PART TWO

# TIME-DEPENDENT ELECTROMAGNETIC EFFECTS

**CHAPTER FIVE**

# THE GENERAL LAWS OF THE ELECTROMAGNETIC FIELD

## § 1. – Electromagnetic induction in a time-constant field.

In Chapter II, we found that an electric charge $e$ (more precisely, an electrified particle) that moves in a time-constant magnetic field $\mathfrak{H}$ with the velocity $\mathfrak{v}$ will experience a force $\mathfrak{f} = \frac{e}{c}\mathfrak{v} \times \mathfrak{H}$. That *electrokinetic* or *electromagnetic force*, when referred to a positive unit of charge ($e = 1$) will then be equal to:

$$\mathfrak{F} = \frac{1}{c}\mathfrak{v} \times \mathfrak{H} \ . \tag{1}$$

Since the force $\mathfrak{f}$ is perpendicular to the velocity of the corresponding charged particle, it can *do no work* under the motion of the latter. However, the expression for $\mathfrak{f}$ above was derived from the change in the potential energy of a linear current under an infinitesimal (virtual) displacement of the current line since that change would be set equal to the work that electromagnetic forces did on the current elements. That apparent contradiction can be resolved as follows:

Under the motion of a current line $\sigma$, the velocity $\mathfrak{v}$ of the electric charges that are instantaneously found in the element $d\sigma$ is composed of *two* components: Its "relative" velocity $\mathfrak{v}'$ relative to $d\sigma$ and the "absolute" velocity $\mathfrak{v}''$ with which $d\sigma$ is displaced (parallel to itself). In Chapter II, when we derived the elementary electromagnetic force, we considered only the velocity $\mathfrak{v}'$, which determined the current strength $i$ according to the formula:

$$\sum \frac{e\mathfrak{v}'}{c} = i\ \tau\, d\sigma$$

(in which the summation extends over all charges that are found in $d\sigma$). That relative, or *longitudinal*, velocity corresponds to *transverse* electromagnetic forces $\mathfrak{f}' = \frac{e}{c}\mathfrak{v}' \times \mathfrak{H}$, whose sum will give the known expression:

$$\sum \mathfrak{f}' = \sum \frac{e}{c}\mathfrak{v}' \times \mathfrak{H} = \left(\sum \frac{e\mathfrak{v}'}{c}\right) \times \mathfrak{H} = i\ \tau\, d\sigma \times \mathfrak{H}$$

for the force that acts upon the element $d\sigma$ of a current line at rest.

As far as the forces $\mathfrak{f}'' = \frac{e}{c}\mathfrak{v}'' \times \mathfrak{H}$ are concerned, which are due to the convection of the charges along the current line element $d\sigma$, their resultant along that element must vanish (as long as the latter is electrically neutral) since the velocity $\mathfrak{v}''$ is the same for the positive and negative charges:

$$\sum \mathfrak{f}'' = \sum \frac{e}{c}\mathfrak{v}'' \times \mathfrak{H} = \left(\sum e\right)\left(\frac{\mathfrak{v}''}{c} \times \mathfrak{H}\right) = 0 .$$

Under a displacement $\mathfrak{v}\, d\tau$ of the current line element $d\sigma$, the transverse electromagnetic forces will do an amount of work equal to:

$$d\,A = \sum \mathfrak{f}'\mathfrak{v}''\, dt = (i\ \tau\, d\sigma \times \mathfrak{H})\ \mathfrak{v}''\, dt .$$

However, in that same time interval $dt$, the charges that were (originally) contained in $d\sigma$ will experience longitudinal displacements $\mathfrak{v}'\, dt$ that must be different for positive and negative charges (since otherwise the current strength would be equal to zero). Those longitudinal displacements correspond to a non-zero work $d\,B$ done by the forces $\mathfrak{f}''$ (which also point longitudinally, i.e., parallel to $d\sigma$, for perpendicular displacements of $d\sigma$.) Although the total force $\sum \mathfrak{f}'' = 0$, we will nonetheless get a non-zero expression for the resultant work done:

$$d\,B = \sum \mathfrak{f}'' \cdot \mathfrak{v}'\, dt = \sum \frac{e}{c}(\mathfrak{v}'' \times \mathfrak{H})\,\mathfrak{v}'\, dt = \frac{1}{c}[(\mathfrak{v}'' \times \mathfrak{H}) \sum e\,\mathfrak{v}']\, dt ,$$

i.e., from (1):

$$d\,B = (\mathfrak{F}'' \cdot c\ i\ \tau)\, d\sigma\, dt = c\, i\, F_\tau'' d\sigma\, dt , \qquad (2)$$

in which $\mathfrak{F}'' = \frac{1}{c}\mathfrak{v}'' \times \mathfrak{H}$, means the component of the electromagnetic force $\mathfrak{F}$ per unit charge that originates in the convective motion of $d\sigma$, and $F_\tau'' = F_\tau$ means its longitudinal projection (onto the direction of the current).

The sum of the works $dA$ and $dB$ must obviously be equal to the total work done by the electromagnetic forces $\mathfrak{f} = \mathfrak{f}' + \mathfrak{f}''$ that act upon the individual charges in $d\sigma$ under the corresponding total displacement $\mathfrak{v}\, dt = \mathfrak{v}'\, dt + \mathfrak{v}''\, dt$. Since the vectors $\mathfrak{f}$ and $\mathfrak{v}$ are perpendicular to each other, it will follow that $dA + dB = 0$.

That relation can be derived immediately from the equation $\mathfrak{f} \cdot \mathfrak{v} = 0$. Since:

$$\mathfrak{f} = \frac{1}{c}\mathfrak{v} \times \mathfrak{H} \quad \text{and} \quad \mathfrak{v} = \mathfrak{v}' + \mathfrak{v}'',$$

one has:

$$\mathfrak{f} \cdot \mathfrak{v} = \left[ \frac{e}{c} (\mathfrak{v}' + \mathfrak{v}'' \times \mathfrak{H}) \right] (\mathfrak{v}' + \mathfrak{v}'') = \frac{e}{c} (\mathfrak{v}' \times \mathfrak{H}) \mathfrak{v}'' + \frac{e}{c} (\mathfrak{v}'' \times \mathfrak{H}) \mathfrak{v}' = \mathfrak{f}'' \cdot \mathfrak{v}'' + \mathfrak{f}'' \cdot \mathfrak{v}',$$

and as a result, $\sum \mathfrak{f}'' \, \mathfrak{v}'' \, dt + \sum \mathfrak{f}'' \, \mathfrak{v}' \, dt = 0$, i.e.:

$$d A + d B = 0 .$$

The results obtained can be formulated as follows:

The motion of a current line in a time-constant magnetic field will generate a longitudinal electromagnetic field ($F_\tau$) that acts upon opposing charges in opposing directions, and the work that it does is equal and opposite to the work done by the transverse electromagnetic forces that act upon the elements of the current line.

The work done by *longitudinal* (or as one often calls them, "electromotive") forces along the entire current line $\sigma$ per unit time is equal to $c\, i \oint F_\tau \, d\sigma$. As far as the work done by the *transverse* (or "ponderomotive") forces is concerned, it can be measured by the decrease in potential energy $U$ of the current line in question (at constant current strength) relative to the external system that generates the field $\mathfrak{H}$. Since one has $U = - \, i \int H_n \, dS$, from (10), Chap. II, then since $\frac{dA}{dt} + \frac{dB}{dt} = 0$, it will follow that:

$$\oint F_\tau \, d\sigma = - \frac{1}{c} \frac{d}{dt} \int H_n \, dS . \tag{3}$$

That equation expresses the law of *electromagnetic induction.* The line integral $\oint F_\tau \, d\sigma$ is called the *electromotive force* that is induced by the motion of the current line in the magnetic field. It should be noted that the vector $\mathfrak{F}$ then plays the role of the ordinary electric field strength in that way. The integral $\int F_\tau \, d\sigma$ along a line that is not closed will then correspond to the potential difference between the endpoints of that line. For a closed line, it must be equal to zero when $\mathfrak{F}$ coincides with the field strength of a time-constant electric field.

The "electromotive force" (3) is completely independent of the current strength. The longitudinal motion of the electric charges that determines the current strength will generate the transverse component of $\mathfrak{F}$. By contrast, the longitudinal component $F_\tau$ is required by only the (transverse) "convectional motion" of the current line.

In the absence of a current ($i = 0$), the induced electromotive force tries to create one. In the general case ($i \neq 0$), it tries to produce a change in the current strength.

## § 2. – Electromagnetic induction in a time-varying magnetic field. The relativity principle.

Let the magnetic field that was considered in the previous section be created by a second linear current $i'(\sigma')$. We shall now show that the two current lines $\sigma$ and $\sigma'$ are coupled by a common translational motion such that they will remain at the same position relative to each other. Experiments show that in this case, *everything happens just as it would in the rest state:* Therefore, only the transverse electromagnetic ("ponderomotive") forces would remain active, while the longitudinal (electromotive) forces would not appear at all ([1]).

It will follow from this that those electromagnetic forces depend upon only the *relative* motion of the current lines in question relative to the other. We would like to refer to that law as the *relativity principle*. It should be pointed out that in that way, one deals with only the relativity of the simplest kinematical quantity, namely, the *velocity*. Indeed, when we determined the electromotive forces that were induced in $\sigma$ using formula (1), we have understood $\mathfrak{v}$ to mean, not the "absolute" velocity of the element of $\sigma$ but its relative velocity relative to a coordinate system that is fixed along the current line $\sigma'$ ([2]).

We must further assume that the magnetic field that is created by $\sigma'$ is carried along by the "absolute" motion of $\sigma'$ without any change to it, i.e., it is constant in time relative to the aforementioned coordinate system that is coupled with $\sigma'$ ([3]). Therefore, the magnetic field strength at a fixed point in space must change in time. In other words, if one considers the magnetic field from the standpoint of a coordinate system "at rest" then one must regard it as *time-varying*.

In the previous section, we regarded the current line $\sigma'$ as being at rest and $\sigma$ as moving. Since the inductive action that is exerted upon $\sigma$ according to the relativity principle depends upon the relative motion, we can switch the roles of the two current lines without altering that action in any way, i.e., we can regard $\sigma$ as being at rest and $\sigma'$ as being in motion (with the opposite velocity). Only the *interpretation* of the electromotive force (3) must change in that way. Indeed, in the case considered, it comes down to an *electric* field that is created by the motion of $\sigma'$, and in that way, the longitudinal component of that "induced" field $E_\tau$ should be identified with the corresponding component $\mathfrak{F}_\tau$ of our previous electromagnetic force per unit charge. The relative character of the velocity then corresponds to a peculiar *relativity of force:* The same force can be interpreted as either an electric or an electromagnetic one.

In what follows, we will denote the time variation of a scalar or vectorial quantity at a *fixed* point in space by the symbol $\frac{\partial}{\partial t}dt$, i.e., by a *partial* derivative with respect to *t*. By contrast, the symbol for a total derivative with respect to time *t*, which appears, e.g., in formula (3), shall refer to the case of a moving point or system. We then get the formula:

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([1]) We imagine, e.g., two coils, one of which is placed inside the other. If a constant electric current flows in one coil (e.g., the outer one) then one will get an induced current as a result of the motion of the inner coil. The same current would be generated by the opposite motion of the outer coil. One will not get any induced current from a common displacement of the two coils.

([2]) We will formulate the relativity principle more precisely later on in Chap. VIII.

([3]) Later on, we will see that this assumption is not fulfilled with complete rigor. It is true only in the limiting case of very small velocities.

$$\oint E_\tau \, d\sigma = -\frac{1}{c}\frac{\partial}{\partial t}\int H_n \, dS$$

or

$$\oint E_\tau \, d\sigma = -\int \frac{1}{c}\frac{\partial \mathfrak{H}}{\partial t}\mathfrak{n} \, dS\,, \tag{4}$$

which is equivalent to the formula (3).

That formula shows that the inductive electric field is required by the time variation of the magnetic field in question. The special circumstances upon which that variation might depend do not enter into (4) explicitly. For that reason, we can assume that those special circumstances are entirely irrelevant. Correspondingly, formula (4) must also be valid when the change in the magnetic field has its origin, not in the motion of one or more current lines at *constant current strength* (which was always assumed up to now), but in the *change in the current strength* in a current line that is either at rest or in motion. If the current strengths $i$ and $i'$ in $\sigma$ ($\sigma'$, resp.) are held constant in some way then the transverse electromagnetic forces that act upon $\sigma$ as a result of $\sigma'$ must do work $A = U_1 - U_2$ under a displacement $\sigma$ from the position (1) to another one (2), where:

$$U = -\,i\int H_n \, dS = -\,i'\int H'_{n'} \, dS' = U'$$

means the mutual potential energy of the two current lines [cf., (20.c), Chap. III]. However, as we saw in **§ 1**, the longitudinal electromagnetic forces that act upon $\sigma$ and require an electromotive force will do the opposite amount of work $B = U_2 - U_1$, such that the sum of the works done by the electromagnetic forces must remain equal to zero.

Furthermore, an electromotive force will be induced in $\sigma'$ due to the motion of $\sigma$ that comes from the time derivative of the magnetic field ($\mathfrak{H}'$) that is created by $\sigma$. According to the relativity principle that was just considered, that electromotive force of electrical origin coincides with the "kinetic" electromotive force that is generated in $\sigma'$ when $\sigma$ is at rest, but $\sigma'$ moves in the opposite way. The work done by that force (at constant field strength $i'$) must be equal to the (algebraic) increase in the potential energy of $\sigma'$ compared to $\sigma$, i.e., the difference $U'_2 - U'_1$, or since $U' = U$, $U_2 - U_1$. It will then follow from this that in the case in question (viz., $\sigma'$ is at rest and $\sigma$ is in motion), the two electromotive forces – viz., the kinetic one $V$ in $\sigma$ and the static one $V'$ in $\sigma'$ – must do the same work $U_2 - U_1$, i.e., their sum is twice that amount of work 2 ($U_2 - U_1$). If one adds that to the work done $A = U_1 - U_2$ on $\sigma$ by the transverse forces that act upon it (the corresponding work done on the current line $\sigma'$ is equal to zero since it is at rest) then one will get the work:

$$2\,(U_2 - U_1) + (U_1 - U_2) = U_2 - U_1 = -\,A$$

in total. That result can be easily generalized to the case in which the two current lines (which are mutually independent) move simultaneously: The work done by the transverse and longitudinal electromagnetic forces remains equal to zero in total. As far as the work done by the two induced electromagnetic forces of *electric* origin is concerned, which is due to the time variation of the

magnetic fields $\mathfrak{H}$ and $\mathfrak{H}'$, it is independent of the "absolute" motion of $\sigma'$ and $\sigma$, so it will always be equal to the increase in the potential energy $U_2 - U_1$.

The energy in such a system will generally be the quantity whose algebraic *decrease* is equal to the work done between by the forces that act between the parts of the system under a transition of that system from one position to another (assuming that the work done us independent of the path taken during that transition). If one would then like to attribute the *total* work that is done by the forces that are active in a system of linear electric currents to an energy function then that function $T$ must be defined in such a way that one has $T_1 - T_2 = U_2 - U_1$. If one establishes that the energy $T$ vanishes when the current lines are separated by an infinite distance, just like $U$, then that will give the following simple relation:

$$T = -U . \tag{5}$$

If one recalls once more that the potential energy $U$ corresponds to only the transverse electromagnetic (i.e., "pondermotive") forces that act upon the elements of the two current lines, while $T$ plays the same role for static (electric) forces of induction that are generated by the time variation of the magnetic fields. The work done by the kinetic forces of induction will be compensated precisely by the work done by the aforementioned transverse forces, and that is why it will remain out of consideration.

We would like to call the quantity $T$ the *electrokinetic* or simply the *magnetic* energy of the current in question. One can also define it to be the work that must be expended (at the expense of any external energy sources) in order to overcome the induced electric forces so that the current strengths would remain constant when the two current lines went to their (relative) configuration considered after starting out at an infinite distance from each other.

Since that work depends upon only the relative motion of the two current lines, one can imagine that one of them, e.g., $\sigma$, is at rest, and that as a result the work done $T$ will be expended completely on that current line (whereas only electrokinetic forces "that do no work" will act upon the moving current line). Moreover, since the induced electromotive force on $\sigma$ is determined completely by the time variation of the corresponding magnetic flux $\int H_n \, dS$ (that emanates from $\sigma'$), and is entirely independent of the special circumstances that caused that variation, we can replace the motion of $\sigma'$ under fixed current strength with a gradual increase in the current strength from 0 to $i'$ while the position is fixed. In both cases, the work that must be expended in order to secure the constancy of the corresponding field strength $i$ in the face of the electromotive force that is induced in $\sigma$ is given by the same quantity $T$. It can also be obtained reversing the roles of the current lines $\sigma$ and $\sigma'$, and furthermore, it is easy to see that it can be obtained in the most-general case when the current strengths in $\sigma$ and $\sigma'$ grow from zero to $i$ ($i'$, resp.) simultaneously. In that way, they can move in a completely arbitrary way, and must arrive at the (relative) configuration in question only at the end of the entire process.

In summation, we can then say that the work that is done by the electric and electromagnetic forces of interaction under an arbitrary change in the configuration of two current lines and the corresponding current strengths is always equal to the (algebraic) decrease in the magnetic energy $T$. It is determined completely by the values of $T$ in the initial and final states and is independent of the intermediary states.

## § 3. – Maxwell's fundamental equations for time-alternating electromagnetic fields.

Obviously, formula (4) must also remain valid when no current circulates in the curve $\sigma$, i.e., one can apply it to any closed curve and the surface that is bounded by the latter. If one transforms the line integral $\oint E_\tau \, d\sigma$ into the surface integral $\int \mathfrak{n}$ rot $\mathfrak{E}\, dS$ using *Stokes's* formula then one will have:

$$\int \mathfrak{n} \operatorname{rot} \mathfrak{E}\, dS = -\int \frac{1}{c}\frac{\partial \mathfrak{H}}{\partial t}\mathfrak{n}\, dS\,.$$

Due to the arbitrariness in the surface *S*, it will then follow that:

$$\operatorname{rot} \mathfrak{E} = -\frac{1}{c}\frac{\partial \mathfrak{H}}{\partial t}\,. \tag{6}$$

That differential equation is the generalization of the equation rot $\mathfrak{E} = 0$ that we exhibited in Chap. I on the basis of the energy principle, and up to now we considered it to be the analytical expression of that principle. We then see that in the general case of time-alternating fields, the energy principle in its usual form will lose its validity, at least for the isolated electric field.

As far as the magnetic field is concerned, one can also adapt the corresponding differential equation div $\mathfrak{H} = 0$ to the general case in question. That already follows from the fact that the energy principle also remains true for alternating currents, although in a somewhat-altered form (viz., electrokinetic energy *T* instead of potential *U*). However, our assertion can be proved with full rigor on the basis of equation (6). That is because when one applies the operation div to the two sides of that equation, that will give (since div rot ≡ 0) div $\dfrac{\partial \mathfrak{H}}{\partial t} = 0$, i.e., $\dfrac{\partial}{\partial t}$ div $\mathfrak{H} = 0$, and as a result div $\mathfrak{H}$ = const. Now, it is always possible to imagine that the magnetic field in its initial or final state remains constant along a time interval. However, one would then need to have div $\mathfrak{H} = 0$. One can then conclude that the equation that was derived from the energy principle for the special case of time-constant fields:

$$\operatorname{div} \mathfrak{H} = 0 \tag{7}$$

is true in general.

In Chap. III, we exhibited the equations div $\mathfrak{E} = 4\pi\rho$ and rot $\mathfrak{H} = 4\pi \mathfrak{j}$ by combining the energy and equivalence principles. The identity div $\mathfrak{j} = 0$ follows from the second of those equations, and in conjunction with the principle of the conservation of electricity, it represents the condition of stationarity of the electrical current that generates the field $\mathfrak{H}$ [Chap. II, formula (4.a)]. Therefore, the equation rot $\mathfrak{H} = 4\pi \mathfrak{j}$ cannot be true in the general case of a non-stationary electrical current since the divergence on its left-hand side will vanish, while the divergence of $\mathfrak{j}$ is non-zero. According to (4), Chap. III (with $\mathfrak{J} = c\,\mathfrak{j}$), the principle of the conservation of electricity is expressed by the equation:

$$\text{div } \mathfrak{j} + \frac{1}{c}\frac{\partial \rho}{\partial t} = 0 \,. \tag{8}$$

On the other hand, the desired generalization of the equation rot $\mathfrak{H} = 4\pi\,\mathfrak{j}$ to the case of time-alternating fields must obviously have the form:

$$\text{rot } \mathfrak{H} = 4\pi\,\mathfrak{j} + \frac{\partial \mathfrak{G}}{\partial t}, \tag{8.a}$$

in which $\mathfrak{G}$ means an initially-undetermined vector that depends upon the distribution of the electrical charges. Upon comparing that equation with (8), we will get:

$$\text{div } \frac{\partial \mathfrak{G}}{\partial t} = \frac{\partial}{\partial t}\,\text{div } \mathfrak{G} = \frac{4\pi}{c}\frac{\partial \rho}{\partial t}, \qquad \text{i.e.,} \qquad \text{div } \mathfrak{G} = \frac{4\pi\,\rho}{c}\,.$$

If we then assume that the first of the aforementioned equations:

$$\text{div } \mathfrak{E} = 4\pi\,\rho\,, \tag{9}$$

just like the corresponding equation for the magnetic field (div $\mathfrak{H} = 0$), is true in general then we must set the vector $\mathfrak{G}$ equal to $\frac{1}{c}\mathfrak{E}$. In that way, (8.a) will assume the following form:

$$\text{rot } \mathfrak{H} = \frac{1}{c}\frac{\partial \mathfrak{E}}{\partial t} + 4\pi\,\mathfrak{j}\,. \tag{10}$$

It should be noted that in "empty space," i.e., for $\rho = 0$ and $\mathfrak{j} = 0$, equations (9) and (10) will be completely analogous to equations (7) and (6), with the single difference that the time derivatives of $\mathfrak{H}$ in (6) and $\mathfrak{E}$ in (10) have opposite signs.

We then see that an alternating electric field will induce a magnetic field in the same way that an electric field will be induced by an alternating magnetic field, but the induced fields will be opposite when the inducing fields are equal. We get the following expression for the total flux of the vector rot $\mathfrak{H}$ through an open surface $S$ from (10):

$$\int \text{rot}_n\, \mathfrak{H}\, dS = \frac{1}{c}\frac{\partial}{\partial t}\int E_n\, dS + 4\pi\, i\,,$$

in which $i = \int j_n\, dS$ means the strength of the electrical current that flows through $S$, or upon applying the *Stokes* transformation formula to the integral on the left-hand side:

$$\oint H_\tau \, d\sigma = \frac{1}{c}\frac{\partial}{\partial t}\int E_n \, dS + 4\pi i \,. \tag{10.a}$$

That formula shows that the time derivative of the "electrical flux" ($\int E_n \, dS$) that flows through $S$, divided by $4\pi c$, plays the same role with respect to the creation of the magnetic field as the corresponding current strength ([1]).

If one represents the vectors $\frac{\partial \mathfrak{E}}{\partial t}$ and $\frac{\partial \mathfrak{H}}{\partial t}$ graphically by a family of parallel lines then the induced fields $\mathfrak{H}$ ($\mathfrak{E}$, resp.) can be represented by ring-like lines of force that surround the aforementioned families of lines, and indeed in the positive sense in the first case and in the negative sense in the negative one (Fig. 29.a,b).

Along with the electric charge and current density ($\rho$, $\mathfrak{j}$), one can introduce (in an entirely formal way) the corresponding *magnetic* quantities $\rho^*$, $\mathfrak{j}^*$, which must be coupled with each other by the "conservation equation":

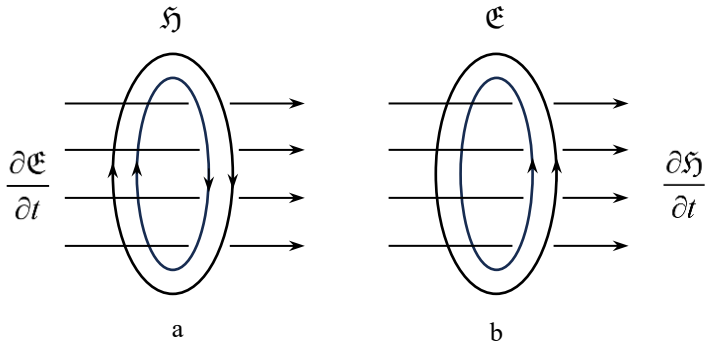

Figure 29.

$$\operatorname{div} \mathfrak{j}^* + \frac{1}{c}\frac{\partial \rho^*}{\partial t} = 0 \,. \tag{11}$$

(That equation is actually fulfilled since $\mathfrak{j}^* = 0$ and $\rho^* = 0$.) Equations (7) and (6) would then assume a form that is entirely analogous to that of (7) and (10), and indeed:

$$\operatorname{div} \mathfrak{H} = 4\pi \rho^* \,, \tag{11.a}$$

$$\operatorname{rot} \mathfrak{E} = -\frac{1}{c}\frac{\partial \mathfrak{H}}{\partial t} - 4\pi \mathfrak{j}^* \,. \tag{11.b}$$

Along with the total force:

$$\mathfrak{f} = e\left(\mathfrak{E} + \frac{1}{c}[\mathfrak{v} \times \mathfrak{H}]\right) \tag{12}$$

that acts upon point-charge $e$ that moves in the electromagnetic field $\mathfrak{E}$, $\mathfrak{H}$ with the velocity $\mathfrak{v}$, one accordingly introduces an analogous (fictitious) force:

([1]) That fact was discovered by *Maxwell*. In so doing, *Maxwell* interpreted the vector $\mathfrak{E}$ as an elastic displacement in the ether and correspondingly interpreted the integral $\frac{1}{4\pi c}\frac{\partial}{\partial t}\int E_n \, dS$ as a "displacement current."

$$\mathfrak{f}^* = e^* \left( \mathfrak{H} - \frac{1}{c} [\mathfrak{v} \times \mathfrak{E}] \right) \tag{12.a}$$

that is supposed to act upon a moving magnetic pole $e^*$.

Those (obviously entirely fictitious) concepts and quantities are often very convenient for the practical calculation of the fields that are created by given electrical motions.

## § 4. – The differential equations for the electromagnetic potentials.

The general differential equations of the electromagnetic field can be summarized in the following two groups ([1]):

$$\left.\begin{aligned} \operatorname{div} \mathfrak{H} &= 0, \\ \operatorname{rot} \mathfrak{E} + \frac{1}{c} \frac{\partial \mathfrak{H}}{\partial t} &= 0, \end{aligned}\right\} \tag{I}$$

and

$$\left.\begin{aligned} \operatorname{div} \mathfrak{E} &= 4\pi \rho, \\ \operatorname{rot} \mathfrak{H} - \frac{1}{c} \frac{\partial \mathfrak{E}}{\partial t} &= 4\pi \mathfrak{j}. \end{aligned}\right\} \tag{II}$$

We shall now turn to the consideration of equations (I). It follows from the first of them that the magnetic field is source-free (or solenoidal), even in the general case, such that $\mathfrak{H}$ can be calculated from a vectorial or magnetic potential $\mathfrak{A}$ according to the formula:

$$\mathfrak{H} = \operatorname{rot} \mathfrak{A} , \tag{13}$$

just as it could in the special case of electrical current. If one substitutes that expression in the equation:

$$\operatorname{rot} \mathfrak{E} + \frac{1}{c} \frac{\partial \mathfrak{H}}{\partial t} = 0$$

then one will have:

$$\operatorname{rot} \mathfrak{E} + \frac{1}{c} \frac{\partial}{\partial t} \operatorname{rot} \mathfrak{A} = \operatorname{rot} \left( \mathfrak{E} + \frac{1}{c} \frac{\partial \mathfrak{A}}{\partial t} \right) = 0 .$$

It follows from that equation that the sum $\mathfrak{E} + \frac{1}{c} \frac{\partial \mathfrak{A}}{\partial t}$ represents a vortex-free vector, i.e., it is equal to the gradient of a scalar quantity – $\varphi$. One can then set:

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([1]) Those equations were first exhibited by *J. C. Maxwell*, and in their ultimate form by *H. A. Lorentz*. For that reason, one ordinarily refers to them as the *Maxwell-Lorentz* fundamental equations.

$$\mathfrak{E} = -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t}. \tag{14}$$

For constant fields, that formula reduces to the formula $\mathfrak{E} = -\operatorname{grad}\varphi$ that was derived in Chap. I, such that $\varphi$ agrees completely with the previously-introduced scalar or electric potential.

If one substitutes the expressions (13) and (14) in equations (II) then that will give:

$$\operatorname{div}\mathfrak{E} \equiv -\nabla^2\varphi - \frac{1}{c}\frac{\partial}{\partial t}\operatorname{div}\mathfrak{A} = 4\pi\rho \tag{14.a}$$

and

$$\operatorname{rot}\mathfrak{H} - \frac{1}{c}\frac{\partial \mathfrak{E}}{\partial t} \equiv \operatorname{rot}\operatorname{rot}\mathfrak{A} + \operatorname{grad}\frac{1}{c}\frac{\partial\varphi}{\partial t} + \frac{1}{c^2}\frac{\partial^2\mathfrak{A}}{\partial t^2} = 4\pi\mathfrak{j},$$

or from the identity:

$$\operatorname{rot}\operatorname{rot}\mathfrak{A} = \operatorname{grad}\operatorname{div}\mathfrak{A} - \nabla^2\mathfrak{A},$$

$$-\nabla^2\mathfrak{A} + \operatorname{grad}\left(\operatorname{div}\mathfrak{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\right) + \frac{1}{c^2}\frac{\partial^2\mathfrak{A}}{\partial t^2} = 4\pi\mathfrak{j}. \tag{14.b}$$

As we have seen before in Chap. II in our consideration of time-constant electromagnetic fields, the vector function $\mathfrak{A}$ that was introduced in (13) is not determined completely. We previously removed that indeterminacy by means of the equation [(12.a), Chap. II]:

$$\operatorname{div}\mathfrak{A} = 0.$$

The relation:

$$\operatorname{div}\mathfrak{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0 \tag{15}$$

would now seem to be a reasonable generalization of that equation, which is completely-analogous to the relation (8) between the right-hand sides of (14.a) and (14.b). That would then make it possible to separate the two unknown functions $\varphi$ and $\mathfrak{A}$ in (14.a) and (14.b) from each other. In fact, if one considers (15) then (14.a) and (14.b) will reduce to two equations *of the same form:*

$$-\nabla^2\varphi + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 4\pi\rho, \tag{16}$$

$$-\nabla^2\mathfrak{A} + \frac{1}{c^2}\frac{\partial^2\mathfrak{A}}{\partial t^2} = 4\pi\mathfrak{j}, \tag{17}$$

which can be regarded as the generalization of equations (8) and (9), Chap. III.

In the case where the positive and negative charges are coupled with each other into the smallest neutral particle (viz., molecules, as in, e.g., dielectrics), the current density can be

represented as in (2), Chap. II as the time derivative of the polarization vector $\mathfrak{P}$, which measures the electric moment per unit volume of the body in question. With the use of the electrokinetic units for the current density [$\mathfrak{j} = (1/c)\,\mathfrak{J}$], one will then have:

$$\mathfrak{j} = \frac{1}{c}\frac{\partial \mathfrak{P}}{\partial t}. \tag{18}$$

Upon substituting that expression in equation (8), the latter will assume the form $\operatorname{div}\frac{1}{c}\frac{\partial \mathfrak{P}}{\partial t} + \frac{1}{c}\frac{\partial \rho}{\partial t} = 0$, i.e.:

$$\frac{1}{c}\frac{\partial}{\partial t}(\operatorname{div}\mathfrak{P} + \rho) = 0\,.$$

It follows from this that the volume density of the electric charge can be represented by means of the polarization vector according to the formula:

$$\rho = -\operatorname{div}\mathfrak{P} \tag{18.a}$$

that we already derived in **§ 10**, Chap. III, in a different way. It should be pointed out that formulas (18) and (19) are not only valid for the case of bound charges, they can also be applied to the case of charges in an arbitrary state of motion. However, in that way, the vector $\mathfrak{P}$ will lose the aforementioned intuitive physical sense and would become merely an auxiliary quantity that is defined by (18) (cf., **10**, Chap. III).

Just as $\mathfrak{j}$ and $\rho$ can both be represented by means of that quantity, one can express the corresponding potentials $\mathfrak{A}$ and $\varphi$ in terms of one and the same quantity $\mathfrak{Z}$ as a result of the relation (15), and that quantity will have the same relation to the vector $\mathfrak{P}$ that $\mathfrak{A}$ has to $\mathfrak{j}$. In analogy with (18) and (19), one sets:

$$\mathfrak{A} = \frac{1}{c}\frac{\partial}{\partial t}\mathfrak{Z} \tag{19}$$

and

$$\varphi = -\operatorname{div}\mathfrak{Z}\,. \tag{19.a}$$

Upon substituting those expressions in equations (16) and (17), we will get:

$$-\nabla^2(-\operatorname{div}\mathfrak{Z}) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(-\operatorname{div}\mathfrak{Z}) = -4\pi\operatorname{div}\mathfrak{P}$$

and

$$\frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla^2\mathfrak{Z} + \frac{1}{c^2}\frac{\partial^2\mathfrak{Z}}{\partial t^2}\right) = \frac{1}{c}\frac{\partial}{\partial t}4\pi\mathfrak{P}\,.$$

It will then follow that the vector $\mathfrak{Z}$ (viz., the so-called *Hertz vector* or *electric polarization potential*) satisfies the equation:

$$-\nabla^2 \mathfrak{Z} + \frac{1}{c^2}\frac{\partial^2 \mathfrak{Z}}{\partial t^2} = 4\pi \mathfrak{P} \,. \tag{20}$$

The form of that equation coincides with that of equations (16) and (17) for the electric and magnetic potential and represents the generalization of equation (36), Chap. III.

