{x - x_0}{v} = \frac{z - z_0}{k}.$$

We fix the coordinates x and z as the *components* (but not the projections!) of the radius vector OQ, which points from the "origin" $x = z = 0$ to the "spacetime point" Q under consideration. In the case where $x_0 = z_0 = 0$, we will have:

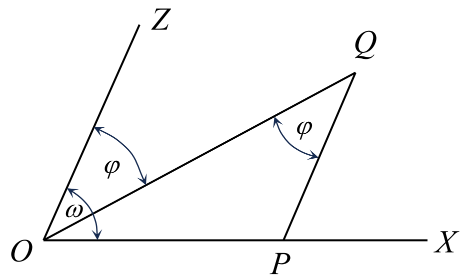

Figure 35.

$$\frac{v}{k} = \frac{x}{z} = \frac{OP}{PQ} = \frac{\sin\varphi}{\sin(\omega-\varphi)},$$

i.e.:

$$v = k\frac{\sin\varphi}{\sin(\omega-\varphi)}, \tag{21}$$

in which φ means the angle of inclination of the "kinematical line" OQ that represents the motion with respect to the time axis OZ. Obviously, that formula also remains valid when the aforementioned line does not go through the origin. All uniform motions that have the same velocity will then be represented by parallel lines whose inclination with respect to the Z-axis will increase with velocity.

The representation of motions in the plane E, and in particular, along the line X_1, corresponds to a certain coordinate system S (X_1, X_2) that is regarded as being *at rest*. When assessed from another coordinate system $S'(X_1', X_2')$ that moves uniformly in the X_1-direction, that same motion will be represented differently relative to the system, of axes X, Y, Z. Indeed, any event that is represented by the point D when assessed in S will be replaced with another point D' when assessed in S'. If, e.g., S' moves relative to S with the same velocity v as the particle considered above then we must replace the line OQ with a line OQ' that coincides with the time axis OZ.

Now, one can easily show that the same events and motions in the plane E can be represented by *the same* points and lines in the coordinate space X, Y, Z when assessed from different kinematical standpoints in the event that one introduces new axes as the representative of the "moving" system S', which are suitably inclined with respect to the old ones and possibly have a new scale of measurement. In fact, the coordinates of the spacetime point D (x, z) and $D'(x', z')$ $(y = y')$ that are referred to the original coordinate system must be coupled with each other by *linear* relations (since a motion that appears uniform and rectilinear in S must also remain so in S', cf., **§ 3**). However, one treats those linear relations as the transformation formulas that determine new axes $X'Z'$, in such a way that the coordinates (x, z) and (x', z') are associated with *the same* point ([1]).

The new Z'-axis must obviously coincide with the line OQ that represents the motion of S' relative to S. Conversely, the old axis OZ must represent the motion of S relative to S', so a motion with the velocity $v' = - v$ (in the X_1-direction). That relationship between v' and v (which can perhaps be regarded as a special "coupling principle"), in conjunction with the formula (21), will imply a relation between the two coordinate angles ω and ω' ($\sphericalangle X'OZ'$). Namely, if one sets $\varphi' = - \varphi$ then, from (21):

([1]) Such a transformation will be called "affine," in general.

$$v' = k' \frac{\sin \varphi'}{\sin(\omega' - \varphi')} = - k' \frac{\sin \varphi}{\sin(\omega' + \varphi)},$$

i.e., since:

$$v' = - v = - \frac{k \sin \varphi}{\sin(\omega' - \varphi)},$$

one will have:

$$\frac{k'}{k} = \frac{\sin(\omega' + \varphi)}{\sin(\omega - \varphi)}.$$

The coefficients k and k' depend upon the units of measurement for the length and time (or more precisely the *ratios* of those units of measurement) that one uses in XZ ($X'Z'$, resp.). However, due to the complete equivalence of the reference systems that are considered to be "at rest" and "moving," we must set $k = k'$. That will lead to the equation:

$$\frac{\sin(\omega' + \varphi)}{\sin(\omega - \varphi)} = 1, \tag{21.a}$$

which has two and only two solutions, namely:

$$\omega' = \omega - 2\varphi \tag{21.b}$$

and

$$\omega' = \omega = \frac{\pi}{2}. \tag{21.c}$$

We must rotate the X-axis through an angle of φ while rotating the Z-axis through the same angle, and indeed in either *the opposite* or the same direction. In the latter case, the original coordinate system, and therefore the new one, as well, must be *rectangular*.

We would first like to consider that second solution. In that way, we will get a very intuitive relationship between the "kinematical" and the "geometric" relativity principle, i.e., the relativity of velocity and the relativity of direction, resp. In the original XZ coordinate system, we have referred to the X-axis as "horizontal." We must accordingly represent the time axis by a vertical line OZ. The vertical direction then has the meaning of "rest," or a "motion in time" for a space at rest. The transition from the "rest" coordinate system X_1 X_2 to the "moving" one $X_1' X_2'$ is expressed geometrically by a rotation of the "vertical-horizontal" coordinate system XZ into the "inclined" configuration $X'Z'$. The question of which of the two systems X_1 and X_1' is *actually* moving then remains just as absurd as the question of which of the two systems XZ and $X'Z'$ is actually inclined. We can just as well treat the new time axis OZ' as vertical and the old one OZ as an inclined line as we can treat the opposite case. That relationship between the concepts of "rest" or "moving," on the one hand, and "vertical" or inclined," on the other, refers to the fact that quantities such as *length* and *time duration*, i.e., the spatial and temporal distances between two events, are regarded as relative or *variant* in the same sense as the horizontal and vertical components of the connecting line between two different points in space.

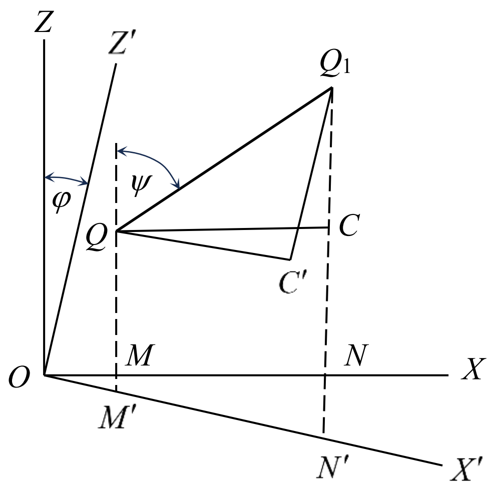

Figure 36.

We imagine any two events, e.g., the throwing and dropping of a stone in the example of the moving ship that was considered above. Let those events be represented graphically by the points Q and Q_1 (Fig. 36) relative to an XZ-coordinate system with a horizontal length axis OX and a vertical time axis OZ. The spatial distance between the two events will then be represented by the line segment $QC = \xi$ (viz., the "horizontal distance" between Q and Q_1). The corresponding time distance τ, i.e., the duration of the flight, is then represented by the "height" of Q_1 relative to Q, multiplied by $1 / k$, so by $\frac{1}{k} CQ_1 = \frac{1}{k}\xi$.

From the standpoint of the comoving observer, those quantities must have different values ξ' and τ', and indeed everything will happen for him as if he had considered the line segment QQ_1 from a different point on the globe where the vertical OZ' was inclined with respect to OZ (in the direction of motion) by an angle of φ. The relationships between ξ and τ, on the one hand, and ξ' and τ', on the other, obviously read [cf., (17) and (17.a)]:

$$\xi' = \xi \cos\varphi - k\,\tau \sin\varphi, \qquad k\tau' = \xi \sin\varphi + k\,\tau \cos\varphi.$$

Now, from (21), and due to the fact that $\omega = \pi / 2$:

$$\tan\varphi = \frac{v}{k}.$$

If one then sets $\cos\varphi = \dfrac{1}{\sqrt{1+v^2/k^2}}$ then one will have:

$$\xi' = \frac{\xi - v\tau}{\sqrt{1+v^2/k^2}}, \qquad \tau' = \frac{\tau + \xi v / k^2}{\sqrt{1+v^2/k^2}}. \tag{22}$$

Let the speed of the stone (or more precisely, its horizontal component, which we assume to be constant) relative to the observer at rest be $u = \xi / \tau$, and let it be $u' = \xi' / \tau'$ relative to the comoving one. Upon dividing both equations (22), we will get:

$$\frac{\xi'}{\tau'} = \frac{\xi - v\tau}{\tau + v\xi / k^2} = \frac{\xi / \tau - v}{1 + v\xi / k^2\tau},$$

i.e.:

$$u' = \frac{u - v}{1 + uv / k^2}. \tag{22.a}$$

One can also derive those relations between the two velocities directly from (21) without the use of the transformation formulas (22). Namely, if one denotes the inclination of the line QQ_1 with respect to OZ by ψ and its inclination with respect to OZ' by ψ' then that will give:

$$\psi' = \psi - \varphi,$$

and as a result:

$$\tan \psi' = \frac{\tan\psi - \tan\varphi}{1 + \tan\psi \tan\varphi},$$

i.e., from (21):

$$\frac{u'}{k} = \frac{u / k - v / k}{1 + uv / k^2}.$$

In order to make formulas (22) and (22.a) consistent with the usual "classical" spacetime picture, one must obviously set $k = \infty$. In that case, they will reduce to the usual formulas $\xi' = \xi - v\,\tau$, $\tau' = \tau$, $u' = u - v$. Recall that the coefficient k means a velocity. The Ansatz $k = \infty$ will then correspond to the "classical" picture in which mechanical action-at-a-distance propagates with an infinitely-large speed. However, in reality, that speed of propagation (viz., c) is *finite*, and *therefore invariant*, according to the relativity principle. If one then considers the propagation of a light or radio signal, instead of the motion of a thrown rock, then one must have $u = u' = c$.

Now, that invariance condition can actually be fulfilled by formula (22.a) when one sets:

$$k^2 = - c^2 \tag{22.b}$$

in it. In that way, formulas (22) will go to:

$$\xi' = \frac{\xi - v\tau}{\sqrt{1 - v^2 / c^2}}, \qquad \tau' = \frac{\tau - \xi v / c^2}{\sqrt{1 - v^2 / c^2}}, \tag{23}$$

i.e., they will be identical to (18), while (22.a) will assume the form:

$$u' = \frac{u - v}{1 - uv / c^2}. \tag{23.a}$$

Note that the imaginary character of the coefficient k is closely linked with the *causality principle* (in its usual conception). That is because although the time interval between events is a variant quantity, the *sequence* of those events *must remain invariant as long as one of them can be regarded as the cause of the other ones.* If k were a real number then one could invert the sequence of two *arbitrary* events by a sufficiently-large relative velocity between two inertial systems. For example, for a sufficiently-large inclination of the Z'-axis with respect to the Z-axis in Fig. 36, the point Q would lie *higher* than Q_1 relative to $X'OZ'$. However, that would mean that the dropping of the stone would precede its being thrown. Such an illogical situation would exclude the possibility that k is imaginary. Namely, if one sets $\xi = u\ \tau$ in the second formula in (23) then one would see that the causality principle, which can be expressed by the inequality:

$$\frac{\tau'}{\tau} > 0 ,$$

will always remain respected if and only if the inequality:

$$\frac{u\,v}{c^2} < 1$$

is fulfilled. That means that the magnitudes of the two velocities u and v must remain smaller than the invariant speed c.

That limiting character of the speed of propagation of electromagnetic actions (or more concisely, the speed of light) is actually ensured by *Einstein*'s superposition law for velocities, which is expressed by formula (23). That is because, from (23), the superposition of two velocities that are not greater than c will yield a resultant velocity that again satisfies that conditions. If one substitutes, e.g., $u = c$ in (23.a) then one will have $\dfrac{c-v}{1-c\,v/c^2} = c$. Note that in *Einstein*'s theory of relativity, velocity is not an additive quantity, as it is in the classical theory. One sees that most clearly when one recalls the second (direct) derivation of formula (22.a): In it, we subtracted the *angles* from each other, but not their *tangents*, which are proportional to the velocity.

Due to the imaginary character of k, the foregoing relations (22) and (22.a) will lose the intuitive geometric meaning that we used as the starting point in their derivation. Nonetheless, we can preserve the geometric way of expressing things above in a purely-formal way. Rather, we can also use the corresponding graphical representation without having to worry about the fact that the coefficient k is imaginary. One needs to consider its imaginary character only in the final result (i.e., one has to set $k = i\ c$ in it).

§ 5. – Spatial and temporal distances in the theory of relativity.

We once more return to Fig. 36 and first note that the length s of the segment QQ_1, which we would like to define to be the *kinematical* or *spacetime* distance between the corresponding events, must be an *invariant* quantity. Since $QQ_1^2 = QC^2 + CQ_1$, it will follow that $s^2 = \xi^2 + k^2\,\tau^2$, i.e.:

$$s^2 = \xi^2 - c^2\,\tau^2\,. \tag{24}$$

The segments QC and CQ_1 represent the horizontal and vertical components (projections), resp., of the vector QQ_1. We can correspondingly treat ξ and $i\,c\,t$ (or simply τ) as the spatial and temporal components or projections of the spacetime vector $\mathfrak{s}$.

For $s^2 < 0$, there is always a speed $u < c$ such that ξ can be represented as a product $u \cdot \tau$. That is the speed of the uniform, rectilinear motion that connects the events in question, i.e., the spacetime points Q and Q_1, so to speak.

The inequality $s^2 < 0$ must always be fulfilled then when Q_1 is a consequence of Q. In the limiting case $u = c$, which occurs only under the propagation of electromagnetic actions, it will go to the equation $s^2 = 0$. If one substitutes $v = u$ $(< c)$ then one will have:

$$\xi' = 0\,, \qquad \tau' = \tau\sqrt{1-u^2/c^2}\,.$$

The first equation means that from the standpoint of the comoving observer, the two events occur at the same spacetime point, which agrees with our usual conception of things. However, in contrast to that picture, the time interval between them has been shortened by the ratio $\sqrt{1-u^2/c^2}:1$. One can then say that the temporal distance between two events (that are causally connected with each other or can be) will appear to be smallest in the inertial system relative to which those events coincide spatially.

By contrast, if $s^2 > 0$ then one cannot speak of a causal relationship between Q and Q_1. It is likewise impossible to make them spatially coincident. However, in that case, one can define an inertial system relative to which the two events appear to be *simultaneous*. Namely, that system S' must move with a velocity relative to the original one that is also determined by the equation $\tau - \xi\,v/c^2 = 0$. If one substitutes the associated value of v in the first of equations (23) then one will get $\xi' = \xi\sqrt{1-v^2/c^2}$, so once more, a smaller value, and indeed it will be the smallest one possible. Obviously, it must coincide with s. Therefore, if $\tau' = 0$ then one will have $\xi' = s$. By contrast, for $s^2 < 0$, τ' will remain non-zero. However, one can set $\xi' = 0$, and one will then get $i\,c\,\tau' = s$.

Those relationships can be summarized in the following formulas:

$$\left.\begin{array}{ll} s = i\,c\,\tau\sqrt{1-u^2/c^2} = i\,c\,\tau_{\min} & (s^2 < 0), \\ s = \xi\;\sqrt{1-u^2/c^2} = \xi_{\min} & (s^2 > 0). \end{array}\right\} \tag{24.a}$$

One must then regard length and time duration as the spatial and temporal projections of a "spacetime" distance and correspondingly treat them as *variant* quantities that depend upon the speed of the observer in the same way as the horizontal and vertical projections of a segment onto the vertical direction of the observer. In both cases, that variance *does not depend upon the choice* *of units at all*. The two observers can use *the same* units of length and time, i.e., they can be provided with identical yardsticks and clocks. A relative motion cannot affect those yardsticks and clocks in any way, just like a rotation of the vertical cannot affect the length of a meter. Nonetheless, the spatial and temporal distances between the same events that are determined by those identical measuring instruments *do not need to be identical*.

One must make it quite clear that one is *not* dealing with a measurement of length by well-defined material bodies or the history of a well-defined clock. In reality, one is dealing with the determination of the distances between various separated *points in empty space* in which certain events have occurred and the intervals between the corresponding time points.

We imagine, e.g., something more concrete, namely, two phenomena, one of which occurs on Jupiter and the other of which occurs on Saturn, and which are observed by terrestrial and Martian astronomers. We will then address the determination of the following quantities: First of all, the *distance* between the spatial point P where Jupiter is found at the moment T when the first event occurs and the spatial point P_1 where Saturn is found at the moment T_1 when the second event occurs, and secondly, the interval between the time-points T and T_1 . However, there are no such things as "well-defined spatial points" (and we should have accepted that fact long ago). They are determined only *relative* to a certain coordinate system, which is a coordinate system that is assumed be "at rest" quite arbitrarily. In astronomy, one ordinarily employs a coordinate system that is linked with the Sun (or the center of mass of the solar system). When assessed from *that* coordinate system, the distance PP_1 must appear to be identical for both terrestrial and Martian astronomers, as long as they both use the same unit of length. However, due to the relativity principle, they would have the same right to assess the events in question *in any other inertial system*, and in particular, the ones relative to which they themselves are at rest. (For the sake of simplicity, we have ignored the rotation of the Earth and Mars and considered their translational motion to be unaccelerated.) Let that terrestrial inertial system be S and let the Martian one be S' . Obviously, it is also clear now that from the standpoint of the classical theory of relativity, the distance PP_1 must appear to be completely different in S and S', in the event that the two events are not simultaneous. However, it was a prejudice of the previous mechanistic picture of the universe that the simultaneity of two *spatially-separated* events was considered to be something that was determined absolutely. We saw above that this prejudice had its roots in the picture of an instantaneous action-at-a-distance. However, physical actions-at-a-distance are retarded. Due to the relativity principle, we must treat their *finite* speed of propagation, and especially the speed of light, as a *relative*, and at the same time *invariant*, quantity. In order to establish the location and time of the events under consideration, the terrestrial and Martian astronomers must not only observe, but also *calculate*. Indeed, they must consider the fact that what they observe *now* on Jupiter and Saturn happened somewhat *earlier*, in reality. However, in order to determine that delay, they must calculate the distance from the effective spatial points P and P_1 to their telescope. If the motion of the two planets (Jupiter and Saturn) is known relative to the inertial systems S and S' then that calculation will encounter no fundamental complications (cf., Chap. VI), but only in the case where the speed of light *is also known* relative to S and S'. It will follow from the relativity principle that they are identical in S and S'. The spatial and temporal distances PP_1 and TT_1 will actually be determined on the basis of that principle. However, it is no wonder that this must produce different results for the latter, as well as for the former.

The time of a certain event is therefore not considered to be something that is given *a priori*, but it must first be *defined* on the basis of the principle of the invariance of the speed of light. The length of the time interval between two spatially-distinct events can be determined only *a posteriori*, accordingly. The spatial and temporal distances between two events are mutually-dependent quantities that can be invariant only when they *both vanish*.

That connection between them is, by its nature, entirely analogous to the connection between the horizontal and vertical measurements of a line segment, e.g., on a building. What does one mean by "the height" of a tower? One often cares to define it to be the distance from its foundation to its top. However, it is clear that this definition would be incorrect when the tower is inclined slightly away from the vertical. One would then define its height to be the projection of the line segment that points from the foundation to the top onto the vertical, and that is the most-general definition. However, the vertical is not an absolute concept. The vertical direction must already be different for neighboring points of the Earth's surface, strictly speaking. That is why the "height" of a tower, in the exact sense of the word, is an ill-defined, variant quantity, just like its length, i.e., its projection onto the horizontal plane. Obviously, one can consider the "natural height" of the tower to be its *maximum* height, which one will get when the line that points from its floor to its top proves to be vertical. Obviously, its length must vanish in that way. We can define the smallest time interval between two events that was defined above to be their "natural" time interval in the same entirely-conventional sense of the term or define the smallest spatial distance between two events that cannot be connected causally to be the "natural" distance. From (24.a), those natural distances are precisely the invariant spacetime distances that can be determined from the formula $s^2 = \xi^2 - c^2 \tau^2$ when one chooses any inertial system as a basis, just like the "natural" height of an inclined tower can be determined from its actual height and length by means of the well-known *Pythagorean* formula.

When one believed some time ago that the Earth was flat and the vertical direction was determined absolutely, one treated height (and correspondingly, the horizontal distance) as an invariant quantity. One continued to believe that about time and spatial distance up to 1905. In the year 1905, that idea was known to be an irrational prejudice for the first time by *Einstein*, and the coupling of time and space determination in different inertial systems was established on the basis of the relativity principle (and the invariance of the speed of light). In that way, the older physics of three-dimensional space and one-dimensional time was converted into the modern physics of four-dimensional spacetime manifolds or the *four-dimensional world*, according to *Minkowski*.

We have often said that in order to treat the determination of spatial distances in the theory of relativity, one must refer them to empty space and not to material objects. Then again, how can one measure the length of a rod, and how can one define it at all? The general definition reads as follows: The length of a rod is the distance between those spatial points where its two ends are found *at the same instant*. However, since the concept of simultaneity is relative, the length of the rod that was determined by the definition above must be different in different inertial systems.

Therefore, we can refer to one of the lengths l that are given in *the* inertial system relative to which the rod is at rest as the "natural" or "rest length." In that way, it is distinguished by the fact that the aforementioned simultaneity condition is inessential for its determination. That is because if one considers two events that occur at the ends of the rod in question at different times then that will always give the same value for their spatial distance. One sees that from Fig. 36, where the aforementioned events might be represented by the points Q and Q_1. The rod shall be at rest relative to the inertial system that goes through XZ such that the "motion" of its ends will be represented by the two (dotted) lines that are parallel to the time axis OZ. The "rest length" of the rod l is then equal to MN (or QC). From the definition that was cited above, its length l' relative to a different inertial system that goes through $X'Z'$ will be determined by the segment $M'N'$ on the new X'-axis (where M' and N' mean its intersection points with the dotted lines). Now, one obviously has $l' = l / \cos\varphi$, i.e., from $\cos\varphi = 1/\sqrt{1+v^2/k^2}$:

$$l' = l\sqrt{1+v^2/k^2}\ ,$$

or ultimately, due to the fact that $k^2 = -\ c^2$:

$$l' = l\sqrt{1-v^2/k^2}\ . \tag{25}$$

The rod that moves with the speed v then seems to have been shortened by the ratio $\sqrt{1-v^2/c^2}:1$ (in the figure, $M'N'$ is bigger than MN since it actually corresponds to a real value of k). That result initially seems to be in complete agreement with the *Lorentz* contraction hypothesis (cf., Chap. VII, **§ 6**). However, that is by no means the case since under that hypothesis, the moving (relative to what?) electron should *actually* be shortened in the direction of motion, while according to the *Einsteinian* relativity principle, that shortening should pertain to not only the electron (the rod, resp.) itself, but also to the "distance between the spatial points where its ends are simultaneously found."

If one considers the time duration of a phenomenon that takes place relative to the inertial system that is represented by XZ at a fixed spatial point (e.g., one end of the rod), instead of the length of a rod, then one will find the following relationship between the "rest duration" τ and the corresponding time interval τ' in the "moving" coordinates system by the same geometric process:

$$\tau' = \frac{\tau}{\sqrt{1-v^2/c^2}}\ , \tag{25.a}$$

which agrees with the first of formulas (24.a), which was derived in a different way, namely, on the basis of the transformation equations (23). Obviously, one can derive it from (25) in the same way. We have preferred the geometric method, due to its intuitiveness.

The problem with this method obviously consists of the fact that the coefficient k is imaginary. However, there is a second method that is free from that drawback. It corresponds to the other solution of equation (21.a), namely, $\omega' = \omega - 2\varphi$. One will have to simply set $k = c$, as is easy to see. In fact, since the two axes X and Z will be rotated in opposite directions, so for positive v, they must coincide for a certain limiting value of v.

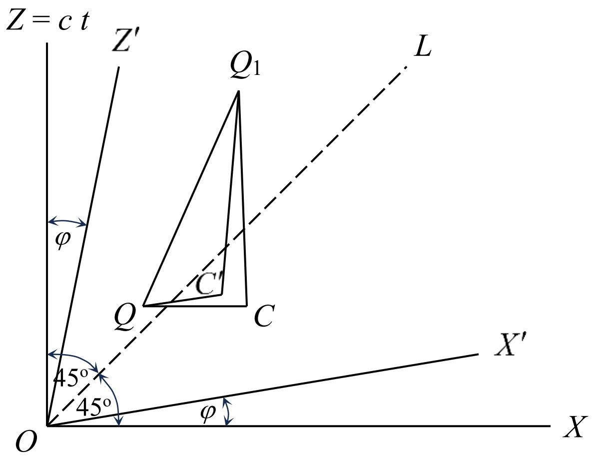

Figure 37.

If the original system is rectangular (ω = 90°) then that will happen for φ = 45°. We will then find from (21) that:

$$\frac{c}{k} = \frac{\sin 45^{\circ}}{\sin(90^{\circ} - 45^{\circ})} = 1 ,$$

i.e., $k = c$.

The spatial and temporal components of the spacetime distance s between the events Q and Q_1 (Fig. 37) from the standpoint of the "moving" inertial system that is represented by $X'Z'$ are, in this case, coupled with the corresponding components of the segment QQ_1 by the formulas:

$$\xi' = \frac{1}{\gamma} QC' , \qquad c\tau' = \frac{1}{\gamma} C'Q ,$$

in which γ means a "gauge factor" that is not equal to 1. Now, it follows from the figure that:

$$QC = QC' \cos\varphi + C'Q_1 \sin\varphi , \qquad CQ_1 = QC' \sin\varphi + C'Q_1 \cos\varphi ,$$

i.e.:

$$\xi = \gamma \cos\varphi(\xi' + v\tau') , \qquad t = \gamma \cos\varphi \left(\tau' + \frac{\xi' v}{c^2} \right) ,$$

since one has:

$$\tan \varphi = \frac{\sin\varphi}{\sin(90^{\circ} - \varphi)} = \frac{v}{c} .$$

In order for those formulas to be identical to the transformation formulas (23) or rather, the reciprocal formulas:

$$\xi = \frac{\xi' + v\tau'}{\sqrt{1 - v^2 / c^2}} , \qquad \tau = \frac{\tau' + v\xi' / c^2}{\sqrt{1 - v^2 / c^2}} ,$$

it will obviously suffice to set $\gamma \cos \varphi = 1/\sqrt{1 - v^2 / c^2}$, or since one has $\cos \varphi = 1/\sqrt{1 + v^2 / c^2}$ in the case considered:

$$\gamma = \sqrt{\frac{1 + v^2 / c^2}{1 - v^2 / c^2}} . \tag{26}$$

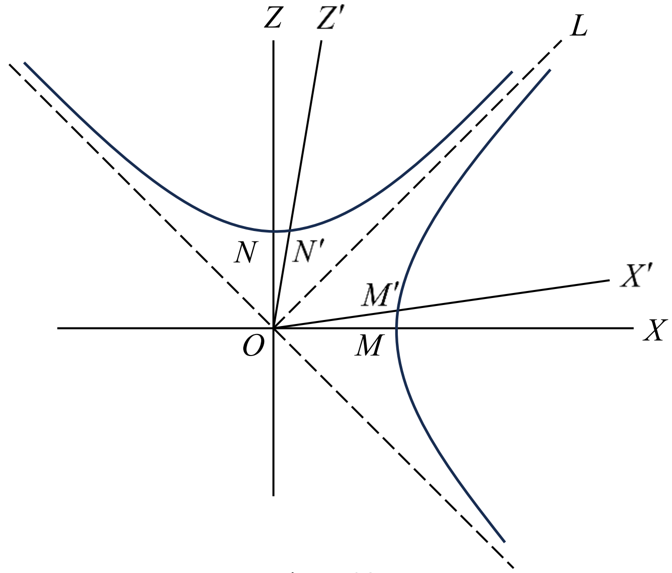

Figure 38.

That coefficient can be determined graphically to be the length of the segment $OM' = ON'$, which is cut out from the new axis $X'Z'$ by the hyperbolas (Fig. 38):

$$x^2 - z^2 = \pm 1\ . \tag{26.a}$$

By solving that equation, in conjunction with the equation $z = (v / c)\, x$ of the X'-axis, we will, in fact, get the coordinates of the point of intersection M':

$$x'^2 = \frac{1}{1 - v^2 / c^2}\,, \qquad z'^2 = \frac{v^2}{c^2} x'^2 .$$

As a result, we will have:

$$OM'^2 = x'^2 + z'^2 = \frac{1 + v^2 / c^2}{1 - v^2 / c^2} = \gamma^2 .$$

In order to represent the events by means of the coordinate system $X'Z'$, which corresponds to the "moving" inertial system, we must then define the unit of length by OM' and the unit of time by $ON' / c = OM' / c$.

The condition that determines the change in the gauge under the transition from the inertial system "at rest" to the moving inertial system with the method of representation that is being considered is analytically identical to the condition that expresses the invariance of that gauge when using the previous method. That is because when one replaces $z = c\, t$ with $z = i\, c\, t$ in (26.a), that equation will assume the form:

$$x^2 + z^2 = 1, \tag{26.b}$$

i.e., it will represent a *circle*.

Note that when the segment QQ_1 in Fig. 37 is inclined with respect to the Z-axis less than the bisector (or asymptote) OL, one can draw the Z'-axis parallel to QQ_1. That means that the two events take place at the same spatial point from the associated "kinematical" standpoint. By contrast, if the angle between QQ_1 and OZ is greater than LOZ then there will be a direction of the X'-axis that is parallel to it, i.e., an inertial system in which those events appear to be simultaneous.

The cited graphical representation can be easily generalized to the case of non-rectilinear motions in a plane. However, we would not like to go into the details of that question here, and we will move on to the most-general case of the four-dimensional "world," while employing geometric terminology.

CHAPTER NINE

APPLYING THE THEORY OF RELATIVITY TO ELECTROMAGNETIC PHENOMENA

§ 1. – Transforming vectors.

The formulas (18) and (18.a) of the last chapter represent a special case of the Lorentz transformation. The general case corresponds to a motion of the spatial coordinate systems $S'(X_1', X_2', X_3')$ relative to the "rest" system $S(X_1, X_2, X_3)$ in an arbitrary direction, and under which the orientations of the two systems can be different. The sixteen coefficients α_{kl} of that general transformation can obviously be determined by composing the special transformation (18) with two ordinary spatial transformations that mean a rotation of the "rest" and "moving" systems of axes.

However, it is simpler and more convenient to represent the aforementioned special transformation in a *coordinate-free* manner as a relationship between the times t and t' and the spatial radius vectors $\mathfrak{r}$ and $\mathfrak{r}'$ whose components are given by x_1, x_2, x_3 (x_1', x_2', x_3', resp.). Recall that $\mathfrak{r}$ and t (or $i\,c\,t$) can be regarded as the spatial and temporal projections of the four-dimensional spacetime vector $\mathfrak{r}$. Therefore, $\mathfrak{r}'$ and t' $(i\,c\,t')$ will mean the corresponding projections of the *same* spacetime vector in the moving," or even better, "primed" inertial system. One must further observe that the "origin" of the two systems O and O' and the initial moments $t = 0$ and $t' = 0$ are chosen in such a way that O and O' will coincide for $t = t' = 0$.

Let the velocity of S' relative to S be $\mathfrak{v}'$. The velocity of S relative to S' will then be:

$$\mathfrak{v}' = -\,\mathfrak{v}\,.$$

We will define the direction of the vector $\mathfrak{v}$ by the unit vector $\mathfrak{v}_0 = \mathfrak{v} / v$.

The formula:

$$t' = \frac{t - x_1\,v/c^2}{\sqrt{1 - v^2/c^2}}\,,$$

can obviously be represented in the form:

$$t' = \frac{t - (\mathfrak{r}\cdot\mathfrak{v})/c^2}{\sqrt{1 - v^2/c^2}}\,, \tag{1}$$

since the product $x_1\ v$ means nothing but the inner product of the vectors $\mathfrak{r}$ and $\mathfrak{v}$, under the assumption that $\mathfrak{v}$ has the same direction as the X_1-axis.