We have seen that in the case of a stationary electric current, the current density can be expressed in terms of the "magnetic polarization vector" $\mathfrak{M}$ and the vector potential can be expressed in terms of the magnetic polarization potential $\mathfrak{Z}^*$ by using the formulas $\mathfrak{j} = \text{rot}\, \mathfrak{M}$ and $\mathfrak{A} = \text{rot}\, \mathfrak{Z}^*$. Obviously, those formulas cannot be adapted to the general case. However, they can be combined with the foregoing ones, and by defining $\mathfrak{j}$ to be the sum:

$$\mathfrak{j} = \text{rot}\, \mathfrak{M} + \frac{1}{c}\frac{\partial \mathfrak{P}}{\partial t} \,. \tag{21}$$

That will correspondingly give $\mathfrak{A}$ as the sum:

$$\mathfrak{A} = \text{rot}\, \mathfrak{Z}^* + \frac{1}{c}\frac{\partial \mathfrak{Z}}{\partial t} \,. \tag{21.a}$$

The original definition of the electric polarization $\mathfrak{P}$ will clearly be altered somewhat by that. However, one can employ that fact in order to replace the actual system with a different one in such a way that the electromagnetic field can be determined as easily as possible in the spatial region considered. It follows from (21) and (21.a), in conjunction with the previous formulas, that:

$$-\nabla^2 \mathfrak{Z}^* + \frac{1}{c^2}\frac{\partial^2 \mathfrak{Z}^*}{\partial t^2} = 4\pi \mathfrak{M} \,, \tag{22}$$

i.e., an equation with the same form as (20).

One refers to such equations as *d'Alembertian* equations. They represent the generalization of the *Laplace* equations for time-varying fields.

## § 5. – Integrating the foregoing differential equations. Retarded potentials.

We would first like to perform the integration of equation (16) for the case in which the electric charge density $\rho$ is zero outside a certain well-defined point $P$. By contrast, we imagine that a charge $e$ of finite and *time-varying* magnitude is concentrated at that point. Therefore, let $e$ be an arbitrary function of time:

$$e = e\,(t)\,.$$

That picture is physically meaningless since it contradicts the principle of the conservation of electricity. However, it has the advantage of the greatest-possible simplicity from the mathematical standpoint. As we will see below, we can easily construct the general solution to equation (17) by means of the corresponding special solution, and in that way, the principle of conservation of electricity will once more come into its own.

We then assume that the equation:

$$-\nabla^2\varphi + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0 \tag{23}$$

is true in all of space, except at the point $P'$, where the right-hand side will become infinite. On the grounds of symmetry, the potential $\varphi$ must have the same value at the same time for all points that are at an equal distance from $P'$. In other words, one can make the Ansatz for $\varphi$:

$$\varphi\,(\mathfrak{r}, t) = \varphi\,(R, t)\,,$$

where $R$ means the distance from the reference point $P$ in question to $P'$ ($P'P = \mathfrak{R} = \mathfrak{r} - \mathfrak{r}'$; as usual, $\mathfrak{r}$ and $\mathfrak{r}'$ are the radius vectors from $P$ and $P'$ relative to any fixed point $O$). From (18), Chap. III, the operation $\nabla^2\varphi$ will then reduce to $\frac{1}{R}\frac{d^2}{dR^2}(R\varphi)$ such that equation (23) will assume the following form:

$$-\frac{1}{R}\frac{\partial^2(R\varphi)}{\partial R^2} + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0\,,$$

or due to the independence of the two variables $R$ and $t$ with respect to each other:

$$-\frac{\partial^2(R\varphi)}{\partial R^2} + \frac{1}{c^2}\frac{\partial^2(R\varphi)}{\partial t^2} = 0\,. \tag{23.a}$$

We introduce the new variable $\tau = R / c$ in place of $R$ and denote the product $R\,\varphi$ by $f$. In that way, (23.a) can be written as follows:

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial \tau^2} = 0\,. \tag{23.b}$$

That equation will be satisfied when one replaces $f$ with any function of the sum $t + \tau$ or the difference $t - \tau$. It will then follow that the most-general solution of (23.b) can be represented in the form:

$$f\,(t,\,\tau) = f_1\,(t - \tau) + f_2\,(t + \tau)\,, \tag{24}$$

in which $f_1(\xi)$ and $f_2(\eta)$ mean two arbitrary functions of the corresponding arguments.

A more rigorous proof of that assertion can be obtained by transforming (23.b) to the new variables:

$$\xi = t - \tau \qquad \text{and} \qquad \eta = t + \tau .$$

Due to the fact that $t = \frac{1}{2}(\xi + \eta)$ and $\tau = \frac{1}{2}(\xi - \eta)$ one will have:

$$\frac{\partial}{\partial \xi} = \frac{1}{2}\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial \eta} = \frac{1}{2}\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial}{\partial \tau},$$

so

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \tau^2} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\right) = 4\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} ,$$

and as a result, from (23.b):

$$\frac{\partial^2 f}{\partial \xi \, \partial \eta} = 0 .$$

The derivative $\partial f / \partial \eta$ is then independent of $\xi$, but it can be an entirely-arbitrary function of $\eta$. Conversely, $\partial f / \partial \eta$ is an arbitrary function of $\xi$ in which $\eta$ does not appear at all. As a result, the function $f$ must be representable as the sum of two arbitrary functions of the individual variables $\xi$ ($\eta$, resp.).

We initially assume that $f_2 = 0$. We will then get the following formula for the potential $\varphi(R, t)$:

$$\varphi = \frac{f_1(t - R/c)}{R} . \tag{24.a}$$

In order to determine the function $f_1$, we observe the fact that equation (23.a) will reduce to the equation for the ordinary *Coulomb* potential that was treated before in Chap. III for $c = \infty$. One must then have:

$$\varphi = \frac{e}{R}$$

in this case, in which $e$ means the quantity of the charge that is concentrated at the point $P'$. In the derivation of that formula in Chap. III, we assumed that $e$ was a constant quantity. However, it is clear that it will still remain valid under the condition that $c = \infty$ when the charge $e$ changes arbitrarily with time. In that way, the value of the potential $\varphi$ at any time-point $t$ must obviously be determined by the *simultaneous* value of the function $e(t)$.

We then see that for $c = \infty$, formula (24.a) must assume the form $\varphi(R, t) = e(t) / R$. We then get from this that $f_1(t - R/c) = e(t - R/c)$, and as a result:

$$\varphi(R, t) = \frac{e(t - R/c)}{R}. \tag{25}$$

That formula shows that the value of $\varphi$ at the points that are at a distance $R$ from the charge $e$ (i.e., they lie on a spherical surface with the radius $R$ and a center at $P'$) and at the moment $t$ are *not* determined by the simultaneous magnitude of that charge, but by its magnitude at the previous moment $t' = t - R/c$.

We must the exhibit the fact that the effect of $e$ at the point $P'$ does not spread instantaneously through the surrounding spatial points, but with a certain *delay* that is directly proportional to its distance to $P'$. In other words, that effect propagates in empty space in the form of a spherical wave, just like the circular waves that are produced on the surface of water by a local perturbation, or even better, like the spherical sound waves in air.

The speed of propagation is obviously equal to $c$, i.e., the quantity $c = 3 \times 10^{10}$ cm/s, which we initially defined to be the ratio of the electromagnetic unit of current strength to the electrostatic one, and later (**§ 6**, Chap. III) to be the "critical speed." We can then regard that critical speed as the speed of propagation of electrical effects in space. It should be recalled once more that it agrees *precisely* with the experimentally-measured speed of light.

For the reasons that were explained above, one refers to the potential that is determined by formula (25) as the delayed or *retarded potential*.

It should be pointed out that this formula is not a *mathematically-necessary* consequence of the original differential equation (23.a). One could just as well set $f_1 = 0$ as $f_2 = 0$ in the general solution of that equation (24), which would give the formula for $\varphi$:

$$\varphi(R, t) = \frac{1}{R} e(t + R/c), \tag{25.a}$$

instead of (25). The most-general form of the solution (24) that is compatible with the physical constraints on our problem, i.e., the one that goes to the usual *Coulomb* potential $\varphi(R, t) = e(t)/R$ when $c = \infty$, is obviously a linear combination of (25) and (25.a):

$$\varphi(R, t) = \frac{1}{R}[\alpha' e(t - R/c) + \alpha'' e(t + R/c)], \tag{25.b}$$

in which $\alpha'$ and $\alpha''$ are two numerical coefficients whose sum is equal to 1. One of those coefficients can then be chosen to be completely arbitrary.

However, one actually sets $\alpha' = 1$, and as a result $\alpha'' = 0$, i.e., one drops the second particular solution (25.a), and indeed because the corresponding *advanced* potential seems to be *physically meaningless*. That is because equation (25.a) says that the effect that would be produced by the charge $e$ at the point $P$ at a moment $t$ would be determined by the magnitude of that charge at a *later* moment $t'' = t + R/c$. In other words, the effect that $e$ produces must be combined with the corresponding point $P'$ according to (25.a), instead of spreading away from it in the sense of formula (25).

Such an "advanced" effect not only seems physically impossible, but also logically inconceivable since it would mean that the cause would follow after the effect, instead of preceding that effect, which would correspond to the usual conception of causality principle.

However, a simple argument will show that in reality the usual conception of things is illusory. In fact, from the standpoint of classical mechanics, which is connected with the picture of an "instantaneous" action-at-a-distance (i.e., instantaneous transfer of force), the *acceleration* of any material particle will depend upon the *simultaneous positions* of all other particles that produce the effect in question. The concept that the motion is delayed somewhat from the force is based upon the fact that we do not perceive the motion by the acceleration, but by the velocity, or rather the corresponding displacement. However, in order to notice a change in velocity or a change of place, one must wait during a certain time interval.

We then see that *causa* and *effectum* are considered to be *simultaneous* in classical mechanics. If that simultaneity were replaced by a delayed force effect, so the temporal unity of cause and effect would be perturbed, then it would not seem impossible to also assume that there would be an advanced force effect. Nonetheless, if there seem to exist definitive grounds for rejecting an advanced force effect then they are not of a logical nature, but of an empirical one: Namely, such an effect would be rejected by the well-known phenomena of the propagation of light (see below).

We must then throw out the solution (25.a) and express the effect that is produced by a charge that is at rest, but time-varying, in terms of the retarded potential (25).

The transition from that simple, but purely fictitious, case to the general case of an arbitrary distribution and current of electricity that is supposed to satisfy the conservation principle can now be easily completed. Due to the linearity of the general equation (16) relative to $\varphi$ and $\rho$, one can represent $\varphi$ as the sum of the elementary potentials $d\varphi$ that are generated by the infinitesimal charges $de' = \rho' dV'$ that are concentrated in the individual volume elements $dV'$. Let $\mathfrak{r}'$ be the radius vector from $dV'$ to any point from $(P')$, and let $\mathfrak{r}$ be the radius vector to the "reference point" $P$ (for which $\varphi$ is to be determined). For finite magnitudes of $P'P = R = |\, \mathfrak{r} - \mathfrak{r}' \,|$ or (what amounts to the same thing) for infinitely-small dimensions of the volume element $dV'$ in comparison to $R$, one can treat the charge $de'$ as a point-charge (whose time variation is compensated for by the corresponding change in the charges of the neighboring volume element) and correspondingly set:

$$d\varphi(\mathfrak{r}, t) = \frac{1}{R}\rho(\mathfrak{r}', t - R/c)\, dV',$$

as in (25). It will then follow that:

$$\varphi(\mathfrak{r}, t) = \int \frac{1}{R}\rho(\mathfrak{r}', t - R/c)\, dV', \tag{26}$$

in which the integration extends over all of space, or really all points $\mathfrak{r}'$ in space that are "charged" at the corresponding "effective" time-point:

$$t' = t - R/c\,. \tag{26.a}$$

In the special case for which $\rho$ is constant in time (electric charges at rest), (26) will reduce to the formula (17.a), Chap. III, for the ordinary *Coulomb* potential.

Formula (26) can be regarded as the solution to the differential equation (16). If follows from the formal identity of that equation with (17) that the solution to the latter must be obtained from (26) by replacing the scalar $\rho$ with the vector j. The vector potential that is created by the electrical current that is determined by the function $\mathfrak{j}'(\mathfrak{r}', t')$ is then expressed by the formula:

$$\mathfrak{A}\,(\mathfrak{r},\, t) = \int \frac{1}{R}\,\mathfrak{j}(\mathfrak{r}', t - R/c)\,dV'\,. \tag{27}$$

Now, it is easy to see that the potential that is defined by (26) and (27) must satisfy the condition (15), as along as the corresponding condition (8) for $\rho$ and j is fulfilled, i.e., the principle of the conservation of electricity.

That is because when one differentiates (26) with respect to *c t* then, since $\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}$ ($t' = t - R/c$), one will have:

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} = \int \frac{1}{R}\frac{1}{c}\frac{\partial}{\partial t'}\rho(\mathfrak{r}', t - R/c)\,dV'\,,$$

and furthermore, with the abbreviation $\mathfrak{j}' = \mathfrak{j}(\mathfrak{r}', t')$:

$$\operatorname{div}\mathfrak{A} = \int \operatorname{div}\frac{\mathfrak{j}'}{R}\,dV' = \int\left(\mathfrak{j}'\operatorname{grad}\frac{1}{R} + \frac{1}{R}\operatorname{div}\mathfrak{j}'\right)dV'\,.$$

Therefore, according to formula (28.a) in the Introduction:

$$\operatorname{div}\mathfrak{j}' = \frac{\partial \mathfrak{i}'}{\partial t'}\operatorname{grad} t' = -\frac{1}{c}\frac{\partial \mathfrak{i}'}{\partial t'}\operatorname{grad} R\,.$$

We now replace the differentiation with respect to $\mathfrak{r}$ with the corresponding differentiation with respect to the radius vector $\mathfrak{r}'$, which plays the role of the independent variable in the foregoing integral. Due to the fact that $\nabla = -\nabla'$ (i.e., grad = – grad′, etc.), that will give:

$$-\frac{1}{c}\frac{\partial \mathfrak{i}'}{\partial t'}\operatorname{grad} R = +\frac{1}{c}\frac{\partial \mathfrak{i}'}{\partial t'}\operatorname{grad}' R = -\{\operatorname{div}'\mathfrak{j}' - (\operatorname{div}'\mathfrak{j}')_{t'=\text{const.}}\}\,,$$

where $(\operatorname{div}'\mathfrak{j}')_{t'=\text{const.}}$ corresponds to a fixed value, and as a result:

$$\operatorname{div}\mathfrak{A} = \int\left\{-\operatorname{div}'\frac{\mathfrak{j}'}{R} + \frac{1}{R}(\operatorname{div}'\mathfrak{j}')_{t'=\text{const.}}\right\}dV'\,.$$

Now, from *Gauss's* formula, one has:

$$\int \operatorname{div}' \frac{\mathfrak{j}'}{R}\, dV' = \int \frac{1}{R}\, j_n'\, dS' .$$

The vector $\mathfrak{j}'$ (or at least its normal component) must vanish on the surface $S'$ that bounds the entire volume in which the electrical current is found (at the corresponding time). One must then have $\int \operatorname{div}' \frac{\mathfrak{j}'}{R}\, dV' = 0$, and as a result:

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} + \operatorname{div} \mathfrak{A} = \int \left\{ \frac{1}{c}\frac{\partial \rho'}{\partial t'} + (\operatorname{div}' \mathfrak{j}')_{t'=\text{const.}} \right\} \frac{dV'}{R} = 0 ,$$

from (8).

The integrals (26) and (27) determine the total effect that will be produced at the point considered in space and time $\mathfrak{r}$, $t$ by all of the charges that are present in the surrounding space. The effect, which converges to $P$ from all directions, can be pictured geometrically by a spherical surface that contracts to its center $P$ with a velocity of $c$ such that its radius $R$ will be equal to zero at the instant $t$. Such a spherical surface that converges to a certain reference point corresponds to the diverging "spherical waves" that spread from any source point, i.e., any charged spatial point at the moment in question, with the same velocity $c$.

The integrals (26) and (27) are composed of the elementary contributions that are, so-to-speak, "grabbed" by the "sphere of action" of the corresponding volume element that converges to the reference point, divided by the respective radius of that sphere.

We consider a cone with an infinitely-small opening angle $d\Omega$ and its vertex at the reference point $P$. In the infinitely-small time interval between the moments $t' = t - R/c$ and $t' + dt' = t - (R + dR)/c$, the volume element $dV = R^2 d\Omega\, dR$ will be cut out of that cone by the sphere of action that converges to $P$. Formulas (26) and (27) can correspondingly be written in the form:

$$\left.\begin{aligned} \varphi &= \int d\Omega \int_0^\infty \rho(\mathfrak{r}', t - R/c)\, R\, dR, \\ \mathfrak{A} &= \int d\Omega \int_0^\infty \mathfrak{j}(\mathfrak{r}', t - R/c)\, R\, dR. \end{aligned}\right\} \qquad (27.a)$$

It should be pointed out that the formulas that were obtained will not yield a *complete* solution to the differential equations (16) and (17). They can be completed by adding any arbitrary solutions to the corresponding *homogeneous* equations:

$$\nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = 0 , \qquad \nabla^2 \mathfrak{A} - \frac{1}{c^2}\frac{\partial^2 \mathfrak{A}}{\partial t^2} = 0$$

that satisfy the condition (15). However, such solutions can always be interpreted as retarded potentials that are required by the *infinitely-distant* electrical charges. Therefore, if the functions $\rho(\mathfrak{r}', t')$ and $\mathfrak{j}(\mathfrak{r}', t')$ are assumed to be known for all of space (including the points at infinity) and for all of time from $-\infty$ to $+\infty$ then one can consider the corresponding solutions to (26) and (27) (which consider those space and time points at infinity) to be complete. We will treat that question in detail below.

## § 6. – The electromagnetic field of an elementary oscillating dipole (oscillator).

If the polarization vector $\mathfrak{P}(\mathfrak{r}', t')$ is known for various points in space and time then one can employ the polarization potential $\mathfrak{Z}$ that was given by the differential equation (22) for the determination of the electromagnetic field. The integral expression for that potential is obviously obtained from formula (27) by replacing the current density $\mathfrak{j}(\mathfrak{r}', t')$ with the vector $\mathfrak{P}(\mathfrak{r}', t')$. One will then have:

$$\mathfrak{Z}\,(\mathfrak{r},\, t) = \int \frac{\mathfrak{P}(\mathfrak{r}', t')}{R}\, dV', \tag{28}$$

in which $t'$ means the "effective time," as before.

We now consider the simplest case of an elementary (mathematical) dipole *at rest* with a *time-varying* electric moment. Obviously, that moment cannot increase or decrease monotonically, but must oscillate about some mean value (otherwise the dipole could not be elementary), i.e., it must continue to have a very small length. For that reason, we shall call such a dipole an *oscillator*. Any neutral electric system of very small linear dimensions whose first-order electric moment is time-varying (e.g., an atom or a molecule) can be treated as an elementary oscillator in the first approximation.

If we imagine that the oscillator is concentrated at a fixed spatial point $P'$ and we denote its moment as a function of time by $\mathfrak{p}\,(t)$ then from (28), and due to the fact that $\int \mathfrak{P}(\mathfrak{r}', t')\, dV' = \mathfrak{p}(t')$:

$$\mathfrak{Z}\,(r,\, t) = \frac{\mathfrak{p}(t')}{R} \tag{28.a}$$

[cf., (32.b), Chap. III].

In order to calculate the potentials $\varphi$ and $\mathfrak{A}$ using (20) and (21), we observe the following formulas, which relate to the differentiation of the vector $\mathfrak{p}' = \mathfrak{p}(t')$, as well as any other vectorial function of the argument $t' = t - R / c$ [cf., Introduction (28) – (28.c)]:

$$\frac{\partial \mathfrak{p}'}{\partial t} = \frac{d\mathfrak{p}'}{dt'} \cdot \frac{\partial t'}{\partial t} = \frac{d\mathfrak{p}'}{dt'},$$

$$\operatorname{div} \mathfrak{p}' = \operatorname{grad} t' \cdot \frac{d\mathfrak{p}'}{dt'}, \quad \operatorname{rot} \mathfrak{p}' = \operatorname{grad} t' \times \frac{d\mathfrak{p}'}{dt'},$$

$$(\mathfrak{k}\ \mathrm{grad})\ \mathfrak{p}' = (\mathfrak{k}\,\mathrm{grad}\,t')\frac{d\mathfrak{p}'}{dt'},$$

in which $\mathfrak{k}$ is an arbitrary vector, and:

$$\mathrm{grad}\,t' = -\ \mathrm{grad}\ \frac{R}{c} = -\ \frac{1}{c}\frac{\mathfrak{R}}{R} = -\ \frac{\mathfrak{R}_0}{c}.$$

By means of those formulas, and with the usual notation for the derivative of any function with respect to time $t'$, we will get:

$$\frac{d\mathfrak{F}(t')}{dt'} = \dot{\mathfrak{F}}(t') = \dot{\mathfrak{F}}',$$

$$\varphi = -\ \mathrm{div}\frac{\mathfrak{p}'}{R} = -\ \mathfrak{p}'\,\mathrm{grad}\frac{1}{R} - \frac{1}{R}\mathrm{div}\,\mathfrak{p}',$$

i.e.:

$$\varphi = \frac{\mathfrak{p}' R_0}{R^2} + \frac{\dot{\mathfrak{p}}' \dot{R}_0}{c\,R} \tag{29}$$

and

$$\mathfrak{A} = \frac{\dot{\mathfrak{p}}'}{c\,R}. \tag{29.a}$$

In the special case $\mathfrak{p}' = \mathfrak{p} = \mathrm{const.}$, (29) reduces to the known expression for the electric potential of an elementary dipole. Formula (29.a) corresponds to the previously-exhibited formula (22), Chap. III, for the vector potential of a moving point charge, with the (very essential) difference that (29.a) does not represent an instantaneous action, but a *retarded* one.

Moreover, the formula $\mathfrak{H} = \mathrm{rot}\,\mathfrak{A}$ gives:

$$\mathfrak{H} = \mathrm{rot}\frac{\dot{\mathfrak{p}}'}{c\,R} = \frac{1}{c}\mathrm{grad}\frac{1}{R}\times\dot{\mathfrak{p}}' + \frac{1}{c\,R}\mathrm{grad}\,\dot{\mathfrak{p}}' = \mathrm{grad}\frac{1}{c\,R}\times\dot{\mathfrak{p}}' + \frac{1}{c\,R}\mathrm{grad}\,t'\times\ddot{\mathfrak{p}}',$$

in which:

$$\ddot{\mathfrak{p}}' = \frac{d^2\mathfrak{p}'}{dt'^2},$$

i.e.:

$$\mathfrak{H} = \frac{\dot{\mathfrak{p}}'\times\mathfrak{R}_0}{c\,R^2} + \frac{\ddot{\mathfrak{p}}'\times\mathfrak{R}_0}{c^2\,R}. \tag{30}$$

In order to calculate the electric field strength $\mathfrak{E} = -\ \mathrm{grad}\ \varphi - \frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t}$, we replace the unit vector $\mathfrak{R}_0$ in (29) with $\mathfrak{R} / R$. One will then have:

$$\operatorname{grad}\varphi = \operatorname{grad}\frac{\mathfrak{p}'\mathfrak{R}}{R^2} + \operatorname{grad}\frac{\dot{\mathfrak{p}}'\mathfrak{R}}{cR^2}$$

$$= \frac{1}{R^2}\operatorname{grad}(\mathfrak{p}'\mathfrak{R}) - 3\frac{(\mathfrak{p}'\mathfrak{R})}{R^4}\operatorname{grad}\mathfrak{R}_0 + \frac{1}{cR^2}\operatorname{grad}(\dot{\mathfrak{p}}'\mathfrak{R}) - \frac{2(\dot{\mathfrak{p}}'\mathfrak{R})}{cR^3}\operatorname{grad}\mathfrak{R}_0 .$$

Moreover, we have [Introduction (27)]:

$$\operatorname{grad}(\mathfrak{p}'\ \mathfrak{R}) = (\mathfrak{p}'\ \operatorname{grad})\ \mathfrak{R} + \mathfrak{p}' \times \operatorname{rot}\mathfrak{R} + (\mathfrak{R}\ \operatorname{grad})\ \mathfrak{p}' + \mathfrak{R} \times \operatorname{rot}\mathfrak{p}'$$

$$= \mathfrak{p}' + (\mathfrak{R}\operatorname{grad}t')\dot{\mathfrak{p}}' + \mathfrak{R} \times (\operatorname{grad}t' \times \dot{\mathfrak{p}}') = \mathfrak{p}' - \frac{R}{c}\dot{\mathfrak{p}}' - \frac{R'}{c} \times (\mathfrak{R}_0 \times \dot{\mathfrak{p}}'),$$

or since:

$$\mathfrak{R} \times (\mathfrak{R}_0 \times \dot{\mathfrak{p}}') = \mathfrak{R}_0(\mathfrak{R}\ \dot{\mathfrak{p}}') - \dot{\mathfrak{p}}'(\mathfrak{R}\ \mathfrak{R}_0) \qquad \text{and} \qquad \mathfrak{R}\ \mathfrak{R}_0 = R ,$$

we will have:

$$\operatorname{grad}(\mathfrak{p}'\ \mathfrak{R}) = \mathfrak{p}' - \frac{\mathfrak{R}_0}{c}(\mathfrak{R}\,\dot{\mathfrak{p}}').$$

If we replace the vector $\mathfrak{p}'$ with $\dot{\mathfrak{p}}'$ here then we will have:

$$\operatorname{grad}(\ \dot{\mathfrak{p}}'\ \mathfrak{R}) = \dot{\mathfrak{p}}' - \frac{\mathfrak{R}_0}{c}(\mathfrak{R}\,\ddot{\mathfrak{p}}').$$

We will then get:

$$\operatorname{grad}\varphi = \frac{1}{R^3}\left[\mathfrak{p}' - \frac{\mathfrak{R}_0}{c}(\mathfrak{R}\,\dot{\mathfrak{p}}')\right] - 3(\mathfrak{p}'\mathfrak{R})\frac{\mathfrak{R}_0}{R^4} + \frac{1}{cR^2}\left[\dot{\mathfrak{p}}' - \frac{\mathfrak{R}_0}{c}(\mathfrak{R}\,\ddot{\mathfrak{p}}')\right] - 2\frac{\mathfrak{R}_0}{cR^3}(\dot{\mathfrak{p}}'\mathfrak{R})$$

$$= \frac{1}{R^3}\left[\mathfrak{p}' - 3\mathfrak{R}_0(\mathfrak{p}'\mathfrak{R}_0)\right] + \frac{1}{cR^2}\left[\dot{\mathfrak{p}}' - 3\mathfrak{R}_0(\mathfrak{R}_0\,\dot{\mathfrak{p}}')\right] - \frac{\mathfrak{R}_0}{c^2 R}(\mathfrak{R}_0\,\ddot{\mathfrak{p}}'),$$

and as a result:

$$\mathfrak{E} = \frac{1}{R^3}\left[3\mathfrak{R}_0(\mathfrak{R}_0\,\mathfrak{p}') - \mathfrak{p}'\right] + \frac{1}{cR^2}\left[3\mathfrak{R}_0(\mathfrak{R}_0\,\dot{\mathfrak{p}}') - \dot{\mathfrak{p}}'\right] + \frac{1}{c^2 R}\left[\mathfrak{R}_0(\mathfrak{R}_0\,\ddot{\mathfrak{p}}') - \ddot{\mathfrak{p}}'\right]. \qquad (30.a)$$

The right-hand side of that formula reduces to the first term when $\mathfrak{p}'$ = const., which agrees with (26), Chap. III. We would like to denote the first term by $\mathfrak{E}^{(0)}$. The second term:

$$\mathfrak{E}^{(1)} = \frac{1}{cR^2}\left[3\mathfrak{R}_0(\mathfrak{R}_0\,\dot{\mathfrak{p}}') - \dot{\mathfrak{p}}'\right]$$

represents an electric field of the same type as the first one, but in which the field strength does not drop off with the third power of distance, but with its *square*, and the role of the electric moment will be played by its time derivative, i.e., the corresponding impulse.

The third term:

$$\mathfrak{E}^{(2)} = \frac{1}{c^2 R}\left[\mathfrak{R}_0(\mathfrak{R}_0 \ddot{\mathfrak{p}}') - \ddot{\mathfrak{p}}'\right]$$

corresponds to an electric field that is inversely proportional to the first power of distance, and it will be required by the second derivative of the moment with respect to time, i.e., the *acceleration* of the oscillating charges in the dipole.

The vector $\mathfrak{R}_0(\mathfrak{R}_0 \ddot{\mathfrak{p}}')$ obviously represents the longitudinal component of $\ddot{\mathfrak{p}}'$, i.e., the component of the vector $\ddot{\mathfrak{p}}'$ in the direction of the radius vector $P'P = \mathfrak{R}$. It will then follow that the difference $\ddot{\mathfrak{p}}' - \mathfrak{R}_0(\mathfrak{R}_0 \ddot{\mathfrak{p}}')$ is equal to the transversal component of $\ddot{\mathfrak{p}}'$, or in other words, *the projection of* $\ddot{\mathfrak{p}}'$ *onto the plane perpendicular to* $\mathfrak{R}$.

$\mathfrak{E}^{(2)}$ can be correspondingly written in the form:

$$\mathfrak{E}^{(2)} = \frac{\mathfrak{R}_0 \times (\mathfrak{R}_0 \times \ddot{\mathfrak{p}}')}{c^2 R}. \tag{31}$$

(Recall that $\mathfrak{R}_0\,\mathfrak{R}_0 = 1$.)

The first term in formula (30):

$$\mathfrak{H}^{(1)} = \frac{\dot{\mathfrak{p}}' \times \mathfrak{R}_0}{c R^2}$$

corresponds to the *Biot-Savart* law, with the aforementioned correction for the retardation of the electromagnetic action-at-a-distance. The second term:

$$\mathfrak{H}^{(2)} = \frac{\ddot{\mathfrak{p}}' \times \mathfrak{R}_0}{c^2 R} \tag{32}$$

represents a magnetic field of the same type as $\mathfrak{H}^{(1)}$, with the differences that the electric impulse $\dot{\mathfrak{p}}'$ is replaced with the "acceleration" $\ddot{\mathfrak{p}}'$, and the second power of the distance is replaced with the *first*, just as one had with $\mathfrak{E}^{(2)}$.

Formulas (31) and (32) show that the vectors $\mathfrak{E}^{(2)}$ and $\mathfrak{H}^{(2)}$ are perpendicular to each other and to the radius vector $\mathfrak{R}$. In that way, $\mathfrak{E}^{(2)}$ lies in the "meridian plane" that is laid through $\mathfrak{R}$ and $\ddot{\mathfrak{p}}'$, while $\mathfrak{H}^{(2)}$ is perpendicular to it. The lines of electric force are then the meridian circles and the magnetic lines are the great circles of a spherical surface that includes the reference point in question and whose polar axis $M'N'$ points in the same direction as the "acceleration vector" $\ddot{\mathfrak{p}}'$ at the corresponding effective moment (Fig. 30).

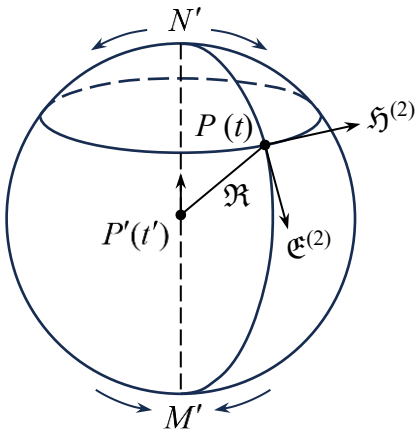

Figure 30.

Moreover, one will find from (31) and (32) that the vectors $\mathfrak{E}^{(2)}$ and $\mathfrak{H}^{(2)}$ are coupled with each other by the relations:

$$\mathfrak{E}^{(2)} = \mathfrak{H}^{(2)} \times \mathfrak{R}_0 , \qquad \mathfrak{H}^{(2)} = \mathfrak{R}_0 \times \mathfrak{E}^{(2)} . \tag{32.a}$$

One sees from this that they have *the same* magnitude (i.e., $| \mathfrak{E}^{(2)} | = | \mathfrak{H}^{(2)} |$), and they are oriented in such a way that the vector product $\mathfrak{E}^{(2)} \times \mathfrak{H}^{(2)}$ will fall in the direction of the radius vector $\mathfrak{R}$.

It should be noted that the electric and magnetic lines of force on a spherical surface with a different radius $\mathfrak{R}_1$ at the same moment $t$ can point in very different directions according to the direction of the "acceleration vector" at the corresponding effective moment $t_1' = t - R_1 / c$.

We can get the common magnitude of $\mathfrak{E}^{(2)}$ and $\mathfrak{H}^{(2)}$ from (31) and (32):

$$E^{(2)} = H^{(2)} = \frac{| \ddot{\mathfrak{p}}' |}{c^2 R} \sin\theta , \tag{32.b}$$

if the angle between the vectors $\mathfrak{R}$ and $\ddot{\mathfrak{p}}'$ is denoted by $\theta$. We see from this that $E^{(2)}$ and $H^{(2)}$ vanish in the direction of the "polar axis," i.e., at the "poles" $M'$ and $N'$, while they achieve their maximum magnitudes at the equator.

In the close vicinity of the oscillator, the strength of the electric field $\mathfrak{E}^{(0)}$ must generally exceed that of the other two ($\mathfrak{E}^{(1)}$ and $\mathfrak{E}^{(2)}$) by a large amount. However, since that field (or more precisely, that part of the electric field) drops off rapidly with distance, the field $\mathfrak{E}^{(1)}$ and the *Biot-Savart* field $\mathfrak{H}^{(1)}$ that is coupled with it, which are inversely proportional to the second power of distance, must dominate for intermediate distances.

Finally, at sufficiently-large distances, the electromagnetic fields $\mathfrak{E}^{(1)}$, $\mathfrak{H}^{(1)}$ that were just examined must take the foreground, such that the other two can be left out of consideration completely in practice.

In summary, we get the following picture for the spreading of the effect of the oscillator in question. At each moment, it defines a new infinitesimal spherical surface of action, or "wave," around the point $P'$ that propagates uniformly in all directions with the speed $c$ such that its radius

will increase by $c$ in a unit time. At the onset, the structure of the electromagnetic field on each such spherical surface will change gradually, but the structure that is suggested in Fig. 30 will soon be achieved, and after that, it will remain unchanged, except that the electric and magnetic field strengths will drop off in inverse proportion to the radius as the sphere expands further.

## § 7. – Electromagnetic waves and the essence of light.

We will now assume that the oscillator performs purely sinusoidal or "harmonic" oscillations with a period of $\tau$. The electric and magnetic field strengths at each fixed point in space must then oscillate with the same period. That temporal periodicity in the oscillations at the same spatial point corresponds to a spatial periodicity along each ray that is drawn through $P'$ at the same moment. Indeed, the electric (magnetic, resp.) field strengths must always have *the same phase* at any points on that ray that lie at a distance of:

$$\lambda = c\ \tau \tag{33}$$

from each other at each point.

If one represents the electric field strength at each point graphically by means of a line segment that is parallel and proportional to it then in the case where the oscillator oscillates linearly and parallel to a certain line *MN*, one will get the picture that is suggested in Fig. 31, i.e., a curve that differs from an ordinary sinusoid only insofar as the amplitude of oscillation will decrease slowly along the ray.

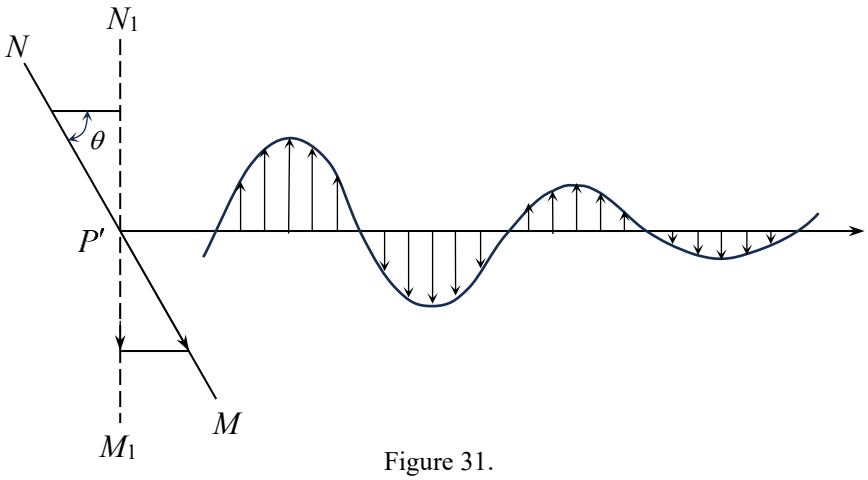


Figure 31.

If the oscillator performs an oscillation of elliptical form then one will get a helical curve of constant pitch $\lambda$ and slowly-decreasing circumference in the individual windings instead of the sinusoid that was suggested above.

Due to the analogy between the process that was described with the known wavelike propagation of mechanical oscillations in material media (or on their separation surfaces), one cares to speak of "waves," and indeed *electromagnetic waves* here, as well. The quantity $\lambda$ is called the *wavelength* of the electromagnetic oscillations in question. In that way, one refers to the latter as linearly, circularly, or elliptically "polarized" according to the form of the curve that is described

at the point in question by the line segment that represents the electric vector. The type of polarization will obviously remain unchanged for all points on the same ray.

It should be remarked that the aforementioned analogy has only an entirely-formal nature. Physically, the process of propagation of electromagnetic effects or "waves" in empty space *has absolutely nothing to do with* the propagation of elastic waves in material bodies. In the latter case, one deals with a process that is based upon the action of neighboring atoms (or molecules) on each other that makes an *oscillatory motion* take place at each location. By contrast, the propagation of electromagnetic waves means nothing but a *retarded action-at-a-distance* of oscillators on the surrounding particles, which is an action-at-a-distance that can be regarded as a "motion" only when a particle or "resonator" that is capable of oscillating actually exists at the spatial point in question. The electromagnetic oscillations are not at all oscillatory *motions*. They must be regarded as *oscillations of force* whose wave-like character arises from the finite speed of propagation of the action that emanates from the oscillator, combined with the periodic character of the motion of the corresponding electric charges. One must sharply distinguish between that *oscillatory motion* of charged material particles that the electromagnetic field generates (viz., "electronic oscillations") and the *oscillations of the field strengths* in the surrounding space ("force oscillations"). Namely, they have the same relationship to each other as the cause does to the effect.

The force oscillations that are generated by certain "primary" electronic oscillations in a certain particle can produce new "secondary" electronic oscillations in other particles in their own right, as long as the latter are found in their neighborhood and are capable of oscillation. Those secondary electronic oscillations must generate new secondary force oscillations with the same period and wavelength as the ones that the primary oscillations generate, and so on. One cares to refer to that phenomenon, i.e., the appearance of second electromagnetic waves, as the "scattering" or "reflection" of the primary waves.

As was suggested before, the theoretically-determined (as the ratio of the electromagnetic unit of charge to the electrostatic one) speed of propagation of electromagnetic effects coincides *precisely* with the empirically-measured speed of light. That single fact already sufficed to define the foundation of the *electromagnetic theory of light*, i.e., the theory that light waves could be regarded as electromagnetic waves of the corresponding (experimentally determined) wavelengths. That concept now seems to be ensured completely by the analysis of the electromagnetic field of an elementary oscillator that was carried out above.

That is because the essential feature of the oscillations of light is known to consist of their *transversality* (which follows experimentally from polarization phenomena). However, we have just seen that the electric and magnetic field strengths at a sufficiently-large distance from the oscillator are *perpendicular* to the corresponding ray. It should be pointed out that this transversality of electromagnetic force oscillations has a double nature, to some degree. That is because it is valid not only in regard to the direction of the field strengths relative to the ray, but also in the sense that those field strengths depend upon only the projection of the corresponding electron oscillations onto the plane that is perpendicular to that ray (e.g., onto the line $M_1 N_1$ in the case of Fig. 31).

Moreover, since the vectors $\mathfrak{E}^{(2)}$ and $\mathfrak{E}^{(2)}$ are inversely proportional to the first power of the distance $R$, the energy of the electromagnetic force oscillations, which is proportional to the square of those vectors (or their product), as we will show below, drops off with the square of $R$, in

agreement with the empirical law for the dependency of the light intensity on the distance from the corresponding point to the light source.

As far as the light source is concerned, it can obviously be regarded as a system of elementary electric oscillators, which are identical to the elementary light sources, viz., the atoms and molecules of the body in question. That offers the simplest proof of the *electric nature of matter*, i.e., the fact that the smallest particles of ordinary neutral matter (i.e., the atoms) are composed of even smaller particles that carry fixed electric charges (and for that reason, are called *electrons*). That is because everything that is *visible*, whether it be "primary" or "secondary," upon being illuminated by foreign light must consist of electrified particles that are capable of oscillation. The electric nature of material bodies then follows immediately from its visibility.

Visible light is known to occupy only a very narrow band of wavelengths that lie between $\lambda = 7.5 \times 10^{-5}$ cm (red light) and $\lambda = 4 \times 10^{-5}$ cm (violet light). The "invisible light rays' lie outside of that spectral region, and indeed on the side of the shorter wavelengths, one has the ultraviolet and Röntgen rays ($\lambda$ down to $10^{-9}$ cm), and on the other side one has the infrared rays (up to about $\lambda = 10^{-2}$ cm) and the long-wave "electrical rays" that are connected with wireless telegraphy and telephony. With the latter, for which the wavelengths can reach up to several hundred meters, one ordinarily speaks of waves, and not of rays. The rays can be defined only formally as the *lines of propagation of the waves.* By contrast, the concept of the ray takes on a so-to-speak physical reality in the realm of shorter wavelengths since it would then be possible to screen out a very-thin, almost-linear bundle of rays from an arbitrarily-wide wave surface. However, the width of such a ray bundle will always be large in comparison to the wavelength, as a more precise examination of that question will show.

## § 8. – The transition from spherical to plane waves.

The electric moment of a harmonically-oscillating oscillator is expressed as a function of time by the real part of the complex quantity:

$$\mathfrak{p}\,(t) = \mathfrak{p}_0\, e^{i\omega t}\,. \tag{34}$$

In that expression, one has:

$$\omega = \frac{2\pi}{\tau} = 2\pi\,\nu\,, \tag{34.a}$$

in which $\nu$ means the *frequency* of the oscillations, i.e., their number per unit time. The amplitude $\mathfrak{p}_0$ must generally be regarded as a *complex vector*, i.e., it is represented in the form:

$$\mathfrak{p}_0 = \mathfrak{a} - i\,\mathfrak{b}\,, \tag{34.b}$$

in which $\mathfrak{a}$ and $\mathfrak{b}$ are two ordinary (real) vectors. The real part of (34) then reads, when written out in detail (since $e^{i\omega t} = \cos\,\omega\,t + i\,\sin\,\omega\,t$):

$$\mathfrak{p}\ (t) = \mathfrak{a} \cos\ \omega\, t + \mathfrak{b}\ \sin\ \omega\, t\ . \tag{34.c}$$

Each term on the right-hand side represents a linear harmonic oscillation with an amplitude of $a$ ($b$, resp.) and the same frequency $\nu$. It is known that by combining such oscillations, one will get *elliptical oscillations*. If one considers $\mathfrak{p}$ to be the radius vector of a moving particle, e.g., the positive end of a dipole with charge 1, while the negative end might be fixed, then from (34.c), that particle must move on an ellipse with conjugate radii $\mathfrak{a}$ and $\mathfrak{b}$. (In particular, the latter can coincide with the semi-axes of the ellipse.)

Upon differentiating (34) twice, we will get:

$$\ddot{\mathfrak{p}}(t)\ = -\ \omega^2\ \mathfrak{p}_0\ e^{i\omega t}\ = -\ \omega^2\ \mathfrak{p}\ ,$$

and as a result, from (31) and (32):

$$\left.\begin{aligned} \mathfrak{E}^{(2)} &= -\mathfrak{N}_0 \times (\mathfrak{N}_0 \times \mathfrak{p}_0)\frac{\omega^2}{c^2\, R} e^{i\omega(t-R/c)}, \\ \mathfrak{H}^{(2)} &= -(\mathfrak{p}_0 \times \mathfrak{N}_0)\frac{\omega^2}{c^2\, R} e^{i\omega(t-R/c)}. \end{aligned}\right\} \tag{35}$$

The quantity in that:

$$\omega\,(t - R\,/\,c) = 2\pi\left(\frac{t}{\tau} - \frac{R}{\lambda}\right) \tag{36}$$

means that *phase* of the electromagnetic oscillations at a distance $R$ from the oscillator. We then see that the wavelength $c\ \tau = \lambda$ plays the same role in relation to $R$ that the period $\tau$ plays relative to time $t$.

As far as the type ("polarization") of the force oscillations considered is concerned, it will be different for different directions of $\mathfrak{N}$. For example, in the directions that lie in the plane of oscillation of $\mathfrak{p}$, the force oscillations will be linearly polarized. In general, they reproduce the projection of $\mathfrak{p}$ onto the plane that is perpendicular to $\mathfrak{N}$. Note that the electric field strengths $\mathfrak{E}$ at each moment $t$ have *the same* direction as the transverse projection of $\mathfrak{p}$ at the corresponding effective moment $t - R\,/\,c$.

The fields $\mathfrak{E}^{(1)}$, $\mathfrak{H}^{(1)}$, and $\mathfrak{E}^{(0)}$ oscillate at each location in a manner that is similar to $\mathfrak{E}^{(2)}$ and $\mathfrak{H}^{(2)}$, so their phase is also given by (36). It will follow from (30.a), (30), and (34) that their amplitudes have roughly the mutual ratios:

$$E^{(0)} : E^{(1)} : E^{(2)} \approx \frac{1}{R^2} : \frac{\omega}{c\,R^2} : \frac{\omega^2}{c^2 R^2}, \qquad\qquad H^{(1)} : H^{(2)} \approx \frac{\omega}{c\,R^2} : \frac{\omega^2}{c^2 R^2},$$

i.e., since $\frac{\omega}{c} = \frac{2\pi}{\lambda} \approx \frac{1}{\lambda}$:

$$E^{(0)} : E^{(1)} : E^{(2)} = 1 : \frac{R}{\lambda} : \frac{R^2}{\lambda^2}, \qquad H^{(1)} : H^{(2)} \approx \frac{R}{\lambda} : \frac{R^2}{\lambda^2}.$$

One must observe in all of this that one has $E^{(2)} = H^{(2)}$ and $E^{(1)} \approx H^{(1)}$.

We then see that in the case of a harmonically-oscillating oscillator, the wavelength can be used as a *unit of length* in order to determine the relative strengths of the three sub-fields, which depend upon $\mathfrak{p}$, $\dot{\mathfrak{p}}$, and $\ddot{\mathfrak{p}}$. Indeed, for "small" distances, i.e., ones that are small compared to $\lambda$, the first field is the strongest. By contrast, for "large" distances (which are, in turn, large compared to the wavelength), one needs to consider only the fields $\mathfrak{E}^{(2)}$, $\mathfrak{H}^{(2)}$. That relationship can be illustrated by the fact that the neighboring molecules of a fixed or fluid body attract each other very powerfully, while they exert only a slight effect on the distant molecules of other bodies. The forces of that type (viz., the so-called *cohesive forces*) originate in the field $\mathfrak{E}^{(0)}$ or the "static" field of the electric moments of order two and higher. By contrast, the forces of the second type must correspond to the fields $\mathfrak{E}^{(2)}$, $\mathfrak{H}^{(2)}$. (Recall that the wavelength of the visible light amounts to around $10^{-5}$ cm, which is then a thousand times bigger than the molecular distances, but very small compared to the usual distance between different "molar" bodies.)

One cares to refer to the spatial region in which the electromagnetic field reduces to the "light field" $\mathfrak{E}^{(2)}$, $\mathfrak{H}^{(2)}$ in practice as the *wave zone* of the oscillators in question. That wave zone thus begins at a certain distance from the oscillator that is sufficiently large relative to the wavelength and extends to infinity. By contrast, the "electrostatic" field $\mathfrak{E}^{(0)}$ originates in the immediate neighborhood of the oscillator. In practice, that field is independent of the finite speed of propagation of electromagnetic effects (i.e., no considerable changes will be required when one sets $c = \infty$).

We now imagine that the oscillator is found at a finitely-distant point $P'$. In that case, we can obviously treat the spherical waves that it emanates into the spatial region that is considered to consist of finite points as *planar* and treat the corresponding radii (i.e., the light rays that emanate from $P'$) as a family of mutually-parallel lines whose direction might be denoted by the *constant* unit vector $\mathfrak{n}$. (That vector will now enter in place of the unit vector $\mathfrak{R}_0$.) We will then have:

$$R = \mathfrak{R}\,\mathfrak{R}_0 = (\mathfrak{r} - \mathfrak{r}')\,\mathrm{n} = \mathfrak{r}\,\mathfrak{n} - \mathfrak{r}'\,\mathfrak{n} = \mathfrak{r}\,\mathfrak{n} + \text{const.},$$

in which $\mathfrak{r}$ means the radius vector to the reference point $P$ in question relative to any point $O$ that lies *at a finite distance*, as it did before.

The general expression for the polarization potential (28.a) assumes the form:

$$\mathfrak{Z}\,(\mathfrak{r},\,t) = \frac{\mathfrak{p}_0}{R}\,e^{i\omega(t-R/c)} \tag{37}$$

in the case of a harmonic oscillator. The distance $R$ appears twice in that expression, and indeed, the first time, it appears in the phase factor, while the second time, it appears in the amplitude factor. As far as the phase factor is concerned, it can be represented in the form:

$$\text{const.}\cdot\, e^{i\omega(t-\mathfrak{r}\,\mathfrak{n}/c)}$$

in the case considered ($R \to \infty$), and it depends upon the radius vector $\mathfrak{r}$ in about the same way as if the oscillator were at a finite distance. However, one can treat the amplitude factor as a quantity that is constant in practice since for a finite value of $\mathfrak{p}_0 / R$, the difference $\Delta\,(\mathfrak{p}_0 / R)$ at any point (e.g., $O$) must be infinitely small for two distinct finite points [since $\Delta\,(\mathfrak{p}_0 / R) = -\,(\Delta R / R^2)\mathfrak{p}_0$].

The formula (37) then reduces to:

$$\mathfrak{Z}\,(\mathfrak{r},\,t) = \mathfrak{Z}_0\,e^{i(\omega t-\mathfrak{k}\,\mathfrak{r})}, \tag{37.a}$$

where:

$$\mathfrak{k} = \frac{\omega}{c}\,\mathfrak{n} = \frac{2\pi}{\lambda}\,\mathfrak{n} \tag{37.b}$$

is a vector that determines the direction of propagation of the waves, and at the same time, their lengths. The constant vector $\mathfrak{Z}_0$ represents the (complex) amplitude of the polarization potential.

One can easily convince oneself that the formula (37.a) that is obtained from the line of reasoning above actually satisfies the differential equation (22). In our case, that equation reduces to the corresponding homogeneous equation:

$$\nabla^2\mathfrak{Z} - \frac{1}{c^2}\frac{\partial^2\mathfrak{Z}}{\partial t^2} = 0\,.$$

If one sets $e^{i(\omega t-\mathfrak{k}\,\mathfrak{r})} = \Phi$, to abbreviate, then from (37.a), one will have:

$$\nabla^2\mathfrak{Z} = \mathfrak{Z}_0\,\nabla^2\Phi = \mathfrak{Z}_0\,\text{div grad}\,\Phi\,,$$

or since:

$$\text{grad}\,\Phi = \Phi\,\text{grad}\,i\,(\omega\,t - \mathfrak{k}\;\mathfrak{r}) = -\,i\,\mathfrak{k}\,\Phi$$

and

$$\text{div}\,\mathfrak{k}\,\Phi = \mathfrak{k}\cdot\text{grad}\,\Phi\,,$$

$$\nabla^2\mathfrak{Z} = \mathfrak{Z}_0\,(-i\,k)^2\Phi = -\,k^2\,\mathfrak{Z}_0\,\Phi = -\,k^2\,\mathfrak{Z}\,.$$

On the other hand, we have:

$$\frac{1}{c^2}\frac{\partial^2 \mathfrak{Z}}{\partial t^2} = \mathfrak{Z}_0 \frac{(i\omega)^2}{c^2}\Phi = -\left(\frac{\omega}{c}\right)^2 \mathfrak{Z},$$

i.e., from (37.b):

$$\frac{1}{c^2}\frac{\partial^2 \mathfrak{Z}}{\partial t^2} = \nabla^2 \mathfrak{Z}.$$

We will get the following expressions for the potentials $\mathfrak{A}$, $\varphi$ and the field strengths $\mathfrak{H}$, $\mathfrak{E}$ from (37.a):

$$\mathfrak{A} = \frac{1}{c}\frac{\partial \mathfrak{Z}}{\partial t} = i\frac{\omega}{c}\mathfrak{Z} = i\,k\,\mathfrak{Z},$$

$$\varphi = -\operatorname{div}\mathfrak{Z} = -\mathfrak{Z}_0 \operatorname{grad}\Phi = i\,\mathfrak{k}\,\mathfrak{Z} = \mathfrak{k}\,\mathfrak{A},$$

$$\mathfrak{H} = \operatorname{rot}\mathfrak{A} = i\,k \operatorname{rot}(\mathfrak{Z}_0\Phi) = i\,k \operatorname{grad}\Phi \times \mathfrak{Z}_0 = k\,\Phi\,\mathfrak{k}\times\mathfrak{Z}_0,$$

$$\mathfrak{E} = -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t} = -i\,(\mathfrak{k}\cdot\mathfrak{Z})\operatorname{grad}\Phi - (i\,k)^2\,\mathfrak{Z}$$

$$= -(\mathfrak{k}\,\mathfrak{Z}_0)\,\Phi + k^2\,\mathfrak{Z} = -(\mathfrak{k}\,\mathfrak{Z})\,\mathfrak{k} + k^2\,\mathfrak{Z},$$

i.e.:

$$\left.\begin{aligned} \mathfrak{E} &= k^2\,\mathfrak{n}\times(\mathfrak{Z}\times\mathfrak{n}), \\ \mathfrak{H} &= k^2\,\mathfrak{n}\times\mathfrak{Z}. \end{aligned}\right\} \tag{37.c}$$

Those formulas show that the vectors $\mathfrak{E}$ and $\mathfrak{H}$ are perpendicular to the vector $\mathfrak{n}$, i.e., to the direction of propagation of the waves. Moreover, they are mutually perpendicular and have identical magnitudes. Those relations are expressed by the following relations, which are easily obtained from (37.c) [cf., (32.a)]:

$$\mathfrak{H} = \mathfrak{n}\times\mathfrak{E}, \quad \mathfrak{E} = \mathfrak{H}\times\mathfrak{n}.$$

We then see that in the case considered (infinitely-distant harmonic oscillator, plane waves), the fields of type $\mathfrak{E}^{(0)}$, $\mathfrak{E}^{(1)}$, and $\mathfrak{H}^{(1)}$ and will vanish. The electromagnetic field reduces to the type $\mathfrak{E}^{(2)}$, $\mathfrak{H}^{(2)}$ that characterizes the wave zone, so the amplitude of the force oscillations will be independent of the distance (which was obviously to be expected).

Those results can be easily generalized to *plane* waves of *arbitrary* form (so they do not need to be periodic). For a *non*-harmonic oscillator at infinity, one must replace (37.a) with the formula:

$$\mathfrak{Z}(\mathfrak{r}, t) = \mathfrak{Z}\left(t - \frac{\mathfrak{n}\mathfrak{r}}{c}\right), \tag{38}$$

in which $\mathfrak{Z}(t')$ is a function of the argument $t' = t - \frac{1}{c}(\mathfrak{n}\ \mathfrak{r})$, i.e., the effective time, that is determined by the type of oscillation of the oscillator from the outset, but otherwise entirely arbitrary.

That formula is implied immediately by (28.a) when $R \to \infty$ and represents the simplest solution to the equation:

$$\nabla^2 \mathfrak{Z} - \frac{1}{c^2}\frac{\partial^2 \mathfrak{Z}}{\partial t^2} = 0 \tag{38.a}$$

that includes no restriction in regard to the dependency of the electromagnetic field on time.

Due to the linear character of equation (38.a), its most-general solution can be represented as a sum of mutually-independent solutions of the simplest harmonic type (37.a). In that way, the "phase vector" $\mathfrak{k}$ might assume arbitrary values (and directions), just like the amplitude $\mathfrak{Z}_0$ (under the condition that $\omega^2 / c^2 = k^2$). If a solution of (38.a) is sought in a finite spatial region that is subject to certain *boundary conditions* on the bounding surface then only certain discrete values (and directions) of $\mathfrak{k}$ will be compatible with those conditions, in general. The corresponding amplitudes also remain completely arbitrary here.

In the first case, one will get an integral for $\mathfrak{Z}$ of the form:

$$\mathfrak{Z}\ (\mathfrak{r}, t) = \iint \mathfrak{Z}_0(\mathfrak{k})\, e^{i(ckt - \mathfrak{k}\mathfrak{r})}\, k^2\, dk\, d\Omega\,, \tag{38.a}$$

in which $d\ \Omega$ means the infinitesimal solid angle (the "direction interval" of the vector $\mathfrak{k}$), and $\mathfrak{Z}_0(\mathfrak{k})$ means an arbitrary vector function of $\mathfrak{k}$. Since the "proper values" of $\mathfrak{k}$ that satisfy the conditions of the problem define a triply-infinite countable set in the second case, one will get a triply-infinite trigonometric series for $\mathfrak{Z}$. However, it should be emphasized once more that such solutions to the homogeneous equation (38.a) can always be interpreted as the potentials of electric charges (in particular, dipoles) that lie outside of the spatial region in question (at finite or infinite points). Therefore, there is no essential difference between the case of time-constant electromagnetic fields that was treated in Chap. IV and the one here in that regard.

## § 9. – Huyghens's principle.

Just as in the aforementioned special case, one can further reduce the *time-varying* field that prevails inside (or outside) a closed surface *S* and satisfies equation (36.a) in the corresponding region, i.e., it is generated by *external* (internal, resp.) electric charges, to a time-varying *distribution of electricity on the boundary surface S itself*. For the sake of simplicity, in what follows, we would not like to consider the polarization potential, but the scalar potential $\varphi$. We then assume that the equation:

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0$$

is fulfilled inside of *S*, and we would like to try to represent the value of the potential $\varphi^*$ for any (internal) point $P^*$ by the boundary values of $\varphi$ on *S* (and possibly the corresponding boundary values of $\nabla\varphi$), which are regarded as known functions of time.

That problem will be solved by means of the general equation (21) in the previous chapter, and indeed the simplest way to solve it is to make the previous Ansatz:

$$\psi = \frac{1}{R}$$

for the auxiliary function $\psi$ (the notations are the same as they were at the time). However, in so doing, one must not consider the potential $\varphi$ at each point *P* at the same moment *t* when its value $\varphi^*$ at $P^*$ is sought, but at the corresponding "effective" moment:

$$t' = t - R/c\ ,$$

where *R* means the distance $P\,P^*$. We set:

$$\varphi' = \varphi(\mathfrak{r}, t')\ .$$

Formula (21) can then be written as follows:

$$\oint\left(\frac{1}{R}\nabla_\nu\varphi' - \varphi'\nabla_\nu\frac{1}{R}\right)d\Sigma = \oint\left(\frac{1}{R}\nabla_m\varphi' - \varphi'\nabla_m\frac{1}{R}\right)dS - \int\frac{1}{R}\nabla^2\varphi'\,dV^*. \tag{39}$$

Along with the operation $\nabla$, which means a complete differentiation with respect to the argument $\mathfrak{r}$, we now introduce the corresponding operation $\nabla'$, which should mean differentiation with respect to $\mathfrak{r}$ *at fixed* $t'$. One will then have:

$$\nabla\varphi' = \nabla'\varphi' + \frac{\partial\varphi'}{\partial t'}\nabla t' \tag{39.a}$$

and

$$\nabla^2\varphi' = \operatorname{div}\nabla\varphi' = \operatorname{div}'(\nabla\varphi') + \frac{\partial}{\partial t'}(\nabla\varphi')\nabla t'\,,$$

i.e., from the foregoing formula:

$$\nabla^2\varphi' = \nabla'^2\varphi' + \operatorname{div}'\left(\frac{\partial\varphi'}{\partial t'}\nabla t'\right) + \frac{\partial}{\partial t}\nabla'\varphi'\cdot\nabla t' + \frac{\partial^2\varphi'}{\partial t'^2}(\nabla t')^2\,.$$

If one further considers the formulas:

$$\nabla t' = -\frac{1}{c}\nabla' R = -\frac{1}{c}\nabla R, \qquad (\nabla R)^2 = 1,$$

$$\mathrm{div}'\left(\frac{\partial \varphi'}{\partial t'}\nabla t'\right) = \nabla'\frac{\partial \varphi'}{\partial t'}\cdot\nabla t' + \frac{\partial \varphi'}{\partial t'}\mathrm{div}'\nabla t' = -\nabla'\frac{1}{c}\frac{\partial \varphi'}{\partial t'}\cdot\nabla R - \frac{1}{c}\frac{\partial \varphi'}{\partial t'}\cdot\nabla R - \frac{1}{c}\frac{\partial \varphi'}{\partial t'}\mathrm{div}\nabla R,$$

and

$$\mathrm{div}\,\nabla R = \mathrm{div}\,\frac{\mathfrak{R}}{R} = \frac{2}{R}$$

then one will have:

$$\nabla^2\varphi' = \nabla'^2\varphi' - \frac{2}{R}\frac{1}{c}\frac{\partial \varphi'}{\partial t'} - 2\nabla' R\cdot\nabla'\frac{\partial \varphi'}{\partial t'} + \frac{1}{c^2}\frac{\partial^2 \varphi'}{\partial t'^2},$$

or finally, since $\varphi'$ satisfies the equation:

$$\nabla'^2\varphi' - \frac{1}{c^2}\frac{\partial^2 \varphi'}{\partial t'^2} = 0,$$

one will have:

$$\nabla^2\varphi' = 2\left\{-\frac{1}{R}\frac{1}{c}\frac{\partial \varphi'}{\partial t'} - \nabla' R\cdot\nabla'\frac{\partial \varphi'}{\partial t'} + \frac{1}{c^2}\frac{\partial^2 \varphi'}{\partial t'^2}\right\}. \tag{39.b}$$

Obviously, one can replace partial differentiation with respect to $t'$ with partial differentiation with respect to *R*, which will make:

$$-\frac{1}{c}\frac{\partial \varphi'}{\partial t'} = \frac{\partial \varphi'}{\partial R} \qquad \text{and} \qquad \frac{1}{c^2}\frac{\partial^2 \varphi'}{\partial t'^2} = \frac{\partial^2 \varphi'}{\partial R^2}.$$

We further introduce the *total* differentiation with respect to *R* according to the following formula:

$$\frac{d\varphi'}{dR} = \frac{\partial \varphi'}{\partial R} + \nabla'\varphi'\cdot\nabla' R. \tag{40}$$

Therefore, $\frac{d\varphi'}{dR}\cdot dR$ will mean the change in $\varphi'$ along an infinitely-small line segment *dR* of the line $P^*P$ for fixed values in the time *t*. We now write (39.b) in the form:

$$\nabla^2\varphi' = 2\left\{\frac{1}{R}\frac{\partial \varphi'}{\partial R} + \frac{\partial^2 \varphi'}{\partial R^2} + \nabla'\frac{\partial \varphi'}{\partial t'}\cdot\nabla' R\right\}.$$

However, from (40), when $\varphi'$ is replaced with $\partial\varphi'/\partial R$, we will have:

$$\frac{\partial^2\varphi'}{\partial R^2}+\nabla'\frac{\partial\varphi'}{\partial R}\cdot\nabla' R = \frac{d}{dR}\frac{\partial\varphi'}{\partial R}.$$

As a result, one will have:

$$\nabla^2\varphi' = \frac{2}{R}\frac{d}{dR}\left(R\frac{\partial\varphi'}{\partial R}\right). \tag{40.a}$$

Now, the volume integral that appears in (39):

$$\int\frac{1}{R}\nabla^2\varphi'\,dV^* = 2\int\frac{1}{R^2}\frac{d}{dR}\left(R\frac{\partial\varphi'}{\partial R}\right)dV^*$$

can be transformed into a surface integral over $S$ and $\Sigma$. If one sets $dV^* = R^2 dR\,d\Omega$, in which $d\Omega$ means an infinitely-small solid angle (with its vertex at $P^*$), then one will have:

$$\int\frac{1}{R^2}\frac{d}{dR}\left(R\frac{\partial\varphi'}{\partial R}\right)dV^* = \int d\Omega\int_{R_1}^{R_2}\frac{d}{dR}\left(R\frac{\partial\varphi'}{\partial R}\right)dR = \int d\Omega\left\{\left(R\frac{\partial\varphi'}{\partial R}\right)_2-\left(R\frac{\partial\varphi'}{\partial R}\right)_1\right\}.$$

The indices 1 and 2 in that refer to the boundary surfaces $\Sigma$ ($S$, resp.). One now introduces the surface elements $d\,\Sigma$ and $dS$ that belong to the angle element $d\,\Omega$. Obviously, since $\cos(n\,R_2) = (\nabla_n R)_2$ and $\cos(n\,R_1) = (\nabla_n R)_1$, one will have the relations:

$$R_2^2\,d\Omega = dS\cdot(\nabla_n R)_2, \qquad R_1^2\,d\Omega = d\,\Sigma\cdot(\nabla_\nu R)_1.$$

One will then have:

$$\int\frac{1}{R^2}\frac{d}{dR}\left(R\frac{\partial\varphi'}{\partial R}\right)dV^* = \oint\frac{1}{R}\frac{\partial\varphi'}{\partial R}\nabla_n R\,dS-\oint\frac{1}{R}\frac{\partial\varphi'}{\partial R}\nabla_\nu R\,d\Sigma,$$

and as a result, from (39):

$$\oint\left(\frac{1}{R}\nabla_\nu\varphi'-\frac{2}{R}\frac{\partial\varphi'}{\partial R}\nabla_\nu R-\varphi'\nabla_\nu\frac{1}{R}\right)d\Sigma = \oint\left(\frac{1}{R}\nabla_n\varphi'-\frac{2}{R}\frac{\partial\varphi'}{\partial R}\nabla_n R-\varphi'\nabla_n\frac{1}{R}\right)dS\ .$$

Due to (39.a), when one recalls that $\frac{\partial\varphi'}{\partial t}\nabla t = \frac{\partial\varphi'}{\partial R}\nabla R$, and:

$$-\frac{1}{R}\frac{\partial\varphi'}{\partial R}\nabla R-\varphi'\nabla\frac{1}{R} = -\left(\frac{1}{R}\frac{\partial\varphi'}{\partial R}-\frac{1}{R^2}\varphi'\right)\nabla R = -\frac{\partial}{\partial R}\left(\frac{\varphi'}{R}\right)\cdot\nabla R,$$

that formula can be written in the form:

$$\oint\left(\frac{1}{R}\nabla_\nu\varphi' - \frac{\partial}{\partial R}\left(\frac{\varphi'}{R}\right)\nabla'_\nu R\right)d\Sigma = \oint\left(\frac{1}{R}\nabla'_n\varphi' - \frac{\partial}{\partial R}\left(\frac{\varphi'}{R}\right)\nabla'_n R\right)dS\,. \tag{40.b}$$

We now imagine that the inner surface Σ contracts to the point $P^*$. Just as in the previously-considered case of a time-constant field (cf., **§ 6**, Chap. IV):

$$\lim_{\Sigma\to 0}\oint\frac{1}{R}\nabla'_\nu\,\varphi'\,d\Sigma = 0\,, \qquad \lim\oint\frac{\partial}{\partial R}\left(\frac{\varphi'}{R}\right)\nabla'_\nu\,R\;d\Sigma = -\varphi^*\oint\frac{\nabla_n R}{R^2}d\Sigma = -\;4\pi\,\varphi^*\,,$$

and ultimately:

$$\varphi^* = \oint\frac{\nabla'_n\varphi'}{4\pi R}dS - \oint\frac{\partial}{\partial R}\left(\frac{\varphi'}{4\pi R}\right)\nabla'_n R\,dS\,. \tag{41}$$

That formula is the desired generalization of formula (21.b), Chap. IV. It represents the potential at the point $P^*$ as the sum of two components, the first of which depends upon a surface charge with the time-varying density $\frac{\nabla'_n\varphi'}{4\pi}$, while the second one corresponds to a double layer. It is easy for one to further see that the right-hand side of (41) will vanish for a point $P^*$ that lies *outside of S*, just as in the static case.