In order to obtain the relation between $\mathfrak{r}$ and $\mathfrak{r}'$, we must combine the formula:

$$x_1' = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}}$$

with the formulas:

$$x_2' = x_2 , \qquad x_3' = x_3 .$$

Now, we obviously have $x_1 = \mathfrak{r} \cdot \mathfrak{v}_0$ and $x_1' = \mathfrak{r}' \cdot \mathfrak{v}_0$. The longitudinal (i.e., parallel to the velocity $\mathfrak{v}$) components of $\mathfrak{r}$ and $\mathfrak{r}'$ are then equal to:

$$(\mathfrak{r}\ \mathfrak{v}_0)\ \mathfrak{v}_0 \qquad [(\mathfrak{r}'\ \mathfrak{v}_0)\ \mathfrak{v}_0, \text{ resp.}] ,$$

and the transverse ones are:

$$\mathfrak{r} - (\mathfrak{r}\ \mathfrak{v}_0)\ \mathfrak{v}_0 = \mathfrak{v}_0 \times (\mathfrak{r} \times \mathfrak{v}_0) \qquad [\mathfrak{r}' - (\mathfrak{r}'\ \mathfrak{v}_0)\ \mathfrak{v}_0 = \mathfrak{v}_0 \times (\mathfrak{r}' \times \mathfrak{v}_0) , \text{ resp.}].$$

The coordinate-wise equations above can then be replaced with the vector equations:

$$(\mathfrak{r}'\ \mathfrak{v}_0)\ \mathfrak{v}_0 = \frac{(\mathfrak{r}\,\mathfrak{v}_0)\mathfrak{v}_0 - \mathfrak{v}t}{\sqrt{1 - v^2/c^2}} ,$$

$$\mathfrak{r}' - (\mathfrak{r}'\ \mathfrak{v}_0)\ \mathfrak{v}_0 = \mathfrak{r} - (\mathfrak{r}\ \mathfrak{v}_0)\ \mathfrak{v}_0 .$$

Adding the latter two will give the desired relation:

$$\mathfrak{r}' = \mathfrak{r} - (\mathfrak{r}\ \mathfrak{v}_0)\ \mathfrak{v}_0 + \frac{(\mathfrak{r}\,\mathfrak{v}_0)\mathfrak{v}_0 - \mathfrak{v}t}{\sqrt{1 - v^2/c^2}} ,$$

or after a slight conversion:

$$\mathfrak{r}' = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)(\mathfrak{r} \times \mathfrak{v}_0) \times \mathfrak{v}_0 + \frac{\mathfrak{r} - \mathfrak{v}t}{\sqrt{1 - v^2/c^2}} , \tag{1.a}$$

in which the vector $-(\mathfrak{r} \times \mathfrak{v}_0) \times \mathfrak{v}_0$ represents simply the component of $\mathfrak{r}$ that is perpendicular to $\mathfrak{v}_0$. One can easily get the coordinate-wise representation of the Lorentz transformation by projecting that vector equation onto the "rest" or "moving" system of axes. In particular, if the two coordinate systems have the same orientation then we will get:

$$x_1' = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)\left\{\frac{(\mathfrak{r}\,\mathfrak{v})}{v^2} v_1 - x_1\right\} + \frac{x_1 - v_1 t}{\sqrt{1 - v^2/c^2}} ,$$

etc.

The transformation equations that are reciprocal to (1) and (1.a) are obtained by simply inverting the sign of $\mathfrak{v}$.

From the covariance principle, the spatial and temporal projections of the other four-vectors, e.g., $\mathfrak{j}$ and $\mathfrak{A}$, must transform according to similar formulas. Namely, if one notes that in the case of the "four-current," the time t is associated with the charge density ρ divided by c, and in the case of the "four-potential," it is associated with the scalar potential, likewise divided by c, then in place of (1) and (1.a), one will have:

$$\rho' = \frac{\rho - (\mathfrak{j}\mathfrak{v})/c}{\sqrt{1-v^2/c^2}}, \tag{2}$$

$$\mathfrak{j}' = \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)(\mathfrak{j}\times\mathfrak{v}_0)\times\mathfrak{v}_0 + \frac{\mathfrak{j}-(\rho/c)\mathfrak{v}}{\sqrt{1-v^2/c^2}}, \tag{2.a}$$

and likewise:

$$\varphi' = \frac{\varphi - (\mathfrak{A}\mathfrak{v})/c}{\sqrt{1-v^2/c^2}}, \tag{3}$$

$$\mathfrak{A}' = \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)(\mathfrak{A}\times\mathfrak{v}_0)\times\mathfrak{v}_0 + \frac{\mathfrak{A}-(\varphi/c)\mathfrak{v}}{\sqrt{1-v^2/c^2}}. \tag{3.a}$$

Note that the *velocity* of a particle, i.e., the differential quotient $\mathfrak{u} = d\,\mathfrak{r}\,/\,dt$, *cannot* be considered to be the spatial projection of a four-vector. In fact, $d\,\mathfrak{r}$ and dt obviously transform in the same way as $\mathfrak{r}$ and t. One will then get the transformed velocity $\mathfrak{u}' = d\mathfrak{r}'/dt'$ from (1) and (1.a):

$$\mathfrak{u}' = \left[1 - \frac{\mathfrak{u}\cdot\mathfrak{v}}{c^2}\right]^{-1}\left[(1-\sqrt{1-v^2/c^2})(\mathfrak{u}\times\mathfrak{v}_0)\times\mathfrak{v}_0 + \mathfrak{u} - \mathfrak{v}\right]. \tag{4}$$

That formula is completely different from (1.a), (2.a), and (3.a). It represents the generalization of the simple formula (23), Chap. VIII, for the superposition of two parallel velocities. By squaring and adding the quantities ($\mathfrak{r}$, ict), ($\mathfrak{j}$, $i\rho$), ($\mathfrak{A}$, $i\varphi$), we will get the squares of the four-vectors that they define, which are *true invariant scalars*. Those invariants then have the following property:

$$\mathfrak{r}^2 = r^2 - c^2 t^2, \tag{5}$$

$$\mathfrak{j}^2 = j^2 - \rho^2, \tag{5.a}$$

$$\mathfrak{A}^2 = A^2 - \varphi^2. \tag{5.b}$$

A third invariant quantity can be defined from the two vectors $\mathfrak{j}$ and $\mathfrak{A}$, namely, the inner product:

$$\mathfrak{A}\ \mathbf{j} = A_1 j_1 + A_2 j_2 + A_3 j_3 + A_4 j_4 \,,$$

or

$$\mathfrak{A}\ \mathbf{j} = \mathfrak{A}\ \mathrm{j} - \varphi\,\rho\,. \tag{5.c}$$

In the case of time-constant fields, one can consider that quantity to be the difference of the magnetic (kinetic) and electric (potential) energy per unit volume. It plays an important role in the mechanics of the theory of relativity (see Chap. X). The other two inner products that can be defined by the four-vectors $\mathfrak{r}$, $\mathbf{j}$, $\mathfrak{A}$, have no physical meaning.

We have already defined the quantity $|\,\mathfrak{r}\,| = \sqrt{r^2 - c^2t^2} = \sqrt{r'^2 - c^2t'^2}$ (for the special case of $r = x_1$) to be a "spacetime" distance. In the case considered, we are dealing with the "distance" between the events that are characterized by ($\mathfrak{r}$, t) [$(\mathfrak{r}', t')$, resp.] and the coincidence of the coordinate origins of the two systems ($r = r' = 0$, $t = t' = 0$). The four-dimensional distance between two arbitrary spacetime points ($\mathfrak{r}_1$, t_1) and ($\mathfrak{r}_2$, t_2) is correspondingly expressed by the formula:

$$s = \sqrt{R^2 - c^2\,(t_2 - t_1)^2}\,, \tag{6}$$

where

$$R = |\,\mathfrak{r}_2 - \mathfrak{r}_1\,| \tag{6.a}$$

means the ordinary spatial distance.

We saw above that we must distinguish two cases, namely, $s^2 < 0$ and $s^2 > 0$. In the first case, the vector $\mathfrak{s}$ is called *time-like* since there is an inertial system S' where its spatial projection $\mathfrak{R}'$ vanishes. In the second case, there exists no such system. However, there is an inertial system relative to which the temporal projection of $\mathfrak{s}$ will vanish. That is why one refers to the vector $\mathfrak{s}$ as *space-like* in this case.

In the limiting case of $s = 0$, which appears in the propagation of an electromagnetic effect, one can make the two projections of $\mathfrak{s}$ (viz., the spatial and temporal ones) vanish *individually*, i.e., one can make the two events (e.g., emission and reception of a light signal) coincide in space and time. However, in that way the new coordinate system must move with the speed of light with respect to the original one, which is physically unrealizable.

The cited considerations obviously remain valid for the arbitrary four-dimensional vectors. In particular, we must then distinguish between space-like and time-like four-currents and four-potentials, i.e., ones that reduce to their spatial or temporal projections for a suitable choice of reference system. Finally, we must make the following remark: We can set $\mathbf{j} = (\rho\,/\,c)\ \mathfrak{u}$ for the current density of a moving electron with a volume charge. In particular, if we are dealing with a pure translational motion with a velocity of $\mathfrak{u}$ then we must have $\mathbf{j}^2 < \rho^2$ since $u < c$. In this case, the vector $\mathbf{j}$ will then be time-like and can be reduced to its temporal projection. Obviously, that will happen for the inertial system S' relative to which the electron is *instantaneously* at rest, i.e., it moves with a velocity of $\mathfrak{v} = \mathfrak{u}$ relative to the original system. The corresponding "rest density" of the electricity ρ_0 is obviously determined from the equation $\rho^2 - j^2 = \rho_0^2$, i.e.:

$$\rho_0 = \rho\sqrt{1-u^2/c^2}\,. \tag{7}$$

It then represents the least-possible value of ρ. u can be larger than c when the electron is rotating (in the event that such a rotation actually takes place). The vector j would then be space-like. Nothing at all can be said about the existence of a rest density for the electricity. By contrast, there is an inertial system S in this case in which ρ vanishes, and j assumes the least-possible "natural" value:

$$j_0 = \sqrt{j^2-\rho^2} = j\sqrt{1-\frac{c^2}{u^2}}\,.$$

However, it is very doubtful whether one can then represent $\mathfrak{j}$ as the product of ρ / c and a "velocity" $\mathfrak{u}$ since superluminal speeds are excluded from the theory of relativity *in principle*.

§ 2. – Transforming six-vectors.

We shall move on to consider the four-dimensional tensors and first investigate the skew-symmetric field tensor or "six-vector," ${}^2\mathfrak{H}$. The general transformation formula for its components in the inertial systems S and S' read:

$$H'_{k'l'} = \sum_{k=1}^{4}\sum_{l=1}^{4}\alpha_{kk'}\,\alpha_{ll'}\,H_{kl}\,. \tag{8}$$

If one replaces t and t' with x_4 / ic and x'_4 / ic, resp., and sets:

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}, \tag{8.a}$$

to abbreviate, then the special Lorentz transformation, which can be expressed by the formulas (18), Chap. VIII, will be written as follows:

$$x'_1 = \gamma\left(x_1 - \frac{v}{i\,c}x_4\right), \qquad x'_2 = x_2\,, \qquad x'_3 = x_3\,, \qquad x'_4 = \gamma\left(x_4 + \frac{v}{i\,c}x_1\right).$$

The coefficients $\alpha_{kk'}$ will then define the following matrix in this case:

$$(\alpha_{kk'}) = \begin{pmatrix} \gamma & 0 & 0 & -\dfrac{v}{ic}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{v}{ic}\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{8.b}$$

If one substitutes that in (8) then when one recalls the skew-symmetric character of ${}^2\mathfrak{H}$ ($H_{kk} = 0$):

$$H'_{23} = H_{23}, \quad H'_{31} = \gamma\left(H_{31} - \frac{v}{ic}H_{34}\right), \quad H'_{12} = \gamma\left(H_{12} - \frac{v}{ic}H_{42}\right),$$
$$H'_{14} = H_{14}, \quad H'_{24} = \gamma\left(\frac{v}{ic}H_{21} + H_{24}\right), \quad H'_{34} = \gamma\left(\frac{v}{ic}H_{31} + H_{34}\right),$$

and as a result, from the substitution matrix [cf., (5), Chap. VIII]:

$$\begin{pmatrix} H_{21} & H_{31} & H_{12} & H_{14} & H_{24} & H_{34} \\ H_1 & H_2 & H_3 & -iE_1 & -iE_2 & -iE_3 \end{pmatrix},$$

one will have:

$$\left.\begin{aligned} H'_1 &= H_1, \quad H'_2 = \gamma\left(H_2 + \frac{v}{c}E_2\right), \quad H'_3 = \gamma\left(H_3 - \frac{v}{c}E_2\right), \\ E'_1 &= E_1, \quad E'_2 = \gamma\left(E_2 - \frac{v}{c}H_3\right), \quad E'_3 = \gamma\left(E_3 + \frac{v}{c}H_2\right). \end{aligned}\right\} \tag{9}$$

Since the velocity $\mathfrak{v}$ has the direction of the first axis in the case considered, the quantities $(v/c)E_3$ and $-(v/c)E_2$ will mean the second (third, resp.) component of the outer product $\mathfrak{E} \times (1/c)\,\mathfrak{v}$, and analogously, the quantities $-(v/c)H_3$ and $(v/c)H_2$ mean the corresponding components of the outer product $(1/c)\ \mathfrak{v} \times \mathfrak{H}$. If one then introduces the longitudinal and transverse components of the vectors $\mathfrak{E}$, $\mathfrak{H}$ ($\mathfrak{E}'$, $\mathfrak{H}'$, resp.), as in the transformation of the vector $\mathfrak{r}$, then the relations (9) can be given the following coordinate-free form:

$$(\mathfrak{H}' \cdot \mathfrak{v}_0)\,\mathfrak{v}_0 = (\mathfrak{H} \cdot \mathfrak{v}_0)\,\mathfrak{v}_0, \qquad \mathfrak{H}' - (\mathfrak{H}' \cdot \mathfrak{v}_0)\,\mathfrak{v}_0 = \gamma[\mathfrak{H} - (\mathfrak{H} \cdot \mathfrak{v}_0)\,\mathfrak{v}_0] - \frac{\gamma}{c}\,\mathfrak{v} \times \mathfrak{E},$$
$$(\mathfrak{E}' \cdot \mathfrak{v}_0)\,\mathfrak{v}_0 = (\mathfrak{E} \cdot \mathfrak{v}_0)\,\mathfrak{v}_0, \qquad \mathfrak{E}' - (\mathfrak{E}' \cdot \mathfrak{v}_0)\,\mathfrak{v}_0 = \gamma[\mathfrak{E} - (\mathfrak{E} \cdot \mathfrak{v}_0)\,\mathfrak{v}_0] - \frac{\gamma}{c}\,\mathfrak{v} \times \mathfrak{H},$$

which will give:

$$\mathfrak{H}' = (1-\gamma)\,(\mathfrak{H}\cdot\mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma\,(\mathfrak{H} - \frac{1}{c}\,\mathfrak{v}\times\mathfrak{E})\,, \tag{9.a}$$

$$\mathfrak{E}' = (1-\gamma)\,(\mathfrak{E}\cdot\mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma\,(\mathfrak{E} + \frac{1}{c}\,\mathfrak{v}\times\mathfrak{H})\,, \tag{9.b}$$

upon addition. One gets the reciprocal transformation formulas, which represent the transition from S' to S, by simply switching the unprimed and primed quantities and inverting the sign of $\mathfrak{v}$. For the limiting case of small velocities ($v / c \ll 1$), the formulas above result to:

$$\left.\begin{aligned} \mathfrak{H}' &= \mathfrak{H} - \frac{1}{c}\,\mathfrak{v}\times\mathfrak{E}, \\ \mathfrak{E}' &= \mathfrak{E} + \frac{1}{c}\,\mathfrak{v}\times\mathfrak{H}. \end{aligned}\right\} \tag{9.c}$$

Note that similar formulas were derived in the foregoing chapters and used many times. In particular, we saw that a system of charges in uniform, rectilinear motion creates a magnetic field strength $\mathfrak{H}$ that is coupled with the corresponding electric field strength by the complete *exact* relation $\mathfrak{H} = (1 / c)\,\mathfrak{v}\times\mathfrak{E}$. That relation will be given by the first of formulas (9.c) when we assume that the charges in question are *at rest* relative to the system S', i.e., they create no magnetic field in that system. The same result also follows from the exact formulas (9.a) and (9.b), and indeed, when $\mathfrak{H}' = 0$, the reciprocal formulas will assume the form:

$$\mathfrak{H} = \frac{\gamma}{c}\,\mathfrak{v}\times\mathfrak{E}', \qquad \mathfrak{E} = (1-\gamma)\,(\mathfrak{E}'\,\mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma\,\mathfrak{E}'.$$

One will then have $\mathfrak{v}\times\mathfrak{E} = \gamma\,\mathfrak{v}\times\mathfrak{E}'$, and as a result:

$$\mathfrak{H} = \frac{1}{c}\,\mathfrak{v}\times\mathfrak{E}.$$

One likewise gets the relation $\mathfrak{E} = -\,(1 / c)\,\mathfrak{v}\times\mathfrak{H}$ for the case in which $\mathfrak{E}' = 0$.

Formulas (9.a) and (9.b) can be combined into a single formula when one multiplies one of them (e.g., the second one) by $i = \sqrt{-1}$ and adds it to the other. That will, in fact, give:

$$\mathfrak{H}' + i\,\mathfrak{E}' = (1-\gamma)\,[(\mathfrak{H} + i\,\mathfrak{E})\cdot\mathfrak{v}_0]\cdot\mathfrak{v}_0 + [\mathfrak{H} + i\,\mathfrak{E} - \frac{1}{i\,c}\;\mathfrak{v}\times(\mathfrak{H} + i\,\mathfrak{E})]\,. \tag{10}$$

If one sets $\mathfrak{H} + i\,\mathfrak{E} = \mathfrak{F}$ here, to abbreviate, and squares that then, due to the facts that:

$$(1/c)\,\mathfrak{v}\times\mathfrak{F} = (v/c)(\mathfrak{v}_0\times\mathfrak{F}) \qquad \text{and} \qquad [\mathfrak{v}_0\times\mathfrak{F}]^2 = F^2 - (\mathfrak{v}_0\,\mathfrak{F})^2,$$

one will get:

$$F'^2 = (1-\gamma)^2\,(\mathfrak{F}\,\mathfrak{v}_0)^2 + \gamma^2\left[F^2 - \frac{v^2}{c^2}(\mathfrak{v}_0\times\mathfrak{F})^2\right] + 2\,(1-\gamma)\,\gamma(\mathfrak{F}\,\mathfrak{v}_0)^2$$

$$= (1-\gamma)^2\,(\mathfrak{F}\,\mathfrak{v}_0)^2 + \gamma^2\left[F^2 - \frac{v^2}{c^2}F^2 + \frac{v^2}{c^2}(\mathfrak{v}_0\,\mathfrak{F})^2\right],$$

i.e., from (8.a):

$$F'^2 = F^2 + (\mathfrak{F}\,\mathfrak{v}_0)^2\left(1-\gamma^2+\gamma^2\frac{v^2}{c^2}\right) = F^2,$$

or ultimately:

$$(\mathfrak{H}' + i\,\mathfrak{E}')^2 = (\mathfrak{H} + i\,\mathfrak{E})^2. \tag{10.a}$$

We have then found a new invariant of the electromagnetic field. Upon setting the real and imaginary parts equal to each other, it will split into two invariants, namely:

$$H'^2 - E'^2 = H^2 - E^2 \tag{10.b}$$

and

$$\mathfrak{H}'\cdot\mathfrak{E}' = \mathfrak{H}\cdot\mathfrak{E}\,. \tag{10.c}$$

Note that those invariants can be obtained more simply. One first takes the square of the tensor ${}^2\mathfrak{H}$ with no concern for its skew-symmetric character:

$${}^2\mathfrak{H}^2 = \sum_k\sum_l H_{kl}^2 = 2(H_{23}^2 + H_{31}^2 + H_{12}^2 + H_{14}^2 + H_{24}^2 + H_{34}^2),$$

i.e.:

$${}^2\mathfrak{H}^2 = 2(H^2 - E^2),$$

and that must obviously be an invariant scalar quantity. However, since ${}^2\mathfrak{H}$ is a skew-symmetric tensor, when one forms its inner product with the "dual" tensor ${}^2\mathfrak{H}^*$, it will follow from (14.a), Chap. VIII that:

$${}^2\mathfrak{H}\cdot{}^2\mathfrak{H}^* = 4\,(H_{23}\,H_{14} + H_{31}\,H_{24} + H_{12}\,H_{34}) = -\,4i\,(\mathfrak{H}\cdot\mathfrak{E}) = \text{invariant}.$$

The invariance of the quantities $H^2 - E^2$ and $\mathfrak{H}\cdot\mathfrak{E}$ shows that the characteristic property of the electromagnetic field in the wave zone (viz., the equality of the magnitudes of the electric and magnetic field strengths and their orthogonality) must be true in all inertial systems. One can then characterize the wave zone from the standpoint of the theory of relativity quite generally by the vanishing of the invariants (10.b) and (10.c).

If both of them are non-zero then one can always choose an inertial system such that the field strengths $\mathfrak{E}$ and $\mathfrak{H}$ (or $\mathfrak{E}'$ and $\mathfrak{H}'$) are *parallel* to each other at the spacetime point in question. However, it is impossible to make one of them vanish or be perpendicular to the other one. One can refer to the electromagnetic field as having "magnetic type" or "electric type" according to whether $H^2 > E^2$ or $H^2 < E^2$, resp. However, it can be *pure* magnetic or *pure* electric only when $\mathfrak{H} \cdot \mathfrak{E} = 0$.

The scalar (10.b), divided by 8π, represents the difference between the magnetic and electric energy densities. It will then correspond completely to the scalar (7).

Entirely-similar formulas and considerations will be true for the vectors $\mathfrak{P}$ and $\mathfrak{M}$, from which the six-vector of electromagnetic polarization ${}^2\mathfrak{P}$ is composed, and for the components $\mathfrak{Z}$ and $\mathfrak{Z}^*$ of the polarization potential ${}^2\mathfrak{Z}$. One must observe only the fact that the vectors $\mathfrak{P}$ and $\mathfrak{Z}$ do not correspond to the quantity $\mathfrak{E}$, but rather – $\mathfrak{E}$. For that reason, instead of (9.a) and (9.b), we will have the following transformation formulas:

$$\left.\begin{aligned} \mathfrak{M}' &= (1-\gamma)(\mathfrak{M}\cdot\mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma\left(\mathfrak{M} + \frac{1}{c}\,\mathfrak{v}\times\mathfrak{P}\right), \\ \mathfrak{P}' &= (1-\gamma)(\mathfrak{P}\cdot\mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma\left(\mathfrak{P} - \frac{1}{c}\,\mathfrak{v}\times\mathfrak{M}\right), \end{aligned}\right\} \tag{11}$$

and there will be identical formulas for the vectors $\mathfrak{Z}^*$, $\mathfrak{Z}$.

Recall that when one treats time-constant fields, the electric and magnetic polarization seem to be entirely independent quantities (cf., **§ 10**, Chap. III), just like the electric and magnetic field strengths. Later on (**§ 4**, Chap. V), when treating time-varying fields, we have composed them in a purely superficial way since the current density $\mathfrak{j}$ was not expressed in terms of merely $\frac{1}{c}\frac{\partial\mathfrak{P}}{\partial t}$, but by the sum rot $\mathfrak{M} + \frac{1}{c}\frac{\partial\mathfrak{P}}{\partial t}$. In that way, we considered both expressions to be completely equivalent and reduced their formal difference to differing definition of the vector $\mathfrak{P}$. In fact, the vector $\mathfrak{P}$ was defined only incompletely by the formula $\rho = -$ div $\mathfrak{P}$ (**§ 10**, Chap. III) that was originally used in its definition, even in conjunction with the boundary condition $P_n = \eta$, and one can use that indeterminacy in order to replace $\frac{1}{c}\frac{\partial\mathfrak{P}}{\partial t}$ with the sum $\frac{1}{c}\frac{\partial\mathfrak{P}}{\partial t}$ + rot $\mathfrak{M}$, along with the additional boundary condition $\mathfrak{M} \times \mathfrak{n} = 0$. Obviously, that definitions of the two vectors $\mathfrak{P}$ and $\mathfrak{M}$ are not free of arbitrariness, either. However, the principle of relativity shows that in the event that they are established in a certain reference system, when one goes to another inertial system, one must treat them as quantities that are closely coupled with each other by the transformation in formulas (11). For example, if $\mathfrak{M}' = 0$ in the "primed" reference system (S'), and $\mathfrak{P}'$ is non-zero, then $\mathfrak{M}$ must also be non-zero relative to *S*, and it must be coupled with the corresponding value of the electric polarization by the relation:

$$\mathfrak{M} = -\frac{1}{c}\,\mathfrak{v}\times\mathfrak{P}\,. \tag{11.a}$$

By contrast, if $\mathfrak{P}' = 0$ and $\mathfrak{M}' \neq 0$ then one will have:

$$\mathfrak{P} = \frac{1}{c}\,\mathfrak{v}\times\mathfrak{M}\,. \tag{11.b}$$

Both relations are true quite rigorously for arbitrarily-large relative velocities $\mathfrak{v}$ ($< c$). If we have a dipole at rest in the system S' or a uniformly-polarized ball with an electric moment $\mathfrak{p}'$ then from the standpoint of the system *S*, that ball will seem to be polarized not only electrically, but also magnetically, with a moment of $\mathfrak{m} = -(1 / c)\,\mathfrak{v}\times\mathfrak{p}$. Therefore, $\mathfrak{p}$ is identical to $\mathfrak{p}'$ in a first approximation. According to (11), they differ by only quantities order two and higher in v / c. Namely, by switching the primed and unprimed quantities and the sign of $\mathfrak{v}$ in (11), we will get:

$$\mathfrak{P} = (1-\gamma)\,(\mathfrak{P}'\cdot\mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma\,\mathfrak{P}'$$

when $\mathfrak{M}' = 0$. In the same way, a ball that is at rest relative to S' and appears to be polarized *only magnetically* with a moment of $\mathfrak{m}'$ ($\mathfrak{P}' = 0$) will also appear to have an electric polarization with a moment of $\mathfrak{p} = +(1 / c)\,\mathfrak{v}\times\mathfrak{m}$ to an observer that is fixed in *S*. Obviously, we will then have:

$$p'^2 = p^2 - m^2 \qquad \text{or} \qquad m'^2 = m^2 - p^2, \qquad \text{resp.,} \tag{11.c}$$

corresponding to (10.c). The inner product $\mathfrak{m}\cdot\mathfrak{p}$ remains equal to zero in both cases, which corresponds to the invariance in (10.c).

It is possible to make a sharp distinction between an electric dipole and an elementary current, or a magnetic dipole that can replace it as the source of an electromagnetic field, only in the corresponding "rest system" then. Just as the electric field partially goes to a magnetic one under the motion of the electric dipole that creates it, the electric moment of that dipole will partially go to a magnetic one, and *vice versa*. Due to the relativity of velocity, the concepts of magnetic and electric will be just as relative as, say, "spatial" and "temporal."

Those considerations can obviously be applied to the case of the rotating electron that was treated in Chap. VII, **§ 8**. In that way, the introduction of the *electric* polarization will be impossible, or better yet, inconvenient, in the "rest system" S', in which the electron has only a rotational motion, since the electron is not a neutral system. That is why one must next neutralize it with an opposite charge (at infinity or anywhere else, e.g., at its center). In that latter case, a radial polarization would arise inside of the electron with a resultant electric moment of zero. By contrast, one can replace the rotational motion of the electron with a homogeneous (as a surface charge) or inhomogeneous (as a volume charge) magnetic polarization with the moment $\mathfrak{m}$ that was calculated in Chap. VII. As we already suggested at the time, when such a magnetic electron moves relative to the observe with a translational velocity of $\mathfrak{v}$, it must appear to be electrically polarized with a moment of $\mathfrak{p} = (1 / c)\, \mathfrak{v} \times \mathfrak{m}$. From (59), Chap. VIII, that additional moment will correspond to an additional scalar potential $+ \mathfrak{A} \cdot \mathfrak{v} / c$. However, that is precisely the value that the transformation formulas (3) will give when we set $\varphi' = 0$ in it. That aforementioned additional electric moment of the rotating electron can also be calculated without introducing the magnetic polarization, and indeed, on the grounds of the transformation formula (2) for the electric charge density. If one ignores the relativistic "contraction" of the electron, i.e., if one restricts oneself to first-order quantities in v / c, then from (2), one will have:

$$\rho = \rho' + \frac{1}{c}\mathbf{j}\mathfrak{v} \approx \rho' + \frac{1}{c}\mathbf{j}'\mathfrak{v} .$$

In that, one has:

$$\mathbf{j}' = \frac{\rho'}{c}(\mathfrak{o} \times \mathfrak{r}) ,$$

where $\mathfrak{o}$ means the rotational velocity of the electron relative to S'. From the definition of the electric moment, that will imply that:

$$\mathfrak{p} = \int \rho\,\mathfrak{r}\, dV = \int \rho'\,\mathfrak{r}\left[\frac{1}{c}\mathfrak{v}\cdot(\mathfrak{o}\times\mathfrak{r})\right] dV ,$$

since $\int \rho'\,\mathfrak{r}\, dV = 0$. Now, we have:

$$\mathfrak{v} \cdot (\mathfrak{o} \times \mathfrak{r}) = \mathfrak{r} \cdot (\mathfrak{v} \times \mathfrak{o}) ,$$

and as a result, in the mean over different directions of $\mathfrak{r}$, we will have:

$$\overline{\mathfrak{r}[\mathfrak{v}\cdot(\mathfrak{o}\times\mathfrak{r})]} = \overline{\mathfrak{r}[\mathfrak{r}\cdot(\mathfrak{v}\times\mathfrak{o})]} = \tfrac{1}{3} r^2 (\mathfrak{v} \times \mathfrak{o}) .$$

We will then have:

$$\mathfrak{p} = \frac{1}{3}\left(\frac{1}{c}\mathfrak{v}\times\mathfrak{o}\right)\int \rho' r^2\, dV = \frac{1}{c}\mathfrak{v}\times\frac{1}{3}\mathfrak{o}\int \rho' r^2\, dV ,$$

i.e., from (45), Chap. VII:

$$\mathfrak{p} = \frac{1}{c}\mathfrak{v} \times \mathfrak{m} .$$

Note that according to the theory of relativity, the angular velocity $\mathfrak{o}$, just like the magnetic moment, is regarded as the spatial part of a six-vector. In order to be consistent, one must treat the rotational motion in an entirely-different way from the translational motion. Rather, it is questionable whether rotational motion can exist at all, in the ordinary sense of the term. That is because a rotation of an ordinary material body can be reduced, in essence, to a translational motion of the electrons that define it. Whether one can speak of a rotation of the individual electrons in the same sense is initially still quite doubtful.

As was suggested before in Chap. VII, **§ 8**, the expressions (57) and (57.a) that were given at the time for the additional torque and force that would appear due to the combination of rotation and translation are *false*, and must be replaced with the formulas:

$$\mathfrak{M}^a = \mathfrak{p} \times \mathfrak{E}^a = \left(\frac{1}{c}\mathfrak{v} \times \mathfrak{m}\right) \times \mathfrak{E}^a$$

and

$$\mathfrak{F}^a = (\mathfrak{p} \text{ grad})\, \mathfrak{E}^a = \left[\left(\frac{1}{c}\mathfrak{v} \times \mathfrak{m}\right) \text{grad}\right] \mathfrak{E}^a .$$

In formulas (67) and (67.b), Chap. VII, which are supposed to characterize the interaction between the two magnetic moments of a rotating electron, one must accordingly replace them with:

$$\mathfrak{M}^a = f \cdot (\mathfrak{p} \times \mathfrak{r})$$

and

$$\mathfrak{M}'^a = \mathfrak{r} \times (\mathfrak{p} \text{ grad})\, f\, \mathfrak{r} = -f \cdot (\mathfrak{p} \times \mathfrak{r}) ,$$

in which one has $\mathfrak{r} \times \mathfrak{r}\,(\mathfrak{p} \text{ grad} f) = 0$. The two torques are then equal and opposite, from which it will follow that the resultant impulse moment of the electron must remain constant. We will return to that topic in the last section of this book.

§ 3. – Transforming the energy tensor. Force and torque.

The various components of the energy tensor ${}^2\Theta$ [cf., the matrix (14.a), Chap. VIII] transform in a rather complicated way. That is why we would like to restrict ourselves to considering the energy density ξ and the momentum $\mathfrak{g}$. As far as the former is concerned, we will have:

$$\xi' = \Theta'_{44} = \sum_k \sum_l \alpha_{k4}\, \alpha_{l4}\, \Theta_{kl} = \alpha_{14}^2\, \Theta_{11} + 2\alpha_{14}\, \alpha_{44}\, \Theta_{14} + \alpha_{44}^2\, \Theta_{44} ,$$

i.e., from (12.a), Chap. VIII and (8.b), Chap. IX:

$$\xi' = \gamma^2\left[-\frac{1}{8\pi}\frac{v^2}{c^2}(H_1^2 - H_2^2 - H_3^2 + E_1^2 - E_2^2 - E_3^2) - 2v\,g_1 + \xi\right],$$

or, since $\xi = (H^2 + E^2)/8\pi$ and $\gamma^2(1 - v^2/c^2) = 1$:

$$\xi' = \xi + 2\gamma^2\left[\frac{1}{8\pi}\frac{v^2}{c^2}(H_2^2 + H_3^2 + E_2^2 + E_3^2) - v\,g_1\right].$$

Now, since the velocity $\mathfrak{v}$ has the direction of the X_1-axis, one will obviously have:

$$v\,g_1 = \mathfrak{v}\cdot\mathfrak{g} \qquad \text{and} \qquad \frac{v^2}{c^2}(H_2^2 + H_3^2 + E_2^2 + E_3^2) = \left(\frac{1}{c}\mathfrak{v}\times\mathfrak{E}\right)^2 + \left(\frac{1}{c}\mathfrak{v}\times\mathfrak{H}\right)^2.$$

The coordinate-free transform of the formula for the energy density will then read:

$$\xi' = \xi + \gamma^2\left\{\frac{1}{4\pi c^2}[(\mathfrak{v}\times\mathfrak{E})^2 + (\mathfrak{v}\times\mathfrak{H})^2] - 2\,\mathfrak{v}\,\mathfrak{g}\right\}. \tag{12}$$

Note that the second term in the bracket corresponds to the classical transformation of kinetic energy for $v \ll c$. Namely, if one considers a particle of the mass m that moves relative to S with a velocity of $\mathfrak{u}$ from the standpoint of the second coordinate system S' then one will get:

$$T' = \tfrac{1}{2}mu'^2 = \tfrac{1}{2}m(\mathfrak{u} - \mathfrak{v})^2 = \tfrac{1}{2}mu^2 - m\,\mathfrak{u}\cdot\mathfrak{v} + \tfrac{1}{2}mv^2,$$

or approximately, for $v \ll c$:

$$2T' = 2T - (m\,\mathfrak{u})\cdot\mathfrak{v}.$$

Here, $m\,\mathfrak{u}$ means the momentum of the particle considered relative to S. The fact that the electromagnetic energy density ($\xi = \mu c^2$) corresponds to twice the kinetic energy of ordinary mechanics was already mentioned in **§ 3**, Chap. VII.

As far as the transformation of the electromagnetic momentum is concerned, we have:

$$-\,i\,c\,g_1' = \Theta_{14}' = \sum_k\sum_l \alpha_{k1}\,\alpha_{l4}\,\Theta_{kl} = \alpha_{11}\,\alpha_{14}\,\Theta_{11} + \alpha_{11}\,\alpha_{44}\,\Theta_{14} + \alpha_{41}\,\alpha_{14}\,\Theta_{41} + \alpha_{41}\,\alpha_{44}\,\Theta_{44}$$

for its first component, i.e., from (8.b):

$$g_1' = \frac{\gamma^2}{-ic}\left[-ic\,g_1\left(1+\frac{v^2}{c^2}\right)+\frac{1}{8\pi}\frac{v}{ic}(H_1^2-H_2^2-H_3^2+E_1^2-E_2^2-E_3^2)-\xi\right],$$

or

$$g_1' = \gamma^2\left[\left(\frac{1+v^2}{c^2}\right)g_1-\frac{2v}{c^2}\xi+\frac{1}{4\pi}\frac{v}{c^2}(E_1^2+H_1^2)\right]. \tag{12.a}$$

We get:

$$-\,ic\,g_2' = \Theta_{24}' = \sum\sum\alpha_{k2}\,\alpha_{l4}\,\Theta_{kl} = \sum_l\alpha_{l4}\,\Theta_{2l} = \alpha_{14}\,\Theta_{21}+\alpha_{44}\,\Theta_{21}\,,$$

i.e.:

$$g_2' = \gamma\left[g_2+\frac{1}{4\pi}\frac{v}{c^2}(E_1\,E_2+H_1\,H_2)\right],$$

for the second component, and likewise:

$$g_3' = \gamma\left[g_3+\frac{1}{4\pi}\frac{v}{c^2}(E_1\,E_3+H_1\,H_3)\right],$$

for the third. Those formulas cannot be simply combined vectorially.

We would like to add the following formula for the linear invariant of the tensor ${}^2\Theta$:

$$\sum_{k=1}^{4}\Theta_{kk} \equiv 0\,.$$

That is given directly by (13), Chap. VIII.

In Chap. VII, **§ 7**, along with the vector $\mathfrak{g}$, we introduced the vector $\mathfrak{r}\times\mathfrak{g}$, which we defined to be the density of the electromagnetic impulse moment. From the standpoint of the theory of relativity, that quantity must be regarded as a component of a rather complicated four-dimensional quantity. The ordinary components of the vector $\mathfrak{r}\times\mathfrak{g}$, namely, $x_k\,g_l - x_l\,g_k$, define three (or six) components of a partially-symmetric, partially skew-symmetric four-dimensional tensor of rank *three*, and the general form of its components is:

$$x_k\,\Theta_{ln} - x_l\,\Theta_{kn}\,.$$

The four-dimensional extension of the usual three-dimensional *torque* $\mathfrak{r}\times\mathfrak{f}$ will be correspondingly represented by the skew-symmetric tensor, or six-vector ${}^2\mathfrak{M}$, with the components:

$$M_{kl} = x_k\,f_l - x_l\,f_k\,. \tag{13}$$

Therefore, the first, second, and third component of the vector will be equal to M_{23}, M_{31}, M_{12}. They will then relate to the angular momentum tensor ${}^2\mathfrak{M}$ in the same way that the magnetic field strengths relate to the field tensor ${}^2\mathfrak{H}$. The associated part of ${}^2\mathfrak{M}$ that corresponds to the electric field strength is a vector with the components:

$$i\,M_{14}\,,\;\; i\,M_{24}\,,\;\; i\,M_{34}\,,$$

i.e., when one recalls $f_4 = i\,l\,/\,c$:

$$\frac{l}{c}\mathfrak{r} - c\,\mathfrak{f}\,. \tag{13.a}$$

We will come back to that vector, and the tensor ${}^2\mathfrak{M}$ in general, later on. It plays a non-trivial role in the theory of rotational and orbital motion. The transformation formulas for the vectors $\mathfrak{r} \times \mathfrak{f}$ and $\frac{l}{c}\mathfrak{r} - c\,\mathfrak{f}$ are completely identical to the transformation formulas for $\mathfrak{H}$ and $\mathfrak{E}$.

Ultimately, it should be pointed out that the components of the "four-force" (or "four-impulse") $\mathfrak{f}$ transform in the same way as any other four-vector (cf., § **1**). For that reason, we will not need to go into the details of that issue. However, it would not seem irrelevant to emphasize that in the theory of relativity, the ordinary force $\mathfrak{f}$ is a *variant* quantity in regard to not only its direction, but also its magnitude, just like the spatial distance between two space-time points. By contrast, the difference:

$$f^2 - \frac{l^2}{c^2} = \mathfrak{f}^2$$

is regarded as an invariant quantity. If one substitutes $\mathfrak{l} = \mathfrak{f} \cdot \mathfrak{u}$ in that then one will have:

$$\mathfrak{f}^2 = \mathfrak{f}^2 - \left(\frac{1}{c}\mathfrak{f}\cdot\mathfrak{u}\right)^2.$$

It follows from the constraint $u < c$ that the vector $\mathfrak{f}$ is *space-like*, and the quantity:

$$f_0 = \sqrt{f^2 - \left(\frac{1}{c}\mathfrak{f}\cdot\mathfrak{v}\right)^2} \tag{13.b}$$

represents its "rest magnitude," which is the "natural" magnitude that is as small as possible. The relativity of force is an immediate consequence of the relativity of velocity since the latter enters explicitly into the expression for the force according to the formula:

$$\mathfrak{f} = \rho\,\mathfrak{E} + \frac{1}{c}\,\mathfrak{v} \times \mathfrak{H}\,.$$

§ 4. – Applying the relativistic transformation formulas to the uniform rectilinear motion of electrons and oscillators.

In **§ 2**, we alluded to the fact that the influence of translation (with constant velocity) on the properties of rotating electrons can be determined by means of the theory of relativity. We would now like to calculate the electromagnetic field of a moving electron (initially with no magnetic moment) in the same way and then the field of an arbitrary oscillator (so the magnetic electron can be regarded as a special case of that). In that way, we will treat the electron, like the oscillator, as point-like, for the sake of simplicity.

The field of those particles relative to the coordinate system S' in which they are at rest must obviously be assumed to be known. As always, we shall denote the velocity of S' relative to S by $\mathfrak{v}$.

a) *Electron* (point-charge). – The electromagnetic potentials in S' are:

$$\mathfrak{A}' = 0\,, \qquad \varphi' = \frac{e}{R'}\,.$$

It will follow from (3) and (3.a) (in which the primed quantities can be switched with the unprimed ones while inverting the sign of $\mathfrak{v}$) that:

$$\mathfrak{A} = \frac{\varphi}{c}\,\mathfrak{v}\,, \qquad \varphi = \frac{\varphi'}{\sqrt{1-\beta^2}} = \frac{e}{R'\sqrt{1-\beta^2}} \qquad \left(\beta = \frac{v}{c}\right). \tag{14}$$

The problem then arises of expressing the distance R' in terms of the "unprimed" quantities. We assume that for $t = t' = 0$, the electron is found at the common origin (O or O', resp.) of the two coordinate systems S and S'. We then, in turn, address the calculation of the quantity R' that is associated with an arbitrary spacetime point Q ($\mathfrak{r}$, t) in S. Let the radius vector of the electron in S be denoted by $\mathfrak{r}^0$. Obviously, we have $\mathfrak{r}^0 = OO' = \mathfrak{v}\,t$, and the distance from the reference point P ($\mathfrak{r}$) considered to the electron at the moment t (in S) will be equal to the magnitude of the vector $\mathfrak{R} = \mathfrak{r} - \mathfrak{r}^0$.

If one introduces the coordinate axes (X_1, X_2, X_3) and (X_1', X_2, X_3), as usual, then after a Lorentz transformation, one will have:

$$x_1' = \frac{x_1 - vt}{\sqrt{1-\beta^2}} = \frac{x_1 - x_1^0}{\sqrt{1-\beta^2}}\,, \qquad x_2' = x_2\,, \qquad x_3' = x_3\,,$$

and

$$R'^2 = x_1'^2 + x_2'^2 + x_3'^2 = \frac{(x_1 - x_1^0)^2 + (1-\beta^2)(x_2^2 + x_3^2)}{(1-\beta^2)} = \frac{R^{*2}}{1-\beta^2}, \tag{14.a}$$

i.e., as a result:

$$\varphi = \frac{e}{R^*}, \tag{14.b}$$

in agreement with the results of Chap. VI.

The electric and magnetic field strengths can be obtained from those formulas by differentiating with respect to the coordinates x_1, x_2, x_3, and time. However, one can determine them *directly* from the corresponding expressions for the rest system S' according to the transformation formulas (9.a) and (9.b) or the formulas that are reciprocal to them, Namely, one has:

$$\mathfrak{E}' = \frac{e}{R^3}\mathfrak{R}', \quad \mathfrak{H}' = 0. \tag{15}$$

As a result, we have:

$$\mathfrak{H} = \frac{1}{c}\mathfrak{v} \times \mathfrak{E}, \mathfrak{E} = (1-\gamma)(\mathfrak{E}'\,\mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma\,\mathfrak{E}' = \frac{e}{R'^3}\{(1-\gamma)(\mathfrak{R}'\,\mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma\,\mathfrak{R}'\}.$$

Now, in coordinate notation, we have:

$$\mathfrak{R}'\,\mathfrak{v}_0 = x_1' = \frac{x_1 - x_1^0}{\sqrt{1-\beta^2}} = \gamma(x_1 - x_1^0) = \gamma(\mathfrak{R}\,\mathfrak{v}_0).$$

The first component (i.e., the one parallel to $\mathfrak{v}_0$) of the vector $(1-\gamma)(\mathfrak{R}'\,\mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma\,\mathfrak{R}'$ is then equal to:

$$(1-\gamma)\,\gamma\,\mathfrak{R}\,\mathfrak{v}_0 + \gamma^2\,\mathfrak{R}\,\mathfrak{v}_0 = \gamma\,\mathfrak{R}\,\mathfrak{v}_0.$$

Since the other components of $\mathfrak{R}$ and $\mathfrak{R}'$ are identical, we will have simply:

$$(1-\gamma)\,\gamma(\mathfrak{R}\cdot\mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma\,\mathfrak{R}' = \gamma\,\mathfrak{R}, \tag{15.a}$$

and therefore, from (14.a):

$$\mathfrak{E} = (1-\beta^2)\,e\frac{\mathfrak{R}}{R^{*3}}. \tag{15.b}$$

That is formula (13), Chap. VI that we know already.

b) *Oscillator*. – We consider the oscillator to be a double dipole whose electric and magnetic moments are given by the vectors $\mathfrak{p}'$ and $\mathfrak{m}'$ in the "rest system" S'. Those vectors can initially be arbitrary (but known) functions of an oscillatory nature in time t'. We can combine the vectors $\mathfrak{p}'$ and $\mathfrak{m}'$ into a skew-symmetric moment tensor ${}^2\mathfrak{p}$ with the components (relative to S'):

$$\begin{aligned} p'_{23} &= m'_1, \quad & p'_{31} &= m'_2, \quad & p'_{12} &= m'_3, \\ p'_{14} &= i\,p'_1, \quad & p'_{24} &= i\,p'_2, \quad & p'_{34} &= i\,p'_3. \end{aligned}$$

The electromagnetic field of that double oscillator in the system S' is determined by the known expressions for the electric and polarization potential:

$$\mathfrak{Z}' = \frac{t' - R'/c}{R'}\,\mathfrak{p}', \qquad \mathfrak{Z}'^* = \frac{t' - R'/c}{R'}\,\mathfrak{m}'. \tag{16}$$

[Cf., (28.a), Chap. V. What was denoted by t' in that expression is now $t' - R'/c$].

According to the formulas that are reciprocal to (11), in which $\mathfrak{P}$ and $\mathfrak{M}$ are replaced with $\mathfrak{Z}$ and $\mathfrak{Z}^*$, the transformed expressions for those potentials (in the system S) will now read:

$$\mathfrak{Z} = (1-\gamma)\,(\mathfrak{Z}' \cdot \mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma(\mathfrak{Z}' + \frac{1}{c}\,\mathfrak{v} \times \mathfrak{Z}'^*)\,,$$

$$\mathfrak{Z}^* = (1-\gamma)\,(\mathfrak{Z}'^* \cdot \mathfrak{v}_0)\,\mathfrak{v}_0 + \gamma(\mathfrak{Z}'^* - \frac{1}{c}\,\mathfrak{v} \times \mathfrak{Z}')\,.$$

If one introduces the expressions above in them and recalls that $R' = \gamma\,R^*$ then one will have:

$$\left.\begin{aligned} \mathfrak{Z} &= \frac{1}{R^*}\left[(\gamma^{-1}-1)(\mathfrak{p}'\cdot\mathfrak{v}_0)\mathfrak{v}_0 + \mathfrak{p}' + \frac{1}{c}\mathfrak{v}\times\mathfrak{m}'\right], \\ \mathfrak{Z}^* &= \frac{1}{R^*}\left[(\gamma^{-1}-1)(\mathfrak{m}'\cdot\mathfrak{v}_0)\mathfrak{v}_0 + \mathfrak{m}' - \frac{1}{c}\mathfrak{v}\times\mathfrak{m}'\right]. \end{aligned}\right\} \tag{16.a}$$

In those formulas, $\mathfrak{p}'$ and $\mathfrak{m}'$ are regarded as *given* vector functions, and we need to express only their *argument*, i.e., effective time relative to S':

$$\tau' = t' - R'/c\,,$$

which plays the role of *phase*, in terms of t and $\mathfrak{r}$ (or $\mathfrak{R}$).

Now, if $R' = \gamma\,R^*$ and:

$$t' = \frac{t - (\mathfrak{r}\cdot\mathfrak{v})/c^2}{\sqrt{1-v^2/c^2}} = \gamma\left(t - \frac{\mathfrak{r}\cdot\mathfrak{v}}{c^2}\right)$$

then, as a result, one will have:

$$\tau' = \gamma\left(t - \frac{\mathfrak{r}\cdot\mathfrak{v}}{c^2} - \frac{R^*}{c}\right) = \gamma\,\tau, \tag{16.b}$$

in which τ has the meaning relative to S that would correspond to τ'. One can calculate the potentials $\mathfrak{A}$, φ, and the field strengths $\mathfrak{E}$, $\mathfrak{H}$ by differentiating (16.a) with respect to x_1, x_2, x_3, t using the last formula. However, as we have remarked before, it is generally much simpler to calculate the desired quantities (e.g., $\mathfrak{E}$ and $\mathfrak{H}$) *directly* (using the formulas that are reciprocal to (9.a) and (9.b)] from the corresponding "primed" quantities, which have the forms:

$$\mathfrak{E}' = \frac{1}{c^2 R}[\mathfrak{R}'_0\times(\mathfrak{R}'_0\times\ddot{\mathfrak{p}}') + \mathfrak{R}'_0\times\ddot{\mathfrak{m}}'], \qquad \mathfrak{H}' = \frac{1}{c^2 R}[\ddot{\mathfrak{p}}'\times\mathfrak{R}'_0 + \mathfrak{R}'_0\times(\mathfrak{R}'_0\times\ddot{\mathfrak{m}}')],$$

in the case considered, according to (31), (32), (51.a), and (51.b).

However, we would not like to do that calculation and will content ourselves with the remark that the electric and magnetic field strengths must also be numerically equal and perpendicular to each other relative to S [according to (10.b) and (10.c)]. It follows from this that the ray vector $\mathfrak{K}$ $= \frac{c}{4\pi}(\mathfrak{E}\times\mathfrak{H})$, just like $\mathfrak{K}'$, can be represented by the product of the energy density times the speed of light. However, the direction of that light velocity must be different in S and S'.

One can determine the relationship between $\mathfrak{K}$ and $\mathfrak{K}'$ directly by means of the transformation formulas (12.a) and the ones that follow from them. However, it is simpler to calculate the change in the two factors ξ and $\mathfrak{c}$ (obviously, the latter relates to only the direction) individually.

For the sake of simplicity, we will assume that the radiated light is "unpolarized," i.e., the oscillations of the oscillator are distributed uniformly in all directions, and the intersections of the vectors $\mathfrak{H}'$ and $\mathfrak{E}'$ with the directions perpendicular to the wave normal $\mathfrak{n}'$ (= $\mathfrak{R}'/R'$) will have the same absolute value correspondingly. Obviously, their mean values $\bar{\mathfrak{E}}$ and $\bar{\mathfrak{H}}$ must vanish then. By contrast, according to the equation $(E'^2 + H'^2)/8\pi = \xi'$, we will have:

$$\overline{E'^2} = \overline{H'^2} = 4\pi\,\xi'$$

for the mean values of their squares. The mean values of $(\mathfrak{v}\times\mathfrak{H}')^2$ and $(\mathfrak{v}\times\mathfrak{E}')^2$ are obviously equal to each other. In order to determine them, we set $\mathfrak{H}' = \mathfrak{n}'\times\mathfrak{E}'$ (or $\mathfrak{E}' = \mathfrak{H}'\times\mathfrak{n}'$). We will then have:

$$\mathfrak{v}\times\mathfrak{H}' = \mathfrak{v}\times(\mathfrak{n}'\times\mathfrak{E}') = (\mathfrak{v}\cdot\mathfrak{E}')\,\mathfrak{n}' - (\mathfrak{n}'\cdot\mathfrak{v})\,\mathfrak{E}',$$

and as a result:

$$\overline{(\mathfrak{v}\times\mathfrak{H}')^2} = \overline{(\mathfrak{v}\cdot\mathfrak{E}')^2} + \overline{E'^2}\,(\mathfrak{n}'\cdot\mathfrak{v}) \qquad (\text{since } \mathfrak{E}'\cdot\mathfrak{n}' = 0),$$

or since:

$$(\mathfrak{v}\,\mathfrak{E}')^2 = v^2\,E'^2 - (\mathfrak{v}' \times \mathfrak{E}')\,,$$

we will have:

$$\overline{(\mathfrak{v}\times\mathfrak{H}')^2} + \overline{(\mathfrak{v}\times\mathfrak{E}')^2} = v^2\,\overline{E'^2}\,(1+\cos^2\theta')\,,$$

in which $\theta' = (\mathfrak{n}' \cdot \mathfrak{v}_0)$ means the angle between the wave normal (i.e., the ray $\mathfrak{R}'$) and the velocity $\mathfrak{v}$.

From (12), when we switch the primed and unprimed quantities and recall the fact that:

$$\mathfrak{v} \cdot \mathfrak{g}' = \frac{v\,\xi'}{c}\cos\theta'\,,$$

we will get the formula:

$$\xi = \xi'\left\{1+\gamma^2\left[\frac{v^2}{c^2}(1+\cos^2\theta') + 2\frac{v}{c}\cos\theta'\right]\right\}.$$

If one sets $v / c = \beta$ in that and observes that $\gamma^2 = 1/(1-\beta^2)$ then one will get:

$$\xi = \xi'[\sin^2\theta' + \gamma^2(\cos\theta' + \beta)^2] = \xi'\,\gamma^2(\beta\cos\theta' + 1)^2\,.$$

The change in direction of the light rays (wave normals) under the transition from the coordinate system S' to S can be derived quite simply from the formula (16.b) for the "phase," when expressed as a function of $\mathfrak{r}$ (and t). That is because the wave surfaces can generally be defined to be the surfaces of *constant phase* at *fixed values of time*. From the standpoint of the coordinate system S', in which the oscillator is "at rest" at its origin O', the surfaces of constant phase τ' for t' = const. are determined by the equation R' = const. They will then be concentric spheres, as is clear from the outset. By contrast, from the standpoint of the system S, according to (16.b), they will be given by the equation:

$$\psi = \frac{\mathfrak{r}\cdot\mathfrak{v}}{c} + R^* = \text{const.} \tag{17}$$

The direction of the wave normal at the reference point ($\mathfrak{r}$, t) is obviously determined by the gradient of ψ. We then have:

$$\operatorname{grad}\psi = \frac{1}{c}\mathfrak{v} + \operatorname{grad} R^*\,,$$

or in coordinate notation:

$$\frac{\partial\psi}{\partial x_1} = \beta + \frac{x_1 - x_1^0}{R^*}\,, \qquad \frac{\partial\psi}{\partial x_2} = (1-\beta^2)\frac{x_2}{R^*}\,, \qquad \frac{\partial\psi}{\partial x_3} = (1-\beta^2)\frac{x_3}{R^*}\,,$$

and as a result, from the known formulas:

$$x_1 - x_1^0 = \gamma\, x_1', \qquad x_2 = x_2', \qquad x_3 = x_3', \qquad R^* = \frac{R'}{\gamma}, \qquad 1-\beta^2 = \frac{1}{\gamma^2},$$

one will have:

$$\frac{\partial \psi}{\partial x_1} = \beta + \frac{x_1'}{R'}, \qquad \frac{\partial \psi}{\partial x_2} = \frac{x_2'}{\gamma R'}, \qquad \frac{\partial \psi}{\partial x_3} = \frac{x_3}{\gamma R'}.$$

If one denotes the angle between the wave normal and the direction of $\mathfrak{v}$ in S and S' by θ and θ', resp., then one will obviously have:

$$\cos\theta = \frac{1}{|\operatorname{grad}\psi|}\frac{\partial \psi}{\partial x_1}, \qquad \sin\theta = \frac{1}{|\operatorname{grad}\psi|}\sqrt{\left(\frac{\partial \psi}{\partial x_1}\right)^2 + \left(\frac{\partial \psi}{\partial x_2}\right)^2},$$

on the one hand, and:

$$\cos\theta' = \frac{x_1'}{R'}, \quad \sin\theta' = \frac{\sqrt{x_2'^2 + x_3'^2}}{R'},$$

on the other. It will then follow that:

$$\tan\theta = \frac{\sin\theta'}{(\gamma\cos\theta' + \beta)}. \tag{18}$$

For example, let S be a coordinate system that is fixed on the Earth. A "fixed star" might serve as an oscillator. That "fixed star" must move relative to S, and indeed with a periodically-alternating velocity (due to the orbital motion of the Earth around the Sun). The direction of its light rays that are observed on Earth, or in other words, its apparent location in the sky, must correspondingly vary periodically. That is the phenomenon of the *aberration of the fixed stars* that was discovered by *Bradley*. Ordinarily, one does not care to speak of the direction of the light ray, but of the direction of observation that is opposite to it, and one does not consider the motion of the star relative to the Earth, but the motion of the Earth relative to star. One must correspondingly invert the sign of θ' (angle of observation) and β (velocity of the Earth divided by the speed of light), which will not alter the formula above.

Note that, in reality, only the *change* in the angle θ can be observed, which is based upon the orbital motion of the Earth around the Sun. Therefore, a *constant* difference must remain between θ and θ', in general, that originates in the relative translational motion of the star and the solar system. That relationship is represented schematically in the accompanying figure (Fig. 39). In that figure, B means the Earth and A' means the so-called true position of the star. A is the "apparent" position (from the standpoint of the system S), while BC is the instantaneous direction of motion of the Earth (as evaluated in S'). The difference $\alpha = \theta' - \theta$ is called the *aberration angle*. From (18), the tangent to that angle is equal to:

$$\tan\alpha = \frac{\tan\theta' - \tan\theta}{1 + \tan\theta'\tan\theta} = \sin\theta'\frac{(\gamma-1)\cos\theta' + \beta}{\gamma\cos\theta'(\cos\theta' + \beta) + \sin^2\theta'}.$$

The velocity of the Earth relative to the stars (and one deals with *only the relative* velocity) is very small compared to the speed of light. That is why one can set $\gamma = 1$, and generally omit terms of second order in β. Therefore, the formula above will assume the well-known simple form:

$$\alpha \approx \beta \sin\, \theta' \approx \beta \sin\, \theta\,.$$

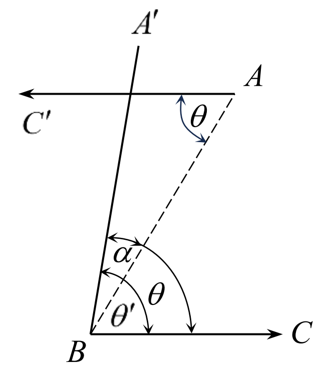

Figure 39.

Up to now, we have made no assumption about the type of light oscillations. We would now like to assume that those oscillations are *harmonic* (i.e., monochromatic light). Let their frequency in the rest system S' (i.e., their "true" frequency) be ν'. The dependency of the oscillations of the oscillator on time is then expressed relative to S' by the phase factor cos $2\pi \nu' t'$, and the corresponding dependency of the light oscillations at the reference point in question is expressed by cos $2\pi \nu' \tau'$, i.e., from the standpoint of the coordinate system S (viz., a terrestrial astronomer), by:

$$\cos\, 2\pi \nu' \gamma \left(t - \frac{\mathfrak{r}\,\mathfrak{v}}{c^2} - \frac{R^*}{c} \right).$$

$\mathfrak{r}$ remains constant for a point that is fixed in S (e.g., a telescope). By contrast, the quantity R^* must vary slowly. For a time interval $t - t_0$ that is not too long, one can represent its dependency upon time by $\left(\frac{dR^*}{dt} \right)_0 (t - t_0)$. Now, one has:

$$\frac{dR^*}{dt} = \frac{(x_1^0 - x_1)}{R^*} \frac{dx_1^0}{dt} = - \frac{\mathfrak{R} \cdot \mathfrak{v}}{R^*}\,.$$

The phase factor above then assumes the following form:

$$\cos \left\{ 2\pi \nu' \gamma \left(1 + \frac{\mathfrak{R}}{R^*} \cdot \frac{\mathfrak{v}}{c} \right) (t - t_0 + \text{const.}) \right\}.$$

The light oscillations that are observed in S are also harmonic then, but they will have a frequency that is different from ν':

$$\nu = \nu' \gamma \left(1 + \frac{\mathfrak{R}}{R^*} \cdot \frac{\mathfrak{v}}{c} \right), \tag{19}$$

or approximately (for very large distances):

$$\nu = \nu' \gamma \left(1 + \beta \cos \theta \right). \tag{19.a}$$

That formula expresses the known *Doppler-Fizeau* principle, but in a form that is sharper than the usual one: Namely, in addition to the "linear" *Doppler* effect, which is found in the factor of first order in β and vanishes for $\theta = \pi / 2$ (viz., motion perpendicular to the direction of observation), there is also a "quadratic" effect that corresponds to the difference between the flow of time in S and S' by way of the factor $\gamma = 1/\sqrt{1-\beta^2}$. One does not need to consider that factor for the speeds that actually occur in nature (for stars, as well as isolated luminous atoms).

The last result regarding the change in the *direction* and *frequency* of light due to the relative motion of the light source and the observer can be derived quite simply when one imagines that the light source (i.e., oscillator) is *infinitely distant* from the outset and correspondingly treats the waves as planar. The dependency of the electromagnetic field of such waves on position and time is known to be determined by a phase factor of the form $\Phi = e^{\pm i\psi}$, in which the phase ψ is expressed linearly in terms of the radius vector $\mathfrak{r}$ and time t by the formula [cf., (37.a) and (37.b), **§ 8**, Chap. V]:

$$\psi = 2\pi(-\nu t + \mathfrak{k}\,\mathfrak{r}) . \tag{20}$$

The vector $\mathfrak{k}$ is numerically equal to the reciprocal wavelength $\frac{\nu}{c} = \frac{1}{\lambda}$, and is directed parallel to the wave normal $\mathfrak{n}$.

The expression (20) refers to the coordinate system S. However, its numerical value must be independent of the choice of coordinate system. In other words, the phase ψ is *invariant under Lorentz transformations*. However, since $\mathfrak{r}$ and t define the spatial and temporal projections of the four-vector $\mathfrak{r}$, $\mathfrak{k}$ and ν (except for certain coefficients) must be the corresponding projections of a certain four-vector $\mathfrak{k}$ such that the phase ψ is equal to the inner product of those four-vectors (times 2π) will be:

$$\psi = 2\pi\,\mathfrak{k}\cdot\mathfrak{r} = 2\pi\sum_{l=1}^{4} k_l\,x_l . \tag{21}$$

On the other hand, from (20), ψ can be represented coordinate-wise in the form:

$$\psi = 2\pi(+k_1 x_1 + k_2 x_2 + k_3 x_3 - \nu t) .$$

Upon comparing that with (21), we will get $-\nu t = k_4 x_4 = k_4\, ic\, t$, i.e.:

$$k_4 = \frac{i}{c}\nu = \frac{i}{\lambda} . \tag{21.a}$$

One must then consider the frequency (multiplied by i / c) to be the fourth component of a four-vector $\mathfrak{k}$ (viz., the *wave vector*). The square of that four-vector is equal to zero identically:

$$\mathfrak{k}^2 = \sum_{l=1}^{4} k_l^2 = 0 , \tag{21.b}$$

which corresponds to the fact that it represents the propagation of an effect with the speed of light.

It follows from the argument above that the spatial and temporal projection of $\mathfrak{k}$ will transform just like the corresponding projections of $\mathfrak{r}$.

From (1) and (1.a) (in which $\mathfrak{r}$ is replaced with $\mathfrak{k}$ and t is replaced with ν / c^2):

$$\nu' = \gamma(\nu - \mathfrak{k} \cdot \mathfrak{v}) \tag{22}$$

and

$$\mathfrak{k}' = \gamma\left(\mathfrak{k} - \frac{\nu}{c^2}\mathfrak{v}\right) + (1-\gamma)\,(\mathfrak{v}_0 \times \mathfrak{k}) \times \mathfrak{v}_0 \,. \tag{22.a}$$

Recall that the vector $(\mathfrak{v}_0 \times \mathfrak{k}) \times \mathfrak{v}_0$ is nothing but the projection of $\mathfrak{k}$ onto a plane that is perpendicular to $\mathfrak{v}$.

We shall once more introduce the coordinate axes (X_1, X_2, X_3), (X_1', X_2, X_3) and assume that the vector $\mathfrak{k}$ lies in the $X_1 X_2$-plane and defines the angle θ with $\mathfrak{v}$. (22) will then assume the form:

$$\nu' = \gamma \nu \left(1 - \frac{v}{c}\cos\theta\right) .$$

That is the formula that is reciprocal to (19.a), i.e., the formula for the *Doppler* principle. By projecting (22.a) onto the first and second axes, we will get:

$$k' \cos\theta' = \gamma\left(k\cos\theta - \frac{v\nu}{c^2}\right) = \gamma k\,(\cos\,\theta - \beta)\,,$$

$$k' \sin\theta' = \gamma k \sin\,\theta + (1-\gamma)\,k \sin\,\theta = k\sin\,\theta\,,$$

from which it will follow immediately that:

$$\tan\,\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}\,,$$

i.e., the formula for the *Bradley* aberration that is reciprocal to (18).

§ 5. – The electromagnetic field of an arbitrarily-moving oscillator.

One can determine the field of an oscillator in *uniform, rectilinear motion* from the field of an oscillator at rest (which is assumed to be known) by means of a Lorentz transformation. We would now like to solve the corresponding problem for an entirely *arbitrary* translational motion. In so doing, we will treat the oscillator as point-like. Note that it can just as well represent a light source (of any type of oscillation) as a "rotating" electron with a time-varying magnetic moment.