If one considers the field outside of *S* that is generated by internal charges then if one is to calculate the potential, one must introduce a surface $S'$ that envelops the two surfaces *S* and Σ and which one can extend to infinity. If one can assume that only the static field prevails at infinitely-distant points (i.e., that the oscillators that are found inside of *S* have begun to oscillate at only a finite time before *t*) then the integral that extends over $S'$ will drop out, and one can again get the formula (41), but $\mathfrak{n}$ shall now mean the *interior* normal to *S*, and not the exterior one.

By a suitable choice of the auxiliary function $\psi$, we can represent the potential $\varphi^*$ in terms of either just the boundary values of $\varphi$ or just the boundary values of $\nabla'_n\varphi$, just as in the static case. However, we would not like to go further into that topic here.

One refers to the possibility of representing the electromagnetic potential inside of an empty spatial region by the boundary values of that potential or the corresponding field strengths on the bounding surface of the region (internally or externally) as *Huyghens's principle*. Originally, that principle was conceived in a somewhat-narrower and more physically intuitive context since one asserted, with *Huyghens*, that every surface element of a light wave can be treated as a "virtual light wave," i.e., as the center of new elementary spherical waves. If one then knows the form of an advancing wave at any moment then one can use those elementary spherical waves to construct the resultant form of the wave for any later moment without having to worry about what the actual source of the "primary" light wave was. As is known, that is the simplest way to solve problems in reflection, refraction, and diffraction of light in wave optics, and with almost no calculation.

However, a complete solution to that problem that considers not only the form of the wave front, but also the intensities and the polarization of the force oscillations, would require complicated methods that we will learn about later (Volume II).

## § 10. – The electromagnetic fields of higher-order oscillators (multipoles).

The results that were obtained in **§ 7** concerning the electromagnetic field of an oscillating dipole can be easily generalized to higher-order multipoles with time-varying moments.

We initially consider the simplest case of *purely-harmonic* (i.e., sinusoidal) oscillations and imagine a *continuous distribution* of electricity inside of a very-small spatial region that might be bounded by a spherical surface $K$ of radius $a$, instead of isolated oscillating point-charges. We shall then assume that at each point $Q$ of that spatial region, the charge and current densities can be represented as functions of time by the formulas:

$$\rho(\mathfrak{r}, t) = \rho_0(\mathfrak{r}) e^{i\omega t}, \quad \mathbf{j}(\mathfrak{r}, t) = \mathbf{j}_0(\mathfrak{r}) e^{i\omega t}, \tag{42}$$

in which the corresponding "amplitudes" $\rho_0$ and $\mathbf{j}_0$ are given functions of position that must be coupled with each other by the equation $\frac{1}{c}\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0$, according to the principle of conservation of electricity, i.e.:

$$i\,k\,\rho_0 + \operatorname{div} \mathbf{j}_0 = 0 \tag{42.a}$$

in the present case. The $k$ in that equation means:

$$k = \frac{\omega}{c} = \frac{2\pi}{\tau\, c} = \frac{2\pi}{\lambda}, \tag{42.b}$$

which is the magnitude of the phase vector that was introduced already in **§ 8**. We assume that the spherical radius $a$, i.e., the linear dimensions of our system are *small compared to the wavelength*.

Under those assumptions, the scalar potential of $S$ at any exterior point $P$ can be developed in a convergent series that is entirely analogous to the series (6) – (7) of **§ 3**, Chap. IV, and can be regarded as a direct generalization of it. From (26) and (42), that potential can be expressed by the formula:

$$\varphi(\mathfrak{r}, t) = \int \frac{\rho_0(\mathfrak{r}')}{R} e^{i(\omega t - kR)} dV' .$$

If one sets:

$$\varphi(\mathfrak{r}, t) = \varphi_0(\mathfrak{r}) e^{i\omega t}, \tag{43}$$

corresponding to (42), then one will have:

$$\varphi_0(\mathfrak{r}) = \int \rho_0(\mathfrak{r}')\,\Psi\,dV', \tag{43.a}$$

in which the function $\Psi$ is defined by:

$$\Psi = \frac{e^{-ikR}}{R}. \tag{43.b}$$

Formula (43.a) has the same form as the usual formula for the scalar potential of a static distribution of charge with the time-constant volume density $\rho_0$, except that the reciprocal radius $1/R$ is replaced by the more-general function $\Psi$. (For infinitely-slow oscillations, i.e., for $k = 0$, $\Psi$ reduces to $1/R$.) If one now develops that function for an arbitrary point $Q$ that lies inside of $K$ in powers of its coordinates $\xi_1'$, $\xi_2'$, $\xi_3'$ relative to the center of the sphere $O$ then one will have:

$$\varphi_0 = \varphi_0^{(0)} + \varphi_0^{(1)} + \varphi_0^{(2)} + \cdots + \varphi_0^{(n)}, \tag{44}$$

just like in **§ 3**, Chap. IV, where:

$$\varphi_0^{(0)} = e_0'\,\psi, \quad \varphi_0^{(1)} = -\sum_i \frac{\partial\psi}{\partial x_i} e_{0i}', \quad \varphi_0^{(2)} = \tfrac{1}{2}\sum_i\sum_k \frac{\partial^2\psi}{\partial x_i\,\partial x_k} e_{0ik}', \quad \ldots \tag{44.a}$$

The $\psi$ in that means the value of $\Psi$ for the point $O$, i.e., (since $OP = \mathfrak{r}$):

$$\psi = \frac{e^{-ikr}}{r}, \tag{44.b}$$

while the quantities:

$$e_0' = \int \rho_0\,dV', \qquad e_{0i}' = \int \rho_0\,\xi_i'\,dV', \qquad e_{0ik}' = \int \rho_0\,\xi_i'\,\xi_k'\,dV', \quad \ldots \tag{44.c}$$

can be defined to be the *amplitudes* of the electric moments of the system in question.

The general term of (44) can also be represented in the form:

$$\varphi_0^{(n)} = (-1)^n \sum_{n_1+n_2+n_3=n} \frac{e_0(n_1,n_2,n_3)}{n_1!\,n_2!\,n_3!} \frac{\partial^n\psi}{\partial x_1^{n_1}\,\partial x_2^{n_2}\,\partial x_3^{n_3}}, \tag{45}$$

with:

$$e_0(n_1, n_2, n_3) = \int \rho_0\,\xi_1'^{n_1}\,\xi_2'^{n_2}\,\xi_3'^{n_3}\,dV'. \tag{45.a}$$

As a special case of that, (45) will imply the expression:

$$\varphi_0^{(n)} = \frac{(-1)^n}{n!}\,p_0^{(n)}\,(\mathfrak{a}_1\nabla)\,(\mathfrak{a}_2\nabla)\,\ldots\,(\mathfrak{a}_n\nabla)\,\psi, \tag{45.b}$$

which one can interpret as the amplitude of the potential of an $n^{\text{th}}$-order multipole with fixed axes $\mathfrak{a}_1$, $\mathfrak{a}_2$, …, $\mathfrak{a}_n$, and a harmonically-oscillating moment $p^{(n)} = p_0^{(n)} e^{i\omega t}$ ([1]). Note that the vectors $\mathfrak{a}_i$ can also be treated as complex quantities. However, that will have an immediate physical sense only when *one* of them (e.g., $\mathfrak{a}_1$) proves to be complex, while the other ones are real. The product $\mathfrak{a}_1 e^{i\omega t}$ will then mean an elliptic oscillation that arises from a rotation of the "first" axis of the multipole in a well-defined plane. If the other axis vectors are also complex then (45.b) will decompose into a sum of components that correspond to a number of multipoles with each rotating axis.

Since the function $\psi e^{i\omega t} = \dfrac{e^{i(\omega t - kr)}}{r}$ satisfies the equation:

$$\nabla^2 \psi e^{i\omega t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\psi e^{i\omega t}) = 0 ,$$

it will follow (as one can also verify by direct calculation) that:

$$\nabla^2 \psi + k^2 \psi = 0 . \tag{46}$$

The derivatives $\dfrac{\partial^n \psi}{\partial x_1^{n_1}\, \partial x_2^{n_2}\, \partial x_3^{n_3}}$, and as a result, the potential (45), must also satisfy that equation [which is regarded as the simplest generalization of the *Laplace* equation $\nabla^2 (1/R) = 0$].

By repeated application of the equation $\nabla^2 \psi = 0$ that is valid for static fields ($k = 0$), one can reduce the number of parameters that appear in (45) to $(2n + 1)$ and correspondingly reduce the expression (45) to the form (46.a). For $k > 0$, such a reduction is *not* possible, in general, except in the simplest case of $n = 1$. Although one can also say here that the external field of a system of electric charges that is included inside of spherical surface is equivalent to a number of multipoles at the center of the sphere, but generally *several* multipoles of the same order must exist that *cannot* be combined into a single multipole for $n > 1$.

The aforementioned assumption that the radius of the sphere is small compared to the wavelength is just as essential for the convergence of the series (44) as the assumption that the point $P$ lies outside the surface of the sphere. That is because in the development of the function $\Psi$ in (43.a) in the coordinates $\xi_1'$, $\xi_2'$, $\xi_3'$, one will get terms of the form:

$$\frac{e^{-ikr}}{r} \cdot \frac{\xi_1'^{n_1}\, \xi_2'^{n_2}\, \xi_3'^{n_3}}{r^{n-p}\, \lambda^p} \quad (n_1 + n_2 + n_3 = n,\ p = 0, 1, \ldots, n).$$

([1]) With the exception of the case $n = 0$ since the resulting charge of the system $e_0'$ must obviously remain constant.

It will then follow that the wavelength $\lambda$ plays an equally-important role in the convergence of the corresponding series.

Obviously, the $n^{\text{th}}$-order potential $\varphi^{(n)} = \varphi_0^{(n)} e^{i\omega t}$ can be represented in the following form:

$$\varphi^{(n)} = \frac{e^{i(\omega t - kr)}}{r} \sum_{p=0}^{n} \frac{K_n^{(p)}}{r^{n-p} \lambda^p}, \tag{46.a}$$

in which the coefficients $K_n^{(p)}$ depend upon only the direction of the radius vector $\mathfrak{r}$. The $K_n^{(0)}$ in that means an ordinary spherical function of order *n* in the angle that determines that direction. In fact, the corresponding "zeroth-order" term in (46) must coincide with (8), Chap. IV, except for the "phase factor" $e^{i(\omega t - kr)}$, which is equal to unity in the static case.

We would like to calculate the $n^{\text{th}}$ term in (46.a) in detail since for large distances (viz., in the wave zone), the potential $\varphi^{(n)}$ must reduce to that term in practice. (All of the other ones will drop off more rapidly.)

In the differentiation of $\psi$ in (45), if we then drop all terms that are proportional to the second and higher powers of $1 / r$ then from (44.b), we will have:

$$\frac{\partial^n \psi}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} \approx (-ik)^n \frac{e^{-ikr}}{r} \left(\frac{\partial r}{\partial x_1}\right)^{n_1} \left(\frac{\partial r}{\partial x_2}\right)^{n_2} \left(\frac{\partial r}{\partial x_3}\right)^{n_3} = (-2\pi i)^n \frac{e^{-ikr}}{r \lambda^n} \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3},$$

in which $\gamma_1$, $\gamma_2$, $\gamma_3$ mean the direction cosines of the radius vector, and as a result:

$$\varphi^{(n)} \approx \frac{e^{i(\omega t - kr)}}{r} \frac{(2\pi i)^n}{\lambda^n} \sum_{n_1+n_2+n_3=n} \frac{e_0(n_1, n_2, n_3)}{n_1! n_2! n_3!} \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3}. \tag{46.b}$$

For $n = 1$ (i.e., the harmonically-oscillating dipole), that expression will be identical to the corresponding (second) term on the right-hand side of formula (29). In fact, (46.b) can then be written in the form:

$$\varphi^{(1)} \approx \frac{e^{i(\omega t - kr)}}{r} ik(\mathfrak{p}_0, \mathfrak{r}_0),$$

in which $\mathfrak{p}_0$ means the vector with the components $e_0$ (1, 0, 0), $e_0$ (0, 1, 0), $e_0$ (0, 0, 1), i.e., since $\omega t - kr = \omega t'$ and $ik\mathfrak{p}_0 e^{i\omega t'} = \frac{1}{c}\dot{\mathfrak{p}}'$:

$$\varphi^{(1)} = \frac{1}{cr} \dot{\mathfrak{p}}' \mathfrak{r}_0.$$

Note that the "main term" in the potential (45.b) under consideration of a simple $n^{\text{th}}$-order multipole is expressed in the formula:

$$\varphi^{(n)} = \frac{e^{i(\omega t - kr)}}{r} \frac{(2\pi i)^n}{n!\lambda^n} (\mathfrak{a}_1\, \mathfrak{r}_0)\, (\mathfrak{a}_2\, \mathfrak{r}_0) \ldots (\mathfrak{a}_n\, \mathfrak{r}_0)\; e_0^{(n)}. \tag{46.c}$$

The vector potential of our system:

$$\mathfrak{A}\,(\mathfrak{r}, t) = \int \frac{\mathfrak{j}_0(\mathfrak{r}')}{R} e^{i(\omega t - kr)}\, dV' = e^{i\omega t} \int \mathfrak{j}_0(\mathfrak{r}')\, \Psi\, dV' = \mathfrak{A}_0\, e^{i\omega t}$$

can be developed into a series:

$$\mathfrak{A}_0 = \mathfrak{A}_0^{(0)} + \mathfrak{A}_0^{(1)} + \mathfrak{A}_0^{(2)} + \cdots + \mathfrak{A}_0^{(n)} + \cdots \tag{47}$$

in a manner that is entirely analogous to what was done for the scalar potential, with the general term:

$$\mathfrak{A}_0^{(n)} = (-1)^n \sum_{n_1+n_2+n_3=n} \frac{\mathfrak{J}_0(n_1, n_2, n_3)}{n_1!\, n_2!\, n_3!} \frac{\partial^n \psi}{\partial x_1^{n_1}\, \partial x_2^{n_2}\, \partial x_3^{n_3}}, \tag{47.a}$$

in which the vector parameter:

$$\mathfrak{J}_0\,(n_1, n_2, n_3) = \int \mathfrak{J}_0\, \xi_1'^{n_1}\, \xi_2'^{n_2}\, \xi_3'^{n_3}\, dV', \tag{47.b}$$

which we can call the "electrokinetic moment" of the system in question, plays the same role in regard to $\mathfrak{A}$ that the electrostatic moment $e_0\,(n_1, n_2, n_3)$ does for $\varphi$.

It follows from the relation (42.a) that (we will drop the index 0 and the prime in what follows):

$$e\,(n_1, n_2, n_3) = \int \rho_0\, \xi_1^{n_1}\, \xi_2^{n_2}\, \xi_3^{n_3}\, dV = -\frac{1}{ik} \int \operatorname{div} \mathfrak{j} \cdot \xi_1^{n_1}\, \xi_2^{n_2}\, \xi_3^{n_3}\, dV$$

$$= -\frac{1}{ik} \int \operatorname{div} (\mathfrak{j} \cdot \xi_1^{n_1}\, \xi_2^{n_2}\, \xi_3^{n_3})\, dV + \frac{1}{ik} \int \mathfrak{j} \cdot \nabla (\xi_1^{n_1}\, \xi_2^{n_2}\, \xi_3^{n_3})\, dV\,.$$

Since we have $j_n = 0$ on the spherical surface that bounds the volume $V$, we have:

$$\int \operatorname{div} (\mathfrak{j} \cdot \xi_1^{n_1}\, \xi_2^{n_2}\, \xi_3^{n_3})\, dV = \oint j_n\, \xi_1^{n_1}\, \xi_2^{n_2}\, \xi_3^{n_3}\, dS = 0\,,$$

and as a result, since:

$$\mathfrak{j} \cdot \nabla (\xi_1^{n_1}\, \xi_2^{n_2}\, \xi_3^{n_3}) = n_1\, j_1\, \xi_1^{n_1-1}\, \xi_2^{n_2}\, \xi_3^{n_3} + n_2\, j_2\, \xi_1^{n_1}\, \xi_2^{n_2-1}\, \xi_3^{n_3} + n_3\, j_3\, \xi_1^{n_1}\, \xi_2^{n_2}\, \xi_3^{n_3-1},$$

one will have:

$$e\,(n_1, n_2, n_3) = \frac{1}{ik} \{n_1\, J_1(n_1 - 1, n_2, n_3) + n_2\, J_2(n_1, n_2 - 1, n_3) + n_3\, J_3(n_1, n_2, n_3 - 1)\}\,. \tag{48}$$

By substituting those expressions in the formula (46.a), we will get:

$$\varphi^{(n)} \approx \frac{e^{i(\omega t - kr)}}{r} \frac{(2\pi i)^{n-1}}{\lambda^{n-1}} \sum_{n_1'+n_2'+n_3'=n-1} \frac{\mathfrak{r}_0 \cdot \mathfrak{J}(n_1', n_2', n_3')}{n_1'! n_2'! n_3'!} \gamma_1^{n_1'} \gamma_2^{n_2'} \gamma_3^{n_3'}.$$

On the other hand, (47.a) will reduce to:

$$\mathfrak{A}_0^{(n)} \approx \frac{e^{-ikr}}{r} \frac{(2\pi i)^n}{\lambda^n} \sum_{n_1+n_2+n_3=n} \frac{\mathfrak{J}(n_1, n_2, n_3)}{n_1! n_2! n_3!} \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3}$$

in the wave zone ($r \gg \lambda$). Thus, we will have:

$$\varphi^{(n)} = \mathfrak{r}_0 \, \mathfrak{A}^{(n-1)}$$

*for the wave zone*, and as a result, since we can always set $\varphi^{(0)} = 0$ ([1]):

$$\varphi = \mathfrak{r}_0 \, \mathfrak{A}. \tag{48.a}$$

That relation between the two potentials agrees with the one that we found in **§ 9** for the electromagnetic field of an infinitely-distant oscillator. Note that it is fulfilled in complete generality for all *plane* (even non-sinusoidal) electromagnetic waves and can be derived from the corresponding general Ansatz (38) for the polarization potential. Moreover, (48.a) implies the known relations between the field strengths in the wave zone and the unit vector $\mathfrak{r}_0$ that determined the direction of propagation of the waves [cf., (32.a)]:

$$\mathfrak{H} = \mathfrak{r}_0 \times \mathfrak{E} \qquad \text{and} \qquad \mathfrak{E} = \mathfrak{H} \times \mathfrak{r}_0 .$$

Instead of characterizing the system in question by giving the electric charge density $\rho$ and current $\mathbf{j}$, to that end, one can obviously introduce the electric polarization $\mathfrak{P}$ and correspondingly develop the polarization potential:

$$\mathfrak{Z}(\mathfrak{r}, t) = \mathfrak{Z}_0 e^{i\omega t} = e^{i\omega t} \int \mathfrak{P}_0(\mathfrak{r}') \psi \, dV'$$

into a series of the same type as the series that was cited above for the vector potential $\mathfrak{A}$. In that way, the "polarization moment" would appear in place of the corresponding electrokinetic moment (47.b).

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([1]) If the system in question is not neutral, i.e., $\varphi^{(0)}$ has a non-zero value, then the latter must remain constant in time, and that is why it is irrelevant to the electric field in the wave zone.

The cited results can be generalized to the case of arbitrary anharmonic oscillations of the charge and current density (the polarization, resp.) in the system considered when those oscillations are not too rapid such that the mean or effective wavelength is large in comparison to the radius of the sphere. One can reduce that case to the foregoing one by decomposing $\rho$ or $\mathbf{j}$ ($\mathfrak{P}$, resp.) as functions of time into *Fourier* series or *Fourier* integrals. The resulting electromagnetic field must then be obtained from the sum (or integral) of the elementary components that originate in the individual harmonic components of $\rho$ and $\mathbf{j}$.

## § 11. – Equivalent magnetic systems. Magnetic oscillators.

In Chap. IV, we saw that a time-constant magnetic field that is created outside of a closed surface $S$ by an internal (or superficial) system of *stationary* currents can be treated in the same way as an electrostatic field when the aforementioned currents are replaced with fictitious magnetic charges (or poles) ([1]). Now, it is easy to see that an analogous "process of replacement" is also possible in the general case of an arbitrary external electromagnetic field. Indeed, we can introduce fictitious magnetic charges in place of the electric current and replace the electric charges with fictitious magnetic currents that are distributed inside of the ball in such a way that the field in question will prevail in the exterior space. Mathematically speaking, that means the following:

Instead of determining the field from the fundamental equations:

$$\left.\begin{aligned} &\operatorname{div}\mathfrak{E} = 4\pi\rho, \quad \operatorname{rot}\mathfrak{H} - \frac{1}{c}\frac{\partial\mathfrak{E}}{\partial t} = 4\pi\mathbf{j}, \\ &\operatorname{div}\mathfrak{H} = 0, \qquad \operatorname{rot}\mathfrak{E} + \frac{1}{c}\frac{\partial\mathfrak{H}}{\partial t} = 0, \end{aligned}\right\} \tag{49}$$

in which $\rho$ and $\mathbf{j}$ are given functions of $\mathfrak{r}$ and $t$ that satisfy the condition $\frac{1}{c}\frac{\partial\rho}{\partial t} + \operatorname{div}\mathbf{j} = 0$, one can also derive it from the "conjugate" equations:

$$\left.\begin{aligned} &\operatorname{div}\mathfrak{E} = 0, \qquad \operatorname{rot}\mathfrak{H} - \frac{1}{c}\frac{\partial\mathfrak{E}}{\partial t} = 0, \\ &\operatorname{div}\mathfrak{H} = 4\pi\rho^*, \quad \operatorname{rot}\mathfrak{E} + \frac{1}{c}\frac{\partial\mathfrak{H}}{\partial t} = 4\pi\mathbf{j}^*, \end{aligned}\right\} \tag{49.a}$$

in which the magnetic charge and current densities $\rho^*$ ($\mathbf{j}^*$, resp.) must be chosen suitably when one considers the condition that $\frac{1}{c}\frac{\partial\rho^*}{\partial t} + \operatorname{div}\mathbf{j}^* = 0$ [cf., (11), (11.a), (11.b), **§ 3**]. In so doing, one

([1]) The surface $S$ must be *simply connected*, i.e., not annular, in order for the magnetic lines of force to not be able to close outside of $S$.

must assume only that the system in question is neutral, i.e., that the volume integral $\int \rho\, dV$ must vanish when it is extended over all of the volume that is bounded by $S$ (just like $\int \rho^* dV = 0$). Obviously, $\rho$ and $\mathbf{j}$ must be equal to zero outside of $S$, just like $\rho^*$ and $\mathbf{j}^*$. The assignment of the functions $\rho^*$ and $\mathbf{j}^*$ to the given functions $\rho$ and $\mathbf{j}$ is therefore not one-to-one, which one can already see from the fact that according to *Huyghens*'s principle, the latter can be replaced with a certain distribution of charge and current on the outer surface $S$.

By means of the substitution:

$$\mathfrak{E} = -\,\mathfrak{H}^*, \qquad \mathfrak{H} = \mathfrak{E}^*, \tag{49.b}$$

(49.a) will imply the equations:

$$\left.\begin{aligned} &\operatorname{div}\mathfrak{E}^* = 4\pi\rho^*, \quad \operatorname{rot}\mathfrak{H}^* - \frac{1}{c}\frac{\partial\mathfrak{E}^*}{\partial t} = 4\pi\mathbf{j}^*, \\ &\operatorname{div}\mathfrak{H}^* = 0, \qquad\quad \operatorname{rot}\mathfrak{E}^* + \frac{1}{c}\frac{\partial\mathfrak{H}^*}{\partial t} = 0, \end{aligned}\right\} \tag{49.c}$$

which coincide with the original equations (49). They can be solved in the same way accordingly, and indeed by introducing the potentials $\varphi^*$, $\mathfrak{A}^*$, or a polarization potential $\mathfrak{Z}^*$, and the corresponding polarization vector $\mathfrak{P}^*$, which is coupled with $\rho^*$ and $\mathbf{j}^*$ by the relations:

$$\rho = -\operatorname{div}\mathfrak{P}^*, \qquad \mathbf{j}^* = \frac{1}{c}\frac{\partial\mathfrak{P}^*}{\partial t}, \tag{50}$$

so $\mathfrak{Z}^*$ must be determined from the equation:

$$-\nabla^2\mathfrak{Z}^* + \frac{1}{c^2}\frac{\partial^2\mathfrak{Z}^*}{\partial t^2} = 4\pi\mathfrak{P}^*. \tag{50.a}$$

If $\mathfrak{Z}^*$ is known then one can calculate $\mathfrak{E}^*$ and $\mathfrak{H}^*$ from the formulas:

$$\mathfrak{E}^* = -\,\nabla\varphi^* - \frac{1}{c}\frac{\partial\mathfrak{A}^*}{\partial t}, \qquad \mathfrak{H}^* = \operatorname{rot}\mathfrak{A}^*,$$

and

$$\varphi^* = -\operatorname{div}\mathfrak{Z}^*, \qquad \mathfrak{A}^* = \frac{1}{c}\frac{\partial\mathfrak{Z}^*}{\partial t},$$

i.e.:

$$\mathfrak{E}^* = \nabla\operatorname{div}\mathfrak{Z}^* - \frac{1}{c^2}\frac{\partial^2\mathfrak{Z}^*}{\partial t^2}, \qquad \mathfrak{H}^* = \operatorname{rot}\frac{1}{c}\frac{\partial\mathfrak{Z}^*}{\partial t}. \tag{50.b}$$

The vector $\mathfrak{Z}^*$ is nothing but the magnetic polarization potential that we introduced already in **§ 4**; $\mathfrak{P}^*$ corresponds to our previous $\mathfrak{M}$. The sense of the formulas above is to be found in the fact that one can replace the actual distribution of electric charge and current density inside of $S$ with a fictitious distribution of infinitely-small magnets with time-varying moments whose resultant per unit volume is equal to $\mathfrak{P}^*$. That representation is very convenient in many cases and allows us to calculate the electromagnetic field of a relatively-complicated system very simply.

We imagine, e.g., an elementary current of time-varying strength. In so doing, the dimensions of the current line ($\sigma$) shall be small in comparison to not only the distance to the reference point $P$, but also the wavelength that corresponds to the frequency of the oscillations in current strength (or also the fluctuations in the orientation of $\sigma$ when such things occur). One calls such a system a *magnetic oscillator* since it must obviously be identical in regard to its external effects to a (fictitious) magnetic dipole whose moment $\mathfrak{m}$ coincides with the magnetic moment of the current at each instant. For instance, if the current line $\sigma$ is planar and the surface that it surrounds is equal to $A$ then, as is known, $m = i\,S$, where $i$ means the current strength at the instant in question.

The magnetic polarization potential of that oscillator must obviously depend upon $\mathfrak{m}$ in the same way that the electric polarization potential $\mathfrak{Z}$ depends upon the electric moment of an oscillating dipole. From (28.a), we will then have:

$$\mathfrak{Z}^*\,(\mathrm{r},\,t) = \frac{1}{R}\,\ddot{\mathfrak{m}}'\,, \tag{51}$$

in which $\mathfrak{m}'$ means the value of $\mathfrak{m}$ at the effective instant $t' = t - R\,/\,c$. That will give the same formulas for the quantities $\mathfrak{E}^*$ and $\mathfrak{H}^*$ that we had posed in **§ 6** for the electric and magnetic field strength of an electric oscillator. In order to get the corresponding field strengths in the case considered (viz., magnetic oscillator!), from (49.b), we must replace $\mathfrak{E}$ (= $\mathfrak{E}^*$) with $\mathfrak{H}$ and $\mathfrak{H}$ (= $\mathfrak{H}^*$) with $-\,\mathfrak{E}$ in those formulas (and obviously replace $\mathfrak{p}$ with $\mathfrak{m}$, in addition). In that way, we will get the following formulas for the electromagnetic field in the wave zone:

$$\mathfrak{E} = \frac{1}{c^2 R}\,\mathfrak{N}_0 \times \ddot{\mathfrak{m}}'\,, \tag{51.a}$$

$$\mathfrak{H} = \frac{1}{c^2 R}\,\mathfrak{N}_0 \times (\mathfrak{N}_0 \times \ddot{\mathfrak{m}}')\,, \tag{51.b}$$

instead of (31) and (32), from which it will follow that the relations (32.a) that we saw must always be fulfilled in the wave zone in **§ 10** will actually remain true in the present case.

## CHAPTER SIX

# THE ELECTROMAGNETIC FIELDS OF MOVING POINT-CHARGES (ELECTRONS)

### § 1. The electromagnetic potential of a moving point charge.

In order to calculate the electromagnetic potential that is created by an electron that moves arbitrarily by using the general formulas (26) and (27), Chap. V, it is necessary, *first of all*, to have a well-defined picture for the "structure" of the electron, i.e., the spatial distribution of the electric charge that belongs to the electron or "defines it," so to speak, and certainly when it is in its *rest state*, and *secondly*, to express the charge density $\rho$ and the current density $\mathbf{j}$ for a moving electron by the aforementioned "rest density" $\rho_0$ and the translational velocity of the electron (or also its angular velocity). In that way, one can either treat the electron as a rigid body or as a deformable body. In the latter case, one must further consider its deformation.

Ordinarily, one considers an electron to be a rigid (or also deformable in a certain sense) ball of a certain radius $a$ and imagines that the electron charge $e$ is distributed uniformly over the outer surface or throughout the volume of that ball.

That "classical" conception of the electron as a spatially-extended "structure" can give rise to some very serious physical and epistemological objections that we will consider in detail later on (Chap. VII). We would first like to try (while completely overlooking the aforementioned objections, merely on the grounds of simplicity) to treat the electron as a *point-charge*, just as we often did before in our investigation of static fields. We then pose the problem of extending the simple formula $\varphi = e'/R$ for the scalar potential of a point-charge at rest to the case of an arbitrarily-moving point-charge and to replace the previously-posed formula for the vector potential of a moving point-charge $\mathfrak{A} = \dfrac{e'\mathfrak{v}'}{cR}$, which is only true approximately, with an exact formula.

To that end, we *temporarily* imagine that the electron is a rigid structure of *very-small* dimensions, and indeed with a volume $V_0$ in which the electron charge is distributed uniformly with the rest density $\rho_0 = e'/V_0$. The case that is of interest to us of a point-like electron will be given later as the limiting case as $V_0 \to 0$. We further imagine that this electron moves with a translational velocity $\mathfrak{v}$ (without rotation), in which $\mathfrak{v}$ does not need to be constant in time.

At each moment, the moving electron will "fill up" a certain "rest volume" $V = V_0$, inside of which the electric charge density $\rho$ will coincide with $\rho_0$, while outside of it, one will have $\rho = 0$. As far as the current density $\mathbf{j}$ is concerned, by definition (as the spatial density of the electrical impulse), it is obviously coupled with $\rho$ by the relation:

$$\mathbf{j} = \frac{\rho}{c}\mathfrak{v}. \tag{1}$$

In order to calculate the scalar potential at a point $P$ (radius vector $\mathfrak{r}$), we would like to use the first of formulas (27.a), Chap. V, and the picture that is connected with it of a sphere of action that contracts to $P$ at a rate of $c$. That sphere shall contract to the point $P$ at time $t$. It must meet the electron somewhat earlier than that, while remaining in contact for a very brief time $\tau'$, or when expressed more precisely, passing through the electron, and then leaving it again. The only exceptional case is defined when the electron moves past $P$ with a speed of $v = c$. We would like to exclude that case, which would correspond to an ongoing contact. For the sake of definiteness, we would like to further assume that:

$$v < c\,, \tag{1.a}$$

i.e., that the electron moves with "subluminal velocity."

For infinitely-small dimensions of the electron, the effective time interval $\tau'$ during which the action of the electron that is received at the point $P$ at time $t$ is emitted will also be infinitely small. For that reason, we can treat the velocity of the electron during that interval $\tau'$ as (= $\mathfrak{v}'$).

We now consider two moments that lie inside of $\tau'$:

$$t_1' = t - R_1'/c \quad \text{and} \quad t_2' = t - R_2'/c \qquad (t_2' > t_1'),$$

and compare the distance $R_1 - R_2$ between the corresponding positions of the sphere of action in the electron with the thickness of the electron layer that is "swept through" in that way being $l$. If one denotes the radial velocity of the electron (relative to the reference point $P$) by $v_R'$ and observes that the radial velocity of the spherical surface element $S$ that cuts the electron *relative to the electron* is equal to the difference $c - v_R'$ then one will get the equations:

$$\tfrac{1}{2}(R_1' - R_2') = l/(c - v_R') = t_2' - t_1',$$

from which it will follow that:

$$dR = dl/(1 - v_R'/c)\,, \tag{2}$$

and from (27.a):

$$\varphi = \frac{1}{1 - v_R'/c}\iint \frac{\rho}{R} R^2\, dl\, d\Omega = \frac{1}{(1 - v_R'/c)R'}\int \rho\, dV_0\,,$$

i.e.:

$$\varphi = \frac{e'}{R'(1 - v_R'/c)}\,. \tag{3}$$

What corresponds to the usual *Coulomb* potential of a point-charge at rest that originates in the motion of the electron then consists of, firstly, replacing the simultaneous position of the latter with the retarded position at the moment $t' = t - R'/c$, and secondly, introducing the factor $1/(1 - v_R'/c)$, which can be interpreted only when one imagines that the electron is not point-like, but infinitely-small, and indeed as the relative change in the effective volume or the effective time

interval that corresponds to the emission of the action that is received at the point *P* at the moment *t*.