The desired solution is given most simply by means of a method that we have applied already in Chap. VI, **§ 4** in the determination of the electromagnetic potential of a moving point-charge. We will now replace that charge with the oscillator in question and replace the four-potential $\mathfrak{A}$ with the polarization potential ${}^2\mathfrak{Z}$. Since the latter satisfies the differential equation $\Box\, {}^2\mathfrak{Z} = 0$, just like $\mathfrak{A}$, in which the oscillator appears merely as a point singularity, one can immediately write down a point solution that corresponds to a singular spacetime point $Q'(\mathfrak{r}')$, and indeed in the form:

$${}^2\mathfrak{Z} = \frac{1}{S^2}\,{}^2\mathfrak{p}\,,$$

in which $S^2 = |\,\mathfrak{r} - \mathfrak{r}'\,|^2$ means the four-dimensional (i.e., spacetime) distance between Q' and the reference point Q ($\mathbf{r}$) in question. Now, the question arises of how one should compose a "line solution" out of such point solutions that would correspond to the actual translational motion of the oscillator and its actual (time-varying) electromagnetic moment.

Since no identity (like say $\Box\,\mathfrak{A} = 0$) exists between the components of the polarization potential, we must look somewhere else for the answer to the question above.

It is initially clear that in the special case of the oscillator at rest, the general solution to be determined must go to the known solution $Z_{kl} = p'_{kl} / R$, in which p'_{kl} are the components of the polarization tensor for the effective moment $t' = t - R / c$. That special solution can be represented in the form:

$$Z_{kl} = \frac{c}{\pi i}\oint \frac{p'_{kl}}{S^2}\,dt'\,.$$

That integral must then be taken along a closed curve in the complete t'-plane around the root of the equation $S^2 = 0$ that corresponds to the retarded action (cf., the corresponding representation of the *Coulomb* potential in **§ 4**, Chap. VI).

If one seeks to adapt the foregoing formula to the general case then one will encounter the following complication from the standpoint of the theory of relativity: The quantities Z_{kl} and p_{kl} are *covariant*, and S^2 is an invariant scalar. However, dt' is *not an invariant scalar*, but only the fourth component of the four-vector $d\mathfrak{r}'$, divided by *ic*, that corresponds to an infinitesimal spacetime displacement of the oscillator along its four-dimensional "world-line."

We now note that this four-vector must have a *time-like* character due to the condition that $v < c$ ($\mathfrak{v}$ = translational velocity of the oscillator). That is why it is possible to replace the time interval of variation dt' with an invariant time-interval $d\tau'$ that is obtained when one considers the oscillator from the standpoint of an inertial system in which it is *instantaneously* at rest, i.e., it moves with the same velocity $\mathfrak{v}$ relative to the original system as the oscillator at the spacetime point considered. According to (24.a), that *natural* time interval, which is as small as possible, is expressed by the formula:

$$d\tau' = dt'\sqrt{1-v'^2/c^2}\,. \tag{23}$$

Note that the corresponding spacetime distance between the points $\mathfrak{r}'$ and $\mathfrak{r}' + d\mathfrak{r}'$, which we would like to denote by ds, is equal to:

$$ds' = \sqrt{dx_1'^2+dx_2'^2+dx_3'^2+dx_4'^2} = i\,c\,dt'\sqrt{1-\left(\frac{dx_1'}{dt'}\right)^2+\left(\frac{dx_2'}{dt'}\right)^2+\left(\frac{dx_3'}{dt'}\right)^2}\,,$$

i.e.:

$$ds' = i\,c\,d\tau\,. \tag{23.a}$$

One calls ds' the arc-length element of the four-dimensional "world-line."

The general covariant formula for Z_{kl} then reads:

$$Z_{kl} = \frac{c}{\pi i}\oint\frac{p_{kl}'\sqrt{1-v'^2/c^2}}{S^2}\,dt'\,. \tag{24}$$

By taking the residue at the "retarded pole," we will get:

$$Z_{kl} = \left[\frac{p_{kl}'\sqrt{1-v'^2/c^2}}{(1-v_R'/c)R}\right]_{t_0=t-R/c}, \tag{24.a}$$

as in **§ 4**, Chap. VI. The method that we used at the time can serve as an illustration of the methodological utility of the theory of relativity.

We would now like to show the simplest way that the ordinary electromagnetic potentials $\mathfrak{A}$, φ can be calculated from Z_{kl} . That will be much simpler on the basis of the four-dimensional formula (24) than it is on the basis of the corresponding three-dimensional formula (24.a). It is generally most convenient to pass to the ordinary three-dimensional space in the *end result* and perform all of the intermediate calculations in four dimensions. From **§ 1**, Chap. VIII., we will have:

$$A_k = \sum_l\frac{\partial Z_{kl}}{\partial x_l} = \frac{c}{\pi i}\oint\sum_l p_{kl}'\sqrt{1-v'^2/c^2}\,\frac{\partial}{\partial x_l}\left(\frac{1}{S^2}\right)dt'\,,$$

or with the abbreviation:

$$p_{kl}^* = p_{kl}'\sqrt{1-v'^2/c^2}\,, \tag{25}$$

and when we recall the facts that $\dfrac{\partial S}{\partial x_l} = \dfrac{x_l - x_l'}{S}$ and $p_{kl} = -\,p_{lk}$, we will get:

$$A_k = \frac{2c}{\pi i}\oint \frac{\sum p_{lk}^*(x_l - x_l')}{S^4}\,dt' . \tag{25.a}$$

For $t' \to t_0'$, we further have $S^4 = a_2(t'-t_0')^2 + a_3(t'-t_0')^3 + \cdots$, with:

$$a_2 = \frac{1}{2}\left[\frac{d^2(S^4)}{dt'^2}\right]_0, \qquad a_3 = \frac{1}{6}\left[\frac{d^3(S^4)}{dt'^3}\right]_0, \qquad \text{etc.}$$

As a result:

$$\frac{1}{S^4} = \frac{1}{a_2(t'-t_0')^2}\left\{1 - \frac{a_3}{a_2}(t'-t_0') + \cdots\right\},$$

and from (25.a):

$$A_k = \frac{4c}{a_2}\frac{1}{2\pi i}\oint \frac{F_k(t')}{(t'-t_0')^2}\,dt' - \frac{4\,a_3\,c}{a_2^2}\frac{1}{2\pi i}\oint \frac{F_k(t')}{t'-t_0'}\,dt',$$

in which we have set:

$$F_k(t') = \sum_l p_{lk}^*(x_l - x_l'),$$

to abbreviate.

Since the function $F_k(t')$ remains finite for $t' = t_0'$, one will have (cf., Chap. VI, § **5**):

$$\frac{1}{2\pi i}\oint \frac{F_k(t')}{t'-t_0'}\,dt' = F_k(t_0') \qquad \text{and} \qquad \frac{1}{2\pi i}\oint \frac{F_k(t')}{(t'-t_0')^2}\,dt' = \left\{\frac{d}{dt'}F_k(t')\right\}_{t'=t_0'} .$$

Moreover, after an elementary calculation, that will imply that:

$$a_2 = 4R^2c^2(1 - v_R'/c)^2$$

and

$$a_3 = -\,4Rc^3(1 - v_R'/c)\left(1 - \frac{v'^2}{c^2} + \frac{\mathfrak{w}'\cdot\mathfrak{R}}{R}\right).$$

It will then follow that:

$$A_k = \frac{1}{(1 - v_R'/c)^2}\sum_{l=1}^{4}\left\{\frac{x_l - x_l'}{c\,R^2}\frac{dp_{lk}^*}{dt'} - \frac{p_{lk}^*}{c\,R^2}\frac{dx_l'}{dt'} + \frac{1 - v'^2/c^2 + w_R'}{1 - v_R'/c}\frac{p_{lk}^*(x_l - x_l')}{R^3}\right\}_{t'=t_0'} . \tag{26}$$

We can now split the four-vector $\mathfrak{A}$ into its spatial and temporal parts. In so doing, we remark that for $k = 1$, we will have:

$$\sum_l p_{lk}^*(x_l - x_l') = p_{21}\,R_2 + p_{31}\,R_3 + p_{41}\,R_4 = m_2\,R_3 - m_3\,R_2 - i\,p_1\,i\,c\,(t - t_0') = (\mathfrak{m}\times\mathfrak{R})_1 + p_1\,R .$$

Likewise:

$$\frac{1}{c}\sum_l p_{l1}^* \frac{dx_l'}{dt'} = \sqrt{1-v'^2/c^2}\left\{\left(\mathfrak{m}\times\frac{1}{c}\mathfrak{v}'\right)_1 + \mathfrak{p}_1\right\}$$

and

$$\sum \frac{dp_{lk}^*}{dt'}(x_l - x_l') = \left[\frac{d}{dt'}\left(\mathfrak{m}\sqrt{1-v'^2/c^2}\right)\times\mathfrak{R}\right]_1 + R\frac{d}{dt'}\left(p_1\sqrt{1-v'^2/c^2}\right).$$

Analogous formulas are true for $k = 2$ and $k = 3$. However, for $k = 4$, we will get:

$$\sum_l p_{l4}^*(x_l - x_l') = i\,(p_1 R_1 + p_2 R_2 + p_3 R_3) = i\,\mathfrak{p}\cdot\mathfrak{R}, \qquad \text{etc.}$$

From (26), the vector potential $\mathfrak{A}$ and the scalar potential $\varphi = A_4 / i$ can then be represented as follows:

$$\mathfrak{A} = \frac{1}{(1-v_R'/c)^2}\left\{\frac{(1-v'^2/c^2+w_R')}{\gamma(1-v_R'/c)}\frac{\mathfrak{m}'\times\mathfrak{R}}{R^3} - \frac{\left(\mathfrak{m}'\times\frac{1}{c}\mathfrak{v}'\right)+\mathfrak{p}'}{\gamma R^2} + \frac{1}{cR^2}\left(\frac{d}{dt'}\frac{\mathfrak{m}'}{\gamma}\right)\times\mathfrak{R} + \frac{1}{cR}\frac{d}{dt'}\frac{\mathfrak{p}'}{\gamma}\right\}, \quad (26.a)$$

$$\varphi = \frac{1}{(1-v_R'/c)^2}\left\{\frac{(1-v'^2/c^2+w_R')}{\gamma(1-v_R'/c)}\frac{\mathfrak{p}'\cdot\mathfrak{R}}{R^3} - \frac{\mathfrak{p}'\cdot\left(\frac{1}{c}\mathfrak{v}'\right)}{\gamma R^2} + \frac{1}{cR^2}\mathfrak{R}\cdot\frac{d}{dt'}\left(\frac{\mathfrak{p}'}{\gamma}\right)\right\}, \quad (26.b)$$

with the usual notation:

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}.$$

The translational velocity of the oscillator is assumed to be a given function of time. As far as the electromagnetic moments $\mathfrak{p}'$ and $\mathfrak{m}'$ are concerned, they can be considered to be immediately given functions of time only in the corresponding *rest system*. Those known "rest values" $\mathfrak{p}^0$ and $\mathfrak{m}^0$ are coupled with the values $\mathfrak{p}'$ and $\mathfrak{m}'$ that are referred to the chosen fixed inertial system by the formulas:

$$\left.\begin{aligned}\mathfrak{m}^0 &= (1-\gamma)(\mathfrak{m}'\cdot\mathfrak{v}_0')\,\mathfrak{v}_0' + \gamma\left(\mathfrak{m}' + \frac{1}{c}\mathfrak{v}'\times\mathfrak{p}'\right),\\ \mathfrak{p}^0 &= (1-\gamma)(\mathfrak{p}'\cdot\mathfrak{v}_0')\,\mathfrak{v}_0' + \gamma\left(\mathfrak{p}' - \frac{1}{c}\mathfrak{v}'\times\mathfrak{m}'\right).\end{aligned}\right\} \quad (27)$$

[Cf., the formulas (11), § **2**. The primed quantities there correspond to $\mathfrak{p}^0$ and $\mathfrak{m}^0$. In our present notation, the prime means that it is not the time *t*, but the effective time $t_0' = t - R_0 / c$ that should serve as the argument.] Obviously, we can also express $\mathfrak{p}'$ and $\mathfrak{m}'$ in terms of $\mathfrak{p}^0$ and $\mathfrak{m}^0$ by means of the reciprocal formulas.

For the "rotating" or (when expressed more carefully and more physically precisely) *magnetic* electron, the electric moment will be equal to zero in the rest system. In that case, we will then have:

$$\mathfrak{p}' = \frac{1}{c}\mathfrak{v}' \times \mathfrak{m}', \tag{27.a}$$

such that the second term in the brackets in (26.a) and (26.b) will vanish identically. The magnetic moment $\mathfrak{m}'$ is then expressed in terms of the corresponding rest moment by the formula:

$$\mathfrak{m}' = (1-\gamma)(\mathfrak{m}^0 \cdot \mathfrak{v}_0')\mathfrak{v}_0' + \gamma\,\mathfrak{m}^0.$$

If one assumes that the vector $\mathfrak{m}^0$ is always perpendicular to the direction of translation, i.e., the velocity vector $\mathfrak{v}$ [which corresponds to the vanishing of the internal torque, according to (53.a), Chap. VII], then one will get simply:

$$\mathfrak{m}' = \gamma\,\mathfrak{m}^0. \tag{28}$$

In that way formulas (26.a) and (26.b) will assume the following forms:

$$\mathfrak{A} = \frac{1}{(1-v_R'/c)^2}\left\{\frac{1-v'^2/c^2+w_R'}{1-v_R'/c}\,\frac{\mathfrak{m}^0\times\mathfrak{R}_0}{R^2} - \frac{\dot{\mathfrak{m}}^0\times\left(\mathfrak{R}_0-\frac{1}{c}\mathfrak{v}\right)+\frac{1}{c}\dot{\mathfrak{v}}\times\mathfrak{m}^0}{c\,R}\right\}, \tag{28.a}$$

$$\varphi = \frac{1}{(1-v_R'/c)^2}\left\{\frac{1-v'^2/c^2+w_R'}{1-v_R'/c}\,\frac{\left(\frac{1}{c}\mathfrak{v}\times\mathfrak{m}^0\right)\mathfrak{R}_0}{R^2} + \frac{\mathfrak{R}_0\left(\frac{1}{c}\mathfrak{v}\times\dot{\mathfrak{m}}^0+\frac{1}{c}\dot{\mathfrak{v}}\times\mathfrak{m}^0\right)}{c\,R}\right\}. \tag{28.b}$$

The dots over $\mathfrak{m}$ and $\mathfrak{v}$ mean the corresponding differential quotients with respect to time t' at the moment $t' = t_0'$. Obviously, one must combine those potentials with the ones that originate in the charge of the electron.

In order to calculate the electric and magnetic field strengths, we must revert to the original four-dimensional expressions since in so doing, we will need to consider the equation $t_0' = t -$

R / c, which determined the effective time, only in the final result, but not in the intermediate calculations.

We would like to explain that situation in the aforementioned example of the potentials that originate in the charge of the electron:

$$\varphi = \frac{e}{(1 - v'_R / c)\,R}, \qquad \mathfrak{A} = \frac{\varphi}{c}\mathfrak{v}'.$$

The determination of the field strengths comes about in the usual way that was described in **§ 2**, Chap. VI, i.e., by means of the formulas:

$$\mathfrak{H} = \text{rot}\,\mathfrak{A}, \qquad \mathfrak{E} = -\,\text{grad}\,\varphi - \frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t}.$$

We now replace those formulas with the four-dimensional ones $H_{kl} = \frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l}$ and employ the integral representation (23), Chap. VI, for A_k:

$$A_k = \frac{e}{\pi i}\oint \frac{dx'_k}{S^2} = \frac{e}{\pi i}\oint \frac{dx'}{dt'}\frac{1}{S^2}dt'.$$

Therefore:

$$H_{kl} = \frac{2e}{\pi i}\oint \left\{\frac{dx'_k}{dt'}(x_l - x'_l) - \frac{dx'_l}{dt'}(x_k - x'_k)\right\}\frac{dt'}{S^4}.$$

That expression can be brought into the form:

$$H_{kl} = \frac{e(1 - v'^2/c^2 + w'_R)}{(1 - v'_R/c)^3 R^3 c}\Phi_{kl}(t'_0) + \frac{e}{(1 - v'_R/c)^2 R^2 c^2}\left\{\frac{d}{dt'}\Phi_{kl}(t')\right\}_{t'=t'_0}$$

in the same way as (25.a), with:

$$\Phi_{kl}(t') \equiv \frac{dx'_k}{dt'}(x_l - x'_l) - \frac{dx'_l}{dt'}(x_k - x'_k),$$

which will immediately give the formulas for $\mathfrak{E}$ and $\mathfrak{H}$ that were derived before in **§ 2**, Chap. VI.

§ 6. – Deriving the basic electromagnetic equations from a variational principle.

In **§ 5**, Chap. IV, we proved the theorem that the differential equation $\nabla^2\varphi = 0$ for the scalar potential inside of a bounded region represents the condition for a *minimum* of the integral:

$$J = \int (\nabla \varphi)^2 \, dV \tag{29}$$

for given boundary values of the function φ on the bounding surface S. We can invert that theorem and consider the differential equation $\nabla^2 \varphi = 0$ to be a *consequence* of the variational equation:

$$\delta J = 0 \tag{29.a}$$

for given boundary values of φ. In it, the quantity δJ (the so-called *first variation* of the integral J) means the change in that integral that is *linear* in the infinitely-small change or *variation* $\delta\varphi$ of the function φ ($\mathfrak{r}$) in question. Equation (29.a) is considered to be only a first approximation for the corresponding change in J then. In **§ 5**, Chap. IV, we also considered the terms that are quadratic in $\delta\varphi$ and showed that the corresponding *second variation* of J (viz., $\delta^2 J$) remains essentially-positive when $\nabla^2 \varphi$ vanishes.

In general, i.e., for a differential expression of arbitrary form under the integral sign in (29), the formula (29.a) can be regarded as only a necessary, but not sufficient, condition for the minimum of J. It can just as well correspond to a maximum or even neither a maximum or a minimum, but a *stationary* value of integral J, i.e., a value that remains constant under an infinitesimal change in the function φ.

The differential equation $\nabla^2 \varphi = 0$ will be true when electricity is absent from the volume considered V. However, it is easy to see that the more general equation:

$$\nabla^2 \varphi = - 4\pi \rho \, ,$$

which is true for a *given, time-constant* distribution of electricity with finite volume density, is also completely equivalent to a variational equation of the form $\delta J = 0$, but in which the integral J is not given by (29), but by the formula:

$$J = \int \left\{ \frac{(\nabla \varphi)^2}{8\pi} - \varphi \rho \right\} dV$$

or

$$J = \int \left\{ \frac{\mathfrak{E}^2}{8\pi} - \varphi \rho \right\} dV \, . \tag{30}$$

$\mathfrak{E} = - \nabla \varphi$ means the electric field strength. The aforementioned boundary condition ($\delta\varphi = 0$) must then remain unchanged by that.

In fact, the first variation of (30) reads:

$$\delta J = \int \left\{ \frac{\mathfrak{E} \cdot \delta \mathfrak{E}}{4\pi} - \delta\varphi \cdot \rho \right\} dV \, .$$

Just as in **§ 6**, Chap. IV, we now have:

$$\mathfrak{E} \cdot \delta \mathfrak{E} = - \mathfrak{E} \cdot \nabla \, \delta\varphi = - \operatorname{div} (\delta\varphi \, \mathfrak{E}) + \delta\varphi \operatorname{div} \mathfrak{E} = - \operatorname{div} \delta\varphi \, \mathfrak{E} + 4\pi \rho \, \delta\varphi ,$$

and as a result:

$$\delta J = - \int \operatorname{div} (\delta\varphi \, \mathfrak{E}) \, dV = \oint \delta\varphi \, E_n \, dS = 0 .$$

Recall that the integral $\int \frac{E^2}{8\pi} dV$, which extends over *all* of space, is considered to be the total electric energy in the charges that create the field $\mathfrak{E}$. As is known, the same energy can also be represented in the form:

$$U = \tfrac{1}{2} \int \varphi \, \rho \, dV .$$

In our case, we then have:

$$J = - U .$$

In that way, we can replace the boundary condition $\delta\varphi = 0$ on S for an *infinitely-distant* surface with the usual condition for the scalar potential $\varphi = 0$.

In an entirely-analogous way, one can prove that the differential equation:

$$\nabla^2 \mathfrak{A} = - 4\pi \, \mathbf{j} ,$$

which determines the vector potential of a given *stationary* current distribution, is completely equivalent to the variational equation $\delta K = 0$, where:

$$K = \int \left\{ \frac{\mathfrak{H} \cdot \delta \mathfrak{H}}{4\pi} - \delta \mathfrak{A} \cdot \mathbf{j} \right\} dV ,$$

and note that:

$$\mathfrak{H} \cdot \delta \mathfrak{H} = \mathfrak{H} \cdot \operatorname{rot} \delta \mathfrak{A} = \operatorname{div} (\delta \mathfrak{A} \times \mathfrak{H}) + \delta \mathfrak{A} \cdot \operatorname{rot} \mathfrak{H} = \operatorname{div} (\delta \mathfrak{A} \times \mathfrak{H}) + 4\pi \, \mathbf{j} \cdot \delta \mathfrak{A} .$$

As a result:

$$\delta K = \int \operatorname{div} (\delta \mathfrak{A} \times \mathfrak{H}) \, dV = \oint (\delta \mathfrak{A} \times \mathfrak{H})_n \, dS = 0 .$$

That result, which can be true for only time-constant fields ([6]), can be immediately generalized to arbitrary fields that correspond to a time-varying distribution of the charge and current density. The general field equations read:

$$\Box^2 \mathfrak{A} = - 4\pi \, \mathbf{j} \qquad \text{and} \qquad {}^2\mathfrak{H} = \operatorname{rot} \mathfrak{A}$$

([6]) Which was first pointed out by *K. Schwarzschild*.

in four-dimensional notation. Let the variational equation that is equivalent to it be:

$$\delta S = 0 \,,$$

in which S means an initially-undetermined integral. In order to determine it, we have the following clues: *First of all*, in the special case of time-constant fields, the equation $\delta S = 0$ must reduce to the equations $\delta J = 0$ and $\delta K = 0$. *Secondly,* S must represent an invariant scalar quantity.

Now, we have seen from **§§ 1** and **2** that the quantities $\mathfrak{A}\,\mathbf{j}$ and $\varphi\,\rho$, on the one hand, H^2 and and E^2, on the other, can actually be combined into the invariant quantities:

$$\mathfrak{A}\,\mathfrak{j} - \varphi\,\rho = \boldsymbol{\mathfrak{A}}\,\mathbf{j}$$

and

$$H^2 - E^2 = \tfrac{1}{2}\,{}^2\mathfrak{H}^2 \,.$$

It will then follow that the desired integral S must include the function:

$$\mathfrak{A}\,\mathfrak{j} - \varphi\,\rho - \frac{1}{8\pi}(H^2 - E^2) = \boldsymbol{\mathfrak{A}}\,\mathbf{j} - \frac{1}{16\pi}\,|\,\mathrm{rot}\,\boldsymbol{\mathfrak{A}}\,|^2 \tag{31}$$

in its integrand.

As far as the domain of integration is concerned, it must be four-dimensional, due to the fact that the space and time coordinates have an equal status. In order to free ourselves from any sort of spacetime boundary conditions, we can extend that integration over the entire space and time manifold, i.e., over the entire four-dimensional "world." If we denote the volume element of "the world" by $d\,\Omega$ then we will have:

$$S = \int \left(\boldsymbol{\mathfrak{A}}\,\mathbf{j} - \frac{1}{16\pi}(\mathrm{rot}\,\boldsymbol{\mathfrak{A}})^2 \right) d\,\Omega \,, \tag{32}$$

or in coordinate notation:

$$S = \iiiint \left[\sum_k A_k\, j_k - \frac{1}{16\pi} \sum_k \sum_l \left(\frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k} \right)^2 \right] dx_1\, dx_2\, dx_3\, dx_4 \,. \tag{32.a}$$

We must still test the tacit assumption that we have made that the expression for the volume of a four-dimensional "world region" that was just cited:

$$\int d\,\Omega = \iiiint dx_1\, dx_2\, dx_3\, dx_4 = \iint i\,c\,dV\,dt \tag{33}$$

is actually invariant under the Lorentz transformation (otherwise the variational equation $\delta S = 0$ would make no sense). If we go from the coordinate system in question X to another system X'

that is in a state of uniform, rectilinear motion then from the well-known theorem of *Jacobi*, the integral (33) will go to the multiple integral:

$$\iiint\int D\,dx_1'\,dx_2'\,dx_3'\,dx_4',$$

in which D means the functional determinant with the elements $\partial x_k / \partial x_l' = \alpha_{kl}$. However, due to the orthogonality of the Lorentz transformation, that determinant will be equal to 1.

The quantity S corresponds to the *Lagrange* function (or more precisely, the *action function*) of ordinary mechanics, i.e., the difference between the kinetic (viz., magnetic) and potential (viz., potential) energy. We will use that fact later when we derive the equations of motion of an electron.

We would now like to calculate the variation δS and convince ourselves that the equation $\delta S = 0$ is actually equivalent to the original field equations. Obviously, in so doing, we must treat the components of the four-current **j** as given functions of the four coordinates that are not to be varied. We will then have $\delta(A_k j_k) = j_k\,\delta A_k$, and furthermore:

$$\delta\left(\frac{\partial A_k}{\partial x_l}-\frac{\partial A_l}{\partial x_k}\right)^2 = 2\left(\frac{\partial A_k}{\partial x_l}-\frac{\partial A_l}{\partial x_k}\right)\left(\frac{\partial\,\delta A_k}{\partial x_l}-\frac{\partial\,\delta A_l}{\partial x_k}\right) = 2H_{lk}\left(\frac{\partial\,\delta A_k}{\partial x_l}-\frac{\partial\,\delta A_l}{\partial x_k}\right).$$

We can convert the last expression as follows:

$$H_{lk}\left(\frac{\partial\,\delta A_k}{\partial x_l}-\frac{\partial\,\delta A_l}{\partial x_k}\right) = \frac{\partial}{\partial x_l}(H_{lk}\,\delta A_k)-\frac{\partial}{\partial x_k}(H_{lk}\,\delta A_l)-\delta A_k\frac{\partial H_{lk}}{\partial x_l}+\delta A_l\frac{\partial H_{lk}}{\partial x_k}.$$

When summing those expression over the two indices k and l, the latter can obviously be switched with each other:

$$\text{[e.g., one has } \sum_k\sum_l\frac{\partial}{\partial x_k}(H_{lk}\,\delta A_l) = \sum_k\sum_l\frac{\partial}{\partial x_l}(H_{lk}\,\delta A_k)\text{].}$$

If one recalls that $H_{kl} = -H_{lk}$ then one will have:

$$\begin{aligned}\delta\left[\sum_k\sum_l\left(\frac{\partial A_k}{\partial x_l}-\frac{\partial A_l}{\partial x_k}\right)^2\right] &= 4\sum_k\sum_l\frac{\partial}{\partial x_l}(H_{lk}\,\delta A_k)+4\sum_k\sum_l\delta A_k\frac{\partial H_{lk}}{\partial x_l}\\ &= 4\sum_l\frac{\partial}{\partial x_l}\left(\sum_k\delta A_k\,H_{lk}\right)+4\sum_k\delta A_k\sum_l\frac{\partial H_{lk}}{\partial x_l}.\end{aligned}$$

The sum $\sum_l \frac{\partial}{\partial x_l}\left(\sum_k \delta A_k\, H_{lk}\right)$ can be treated as a four-dimensional divergence of a four-vector with the components $\sum_k \delta A_k\, H_{lk}$ ($l = 1, 2, 3, 4$). It must vanish under the integration over the entire "world." We will then get:

$$\delta S = \iiint\int \sum_k \delta A_k \left(j_k - \frac{1}{4\pi}\sum_l \frac{\partial H_{kl}}{\partial x_l} \right) dx_1\, dx_2\, dx_3\, dx_4 \,.$$

In order for that integral to vanish for arbitrary (infinitely-small) values of δA_k , the equations:

$$\sum_l \frac{\partial H_{kl}}{\partial x_l} = -\sum_l \frac{\partial^2 A_k}{\partial x_l^2} + \frac{\partial}{\partial x_k}\sum_l \frac{\partial A_k}{\partial x_l} = 4\pi j_k$$

must be fulfilled. However, they are precisely equations (7) from Chap. VIII. It will follow those equations that:

$$4\pi \sum \frac{\partial j_k}{\partial x_k} = -\sum_l \frac{\partial^2}{\partial x_l^2}\sum_k \frac{\partial A_k}{\partial x_k} + \sum_k \frac{\partial^2}{\partial x_k^2}\sum_l \frac{\partial A_l}{\partial x_l} \equiv 0 \,,$$

i.e., the conservation law for electricity, as well as the relation:

$$\sum_{l=1}^{4} \frac{\partial A_l}{\partial x_l} = 0 \,,$$

and the usual field equation: $\Box^2 A_k = -4\pi j_k$.

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CHAPTER TEN

RELATIVISTIC MECHANICS

§ 1. – The elementary theory of translational motion.

When investigating the motion of a material particle in classical mechanics, one prefers to choose time to be the independent variable, i.e., to represent the spatial coordinates x_1, x_2, x_3 as functions of time t. In reality, due to its varying character, time is just as unsuited to the role of an independent variable as the coordinates themselves. In the "graphical" representation of events, i.e., spacetime points, by points in four-dimensional space, time, multiplied by ic, will simply become a fourth coordinate on a par with the other three. *Kinematics*, i.e., the study of motion in ordinary three-dimensional space, will then be reduced to a pure *geometry* of four-dimensional space. In that way, only the condition that was suggested in the last chapter (§ **5**) of the *time-like character* of all four-dimensional lines that represent the motion of a particle (electrons, oscillators) must be fulfilled. One cares to refer to such time-like four-dimensional lines as the *world-lines* of the corresponding particles. That is why it would seem convenient to define the position of the particle in question along its world-line, not by time, but by the *length s* of that world-line, as measured from a certain point. The element of length of the world-line ds and the corresponding smallest-possible time interval $d\tau$, as we have seen before, is determined by the formulas:

$$ds = ic\, d\tau = ic\, dt \sqrt{1-v^2/c^2}\ . \tag{1}$$

If the motion of the particle is known as a function of the associated time in any coordinate system then one calculates the invariant *proper time* $t = \int dt$ of that particle, and as a result, the arc-length $s = ic\tau$, according to the formula:

$$\tau = \int \sqrt{1-v^2/c^2}\, dt\ . \tag{1.a}$$

Dividing the four-vector $d\,\mathfrak{r}$ that determines the elementary spacetime displacement by its magnitude $ds = |\, d\,\mathfrak{r}\,|$ will yield a unit vector with the components dx_k / ds. From the standpoint of "world geometry," that vector defines the *tangent direction* to the world-line of the particle in question, and from the kinematical standpoint, it represents the direction and magnitude of its velocity. That is because one has:

$$\frac{dx_1}{ds} = \frac{(dx_1/dt)}{(ds/dt)} = \frac{v_1}{ic\sqrt{1-v^2/c^2}}\,, \qquad \frac{dx_2}{ds} = \frac{v_2}{ic\sqrt{1-v^2/c^2}}\,,$$

$$\frac{dx_3}{ds} = \frac{v_3}{ic\sqrt{1-v^2/c^2}}\,, \qquad \frac{dx_4}{ds} = \frac{1}{\sqrt{1-v^2/c^2}}\,.$$

It is often much more convenient to use the proper time as the independent variable, and not the arc-length.

The quantities:

$$\frac{dx_1}{d\tau} = \frac{v_1}{\sqrt{1-v^2/c^2}}, \quad \frac{dx_2}{d\tau} = \frac{v_2}{\sqrt{1-v^2/c^2}}, \quad \frac{dx_3}{d\tau} = \frac{v_3}{\sqrt{1-v^2/c^2}}, \quad \frac{dx_4}{d\tau} = \frac{ic}{\sqrt{1-v^2/c^2}}$$

define the components of a four-vector $\mathfrak{v}$ that we will refer to as the *four-velocity*. Its spatial and temporal projections are equal to $\dfrac{\mathfrak{v}}{\sqrt{1-v^2/c^2}}$ and $\dfrac{ic}{\sqrt{1-v^2/c^2}}$, resp. For $v = 0$, the first one vanishes, while the second one assumes the constant value ic: That means that a particle that is at rest in the chosen coordinate inertial system will displace in time, i.e., in the direction of the fourth axis, with a constant velocity of ic. If the ratio v / c is small compared to 1 then the spatial projection of $\mathfrak{v}$ will reduce to the ordinary velocity $\mathfrak{v}$. A uniform, rectilinear motion in three-dimensional space will be represented by a "world-line." In that way, the four-vector $\mathfrak{v}$ will remain "constant," i.e., independent of τ or s. Any deviation of the motion from rectilinearity or uniformity would correspond to a *curvature* of the world line. That curvature is determined by the change in velocity of the vector dx_k / ds (tangent direction) relative to s, i.e., by the second derivatives of the coordinates with respect to s. Those second derivatives d^2x_k / ds^2 define the components of the four-dimensional *curvature vector*, which determine the magnitude and direction of the *acceleration* of the particle from the kinematical standpoint. We define that four-acceleration $\mathfrak{w}$ to be the derivative of $\mathfrak{v}$ with respect to τ (or the second derivative of $\mathfrak{r}$ with respect to τ). As a result, its components are equal to:

$$\frac{d^2x_1}{d\tau^2} = \frac{1}{\sqrt{1-v^2/c^2}}\frac{d}{dt}\frac{v_1}{\sqrt{1-v^2/c^2}}, \qquad \frac{d^2x_2}{d\tau^2} = \frac{1}{\sqrt{1-v^2/c^2}}\frac{d}{dt}\frac{v_2}{\sqrt{1-v^2/c^2}},$$

$$\frac{d^2x_3}{d\tau^2} = \frac{1}{\sqrt{1-v^2/c^2}}\frac{d}{dt}\frac{v_3}{\sqrt{1-v^2/c^2}}, \qquad \frac{d^2x_4}{d\tau^2} = \frac{1}{\sqrt{1-v^2/c^2}}\frac{d}{dt}\frac{ic}{\sqrt{1-v^2/c^2}}.$$

The spatial projection of that four-vector can also differ from the ordinary acceleration $\mathfrak{w} = d\mathfrak{w}/dt$ when the ratio v / c is very small, in the event that the magnitude of $\mathfrak{w}$ is very large. Upon differentiating the equation $\sum\left(\dfrac{dx_k}{ds}\right)^2 = 1$ with respect to s, that will give:

$$\sum \frac{dx_k}{ds}\frac{d^2x_k}{ds^2} = 0 . \tag{2}$$

Geometrically, that means that the vectors $\frac{d\mathfrak{v}}{ds}$ and $\frac{d^2\mathfrak{v}}{ds^2}$ are perpendicular to each other. If one replaces s with τ in that then one will get the equations $\mathfrak{v}^2 = -\, c^2$, and:

$$v_1 \frac{d}{dt}\frac{v_1}{\sqrt{1-v^2/c^2}} + v_2 \frac{d}{dt}\frac{v_2}{\sqrt{1-v^2/c^2}} + v_3 \frac{d}{dt}\frac{v_3}{\sqrt{1-v^2/c^2}} = c^2 \frac{d}{dt}\frac{1}{\sqrt{1-v^2/c^2}} ,$$

or with the usual vector notations:

$$\mathfrak{v} \cdot \frac{d}{dt}\frac{\mathfrak{v}}{\sqrt{1-v^2/c^2}} = \frac{d}{dt}\frac{c^2}{\sqrt{1-v^2/c^2}} . \tag{2.a}$$

That equation then expresses the fact that the magnitude of the four-velocity will always remain equal to *ic*.

The results above define the mathematical foundation for the structure of the *mechanics* of material particles, i.e., for exhibiting the equation that will determine their translational motion under the action of given external forces. In that way, we will not need to speak of the internal forces that *Lorentz*'s theory operates with (not even when they are actually present), e.g., the forces of interaction between different electrons in a moving atom (in the event that the latter can be regarded as isolated particles).

As a starting point, we would like to take the classical *Newtonian* equation motion:

$$m_0 \frac{d\mathfrak{v}}{dt} = \mathfrak{f} ,$$

which is certainly true in the limiting case of very small speeds.

From the standpoint of the theory of relativity, one can consider the *Newtonian* law to be an approximate and incomplete form of a law of motion that represents a relation between two four-dimensional vectors. If one would not like to alter *Newton*'s law fundamentally, but replace it with a four-dimensional equation that reduces to $m_0 \frac{d\mathfrak{v}}{dt} = \mathfrak{f}$ in the limiting case $\mathfrak{v} \to 0$, then one must obviously do the following:

First of all, replace the three-dimensional acceleration $\frac{d\mathfrak{v}}{dt}$, with the four-dimensional one $\frac{d\mathfrak{v}}{d\tau}$, and *secondly*, replace the force $\mathfrak{f}$ with the four-vector of impulse and work $\mathfrak{F}$, in which those quantities must not be referred to the usual unit of time, but to the unit of *proper time*. In that way, we will get the following relativistic equation of motion:

$$m_0 \frac{d\mathfrak{v}}{d\tau} = \mathfrak{F} , \tag{3}$$

The spatial projection of the vector $\mathfrak{F}$ is equal to the impulse of the force $\mathfrak{f}$ per unit proper time, i.e.:

$$\mathfrak{f}\frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2/c^2}}\mathfrak{f}\,.$$

The corresponding projection of the vector $\dfrac{d\mathfrak{v}}{d\tau}$ is $\dfrac{dt}{d\tau}\dfrac{d}{dt}\dfrac{\mathfrak{v}}{\sqrt{1-v^2/c^2}}$. One will then have:

$$\frac{d}{dt}\frac{m_0\mathfrak{v}}{\sqrt{1-v^2/c^2}} = \mathfrak{f}\,. \tag{3.a}$$

The temporal projection of equation (3) reads:

$$m_0\frac{dt}{d\tau}\frac{d}{dt}\frac{ic}{\sqrt{1-v^2/c^2}} = \frac{dt}{d\tau}\frac{i}{c}l\,,$$

in which l means the work done by the force $\mathfrak{f}$ per unit of ordinary time, i.e.:

$$\frac{d}{dt}\frac{m_0\,c^2}{\sqrt{1-v^2/c^2}} = l\,. \tag{3.b}$$

One can easily convince oneself of the validity of the definition of l above by means of equation (2.a). Namely, if one multiplies the latter by m_0 then one will get:

$$l = \mathfrak{f}\cdot\mathfrak{v}$$

when one recalls (3.a). According to *Einstein*, when one applies the theory of relativity to *Newton*'s theory, equation (3.a) will agree completely with equation (42), Chap. VII, which we derived for an electron on the basis of *Lorentz*'s principle. We then see that *Einstein*'s equation is much more general, and must be true for not only a free electron, but for any material particle (atom, molecule, as well as celestial bodies), in the event that *Newton*'s law is confirmed the limiting case $v / c \ll 1$, and entirely independent of any hypothesis about the origin of mass or the force of inertia.

The three-dimensional vector:

$$\mathfrak{G} = \frac{m_0\mathfrak{v}}{\sqrt{1-v^2/c^2}} = m\,\mathfrak{v} \tag{4}$$

represents the mechanical momentum of the particle (which we have previously defined to be the electromagnetic momentum of the electron), while its mass is:

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}\,. \tag{4.a}$$

The quantity:

$$W = m\,c^2 \tag{4.b}$$

is ordinarily considered to be the *complete* internal energy of the particle. However, it is not entirely clear what that means because a particle at rest must have an energy of $m_0\,c^2$ according to that definition. Now, the mass of a particle (or body) is mainly composed of the masses of the electrons that define it. The mutual mass, which originates in its interactions and corresponds to the sum of its mutual (electric and magnetic) energies, has a relatively negligible value. We then meet up with the following question again: What should the term "internal" or "proper" energy mean for an electron at rest?

We would not like to go into a detailed discussion of that question here.

The momentum (4) and the energy (4.b) multiplied by i / c obviously define the spatial and temporal projections, resp., of the four-vector:

$$\mathfrak{G} = m_0\,\mathfrak{v} \qquad \left(G_k = m_0\,\frac{dx_k}{d\tau}\right), \tag{5}$$

which corresponds to the ordinary momentum (or "impulse") and is referred to as the impulse-energy vector. The equations of motion (3) can be written in the form:

$$\frac{d}{d\tau}\,\mathfrak{G} = \mathfrak{F} \tag{5.a}$$

by means of that four-vector.

In the treatment of the translational motion of an electron on the grounds of the *Lorentz* principle, along with the "inertial force" $-\dfrac{d}{d\tau}(m\,\mathfrak{v})$, we also found a second derivative of $\mathfrak{v}$ with respect to t that is proportional to "a force of friction" (viz., radiation damping) and referred to the fact that even higher derivatives of $\mathfrak{v}$ must be added in order to get the precise expression for the self-force (and as a result, in the exact equation of motion). The formal nature of the theory of relativity gives no reference point for assessing the question of whether the simple equations of motion (3) [or (3.a)] must actually be extended by such terms or not. The theory of relativity asserts only that all of those terms must be *four-vectors* (or their spatial projections). If we assume that, e.g., in connection with equation (24.b), Chap. VII, the equation of motion of the electron has the form:

$$m_0\,\frac{d\mathfrak{v}}{d\tau} - \frac{2e^2}{3c^3}\,\frac{d^2\mathfrak{v}}{d\tau^2} = \mathfrak{f} \tag{6}$$

for very small speeds, then we can conclude immediately from the theory of relativity that the exact (i.e., also valid for large speeds) equation of motion must have the following form:

$$m_0 \frac{d\mathfrak{v}}{d\tau} - \frac{2e^2}{3c^3}\frac{d^2\mathfrak{v}}{d\tau^2} = \mathfrak{F}\,, \tag{6.a}$$

or when one takes the spatial projection of that and multiplies by $\sqrt{1-v^2/c^2}$:

$$\frac{d}{dt}\frac{m_0\mathfrak{v}}{\sqrt{1-v^2/c^2}} - \frac{2e^2}{3c^3}\frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}}\frac{d}{d\tau}\frac{\mathfrak{v}}{\sqrt{1-v^2/c^2}}\right) = \mathfrak{f}\,. \tag{6.b}$$

However, that equation can be completely false, and there are actually many reasons for saying that the simple equation of motion (3.a) is valid *precisely* in the case of an isolated electron.

§ 2. – The variational theory of the translational motion of an electron in a given electromagnetic field.

We assume that the external fields $\mathfrak{E}$, $\mathfrak{H}$ in which the electron in question moves are known (in which $\mathfrak{E}$ and $\mathfrak{H}$ do not need to be constant in time.). The external force:

$$\mathfrak{f} = e\left(\mathfrak{E} + \frac{1}{c}\mathfrak{v}\times\mathfrak{H}\right) \tag{7}$$

cannot be treated as known in that way because it includes the velocity of the electron *explicitly*. If one multiplies (7) by $\dfrac{dt}{d\tau} = \dfrac{1}{\sqrt{1-v^2/c^2}}$ then that will give the spatial projection of the four-vector $\mathfrak{F}$. Its first component (in any coordinate system) reads:

$$F_1 = \frac{e}{c\sqrt{1-v^2/c^2}}(v_2\,H_3 - v_3\,H_2 + c\,E) = \frac{e}{c\sqrt{1-v^2/c^2}}(v_2\,H_{12} + v_3\,H_{13} + ic\,H_{14})\,,$$

i.e.:

$$R_1 = \frac{e}{c}\left(H_{12}\frac{dx_2}{d\tau} + H_{13}\frac{dx_3}{d\tau} + H_{14}\frac{dx_4}{d\tau}\right).$$

Obviously, analogous expressions are true for the other components. As a result, one will have:

$$F_k = \frac{e}{c}\sum_l H_{kl}\frac{dx_l}{d\tau}\,, \tag{7.a}$$

or in coordinate-free notation:

$$\mathfrak{F} = \frac{e}{c}\frac{d\mathfrak{r}}{d\tau}{}^{2}\mathfrak{H} .$$

We also point out the following expression:

$$F_k = \frac{e}{m_0\,c}\sum_l H_{kl}\,G_l\,, \tag{7.b}$$

which we will get from (7.a) when we replace dx_l / dt with G_l / m_0 using (5).

Formula (7.a) corresponds completely to formula (10.a), Chap. VIII for the components of the impulse-work four-vector per unit volume and time. It is easy to derive the last formula from the first when one represents the electron as an infinitely-small body, and not merely a point. Let the volume of electron be V and let the charge density be ρ. One can then set the current density equal to $(\rho / c)\,\mathfrak{v}$. The work done and the impulse (with the former multiplied by i / c) contained in the volume element dV of the electron during the time interval dt are expressed by the four-vector with the components:

$$\sum_l H_{kl}\;j_l\,dV\,dt \;=\; \sum_l H_{kl}\;\rho\,dV\,\frac{v_l}{c}\,dt$$

(in which $v_4 = ic$) ([1]). Integrating that expression over the volume of the electron will run into very great difficulties, in principle, due to the appearance of time, which does not need to have the same value for different elements of the electron. Those difficulties will go away in the limiting case considered of infinitely-small electrons, and we will get simply:

$$\frac{e}{c}\sum_l H_{kl}\;v_l\;dt$$

for the total impulse (work, resp.) during the time dt. Dividing by $d\tau = dt\sqrt{1-v^2/c^2}$ will ultimately give:

$$\frac{e}{c}\sum_l H_{kl}\,\frac{v_l}{\sqrt{1-v^2/c^2}} = \frac{e}{c}\sum_l H_{kl}\,\frac{dx_l}{d\tau}\;,$$

i.e., the expression (7.a).

We can then write out the equation of motion (3) in detail in the form:

$$\frac{d^2x_k}{d\tau^2} = \kappa\sum_l H_{kl}\,\frac{dx_l}{d\tau} \tag{8}$$

or

([1]) Note that the products $dV\,dt$ and $\rho\,dV$ are invariant quantities. The former means an element of the world-volume, while the latter means the element of electron charge de.

$$\frac{dG_k}{d\tau} = \kappa \sum_l H_{kl} G_l \,, \tag{8.a}$$

with the abbreviation:

$$\kappa = \frac{e}{m_0 \, c} \,. \tag{8.b}$$

Now, it is easy to show ([1]) that equation (8) is a necessary consequence of the *Schwarzschild* variational equation:

$$\delta S \equiv \delta \int \left[\sum_k A_k \, j_k - \frac{1}{16\pi} \sum_k \sum_l \left(\frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k} \right)^2 \right] d\Omega = 0 \,,$$

when one considers the four-potential (A_k) in the latter to be a given function of the coordinates (and time) that determines the external field and considers the *current density* (j_k) to be the *desired function* that corresponds to the motion of the electron. Therefore, $\delta A_k = 0$ in all of space (and for all time) such that the variational equation above will reduce to:

$$\delta S^* = 0 \,, \tag{9}$$

with

$$S^* = \delta \int \sum_k A_k \, j_k \, d\Omega \,,$$

or when one drops the irrelevant factor *ic* :

$$S^* = \delta \iint \sum_k A_k \, j_k \, dV \, dt \,. \tag{9.a}$$

We consider the electron to be (spatially) infinitely small. That is why we can treat the factor $A_k \, dt$ as constant when performing the volume integration. We already calculated the integral $\int j_k dV$ before; it is equal to: $\dfrac{e}{c} v_k = \dfrac{e}{c} \dfrac{dx_k}{dt}$.

We then get:

$$S^* = \frac{e}{c} \int \sum_k A_k \, dx_k = \frac{e}{c} \int \sum_k A_k \frac{dx_k}{d\tau} d\tau \,, \tag{10}$$

in which the proper time of the electron should serve as the independent variable. We can imagine that the integration extends over an arbitrary piece of the world-line of the electron. Therefore, that line must be chosen in such a way that the integral (10) will remain constant in the first approximation under infinitely-small changes to its form (that take the form of motions). In other words, the coordinates x_k of the electron (k = 1, 2, 3, 4) must be determined as functions of the

([1]) According to *M. Born*.

parameter τ in such a way that the first variation of (10) will vanish under an infinitely-small variation of those functions. However, we should observe that, *first of all*, the x_k are not completely independent of each other, but must fulfill the relation:

$$\sum_k \left(\frac{dx_k}{d\tau}\right)^2 = -c^2 \tag{10.a}$$

identically (for all τ), and *secondly*, that certain conditions must be posed for the variations δx_k at the limits of integration $\tau = \tau_1$ and $\tau = \tau_2$ since otherwise the entire problem would lose any well-defined meaning. We would initially like to leave those boundary conditions indeterminate and address the solution of the variational problem that was defined by (9), (10), and (10.a).

As was mentioned before, the components of the four-potential are regarded as known functions of the four coordinates x_k. Now, their values in (10) refer to those spacetime points that lie along the world-line of the electron. As a result, while varying that line, they must suffer a variation of $\delta A_k = \sum_l \frac{\partial A_k}{\partial x_l}\delta x_l$, just as when the electron is displaced along a certain world-line. In the latter case, we will have $\frac{dA_k}{d\tau} = \sum_l \frac{\partial A_k}{\partial x_l}\frac{dx_l}{d\tau}$.

The variation of (10) reads:

$$\delta S^* = \frac{e}{c}\int_{\tau_1}^{\tau_2}\sum_k\left(\delta A_k \cdot \frac{dx_k}{d\tau} + A_k\,\delta\frac{dx_k}{d\tau}\right)d\tau\,,$$

or due to the commutability of the operations δ and d:

$$\delta S^* = \frac{e}{c}\int_{\tau_1}^{\tau_2}\sum_k\left(\delta A_k \cdot \frac{dx_k}{d\tau} + A_k\,\frac{d\delta x_k}{d\tau}\right)d\tau\,.$$

Now, one has:

$$\sum_k \delta A_k \cdot \frac{dx_k}{d\tau} = \sum_k\sum_l \delta x_k \cdot \frac{\partial A_k}{\partial x_l}\frac{dx_k}{d\tau}$$

and

$$\sum_k A_k \frac{d\delta x_k}{d\tau} = \frac{d}{d\tau}\left(\sum_k A_k\,\delta x_k\right) - \sum_k\sum_l \delta x_k \frac{\partial A_k}{\partial x_l}\frac{dx_l}{d\tau}.$$

Upon switching the indices k and l in the double sum, one will get:

$$\sum_k A_k \frac{d\delta x_k}{d\tau} = \frac{d}{d\tau}\left(\sum_k A_k\,\delta x_k\right) - \sum_l\sum_k \delta x_l \frac{\partial A_l}{\partial x_k}\frac{dx_k}{d\tau}.$$

As a result, one will have:

$$\delta S^* = \left[\sum_k \frac{e}{c} A_k\, \delta x_k\right]_{\tau_1}^{\tau_2} + \frac{e}{c}\int_{\tau_1}^{\tau_2} \sum_l \sum_k \delta x_k \left(\frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k}\right)\frac{dx_k}{d\tau}\, d\tau\,,$$

or when one introduces the notations:

$$\left(\frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k}\right) = H_{lk}\,, \qquad \frac{e}{c}\sum_k H_{lk}\frac{dx_k}{d\tau} = F_l\,, \tag{11}$$

with no concern for the previous formulas, but merely for the sake of brevity, one will have:

$$\delta S^* = \int \sum_l F_l\, \delta x_k\, d\tau + \frac{e}{c}\left[\sum_k A_k\, \delta x_k\right]_{\tau_1}^{\tau_2}. \tag{11.a}$$

If the variations of the four coordinates δx_k are independent of each other then the condition for the vanishing of δS^* will read simply:

$$F_l = 0 \qquad \text{and} \qquad \delta x_k^{(1)} = \delta x_k^{(2)} = 0\,.$$

However, in reality, the x_k are coupled with each other by the relation (10.a). That situation can be considered by way of the known method of *Lagrange* multipliers. We next construct the variation of (10.a). In that way, we will get the equation:

$$\sum_l \frac{dx_l}{d\tau}\,\delta\frac{dx_l}{d\tau} \equiv \sum_l \frac{dx_l}{d\tau}\frac{d\delta x_l}{d\tau} = 0\,.$$

We multiply that equation by a factor μ that is initially completely undetermined and can be an arbitrary function of τ, and extend the integrand in the original expression for δS^* by:

$$\sum_l \mu\frac{dx_l}{d\tau}\frac{d\delta x_l}{d\tau} = \sum_l \frac{d}{d\tau}\left(\mu\frac{dx_l}{d\tau}\delta x_l\right) - \sum_l \delta x_l\,\frac{d}{d\tau}\left(\mu\frac{dx_l}{d\tau}\right).$$

Instead of (11.a), we will then get:

$$\delta S^* = \int \sum_l \delta x_k\left[F_l - \frac{d}{d\tau}\left(\mu\frac{dx_l}{d\tau}\right)\right] d\tau + \left[\sum_l\left(\frac{e}{c}A_l + \mu\frac{dx_l}{d\tau}\delta x_k\right)\right]_{\tau_1}^{\tau_2}. \tag{11.b}$$

One can always choose the boundary condition for the x_l, $dx_l / d\tau$, and dx_l (and in fact, in different ways) such the second terms in 4.b) vanishes. If one then sets:

$$\left[\sum_l \left(\frac{e}{c} A_l + \mu \frac{dx_l}{d\tau} \delta x_k\right)\right]_{\tau_1}^{\tau_2} = 0 \tag{12}$$

then the necessary and sufficient condition for the vanishing of δS^* will reduce to the four differential equations:

$$F_l - \frac{d}{d\tau}\left(\mu \frac{dx_l}{d\tau}\right) = 0 . \tag{12.a}$$

Indeed, one can use the arbitrariness of the function $\mu(\tau)$ in order to treat the four variations δx_l as independent arbitrary quantities, despite the relation (10.a) between them (that is exactly the essence of the *Lagrange* method). However, the function μ must ultimately be determined in such a way that the expressions for the derivatives $dx_l / d\tau$ that follow from the differential equations will actually satisfy the condition. To that end, one multiplies (12.a) by $dx_l / d\tau$ and sum over l. In that way, one will have:

$$\sum_l F_l \frac{dx_l}{d\tau} - \mu \sum_l \frac{dx_l}{d\tau}\frac{d^2 x_l}{d\tau^2} - \frac{d\mu}{d\tau}\sum_l \left(\frac{d\mu}{d\tau}\right)^2 = 0 . \tag{12.b}$$

The first sum must vanish *identically* from the definition of the quantities F_l. In fact, from (11), one will have $H_{lk} = - H_{kl}$, and:

$$\sum_l F_l \frac{dx_l}{d\tau} = \frac{e}{c}\sum_l \sum_k H_{kl} \frac{dx_k}{d\tau}\frac{dx_l}{d\tau} = \frac{e}{c}\sum\sum_{l<k} (H_{kl} + H_{lk}) \frac{dx_k}{d\tau}\frac{dx_l}{d\tau} \equiv 0 .$$

From (10.a), the third sum in (12.b) is equal to the constant quantity $-c^2$, and as a result, the second one will be equal to zero [cf., (2)]. It will then follow that:

$$\frac{d\mu}{d\tau} c^2 = 0 , \qquad \text{i.e.,} \qquad \mu = \text{const.}$$

The differential equations (12.a) are then completely identical to the equations of motion (8) that were derived above when $\mu = m_0$. At the same time, in that way, we will get the general expression for the electromagnetic four-force from the *Schwarzschild* variational principle. That is because the quantities F_l that are defined by (11) play the same role in the components of the four-force in (12.a) that we have previously described in an entirely-different way.

We then see that the two groups of the fundamental electrodynamical laws, one of which determines the electromagnetic of moving electrons, while the other defines the motion of an electron in a given electromagnetic field, can be combined into *a single equation*, namely, the equation of variation:

$$\delta S = 0 .$$

One can say that all of electrodynamics is included in that equation.

In that way, one must note the following: In the derivation of the equations of the electromagnetic field, we have treated the current distribution as continuous and found the differential equation $\sum_k \frac{\partial j_k}{\partial x_k} = 0$ as the law of conservation of electricity. In so doing, when deriving the equations of motion, we regarded the electron as a point-charge, and employed the equation $\sum_k \left(\frac{dx_k}{d\tau}\right)^2 = -c^2$ in place of the one above. It is easy to see that those two equations are completely equivalent (when the charge of the electron can be considered to be constant). We would not like to go into the details of the proof of that here, which offers only a purely-mathematical interest. Moreover, in the first case, we considered the *complete* field of the charges in question, while in the second case, we considered only the *external* field. In that sense, the two forms of the *Schwarzschild* variational principle are not completely equivalent from the physical standpoint. Under the transition from the original integral S to the integral (9.a), we have tacitly ignored the field that the electron itself creates since otherwise we would not be able to assume that the components of the four-potentials A_k are known quantities.

§ 3. – Three-dimensional form of the variational principle.

One can *formally* replace the variational equation $\delta S^* = 0$, together with the auxiliary condition:

$$\sum_k \left(\frac{dx_k}{d\tau}\right)^2 = -c^2 ,$$

with the variational equation:

$$\delta \int \sum_k \left\{ \tfrac{1}{2} m_0 \left(\frac{dx_k}{d\tau}\right)^2 + \frac{e}{c} A_k \frac{dx_k}{d\tau} \right\} d\tau = 0 \tag{13}$$

that is free of that auxiliary condition. In fact, after performing the variation and converting the corresponding expressions, the left-hand side of (13) will again reduce to (11.b), except that μ is replaced with m_0. However, in order to actually liberate oneself from the aforementioned auxiliary condition, one must split the four-dimensional world into space and time, and reintroduce the ordinary time t in place of the proper time as the independent variable, and correspondingly set $\delta t = 0$, *and not* $\delta\tau = 0$. The derivatives of the spatial coordinates with respect to time, like the variations δx_1, δx_3, δx_3, are obviously treated as quantities that are completely independent of each other, such that one does not need to bring any auxiliary conditions into the picture.

The transition from τ to t, to the extent that it concerns the integral (10), happens quite simply: Namely, we set:

$$S^* = \frac{e}{c}\int\sum_k A_k\, dx_k = \int\sum_k \frac{e}{c} A_k \frac{dx_k}{dt}\, dt\,,$$

i.e., in ordinary three-dimensional notation:

$$S^* = \int\left(\frac{e}{c}\mathfrak{v}\cdot\mathfrak{A} - e\,\varphi\right)dt\,. \tag{13.a}$$

It is easy to see that setting the variation of that function to zero will not lead to equations of motion for the electron, but to the equation $\mathfrak{f} = 0$ for the external force $\mathfrak{f} = e\,(\mathfrak{E} + \frac{1}{c}\,\mathfrak{v}\times\mathfrak{H})$ that acts on it. The aforementioned equation of motion reads:

$$\frac{d}{dt}\frac{m_0\,\mathfrak{v}}{\sqrt{1-v^2/c^2}} = e\,(\mathfrak{E} + \frac{1}{c}\,\mathfrak{v}\times\mathfrak{H})\,. \tag{13.b}$$

We would like to show directly that this equation will be implied when one starts from the integral (1):

$$V = \int\left(-m_0\,c^2 + \frac{e}{c}\sum_k A_k\frac{dx_k}{d\tau}\right)d\tau\,, \tag{14}$$

instead of the integral (10). As long as one treats τ as the independent variable, i.e., one sets $\delta\,d\tau = 0$, the variational equation $\delta V = 0$ will be completely equivalent to $\delta S^* = 0$. However, if one introduces the usual time as the argument in place of τ then one will have $\delta t = 0$ and $\delta\,dt = 0$, while $\delta\,d\tau = \delta\,dt\sqrt{1-v^2/c^2} \neq 0$, and the equation $\delta V = 0$ differs essentially from the equation $\delta S^* = 0$. Therefore, the integral (14) will assume the form:

$$V = \int\left(-m_0\,c^2\sqrt{1-v^2/c^2} + \frac{e}{c}\,\mathfrak{v}\mathfrak{A} - e\,\varphi\right)dt\,, \tag{14.a}$$

and differs from (13.a) by the term $-m_0\,c^2\sqrt{1-v^2/c^2}$, which implies precisely the left-hand side of the equation of motion (13.b).

The boundary conditions can be chosen in the simplest way here (due to the independence of the variations δx_1, δx_3, δx_3). We would like to assume that the initial and final points of the path

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(1) The additional term $-\,m_0\,c^2$ is twice as large as the corresponding additional term $\frac{1}{2}m_0\sum_k\left(\frac{dx_k}{d\tau}\right)^2$ in (13).

electron are fixed spatially and temporally. The variation of its radius vector $\delta\mathfrak{r}$ must then vanish at the limits of the integral (14.a) ($t = t_1$, $\mathfrak{r} = \mathfrak{r}_1$, and $t = t_2$, $\mathfrak{r} = \mathfrak{r}_2$). In order to calculate the variation of (14.a), we note the following formulas:

$$\delta\varphi = \delta\mathfrak{r}\cdot\nabla\varphi, \qquad \delta\mathfrak{A} = (\delta\mathfrak{r}\,\nabla)\,\mathfrak{A}, \qquad \frac{d}{dt}\mathfrak{A} = \left(\frac{d\mathfrak{r}}{dt}\cdot\nabla\right)\mathfrak{A} = (\mathfrak{v}\,\nabla)\,\mathfrak{A}.$$

Moreover:

$$\delta(\mathfrak{v}\,\mathfrak{A}) = \delta\frac{d\mathfrak{r}}{dt}\cdot\mathfrak{A} + \mathfrak{v}\cdot(\delta\mathfrak{r}\,\nabla)\,\mathfrak{A} = \left(\frac{d}{dt}\delta\mathfrak{r}\right)\cdot\mathfrak{A} + \mathfrak{v}\cdot(\delta\mathfrak{r}\,\nabla)\,\mathfrak{A}$$

$$= \frac{d}{dt}\left(\delta\mathfrak{r}\,\mathfrak{A}\right) - \delta\mathfrak{r}\cdot\frac{d}{dt}\mathfrak{A} + \delta\mathfrak{r}\cdot\nabla\,(\mathfrak{A}\,\mathfrak{v}),$$

and ultimately:

$$-m_0\,\delta\sqrt{1-v^2/c^2} = \frac{m_0\,\mathfrak{v}\,\delta\mathfrak{v}}{\sqrt{1-v^2/c^2}} = m\,\mathfrak{v}\cdot\frac{d\,\delta\mathfrak{v}}{dt} = \frac{d}{dt}(\delta\mathfrak{r}\cdot m\mathfrak{v}) - \delta\mathfrak{r}\cdot\frac{d}{dt}(m\mathfrak{v}).$$

We will then have:

$$\delta V = \left[\delta\mathfrak{r}\cdot\left(m\,\mathfrak{v} + \frac{e}{c}\mathfrak{A}\right)\right]_1^2 + \int\delta\mathfrak{r}\cdot\left\{-\frac{d}{dt}\left(m\,\mathfrak{v} + \frac{e}{c}\mathfrak{A}\right) + \nabla\left(\frac{e}{c}\mathfrak{v}\cdot\mathfrak{A} - e\,\varphi\right)\right\} = 0\,.$$

Due to the arbitrariness in $\delta\mathfrak{r}$ between the limits of integration and its vanishing at those limits, we will get the following differential equation as the necessary and sufficient condition for the vanishing of δV:

$$\frac{d}{dt}\left(m\,\mathfrak{v} + \frac{e}{c}\mathfrak{A}\right) = \operatorname{grad}\left(\frac{e}{c}\mathfrak{v}\cdot\mathfrak{A} - e\,\varphi\right). \tag{15}$$

We consider the relations – grad $\varphi = \mathfrak{E}$ and rot $\mathfrak{A} = \mathfrak{H}$ then from the known formula:

$$\operatorname{grad}(\mathfrak{v}\cdot\mathfrak{A}) = (\mathfrak{v}\,\operatorname{grad})\,\mathfrak{A} + \mathfrak{v}\times\operatorname{rot}\mathfrak{A},$$

that equation can be brought into the form (13.b).

The vector:

$$\mathfrak{G}^* = m\,\mathfrak{v} + \frac{e}{c}\mathfrak{A} \tag{15.a}$$

plays the role of total momentum in equation (15). One can define it to be the sum of the ordinary mechanical momentum and the “mutual” momentum relative to the electrons that create the vector potential $\mathfrak{A}$. One can correspondingly interpret the right-hand side as the gradient of a mutual potential energy that equals $e\,\varphi - (e / c)\,\mathfrak{v}\,\mathfrak{A}$. Upon multiplying equation (15) by $dt / d\tau$, that will give:

$$\frac{d}{d\tau}\mathfrak{G}^* = \text{grad} \sum \frac{e}{c} A_k \frac{dx_k}{d\tau} = \text{grad}\ \kappa\,\mathfrak{A}\,\mathfrak{G}.$$

That is the spatial projection of a four-dimensional vector equation:

$$\frac{d}{d\tau}\mathfrak{G}^* = \kappa\,\square\,\mathfrak{A}\,\mathfrak{G}\ ,$$

for which the temporal projection of four-vector $\mathfrak{G}^*$ will be equal to:

$$G_4^* = i\,c\,(m + \frac{e}{c^2}\,\varphi) = i\,c\,m^*\,. \tag{15.b}$$

The quantity $m^* = m + (e/c^2)\varphi$ plays the role of the total mass, i.e., the sum of $m = m_0 / \sqrt{1 - v^2/c^2}$ (proper mass of the electron) and a "mutual" mass $(e/c^2)\varphi$.

When one substitutes the difference $\mathfrak{G}^* - (e / \tau)\,\mathfrak{A}$ for $\mathfrak{G}$, the foregoing equation will assume the following form:

$$\frac{d}{d\tau}\mathfrak{G}^* = \kappa\,\square\,(\mathfrak{G}^*\mathfrak{A} - \frac{e}{c}\mathfrak{A}^2)\ . \tag{16}$$

One can easily derive that four-dimensional form of the equations of motion from the original one (8) when one sets:

$$H_{kl} = \frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l}$$

in it. Namely, one will get:

$$\frac{d^2 x_k}{d\tau^2} = \kappa \sum_l \left(\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l}\right)\frac{dx_l}{d\tau} = \kappa\,\frac{\partial}{\partial x_k}\left(\sum_l A_k \frac{dx_l}{d\tau}\right) - \kappa\,\frac{dA_k}{d\tau}\,,$$

i.e.:

$$\frac{d}{d\tau}\left(m_0 \frac{dx_k}{d\tau} + \frac{e}{c} A_k\right) = \kappa\,\frac{\partial}{\partial x_k}\left(\sum_l A_l\, m_0 \frac{dx_l}{d\tau}\right). \tag{16.a}$$

However, that is nothing but equation (16).