One will then have:

$$\mathfrak{A} = \frac{e'}{c\,R'(1-v'_R/c)}\mathfrak{v}' \tag{4}$$

or

$$\mathfrak{A} = \frac{1}{c}\varphi\,\mathfrak{v}'\,. \tag{4.a}$$

Upon differentiating the expressions (3) and (4) directly, one can easily prove that it is actually enough for them to satisfy the differential equations:

$$-\nabla^2\varphi + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0\,, \qquad -\nabla^2\mathfrak{A} + \frac{1}{c^2}\frac{\partial^2\mathfrak{A}}{\partial t^2} = 0\,, \qquad \text{resp.,}$$

and indeed, for all points in space and time except for those spacetime points where the electron itself is found and which are characterized by $R' = 0$ ([1]). In so doing, one must treat $R' = R(t')$ as the function of time that is defined by the equation:

$$t' = t - R(t')/c \tag{5}$$

and corresponds to the motion of the electron in question, i.e., the dependency of its radius vector $\mathfrak{r}'$ on time $t'$, which is assumed to be known. From the usual definition, one has:

$$\mathfrak{R}(t') = \mathfrak{r} - \mathfrak{r}'(t')\,, \tag{5.a}$$

in which $\mathfrak{r}$ means the radius vector of the reference point.

## § 2. The electric and magnetic field strengths.

In order to calculate the electric and magnetic field strengths that belong to the potentials (3) and (4), we must first derive some auxiliary formulas that are obtained by differentiating equations (5) and (5.a) with respect to the two independent variables *t* and $\mathfrak{r}$. For the sake of abbreviation, we then introduce the following notations:

$$\frac{1}{1-v'_R/c} = \gamma \tag{6}$$

([1]) We will refer to this proof later on in the context of a new derivation of formulas (3) and (4). Note that those formulas were first derived in the way that was described above by *Liénard* and *Wiechert*. For that reason, the potentials (3) and (4) are usually referred to as the *Liénard-Wiechert* potentials.

and

$$R'(1-v'_R/c) = R' - \frac{1}{c}\mathfrak{R}'\mathfrak{v}' = R^*, \tag{6.a}$$

or

$$R' = \gamma R^*. \tag{6.b}$$

From (5.a), one has:

$$\frac{d\mathfrak{R}'}{dt'} = -\frac{d\mathfrak{r}'(t')}{dt'} = -\mathfrak{v}',$$

and as a result, since:

$$\mathfrak{R}'\frac{d\mathfrak{R}'}{dt'} = R'\frac{dR'}{dt'},$$

one will have:

$$\frac{dR'}{dt'} = -\frac{(\mathfrak{R}'\mathfrak{v}')}{R'} = -v'_R.$$

Upon differentiating (5) with respect to $t$ and $\mathfrak{r}$, we will get:

$$\frac{\partial t'}{\partial t} = 1 - \frac{1}{c}\frac{dR'}{dt'}\frac{\partial t'}{\partial t} = 1 + \frac{v'_R}{c}\frac{\partial t'}{\partial t}$$

and

$$\nabla t' = -\frac{1}{c}\nabla R' = -\frac{1}{c}(\nabla R')_{t'=\text{const.}} - \frac{1}{c}\frac{dR'}{dt'}\nabla t' = -\frac{R'_0}{c} + \frac{v'_R}{c}\nabla t',$$

where $\mathfrak{R}'_0 = \mathfrak{R}'/R$ means the unit vector on the direction of the electron reference point (effective position). It will then follow that:

$$\frac{\partial t'}{\partial t} = \gamma, \qquad \nabla t' = -\frac{\gamma}{c}\mathfrak{R}'_0, \tag{7}$$

and furthermore:

$$\frac{\partial t'}{\partial t} = -\gamma v'_R, \quad \nabla R' = \gamma \mathfrak{R}'_0. \tag{7.a}$$

In what follows, we will denote differentiation with respect to $\mathfrak{r}$ for fixed $t'$ by $\nabla'$ [$= (\nabla)_{t'=\text{const.}}$].

We will then have:

$$\nabla\varphi = \nabla'\varphi + \frac{\partial\varphi}{\partial t'}\nabla t' = -\frac{e'}{R^{*2}}\left[\nabla R' - \nabla'\left(\mathfrak{R}'\frac{\mathfrak{v}'}{c}\right)\right] + \frac{e'}{R^{*2}}\frac{\gamma\mathfrak{R}'_0}{c}\left[-v'_R + \frac{v'^2_R}{c} - \frac{1}{c}\mathfrak{R}'\frac{d\mathfrak{v}'}{dt'}\right],$$

i.e., due to the fact that:

$$\nabla'\mathfrak{R}' = \mathfrak{R}'_0, \quad \nabla'(\mathfrak{R}'\mathfrak{v}') = \mathfrak{v}',$$

and when we denote the acceleration of the electron $d\,\mathfrak{v}'/dt$ by $\mathfrak{w}'$:

$$\nabla\varphi = -\frac{e'}{R^{*2}}\left\{\mathfrak{R}'_0 - \mathfrak{v}'/c + \gamma\,\mathfrak{R}'_0\left[v'_R/c - (v'/c)^2 + \frac{1}{c}\mathfrak{R}'\mathfrak{w}'\right]\right\}. \tag{7.b}$$

Moreover, we have:

$$\frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\left(\varphi\frac{\mathfrak{v}'}{c}\right) = \left(\frac{\partial\varphi}{\partial t'}\frac{\mathfrak{v}'}{c^2} + \frac{\varphi}{c^2}\frac{d\mathfrak{v}'}{dt'}\right)\frac{\partial t'}{\partial t} = -\frac{e'}{R^{*2}}\frac{\gamma}{c^2}\left[v'_R/c - (v'/c)^2 + \frac{1}{c}\mathfrak{R}'\mathfrak{w}'\right] + \frac{e'}{c^2R^*}\gamma\,\mathfrak{w}'$$

and

$$\text{rot}\,\mathfrak{A} = \text{rot}\left(\varphi\frac{\mathfrak{v}'}{c}\right) = \text{rot}'\left(\varphi\frac{\mathfrak{v}'}{c}\right) + \nabla t'\times\frac{\partial}{\partial t'}\left(\varphi\frac{\mathfrak{v}'}{c}\right) = \nabla'\varphi\times\frac{\mathfrak{v}'}{c} - \frac{\gamma}{c}\mathfrak{R}'_0\times\frac{\partial}{\partial t'}\left(\varphi\frac{\mathfrak{v}'}{c}\right).$$

From those formulas, and the facts that $\mathfrak{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t}$ and $\mathfrak{H} = \text{rot}\,\mathfrak{A}$, one will get:

$$\mathfrak{E} = \frac{e'}{R'^2}\gamma^3\left(1 - \frac{v'^2}{c^2} + \frac{\mathfrak{R}'\mathfrak{w}'}{c^2}\right)\left(\mathfrak{R}'_0 - \frac{1}{c}\mathfrak{v}'\right) - \frac{e'}{c^2R'}\gamma^2\mathfrak{w}', \tag{8}$$

$$\mathfrak{H} = \frac{e'}{c\,R'^2}\gamma^3\left(1 - \frac{v'^2}{c^2} + \frac{\mathfrak{R}'\mathfrak{w}'}{c^2}\right)\mathfrak{v}'\times\mathfrak{R}'_0 + \frac{e'}{c^2R'}\gamma^2\mathfrak{w}'\times\mathfrak{R}'_0, \tag{9}$$

when one replaces $1/R^*$ with $\gamma/R'$. The magnetic and electric field strengths are then coupled with each other by the simple relation:

$$\mathfrak{H} = \mathfrak{R}'_0\times\mathfrak{E}\,. \tag{9.a}$$

Note that this relation coincides with one of the relations (32.a) in Chap. V. It shows that the magnetic field strength always points perpendicular to the radius vector to the respective effective position of the electron from the reference point and that its magnitude can never exceed the magnitude of the electric field strength. However, the second of the aforementioned relations ($\mathfrak{E}^{(2)} = \mathfrak{H}^{(2)}\times\mathfrak{R}'_0$) is generally fulfilled only in the wave zone, i.e., for the part of the two field strengths that is proportional to the first power of $1/R'$. Namely, those parts of (8) and (9) are equal to:

$$\mathfrak{E}^{(2)} = \frac{e'}{c^2R'}\gamma^2\left\{\gamma(\mathfrak{R}'_0\,\mathfrak{w}')\left(\mathfrak{R}'_0 - \frac{1}{c}\mathfrak{v}'\right) - \mathfrak{w}'\right\}, \tag{10}$$

$$\mathfrak{H}^{(2)} = \frac{e'}{c^2R'}\gamma^2\left\{\gamma(\mathfrak{R}'_0\,\mathfrak{w}')\left(\frac{1}{c}\mathfrak{v}'\times\mathfrak{R}'_0\right) + \mathfrak{w}'\times\mathfrak{R}'_0\right\}, \tag{10.a}$$

such that:

$$\mathfrak{H}^{(2)} \times \mathfrak{R}'_0 = \frac{e'}{c^2 R'} \gamma^2 \left\{ \gamma (\mathfrak{R}'_0 \mathfrak{w}') \left( \frac{v'_R}{c} \mathfrak{R}'_0 - \frac{1}{c} \mathfrak{v}' \right) + (\mathfrak{R}'_0 \mathfrak{w}') \mathfrak{R}'_0 - \mathfrak{w}' \right\} .$$

If one now observes the identity:

$$\gamma \frac{v'_R}{c} \mathfrak{R}'_0 = \gamma \left( \frac{v'_R}{c} \mathfrak{R}'_0 - \mathfrak{R}'_0 \right) + \gamma \mathfrak{R}'_0 = - \mathfrak{R}'_0 + \gamma \mathfrak{R}'_0$$

then the expression in curly brackets in the previous formula will assume the form:

$$\gamma (\mathfrak{R}'_0 \mathfrak{w}') (\mathfrak{R}'_0 - \frac{1}{c} \mathfrak{v}') - \mathfrak{w}' ,$$

which is, consequently, identical to the corresponding expression in (10). Thus, one has the following two relations in this case:

$$\left. \begin{aligned} \mathfrak{H}^{(2)} &= \mathfrak{R}'_0 \times \mathfrak{E}^{(2)}, \\ \mathfrak{E}^{(2)} &= \mathfrak{H}^{(2)} \times \mathfrak{R}'_0 , \end{aligned} \right\} \tag{10.b}$$

which are characteristic of the electromagnetic field in the wave zone.

We now imagine that the electron in question performs regular or irregular oscillations of very small amplitude about a certain fixed equilibrium position $P'$. We further imagine that an electron with the opposite charge $- e'$ rests at that point. We then get an elementary oscillator whose field was examined before in the previous chapter in **§ 6**. In that case, the formulas (10), (10.a) must obviously coincide with the previous formulas (31), (32). As is easy to see, that is, in fact fulfilled. Indeed, one can neglect the quantity $v' / c$ ($v'_R / c$, resp.), due to the smallness of the oscillations (since the speed of the electron must remain very small compared to the speed of light), and set $R' = R = P' P$ (= const.), moreover. Formulas (10) and (10.a) will then reduce to:

$$\mathfrak{E}^{(2)} = \frac{e'}{c^2 R} \{ (\mathfrak{R} \mathfrak{w}') \mathfrak{R}_0 - \mathfrak{w}' \} = \frac{e'}{c^2 R} \mathfrak{R}_0 \times (\mathfrak{R}_0 \times \mathfrak{w}') , \tag{11}$$

$$\mathfrak{H}^{(2)} = \frac{e'}{c^2 R} \mathfrak{w}' \times \mathfrak{R}_0 . \tag{11.a}$$

However, those expressions will be identical to (31) and (32), Chap. V, when one observes that $\ddot{\mathfrak{p}}' = e' \mathfrak{w}'$ in the case considered.

Formulas (11), (11.a) correspond to the extreme case in which the velocity of the electron vanishes in comparison to the acceleration. The general formulas (8), (9) will also take on an especially-simple form in the opposite extreme case when the acceleration vanishes in comparison to the velocity, i.e., when the electron (practically) moves in a uniform, rectilinear way.

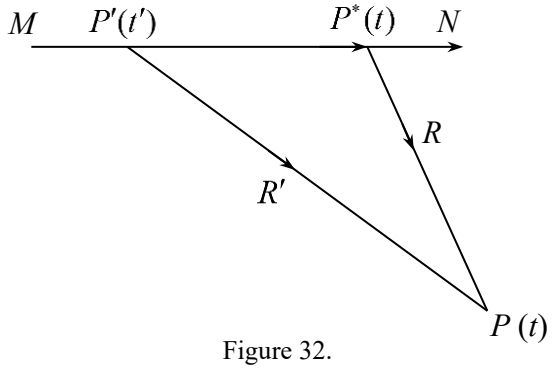

Figure 32.

Let the path of the electron be represented by the line *MN* (Fig. 32). Let its effective position at the instant $t' = t - R'/c$ be $P'$. While the action that is emitted from $P'$ runs to the reference point $P$ along the path $P'P$, the electron will displace from $P'$ to $P^*$. $P^*$ will then be its "instantaneous" position at the moment *t*. The equation $P'P^* = P'P - P^*P$ is obviously true, i.e.:

$$\mathfrak{v}'(t-t') = \mathfrak{R}' - \mathfrak{R},$$

and since $t - t' = R'/c$, it will follow from this that:

$$\mathfrak{R} = \mathfrak{R}' - \frac{R'}{c}\mathfrak{v}' = R'\left(\mathfrak{R}'_0 - \frac{1}{c}\mathfrak{v}'\right). \tag{12}$$

The vector $\mathfrak{R}'_0 - \frac{1}{c}\mathfrak{v}'$ that appears in (7) is then equation to $(1/R')\mathfrak{R}$. If one replaces $\gamma/R'$ with $1/R^*$ in (8) and introduces the notation:

$$\frac{v'}{c} = \beta', \tag{12.a}$$

to abbreviate, then one will have:

$$\mathfrak{E} = \frac{e'}{R^{*3}}(1-\beta'^2)\mathfrak{R}. \tag{13}$$

Moreover, when one observes the relation:

$$\mathfrak{v}'\times\mathfrak{R}'_0 = \frac{1}{R'}\mathfrak{v}'\times\mathfrak{R}' = \frac{1}{R'}\mathfrak{v}'\times\left(\mathfrak{R} + \frac{R'}{c}\mathfrak{v}'\right) = \frac{1}{R'}\mathfrak{v}'\times\mathfrak{R},$$

which follows from (12), one will get from (9) that:

$$\mathfrak{H} = \frac{e'}{c\,R^{*3}}(1-\beta'^2)\mathfrak{v}'\times\mathfrak{R}. \tag{13.a}$$

It is very remarkable that the "instantaneous" radius vector $\mathfrak{R}$ enters in place of the effective radius vector $\mathfrak{R}'$ in those formulas. That means that the *forms* of the lines of force of the electric and magnetic fields of a point-charge in uniform, rectilinear motion will behave as if the electromagnetic action-at-a-distance were *instantaneous*. For example, the electric lines of force are straight lines that emanate from $P'$, just as in the case of a charge at rest at $P^*$. However, in contrast to the static case, the density of that family of lines must vary in different directions, such

that the electric field will exhibit no radial symmetry (relative to $P^*$), but only an axial symmetry relative to the path-line *MN*. (13) and (13.a) imply the relation:

$$\mathfrak{H} = \frac{1}{c}\mathfrak{v}' \times \mathfrak{E} \ , \tag{13.b}$$

which is true for only constant velocity of the electron, in contrast to (9.a). Obviously, formulas (13) to (13.b) will also remain approximately true when that velocity varies relatively slowly (quasi-stationary motion).

## § 3. – Special consideration of uniform rectilinear motion.

We now imagine that a second electron with charge *e* moves with a constant velocity $\mathfrak{v}$ and is found at the point *P* at the moment *t*. As is known, the force that acts on that electron as a result of the first one $(e')$ is expressed by the general formula:

$$\mathfrak{F} = e\,(\mathfrak{E} + \frac{1}{c}\mathfrak{v} \times \mathfrak{H}) \ ,$$

which will assume the form:

$$\mathfrak{F} = \frac{e'e}{R^{*3}}(1-\beta'^2)\left\{\left(1-\frac{\mathfrak{v}\mathfrak{v}'}{c^2}\right)\mathfrak{R} + \frac{1}{c}\left(\frac{1}{c}\mathfrak{R}\,\mathfrak{v}\right)\mathfrak{v}'\right\} \tag{14}$$

in our case ($\mathfrak{v}'$ = const.), from (13) and (13.a) [or (13.b)]. (Cf., the approximate expression (23.b), Chap. III for the electromagnetic force $\mathfrak{f} = \frac{1}{c}\mathfrak{v} \times \mathfrak{H}$.)

The quantities that characterize the two electrons appear asymmetrically in that formula. That is why one cannot speak of the equality of action and reaction for arbitrary $\mathfrak{v}$ and $\mathfrak{v}'$. The only exception to that is the case in which the two electrons move with the same velocity, i.e., they remain at rest relative to each other. In that case ($\mathfrak{v} = \mathfrak{v}'$, $\beta' = \beta$), (14) reduces to:

$$\mathfrak{F} = \frac{e\,e'}{R^{*3}}(1-\beta^2)\left\{(1-\beta^2)\,\mathfrak{R} + \frac{1}{c}\left(\frac{1}{c}\mathfrak{R}\,\mathfrak{v}\right)\mathfrak{v}\right\}. \tag{14.a}$$

The quantity $R^*$ has the same value for the two electrons. As a result, one will get the force $\mathfrak{F}'$ that acts on the first electron on the part of the second one from (14.a) by simply switching the sign of the vector $\mathfrak{R}$, such that one will have $\mathfrak{F}' = -\mathfrak{F}$.

Formula (14.a) can be converted as follows: From (12), when we recall (6.a), we will have:

$$(1-\beta^2)\,\mathfrak{R}+\frac{1}{c}\left(\frac{1}{c}\mathfrak{R}\,\mathfrak{v}\right)\mathfrak{v} = (1-\beta^2)\,\mathfrak{R}'+\frac{1}{c}\left[R'(1-\beta^2)-\frac{1}{c}\mathfrak{R}'\mathfrak{v}+\frac{v^2}{c^2}R'\right]\mathfrak{v}$$
$$= (1-\beta^2)\,\mathfrak{R}'+\frac{1}{c}R^*\,\mathfrak{v}\,.$$

On the other hand, from (7.b) (with $\mathfrak{w}' = 0$):

$$\nabla R^* = \mathfrak{R}'_0-\frac{1}{c}\mathfrak{v}'+\gamma\left(\frac{v'_R}{c}-\frac{v'^2}{c^2}\right)\mathfrak{R}'_0,$$

and as a result, since $\gamma R^*\,\mathfrak{R}'_0 = \mathfrak{R}$:

$$R^*\,\nabla R^* = -\frac{1}{c}R^*\mathfrak{v}'+\left(1-\frac{v'^2}{c^2}\right)\mathfrak{R}'.$$

One then has (for $\mathfrak{v}' = \mathfrak{v}$):

$$(1-\beta^2)\,\mathfrak{R}+\frac{1}{c}\left(\frac{1}{c}\mathfrak{R}\,\mathfrak{v}\right)\mathfrak{v} = R^*\,\nabla R^*,$$

which will make formula (14.a) assume the form:

$$\mathfrak{F} = \frac{e\,e'(1-\beta^2)}{R^{*3}}\nabla R^*,$$

or

$$\mathfrak{F} = -\,e\,\nabla\psi, \tag{15}$$

with:

$$\psi = \frac{e'(1-\beta^2)}{R^*} = (1-\beta^2)\,\varphi\,. \tag{15.a}$$

One can derive that result in a simpler way when one considers the situation in which the two electrons move with the same constant velocity from the outset. Indeed, the potentials $\varphi$ and $\mathfrak{A}$ that determine the action of $e'$ on $e$ must remain constant in time. That follows from the fact that those potentials depend upon only the velocity $\mathfrak{v}$ and the relative position of the two electrons (at the same moment), i.e., the radius vector $\mathfrak{R}$. However, $\mathfrak{v}$ and $\mathfrak{R}$ are constant in our present case.

One then has $d\varphi / dt = 0$ and $d\,\mathfrak{A} / dt = 0$, in which the symbol $d / dt$ means the *complete* differentiation with respect to time, i.e., the rate of change of the corresponding quantity at a reference point that moves with $e$. Now, when the time derivative at a fixed spatial point is denoted by $\partial / \partial t$, one will have:

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t}+\frac{d\mathfrak{r}}{dt}\cdot\nabla\varphi = \frac{\partial\varphi}{\partial t}+\mathfrak{v}\cdot\nabla\varphi, \tag{16}$$

and likewise:

$$\frac{d\mathfrak{A}}{dt} = \frac{\partial \mathfrak{A}}{\partial t} + (\mathfrak{v}\cdot\nabla)\,\mathfrak{A}\,. \tag{16.a}$$

As a result, the relation:

$$-\frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t} = \left(\frac{1}{c}\mathfrak{v}\cdot\nabla\right)\mathfrak{A}$$

will exist in the case considered, or since $\mathfrak{A} = (\varphi / c)\,\mathfrak{v}$ :

$$-\frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t} = \frac{1}{c}\left(\frac{1}{c}\mathfrak{v}\cdot\nabla\varphi\right)\mathfrak{v}\,.$$

By means of that relation, we will get:

$$\mathfrak{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t} = -\nabla\varphi + \frac{1}{c}\left(\frac{1}{c}\mathfrak{v}\cdot\nabla\varphi\right)\mathfrak{v}\,.$$

Moreover, one has:

$$\mathfrak{H} = \text{rot}\,\mathfrak{A} = \frac{1}{c}\nabla\varphi\times\mathfrak{v}\ .$$

As a result, one will have:

$$\mathfrak{F} = e\,(\mathfrak{E} + \frac{1}{c}\mathfrak{v}\times\mathfrak{H}) = e\left[-\nabla\varphi + \frac{1}{c}\left(\frac{1}{c}\mathfrak{v}\nabla\varphi\right)\mathfrak{v} + \left(\frac{\mathfrak{v}}{c}\right)^2\nabla\varphi - \frac{1}{c}\left(\frac{1}{c}\mathfrak{v}\cdot\nabla\varphi\right)\mathfrak{v}\right]$$
$$= e\,(1 - v^2/c^2)\,\nabla\varphi\,.$$

We have then found the previous result once more without explicitly using the expression for $\varphi$.

That result can obviously be generalized to an arbitrary system of electrons that possess a common constant translational velocity. The interaction of the electrons then preserves the same character that it has in a state of absolute rest ($v$ = 0). The electric and electromagnetic force combine into a force $\mathfrak{F}$ that one can interpret as an ordinary electric force that corresponds to the scalar potential $\psi$.

One cares to refer to that potential as a *convective* potential. One can correspondingly refer to the vector field $-\nabla\psi$ as a convective force field or also as the effective electric field strength.

The lines of the vector $-\nabla\psi$ are not straight lines, in contrast to the electric lines of force, since the level surfaces of the convective potentials, i.e., the surfaces $\psi$ = const. are not spherical as they are for a charge at rest, but *spheroidal*.

In order to see that, we introduce a rectangular coordinate system $X_1$, $X_2$, $X_3$ with its origin at the point $P^*$ (where the electron $e'$ is found at the moment in question). Let the $X_1$-axis of that system then be the line *MN* (Fig. 32), i.e., it shall point in the direction of motion.

If one denotes the components of the vectors $\mathfrak{R}$ and $\mathfrak{R}'$, i.e., the coordinates of the reference point $P$ relative to $P^*$ and $P'$, resp., by $x_k$ ( $x'_k$, resp.) ($k$ = 1, 2, 3) then one can replace the vector equation (12):

$$\mathfrak{R} = \mathfrak{R}' - \frac{R'}{c}\mathfrak{v}$$

with three scalar equations:

$$x_1 = x'_1 - \frac{v}{c}R', \qquad x_2 = x'_2, \qquad x_3 = x'_3. \tag{17}$$

Due to the fact that $R' = \sqrt{x'_1 + x'_2 + x'_3}$, it will follow from this that:

$$(x'_1 - x_1)^2 = \beta^2(x'^2_1 + x^2_2 + x^2_3) \qquad (\beta = v/c)$$

or

$$x'^2_1(1-\beta^2) - 2x_1 x'_1 + [x^2_1 - \beta^2(x^2_2 + x^2_3)] = 0 .$$

Upon solving that equation (for $x'_1$, and recalling the condition that $x'_1 - x_1 = \beta R' > 0$), we will get:

$$x'_1 = \frac{x_1 + \beta\sqrt{x^2_1 + (1-\beta^2)(x^2_2 + x^2_3)}}{1-\beta^2}. \tag{17.a}$$

Now, one has:

$$R^* = R' - \frac{1}{c}(\mathfrak{R}'\mathfrak{v}) = R' - \beta x'_1,$$

or from (17):

$$R^* = \frac{x'_1 - x_1}{\beta} - \beta x'_1 = \frac{x'_1(1-\beta^2) - x_1}{\beta},$$

and as a result, from (17.a):

$$R^* = \sqrt{x^2_1 + (1-\beta^2)(x^2_2 + x^2_3)}. \tag{17.b}$$

We then see that the surfaces $\psi$ = const., just like the surfaces $\varphi$ = const. and $\mathfrak{A}$ = const. are determined by the equation:

$$x^2_1 + (1-\beta^2)(x^2_2 + x^2_3) = R^{*2} = \text{const.}$$

or

$$\frac{x^2_1}{(R^*\sqrt{1-\beta^2})^2} + \frac{x^2_2}{R^{*2}} + \frac{x^2_3}{R^{*2}} = 1 .$$

As a result, they can be represented as flattened ellipsoids that arise from a ball by longitudinal contraction with a ratio of $\sqrt{1-\beta^2} : 1$ (Fig. 33). The lines of the effective electric field of force $(-\nabla\psi)$, just like the electric and magnetic lines of force, will correspondingly concentrate in the neighborhood of the equatorial plane, while the field strengths will be diminished in the neighborhood of the line of motion.

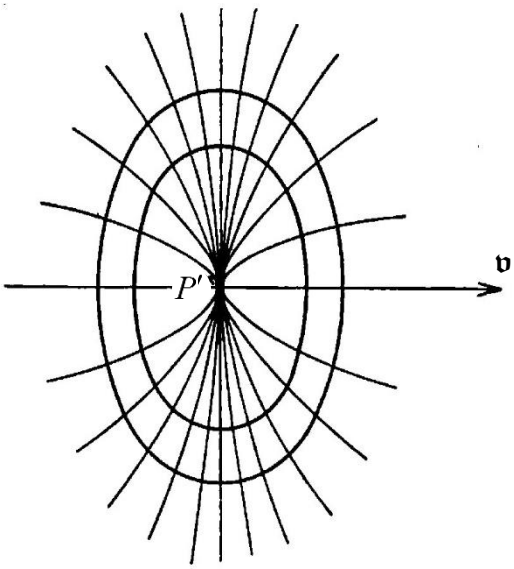

Figure 33.

That flattening, or "contraction," of the electromagnetic field in the direction of motion will initially increase slowly with the increase in velocity, but then very rapidly when one is close to the critical velocity of $c$. In the limiting case $v = c$, the field of the electron must contract completely in the equatorial plane. We shall consider that fact to be a proof of the idea that the critical velocity (or, as one ordinarily says, the speed of light) *can never be achieved* in reality. According to (17.b), for $v > c$ (i.e., $\beta > 1$), the quantity $R^*$ will become imaginary outside of the cone:

$$x_1^2 = (\beta^2 - 1)(x_2^2 + x_3^2), \tag{17.c}$$

and therefore, the potentials and field strengths, as well.

However, we have already excluded that case, i.e., the case of "superluminal velocity," from the outset [see (1.a)]. We now ask whether we would arrive at the same results under the assumption that $v > c$. We would then have to distinguish two cases according to whether the electron pierces the sphere of action that converges to the reference point from inside to outside or from outside to inside ([1]). In the first case ($v_R < 0$), what was done in **§ 1** will remain completely unchanged. In the second case ($v_R > 0$), we must replace $c - v_R$ with $v_R - c$ in formula (2) and correspondingly invert the signs in the following formulas that include that expression. Therefore, when an electron is in uniform, rectilinear motion, only the sign of (i.e., of $\varphi$ and $\mathfrak{A}$) will change, but formulas (16) to (17.c) will suffer no change whatsoever. We would then see that the potentials and field strengths inside of the cone (17.c) would assume imaginary values in both cases. A motion with constant superluminal velocity would then seem to be physically impossible.

([1]) In general, both cases can happen at the same time. If the electron moves with, e.g., constant superluminal velocity then it must initially dive into the sphere of action and then fly away from it.

## § 4. – Electrons as point-singularities in the spacetime continuum. New derivation of the electromagnetic potential of a moving point-charge.

In **§ 1**, we have derived the formulas (3), (4) for the potential of a moving point-charge (of constant magnitude) on the basis of formulas (26) and (27), Chap. V, which referred to the case of charges of varying magnitude at rest, strictly speaking. However, one can get them in an entirely different, so-to-speak direct, way by a process of integration that is a generalization of the one that we had used in order to solve the *Laplace* equation for a point-charge *at rest* ([1]).

We introduce a rectangular coordinate system with an entirely-arbitrary *fixed* origin *O*. Let the coordinates of the reference point *P* be $x_1, x_2, x_3$. We shall denote the coordinates of the electron at an arbitrary moment $t'$ (which does not necessarily coincide with the effective moment) by $x_1'$, $x_2'$, $x_3'$.

We would like to combine a spatial point and the moment at which we consider it into a "spacetime point" in a four-dimensional spacetime continuum. Instead of speaking of the reference point *P* ($x_1, x_2, x_3$) at the moment *t*, we will simply say the reference point *P* ($x_1, x_2, x_3, t$). We will likewise refer to the position of the electron at the moment $t'$ as the source point $P'(x_1', x_2', x_3', t')$.

The electrostatic potential $\varphi$ and the components $A_1, A_2, A_3$ of the vector potential satisfy the differential equation:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

in all of the spacetime continuum, with the exception of the source point $P'$, and in the special case of a point-charge at rest, it will reduce to the *Laplace* equation $\nabla^2\psi = 0$. Therefore, we will have $\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$, and as a result, when we introduce the notation:

$$x_4 = i\,c\,t \qquad (i = \sqrt{-1}) \tag{18}$$

(and correspondingly $x_4' = i c t'$), we will have:

$$\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2} + \frac{\partial^2\psi}{\partial x_4^2} = 0\,. \tag{18.a}$$

That equation has the same form as the *Laplace* equation, but it includes four variables (viz., spacetime coordinates) $x_1, x_2, x_3, x_4$ in a completely-symmetric way, instead of three. We must further consider the fact that $\varphi$ and $\mathfrak{A}$ are coupled with each other by the relation:

$$\operatorname{div}\mathfrak{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0\,.$$

([1]) That derivation goes back to *Herglotz*.

That relation can be written in coordinates as follows:

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0 .$$

If one now introduces the relation that corresponds to (18):

$$A_4 = i\,\varphi \tag{19}$$

then that equation will assume the symmetric form:

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0 . \tag{19.a}$$

In place of spatial coordinates and time, we must then use four equally-justified "coordinates" in (18.a) and (19.a) and correspondingly replace the vector potential $\mathfrak{A}$ and the scalar $\varphi$ with four components of a "four-dimensional" vector potential (which will make $\psi = A_1, A_2, A_3, A_4$)

One can likewise treat the equations of motion of the electron (which are regarded as known):

$$x_1' = f_1(t'), \quad x_2' = f_2(t'), \quad x_3' = f_3(t'), \quad x_4' = i\,c\,t' \tag{20}$$

as the equations of a "line" in a four-dimensional "space" ($x_1$, $x_2$, $x_3$, $x_4$), and indeed as the set of all source-points of the electromagnetic field that the electron creates. For that reason, we would like to refer to that line as the *singular line* of the functions $A_k$ ($k = 1, \ldots, 4$).

We next consider the following hypothetical problem: Find a function that is symmetric in the four "coordinates" $x_1$, $x_2$, $x_3$, $x_4$, satisfies equation (18.a), and possesses *only one* singular point $P'(x_1', x_2', x_3', t')$. From a purely-formal standpoint, that problem corresponds completely to the integration of the usual three-dimensional *Laplace* equation for a point-charge at rest. We correspondingly replace the ordinary three-dimensional distance:

$$R = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}$$

with the four-dimensional distance:

$$S = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 + (x_4 - x_4')^2} = \sqrt{R^2 - c^2(t - t')^2}\,, \tag{21}$$

and consider $\psi$ to be a function of $S$.

We will then get:

$$\frac{\partial \psi}{\partial x_k} = \frac{d\psi}{dS}\frac{\partial \psi}{\partial x_k} = \frac{d\psi}{dS}\frac{x_k - x_k'}{S},$$

$$\frac{\partial^2\psi}{\partial x_k^2} = \frac{d^2\psi}{dS^2}\frac{(x_k - x_k')^2}{S^2} + \frac{d\psi}{dS}\frac{S^2 - (x_k - x_k')^2}{S^3},$$

and as a result:

$$\sum_{k=1}^{4}\frac{\partial^2\psi}{\partial x_k^2} = \frac{d^2\psi}{dS^2} + \frac{3}{S}\frac{d\psi}{dS} = 0,$$

from which, we will easily get:

$$\psi = \frac{\alpha}{S^2}. \tag{22}$$

Here, $\alpha$ means an arbitrary integration constant. (The second additive integration constant can be set equal to zero.)