§ 4. – The action function and the *Hamilton-Jacobi* differential equation.

The variational equation $\delta V = 0$ represents the refined form of *Hamilton*'s principle of classical mechanics for the case of a single electron. The integral V is the so-called *action function*, and the integrand:

$$L^* = -\, m_0\, c^2 \sqrt{1 - v^2 / c^2} + \frac{e}{c} \mathfrak{v} \mathfrak{A} - e \varphi \tag{17}$$

is the *Lagrangian* function. One ordinarily defines the latter to be the difference between the kinetic and potential energy. For the limiting case of very small speeds, one will get:

$$- m_0\, c^2 \sqrt{1 - v^2 / c^2} \approx - m_0\, c^2 + \tfrac{1}{2} m_0\, v^2 ,$$

and as a result:

$$L^* + m_0\, c^2 \equiv L \approx \tfrac{1}{2} m_0\, v^2 + \frac{e}{c} \mathfrak{v} \mathfrak{A} - e \varphi \,. \tag{17.a}$$

For $\mathfrak{A} = 0$, i.e., in the absence of an external magnetic field, L will correspond completely to the usual definition of the *Lagrangian* function. However, if a magnetic field is present then one must either add the quantity $(e / c)\, \mathfrak{v}\, \mathfrak{A}$ to the kinetic energy or subtract it from the potential energy. In other words, when one treats the (mutual) magnetic energy as a potential energy (as was done in, e.g., Part One), it will be equal to $-\,(e / c)\, \mathfrak{v}\, \mathfrak{A}$. By contrast, if one treats it as kinetic energy then one must set it equal to $+\,(e / c)\, \mathfrak{v}\, \mathfrak{A}$.

The derivatives of the *Lagrangian* function with respect to the components of the velocity:

$$p_k = \frac{\partial L}{\partial v_k} \qquad \left(\mathfrak{p} = \frac{\partial L}{\partial \mathfrak{v}} \right) \tag{18}$$

are called the *momenta*, as is known. The function that is expressed in terms of the coordinates and momenta:

$$H = \sum_{k=1}^{4} \frac{\partial L}{\partial v_k} v_k \; = \mathfrak{p} \cdot \mathfrak{v} - L \tag{18.a}$$

will be called the *Hamiltonian* function. For the *Lagrangian* function (17.a), it is equal to:

$$H = \tfrac{1}{2} m_0\, v^2 + e\, \varphi \,, \tag{19}$$

i.e., the sum of the kinetic and electric energy, in which the mutual magnetic energy makes no contribution to H. That corresponds to the fact that magnetic forces do no work. The function that appears on the right-hand side of (19) is still not a *Hamiltonian* function since the latter must not be expressed in terms of velocity, but in terms of the "momenta," as was just mentioned. In the case considered, those moments are equal to:

$$p_k = m_0\, v_k + \frac{e}{c}\, A_k\,. \tag{19.a}$$

In the first approximation, they will then coincide with the components of the vector $\mathfrak{G}^*$ that is defined by (15.a).

The expression for the *Hamiltonian* function (19) in the absence of a magnetic force will then be equal to:

$$H = \frac{1}{2m_0}(\mathfrak{p} - \frac{e}{c}\mathfrak{A})^2 + e\,\varphi\,. \tag{19.b}$$

However, that expression is valid for only small speeds of the electron ($v / c \ll 1$). In the general case, the momenta that are defined by formulas (17) and (18) are equal to precisely the components of the vector $\mathfrak{G}^*$, which we can write vectorially as follows:

$$\mathfrak{G}^* = \frac{\partial L^*}{\partial \mathfrak{v}}\,. \tag{20}$$

The *Hamiltonian* function is:

$$H^* = \mathfrak{G}^*\, \mathfrak{v} - L^*\,. \tag{20.a}$$

Its magnitude is then equal to:

$$H^* = \frac{m_0\, c^2}{\sqrt{1 - v^2 / c^2}} + e\,\varphi = m\,c^2 + e\,\varphi = -\, i\,c\,G_4^*\,. \tag{20.b}$$

It will differ from the fourth component of the four-vector $\mathfrak{G}^*$ by only the factor i / c then. Ordinarily, one subtracts the "rest energy" of the electron $m_0\, c^2$ from that and defines the magnitude of the *Hamiltonian* function to be the quantity:

$$H^* - m_0\, c^2 \equiv H = m_0\, c^2 \left(\frac{1}{\sqrt{1 - v^2 / c^2}} - 1\right) + e\,\varphi\,, \tag{20.c}$$

i.e., just as in classical mechanics, it is the sum of the kinetic energy $m_0\, c^2 \left(\frac{1}{\sqrt{1 - v^2 / c^2}} - 1\right)$ (in the limiting case of $v / c \ll 1$, it coincides with $\frac{1}{2} m_0\, v^2$) and the electrical energy $e\,\varphi$. In order to express H^* as a function of the momenta $p_k = G_k^*$, we must employ the relation (15.a), or when written out in detail:

$$\mathfrak{G}^* = \frac{m_0\,\mathfrak{v}}{\sqrt{1-v^2/c^2}} + \frac{e}{c}\mathfrak{A}\,. \tag{21}$$

It follows from this relation that:

$$\frac{m_0^2\,v^2}{1-v^2/c^2} = \left(\mathfrak{G}^* - \frac{e}{c}\mathfrak{A}\right)^2 = \frac{m_0^2\,c^2}{1-v^2/c^2} - m_0^2\,c^2 = m^2\,c^2 - m_0^2\,c^2\;,$$

or

$$m = \sqrt{m_0 + \frac{1}{c^2}\left(\mathfrak{G}^* - \frac{e}{c}\mathfrak{A}\right)^2}\;. \tag{21.a}$$

Therefore, from (20.b), one will have:

$$H^* = \sqrt{m_0\,c^2 + \left(\mathfrak{G}^* - \frac{e}{c}\mathfrak{A}\right)^2} + e\,\varphi. \tag{21.b}$$

When one replaces H^* with $-\,i\,c\,G_4^*$ and φ with $-\,i\,A_4$, that formula can be written in the following symmetric four-dimensional form:

$$\left(\mathfrak{G}^* - \frac{e}{c}\mathfrak{A}\right)^2 = \sum_{k=1}^{4}\left(G_k^* - \frac{e}{c}A_k\right)^2 = -\,m_0^2\,c^2\,, \tag{22}$$

which one will obtain immediately when one observes that, by definition:

$$G_k^* - \frac{e}{c}A_k = G_k = m_0\,\frac{dx_k}{d\tau}\,. \tag{22.a}$$

One can derive the motion of an electron in a given field that is characterized by the four-potential $\mathfrak{A}$ with no further analysis by immediately integrating its equations of motion in the form that was quoted above. However, it is possible and often much more convenient to solve that problem by a direct method that was introduced into classical mechanics by *Hamilton* and *Jacobi*. That method is described most simply as follows: One can also write the relation (20.a) between the *Lagrangian* and *Hamiltonian* function in the form:

$$L^* = \mathfrak{G}^*\cdot\mathfrak{v} + i\,c\,G_4^* = \sum_{k=1}^{4} G_k^*\,\frac{dx_k}{dt}\,.$$

By the definition of the *Lagrangian* function, we correspondingly get:

$$V = \int L^* dt = \int \sum_k G_k^* \, dx_k \,. \tag{23}$$

If the spatial coordinates of the electron are actually known as functions of time then V can be represented as a function of the initial and final values of time t. That representation is not invariant. However, one can replace it with an invariant representation (or imagine replacing it) since V is considered to be a scalar function of the initial and final values of the four-dimensional spacetime vectors $\mathfrak{r}$. In so doing, one must observe that the change in V under an infinitely-small change in the path of integration *will be equal to zero* as long as the endpoints are fixed. If one imagines that the initial value $\mathfrak{r} = \mathfrak{r}^0$ is fixed and the final value is variable then one can, as a result, treat the quantity V as a completely *determined* function of those variable final values. According to (23), the change in V that corresponds to an infinitely-small change in $\mathfrak{r}$ is then expressed by the formula:

$$dV = \sum_{k=1}^{4} G_k^* \, dx_k = \mathfrak{G}^* \cdot d\,\mathfrak{r} \,.$$

It obviously follows from this:

$$G_k^* = \frac{\partial V}{\partial x_k} \qquad (\mathfrak{G}^* = \Box\, V), \tag{23.a}$$

i.e., when one compares different motion with the same starting point, the vector $\mathfrak{G}^*$ (energy-momentum) can be treated as the gradient of a scalar function V.

That function will be called the *action function*. In will be, in fact, unknown, as long as the equations of motion of the electron have not been integrated. However, one does not need to integrate the equations of motion in order to determine it. To that end, one can use equation (22). That is because, from (23.a), the latter will assume the following form:

$$\left(\Box V - \frac{e}{c}\mathfrak{A}\right)^2 \equiv \sum_{k=1}^{4}\left(\frac{\partial V}{\partial x_k} - \frac{e}{c}A_k\right)^2 = -\, m_0^2\, c^2 \,, \tag{24}$$

i.e., it will be converted into a first-order partial differential equation of degree two for the action function V.

The complete integral of (24) includes four arbitrary constants that can be characterized by, e.g., the initial value of the spacetime vector $\mathfrak{r}^0$, which was left indeterminate. If one now considers V to be a function of $\mathfrak{r}$ *and* $\mathfrak{r}^0$, i.e., of x_k and x_k^0 ($k = 1, \ldots, 4$) then one will have:

$$V = \int_{\mathfrak{r}_0}^{\mathfrak{r}} \sum_{k=1}^{4} G_k^* \, dx_k = \int_{\mathfrak{r}_0}^{\mathfrak{r}} \mathfrak{G}^* \cdot d\mathfrak{r} \,. \tag{25}$$

It will then follow that along with equations (23.a), which are obtained by differentiating V with respect to the upper limit of that integral, the following equations must also exist:

$$\mathfrak{G}_k^{*0} = -\frac{\partial V}{\partial x_k^0} \qquad (\mathfrak{G}^{*0} = -\,\square^0\, V)\,, \tag{25.a}$$

which correspond to differentiating with respect to the lower limit [1].

If the initial values of the "momenta" G_k^*, i.e., the (total) momenta and energy, are given along with the initial coordinates x_k^0 then the world-line of the electron will be determined completely by equations (24.a). In fact, only three of the four equations (25.a) are mutually independent, but the fourth one must be a consequence of those three since the four quantities G_k^{*0} ($\partial V / \partial x_k^0$, resp.) must satisfy equation (22) [(24), resp.] *identically*, by their very nature. By solving equations (25.a), one will then get only *three* relations between the four coordinates x_1, x_2, x_3, x_4 that determine a *line* in four-dimensional space. The equation of that line includes seven arbitrary constants: Namely, the four initial coordinates x_k^0 and three of the "initial momenta" G_k^{*0}, or any three arbitrary constants by which the initial momenta can be expressed consistently with equation (22). Obviously, the initial momenta determine the *initial direction* of the world-line. By varying G_k^{*0} with fixed values of the initial coordinates, one will then get a bundle of world-lines that go through the same initial point (and which can obviously be extended in the opposite direction) Each of those lines is orthogonal to the *associated* surface V = const., but not to the other ones, which indeed correspond to the same starting point, but have different final points.

The surfaces that are orthogonal to *all* lines of the bundle considered obviously define the *envelope* of the surfaces V = const. One can get the equation of such an envelope as follows: One expresses the initial coordinates x_k^0 as functions of the x_k and the initial momenta G_k^{*0} by means of equations (25.a) and substitutes those expressions in the action function V. If the quantities G_k^{*0} are independent of each other then the desired equation of the envelope would be given by eliminating the G_k^{*0} from the equations V = const. and $\dfrac{\partial V}{\partial G_k^{*0}} = 0$. However, as a result of the relation:

$$\sum \left(G_k^{*0} - \frac{e}{c} A_k^0 \right)^2 = -\, m_0^2\, c^2\,, \tag{25.b}$$

the equations above must be replaced with:

---

[1] The quantities $\left(\dfrac{\partial V}{\partial x_k^0}\right)$ should not be confused with the quantities $\left(\dfrac{\partial V}{\partial x_k}\right)_0$, which are given for the values $x_k = x_k^0$ of the coordinates; namely, the latter are equal to + G_k^{*0}.

$$V = \text{const.}, \qquad \frac{\partial V}{\partial G_k^{*0}} - \lambda\left(G_k^{*0} - \frac{e}{c}A_k^0\right) = 0 \tag{26}$$

from the well-known method of multipliers, in which λ means an undetermined factor. If one now introduces a new function of $\mathfrak{r}$ and $\mathfrak{r}^0$ according to the formula:

$$V^* = V + \mathfrak{G}^{*0} \cdot \mathfrak{r}^0 \tag{26.a}$$

then the elimination equations (26) can be regarded as *determining equations* for the initial coordinates:

$$V^* = \text{const.}, \qquad x_k^0 = \frac{\partial V}{\partial G_k^{*0}} - \lambda\left(G_k^{*0} - \frac{e}{c}A_k^0\right).$$

One can also represent those equations in a purely-analytical way. Namely, one considers an infinitely-small variation of the action function that is defined by (25) that corresponds to an infinitesimal displacement of the initial and final points:

$$\delta V = \mathfrak{G}^* \cdot \delta\,\mathfrak{r} - \mathfrak{G}^{*0} \cdot \delta\,\mathfrak{r}^0,$$

and replaces $\mathfrak{G}^{*0} \cdot \delta\,\mathfrak{r}^0$ with $\delta(\mathfrak{G}^{*0} \cdot \mathfrak{r}^0) - \mathfrak{r}^0 \cdot \delta\,\mathfrak{G}^{*0}$. Hence:

$$\delta(V + \mathfrak{r}^0 \cdot \mathfrak{G}^{*0}) \equiv \delta V^* = \mathfrak{G}^*\ \delta\,\mathfrak{r} + \mathfrak{r}^0 \cdot \delta\,\mathfrak{G}^{*0}.$$

When one recalls (25.b), it will then follow that:

$$V^* = \text{const.}, \quad \mathfrak{G}_k^* = \frac{\partial V^*}{\partial x_k}, \quad x_k^0 = \frac{\partial V}{\partial G_k^{*0}} - \lambda\left(G_k^{*0} - \frac{e}{c}A_k^0\right). \tag{26.b}$$

The function V^*, like V, will be determined *directly* as the solution to the differential equation (24). The constants that appear in that solution must not be expressed in terms of the components of $\mathfrak{r}^0$ then, but in terms of the components of $\mathfrak{G}^{*0}$.

In order to explain that concept, we would like to examine the simplest case of force-free initial motion. Since $\mathfrak{A} = 0$, it will follow from (24):

$$\square\, V = \text{const.}, \quad \text{and} \qquad V = \mathfrak{B} \cdot \mathfrak{r} + C,$$

in which $\mathfrak{B}$ is a constant four-vector, and C is an ordinary constant. The constant C is determined by the condition that $V = 0$ for $\mathfrak{r} = \mathfrak{r}^0$. One will then have $V = \mathfrak{B} \cdot (\mathfrak{r} - \mathfrak{r}^0)$, and as a result, from (22) and (25), $\mathfrak{G}^* = \mathfrak{G}^{*(0)} = \mathfrak{B}$.

Those equations mean that the world-line is a line that goes through the point $\mathfrak{r}^0$ perpendicular to the "plane" V = const. However, the relation between the coordinates that we should obtain from equations (25) is not to be found in that way, in general. In order to obtain that relation, we can use equations (26.b).

The function V^* can be represented in the same form as V formally. However, the initial point will remain undetermined. One can get the initial momenta from the formula $\mathfrak{G}_k^{*0} = \left(\frac{\partial V^*}{\partial x_k}\right)_0 = B_k$. From (26.b), it will follow that:

$$x_k^0 = \frac{\partial V^*}{\partial B_k} - \lambda B_k = x_k - \lambda B_k ,$$

i.e.:

$$\frac{x_1 - x_1^0}{B_1} = \frac{x_2 - x_2^0}{B_2} = \frac{x_3 - x_3^0}{B_3} = \frac{x_4 - x_4^0}{B_4} = \lambda .$$

Upon replacing the expressions $x_k - x_k^0 = \lambda B_k$ in the function V^*, we will get: $V^* = \lambda + B_k x_k^0$, or $\lambda = V$.

We would like to consider the case that appears most frequently in which the external electromagnetic field is constant in time. In that way, we can set $\partial V / \partial x_4$ = const. = B_4 in equation (24), i.e., we can put the action function into the form:

$$V^* = B_4 x_4 + \Phi (x_1, x_2, x_3) , \tag{27}$$

in which the function Φ depends upon only the spatial coordinates. As long as we are not dealing with very rapid motions, we can determine Φ as the integral of the approximate three-dimensional equation:

$$\frac{1}{2m_0}\left(\nabla\Phi - \frac{e}{c}\mathfrak{A}\right)^2 + e\varphi = H = \text{const.}, \tag{27.a}$$

that is obtained from (19) when we recall that $\mathfrak{G}^* = \nabla\Phi$ and $B_4 x_4 = - H t$.

While integrating that equation, one can treat H as an entirely arbitrary parameter without considering its relationship to the quantities G_1^{*0}, G_2^{*0}, G_3^{*0}. The action function $V^* = - H t + \Phi$ can correspondingly be represented as a function of four *independent* parameters G_1^{*0}, G_2^{*0}, G_3^{*0}, H (so the latter appear in Φ explicitly). That is why it is unnecessary in this case to introduce the additional term that is multiplied by λ. The equations of motion (26.b) will then reduce to the known equations of classical mechanics:

$$G_k^{*0} = \frac{\partial \Phi}{\partial x_k}, \qquad x_k^0 = \frac{\partial \Phi}{\partial G_k^{*0}} \qquad (k = 1, 2, 3)$$

and

$$t_0 = t + \frac{\partial \Phi}{\partial H}.$$

§ 5. – The simplest examples of the motion of a free electron.

We would now like to consider some concrete examples of the motion of an electron as it is determined from the relativistic equations of motion. As a first example, we take the simplest case of the motion in an electromagnetic field that is constant *in space and time*.

Therefore, from (8), one can set:

$$\frac{d^2}{d\tau^2}\mathfrak{r} = \kappa^2\, \mathfrak{H} \cdot \frac{d\mathfrak{r}}{d\tau} = \frac{d}{d\tau}(\kappa^2\, \mathfrak{H} \cdot \mathfrak{r}),$$

from which it will follow that:

$$\frac{d}{d\tau}\mathfrak{r} = \mathfrak{v} = \kappa^2\, \mathfrak{H} \cdot \mathfrak{r} + \mathfrak{a}, \tag{28}$$

in which $\mathfrak{a}$ is a constant four-vector.

Upon multiplying (28) by $\mathfrak{r}$, since the product $({}^2\mathfrak{H} \cdot \mathfrak{r}) \cdot \mathfrak{r}$ will vanish as a result of the skew-symmetric character of the tensor ${}^2\mathfrak{H}$, we will get:

$$\mathfrak{r} \cdot \frac{d}{d\tau}\mathfrak{r} \equiv \frac{1}{2}\frac{d}{d\tau}\mathfrak{r}^2 = \mathfrak{a} \cdot \mathfrak{r}. \tag{29}$$

In the special case in which the product $\mathfrak{a} \cdot \mathfrak{r}$ is equal to zero, that will give $\mathfrak{r}^2$ = const., i.e.:

$$r^2 - c^2 t^2 = \text{const.} \tag{29}$$

However, in that way, $\mathfrak{r}$ and t must satisfy the condition:

$$\mathfrak{a} \cdot \mathfrak{r} = a_t \cdot t, \tag{29.a}$$

in which $\mathfrak{a}$ is the spatial projection of the four-vector $\mathfrak{a}$, and $(i / c)\, a_t$ is its temporal projection.

The various types of motion can be systematized most simply when one considers the invariants $H^2 - E^2$ and $\mathfrak{H} \cdot \mathfrak{E}$. Obviously, one must distinguish four cases, namely, $H^2 - E^2 > 0$ or $H^2 - E^2 < 0$, while $\mathfrak{H} \cdot \mathfrak{E} = 0$ or $\mathfrak{H} \cdot \mathfrak{E} \neq 0$. (We can overlook the case $H = E$ for a constant field.)

First case: $H^2 - E^2 > 0$, $\mathfrak{H} \cdot \mathfrak{E} = 0$. – One can imagine that the magnetic field is eliminated by a suitably-chosen Lorentz transformation. From § **2**, Chap. IX, the new reference system S' must move with a velocity that is determined from the formula $\mathfrak{H} = (1 / c)\, \mathfrak{v} \times \mathfrak{E}$. Moreover, since $\mathfrak{H}$ is perpendicular to $\mathfrak{E}$, that additional velocity must lie in the place that is perpendicular to $\mathfrak{E}$ and $\mathfrak{H}$ and have a magnitude of $c\,H / E$. It then remains for one to consider the motion of the electron in a constant electric field $\mathfrak{E}'$.

For the sake of simplicity, we would like to drop the prime, which refers to the "moving" coordinate system S'. The spatial projection of equation (28) reduces to:

$$m\,\mathfrak{v} = e\,\mathfrak{E}\,t + \mathfrak{a} \tag{30}$$

for $H = 0$. If $\mathfrak{a} = 0$ then from (29), we will have $\mathfrak{r} \cdot d\,\mathfrak{r} / dt - c^2 t = 0$, i.e., $\mathfrak{r} \cdot \mathfrak{v} = c^2 t$, and as a result, since $m\,\mathfrak{r} \cdot \mathfrak{v} = e\,\mathfrak{E}\,\mathfrak{r}\,t$:

$$m c^2 = \frac{m_0\,c^2}{\sqrt{1 - v^2 / c^2}} = e\,\mathfrak{E} \cdot \mathfrak{r}\,.$$

That equation is a special case of the energy equation $m c^2 + e\,\varphi = \text{const.}$

If one sets $r = x_1 - x_1^0$, $x_2 = 0$, $x_3 = 0$, then from (29), one will get the equation:

$$(x_1 - x_1^0)^2 - c^2 t^2 = b^2\,,$$

in which b^2 is an essentially-positive constant (since otherwise r would be imaginary for sufficiently-small t). The motion that is represented by the equation above is called *hyperbolic* since it can be represented graphically by a branch of a hyperbola. It corresponds to the uniformly-accelerated motion of ordinary mechanics, e.g., the motion of a stone that is thrown upwards (viz., the negative X_1-direction) under the action of gravity. The difference originates in the dependency of mass on velocity. When t increases from $-\infty$ to $t = 0$, x_1 will decrease from $+\infty$ to $x_1^0 + b$, but it will then increase to $+\infty$ again. As the electron goes to infinity, its velocity will asymptotically tend to the limiting value c. In the vicinity of the inflection point ($x = x_1^0 + b$), its motion will approach the ordinary uniform rectilinear motion. In fact, for $ct \ll b$, one will have:

$$x_1 - x_1^0 = \sqrt{b^2 + c^2 t^2} \approx b\left[1 + \frac{1}{2}\left(\frac{c\,t}{b}\right)^2\right] = b + \tfrac{1}{2} w t^2\,,$$

in which $w = c^2 / b$ means the acceleration. Obviously, since $w = e\,E / m_0$, that will give the value of $b = m_0\,c^2 / (e\,E)$ to the parameter b.

When the vector $\mathfrak{a}$ is non-zero and has a different direction from $\mathfrak{E}$, one will get a motion that is somewhat more complicated, but we will not investigate that here.

Second case: $H^2 - E^2 > 0$, $\mathfrak{H} \cdot \mathfrak{E} = 0$. – Therefore, the electric field can be transformed away by an additive constant velocity with a magnitude of $c\,E / H$. Thus, only motion in a homogeneous magnetic field will remain. Equation (28), which reduces to:

$$m\,\mathfrak{v} = \frac{e}{c}\mathfrak{r} \times \mathfrak{H}$$

in this case, shows that (for $\mathfrak{a} = 0$) the velocity of the electron will always remain perpendicular to the radius vector and the magnetic field strength. The electron must then move in a circle with its center ($r = 0$) in a plane that is perpendicular to $\mathfrak{H}$. If one denotes the angular velocity by $\mathfrak{o}$ then one will have $\mathfrak{v} = \mathfrak{o} \times \mathfrak{r}$, i.e.:

$$\mathfrak{v} = -\frac{e}{mc}\mathfrak{H} = -\kappa\,\mathfrak{H}\ . \tag{31}$$

That velocity is twice as great as the mean precessional velocity that is required by the orbital motion of an electron around a fixed attracting center (*Larmor* precession, Chap. VII, **§ 9**). However, it is experimentally identical to the precessional velocity of the magnetic or rotational axis of a free electron.

Due to the constancy of the mass, if the vector $\mathfrak{a}$ is non-zero then one sets $\mathfrak{a} = m\,\mathfrak{v}^0$, which will then imply: $\mathfrak{v} = \mathfrak{v}^0 + \mathfrak{o} \times \mathfrak{r}$. The general type of motion of an electron in a constant magnetic field will then be a combination of the circular motion that was considered above with a uniform, rectilinear motion.

Third and fourth case: $\mathfrak{H} \cdot \mathfrak{E} \neq 0$. – The electric and magnetic field strengths will remain non-zero in all inertial systems. However, there will be a "canonical" reference system in which they are parallel to each other. When the electron moves in that distinguished direction, it will not "feel" the magnetic field at all. We will then get the hyperbolic motion that we examined before. As long as the mass of the electron remains almost constant, i.e., for low speeds, the electron will generally move along a helix, and indeed with a constant orbital velocity around the common $\mathfrak{E}$-$\mathfrak{H}$-direction and with uniformly-increasing velocity ($v / m_0\,\mathfrak{E}\,t$) in that direction. For high speeds, the motion will become rather complicated.

We would not like to analyze that motion here and will be content to refer to the general solution of the equations of motion (28). Since that equation is linear when we use t as its argument, we can easily determine $\mathfrak{r}$ as a function of the proper time τ. When written in coordinate form, (28) will produce the system of equations:

$$\frac{dx_k}{d\tau} = \kappa \sum_n H_{kn}\, x_n + a_k \,, \tag{32}$$

which can be solved by the Ansatz:

$$x_k = \sum_{l=1}^{4} \xi_{kl}\, e^{\alpha_l \tau} + a_k \tau \,, \tag{32.a}$$

as is known. The functions $\xi_{kl}\, e^{\alpha_l \tau}$ in that are different solutions of the homogeneous system of equations that are obtained from (32) when $\mathfrak{a} = 0$. The coefficients a are the roots of the characteristic equation:

$$\begin{vmatrix} H_{11}-\alpha & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22}-\alpha & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33}-\alpha & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44}-\alpha \end{vmatrix} = 0 \,,$$

i.e.:

$$\begin{vmatrix} -\alpha & H_3 & -H_2 & -i\,E_1 \\ -H_3 & -\alpha & H_1 & -i\,E_2 \\ +H_2 & -H_1 & -\alpha & -i\,E_3 \\ i\,E_{41} & i\,E_2 & i\,E_3 & -\alpha \end{vmatrix} = 0 \,,$$

or, after calculating the determinant:

$$\alpha^4 + (\mathfrak{E}^2 - \mathfrak{H}^2)\,\alpha^2 - (\mathfrak{E}\mathfrak{H})^2 = 0 \,. \tag{32.b}$$

The roots of that equation are then determined in a simple way in terms of the two invariants $\mathfrak{E}^2 - \mathfrak{H}^2$ and $\mathfrak{E} \cdot \mathfrak{H}$. The real roots correspond to “hyperbolic” asymptotic motions that are required by the electric field, while the imaginary roots correspond to periodic orbital motions that originate in the magnetic force.

Each root α_l belongs to a system of solutions that is given by the formulas $x_k = \xi_{kl}\, e^{\alpha_l \tau}$. The coefficients ξ_{kl}, can be determined up to an arbitrary factor from the equations:

$$\alpha_l\, \xi_{kl} = \kappa \sum_{n=1}^{4} H_{kn}\, \xi_{nl} \,.$$

If one would ultimately like to replace proper time with ordinary time as an argument then one would have to solve a transcendental equation [$x_4 = f(\tau)$], which is possible only approximately for very small or very large velocities.

As a second example, we consider the motion of a free electron in the electromagnetic field of a *plane light wave*, i.e., in the wave zone of an arbitrary electric system. In that case, the two invariants $H^2 - E^2$ and $\mathfrak{E} \cdot \mathfrak{H}$ are equal to zero, which is why it seems especially simple from the standpoint of the theory of relativity. We would not like to make any restricting assumptions about the type of oscillations, and we will characterize them by the phase factor:

$$\Phi(t') = \Phi\left(t - \frac{1}{c}\mathfrak{n}\mathfrak{r}\right). \tag{33}$$

The $\mathfrak{n}$ in that means the wave normal, and t' means the "phase time," i.e., the quantity that determines the phase of the oscillation of the waves at the location in question. For the special case of harmonic oscillations, that phase factor will assume the form:

$$e^{2\pi i\nu\,(t - \mathfrak{n}\mathfrak{r}/c)},$$

in which the exponent $\nu(t - \mathfrak{n}\,\mathfrak{r}/c)$ can also be represented in the form $-\mathfrak{k}\cdot\mathfrak{r} = -\sum_{l=1}^{4} k_l\,x_l$. $\mathfrak{k}$ means the phase vector that was considered in **§ 4**, Chap. IX here.

The integration of the equations of motion is quite easy to perform. Namely, due to the facts that:

$$\mathfrak{H} = \mathfrak{n} \times \mathfrak{E} \qquad \text{and} \qquad \mathfrak{E}\cdot\mathfrak{n} = 0\,,$$

we will have:

$$\frac{d}{dt} m\mathfrak{v} = e\,\mathfrak{E} + \frac{e}{c}\mathfrak{v}\times\mathfrak{H} = e\left(1 - \frac{v_n}{c}\right)\mathfrak{E} + \frac{e}{c}\mathfrak{n}\,(\mathfrak{E}\cdot\mathfrak{v})\,. \tag{33.a}$$

Moreover, upon inner multiplying by the constant unit vector $\mathfrak{n}$, we will get:

$$\frac{d}{dt}(\mathfrak{n}\cdot m\,\mathfrak{v}) = \frac{d}{dt}(m\,v_n) = \frac{e}{c}\mathfrak{E}\cdot\mathfrak{v}\,.$$

The quantity $e\,\mathfrak{E}\cdot\mathfrak{v}$ is the work done by the electric (or electromagnetic) force per unit time. One then has $e\,\mathfrak{E}\cdot\mathfrak{v} = \frac{d}{dt}(c^2 m)$, from the general relation between energy and mass. As a result, we have:

$$m\,v_n = c\,m + \text{const.}$$

If we measure the time from a moment $t = 0$ when the electron was at rest then we will have:

$$m\,v_n = c\,(m - m_0) = \frac{T}{c}\,, \tag{34}$$

in which $T = c^2 (m - m_0) = m_0 c^2 \left(\dfrac{1}{\sqrt{1-v^2/c^2}} - 1 \right)$ is the kinetic energy of the electron.