We would now like to use the "point solution" (22) to (15.a) in order to construct functions $A_k$ that satisfy not only equation (18.a), but also the relation (19.a) We shall overlook the physical sense of those functions completely, for the time being, and pose the problem from a purely-analytical standpoint.

To that end, we shall treat the time $t'$ ($t$, resp.) as a *complex* variable, not a real one, and then imagine functions $f_1$, $f_2$, $f_3$ that are determined for only real values of $t'$ being continued analytically into the entire complete $t'$-plane. (In so doing, we must obviously assume that such an analytic continuation is actually permissible.)

We further draw an entirely-arbitrary line in that place and combine the "point solutions" $dA_k = \alpha_k' dt'/S^2$, which correspond to the isolated points (or more precisely, the infinitely-small elements $dt'$ of that line) according to (22), into a "line solution":

$$A_k = \int\frac{\alpha_k' dt'}{S^2}. \tag{22.a}$$

Therefore, the quantities $x_k'$ ($k = 1, \ldots, 4$) that appear in $S^2$ are treated as the functions of the complex argument $t'$ that are defined by (20) (by means of the aforementioned analytic continuation). By contrast, the coefficients $\alpha_k$ are initially undetermined functions of $t'$.

We must now choose those functions and the path of integration in such a way that the quantities (22.a) will satisfy the relation (19.a) when regarded as functions of $x_k$ ($k = 1, \ldots, 4$). Since the $x_k$ appear as parameters in the integrals (22.a), from (21), we will have:

$$\frac{\partial A_k}{\partial x_k} = \int\alpha_k'\frac{\partial}{\partial x_k}\left(\frac{1}{S^2}\right)dt' = -\int\alpha_k'\frac{\partial}{\partial x_k'}\left(\frac{1}{S^2}\right)dt',$$

and as a result:

$$\sum_{k=1}^{4}\frac{\partial A_k}{\partial x_k} = -\int\sum_{k=1}^{4}\frac{\partial}{\partial x_k'}\left(\frac{1}{S^2}\right)\alpha_k'\,dt'.$$

In order for the integral on the right-hand side to vanish identically, i.e., independently of the form of the functions $f_1$, $f_2$, $f_3$, it is obviously necessary and sufficient that, *first of all*, the coefficients $\alpha'_k$ can be expressed by the formulas:

$$\alpha'_k = \alpha \frac{dx'_k}{dt'}, \tag{22.b}$$

in which $\alpha$ means a constant, and *secondly*, that the path of integration must be represented by a *closed* curve. Namely, we will then get:

$$\sum_{k=1}^{4} \frac{\partial}{\partial x'_k}\left(\frac{1}{S^2}\right)\alpha'_k\, dt' = \alpha \sum_{k=1}^{4} \frac{\partial}{\partial x'_k}\left(\frac{1}{S^2}\right) dt' = \alpha\, d\left(\frac{1}{S^2}\right),$$

and as a result:

$$\sum_{k=1}^{4} \frac{\partial A_k}{\partial x_k} = -\,\alpha \oint d\left(\frac{1}{S^2}\right) \equiv 0\,.$$

One will then have:

$$A_k = \alpha \oint \frac{(dx'_k / dt')}{S^2}\, dt' = \alpha \oint \frac{dx'_k}{S^2}. \tag{23}$$

If the integration curve includes no pole of the integrand, i.e., no root of the function $S^2$, then the integral $\oint dx_k / S^2$ must vanish. In that way, we get a "trivial" solution of (18.a) and (19.a), namely, $A_k = 0$. By contrast, if a root $t' = t'_0$ of $S^2$ lies inside the integration curve then, as is known from *Cauchy*'s theorem ([1]):

$$\oint \frac{\alpha'_k}{S^2}\, dt' = 2\pi i\, \mathrm{Res}\left(\frac{\alpha'_k}{S^2}\right)_{t'=t'_0} = 2\pi i \left(\frac{\alpha'_k}{\dfrac{dS^2}{dt'}}\right)_{t'=t'_0}, \tag{23.a}$$

or since (21) implies that:

$$\frac{dS^2}{dt'} = 2\left[R\frac{dR}{dt'} + c^2(t-t')\right] \quad \text{and} \quad R_0 = \pm c\,(t-t'_0)\,,$$

---

([1]) Namely, we can contract the path of integration to an infinitely-small circle with its center at $t'_0$ in that case. Along that circle, it is sufficient to consider only the first two terms in the development of $S^2$ in powers of the difference $t'-t'_0$. However, since $S_0 = 0$, we will have $S^2 = \left(\dfrac{dS^2}{dt'}\right)_0 (t' = t'_0)$, and as a result:

$$\oint \frac{\alpha'_k\, dt'}{S^2} = \left(\frac{\alpha'_k}{\dfrac{dS^2}{dt'}}\right)_0 \oint \frac{dt'}{t'-t'_0} = 2\pi i \left(\frac{\alpha'_k}{\dfrac{dS^2}{dt'}}\right)_0.$$

one will have:

$$\oint \frac{\alpha'_k}{S^2}\, dt' = \pi i \left[ \frac{\alpha'_k}{c\, R\left( \frac{1}{c}\frac{dR}{dt'} \pm 1 \right)} \right]_{t'=t'_0}, \tag{23.b}$$

in which the upper sign corresponds to the equation:

$$R - c(t - t') = 0 \tag{23.c}$$

for a positive sense of traversal, and the lower one, to the equation:

$$R + c(t - t') = 0 \tag{23.d}$$

for the negative sense.

If the electron moves with a superluminal velocity then those equations will *each have one real root*. Therefore, the real root of (23.c) means nothing but the effective time that is characteristic of the retarded potential. Now, it is easy to see that for that value of $t'_0$, we will get the components of the potential $A_k$ that we found before in the (*Liénard-Wiechert*) formulas (3), (4). That is because we have:

$$\frac{1}{c}\frac{dR}{dt'} = -\frac{v'_R}{c},$$

and as a result, from (23) and (23,b):

$$A_k = \alpha \pi i \frac{dx_k / dt'}{c\, R(1 - v'_R / c)} \qquad (t' = t'_0),$$

i.e.:

$$\mathfrak{A} = \alpha \pi i \frac{1}{c\, R(1 - v'_R / c)} \mathfrak{v}',$$

and from (19):

$$\varphi = \frac{\alpha \pi i}{R(1 - v'_R / c)}.$$

Those formulas will coincide with (4) [(3), resp.] when one sets:

$$\alpha = \frac{e'}{\pi i}. \tag{24}$$

In the special case of an electron at rest or one in uniform, rectilinear motion, the equation $S^2 = 0$ will have no imaginary roots. One can then deform the path of integration into an arbitrary, closed or infinite, curve that runs between the two real roots $t' = t'_0$ and $t' = t''_0$ from below to above (Fig. 34). The simplest of those curves is the line that goes in the direction of the imaginary

axis through the point $t' = t$. If it were chosen to be the path of integration then, according to (23) and (24), one would get:

$$\varphi = \frac{A_4}{i} = -\frac{e'}{\pi}\int_{t'-t=-i\infty}^{t'-t=+i\infty}\frac{i\,c\,dt'}{R^2+(x_4'-x_4)^2} = \frac{e'}{\pi}\int_{-\infty}^{+\infty}\frac{d\,(x_4'-x_4)}{R^2+(x_4'-x_4)^2} = \frac{e'}{R}$$

in the case of an electron at rest ($R$ = const.), so the *Coulomb* potential.

If the electron moves in the positive $X_1$-direction with a velocity $v < c$ then when one sets $x_1' = vt' = -i\beta x_4'$ ($\beta = v/c$), $x_1' = x_2' = 0$, and $t = 0$ :

$$S^2 = x_1^2 + x_2^2 + x_3^2 + 2i\beta x_1 x_4' + x_4'^2(1-\beta^2) = (1-\beta^2)\left(x_4' + \frac{i\beta x_1}{1-\beta^2}\right)^2 + \frac{x_1^2 + (1-\beta^2)(x_2^2+x_3^2)}{1-\beta^2}.$$

If one denotes $x_1^2 + (1-\beta^2)(x_2^2+x_3^2)$ by $R^{*2}$ [as in (17.b)] and introduces the variable:

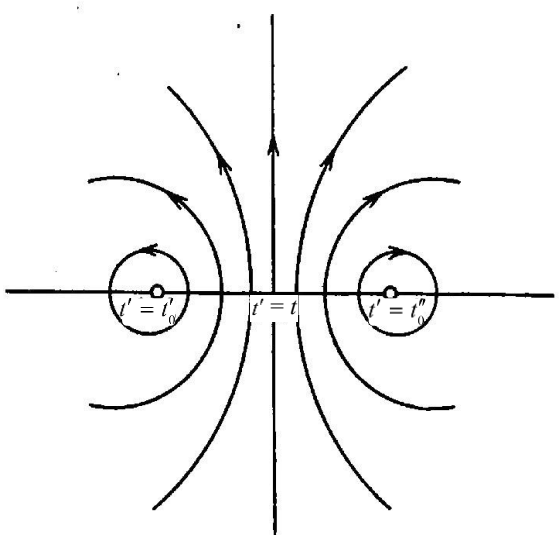


Figure 34.

$$u = (1-\beta^2)\left(x_4' + \frac{i\beta x_1}{1-\beta^2}\right)$$

in place of $t'$ then one will get:

$$\varphi = \frac{e'}{\pi}\int_{-\infty}^{+\infty}\frac{du}{R^{*2}+u^2} = \frac{e'}{R^*} \text{ and } \qquad \mathfrak{A} = \frac{1}{c}\varphi\mathfrak{v},$$

which agrees with the previous results.

Note that in those cases, the usual "retarded" potential, which corresponds to the moment $t' = t_0'$, corresponds to the "advanced" one, which is given by the negative residue relative to the point $t' = t_0''$.

In the general case of an arbitrarily-moving electron, the advanced potential will be completely different from the retarded one. It is ignored as "physically absurd." However, in that way, one could bring a whole set of potentials under consideration that belong to the *complex* roots of (23.c) and (23.d) and depend upon the type of motion in a very special "singular" way. Those singular potentials correspond to an electric action-at-a-distance that is neither retarded nor delayed, and which can still be possibly found in nature. However, that question has still not been investigated up to now.

**§ 5. – Formal reduction of the retarded action-at-a-distance to an instantaneous one.**

By means of the identity:

$$\frac{1}{S^2} = \frac{1}{[R+c(t-t')][R-c(t-t')]} = \frac{1}{2R}\left[\frac{1}{R-c(t-t')}+\frac{1}{R+c(t-t')}\right],$$

formula (23), with the value (24) for the constant $\alpha$, can be written in the form:

$$A_k = \frac{e'}{2\pi i}\oint \frac{dx'_k}{R[R-c(t-t')]} + \frac{e'}{2\pi i}\oint \frac{dx'_k}{R[R+c(t-t')]}. \tag{25}$$

We choose the path of integration $(l')$ in such a way that it includes *only one* root $S^2 = 0$, and indeed the effective time $t'_0$ that corresponds to the retarded action, and *in addition, the time-point in question* $t' = t$, at which the potentials are to be calculated. In that way, the second integral in the previous formula will drop out since the function $1/R[R+c(t-t')]$ will remain regular inside of $l'$. We further assume that the condition:

$$|R| < |c(t-t')| \tag{25.a}$$

is fulfilled for all time-points $(t')$ that lie along $l'$. Note that the inequality (25.a) will always be true for the two *real* points of $l'$ (where that curve intersects the real axis), as long as the velocity of the electron remains smaller than the speed of light [see the inequality (1.a)]. That is why it seems possible to assume the validity of (25) for other complex points of $l'$, as well.

Under that assumption (which might also be inapplicable, though), one develops the function $1/[R+c(t-t')]$ into a series in increasing positive powers of $R/[c(t'-t)]$ that converges (inside and *on* the curve $l'$).

One will have:

$$\frac{1}{R[R-c(t-t')]} = \frac{1}{Rc(t-t')\left[1+\dfrac{R}{c(t-t')}\right]} = \frac{1}{Rc(t'-t)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{R}{c(t'-t)}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{R^{n-1}}{c^{n+1}(t'-t)^{n+1}},$$

and correspondingly, from (25):

$$A_k = \frac{e'}{2\pi i}\sum_{n=0}^{\infty}(-1)^n\oint\frac{1}{c}\frac{dx'_k}{dt'}\frac{R^{n-1}c\,dt'}{c^{n+1}(t'-t)^{n+1}}.$$

If $F(t')$ means any function that is regular inside of the integration curve then, as is known, one will have the formula:

$$\frac{1}{2\pi i}\oint \frac{F(t')\,dt'}{(t'-t)^{n+1}} = \frac{1}{n!}\left[\frac{d^n F(t')}{dt'^n}\right]_{t'=t} = \frac{F^{(n)}(t)}{n!}\,.$$

We then get the following (so-called *Lagrange*) series for $A_k$ :

$$A_k = e'\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{c^n}\frac{d^n}{dt'^n}\left[\frac{1}{c}\frac{dx'_k}{dt'}R^{n-1}\right] \qquad (t'=t)\,, \qquad (26)$$

i.e., the series:

$$\varphi = e'\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{c^n}\frac{d^n(R^{n-1})}{dt'^n} \qquad (t'=t)\,, \qquad (26.a)$$

for the scalar potential, and:

$$\mathfrak{A} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{c^n}\frac{d^n}{dt'^n}\left(\frac{e'}{c}R^{n-1}\,\mathfrak{v}'\right) \qquad (t'=t) \qquad (26.a)$$

for the vector potential.

In those formulas, the quantities $\mathfrak{v}$ and $R$, i.e., the velocity of the electron and its distance from the reference point $P(x_1, x_2, x_3) = P(\mathfrak{r})$, refer to *the same moment* $t' = t$ at which its action at $P$ is calculated.

That action can then be treated *formally* as an instantaneous or *momentary action-at-a-distance* that is nonetheless completely equivalent to the *retarded action-at-a-distance* that was considered before and was expressed by formulas (3), (4).

The formal reduction of the actual retarded action-at-a-distance to a fictitious momentary one is obviously permissible for only those motions of the electron for which its velocity does not satisfy the condition (1.a), but also the condition that all of its derivatives with respect to time must remain finite and continuous. Moreover, the series (26) to (26.b) are generally unconditionally convergent only for sufficiently-small distances $R$. By contrast, it will seem to be divergent (or at best semi-convergent) for large distances. (The wavelength can then be regarded as the most-natural unit of length for periodic motion, as was stated before in Chap. V.) Obviously, that is connected with the fact that our assumption above regarding the validity of the inequality (25.a) along the *entire* integration curve $l'$ (which must initially include the two points $t'$ and $t$) is not fulfilled, in general.

Formulas (26.a) and (26.b) are (in the corresponding domain of convergence) very convenient for investigating the *interaction* between two or more electrons, since they allow all of those electrons (just like in the ordinary mechanics of a system of material points) to be considered at the same moment, instead of introducing two different "effective" times for each electron-pair. One can also apply them to the consideration of the interaction between the various elements of the same electron (as long as the latter is treated as a spatially-extended structure).

If one restricts oneself to the first two terms in the series (26.b) then one will have:

$$\mathfrak{A} = \frac{e'}{c\,R}\mathfrak{v} - \frac{e'}{c^2}\mathfrak{w} \tag{27}$$

(in which the primes on $\mathfrak{v}$ and $\mathfrak{w} = d\,\mathfrak{v} / dt$ have been dropped since those quantities refer to the moment $t$).

The second term in the series (26.a) is obviously equal to zero. If one considers the next term then one will have:

$$\varphi = \frac{e'}{R} + \frac{e'}{2c^2}\frac{d^2R}{dt^2}\,.$$

Now:

$$\frac{dR}{dt} = -\,v_R = -\,\frac{\mathfrak{R}\,\mathfrak{v}}{R},$$

and as a result:

$$\frac{d^2R}{dt^2} = -\,\frac{\mathfrak{R}\,\mathfrak{v}}{R} - \left(\frac{1}{R}\frac{d\,\mathfrak{R}}{dt} - \frac{\mathfrak{R}}{R^2}\frac{dR}{dt}\right)\mathfrak{v} = -\,\mathfrak{R}_0\,\mathfrak{w} + \frac{1}{R}\,[\mathfrak{v}^2 - (\mathfrak{R}_0\,\mathfrak{v})^2]\,.$$

We then get:

$$\varphi = \frac{e'}{R}\left\{1 + \frac{1}{2c^2}[\mathfrak{v}^2 - (\mathfrak{R}_0\,\mathfrak{v})^2]\right\} - \frac{e'}{2c^2}\mathfrak{R}_0\,\mathfrak{w}\,. \tag{27.a}$$

Note that the second terms on the right-hand sides of (27) and (27.a) represent (momentary) actions-at-a-distance whose intensities are *completely independent* of the distance. The omitted higher-order terms in (26.a) and (26.b) generally lead to action-at-a-distance that does not decrease with distance, but even *increases*, and indeed as a positive power of that distance.

Such momentary actions-at-a-distance that collectively replace the retarded actions-at-a-distance that correspond to the potentials (3) and (4) are obviously merely mathematical fictions that would have no physical meaning whatsoever when taken *individually*.

According to the general equations:

$$\mathfrak{H} = \text{rot}\,\mathfrak{A} \qquad \text{and} \qquad \mathfrak{E} = -\,\nabla\varphi - \frac{1}{c}\frac{\partial\mathfrak{A}}{\partial t},$$

formulas (27) and (27.a) will imply the following expressions for the magnetic and electric field strengths:

$$\mathfrak{H} = \frac{e'}{c\,R^2}\mathfrak{v}\times\mathfrak{R}_0 \tag{28}$$

(viz., the *Biot-Savart* law) and:

$$\mathfrak{E} = \frac{e'}{R^2}\left\{1-\frac{1}{2}\frac{(\mathfrak{v}\mathfrak{R}_0)^2}{c^2}\right\}\mathfrak{R}_0 - \frac{e'}{2c^2 R}[\mathfrak{w}+(\mathfrak{R}_0\,\mathfrak{w})\mathfrak{R}_0]+\frac{e'}{c^3}\frac{d\mathfrak{w}}{dt}\,. \tag{28.a}$$

Those approximate formulas deviate from the exact formulas (8) and (9) rather strongly.
The last term in the foregoing expression for $\mathfrak{E}$ :

$$\mathfrak{E}^{(3)} = \frac{e'}{c^3}\frac{d\mathfrak{w}}{dt} \tag{28.b}$$

plays an important role in the theory of radiation, as we will see below.

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**CHAPTER SEVEN**

# ENERGY AND MOMENTUM FOR TIME-VARYING ELECTROMAGNETIC PHENOMENA. DYNAMICS OF ELECTRONS.

## § 1. – The electric energy of a system of charges at rest.

As is known, the mutual (or "relative") energy of two point-charges $e'$ and $e$ is expressed by the formula:

$$U = \frac{e'e}{R}, \tag{1}$$

in which $R$ means the distance between the corresponding points $P'$ and $P$, as always.

Since $e'/R = \varphi$ represents the potential of $e'$ at the point $P$, and likewise $e/R = \varphi'$ represents the potential of $e$ at the point $P'$, one can also set:

$$U = e\,\varphi = e'\varphi' = \tfrac{1}{2}(\varphi e + \varphi' e') \ . \tag{1.a}$$

Those formulas can be easily generalized to the case of a system of arbitrarily-many charges $e_1, e_2, \ldots, e_n$ . Let those charges be originally concentrated at the points $P_1, P_2, \ldots, P_n$ . We now imagine that they are displaced from that configuration into the new one $P_1', P_2', \ldots, P_n'$. The work done by that, namely $A$, is defined to be equal to the algebraic decrease in the mutual potential energy of the charges in question, i.e., the electric energy of the system that they define. If one denotes the value of that energy in the initial and final configurations by $U$ and $U'$, resp., then one will have:

$$A = U - U' \ .$$

The work done $A$ is obviously composed of the individual contributions that originate in the action of each charge on the other ones. If one then denotes the work done by the force $e_\alpha$ exerts upon $e_\beta$ by $A_{\alpha\beta}$ then one will have:

$$A = \sum\sum_{\alpha\neq\beta} A_{\alpha\beta} = \sum\sum_{\alpha<\beta} (A_{\alpha\beta} + A_{\beta\alpha}) = \tfrac{1}{2}\sum\sum_{\alpha\neq\beta} (A_{\alpha\beta} + A_{\beta\alpha}) \ .$$

Now, the sum $A_{\alpha\beta} + A_{\beta\alpha}$ is obviously the algebraic decrease in the mutual energy of $e_\alpha$ and $e_\beta$, i.e., the difference $U_{\alpha\beta} - U'_{\alpha\beta}$ . As a result, one will have:

$$U - U' = \sum\sum_{\alpha<\beta} U_{\alpha\beta} - \sum\sum_{\alpha<\beta} U'_{\alpha\beta},$$

i.e.:

$$U = \sum\sum_{\alpha<\beta} U_{\alpha\beta},$$

or from (1):

$$U = \sum\sum_{\alpha<\beta} \frac{e_\alpha e_\beta}{R_{\alpha\beta}} = \tfrac{1}{2}\sum\sum_{\alpha\neq\beta} \frac{e_\alpha e_\beta}{R_{\alpha\beta}}. \tag{2}$$

If we now introduce the potential at the point $P_\beta$ where the $\beta^{\text{th}}$ charge is found that is generated by all other charges:

$$\varphi^*_\beta = \sum_{\alpha\neq\beta} \frac{e_\alpha}{R_{\alpha\beta}} \tag{2.a}$$

then we can represent the energy $U$ in the form:

$$U = \tfrac{1}{2}\sum_{\beta=1}^{n} \varphi^*_\beta e_\beta. \tag{2.b}$$

One can now transform that energy expression in an entirely-different way that will have fundamental significance in what follows:

Let $\varphi_\alpha$ and $\mathfrak{E}_\alpha$ be the potential (electric field strength, resp.) of the $\alpha^{\text{th}}$ charge at *any* point $P$, while $\varphi'_\alpha$ and $\mathfrak{E}'_\alpha$ are the contributions at that point that originate in all of the other charges ($\varphi'_\alpha = \varphi - \varphi_\alpha$, $\mathfrak{E}'_\alpha = \mathfrak{E} - \mathfrak{E}_\alpha$). One has $\varphi'_\alpha = \varphi^*_\alpha$ at the point $P_\alpha$ where the charge $e_\alpha$ is found. Moreover, let $S$ be a surface that includes all of those charges. By means of the formula:

$$\frac{1}{4\pi}\oint (\mathfrak{E}_\alpha)_n \, dS,$$

one will get:

$$\varphi^*_\alpha e_\alpha = \frac{1}{4\pi}\oint (\varphi^*_\alpha \mathfrak{E}_\alpha)_n \, dS = \frac{1}{4\pi}\oint (\varphi'_\alpha \mathfrak{E}_\alpha)_n \, dS - \frac{1}{4\pi}\oint [(\varphi' - \varphi^*_\alpha)\mathfrak{E}_\alpha]_n \, dS.$$

We use *Gauss's* theorem to transform the last integral into:

$$\oint [(\varphi' - \varphi^*_\alpha)\mathfrak{E}_\alpha]_n \, dS = \int \operatorname{div}(\varphi' - \varphi^*_\alpha)\mathfrak{E}_\alpha \cdot dV = \int (\varphi' - \varphi^*_\alpha)\operatorname{div}\mathfrak{E}_\alpha \, dV + \int [\mathfrak{E}_\alpha \cdot \operatorname{grad}(\varphi' - \varphi^*_\alpha)]\, dV.$$

Since $\operatorname{div}\mathfrak{E}_\alpha = 0$ outside of the point $P_\alpha$, while:

$$\int \operatorname{div}\mathfrak{E}_\alpha \, dV = \oint (\mathfrak{E}_\alpha)_n \, dS = 4\pi e_\alpha$$

has a *finite* magnitude, we will have:

$$\int (\varphi' - \varphi^*_\alpha) \operatorname{div} \mathfrak{E}_\alpha \, dV = 0 \, .$$

Moreover, we have:

$$\operatorname{grad} (\varphi' - \varphi^*_\alpha) = \operatorname{grad} \varphi'_\alpha = - \mathfrak{E}'_\alpha \, ,$$

and as a result:

$$\int \mathfrak{E}_\alpha \operatorname{grad} (\varphi' - \varphi^*_\alpha) \, dV = - \int \mathfrak{E}_\alpha \, \mathfrak{E}'_\alpha \, dV \, .$$

We then come to the following formula:

$$U = \tfrac{1}{2} \sum_\alpha \varphi^*_\alpha \, e_\alpha = \frac{1}{8\pi} \oint \left( \sum_\alpha \varphi'_\alpha \, \mathfrak{E}_\alpha \right)_n dS + \frac{1}{8\pi} \int \sum_\alpha (\mathfrak{E}_\alpha \, \mathfrak{E}'_\alpha) \, dV \, ,$$

or when the surface $S$ is shifted to infinity, which will make the surface integral vanish (since $\sum \varphi'_\alpha \, \mathfrak{E}_\alpha$ decreases in inverse proportion to the third or higher power):

$$U = \frac{1}{8\pi} \int \sum_\alpha (\mathfrak{E}_\alpha \, \mathfrak{E}'_\alpha) \, dV = \frac{1}{4\pi} \int \sum \sum_{\alpha < \beta} \mathfrak{E}_\alpha \, \mathfrak{E}_\beta \, dV \, . \tag{3}$$

As one approaches each of the points $P_\alpha$ ($\alpha = 1, 2, \ldots$), the corresponding field strength $\mathfrak{E}_\alpha$ will become infinite, and indeed in inverse proportion to the square of the distance $l_\alpha$ from $P_\alpha$. However, since the size of a very small volume $V_\alpha$ that includes $P_\alpha$ is proportional to the third power of its linear dimensions, the integral $\int \mathfrak{E}_\alpha \, dV_\alpha$ must vanish as $V_\alpha \to 0$. In fact, if one introduces an infinitely-small solid angle $d\,\Omega_\alpha$ with its vertex at $P_\alpha$, along with the distance $l_\alpha$, then since $dV_\alpha = l^2 \, d\Omega_\alpha \, dl_\alpha$ and $E_\alpha = e_\alpha / l^2_\alpha$, one will have:

$$\int E_\alpha \, dV_\alpha = e_\alpha \iint dl_\alpha \, d\,\Omega_\alpha = e_\alpha [V^{1/3}_\alpha] \, , \tag{3.a}$$

in which the symbol $[A]$ means a number with the same order of magnitude as $A$. One will then have $\int E_\alpha \, dV_\alpha \to 0$ as $V_\alpha \to 0$ . That explains the fact that the integral (3) keeps a finite value despite the fact that the quantity $E_\alpha$ will become infinite at the individual points.

When we recall that $\mathfrak{E}'_\alpha = \mathfrak{E} - \mathfrak{E}_\alpha$ , we will have:

$$\sum_\alpha (\mathfrak{E}_\alpha \, \mathfrak{E}'_\alpha) = \left( \sum_\alpha \mathfrak{E}_\alpha \right) \mathfrak{E}^2 - \sum_\alpha E^2_\alpha = E^2 - \sum_\alpha E^2_\alpha \, .$$

We can correspondingly write the formula (3) as follows:

$$U = \frac{1}{8\pi}\int E^2 dV - \frac{1}{8\pi}\sum_\alpha \int E_\alpha^2\, dV\,. \tag{4}$$

However, that representation of the energy makes no sense since the integrals $\int E^2 dV$ and $\int E_\alpha^2\, dV$ are infinite when taken individually. For example, from (3.a), we have:

$$\int E_\alpha^2\, dV_\alpha \;=\; e_\alpha^2 \iint \frac{dl_\alpha}{l_\alpha^2}\, d\,\Omega_\alpha \;=\; e_\alpha^2\,[V_\alpha^{-1/3}] \to \infty\,. \tag{4.a}$$

We now imagine a *continuous* distribution of electricity inside of a finite volume $V_0$ with a *finite* volume density $\rho$ instead of a system of point charges. If we let $\mathfrak{E}_\alpha$ denote the electric field strength that originates in the electric charge $e_\alpha = \int \rho\, dV_\alpha \approx \rho\, V_\alpha$ that is contained in the very small volume $V_\alpha$ then the integral $\int E_\alpha^2\, dV_\alpha$ must not become infinitely large as $V_\alpha \to 0$ but become *infinitely small*. In fact, if the charge $e_\alpha$ were concentrated at a point $P_\alpha$ (which would not alter the integral $\int E_\alpha^2\, dV_\alpha$ appreciably) then, from (4.a), we would have:

$$\int E_\alpha^2\, dV_\alpha \;=\; \rho^2\,[V_\alpha^{5/3}] \to 0 \tag{4.b}$$

upon contracting $V_\alpha$ to $P_\alpha$. The order of magnitude of the integral $\int E_\alpha^2\, dV_\alpha'$, which extends over the whole volume $V_0$, except for $V_\alpha$ ($V_\alpha' = V - V_\alpha$), must obviously be infinitesimal of order higher than (4.b) (since one has $\int E_\alpha^2\, dV_\alpha' \;=\; \rho^2\,[V_\alpha^2][V_0^{1/3}]$). We will then have:

$$\int E_\alpha^2\, dV \;=\; \rho^2\,[V_\alpha^{5/3}]\,,$$

and as a result, when the volume $V$ is subdivided into elementary volumes of the same size (or order of magnitude):

$$\sum_\alpha \int E_\alpha^2\, dV \;=\; \rho^2 \left[\frac{V_0}{V_\alpha}\right][V_\alpha^{5/3}] \;=\; \rho^2\, V_0\,[V_\alpha^{5/3}] \to 0\,. \tag{4.c}$$

One will also arrive at the same result for the case of the distribution of electricity over a surface $S_0$ with the finite *surface density* $\eta$. If one subdivides $S_0$ into infinitely-small surface elements $S_\alpha$ that might contain infinitely-small volume elements $V_\alpha$ of the same linear dimensions then in place of (4.b), one will have:

$$\int E_\alpha^2\, dV_\alpha \;=\; \eta^2\,[S_\alpha^2][V_\alpha^{-1/3}] \;=\; \eta^2\,[V_\alpha^{4/3}][V_\alpha^{-1/3}] \;=\; \eta^2\,[V_\alpha]\,, \tag{4.d}$$

and instead of (4.c):

$$\sum_{\alpha} \int E_{\alpha}^2 \, dV \; = \eta^2 [V_{\alpha}] \left[ \frac{S_0}{S_{\alpha}} \right] = \eta^2 \, S_0 \, [V_{\alpha}^{1/3}] \to 0 \, . \tag{4.e}$$

However, it must be emphasized that with a distribution of electricity along a *line* $\sigma_0$ with finite *linear density* $\zeta \left( = \frac{de}{d\sigma} \right)$, the sum $\sum_{\alpha} \int E_{\alpha}^2 \, dV$ will remain *finite* as $\sigma_{\alpha} \to 0$ and $V_{\alpha} \to 0$, and indeed with the order of magnitude $\zeta^2 \sigma_0$.

We then see that with a continuous distribution of electricity with finite volume or surface density (but *not* linear density!), the mutual electric energy of the infinitely-small charge elements that are found in the infinitely-small volume (surface, resp.) elements of the system considered can be expressed by the integral:

$$U = \frac{1}{8\pi} \int E^2 dV \, . \tag{5}$$

That result will also be valid for infinitely-small charged volumes $V_0$ or surfaces $S_0$ as long as the total charge $\int \rho \, dV_0$ ( $\int \eta \, dS_0$, resp.) is finite.

From the formula (5), one can treat the electric energy as a spatially-localized quantity, say, of the same type as the electric charge (in the case of the volume distribution). In that way, one can show that the total energy $\frac{E^2}{8\pi} dV$ is stored in the volume element $dV$, i.e., that the energy is distributed in space with the volume density:

$$\xi = \frac{E^2}{8\pi} \, . \tag{5.a}$$

In that way, the "volume density" of energy at each point is determined by the corresponding *resultant* field strength.

We would now like to transform formula (5) "in reverse," i.e., reduce the energy that it represents into the form that corresponds to our original formula (26) for the mutual energy of a system of point charges.

To that end, we shall first perform the integration that was suggested in (5) over a bounded volume ($V$). In so doing, we would like to confine ourselves to the case of a volume distribution. That will then give div $\mathfrak{E} = 4\pi\rho$, and as a result, due to the identity:

$$\text{div } \rho \, \mathfrak{E} = \varphi \, \text{div } \mathfrak{E} + \mathfrak{E} \, \text{grad } \varphi \, ,$$

and the relation $\mathfrak{E} = -$ grad $\varphi$:

$$E^2 \; = \varphi \, \text{div } \mathfrak{E} - \text{div } \varphi \, \mathfrak{E} = 4\pi \, \varphi \, \rho - \text{div } \varphi \, \mathfrak{E} \, .$$

When one denotes the surface that bounds the volume $V$ by $S$, one will have:

$$\frac{1}{8\pi}\int_{(V)} E^2\, dV = \frac{1}{2}\int_{(V)} \varphi\rho\, dV - \frac{1}{8\pi}\oint \varphi E_n\, dS\,.$$

We now shift $S$ to infinity. In that way, the surface integral will vanish (as long as the electric charge is still distributed over finite points), so from (5):

$$U = \tfrac{1}{2}\int \varphi\rho\, dV\,. \tag{6}$$

For a surface distribution of electricity, one will get:

$$U = \tfrac{1}{2}\int \varphi\eta\, dS \tag{6.a}$$

by an entirely-analogous transformation of (5) using equation (23.b) in Chap. IV. Those formulas seem to correspond completely to formula (2.b) on first glance. However, in reality, there is a very crucial difference between them. That difference already asserts itself in the fact that the energy that is defined by (6) and 6.a) must always be positive [since the aforementioned formulas are identical to (5)], while the energy of a system of electric *point* charges will prove to be positive or negative according to whether the charges with the same or opposite signs lie closer to each other, resp.

The quantity $\varphi_\alpha$ that enters into the formula (2.b) represents the potential at the point $P_\alpha$ of all charges, except for the charge $e_\alpha$, which is itself found at $P_\alpha$. By contrast, the quantity $\varphi$ in formula (6) [or (6.a)] means the *total* value of the potential at the point in question, so the value that originates in not just the charges "at a distance," but also in the charge that belongs to that point. Due to the assumed finitude of the charge density $\rho$ ($\eta$, resp.), the aforementioned charge will actually be equal to zero. More precisely: The part of the potential at any point $P_\alpha$ in a volume element $V_\alpha$ that originates in the charge $\rho\, V_\alpha$ (or $\eta\, S_\alpha$) that is found inside of it is an infinitely-small quantity of order $V_\alpha^{2/3}$ ($V_\alpha^{1/3}$, resp.). For that reason, there is no difference in practice between the total value $\varphi$ of the potential at $V_\alpha$ and the value $\varphi_\alpha^*$ that is required by the charges that lie outside of $V_\alpha$.

The formula (2.b) corresponds to the case in question of the sum $\frac{1}{2}\sum \varphi_\alpha^*\, \rho V_\alpha$, which represents the *mutual* energy of the elementary charges that are found in the infinitely-small volume element. However, that sum will drop off, together with the integral (6), in the limit as $V_\alpha \to 0$. One can then define that the integral to be the *mutual* electric energy of the various infinitely-small elements of the electric charge. However, one can also refer to it as the *total* electric energy of the system in question since the sum of the "internal" energies of the individual charge elements (i.e., the "mutual" energy of the even-smaller elements of which they are composed) will vanish in the limit.

The fact that the energy that is represented by (6) and (6.a) is always *positive* is explained by the fact that in that way the closest element of electric charge, which contributes the most to the "mutual" energy of the whole system, due to the (presumed) continuity of the density function $\rho$

(or $\eta$), must have *the same sign*. By contrast, in the case of a system of point charges, those closest charges might just as well possess the opposite sign to it (see above).

That analysis shows that a crucial principal difference exists between systems of point charges ("point-like electrons"), on the one hand, and the continuous distributions of electricity with a finite volume or surface density in regard to the concept of energy ([1]). That difference has its physical roots in the form of the fundamental (viz., *Coulomb*) law of electrostatic interaction. For example, if the force of attraction or repulsion between two charges were not inversely proportional to the square of the distance between them, but directly proportional to the first power, then an irreducibility of the aforementioned type would not exist between point-systems and continuous distributions.

That irreducibility asserts itself most clearly in the fact that the representation of energy in the form (5), in which it appears in the form of a "quasi-material" quantity with volume density that is determined from the resultant electric field strength according to (5.a), is impossible for a system of point-charges and must be replaced with the original representation (3) in which the elementary field strengths that originate in the individual point-charges appear explicitly.

We will represent the electric energy of a system of electrons at rest by the formula (5) accordingly. That energy is obviously composed of two parts, and indeed, they would be the *mutual* energy of the various electrons and the "internal" energy of the individual electrons, i.e., the energy of interaction of the infinitely-small elements of each individual electron. We would like to denote the first type of energy by $U^g$ and the second one by the sum $\sum_\alpha U_\alpha$, in which the index $\alpha$ (= 1, 2, …) should refer to the individual electrons. We will then have:

$$U = U^g + \sum_\alpha U_\alpha \ . \tag{7}$$

The cited subdivision of the "total" energy $U$ corresponds to the splitting of the integral (5) that the identity:

$$E^2 = \left( \sum_\alpha \mathfrak{E}_\alpha \right)^2 = \sum\sum_{\alpha \neq \beta} (\mathfrak{E}_\alpha \mathfrak{E}_\beta) + \sum_\alpha E_\alpha^2$$

implies, which splits it into the following sub-integrals:

$$\frac{1}{8\pi} \int E^2 dV = \frac{1}{8\pi} \sum\sum_{\alpha \neq \beta} \int \mathfrak{E}_\alpha \mathfrak{E}_\beta \, dV + \sum_\alpha \frac{1}{8\pi} \int E_\alpha^2 \, dV \, ,$$

in which one obviously has:

$$U^g = \frac{1}{8\pi} \sum\sum_{\alpha \neq \beta} \int \mathfrak{E}_\alpha \mathfrak{E}_\beta \, dV \tag{7.a}$$

and

([1]) We will overlook the case of a finite linear density in what follows.

$$U_\alpha = \frac{1}{8\pi}\int E_\alpha^2\, dV\,. \tag{7.b}$$

Formula (7.a) is externally identical to our original formula (3) for the mutual energy of a system of point-charges. However, the elementary field strengths $\mathfrak{E}_\alpha$ must actually be different from the usual *Coulomb* expressions for point-charges, at least inside the electrons.

If one considers the electron to be a ball of radius $a$ on whose surface or in whose interior the electron charge $e$ (we shall drop the index $\alpha$) is distributed *uniformly*, and indeed with a surface density of $\eta = \frac{e}{4\pi a^2}$ (volume density of $\rho = \frac{3e}{4\pi a^3}$, resp.), then one can state the following: "Outside of" the electron, its electric field will coincide with the field of a point-charge of the same magnitude that is concentrated at its center (cf., Chap. IV, **§ 5**). Now, as far as the "electron interior" is concerned, one must distinguish between the aforementioned two cases. Indeed, the electric field strength inside of the ball must vanish in the case of a uniform surface charge. By contrast, if the charge is distributed uniformly over the entire volume of the ball then the electric field strength $E$ at a distance $r$ from the center will depend upon only the charge that is found inside of a ball of radius $r$. In that way, that charge will obviously act as if it were concentrated at the center of the ball. In the case considered, one will then have:

$$E = \frac{4\pi\rho r^3}{3r^2} = \frac{4\pi}{3}\rho r = \frac{er}{a^3}$$

for $r < a$. We can now calculate the "internal," or proper, energy of our electron by means of the formula (7.b) ([1]). In the case of the surface-charged electron, that will give $U = \frac{1}{8\pi}\int\limits_{r=a}^{\infty}\frac{e^2}{r^4}dV$, i.e., with $dV = 4\pi r^2\, dr$:

$$U = \frac{e^2}{2a}. \tag{8}$$

For the volume-charged electron, one will get:

$$U = \frac{1}{8\pi}\int_0^a \frac{e^2}{a^6} r^2 dV + \frac{1}{8\pi}\int_a^\infty \frac{e^2}{r^4} dV\,,$$

i.e.:

$$U = \frac{3}{5}\frac{e^2}{a}. \tag{8.a}$$

([1]) Obviously, one can also employ the formula (6) or (6.a) in order to do that.

It should be noted that the proper energy of an electron, i.e., the mutual energy of the infinitely-small elements of its electric charge, represents the work that must be done by the forces of repulsion between those elements when they fly off to infinity, so when the electron *explodes*. However, insofar as such an explosion seems to be impossible, and as long as the structure of the electron remains invariable, its proper energy will generally play the role of only an inessential additive constant in the expression (7) for the total energy of a system of electrons. Thus, when one considers the electrostatic interactions, it is irrelevant in practice whether one calculates with the total or only the mutual energy of the electrons. However, we will see that the state of affairs will be entirely different as soon as one passes from the rest state of the electron to a state of motion.

## § 2. – The magnetic energy of electric currents.

In **§ 2**, Chap. V, we saw that the mutual magnetic energy $T$ of two linear currents, which is defined to be the quantity whose decrease under any change in the configuration of the current lines or current strengths would be equal to the total work done by the transverse ("ponderomotive") and longitudinal ("electromotive") forces, is expressed by the formula:

$$T = i\int H_n\, dS = i'\int H'_{n'}\, dS' = T', \tag{9}$$

in which $\mathfrak{H}$ means the magnetic field strength that is due to $i'$, and $\int H_n\, dS = \Phi$ means its flux through the current line $\sigma(i)$. The quantities $\mathfrak{H}'$ and $\int H'_{n'}\, dS' = \Phi'$ have the same meanings for the second current line $\sigma'(i')$. If one introduces the corresponding vector potentials $\mathfrak{A}$ and $\mathfrak{A}'$ instead of the magnetic field strength then, from (20.b), Chap. III, the energy $T = T'$ can be put into a form that is symmetric in the two current lines, namely:

$$T = L i i', \tag{9.a}$$

with

$$L = \oint\oint \frac{\tau\,\tau'}{R}\, d\sigma\, d\sigma'. \tag{9.b}$$

The formula (9.a) is entirely analogous to the formula (1) for the mutual electric energy of two point-charges. The current strengths in it play the role of charges ($e$ or $e'$, resp.), and the coefficient $L$ plays the role of the inverse distance between the latter. That coefficient, which depends upon only the relative positions of the two current lines, is called the *coefficient of mutual induction*. From (9.a), it can be defined to be the mutual magnetic energy of the current in question when the current strength is *unity* ($i = i' = 1$). One can also define it to be the magnetic flux that links a current line as a result of the presence of the other when the current strength in the latter one is equal to 1. In fact, for arbitrary current strengths, from (9) and (9.a), one will have:

$$\Phi = i' L\,, \qquad \Phi' = i\, L\,. \tag{9.c}$$

It should be observed that the formula (9) can be written in the form:

$$T = i\,\Phi = i'\Phi' = \tfrac{1}{2}(i\,\Phi + i'\Phi')\,.$$

Upon comparing that with (1.a), it will follow that the magnetic flux plays the same role for the magnetic energy of linear currents that the scalar potential does for the electric energy of point-charges. The result above can be generalized to a system of arbitrarily many linear currents, and indeed in the same way as what was done in the case of a system of point-charges that was treated above. Due to the analogy between electric and magnetic quantities that was just described, with no further analysis, we can then set:

$$T = \sum_{\alpha<\beta}\sum L_{\alpha\beta}\, i_\alpha\, i_\beta = \tfrac{1}{2}\sum_{\alpha\neq\beta}\sum L_{\alpha\beta}\, i_\alpha\, i_\beta\,, \tag{10}$$

in conjunction with (2), or correspondingly:

$$\Phi^*_\alpha = \sum_{\alpha\neq\beta} L_{\alpha\beta}\, i_\alpha \tag{10.a}$$

and

$$T = \tfrac{1}{2}\sum_\alpha \Phi^*_\alpha\, i_\alpha\,, \tag{10.b}$$

in conjunction with (2.a) and (2.b). The $\Phi^*_\alpha$ in that means the magnetic flux through the current line $\sigma_\alpha$ that is created by the other current lines. That flux is obviously equal to the algebraic sum of the fluxes $\Phi_{\alpha\beta} = L_{\alpha\beta}\, i_\beta$, The coefficients of induction are determined by formulas of type (9.b) and satisfy the symmetry condition:

$$L_{\alpha\beta} = L_{\beta\alpha}\,.$$

We would now like to turn our attention from linear currents, which represent a mathematical fiction, to stationary currents that circulate in the interior or on the outer surface of any sort of body (say, a metal conductor). The generalization of the concept of energy to that case comes about in the following way: We imagine that the electrical current in question is subdivided into infinitely-thin closed current filaments (which is always possible for stationary currents). We treat those elementary current filaments as linear currents and define the limit of their mutual energy according to (10.b) to be the total magnetic energy in the system of currents under consideration.

The magnetic flux $\Phi^*_\alpha = \int \mathfrak{H}^*_\alpha\, \mathfrak{n}_\alpha\, dS_\alpha$ can be represented in the form $\Phi^*_\alpha = \int \mathfrak{A}^*_\alpha\, \tau_\alpha\, d\sigma_\alpha$ by introducing the corresponding vector potential $\mathfrak{H}^*_\alpha$ ($\mathfrak{H}^*_\alpha$ = rot $\mathfrak{A}^*_\alpha$). Moreover, in the case of a current distribution with a finite volume density $\mathbf{j}$, one can replace $i_\alpha\, \tau_\alpha\, d\sigma_\alpha$ with $\mathbf{j}_\alpha\, dV_\alpha$, where

$dV_\alpha$ means the volume element that is equivalent to the element $d\sigma_\alpha$ of the current line that represents it. From (10.b), one will then have:

$$T = \tfrac{1}{2}\sum_\alpha \int \mathfrak{A}^*_\alpha \, \mathfrak{j}\, dV_\alpha \,.$$

In the limit of infinitely-small cross-sections of the elementary current filaments, the contribution of the vector potential, which is itself generated by each of those current filaments, will vanish, as is easy to see. For that reason, in the passage to the limit that was suggested, we can replace the "mutual" potential $\mathfrak{A}^*_\alpha$ that appears in the foregoing formula with the *total* vector potential $\mathfrak{A}$ at the point in question. Since the subdivision of the electrical current into individual filaments would become absurd then, and the combination of the volume integration over a current filament ($\int dV_\alpha$) with the summation over all current filaments would be equivalent to an integration over the total region that the current goes through, we would get:

$$T = \tfrac{1}{2}\int \mathfrak{A}\, \mathfrak{j}\, dV \,. \tag{11}$$

For the case of a surface-distributed current, that will give:

$$T = \tfrac{1}{2}\int \mathfrak{A}\, \mathfrak{k}\, dS \tag{11.a}$$

in the same way, in which $\mathfrak{k}$ means the (finite) surface density of the current. Thus, the vector potential at any external or internal point will be determined by (20), Chap. III to be:

$$\mathfrak{A} = \int \frac{\mathfrak{j}'\, dV'}{R} \tag{11.b}$$

or

$$\mathfrak{A} = \int \frac{\mathfrak{k}'\, dS'}{R}\,, \tag{11.c}$$

resp. As a result, one can represent the magnetic energy of the electrical current in question in the form of a double integral over the spatial region that the current flows through, namely:

$$T = \tfrac{1}{2}\int\int \frac{\mathfrak{j}\,\mathfrak{j}'}{R}\, dV\, dV' \tag{12}$$

for the volume current and:

$$T = \tfrac{1}{2}\int\int \frac{\mathfrak{k}\,\mathfrak{k}'}{R}\, dS\, dS' \tag{12.a}$$

for the surface current.

One can get those expressions directly by applying the formula (23) that was exhibited in **§ 6**, Chap. III for the "effective" mutual energy of two moving point-charges. That is because if one

switches the sign in that formula, corresponding to the general relation (5), Chap. V, then one will get an "effective" elementary magnetic energy for the charges in question:

$$T = \frac{1}{R}\frac{e\mathfrak{v}}{c}\cdot\frac{e'\mathfrak{v}'}{c}. \tag{12.b}$$

That energy can be defined in the spirit of an "effective" mutual energy, as in the case of stationary or quasi-stationary currents in *closed* current filaments, so the sum of the expressions (12.b) over all charge-pairs that are found in different current filaments must give the correct expression for the mutual magnetic energy of those current filaments. By contrast, no summand of the type (12.b) has any immediate physical meaning since the electromagnetic forces that act between two moving point-charges cannot be derived from any energy function. The formula (12.b) is completely analogous to the corresponding formula (1) for electrostatic energy. Its generalization for arbitrary current systems, and especially to ones with continuous volume or surface distributions, then comes about in the same way as the corresponding generalization of (1). One thus proves the admissibility of the replacement of the mutual potential $\mathfrak{A}^*_\alpha$ with the total one $\mathfrak{A}$ that was made above. We can further assert that such a replacement would be impossible in the case of *linear* currents with finite current strengths, i.e., with finite "linear density" of the electrical impulse.

In the case of a *stationary* volume current that can be characterized by the relation rot $\mathfrak{H} = 4\pi\mathfrak{j}$, the formula (11) can be transformed into the corresponding formula (6) for the electric energy in the same way. Indeed [from formula (25), Introduction], we will have:

$$\mathfrak{A}\,\mathfrak{j} = \frac{1}{4\pi}\mathfrak{A}\ \text{rot}\ \mathfrak{H} = \frac{1}{4\pi}\text{div}\,(\mathfrak{H}\times\mathfrak{A}) + \frac{1}{4\pi}\mathfrak{H}\ \text{rot}\ \mathfrak{A}\,,$$

i.e., due to the fact that $\mathfrak{H} = \text{rot}\ \mathfrak{A}$ :

$$\mathfrak{A}\,\mathfrak{j} = \frac{1}{4\pi}\text{div}\,(\mathfrak{H}\times\mathfrak{A}) + \frac{1}{4\pi}H^2,$$

and as a result:

$$T = \tfrac{1}{2}\int\mathfrak{A}\,\mathfrak{j}\,dV = \frac{1}{8\pi}\oint(\mathfrak{H}\times\mathfrak{A})\,dS + \frac{1}{8\pi}\int_{(V)}H^2dV\,.$$

If one shifts the outer surface $S$ to infinity and correspondingly extends the volume integral over all of space then that will give:

$$T = \frac{1}{8\pi}\int_{(V)}H^2dV\,. \tag{13}$$

That formula can also be obtained from (11.a) [by means of relation (23), Chap. IV]. However, one must observe that it will lose all meaning in the case of a system of *linear* currents, just like the corresponding formula (5) in **§ 1** for continuous distributions of electric charge with finite linear density.

For a system that consists of two or more filamentary currents (e.g., in metal wires) that are separate from each other, the energy $T$ decomposes into the *mutual* magnetic energy of those currents $i_\alpha$ ($\alpha = 1, 2, \ldots$):

$$T^g = \frac{1}{8\pi}\int \sum\sum_{\alpha\neq\beta} \mathfrak{H}_\alpha\, \mathfrak{H}_\beta\, dV = \frac{1}{4\pi}\int \sum\sum_{\alpha<\beta} \mathfrak{H}_\alpha\, \mathfrak{H}_\beta\, dV\,, \tag{13.a}$$

and the "internal" or "proper energy" for each of them:

$$T_\alpha = \frac{1}{8\pi}\int H_\alpha^2\, dV\,. \tag{13.b}$$

In that way, the expression (13.a) will also exist for linear currents, while the expression (13.b) will become infinite in that case.

Since the "elementary field strengths" $\mathfrak{H}_\alpha$ are proportional to the corresponding current strengths, one can write $T^g$ in the form:

$$T^g = \tfrac{1}{2}\sum\sum_{\alpha\neq\beta} L_{\alpha\beta}\, \boldsymbol{i}_\alpha\, \boldsymbol{i}_\beta\,,$$

which agrees with the formula (10). The coefficients of mutual induction $L_{\alpha\beta}$ then depend upon only the configuration and form of the current filaments. One can easily convince oneself that in the limiting case of infinitely-thin current filaments, the new (general) expression for the coefficients of mutual induction:

$$L_{\alpha\beta} = \frac{1}{4\pi}\int \mathfrak{H}_\alpha\, \mathfrak{H}_\beta\, dV \qquad (\boldsymbol{i}_\alpha = \boldsymbol{i}_\beta = 1) \tag{14}$$

can be reduced to the original form (9.b). Rather, we define the *coefficients of self-induction* of the individual currents (or more precisely, current conductors) in a manner that corresponds to (14). Namely, if we set:

$$L_{\alpha\alpha} = \frac{1}{4\pi}\int H_\alpha^2\, dV \qquad (\boldsymbol{i}_\alpha = 1)$$

then that will give the following expression for the "proper energy" of the current $\boldsymbol{i}_\alpha$:

$$T_\alpha = \tfrac{1}{2} L_{\alpha\alpha}\, \boldsymbol{i}_\alpha^2\,. \tag{14.a}$$

One can accordingly define the quantity:

$$\Phi_{\alpha\alpha} = L_{\alpha\alpha}\, i_\alpha$$

to be the magnetic "self-flux" of the current in question. However, the latter can by no means be regarded as linear in form, not even approximately, so that concept will lose the sharpness that it

gave to the original definition that related to the mutual energy of linear currents. The magnetic flux of the current ($\beta$) through the current filament ($\alpha$) that is defined by the formula:

$$\Phi_{\alpha\beta} = L_{\alpha\beta}\, i_\beta\,, \tag{14.b}$$

in conjunction with (14), can be represented in the form of the surface integral $\int \mathfrak{H}_\beta\, \mathfrak{n}_\alpha\, dS_\alpha$ only approximately for a vanishingly-small cross-section of that filament.

In essence, the proper energy of an electric current $T = \frac{1}{2} L i^2$ (we will drop the index $\alpha$ in what follows) is nothing but the *mutual* magnetic energy of the infinitely-thin current filaments into which the current in question can be decomposed. From what was said in **§ 2**, Chap. V, that energy is equal to the work that must be done against the forces of mutual electric induction that are created by the simultaneous increase in the elementary current strengths in the individual sub-filaments when the current strength rises from zero to *i*. Those elementary forces of induction can be combined into a total electromotive force that will be given by the expression:

$$\psi = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{1}{c} L \frac{di}{dt} \tag{15}$$

in the limit. It corresponds to a scalar potential difference of the same magnitude (in electrostatic units), such that an amount of work equal to $\psi c\, i\, dt = L \frac{di}{dt} i\, dt = L\, i\, di$ would be required in order to overcome it. Integrating that expression will, in fact, give the magnetic energy of the current in question $\frac{1}{2} L i^2$.

## § 3. – The electromagnetic theory of mass.

The analogy between the magnetic energy of a current loop that is defined and expressed in that way with the usual *kinetic energy* of a material particle comes to mind. In that way, the current strength corresponds to the velocity of the particle $\mathfrak{v}$ and the coefficient of self-induction corresponds to its mass *m*. The force of self-induction (15) is completely analogous to the "inertial force":

$$-\, m \frac{d\mathfrak{v}}{dt} = -\, m\, \mathfrak{w}\,,$$

that must be overcome under an *acceleration* of the particle.

The analogy will be converted into an identity when we consider the translational motion of a body that carries a charge that is distributed over its surface or throughout its volume instead of a closed current loop.

Imagine, e.g., a rigid spherical electron in a state of uniform rectilinear motion with a velocity $\mathfrak{v}$. Since that motion is not stationary, strictly speaking, one cannot apply the concept of magnetic

energy to case in question. Nonetheless, if one attempts to calculate the corresponding magnetic energy $T$ then that will give:

$$T = U\frac{v^2}{c^2},$$

based upon the formula (12.b), in which $U$ means the electric energy of the electron in the rest state. That expression is identical to the kinetic energy of a particle with a mass of:

$$m = \frac{2U}{c^2}.$$

However, one must observe that the formula (12.b) is certainly inapplicable in the present case. That is why we would like to calculate the energy $T$ by means of the formula (13). (Later on, we will see that it is valid in general.) In regard to that, one should note that the relation:

$$\mathfrak{H} = \frac{1}{c}\mathfrak{v} \times \mathfrak{E},$$

which exists between the magnetic and electric field strengths of a point-charge in uniform rectilinear motion, according to (13.b), Chap. VI, must obviously be true for the corresponding *resultant* field strengths of any system of electric charges that moves with the same constant velocity, as well. One must then have:

$$H^2 = \frac{1}{c^2}\{E^2 v^2 - (\mathfrak{E}\mathfrak{v})^2\},$$

and as a result, from (13) and (5), one will have:

$$T = U\frac{v^2}{c^2} - \frac{1}{8\pi c^2}\int (\mathfrak{E}\mathfrak{v})^2\, dV. \tag{16}$$

In the case considered (viz., an electron with a spherically-symmetric distribution of charge), the electric field must also be spherically or radially-symmetric in the rest state, and approximately so for small velocities of motion ($v \ll c$). If one then denotes the angle between $\mathfrak{v}$ and the radius vector $\mathfrak{r}$ of the volume element (relative to the center of the electron) by $\theta$ then $|\,\mathfrak{E}\ \mathfrak{v}\,| = E v \cos\theta$, and as a result:

$$\frac{1}{8\pi c^2}\int (\mathfrak{E}\mathfrak{v})^2\, dV = \left(\frac{1}{8\pi}\int E^2 dV\right)\frac{v^2}{c^2}\,\overline{\cos^2\theta}\ .$$

The $\overline{\cos^2\theta}$ in that means the mean value of $\cos^2\theta$ over all directions of $\mathfrak{r}$. It is known that one has $\overline{\cos^2\theta} = \frac{1}{3}$, such that from (16), one will have:

$$T = \frac{2}{3} U \frac{v^2}{c^2}. \tag{16.a}$$

That expression is then calculated to be one-third smaller than the previous one, and corresponds to an electromagnetic mass of:

$$m = \frac{4}{3}\frac{U}{c^2}, \tag{16.a}$$

i.e., $m = \frac{2}{3}\frac{e^2}{a c^2}$, in the case of the surface charge, and $m = \frac{4}{5}\frac{e^2}{a c^2}$ in the case of the volume charge [cf., (8) and (8.a)]

One can convince oneself of the validity of that formula ([1]) directly by calculating the force of electric induction with which the electron must act upon itself under an *accelerated* motion. We saw in Chap. VI, **§ 2**, that *for low speeds*, the part of its electric field strength that depends upon the acceleration of a point-charge is expressed by:

$$d\,\mathfrak{E}^{(2)} = -\frac{de'}{c^2 R}\{\mathfrak{w}' - (\mathfrak{R}_0\,\mathfrak{w}')\,\mathfrak{R}_0\}\ . \tag{17}$$

It will follow from that this that the force with which an infinitely-small element $de'$ of the electron acts on another one *de* can be represented by the product of the mutual electrical energy of the two elements $\frac{de\,de'}{c^2 R}$ with the vector $-\{\mathfrak{w} - (\mathfrak{R}_0\,\mathfrak{w})\,\mathfrak{R}_0\}$ (we drop the prime and consider the interaction of the various elements of the electron to be instantaneous, cf., *infra*, **§ 3**), and divided by $c^2$. Upon summing that product over all elements *de* and $de'$, one will obviously get:

$$-\frac{2U}{c^2}\{\mathfrak{w} - \overline{(\mathfrak{R}_0\,\mathfrak{w})\,\mathfrak{R}_0}\}\ ,$$

in which $\overline{(\mathfrak{R}_0\,\mathfrak{w})\,\mathfrak{R}_0}$ means the mean value of the vector $(\mathfrak{R}_0\,\mathfrak{w})\,\mathfrak{R}_0$ for various directions of the radius vector $\mathfrak{R}$ ($\mathfrak{R}_0 = \mathfrak{R} / R$) that points from $de'$ to *de*. Due to the spherically-symmetric structure of the electron, the component of the vector $\overline{(\mathfrak{R}_0\,\mathfrak{w})\,\mathfrak{R}_0}$ that is perpendicular to $\mathfrak{w}$ must vanish. By contrast, one will get:

([1]) For small velocities. We will derive the formulas that are valid for arbitrary velocities later on.

$$\overline{\mathfrak{R}_0\, \mathfrak{w} \cos\theta'} = w\overline{\cos^2\theta'} = \tfrac{1}{3} w ,$$

for the component that is parallel to it, when the angle between $\mathfrak{R}_0$ and $\mathfrak{w}$ is denoted by $\theta'$.

The desired "force of self-induction" that is required by the acceleration of the electron is then expressed by the formula:

$$\mathfrak{F} = -\frac{4}{3}\frac{U}{c^2}\mathfrak{w} . \tag{17.a}$$

As a result, that force is identical (at least *formally*) to the usual force of inertia of a particle of mass (16.b).

It should be noted that the part of the electric field strength that was overlooked in the foregoing calculation, just like the force that is implied by the magnetic field strengths, must make no contribution to the resultant "self-force" when one (doubly) sums over all elements of the electron. Therefore, that "self-force" must vanish for an electron in uniform, rectilinear motion.

The result above ([1]) defines the foundation for the so-called *electromagnetic theory of mass*. According to that theory, the mass of an electron is not a primitive property (like, say, its charge), but a property that is determined by the formula (16.b). *The mass of an electron can then be defined to be its coefficient of self-induction*.

The actual physical sense of that statement consists of the assertion that under an (accelerated) motion of the electron, the *external* force $\mathfrak{F}^{(a)}$ that acts on it must be compensated by the corresponding self-force $\mathfrak{F}$ at each moment. The equation of motion of an electron in a given external electromagnetic field can be exhibited using that equilibrium principle, or even better, *principle of motion*, which goes back to *H. A. Lorentz*. If one then sets:

$$\mathfrak{F} + \mathfrak{F}^{(a)} = 0 , \tag{17.b}$$

in general, and one employs the expression for $\mathfrak{F}^{(a)}$ ($= -m\,\mathfrak{w}$) that follows from (17.a) and (17) then one will get the known Newtonian equation of motion:

$$m\,\mathfrak{w} = \mathfrak{F} . \tag{17.c}$$

It will follow from this that since the cited expression for the self-force presents only an approximate value for small speeds, the equation of motion above will also be inaccurate, and must be replaced with a more-complicated equation for high speeds, in any event.

Since the charge and mass of the electron can be established experimentally, under certain assumptions about its "structure" (i.e., the charge distribution), it is possible to assess its geometric dimensions. The simplest assumption, namely, that the electron is a rigid ball with a uniformly-distributed surface or value charge, will give the radius of that ball in the case of negative electrons ($e = -4.77 \times 10^{-10}$, $m = 9.8 \times 10^{-27}$ g) as:

([1]) Which was exhibited for the first time by *J. J. Thomson*.

$$a \approx 2 \times 10^{-13} \text{ cm},$$

and in the case of positive electron or "protons" ($e = +4.77 \times 10^{-10}$, $m = 1.7 \times 10^{-27}$ g):

$$a \approx 2 \times 10^{-16} \text{ cm}.$$

Finally, it should be observed that the mass of any system of electrons, e.g., an atom or material body, is generally different from the sum of the masses of those electrons, in general. That is because one must then turn one's attention to the "mutual masses" of the individual electrons, along with their "proper masses." That mutual mass of the various electrons is analogous to the coefficients of mutual induction of different currents. It will correspond to the mutual magnetic energy or the force of mutual induction that must appear when the system of electrons in question moves uniformly (accelerates, resp.) as a whole. According to (16), its order of magnitude is equal ([1]) to the mutual *electric* energy of the various electrons, divided by $c^2$, i.e., the square of the speed of light. As long as the distances between the electrons remain large in comparison to their dimensions (i.e., radii), their mutual mass must also be small in comparison to the sum of their proper masses. We would like to emphasize once more that, according to the electromagnetic theory that was sketched out here, the proper mass of an electron can be represented as the mutual mass of the mutually-inseparable elements into which one imagines that the electron can be subdivided.

Strictly speaking, there is no such thing as a "proper" mass, but only a mutual mass. The concept of mass has no simple and direct relationship to the concept of *matter*, *but only to energy*. Namely, matter is nothing but a collection of electrons, which are regarded as indestructible and invariable things. However, the mass of a material body is, in part, a variable, and at the same time non-additive, quantity that depends upon the relative configuration of the electrons that comprise that body, and indeed in the same way as its mutual electric energy. That accordingly raises the question of the *localization* of mass. Obviously, that question must be resolved at the same time as the question of the localization of energy.

Insofar as the electromagnetic energy of a system of electrons can be expressed as a function of their reciprocal distances (and possibly also their velocities), one cannot speak of such a localization. However, we have seen that in the case of time-constant electric and magnetic fields, the "total" energy can be regarded as a "quasi-material" quantity that is distributed over all space with the volume density $\frac{1}{8\pi} E^2$ ($\frac{1}{8\pi} H^2$, resp.). Now, we would like to try to generalize that representation of the electromagnetic energy as a quasi-material, spatially-localizable quantity to arbitrary time-varying phenomena.

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([1]) One observes that the formula (16), just like the fundamental relation $\mathfrak{H} = \frac{1}{c} \mathfrak{v} \times \mathfrak{E}$, must be true for not only an isolated electron, but for arbitrary electric systems that move uniformly and rectilinearly as a whole.

## § 4. – Electromagnetic energy and radiation.

We shall consider a system of electrons that might move in a completely-arbitrary way and characterize them by an initially-arbitrary spatial distribution of electric charge and impulse with a finite volume density $\rho$ ($\mathbf{j}$, resp.). We can then set:

$$\mathbf{j} = \rho \frac{\mathfrak{v}}{c} \tag{18}$$

(cf., **§ 1**, Chap. VI) and correspondingly write the conservation of electricity in the form:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho\, \mathfrak{v} = 0 \,. \tag{18.a}$$

In order to obtain a definition of energy that is generally valid, we must return to the *source* of that concept, so we must consider the *work* that is performed by electric and electromagnetic forces. Therefore, we will consider not only the "external" forces that act upon an electron as a result of the other ones, but also the corresponding *self-forces*. In other words, we will understand the *total force* to mean the force $d\,\mathfrak{F}$ that acts upon a charge $de = \rho\, dV$ that is found in the volume in element $dV$, including the part that originates in the neighboring elements of the electron in question, and mainly in the element $\rho\, dV$ itself. (However, the latter must vanish in the limiting case $dV \to 0$.)