The projection of the mechanical momentum of the electron $\mathfrak{G} = m\,\mathfrak{v}$ onto the wave normal then gives an essentially-positive quantity that is proportional to the kinetic energy (that is communicated by the waves). One can then say that the electron experiences a type of *light pressure*, and as a result, it will have a velocity that is parallel to the light ray and has a magnitude of:

$$v_n = \frac{c(m-m_0)}{m} = c(1-\sqrt{1-\beta^2}) \qquad \left(\beta = \frac{v}{c}\right).$$

The relation between the normal and total velocity can also be written as follows:

$$\sqrt{1-\beta^2} = 1 - \frac{v_n}{c}. \tag{34.a}$$

From the definition of proper time τ, we have $\sqrt{1-\beta^2} = d\tau / dt$. On the other hand, from the definition of "phase time" t':

$$\frac{dt'}{dt} = \frac{d}{dt}\left(t - \frac{1}{c}\mathfrak{r}\cdot\mathfrak{n} \right) = 1 - \frac{v_n}{c}.$$

It follows from (34.a) that t' and τ *are identical* in the case considered:

$$t' = t - \frac{1}{c}\mathfrak{r}\cdot\mathfrak{n} = \tau = \int_0^1 \sqrt{1-v^2/c^2}\, dt\,. \tag{35}$$

By means of those formulas, equation (33.a) can be put into the form:

$$\frac{d}{dt}\left(m_0 \frac{d\mathfrak{r}}{dt'} \right) = e\,\frac{dt'}{dt}\,\mathfrak{E} + \frac{e}{c}(\mathfrak{E}\ \ \mathfrak{v})\,\mathfrak{v}$$

or

$$\frac{d^2\mathfrak{r}}{dt'^2} = \frac{e}{m_0}\mathfrak{E} + \frac{e}{m_0 c}\left(\mathfrak{E}\cdot\frac{d\mathfrak{r}'}{dt'} \right)\mathfrak{n}\,. \tag{36}$$

Projecting that equation onto the wave plane and the wave normal will give:

$$\frac{d^2\mathfrak{r}'}{dt'^2} = \frac{e}{m_0}\mathfrak{E}(t')\,, \tag{36.a}$$

with the abbreviation $\mathfrak{r}' = \mathfrak{r} - r_n\,\mathfrak{n}$, and due to the facts that $\mathfrak{E}\cdot\mathfrak{n} = 0$ and $\mathfrak{E}\cdot\dfrac{d\mathfrak{r}}{dt'} = \mathfrak{E}\cdot\dfrac{d\mathfrak{r}'}{dt'}$:

$$\frac{d^2 r_n}{dt'^2} = \frac{e}{m_0\,c}\left(\mathfrak{E}\cdot\frac{d\mathfrak{r}}{dt'}\right) = \frac{e}{m_0\,c}\,\mathfrak{E}\cdot\frac{d\mathfrak{r}'}{dt'}\,. \tag{36.b}$$

If one sets:

$$\mathfrak{E}^{(-1)} = \int_0^{t'}\mathfrak{E}(t')\,dt' \qquad \text{and} \qquad \mathfrak{E}^{(-2)} = \int_0^{t'}\mathfrak{E}^{(-1)}(t')\,dt'\,, \tag{37}$$

to abbreviate, then from (36.a), one will have:

$$\frac{d\mathfrak{r}'}{dt'} = \frac{e}{m_0}\,\mathfrak{E}^{(-1)} \qquad \text{and} \qquad \mathfrak{r}' = \frac{e}{m_0}\,\mathfrak{E}^{(-1)}\,. \tag{38}$$

Moreover, we have:

$$\mathfrak{E}\cdot\frac{d\mathfrak{r}'}{dt'} = \frac{e}{m_0}\,\mathfrak{E}\cdot\mathfrak{E}^{(-1)} = \frac{e}{m_0}\,\mathfrak{E}^{(-1)}\cdot\frac{d\mathfrak{E}^{(-1)}}{dt'} = \frac{e}{2m_0}\frac{d}{dt'}(E^{(-1)})^2\,,$$

and as a result, from (36.b):

$$r_n = \frac{e^2}{2m_0\,c}\int_0^{t'} E^{(-1)2}\,dt'\,. \tag{38.a}$$

Finally, we will get the following expression for the radius vector of the electron:

$$\mathfrak{r} = \frac{e}{m_0}\left[\mathfrak{E}^{(-2)} + \frac{e}{2m_0\,c}\left(\int_0^{t'} E^{(-1)2}\,dt'\right)\mathfrak{n}\right]. \tag{38.b}$$

The first term represents the action of the lateral electric force, while the second one represents the combined effect of it with the magnetic force, which can be regarded as light pressure.

The dependency of the radius vector on proper or phase time is determined by the formula (38.b). In order to represent $\mathfrak{r}$ as a function of the ordinary time, one must eliminate from (38.b) and the equation:

$$t = t' + \frac{1}{c}\,\mathfrak{r}\,\mathfrak{n} = t' + \frac{e^2}{2m_0^2\,c^2}\int_0^{t'} E^{(-1)2}\,dt'\,.$$

§ 6. – Systems of electrons. Virial theorem and mass defect.

The general problem of the motion of a system of material particles already raises very significant complications in classical mechanics. One can say that only the problem of the motion of two mass-points can be solved simply and completely in classical mechanics. However, up to now, the famous "three-body problem" possesses no complete solution. In the relativistic theory of mechanics, the state of affairs is much narrower in scope since the "two-body" problem has not been solved completely either in that context. In addition to the complications that originate in the variability of mass, a much more essential complication enters in that is required by the *retarded character* of the electromagnetic action-at-a-distance. We must defer a thorough discussion of those questions to Vol. III of this book, since they define the essential content of the mechanics and electrodynamics of atoms. At this point, we will restrict ourselves to considering an important general theory that relates to the motion of an arbitrary number of electrons and the consideration of a special case of the two-body problem that is connected with the simple problems that were treated above.

We imagine a number of electrons (some of which are positive, some of which are negative) that define a *closed* system, i.e., they move under the action of their mutual forces (in the absence of any "external" forces) and indeed in such a way that they always remain a finite distance from each other. Obviously, the forces of attraction between opposing electrons must exceed the forces of repulsion that act between electrons of the same type.

We shall initially overlook the aforementioned complications in relativistic mechanics. The error that is made in so doing will not be too great as long as the speed of the electrons is small compared to the speed of light and their distances from each other are small compared to the wavelength of the light that is radiated. We will correspondingly consider only the electrostatic forces.

From **§ 2**, Chap. VII, the mutual potential energy of our system is expressed by:

$$U = \sum_{\alpha<\beta}\sum \frac{e_\alpha\, e_\beta}{R_{\alpha\beta}} .$$

In that way, the equations of motion of an electron (in the approximation that we are considering) will read:

$$m^{(\alpha)}\,\frac{d^2 x_k^{(\alpha)}}{dt^2} = -\,\frac{\partial U}{\partial x_k^{(\alpha)}} \qquad (k = 1, 2, 3). \quad \text{`} \tag{39}$$

For the sake of simplicity, we will drop the indices α and k, and the summation over them will be denoted by simply Σ. It follows from equation (39) upon multiplying by x (i.e., $x_k^{(\alpha)}$) that:

$$m\,x\,\frac{d^2 x}{dt^2} = -\,x\,\frac{\partial U}{\partial x} ,$$

or after converting the left-hand side and summing:

$$\frac{d}{dt}\left(\sum m\,x\,\frac{dx}{dt}\right)-\sum m\left(\frac{dx}{dt}\right)^2=-\sum x\frac{\partial U}{\partial x}\,. \tag{39.a}$$

The sum $\sum m\left(\frac{dx}{dt}\right)^2$ is obviously equal to twice the kinetic energy of the total system (2 T). Moreover, one can easily show that the sum $\sum\frac{\partial U}{\partial x}x$ is equal to simply the negative potential energy. In fact, if one combines the terms pair-wise that express the interaction of two electrons (α) and (β) and observes that:

$$R_{\alpha\beta}^2=\sum_{k=1}^{3}(x_k^\alpha-x_k^\beta)^2$$

then one will have:

$$e_\alpha\,e_\beta\sum_{k=1}^{3}\left(x_k^{(\alpha)}\frac{\partial}{\partial x_k^{(\alpha)}}\frac{1}{R_{\alpha\beta}}+x_k^{(\beta)}\frac{\partial}{\partial x_k^{(\beta)}}\frac{1}{R_{\alpha\beta}}\right)$$

$$=-\,e_\alpha\,e_\beta\frac{\sum[x_k^{(\alpha)}(x_k^{(\alpha)}-x_k^{(\beta)})+x_k^{(\beta)}(x_k^{(\beta)}-x_k^{(\alpha)})]}{R_{\alpha\beta}^3}=-\,e_\alpha\,e_\beta\frac{\sum(x_k^{(\alpha)}-x_k^{(\beta)})^2}{R_{\alpha\beta}^3}$$

$$=-\,\frac{e_\alpha\,e_\beta}{R_{\alpha\beta}}$$

It will then follow that:

$$-\sum\frac{\partial U}{\partial x}x=+U\,.$$

Equation (39.a) can then be written as follows:

$$2T=-\,U+\frac{d}{dt}Q\,,$$

in which Q means the sum $\sum m\,x\frac{dx}{dt}=\frac{d}{dt}\sum\frac{1}{2}m\,x^2$. From our assumption that the electrons will always remain at a finite distance from each other, the coordinates x can oscillate only within certain limits (but not vary monotonically). The mean value of $\frac{d}{dt}Q$ over a length of time that is sufficiently long in comparison to the period of such oscillations (but is still very small, such as when one treats the motion of electrons in an atom or molecule) must vanish as a result. The corresponding mean values of the kinetic and potential energy must then be coupled with each other by the formula [viz., the *virial theorem* ([1])]:

([1]) The virial theorem for the general case of arbitrary conservative forces was first derived by *Clausius*.

$$2\bar{T} = -\bar{U}\,. \tag{40}$$

In order for it to be possible for the system in question to actually exist, its potential energy must be *negative*, i.e., the effect of the forces of attraction must exceed the effect of the forces of repulsion. Under a contraction of the system, i.e., a reduction of all distances (while preserving the configuration), the potential energy must decrease (algebraically), and the kinetic energy must increase by half as much (as a result of the increase in the forces of interaction). From (40), the total energy of the system:

$$W = T + U = \bar{T} + \bar{U} \tag{40.a}$$

must be equal and opposite to the mean kinetic energy:

$$W = -\bar{T}\,. \tag{40.b}$$

We would like to illustrate that remarkable relationship by the following example: When a material body goes from the gaseous state to the fluid or the solid one (at the same temperature) then it will lose a certain amount of its internal energy (viz., latent heat). Based upon formula (40.b), we can now assert that the kinetic energy of the electrons in the atoms or molecules of that body must increase by the same amount. If the mechanical energy in an atom or a heavenly body is lost due to radiation [cf., Chap. VII, **§ 3**] then the kinetic energy of the electrons will increase through the same amount.

We saw that the mass of any material body is proportional to its energy according to the theory of relativity. We would like to defer the question of the intrinsic nature of the energy of an isolated electron at rest and simply define it to be the product $m_0 c^2$. However, we can assert that the mutual potential energy of all of those electrons U corresponds to an additional *mutual "electromagnetic" mass* U / c^2, and likewise the kinetic energy will correspond to an additional mass of magnitude T / c^2. The total mass of a material body, whether molecules or atoms, is then equal to the sum of the "rest masses" of all electrons that comprise that body, whether molecules or atoms, and it will increase by the amount $(T + U) / c^2 = W / c^2$, which represents the *mechanical equivalent of that body*. *That additional mass is always negative* since W is equal to $-\bar{T}$, from (40.b). That is why the opposite quantity:

$$\mu = \frac{\bar{T}}{c^2} = -\frac{W}{c^2} \tag{41}$$

will be referred to as the *mass defect* of the material system considered. That mass defect, multiplied by c^2, is the work that must be done in order to disassemble the system completely, i.e., to separate all electrons from each other and bring them to rest. That is because the mechanical energy of such a disassembled system will be equal to zero, so the increase in the energy under disassembly (= work expended) will be equal to $0 - W = -W = \bar{T}$.

Insofar as the electrons are indivisible and a conversion of their proper energy $m_0 c^2$ into another "useful" form of energy would then seem impossible, one must not consider the material bodies to be energy reservoirs (as is often done), but as structures devoid of energy that can only serve as sources of energy when they sink even deeper below the zero level.

The cited argument in regard to the mass defect can initially seem somewhat inconsistent since we have coupled the kinetic energy with an increase in mass, while we have left the derivation of equation (40) for that mass increase out of consideration.

However, one can easily show that the argument remains valid in the first approximation. Namely, if one replaces the classical equations of motion (39) with the relativistic ones (or rather semi-relativistic ones, since one would then avoid the problem of finding an exact expression for the force and would consider only the velocity dependency of the mass):

$$\frac{d}{dt}(m^\alpha \mathfrak{v}^\alpha) = -\nabla^{(\alpha)} U \qquad \left(m^\alpha = \frac{m_0^\alpha}{\sqrt{1 - v^{(\alpha)2} / c^2}} \right), \tag{42}$$

then that will give:

$$\sum \mathfrak{r} \cdot \frac{d}{dt}(m\mathfrak{v}) = \frac{d}{dt}\left(\sum m\,\mathfrak{r} \cdot \mathfrak{v}\right) - \sum m v^2 = -\sum \mathfrak{r} \cdot \nabla U = +\bar{U},$$

in the same way as before, or when one goes over to mean values:

$$\sum \overline{m v^2} = -\bar{U}. \tag{42.a}$$

In the case in question, the quantity $\frac{1}{2} m v^2$ is somewhat different from the kinetic energy of an electron, but only by a term of order of magnitude $\left(\frac{v}{c}\right)^2$, as is easy to see. When written in the form:

$$m v^2 = m_0 \frac{v^2}{\sqrt{1 - v^2 / c^2}} = m_0 c^2 \left(\frac{1}{\sqrt{1 - v^2 / c^2}} - \sqrt{1 - v^2 / c^2} \right), \tag{42.b}$$

the quantity will be defined to be the sum of the energy of the electron $\dfrac{m_0 c^2}{\sqrt{1 - v^2 / c^2}}$ and the corresponding part of its *Lagrangian* function [$L^* = -m_0 c^2 \sqrt{1 - v^2 / c^2}$, cf., (17), **§ 3**].

From (15), **§ 3**, the exact and complete equations of motion of the electrons in our system read:

$$\frac{d}{dt}\left(m^\alpha \mathfrak{v}^\alpha + \frac{e_\alpha}{c} \mathfrak{A}^\alpha \right) = -\nabla^{(\alpha)} \left(e_\alpha \varphi^\alpha - \frac{e_\alpha}{c} \mathfrak{A}^\alpha \mathfrak{v}^\alpha \right),$$

in which $\mathfrak{A}^\alpha$ and φ^α are the potentials that are created by all other electrons at the spacetime point in question ([1]). In the event that one *neglects the retardation of the electric action-at-a-distance*, one can express those potentials by the simple formulas:

$$\varphi^\alpha = \sum_\beta \frac{e_\beta}{R_{\alpha\beta}}, \qquad \mathfrak{A}^\alpha = \sum_\beta \frac{e_\beta\,\mathfrak{w}^\beta}{c\,R_{\alpha\beta}}, \tag{43}$$

and corresponding to the mutual electric or potential energy $U = \frac{1}{2}\sum_\alpha e_\alpha\,\varphi^\alpha = \sum\sum_{\alpha<\beta} \frac{e_\alpha\,e_\beta}{R_{\alpha\beta}}$, one introduces the mutual *magnetic* or kinetic energy (cf., Chap. VII, **§ 2**):

$$T^* = \frac{1}{2}\sum_\alpha \frac{e_\alpha}{c}\mathfrak{v}^\alpha\,\mathfrak{A}^\alpha . \tag{43.a}$$

Obviously, one will then have:

$$\nabla^{(\alpha)}\left(e_\alpha\,\varphi^\alpha - \frac{e_\alpha}{c}\mathfrak{v}^\alpha\cdot\mathfrak{A}^\alpha\right) = \nabla^{(\alpha)}(U - T^*)$$

and

$$\sum_\alpha \mathfrak{r}^\alpha\cdot\nabla^{(\alpha)}(U - T^*) = -\,(U - T^*)\,. \tag{43.b}$$

We can then write the equations of motion, which consider not only the electric, but also the magnetic, interactions of the electrons, and at the same time, the variability of their mass, in the following form, which is entirely analogous to (42) (when one drops the indices α):

$$\frac{d}{dt}\left(m\,\mathfrak{v} + \frac{e}{c}\mathfrak{A}\right) = -\,\nabla(U - T^*)\,. \tag{44}$$

From (43.b), that will imply:

$$\sum\mathfrak{r}\,\frac{d}{dt}\left(m\,\mathfrak{v} + \frac{e}{c}\mathfrak{A}\right) = \sum\frac{d}{dt}\left[\mathfrak{r}\left(m\,\mathfrak{v} + \frac{e}{c}\mathfrak{A}\right)\right] - \sum\left(m\,\mathfrak{v} + \frac{e}{c}\mathfrak{A}\right)\mathfrak{v} = +(U - T^*)\,,$$

as before, i.e., from (43.a):

$$\frac{d}{dt}\sum\left[\mathfrak{r}\cdot\left(m\,\mathfrak{v} + \frac{e}{c}\mathfrak{A}\right)\right] - \sum m\,v^2 = U + T^*,$$

or for the mean values:

$$\sum\overline{m\,v^2} = -\,\bar{U} - \bar{T}^*. \tag{44.a}$$

([1]) Note that the differentiation with respect to time must also involve the positions and velocities of those electrons, insofar as the vector potential $\mathfrak{A}^\alpha$ depends upon them.

In the limiting case of low speeds, one can once more set $\sum \overline{m v^2} = \bar{T}$ here, and as a result, when one defines the total mechanical energy W to be the sum $T + T^* + U$:

$$\bar{W} = - \bar{T} .$$

Recall that the *Hamiltonian* function H is independent of the magnetic forces and is expressed simply by the sum of T and U. If one introduces that *Hamiltonian* function in place of W then one will have:

$$\bar{T} + \bar{T}^* = - H . \tag{44.b}$$

That formula is the generalization of the virial theorem (40). In the absence of magnetic forces, the *Hamiltonian* function will remain constants, but not the quantity $W = H + T^*$.

§ 7. – The orbital motion of an electron.

We would now like to treat the two-body problem under the simplifying assumption that the mass of one particle is much larger than the mass of the second one. That corresponds to the actual relationships in material atoms, which are known to consist of a heavy, positively-charged nucleus and a number of light negative electrons that orbit around it. We shall consider the case of a single electron and ignore the associated motion of the nucleus. The actually reduction of the two-body problem to an ordinary one-body problem will then become especially simple since the nucleus at rest will create a constant electrostatic field. Let the charge of the nucleus be – $Z\,e$, so the scalar potential of the field will be:

$$\varphi = - \frac{Z\,e}{r}$$

(+ e = electron charge).

The equation of motion of the electron will then read:

$$\frac{d}{dt} m \mathfrak{v} = - \operatorname{grad} (e\,\varphi) = - \frac{Z\,e^2}{r^2} \mathfrak{r}_0 \qquad \left(\mathfrak{r}_0 = \frac{\mathfrak{r}}{r} \right). \tag{45}$$

It will then follow immediately that its angular momentum:

$$\mathfrak{J} = m\,\mathfrak{r} \times \mathfrak{v}$$

remains constant in time. (That theorem is known to be true for all central force motions; cf., Chap. VII, **§ 9**). If one introduces polar coordinates r, θ in the plane of motion (θ = angle between $\mathfrak{r}$ and a fixed OX-axis) then the magnitude of $\mathfrak{J}$ can be written in the form $mr^2 \frac{d\theta}{dt}$. One will then have:

$$mr^2 \frac{d\theta}{dt} = J = \text{const.} \tag{45.a}$$

Moreover, we have the energy theorem:

$$mc^2 - \frac{Ze^2}{r} = H = \text{const.}, \tag{45.b}$$

with

$$m = \frac{m_0}{\sqrt{1 - v^2 / c^2}}.$$

Upon multiplying (45) by the unit vector $\mathfrak{r}_0$ and recalling the facts that $\left| \frac{d\mathfrak{r}_0}{dt} \right| = \frac{d\theta}{dt}$, $\mathfrak{v} \cdot \mathfrak{r}_0 = \frac{dr}{dt}$, we will get $\mathfrak{v} \cdot \frac{d\mathfrak{r}_0}{dt} = r\left(\frac{d\theta}{dt}\right)^2$ (the vector $\frac{d\mathfrak{r}_0}{dt}$ is perpendicular to $\mathfrak{r}$; $r\frac{d\theta}{dt}$ is the corresponding azimuthal component of the velocity):

$$\frac{d}{dt}(m\mathfrak{v}\,\mathfrak{r}_0) - m\mathfrak{v}\frac{d\mathfrak{r}_0}{dt} = -\frac{Z\,e^2}{r^2},$$

i.e.:

$$\frac{d}{dt}\left(m\,\frac{dr}{dt}\right) - m\,r\left(\frac{d\theta}{dt}\right)^2 = -\frac{Z\,e^2}{r^2}.$$

If one introduces $\frac{d(\cdot)}{dt} = \frac{J}{mr^2}\frac{d(\cdot)}{d\theta}$ into that, according to (45.a), then one will have:

$$\frac{J}{mr^2}\frac{d}{d\theta}\frac{J}{r^2}\frac{dr}{d\theta} - \frac{J^2}{mr^3} = -\frac{Z\,e^2}{r^2},$$

i.e., with the notation $1 / r = \sigma$:

$$\frac{d^2\sigma}{d\theta^2} + \sigma = \frac{Z\,e^2}{J^2}\,m,$$

or ultimately, since from (45.b), one has $m = (H + Z\,e^2\sigma) / c^2$:

$$\frac{d^2\sigma}{d\theta^2} + \left(1 - \frac{Z^2e^4}{J^2c^2}\right)\sigma = \frac{Z\,e^2H}{J^2c^2} \equiv \frac{1}{p}. \tag{46}$$

That equation can be integrated directly. Namely, from the general theory of linear equations with constant coefficients, we can set:

$$\sigma = \frac{1}{p}\left(1+\varepsilon\cos\sqrt{1-\frac{Z^2e^4}{J^2c^2}}\theta\right), \tag{46.a}$$

in which ε means an initially-undetermined integration constant. If the parameter:

$$\alpha = \frac{Z\,e^2}{J\,c} \tag{46.b}$$

is very small compared to 1 then the formula above will, in practice, reduce to the known equation for a conic section:

$$r = \frac{p}{1+\varepsilon\cos\theta},$$

in which ε means its eccentricity.

The path that is represented by (46.a) then differs (when $\alpha < 1$) from the ordinary elliptical or hyperbolic paths of celestial mechanics by the fact that the product $\theta\sqrt{1-\alpha^2}$ appears in place of θ. For $\varepsilon < 1$, the electron will describe a quasi-elliptical path that will yield an ordinary ellipse after an additional rotation (viz., precession in the orbital plane). A complete oscillation of r (from the minimum value $\frac{p}{1+\varepsilon}$ to the maximum $\frac{p}{1-\varepsilon}$ and then back to the minimum again) will then require an increase in θ of $\frac{2\pi}{\sqrt{1-\alpha^2}}$, not 2π. The major axis of the ellipse will then shift by an angle of $2\pi\left(\frac{1}{\sqrt{1-\alpha^2}}-1\right)$ during each orbit, or approximately $\pi\,\alpha^2$ by when $\alpha \ll 1$. One can easily calculate the eccentricity in the path from the equations $J = m\,r_{\min}\,v$ and $H = mc^2 - \frac{Ze^2}{r_{\min}}$, which correspond to passing through the perihelion. (Thus, $\mathfrak{v}$ will be perpendicular to $\mathfrak{r}$, and as a result $|\mathfrak{r}\times\mathfrak{v}| = rv$). Namely, if one eliminates the mass m and velocity (by means of the relation $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$) and considers the fact that $r_{\min} = \frac{p}{1+\varepsilon}$ then that will give:

$$\varepsilon = \sqrt{1+\frac{2J^2(H-m_0\,c^2)}{m_0\,Z^4e^4}}\,. \tag{46.c}$$

One will then get the quasi-elliptic motion in question only when $H - m_0\,c^2 < 0$, i.e., for negative mechanical energy of the electron, which was clear from the outset. By contrast, for $H - m_0\,c^2 >$

0, the electron moves on a quasi-hyperbolic path that goes from infinity to infinity. (That motion should not be confused with the uniform hyperbolic motion in **§ 5**!)

However, those two types of paths are obtained only when the parameter $\alpha < 1$. Note that in the simplest case of a circular motion ($\varepsilon = 0$), that parameter will be equal to the ratio of the speed of the electron v to the speed of light. Thus, if $\alpha > 1$ then the type of motion must change completely. In fact, instead of the trigonometric function in (46.a), we will get an exponential function according to the formula:

$$\sigma = \frac{1}{r} = \frac{1}{p}(1 + \varepsilon\, e^{\sqrt{\alpha^2 - 1}\theta}) \ . \tag{47}$$

That is the equation of a spiral that always winds closer to the origin. The speed of the electron will increase continuously up to the limiting value $r = c$.

This is not the place for a detailed discussion of the results above from the physical standpoint. However, we would like to illustrate the relativistic conversion of the classical-mechanical concepts of angular momentum ($\mathfrak{J}$) and torque ($\mathfrak{M}$) in this example of the orbital motion of an electron.

Up to now, we have assumed that the nucleus is at rest. However, we can go from the original coordinate system S to another one S' that moves with a velocity $\mathfrak{v}'$. What form will the orbital motion of the electron take when we consider it from the standpoint of the system S', in which the entire "atom" (nucleus + electron) has a constant translational velocity $-\mathfrak{v}'$?

In order to answer that question, we introduce the coordinates of the nucleus and electron and denote them by x_k^0 (x_k, resp.) for the system S and by $x_k'^0$ (x_k', resp.) for S'. In the original system, we have made no distinction between the fourth coordinates x_4^0 and x_4, i.e., We must now recall that the concept of simultaneity has no absolute meaning. That is because after the Lorentz transformation that determines the transition from S to S', we will get different values for $x_4'^0$ and x_4 when $x_4^0 = x_4$, in general. That shift in the time-point that is ascribed to the nucleus and the electron is closely connected with a change in the vector $\mathfrak{J}$ under the transition from S to S'. In classical mechanics, that vector, like any other three-dimensional vector, is considered to be an invariant quantity. In relativistic mechanics, one must treat it as a varying quantity (just like any other three-dimensional vector), and indeed as is easy to see, as the *spatial part of a six-vector*. In fact, the components of $\mathfrak{J}$ are expressed in S by the formulas:

$$J_1 = m[(x_2 - x_2^0)v_3 - (x_3 - x_3^0)v_2] = m\left[(x_2 - x_2^0)\frac{dx_3}{dt} - (x_3 - x_3^0)\frac{dx_2}{dt}\right],$$

etc. If one introduces the proper time *of the electron* $d\tau = \sqrt{1 - v^2/c^2}\, dt = (m_0 / m)\, dt$ in place of the ordinary time t then one will get the three components of $\mathfrak{J}$ from:

$$J_{kl} = m_0 \left[(x_2 - x_2^0)\frac{dx_3}{d\tau} - (x_3 - x_3^0)\frac{dx_2}{d\tau} \right] \qquad (k, l = 1, 2, 3). \tag{48}$$

However, those three quantities must obviously be the three spatial components of a four-dimensional skew-symmetric tensor $J_{kl} = - J_{lk}$. The other components will be obtained when one introduces the fourth index, along with the first three. One can then set:

$$\left.\begin{array}{lll} J_1 = J_{23}, & J_2 = J_{31}, & J_3 = J_{12}, \\ i\,J_1^* = J_{14}, & i\,J_2^* = J_{24}, & i\,J_3^* = J_{34}, \end{array}\right\} \tag{48.a}$$

in which $\mathfrak{J}^*$ represents the temporal part of ${}^2\mathfrak{J}$ that belongs to $\mathfrak{J}$. *Due to the condition that* $x_4^0 = x_4$, from (48), we have:

$$J_{k4} = m_0\,(x_k - x_k^0)\frac{i\,c\,dt}{d\tau} = i\,c\,m(x_k - x_k^0) \qquad (k = 1, 2, 3),$$

i.e., with the abbreviation $x_k - x_k^0 = R_k$:

$$\mathfrak{J}^* = c\,m\,\mathfrak{R}\,. \tag{48.b}$$

Under the transition to the system S', the vectors $\mathfrak{J}$ and $\mathfrak{J}^*$ must obviously transform under the same formulas as the magnetic and electric polarization, i.e., from the formulas (11), Chap. IX. For low speeds, one has the approximate formulas:

$$\mathfrak{J}' = \mathfrak{J} + \frac{1}{c}\mathfrak{v}' \times \mathfrak{J}^*, \qquad \mathfrak{J}^{*\prime} = \mathfrak{J}^* - \frac{1}{c}\mathfrak{v}' \times \mathfrak{J}\,. \tag{49}$$

Note that under the aforementioned restriction ($v / c \ll 1$), one regards the mass of the electron as constant, and its magnetic moment (i.e., the magnetic moment that originates in the orbital motion) $\mathfrak{m} = \dfrac{e}{2c}\mathfrak{r} \times \mathfrak{v}$ can be determined from the formula:

$$\mathfrak{m} = \frac{1}{\kappa'}\mathfrak{J} \qquad \left(\kappa' = \frac{e}{2m_0\,c}\right). \tag{49.a}$$

We will then get the following expression for the associated electric moment $\mathfrak{p}$:

$$\mathfrak{p} = \tfrac{1}{2}e\,\mathfrak{R}\,. \tag{49.b}$$

If the nucleus has a charge – e (Z = 1) then it will define a dipole with the electron that has a moment of $e\,\mathfrak{R}$, i.e., twice as large as (49.b). However, if Z is not equal to 1 then it will be impossible to ascribe a *well-defined* electric moment to the nucleus-electron system on the basis of the usual definition of a dipole moment. That is because one can just as well think of that system as a combination of a dipole with a moment $e\,\mathfrak{R}$ and a resultant nuclear charge – $(Z - 1)\,e$ or a dipole with the moment $Z\,e\,\mathfrak{R}$, and a resultant electron charge + $(Z - 1)\,e$.

That indeterminacy in the electric moment is obviously closely connected with the arbitrariness that adheres to the normalization of the simultaneity of the nucleus and electron above. That is because it is not clear from the outset (and in fact, it is impossible to establish) why one should have the equation $x_4 = x_4^0$ in the system S and not in the system S'. The changes in $\mathfrak{J}$ and $\mathfrak{J}^*$ that are determined by (49) (and obviously the same formulas are true for $\mathfrak{m}$ and $\mathfrak{p}$) originates in precisely the changes in the spatial and temporal distances between those "events" that are defined by the presence of the nucleus and electron at the spacetime points considered. One can easily show that the change in $\mathfrak{J}$ (or $\mathfrak{m}$) is mainly implied by the change in the spatial distance, while the change in $\mathfrak{J}^*$ ($\mathfrak{p}$, resp.) is implied by the change in the temporal distance.

The same difficulties will arise when one seeks to formulate the concept of the *torque* for a system that consists of two or more electrons in a four-dimensional way while considering the relativity of time. If the electron is subject to an external force $\mathfrak{f}$, but not the nucleus (which is what happens, e.g., in an external magnetic field), then one regard can the moment of that force relative to the nucleus:

$$\mathfrak{M} = \mathfrak{R} \times \mathfrak{f}$$

as the spatial part of a six-vector with the components:

$$M_{ik} = R_i\,F_k - R_k\,F_i \qquad (i, k = 1, 2, 3, 4),$$

in which $\mathfrak{F}$ means the *four-impulse* (per unit proper time of the electron) that corresponds to that force. However, the same arbitrariness will adhere to the definition of the temporal part of the tensor M_{ik} that adheres to the tensor ${}^2\mathfrak{J}$.

For a rigorous and consistent treatment of the two-body or many-body problem according to the theory of relativity, one must then consider the motion of each electron in isolation and consider the other ones only to the extent that they are the sources of the external field that acts upon the electron in question. The relativity of simultaneity will then do no harm due to the invariance of the electrodynamical equations.

§ 8. – Rotational motion. Equations of motion of the magnetic electron.

The complications that were considered in the foregoing section in regard to the definition of the angular momentum and torque on a system of two or more electrons will drop away when one refers those quantities to an isolated electron. In so doing, it is not necessary, nor even preferable, to decompose the element into elements and to reduce its "rotational motion" to an orbital motion of the latter around a certain axis that goes through the center of the electron. One can (and this is the path that we shall go down) treat the electron as simply a *point* whose properties are characterized by certain scalar, vector, and tensor quantities. If one assumes the validity of the usual mechanical relations spatial projections of those quantities for the limiting case of vanishing translational velocity then the theory of relativity will permit one to ascertain the corresponding exact relations that are true for arbitrary translational velocities directly without needing to go into any detailed consideration of the "structure" of the electron. In that way, as we will see, the difficulties that are implied by the usual theory of an extended rotating electron in regard to its mass that were presented in **§ 8**, Chap. VII, will also vanish.

We define the magnetic moment of the electron $\mathfrak{m}$ to be the spatial component of a skew-symmetric tensor $\mu_{\alpha\beta} = -\mu_{\beta\alpha}$:

$$\begin{pmatrix} \mu_{23} & \mu_{31} & \mu_{12} & \mu_{14} & \mu_{24} & \mu_{34} \\ m_1 & m_2 & m_3 & i\,p_1 & i\,p_2 & i\,p_3 \end{pmatrix},$$

in which p_1, p_2, p_3 are the spatial components of the associated electron moment.

We would like to determine that dipole moment from the condition that *it should vanish* ($\mathfrak{p}' = 0$) in the coordinate system S' where the electron is instantaneously at rest. It will then follow (cf., Chap. IX, **§ 2**) that it is known for an arbitrary coordinate system S relative to which the electron has the translational velocity $\mathfrak{v}$:

$$\mathfrak{p} = \frac{\mathfrak{v}}{c} \times \mathfrak{m} . \tag{50}$$

One can also derive that result in the following way: Let x_α be the coordinates of the electron and time, multiplied by $i\,c$ ($i\,c\,t = x_4$) relative to the system S. One then uses $\mu_{\alpha\beta}$ and $\dot{x}_\alpha = dx / d\tau$, where $d\tau = dt\sqrt{1-v^2/c^2}$ means the proper time of the electron, to define the four-dimensional vector $\sum_{\beta=1}^{4} \mu_{\alpha\beta}\,\dot{x}_\beta$, or $\mu_{\alpha\beta}\,\dot{x}_\beta$ when written more simply (the summation sign will always be omitted for *pairs of equal indices* in what follows). The components of that vector $\mu'_{\alpha\beta}\,\dot{x}'_\beta$ in the "rest system" S' must vanish since one would have $\dot{x}'_1 = \dot{x}'_2 = \dot{x}'_3 = 0$, and by assumption, one would then have $\mu'_{14} = \mu'_{24} = \mu'_{34} = 0$. However, it would follow from this that the equations:

$$\mu_{\alpha\beta}\,\dot{x}_\beta = 0 , \tag{50.a}$$

which express the vanishing of the vector above, will be fulfilled in any other coordinate system S. If one replaces $\mu_{\alpha\beta}$ and $\dot{x}_\beta$ with their corresponding three-dimensional expressions then one will get the spatial components of the vector:

$$\frac{c}{\sqrt{1-v^2/c^2}}\left(\frac{\mathfrak{v}}{c}\times\mathfrak{m}-\mathfrak{p}\right) \tag{51}$$

for $\alpha = 1, 2, 3$, while one will have:

$$\mu_{4\beta}\,\dot{x}_\beta = -\frac{i}{\sqrt{1-v^2/c^2}}(\mathfrak{v}\,\mathfrak{p})$$

for $\alpha = 4$. The vanishing of the second expression follows immediately from the vanishing of the first one, i.e., equation (50).

It is known that one can form the following two invariant quantities from the tensor components:

$$m^2 - p^2 = \tfrac{1}{2}\,\mu_{\alpha\beta}\,\mu_{\alpha\beta}$$

and

$$\mathfrak{m}\;\mathfrak{p} = i\,(\mu_{23}\,\mu_{14} + \mu_{31}\,\mu_{24} + \mu_{12}\,\mu_{34})\,.$$

In that way, from (50) (i.e., due to the fact that $\mathfrak{p}' = 0$):

$$\mathfrak{m}\;\mathfrak{p} = \mathfrak{m}'\;\mathfrak{p}' = 0$$

and

$$m^2 - p^2 = m^2 - \left(\frac{\mathfrak{v}}{c}\times\mathfrak{m}\right)^2 = m'^2. \tag{51}$$

The last equation determines the dependency of the magnetic moment of the electron on its translational velocity $\mathfrak{v}$. One can rewrite it in the form:

$$m = \frac{\mu}{\sqrt{1-v_\perp^2/c^2}}\ , \tag{51.a}$$

in which $v_\perp$ means the component of $\mathfrak{v}$ that is perpendicular to $\mathfrak{m}$. $m' = \mu$ is the magnitude of the magnetic moment in the "rest system."

We now introduce the four-dimensional quantities that correspond to the magnetic energy $-\mathfrak{m}\,\mathfrak{H} = -\,m_\alpha\,H_\alpha$ and the magnetic torque $\mathfrak{m}\times\mathfrak{H}$, i.e., the skew-symmetric three-dimensional tensor with the components $m_\alpha\,H_\beta - m_\beta\,H_\alpha$. The four-dimensional "extension" of the energy function is obviously the scalar:

$$U = -\tfrac{1}{2}\,\mu_{\alpha\beta}\,H_{\alpha\beta} = -(\mathfrak{m}\;\mathfrak{H}) - (\mathfrak{p}\;\mathfrak{E})\,. \tag{52}$$

It is easy to see that the corresponding "extension" of the torque is given by the skew-symmetric four-dimensional tensor (six-vector):

$$M_{\alpha\beta} = \mu_{\alpha\gamma} H_{\beta\gamma} - \mu_{\beta\gamma} H_{\alpha\gamma} \tag{53}$$

with the spatial components:

$$(M_{23}, M_{31}, M_{12}) = \mathfrak{m} \times \mathfrak{H} + \mathfrak{p} \times \mathfrak{E} ,$$

and the temporal components:

$$- i\,(M_{14}, M_{24}, M_{34}) = - \mathfrak{m} \times \mathfrak{E} + \mathfrak{p} \times \mathfrak{H} .$$

We define the angular momentum of the electron to be the spatial part of the tensor:

$$\frac{1}{\kappa}\mu_{\alpha\beta} ,$$

with $\kappa = e / c\, m_0$.

The simplest four-dimensional "extension" of the usual differential equation for the time variation of $\mu_{\alpha\beta}$ will then read:

$$\frac{1}{\kappa}\dot{\mu}_{\alpha\beta} = M_{\alpha\beta} , \tag{54}$$

i.e.:

$$\frac{1}{\kappa}\dot{\mathfrak{m}} = \mathfrak{m} \times \mathfrak{H} + \mathfrak{p} \times \mathfrak{E} \tag{54.a}$$

and

$$\frac{1}{\kappa}\dot{\mathfrak{p}} = \mathfrak{p} \times \mathfrak{H} - \mathfrak{m} \times \mathfrak{E} , \tag{54.b}$$

in which the dot means differentiation with respect to proper time.

However, equations (54.a) and (54.b) can be fulfilled simultaneously only in the case where the vectors $\mathfrak{m}$ and $\mathfrak{p}$ are mutually independent (*a priori*). However, the relation (50) must exist between them, which will make equations (54.a), (54.b) incompatible. Now, it is easy to modify the combined equation (11) in such a way that the condition (7.a) will be fulfilled. To that end, we introduce an initially-indeterminate four-dimensional vector a_α and define the invariant scalar:

$$- \mu_{\alpha\beta}\, a_\alpha\, \dot{x}_\beta = - \tfrac{1}{2}\mu_{\alpha\beta}\,(a_\alpha\, \dot{x}_\beta - a_\beta\, \dot{x}_\alpha) , \tag{55}$$

which vanishes identically, according to (50.a). We add that scalar to the "energy function" U, i.e., we replace the latter with:

$$U' = - \tfrac{1}{2}\mu_{\alpha\beta}\,(H_{\alpha\beta} + a_\alpha\, \dot{x}_\beta - a_\beta\, \dot{x}_\alpha) = - \tfrac{1}{2}\mu_{\alpha\beta}\, H'_{\alpha\beta} . \tag{55.a}$$

We correspondingly replace the tensor $M_{\alpha\beta}$ with:

$$M'_{\alpha\beta} = \mu_{\alpha\gamma} H'_{\beta\gamma} - \mu_{\beta\gamma} H'_{\alpha\gamma} , \tag{55.b}$$

i.e.:

$$M'_{\alpha\beta} = M_{\alpha\beta} + a_\gamma (\dot{x}_\alpha \mu_{\beta\gamma} - \dot{x}_\beta \mu_{\alpha\gamma}) , \tag{55.c}$$

and the "equations of motion" (54) with:

$$\frac{1}{\kappa} \dot{\mu}_{\alpha\beta} = M'_{\alpha\beta} , \tag{56}$$

or when written out in full:

$$\frac{1}{\kappa} \dot{\mu}_{\alpha\beta} = \mu_{\alpha\gamma} H_{\beta\gamma} - \mu_{\beta\gamma} H_{\alpha\gamma} + a_\gamma (\dot{x}_\alpha \mu_{\beta\gamma} - \dot{x}_\beta \mu_{\alpha\gamma}) . \tag{56.a}$$

We now determine the vector a_α in such a way that this equation will agree with the relation (50.a). Indeed, when we consider (50.a) and the identity:

$$\dot{x}_\alpha \dot{x}_\alpha = - c^2 ,$$

it will follow from (56.a) that:

$$\frac{1}{\kappa} \dot{\mu}_{\alpha\beta} \dot{x}_\beta = - \frac{1}{\kappa} \mu_{\alpha\beta} \ddot{x}_\beta = \mu_{\alpha\gamma} H_{\beta\gamma} \dot{x}_\beta - a_\gamma \mu_{\alpha\gamma} \dot{x}_\beta \dot{x}_\beta = \mu_{\alpha\gamma} (H_{\beta\gamma} \dot{x}_\beta + a_\gamma c^2) ,$$

or

$$\mu_{\alpha\beta} \left(\frac{1}{\kappa} \ddot{x}_\beta + H_{\beta\gamma} \dot{x}_\beta + a_\gamma c^2 \right) = 0 .$$

That will then give:

$$a_\gamma = \frac{1}{\kappa c^2} \left(\kappa H_{\gamma\beta} \dot{x}_\beta - \ddot{x}_\gamma \right). \tag{57}$$

Independently of that expression for a_γ, when one considers (50.a), one will get from (56.a) that:

$$\frac{1}{\kappa} \dot{\mu}_{\alpha\beta} \dot{\mu}_{\alpha\beta} = \mu_{\alpha\beta} \mu_{\alpha\gamma} H_{\beta\gamma} - \mu_{\alpha\beta} \mu_{\beta\gamma} H_{\alpha\gamma} = 2 \mu_{\alpha\beta} \mu_{\alpha\gamma} H_{\beta\gamma} = 0$$

(due to the skew-symmetric character of $H_{\beta\gamma}$), i.e.:

$$\frac{d}{d\tau} \mu_{\alpha\beta}^2 = 0$$

or

$$\tfrac{1}{2} \mu_{\alpha\beta}^2 = m^2 - p^2 = \mu^2 = \text{const.} \tag{58}$$

As is known, the equations of motion of a non-magnetic electron [cf., (8), **§ 2**] are:

$$m_0\,\ddot{x}_\alpha = \frac{e}{c} H_{\alpha\beta}\,\dot{x}_\beta\ ,$$

or, with $e / m_0\,c = \kappa$:

$$\ddot{x}_\alpha = \kappa H_{\alpha\beta}\,\dot{x}_\beta\ . \tag{58.a}$$

If one neglects the force that originates in the magnetic moment in comparison to the force $e\left(\mathfrak{E}+\frac{1}{c}\mathfrak{v}\times\mathfrak{H}\right)$, which corresponds to the four-vector $\frac{e}{c}F_{\alpha\beta}\,\dot{x}_\beta$, then from (58) and (58.a), one will have:

$$a_\gamma \sim 0\ . \tag{58.b}$$

In that approximation (i.e., when one neglects the perturbation of the translational motion of the electron that is required by the magnetic force), one then determines its "rotational motion," i.e., the time variation of the vector $\mathfrak{m}$, from the simple equations (54) or (54.a).

If one substitutes $\mathfrak{p} = (1 / c)\ \mathfrak{v} \times \mathfrak{m}$ in (54.a) using (50) then one will have:

$$\frac{1}{\kappa}\dot{\mathfrak{m}} \approx \mathfrak{m}\times\mathfrak{H} + \left(\frac{1}{c}\mathfrak{v}\times\mathfrak{m}\right)\times\mathfrak{E}\ . \tag{59}$$

One now considers the case in which the electron moves around the nucleus in a weak external magnetic field $\mathfrak{H}$. One can then set:

$$\mathfrak{E} \approx \frac{m_0}{e}\frac{d\mathfrak{v}}{dt}, \tag{59.a}$$

to an even-coarser degree of approximation (viz., one ignores the terms that are quadratic in v / c). In that way, the second term on the right-hand side of (59) will assume the form:

$$\frac{m_0}{e\,c}(\mathfrak{v}\times\mathfrak{m})\times\frac{d\mathfrak{v}}{dt}.$$

We would now like to calculate the mean value of that expression for the unperturbed motion.

One has (for the unperturbed motion, cf., **§ 9**, Chap. VII):

$$\overline{\frac{d}{dt}(\mathfrak{v}\times\mathfrak{m})\times\mathfrak{v}} = \overline{(\mathfrak{v}\times\mathfrak{m})\times\frac{d\mathfrak{v}}{dt}} + \overline{\left(\frac{d\mathfrak{v}}{dt}\times\mathfrak{m}\right)\times\mathfrak{v}} = 0\ .$$

It then follows from this that:

$$\overline{(\mathfrak{v}\times\mathfrak{m})\times\frac{d\mathfrak{v}}{dt}} = \overline{\tfrac{1}{2}\mathfrak{m}\times\left(\frac{d\mathfrak{v}}{dt}\times\mathfrak{v}\right)}\ ,$$

or from (59.a):

$$\overline{\left(\frac{1}{c}\mathfrak{v}\times\mathfrak{m}\right)\times\mathfrak{E}} \approx \frac{1}{2c}\mathfrak{m}\times\overline{(\mathfrak{E}\times\mathfrak{v})} = \frac{1}{2}\mathfrak{m}\times\overline{\mathfrak{H}'} \; . \tag{59.c}$$

As a result, the mean variation of the vector $\mathfrak{m}$ is determined, to the degree of approximation above, by the equation:

$$\frac{1}{\kappa}\overline{\frac{d\mathfrak{m}}{dt}} \approx \mathfrak{m}\times\mathfrak{H} + \frac{1}{2}\mathfrak{m}\times\overline{\mathfrak{H}'}, \tag{60}$$

in which $\mathfrak{H}' = -\frac{1}{2}\mathfrak{v}\times\mathfrak{E}' \approx -\frac{1}{2}\mathfrak{v}\times\mathfrak{E}$ means the magnetic field strength in the coordinate system S', in which the electron is instantaneously at rest, and $\overline{\mathfrak{H}'}$ means the mean value of that field strength.

We will now cite a more rigorous derivation of the differential equation (56) for the "rotational motion" of the electron on the basis of the *Hamilton-Schwarzschild* principle. At the same time, that will yield the precise differential equations for translational motion.

We then set:

$$\delta\int L\,d\tau = 0\,, \tag{61}$$

as usual (cf., **§ 3**), with the auxiliary conditions:

$$\dot{x}_\alpha^2 = -\,c^2\,, \tag{61.a}$$

$$\mu_{\alpha\beta}\,\dot{x}_\beta = 0\,. \tag{61.b}$$

We then write the *Lagrangian* function in the form:

$$L = \frac{e}{c}A_\alpha\,\dot{x}_\alpha + T^* + \tfrac{1}{2}\mu_{\alpha\beta}\,H_{\alpha\beta}\,, \tag{62}$$

in which T^* means the "kinetic energy" of the rotational motion.

We consider that energy in conjunction with the usual three-dimensional mechanics as a function of the "angular velocity," which we characterize by the skew-symmetric tensor $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. In that way, we set:

$$\delta T^* = \frac{1}{2\kappa}\mu_{\alpha\beta}\,\delta\omega_{\alpha\beta}\,, \tag{62.a}$$

by definition. In order to determine the variation of $\mu_{\alpha\beta}$, we next observe the corresponding operation in ordinary mechanics. The work done by the magnetic torque $\mathfrak{m}\times\mathfrak{H}$ under a virtual infinitesimal rotation $\delta\mathfrak{v}$ is equal to the inner product $\delta\mathfrak{v}\cdot(\mathfrak{m}\times\mathfrak{H})$. On the other hand, it must be equal to the decrease in magnetic energy $-\delta(-\mathfrak{m}\ \mathfrak{H}) = \delta\mathfrak{m}\cdot\mathfrak{H}$. We will then have $\delta\mathfrak{m}\cdot\mathfrak{H} = \delta\mathfrak{v}\cdot(\mathfrak{m}\times\mathfrak{H})$ or $(\delta\mathfrak{m}\ \mathfrak{H}) = (\delta\mathfrak{v}\times\mathfrak{m})\cdot\mathfrak{H}$, and as a result:

$$\delta\mathfrak{m} = \delta\mathfrak{v}\times\mathfrak{m}\ .$$

The corresponding four-dimensional variational formula must be obtained from that in the same way that formula (53) was obtained from the three-dimensional expression for the torque $\mathfrak{m}\times\mathfrak{H}$. If one then introduces the four-dimensional skew-symmetrical "rotation tensor" $\delta\Omega_{\alpha\beta}$ whose spatial part is equal to the vector $\delta\mathfrak{v}$ then one will have:

$$\delta\mu_{\alpha\beta} = \delta\Omega_{\alpha\beta}\cdot\mu_{\beta\gamma} - \delta\Omega_{\beta\gamma}\cdot\mu_{\alpha\gamma}\,. \tag{62.b}$$

Obviously, the quantities $\delta\Omega_{\alpha\beta}$ (just like $\delta\omega_{\alpha\beta}$) do not represent complete differentials, i.e., there is no "angular coordinate" $\Omega_{\alpha\beta}$ (non-holonomic system) that would correspond to x_α. Nonetheless, we shall assume that along with the relations:

$$\delta\dot{x}_\alpha = \frac{d}{d\tau}\delta x_\alpha\,, \tag{63}$$

the corresponding commutation relations for $\delta\Omega_{\alpha\beta}$ and $d\,\Omega_{\alpha\beta} = \omega_{\beta\gamma}\,d\tau$ are also true, i.e.:

$$\delta\,\omega_{\alpha\beta} = \frac{d}{d\tau}\delta\Omega_{\alpha\beta}\,. \tag{63.a}$$

By means of the formulas and the relations that were partially employed before:

$$\delta A_\alpha = \frac{\partial A_\alpha}{\partial x_\gamma}\delta x_\gamma\,, \qquad \dot{A}_\alpha = \frac{\partial A_\alpha}{\partial x_\gamma}\dot{x}_\gamma\,,$$
$$\delta H_{\alpha\beta} = \frac{\partial H_{\alpha\beta}}{\partial x_\gamma}\delta x_\gamma\,, \qquad \dot{H}_{\alpha\beta} = \frac{\partial H_{\alpha\beta}}{\partial x_\gamma}\dot{x}_\gamma\,,$$

we will get:

$$\delta L = \frac{e}{c}\frac{\partial A_\alpha}{\partial x_\gamma}\dot{x}_\alpha\,\delta x_\gamma - \frac{e}{c}\frac{\partial A_\alpha}{\partial x_\gamma}\dot{x}_\gamma\,\delta x_\alpha + \frac{d}{d\tau}\left(\frac{e}{c}A_\alpha\,\delta x_\alpha\right) - \frac{1}{2\kappa}\dot{\mu}_{\alpha\beta}\,\delta\Omega_{\alpha\beta} + \frac{d}{d\tau}\left(\frac{1}{2\kappa}\mu_{\alpha\beta}\,\delta\Omega_{\alpha\beta}\right)$$
$$+\ \tfrac{1}{2}\mu_{\alpha\beta}\frac{\partial H_{\alpha\beta}}{\partial x_\gamma}\delta x_\gamma + \tfrac{1}{2}H_{\alpha\beta}\,(\delta\Omega_{\alpha\gamma}\,\mu_{\beta\gamma} - \delta\Omega_{\beta\gamma}\,\mu_{\alpha\gamma}),$$

or since:

$$\frac{\partial A_\alpha}{\partial x_\beta} - \frac{\partial A_\beta}{\partial x_\alpha} = H_{\beta\alpha} ,$$

$$\delta L$$

$$= \left(\frac{e}{c} H_{\alpha\beta}\, \dot{x}_\alpha + \tfrac{1}{2} \mu_{\beta\gamma} \frac{\partial H_{\beta\gamma}}{\partial x_\alpha} \right) \delta x_\alpha + \tfrac{1}{2} \left(-\frac{1}{\kappa} \dot{\mu}_{\alpha\gamma} + \mu_{\beta\gamma}\, H_{\beta\gamma} - \mu_{\beta\gamma}\, H_{\alpha\gamma} \right) \delta \Omega_{\alpha\beta} + \frac{d}{d\tau} \left(\frac{e}{c} A_\alpha\, \delta x_\alpha + \frac{1}{\kappa} \mu_{\alpha\beta}\, \delta \Omega_{\alpha\beta} \right).$$

Similarly, from (61.a) and (61.b), when one adds the undetermined *Lagrange* multipliers λ and a_α ($\alpha = 1, 2, 3, 4$):

$$\lambda\, \dot{x}_\alpha\, \delta\, \dot{x}_\alpha = -\, \delta x_\alpha \frac{d}{d\tau} (\lambda\, \dot{x}_\alpha) + \frac{d}{d\tau} (\lambda\, \dot{x}_\alpha\, \delta\, x_\alpha) = 0$$

and

$$a_\alpha\, \delta (\mu_{\alpha\beta}\, \dot{x}_\beta) = \tfrac{1}{2} [a_\alpha\, \delta (\mu_{\alpha\beta}\, \dot{x}_\beta) - a_\beta\, \delta (\mu_{\alpha\beta}\, \dot{x}_\alpha)]$$

$$= \frac{d}{d\tau} (a_\alpha\, \mu_{\alpha\beta}\, \delta x_\beta) - \delta x_\alpha \frac{d}{d\tau} (\mu_{\alpha\beta}\, a_\beta) + \tfrac{1}{2} (a_\alpha\, \dot{x}_\beta - a_\beta\, \dot{x}_\alpha)(\delta \Omega_{\alpha\gamma} \cdot \mu_{\beta\gamma} - \delta \Omega_{\beta\gamma} \cdot \mu_{\alpha\gamma}) = 0 ,$$

or since:

$$\dot{x}_\beta\, \mu_{\beta\gamma} = \dot{x}_\alpha\, \mu_{\alpha\gamma} = 0 ,$$

one will have:

$$a_\alpha\, \delta (\mu_{\alpha\beta}\, \dot{x}_\beta) = \frac{d}{d\tau} (a_\alpha\, \mu_{\alpha\beta}\, \delta x_\beta) - \delta x_\alpha \frac{d}{d\tau} (\mu_{\alpha\beta}\, a_\beta) + \tfrac{1}{2} \delta \Omega_{\alpha\gamma} \cdot a_\gamma (\dot{x}_\alpha\, \mu_{\beta\gamma} - \dot{x}_\beta \cdot \mu_{\alpha\gamma}) = 0 .$$

With the usual assumption that the variations δx_α, $\delta \Omega_{\alpha\beta}$ vanish at the limits of the integral (61), it will then follow from (61), (61.a), (61.b) (upon adding the expressions above and setting the coefficients of δx_α and $\delta \Omega_{\alpha\beta}$ equal to zero) that:

$$\frac{d}{d\tau} (\lambda\, \dot{x}_\alpha + \mu_{\beta\alpha}\, a_\beta) = \frac{e}{c} H_{\alpha\beta}\, \dot{x}_\beta + \tfrac{1}{2} \mu_{\beta\gamma} \frac{\partial H_{\beta\gamma}}{\partial x_\alpha} \tag{64}$$

and

$$\frac{1}{\kappa} \dot{\mu}_{\alpha\beta} = \mu_{\alpha\gamma} H_{\beta\gamma} - \mu_{\beta\gamma} H_{\alpha\gamma} + a_\gamma (\dot{x}_\alpha\, \mu_{\beta\gamma} - \dot{x}_\beta\, \mu_{\alpha\gamma}) .$$

The last equation coincides with (56.a). The first of them is the generalization of the ordinary equations of motion (58.a) for a non-magnetic electron.

We correspondingly set:

$$\lambda = m_0 + \lambda' , \tag{64.a}$$

in which λ' means an additional term that depends upon the moment of the electron. After performing the differentiation on the left-hand side of (64), we will get from (57) that:

$$\lambda' \ddot{x}_\alpha + \dot{\lambda}' \dot{x}_\alpha + \mu_{\beta\alpha} \dot{a}_\beta + \dot{\mu}_{\beta\alpha} a_\beta = \kappa m_0 c^2 a_\alpha + \tfrac{1}{2} \mu_{\beta\gamma} \frac{\partial H_{\beta\gamma}}{\partial x_\alpha} .$$

Due to the relations $\dot{x}_\alpha^2 = - c^2$, $\dot{x}_\alpha \ddot{x}_\alpha = 0$, and $a_\alpha \dot{x}_\alpha = 0$, it will follow from that upon multiplying by $\dot{x}_\alpha$ that:

$$- c^2 \dot{\lambda}' + \dot{\lambda}' \dot{x}_\alpha + \dot{\mu}_{\beta\alpha} a_\beta \dot{x}_\alpha = \tfrac{1}{2} \mu_{\beta\gamma} \frac{\partial H_{\beta\gamma}}{\partial x_\alpha} \dot{x}_\alpha = \tfrac{1}{2} \mu_{\beta\gamma} \frac{dH_{\beta\gamma}}{d\tau}$$

or

$$- c^2 \dot{\lambda}' = \frac{d}{d\tau}\left(\tfrac{1}{2} \mu_{\beta\gamma} H_{\beta\gamma}\right) - \tfrac{1}{2}(\dot{\mu}_{\alpha\beta} H_{\alpha\beta} + a_\alpha \dot{x}_\beta - a_\beta \dot{x}_\alpha) .$$

From (55.a), (55.b), and (56), we have:

$$\tfrac{1}{2} \dot{\mu}_{\alpha\beta} (H_{\alpha\beta} + a_\alpha \dot{x}_\beta - a_\beta \dot{x}_\alpha) = \tfrac{1}{2} \dot{\mu}_{\alpha\beta} H'_{\alpha\beta} = \frac{\kappa}{2}(\mu_{\alpha\beta} H'_{\beta\gamma} - \mu_{\beta\gamma} H'_{\alpha\gamma}) H'_{\alpha\beta}$$

$$= \frac{\kappa}{2}(\mu_{\alpha\beta} H'_{\gamma\beta} H'_{\alpha\gamma} - \mu_{\beta\alpha} H'_{\gamma\alpha} H'_{\gamma\beta}) = \kappa \mu_{\alpha\beta} H'_{\alpha\gamma} H'_{\beta\gamma} = 0 ,$$

due to the skew-symmetric character of the tensor μ_α . As a result, we will have:

$$\lambda' = - \frac{1}{2c^2} \mu_{\alpha\beta} H_{\alpha\beta} . \tag{64.b}$$

The increase in mass m_0 is then equal to the "*relative* magnetic energy" of the electron (relative to the nucleus and other particles that create the field $H_{\alpha\beta}$) divided by the square of the speed of light.

One can interpret the expression $\mu_{\alpha\beta} a_\alpha$ in (64) as the α-component of the additional impulse that originates in the *absolute* energy of the electron, i.e., the kinetic energy of its rotation.

Due to (57), upon substituting (64.a) in (64), that will give:

$$\frac{d}{d\tau}(\lambda' \dot{x}_\alpha + \mu_{\beta\alpha} a_\beta) = c^2 m_0 \kappa a_\alpha + \tfrac{1}{2} \mu_{\beta\gamma} \frac{\partial H_{\beta\gamma}}{\partial x_\alpha} . \tag{65}$$

One can employ that equation in order to determine a_α for an approximately, and indeed, in the first approximation, when one neglects the left-hand side of (65), one will get:

$$a_\alpha = - \frac{1}{2 e c^2} \mu_{\beta\gamma} \frac{\partial H_{\beta\gamma}}{\partial x_\alpha} . \tag{65.a}$$

It will follow from equation (64) that:

$$x_\alpha \frac{d}{d\tau}(\lambda' \dot{x}_\beta + \mu_{\gamma\beta}\, a_\gamma) - x_\beta \frac{d}{d\tau}(\lambda' \dot{x}_\alpha + \mu_{\gamma\alpha}\, a_\gamma)$$

$$= \frac{e}{c}\left(H_{\beta\gamma}\, x_\alpha\, \dot{x}_\gamma - H_{\alpha\gamma}\, x_\beta\, \dot{x}_\gamma\right) + \tfrac{1}{2}\,\mu_{\rho\sigma}\left(x_\alpha \frac{\partial H_{\rho\sigma}}{\partial x_\beta} - x_\beta \frac{\partial H_{\rho\sigma}}{\partial x_\alpha}\right)$$

or

$$\left.\begin{aligned} &\frac{d}{d\tau}\left\{\lambda\,(x_\alpha\, \dot{x}_\beta - x_\beta\, \dot{x}_\alpha) + a_\gamma\,(x_\alpha \mu_{\gamma\beta} - x_\beta \mu_{\gamma\alpha})\right\} \\ &= \frac{e}{c}\left(H_{\beta\gamma}\, x_\alpha\, \dot{x}_\gamma - H_{\alpha\gamma}\, x_\beta\, \dot{x}_\gamma\right) + \tfrac{1}{2}\,\mu_{\rho\sigma}\left(x_\alpha \frac{\partial H_{\rho\sigma}}{\partial x_\beta} - x_\beta \frac{\partial H_{\rho\sigma}}{\partial x_\alpha}\right) - a_\gamma\,(\dot{x}_\alpha \mu_{\beta\gamma} - \dot{x}_\beta \mu_{\alpha\gamma}). \end{aligned}\right\} \tag{66}$$

That equation can be regarded as the generalization of the "law of areas," i.e., the usual formula for the rate of change of the ordinary angular momentum of the translational motion $m_0\, \mathfrak{r} \times \dfrac{d\mathfrak{r}}{d\tau}$. In that way, that angular momentum is replaced with the skew-symmetric tensor:

$$I_{\alpha\beta} = \lambda\,(x_\alpha\, \dot{x}_\beta - x_\beta\, \dot{x}_\alpha) + a_\gamma\,(x_\alpha \mu_{\gamma\beta} - x_\beta \mu_{\gamma\alpha}), \tag{66.a}$$

whose spatial part coincides with $m_0\, \mathfrak{r} \times \mathfrak{v}$ *in the first approximation*. It should be further noted that the second term on the right-hand side of (66) is equal and opposite to the corresponding additional term in formula (56.a) for the rate of change of the angular momentum of rotational motion. If one sets:

$$\frac{1}{\kappa}\,\mu_{\alpha\beta} = i_{\alpha\beta}$$

then one will have:

$$\frac{d}{d\tau}(i_{\alpha\beta} + I_{\alpha\beta}) = \mu_{\alpha\gamma}\, H_\beta - \mu_{\beta\gamma}\, H_\alpha + \frac{e}{c}\left(H_{\beta\gamma}\, x_\alpha\, \dot{x}_\gamma - H_{\alpha\gamma}\, x_\beta\, \dot{x}_\gamma\right) + x_\beta \frac{\partial U}{\partial x_\alpha} - x_\alpha \frac{\partial U}{\partial x_\beta} \tag{66.b}$$

for the sum of the two moments, according to (56.a) and (70), and in which U means the "relative energy":

$$U = -\ \tfrac{1}{2}\,\mu_{\sigma\rho}\, H_{\rho\sigma}\,.$$

We now consider the case in which the electron moves in a radially-symmetric electric field $\mathfrak{E} = f(r)\, \mathfrak{r}$, although an (external) magnetic field is absent. In that case, one will have $U = -\, \mathfrak{p}\ \mathfrak{E} = -f\ \mathfrak{p}\mathfrak{r}$, and as a result, for $\alpha, \beta = 1, 2, 3$:

$$x_\beta \frac{\partial U}{\partial x_\alpha} - x_\alpha \frac{\partial U}{\partial x_\beta} = f(x_\alpha p_\beta - x_\beta p_\alpha) = E_\alpha p_\beta - E_\beta p_\alpha .$$

The resulting torque, which corresponds to the spatial part of the tensor on the right-hand side of (66.a), will then be equal to zero (cf., **§ 9**, Chap. VII and **§ 2**, Chap. IX).

It will then follow that magnitude and direction of *the resultant angular moment of the electron* in the case in question must *remain constant*.

As we have already seen above [equation (15)], the magnitude of the tensor $\mu_{\alpha\beta}$, and as a result, $i_{\alpha\beta}$, as well, is constant in time. As a result, in the first approximation (when one drops the terms that are quadratic in $1 / c$), one can consider the magnitude of the angular momentum of the rotational motion $\mathfrak{i} = \mathfrak{m} / \kappa$ to be constant in time. If one denotes the angular momentum of the translational motion (i.e., the spatial part of the tensor $I_{\alpha\beta}$) by $\mathfrak{J}$ then, due to the condition $\mathfrak{i} + \mathfrak{J}$ = const., it will follow that the magnitude of $\mathfrak{J}$ must also remain constant and that both vectors $\mathfrak{i}$ and $\mathfrak{J}$ must precess around their resultant with the same angular velocity.

If we introduce the corresponding *magnetic* moments $\mathfrak{m} = \kappa\,\mathfrak{i}$ and $\mathfrak{M} = (\kappa / 2)\,\mathfrak{J}$ in place of the angular momenta $\mathfrak{i}$ and $\mathfrak{J}$ then we will see that the sum $\mathfrak{m} + \mathfrak{M} = \frac{1}{2}\kappa\,(\mathfrak{i} + \mathfrak{J}) + \frac{1}{2}\kappa\,\mathfrak{i}$ does not represent a constant vector. Indeed, the magnitude of that vector will remain constant, but its direction must precess around the atomic axis with the aforementioned angular velocity. There is no simple way of determining that angular velocity. However, formula (64) shows that its *mean value* coincides with the usual *Larmor* velocity of the electron orbit in an external magnetic field $\bar{\mathfrak{H}}'$.

In the foregoing discussion, we have left the *internal* torque that was found in **§ 8**, Chap. VII, completely out of consideration. That torque (when it actually does exist) would represent the spatial part of a six-vector and should reduce to the form (63.a), Chap. VII, in the limiting case of small translational velocities. Now, it is easy to convince oneself that such a six-vector, with its components that are quadratic in $\mu_{\alpha\beta}$ and $\dot{x}_\gamma$, can indeed be constructed, but it must vanish identically due to the condition that $\mu_{\alpha\beta}\,\dot{x}_\beta = 0$.

That is why the existence of an internal torque seems to be incompatible with the theory of relativity.