We will correspondingly consider the *total* electric and magnetic field strengths, which are coupled with $\rho$ and $\mathbf{j}$ by the equations:

$$\operatorname{div} \mathfrak{E} = 4\pi\rho \,, \qquad \operatorname{rot} \mathfrak{H} - \frac{1}{c}\frac{\partial \mathfrak{E}}{\partial t} = 4\pi \mathbf{j} \,.$$

The force that acts upon the charge $de = \rho\, dV$ is expressed by the formula:

$$d\,\mathfrak{F} = de\left( \mathfrak{E} + \frac{1}{c}\mathfrak{v} \times \mathfrak{H} \right) . \tag{18.b}$$

We will get the work that it does during the time $dt$ by internally multiplying $d\,\mathfrak{F}$ by the corresponding displacement of the charge element $\mathfrak{v}\, dt$ in question. That work is then equal to:

$$d\,\mathfrak{F}\, \mathfrak{v}\, dt = \rho\, \mathfrak{E}\, \mathfrak{v}\, dV\, dt$$

since the product $(\mathfrak{v} \times \mathfrak{H})\, \mathfrak{v}$ vanishes. (The electromagnetic force always "does no work".) If we denote the work done per unit volume and time by $l$ then we will have:

$$l = \rho\, \mathfrak{E}\, \mathfrak{v}\ ,$$

or from (18):

$$l = c\, \mathfrak{E}\, \mathfrak{j}\ . \tag{19}$$

We shall now replace j with $\frac{1}{4\pi}\left(\operatorname{rot}\mathfrak{H} - \frac{1}{c}\frac{\partial \mathfrak{E}}{\partial t}\right)$ in that expression and add the conjugate expression:

$$l^* = -\frac{c}{4\pi}\mathfrak{H}\left(\operatorname{rot}\mathfrak{E} + \frac{1}{c}\frac{\partial \mathfrak{H}}{\partial t}\right) \tag{19.a}$$

to it, and according to (49.a), Chap. V, that would represent the work done by the magnetic forces for non-vanishing "magnetic" current density $\mathfrak{j}^*$. However, $\mathfrak{j}^*$ is, in fact, zero, and as a result, so is $l^*$, such that adding $l^*$ would be pointless. (However, it does allow us to put the work in question into a symmetric form that is easy to transform in the desired way.) We then have:

$$l = l + l^* = \frac{c}{4\pi}\{\mathfrak{E}\operatorname{rot}\mathfrak{H} - \mathfrak{H}\operatorname{rot}\mathfrak{E}\} - \frac{1}{4\pi}\left(\mathfrak{E}\frac{\partial \mathfrak{E}}{\partial t} + \mathfrak{H}\frac{\partial \mathfrak{H}}{\partial t}\right)\ ,$$

i.e.:

$$l = -\frac{c}{4\pi}\operatorname{div}(\mathfrak{E}\times\mathfrak{H}) - \frac{\partial}{\partial t}\frac{E^2 + H^2}{8\pi}\ . \tag{19.b}$$

If one now multiplies that equation by $dV$ and integrates over a volume ($V$) that is bounded by the surface $S$ then the work done per unit time for the total volume $A = \int l\, dV$ will be:

$$A = -\frac{dW}{dt} - \oint K_n\, dS\ , \tag{20}$$

in which one sets:

$$W = \frac{1}{8\pi}\int (E^2 + H^2)\, dV \tag{20.a}$$

and

$$\mathfrak{K} = \frac{c}{4\pi}\mathfrak{E}\times\mathfrak{H}\ . \tag{20.b}$$

From the results of the previous section, the quantity $W$ can be interpreted as the electromagnetic energy that is stored in the volume $V$. The interpretation corresponds completely to the role that $W$ plays in the general formula (20). That is because if we imagine, for the moment, that the surface integral $\int K_n\, dS$ vanishes then formula (20) will mean that the work $K$ will be done *at the expense* of the quantity $W$. However, that is precisely the characteristic feature of energy.

In general, the integral $\oint K_n\, dS$ will still remain non-zero when we displace the surface $S$ to infinity. By contrast, the work done $A$ when the motion of the isolated electrons takes place in such

a way that the external force that acts upon each of them will be cancelled by the corresponding self-force [cf., (17.b)] and will vanish for an arbitrary closed surface that does not intersect any electron. If we then assume that this *Lorentzian* principle of motion is valid then equation (20) will reduce to:

$$-\frac{dW}{dt} = \oint K_n \, dS \,. \tag{21}$$

It should be pointed out that the *Lorentzian* principle has no direct relationship to equation (20). That equation can always be reduced to the form (21) in any event when no electrons at all are inside of the surface *S*.

The equation (21) is completely analogous to the equation:

$$-\frac{d}{dt}\int \rho \, dV = \oint c \, j_n \, dS$$

(Chap. II, **§ 2**), which expresses the principle of conservation of electricity. In that way, the charge density $\rho$ will correspond to the *energy density:*

$$\xi = \frac{1}{8\pi}(E^2 + H^2), \tag{21.a}$$

and the current density $c\,\mathbf{j} = \mathfrak{J}$ (in electrostatic units) will correspond to the vector $\mathfrak{K}$. That is why it is reasonable to treat electromagnetic energy as a type of substance (in the same way as electricity) and to regard equation (21) as the statement of the principle of the conservation of the substance. One correspondingly refers to the vector $\mathfrak{K}$ as the *density of energy current.* If the analogy between energy and electricity were complete then a relationship of the form:

$$\mathfrak{K} = \xi \, \mathfrak{E} \tag{21.b}$$

would have to exist between $\mathfrak{K}$ and $\xi$, which would correspond to the relation (18) between *J* and $\rho$. In that way, the vector $\mathfrak{E}$ would have the meaning of the *velocity of the energy current* at the point in question. Obviously, such a vector can always be *defined* to be simply the quotient of $\mathfrak{K}$ by $\xi$. However, one must ask whether, and to what extent, it can be assigned any sort of physical meaning. That is also closely connected with the question of whether, and to what extent, the "substance picture" for electromagnetic energy that was depicted above is permissible.

In order to resolve that question, we would like to calculate the vector $\mathfrak{K}$ for the simplest case of the electromagnetic field of an elementary oscillator. In that way, we will initially consider only the field in the wave zone, so we will identify $\mathfrak{E}$ and $\mathfrak{H}$ with the field strengths $\mathfrak{E}^{(2)}$ and $\mathfrak{H}^{(2)}$ that are determined by the formulas (31) and (32) of Chap. V. By means of the relations:

$$\mathfrak{H} = \mathfrak{N}_0 \times \mathfrak{E}, \quad \mathfrak{E} = \mathfrak{H} \times \mathfrak{N}_0 , \tag{22}$$

we will immediately get from (20.b):

$$\mathfrak{K} = \frac{c}{4\pi} E^2 \mathfrak{N}_0 \; = \frac{c}{4\pi} H^2 \mathfrak{N}_0 \; = \frac{E^2 + H^2}{8\pi} c \mathfrak{N}_0 \; . \tag{22.a}$$

It will follow upon comparing (21.a) with (21.b) that:

$$\mathfrak{C} = c \, \mathfrak{N}_0 \, . \tag{22.b}$$

In the present case, that velocity $\mathfrak{C}$ is nothing but the speed of propagation of the electromagnetic waves at the point in question then. That result can obviously be generalized to arbitrary systems of electric (or magnetic) oscillators, and above all, to arbitrary electric systems that are always found in a bounded region of space because at very large distances from that region, the relation (22) (cf., Chap. V, **§ 8**) that is characteristic of the wave zone must always exist.

With that, the question that was posed above is resolved for the wave zone in complete generality, and even in the affirmative sense.

However, it is easy to convince oneself that the velocity that was defined by (21.b) will become physically meaningless for the other components of the electromagnetic field that were left out of consideration in the analysis above, which play the principal role for small distances (obviously, "small" means relative to the wavelength). For example, we shall present a system that consists of one charge at rest and one elementary current. For distances that are not-too-small, one can imagine that both of them are combined at the same point. Since the electrical force lines are directed radially, while the magnetic ones lie in the meridian planes that are determined by magnetic axis of the current, the lines that represent the vector $\mathfrak{E} \times \mathfrak{H}$ must be coaxial circles (parallels). In that case, we will then have an "energy flux" around the axis of the electrical current in a direction that depends upon the sign of the aforementioned charge and has a velocity [in the sense of formula (21.b)] that will become zero for vanishing charge. Obviously, that process represents a mere fiction since the charge and the current that we have combined into a system have no reciprocal influence on each other at all.

In the case that was considered above of the wave zone of an (elementary) oscillator, according to (31) and (32), Chap. V, the electromagnetic energy density will be given by the expression:

$$\xi = \frac{| \ddot{\mathfrak{p}}' \times \mathfrak{N}_0 |^2}{4\pi c^4 R^2} \, .$$

If one lets $\theta$ denote the angle between $\mathfrak{N}$ and $\ddot{\mathfrak{p}}'$ (in which the prime suggests that the quantity refers to the effective time $t' = t - R / c$) then one will get the following expression for the vector $\mathfrak{K}$ according to (22.a):

$$\mathfrak{K} = \frac{1}{4\pi R^2}\frac{|\ddot{\mathfrak{p}}'|^2}{c^3}\sin^2\theta\,\mathfrak{N}_0\,. \tag{23}$$

That will then give the following formula for the total energy flux through a spherical surface of radius *R*:

$$Q = \oint K_n\,dS = \frac{1}{c^3}|\ddot{\mathfrak{p}}'|^2\,\overline{\sin^2\theta},$$

or since $\overline{\sin^2\theta} = 1 - \overline{\cos^2\theta} = 2/3$:

$$Q = \frac{2}{3}\frac{|\ddot{\mathfrak{p}}'|^2}{c^3}\,. \tag{23.a}$$

That energy flux is independent of the size of the sphere when one restricts oneself to the consideration of a certain *effective* time-point. One can then state the following: The *radiation* of a certain amount of energy $Q\,dt'$ will be required by the "accelerated" motion of the oscillator during an infinitely-small time interval between $t'$ and $t'+dt'$. That energy will propagate with the speed of light in all directions in the wave zone, and it will remain localized inside a spherical shell of thickness $c\,dt'$ that is bounded by two spherical surfaces with radii $c(t-t')$ and $c(t-t'-dt')$ at each succeeding moment *t*.

In the case of light waves, the vector $\mathfrak{K}$ determines the direction, and at the same time, the intensity, of the light ray. For that reason, one cares to refer to it as the "ray vector" ([1]).

From formula (21), the radiation of energy through any closed surface *S* must occur at the expense of the energy that is stored inside of that surface. If one then thinks of *S* as the separating surface between the wave zone and the "internal" spatial region, in which the field strengths $\mathfrak{E}^{(0)}$, $\mathfrak{E}^{(1)}$, and $\mathfrak{H}^{(1)}$ play the main role, then one can consider the process of energy radiation that was described above to be a conversion of the energy $W^{(0)}+W^{(1)}$ that corresponds to the aforementioned field strengths into the energy $W^{(2)}$ of the electromagnetic field $\mathfrak{E}^{(2)}$, $\mathfrak{E}^{(2)}$ that prevails in the wave zone and spreads out every further in all directions with time.

The energy $W^{(0)}+W^{(1)}$ is composed of the electric and magnetic energy of the electrons that define the oscillator in question. In that way, the sum of the proper electric energies of those electrons will remain constant, and that is why it will not be considered. By contrast, their mutual electric energy must decrease. That decrease can be partially compensated by an increase in the magnetic energy of the electrons, i.e., some of their kinetic energy. (Recall that the magnetic field strength $\mathfrak{H}^{(1)}$ follows the *Biot-Savart* law, so it will be proportional to the velocity of the electrons.) We would not like to go into the details of that question since the only stated goal of the present analysis was to explain the relationship between the energies $W^{(0)}+W^{(1)}$, on the one hand, and $W^{(2)}$, on the other.

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([1]) It is also frequently referred to as the *Poynting* vector since it was first introduced by *J. Poynting*.

The energy $W^{(0)}+W^{(1)}$ corresponds to the ordinary "mechanical" energy of a system of material particles. One can represent it approximately as the sum of potential (electric) and kinetic (magnetic) energy of the electrons, the former of which will depend upon their relative positions, while the latter will depend upon their velocities. (The mutual kinetic energy will remain quite small comparatively.) By contrast, in classical mechanics, there does not exist a quantity that corresponds to the energy $W^{(2)}$. One refers to that energy, which is required by the acceleration of the electrons and propagates to infinity with the electromagnetic waves, as the *energy of radiation* (or "radiated energy"). One can crudely define the process of energy radiation as the conversion of mechanical energy into energy of radiation then. However, one must not forget that the cited division of electromagnetic energy into "mechanical" and "radiant" energy *is not* precise. Indeed, a precise division of that kind is largely impossible.

One generally refers to the reduction of the "mechanical" energy of a system of electrons by radiation as *radiation damping*. That radiation damping can be reduced (at least formally) to certain forces of the same type as the *forces of friction* in ordinary mechanics. For the sake of simplicity, imagine an oscillator that includes *only one* moving (oscillating or orbiting) electron. In that way, one can set the electric moment $\mathfrak{p}$ of that oscillator equal to $e\,\mathfrak{r}$, where $e$ means the charge of the moving electron, and $\mathfrak{r}$ refers to its radius vector relative to a fixed point (e.g., the opposite fixed electron). One correspondingly has $\ddot{\mathfrak{p}} = e\dfrac{d^2\mathfrak{r}}{dt^2} = e\dfrac{d\mathfrak{v}}{dt} = e\,\mathfrak{w}$, and as a result one has, from (23.a):

$$Q = \frac{2}{3}\frac{e^2 w^2}{c^3}\,. \tag{23.b}$$

That is therefore the energy that the moving electron (or more correctly, *the oscillator*) loses per unit time to radiation. That loss of energy must be produced by an additional "frictional force" $\mathfrak{f}$ (that has remained out of consideration up to now). In order to determine that force, we set the work that it does $\int_{t_1}^{t_2} \mathfrak{f}\,\mathfrak{v}\,dt$ during the time interval ($t_1$, $t_2$) equal to minus the energy $\int_{t_1}^{t_2} Q\,dt$ that was radiated during that same interval. From (12.b), that would yield:

$$\int_{t_1}^{t_2} \mathfrak{f}\,\mathfrak{v}\,dt = -\frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2} w^2\,dt\,. \tag{24}$$

We convert the integral on the right by partial integration. We will then have:

$$\int_{t_1}^{t_2} w^2\,dt = \int_{t_1}^{t_2} \mathfrak{w}\,d\mathfrak{v} = [\mathfrak{w}\cdot\mathfrak{v}]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathfrak{v}\frac{d\mathfrak{v}}{dt}\,dt\,.$$

The quantity:

$$[\mathfrak{w}\cdot\mathfrak{v}]_{t_1}^{t_2} = \frac{1}{2}\left(\frac{dv^2}{dt}\right)_2 - \frac{1}{2}\left(\frac{dv^2}{dt}\right)_1$$

must obviously remain practically (i.e., in the mean) zero for stationary or weakly-damped oscillations, at least in comparison to the quantity $\int_{t_1}^{t_2} w^2\,dt$ . For that reason, we can set:

$$\int_{t_1}^{t_2} w^2\,dt \cong -\int_{t_1}^{t_2} \mathfrak{v}\frac{d\mathfrak{v}}{dt}dt$$

for time intervals ($t_1$, $t_2$) that are sufficiently large relative to the period of oscillation (but can be otherwise very short), and correspondingly, from (24), we will have:

$$\int_{t_1}^{t_2} \mathfrak{f}\,\mathfrak{v}\,dt = \frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2}\frac{d\mathfrak{w}}{dt}\mathfrak{v}\,dt\,,$$

or ultimately:

$$\mathfrak{f} = \frac{2}{3}\frac{e^2}{c^3}\frac{d\mathfrak{w}}{dt}\ . \qquad (24.a)$$

That force of friction, which is usually referred to as *radiation damping* ([1]), must obviously represent part of the "self-force" $\mathfrak{F}$ that was left unconsidered in the last section, i.e., the resultant of the elementary forces with which the various infinitely-small elements of the electron $(de, de')$ act upon each other. In fact, when one calculates that force on the basis of formula (17), one can consider the *retardation* of the corresponding elementary force effects, so one must understand $\mathfrak{w}'$ to mean the acceleration of the electron, not at the moment $t$ in question, but at the earlier moment $t' = t - R/c$, in which $R$ means the distance from the "active" charge element $de'$ to the "passive" $de$. In so doing, one assumes that *the electron moves like a rigid body*, i.e., that the various electron elements have the same velocity at the same moments, and as a result, the same accelerations ([2]).

If one now develops the acceleration $\mathfrak{w}'$ as a function of $t'$ in powers of the difference $t' - t$ then one will have:

$$\mathfrak{w}' = \mathfrak{w} + (t' - t)\frac{d\mathfrak{w}}{dt}$$

in the first approximation, in which $\mathfrak{w}$ and $d\,\mathfrak{w}\,/\,dt$ refer to the time-point $t$, or since $t' - t = -R/c$:

([1]) And which was first introduced by *M. Planck*.
([2]) Cf., *infra*, in the theory of the deformable (*Lorentz*) electron.

$$\mathfrak{w}' = \mathfrak{w} - \frac{R}{c}\frac{d\mathfrak{w}}{dt} = \mathfrak{w} - \frac{R}{c}\dot{\mathfrak{w}}.$$

Upon substituting that expression in (17), we will get:

$$\mathfrak{E}^{(2)} = -\frac{de'}{c^2 R}\{\mathfrak{w} - (\mathfrak{R}_0\,\mathfrak{w})\,\mathfrak{R}_0\} + \frac{de'}{c^2}\{\dot{\mathfrak{w}} - (\mathfrak{R}_0\,\dot{\mathfrak{w}})\,\mathfrak{R}_0\},$$

from which, after multiplying by *de* and summing twice of the various elements of the (spherically-symmetric) electron, we will get [cf., the derivative of (17.a)]:

$$\mathfrak{F} = -m\,\mathfrak{w} + \frac{2}{3}\frac{e^2}{c^3}\dot{\mathfrak{w}}. \tag{24.b}$$

That formula can be regarded as the completion of our previous formula (17.a) for the self-force of the electron. In that way, we have introduced the electron mass *m* according to (16.b) into the expression for that force. The self-force is then composed of two components: The inertial force $-m\mathfrak{w}$ and the damping force $\mathfrak{f} = \frac{2}{3}\frac{e^2}{c^3}\dot{\mathfrak{w}}$ that was calculated above in a different way.

At the end of Chapter VI, it was pointed out that when one considers the interaction between different parts of a system electric charges, the usual expressions for the *retarded* electromagnetic potentials and field strengths are to be preferred over the corresponding series developments that represent them as instantaneous actions at a distance. In fact, one can derive formula (24.b) directly from the slightly-corrected formula (28.a), in Chap. VI. In that way, the first term in (28), which is quadratic in 1 / *R*, will not correspond to any resultant self-force. The second term, which is linear in 1 / *R* will correspond to a self-force of:

$$-\frac{U}{c^2}\{\mathfrak{w} - \overline{(\mathfrak{R}_0\,\mathfrak{w})\,\mathfrak{R}_0}\} = -\frac{4}{3}\frac{U}{c^2}\mathfrak{w} = -m\,\mathfrak{w},$$

which coincides with (17.a), and the third term, which is independent of *R*, corresponds to the self-force:

$$+\frac{e^2}{c^3}\frac{d\mathfrak{w}}{dt}.$$

The factor of 2/3 is missing from that expression, which is explained by the imprecision in the formula (28.a). Had we considered one more term in the series (27) and (27.a), Chap. VI, then that would have given the part of the self-force in question with the correct factor.

In conclusion, let us point out the following fact: We can replace equation (21), which can be regarded as the integral expression for the law of the conservation of energy, with a differential equation, just like the corresponding equation for the principle of the conservation of electricity, and it will obviously read:

$$\frac{\partial \xi}{\partial t} + \operatorname{div} \mathfrak{K} = 0 \,. \tag{25}$$

That is nothing but equation (19.b), but in which we have set $l = 0$. Now, it must be observed that the last equation is not valid in general. That is because one cannot conclude from the vanishing of the work $\int l \, dV$ for a *complete* electron (according to the *Lorentzian* principle of motion) that the elementary work $l \, dV$ must vanish for each *individual* element of the electron. The development of that question is not possible until one goes deeper into the details of the structure of the electron.

## § 5. – Transforming the electric and magnetic forces. Electromagnetic momenta.

The total force $d\,\mathfrak{F}$ that acts on a charge element $de = \rho \, dV$ can be represented in the form $d\,\mathfrak{F} = \mathfrak{f} \, dV$. According to (18.b), the vector $\mathfrak{f}$ is expressed by the formula:

$$\mathfrak{f} = e\,\mathfrak{E} + \mathfrak{i} \times \mathfrak{H} \,, \tag{26}$$

and can be defined to be the force per unit volume or also the *impulse* per unit volume of the force. In that conception of things, the vector $\mathfrak{f}$ corresponds to the scalar $l$ : The latter means the work done per unit time, while the former means the corresponding impulse.

We would now like to show that the expression (26) can be transformed in a way that is completely analogous to the transformation of the expression (19) that was performed in the previous section.

We first consider the case in which the electric and magnetic field are constant in time. The two components of (26) can then be transformed independently of each other.

When one recalls the differential equations:

$$\operatorname{div} \mathfrak{E} = 4\pi\rho \quad \text{and} \quad \operatorname{rot} \mathfrak{E} = 0 \,,$$

then resulting electric force:

$$\mathfrak{F}^{(e)} = \int \mathfrak{E} \, \rho \, dV$$

that acts upon all of the charges that are found in the volume $V$ can initially be written in the form:

$$\mathfrak{F}^{(e)} = \frac{1}{4\pi} \int \mathfrak{E} \operatorname{div} \mathfrak{E} \, dV \,.$$

Moreover, from the general formula (16.c), Introduction, we have:

$$\int \mathfrak{E} \operatorname{div} \mathfrak{E} \, dV = \oint (\mathfrak{E}\,\mathfrak{n})\,\mathfrak{E} \, dV - \int (\mathfrak{E} \operatorname{grad})\,\mathfrak{E} \, dV \,,$$

and from (27) (*ibid.*), since rot $\mathfrak{E} = 0$ , we will have:

$$(\mathfrak{E}\ \text{grad})\ \mathfrak{E} = \tfrac{1}{2}\,\text{grad}\,E^2 .$$

It will follow from this that:

$$\mathfrak{F}^{(e)} = \frac{1}{4\pi}\oint (\mathfrak{E}\,\mathfrak{n})\,\mathfrak{E}\,dV - \frac{1}{8\pi}\int \text{grad}\,E^2\,dV ,$$

and finally, from (16), Introduction:

$$\mathfrak{F}^{(e)} = \frac{1}{4\pi}\oint \left(E_n\,\mathfrak{E} - \tfrac{1}{2}E^2\mathfrak{n}\right) dS . \tag{27}$$

With that formula, the force $\mathfrak{F}^{(e)}$ can be represented as the resultant of the quasi-elastic *stresses* that act in all of the electric field (even where no charge is present). In that way, we can use (27) to say that a certain force $-\ T_n^{(e)}\,dS$ acts on any surface element *dS*, where $T_n^{(e)}$ – viz., the "stress" per unit area – is given by the formula:

$$\mathfrak{T}_n^{(e)} = \frac{E^2}{8\pi}\mathfrak{n} - \frac{E_n}{4\pi}\mathfrak{E} . \tag{27.a}$$

One considers such stresses, or surface forces, in the theory of elasticity, where they characterize the interaction of two parts of a body that bound each other along the surface in question. In our case, those stresses are mere mathematical fictions that represent the interaction of two electric charges that are found on the two sides of any surface *S* in a very convenient and intuitive way. If the electric field strength is parallel to the surface normal $\mathfrak{n}$ ($E_n\,\mathfrak{E} = E^2\mathfrak{n}$ ) then the stress (27.a) will reduce to $-\ \frac{E^2}{8\pi}\mathfrak{n}$ . In that case, we will then have a *tension* that acts from the inside to the outside. By contrast, if the vectors $\mathfrak{n}$ and $\mathfrak{E}$ are perpendicular to each other then we will get a *compression* with the same magnetic that points from the outside to the inside (in the usual manner of hydrostatic pressure). In other words, we can represent the electric lines of force as stressed wires that are subject to a pull in the longitudinal direction and a push in the lateral direction. Therefore, the aforementioned "principal stresses" – viz., the longitudinal pull and lateral push per unit area – will be equal to the volume density of the electric energy (*Maxwellian* stresses). From the mathematical viewpoint, we can treat the stress vector $\mathfrak{T}_n^{(e)}$ as the inner product of a certain *tensor quantity* ${}^2\mathfrak{T}^{(e)}$, namely, the so-called electric *stress tensor*, with the normal $\mathfrak{n}$. Upon multiplying $\mathfrak{T}_n$ with any other unit vector $\mathfrak{k}$, we will get a scalar quantity:

$$T_{nk}^{(e)} = (\mathfrak{n}\,\mathfrak{k})\frac{E^2}{8\pi} - \frac{E_n\,E_k}{4\pi} \tag{27.b}$$

that we can refer to as the *component* of the tensor ${}^2\mathfrak{T}^{(e)}$ relative to the two directions $\mathfrak{n}$ and $\mathfrak{k}$. We must conclude from the fact that $\mathfrak{n}$ and $\mathfrak{k}$ enter into (27.b) in a completely-symmetric way that the electric stress tensor is a symmetric tensor. According to (27.b) (in which $\mathfrak{n}$ and $\mathfrak{k}$ should mean two coordinate vectors), its components relative to any rectangular coordinate system $X_1$, $X_2$, $X_3$ are expressed by the formulas:

$$\left.\begin{aligned} &T_{11}^{(e)} = \frac{1}{8\pi}(-E_1^2 + E_2^2 + E_3^2), \quad T_{22}^{(e)} = \frac{1}{8\pi}(-E_2^2 + E_3^2 + E_1^2), \quad T_{33}^{(e)} = \frac{1}{8\pi}(-E_3^2 + E_1^2 + E_2^2), \\ &T_{23}^{(e)} = T_{32}^{(e)} = -\frac{E_2 E_3}{4\pi}, \qquad T_{31}^{(e)} = T_{13}^{(e)} = -\frac{E_1 E_3}{4\pi}, \qquad T_{12}^{(e)} = T_{21}^{(e)} = -\frac{E_1 E_2}{4\pi}. \end{aligned}\right\} \quad (27.c)$$

Entirely-analogous results can be obtained by transforming the magnetic (or electromagnetic) force:

$$\mathfrak{F}^{(m)} = \int \mathfrak{j} \times \mathfrak{H}\, dV$$

when the electric current is assumed to be *stationary* (and as a result, the magnetic field strength is assumed to be constant in time). In that case, we will have:

$$\text{rot}\ \mathfrak{H} = 4\pi \mathfrak{j}\ , \ \ \text{div}\ \mathfrak{H} = 0\ .$$

We will then have $\mathfrak{j} \times \mathfrak{H} = \dfrac{1}{4\pi}\ \text{rot}\ \mathfrak{H} \times \mathfrak{H}$, and as a result, due to the identity:

$$\text{grad}\ \tfrac{1}{2}H^2 = (\mathfrak{H}\ \text{grad})\ \mathfrak{H} + \mathfrak{H} \times \text{rot}\ \mathfrak{H}\ ,$$

we will then have:

$$\int \mathfrak{j} \times \mathfrak{H}\, dV\ = \frac{1}{4\pi}\int (\mathfrak{H}\,\text{grad})\,\mathfrak{H}\, dV - \frac{1}{8\pi}\int \text{grad}\, H^2\, dV\ ,$$

i.e.:

$$\mathfrak{F}^{(m)} = \frac{1}{4\pi}\oint \left( H_n\, \mathfrak{H} - \tfrac{1}{2} H^2 \mathfrak{n} \right) dS\ . \quad (28)$$

That formula is completely-identical to (27). That is what we do not need to go into the details of its interpretation. We can define the vector:

$$\mathfrak{T}_n^{(m)} = \frac{H^2}{8\pi}\mathfrak{n} - \frac{H_n}{4\pi}\mathfrak{H} \quad (28.a)$$

to be the normal component of the *magnetic stress tensor* ${}^2\mathfrak{T}^{(m)}$. Its rectangular components are expressed by formulas that are entirely analogous to (27.c):

$$\left.\begin{aligned} &T_{11}^{(m)} = \frac{1}{8\pi}(-H_1^2 + H_2^2 + H_3^2), \quad T_{22}^{(m)} = \frac{1}{8\pi}(-H_2^2 + H_3^2 + H_1^2), \quad T_{33}^{(m)} = \frac{1}{8\pi}(-H_3^2 + H_1^2 + H_2^2), \\ &T_{23}^{(m)} = T_{32}^{(m)} = -\frac{H_2 H_3}{4\pi}, \qquad T_{31}^{(m)} = T_{13}^{(m)} = -\frac{H_1 H_3}{4\pi}, \qquad T_{12}^{(m)} = T_{21}^{(m)} = -\frac{H_1 H_2}{4\pi}. \end{aligned}\right\} \quad (28.b)$$

We refer to the sum:

$$^2\mathfrak{T} = {}^2\mathfrak{T}^{(e)} + {}^2\mathfrak{T}^{(m)} \quad (28.c)$$

as the *electromagnetic stress tensor.*

We now go on to our general problem: Transforming the expression (26), or the integral:

$$\mathfrak{F} = \int \mathfrak{f}\, dV\,,$$

for an arbitrary electric system, i.e., for time-varying electromagnetic fields.

We therefore write the fundamental equations of the electromagnetic field in the form:

$$\left.\begin{aligned} &\operatorname{div}\mathfrak{E} = 4\pi\rho\,, \quad \operatorname{rot}\mathfrak{H} - \frac{1}{c}\frac{\partial\mathfrak{E}}{\partial t} = 4\pi\mathfrak{j}, \\ &\operatorname{div}\mathfrak{H} = 4\pi\rho^*\,, \quad \operatorname{rot}\mathfrak{E} + \frac{1}{c}\frac{\partial\mathfrak{H}}{\partial t} = -4\pi\mathfrak{j}^*\,, \end{aligned}\right\} \quad (29)$$

in which the magnetic charge and current densities are obviously equal to zero ( $\rho^* = 0,\ \mathfrak{j}^* = 0$), and add the conjugate magnetic expression to the expression (26):

$$\mathfrak{f}^* = \rho^*\mathfrak{H} - \mathfrak{j}^* \times \mathfrak{E}$$

[cf., (12.a), Chap. V]. We will then have:

$$\mathfrak{f} = \mathfrak{f} + \mathfrak{f}^* = \rho\,\mathfrak{E} + \rho^*\mathfrak{H} + \mathfrak{j} \times \mathfrak{H} - \mathfrak{j}^* \times \mathfrak{E}\,,$$

or from (29):

$$\begin{aligned} \mathfrak{f} &= \frac{1}{4\pi}\{\mathfrak{E}\operatorname{div}\mathfrak{E} + \mathfrak{H}\operatorname{div}\mathfrak{H} + \operatorname{rot}\mathfrak{H}\times\mathfrak{H} + \operatorname{rot}\mathfrak{E}\times\mathfrak{E} - \frac{1}{c}\frac{\partial\mathfrak{E}}{\partial t}\times\mathfrak{H} + \frac{1}{c}\frac{\partial\mathfrak{H}}{\partial t}\times\mathfrak{E}\} \\ &= \frac{1}{4\pi}(\mathfrak{E}\operatorname{div}\mathfrak{E} - \mathfrak{E}\times\operatorname{rot}\mathfrak{E} + \mathfrak{H}\operatorname{div}\mathfrak{H} - \mathfrak{H}\times\operatorname{rot}\mathfrak{H}) - \frac{1}{4\pi c}\frac{\partial}{\partial t}(\mathfrak{E}\times\mathfrak{H})\,. \end{aligned}$$

When we subtract the formulas that were already used above:

$$\oint(\mathfrak{E}\,\mathfrak{n})\,\mathfrak{E}\,dS = \int(\mathfrak{E}\operatorname{grad})\,\mathfrak{E}\,dV + \int\mathfrak{E}\operatorname{div}\mathfrak{E}\,dV$$

and

$$\oint\tfrac{1}{2}E^2\,\mathfrak{n}\,dS = \int\tfrac{1}{2}\operatorname{grad}E^2\,dV = \int(\mathfrak{E}\operatorname{grad})\,\mathfrak{E}\,dV + \int\mathfrak{E}\times\operatorname{rot}\mathfrak{E}\,dV\,,$$

that will give the identity:

$$\int (\mathfrak{E}\,\mathrm{div}\,\mathfrak{E} - \mathfrak{E}\times\mathrm{rot}\,\mathfrak{E})\,dV = \oint \left\{(\mathfrak{E}\,\mathfrak{n})\,\mathfrak{E} - \tfrac{1}{2}E^2\,\mathfrak{n}\right\} dS\,.$$

From this identity and the corresponding one for $\mathfrak{H}$, we get:

$$\int \mathfrak{f}\,dV = \frac{1}{4\pi}\oint \left\{(\mathfrak{E}\,\mathfrak{n})\,\mathfrak{E} + (\mathfrak{H}\,\mathfrak{n})\,\mathfrak{H} - \tfrac{1}{2}(E^2+H^2)\,\mathfrak{n}\right\} dS - \frac{1}{4\pi c}\int \frac{\partial}{\partial t}(\mathfrak{E}\times\mathfrak{H})\,dV\,,$$

i.e.:

$$\mathfrak{F} = -\frac{\partial \mathfrak{G}}{\partial t} - \oint \mathfrak{T}_n\,dS\,, \tag{30}$$

in which:

$$\mathfrak{G} = \int \frac{1}{4\pi c}(\mathfrak{E}\times\mathfrak{H})\,dV = \frac{1}{c^2}\int \mathfrak{K}\,dV\,. \tag{30.a}$$

$\mathfrak{K}$ means the energy flux vector (20.b), and $\mathfrak{T}_n$ means the normal component of the stress tensor:

$$\mathfrak{T}_n = \frac{1}{4\pi}\left[E_n\,\mathfrak{E} + H_n\,\mathfrak{H} - \tfrac{1}{2}(E^2+H^2)\,\mathfrak{n}\right]\,. \tag{30.b}$$

The formula (30) is entirely analogous to the formula (20). In that way, the scalars $A$ (work per unit time) and $W$ (energy) correspond to the vectors $\mathfrak{F}$ and $\mathfrak{G}$, resp. The vector $\mathfrak{K}$ in (20) corresponds to the *tensor* $^2\mathfrak{T}$ in (30). It follows from this purely-formal relationship between the two expressions that they must admit a corresponding physical interpretation. In fact, if one considers the vector $\mathfrak{K}$, not as the *force* per unit time, but as the *impulse* delivered by that force per unit time (see above), and employs the usual mechanical relationship between the impulse delivered by a force and the resultant change in the mechanical *momentum* (which corresponds completely to the relationship between work and the change in kinetic energy) then one can define the vector $\mathfrak{G}$ to be the *electromagnetic momentum* that is stored in the volume $V$ and define the tensor $^2\mathfrak{T}$ to be the *volume density of the momentum flux*.

If the impulse $\mathfrak{F}$ were non-zero then one could say that it was gained at the expense of the electromagnetic momentum. However, since that impulse vanishes for any isolated electron, according to the *Lorentz* principle of motion, the formula (30) will reduce to (in the event that the surface in question does not intersect any electron):

$$-\frac{d\mathfrak{G}}{dt} = \int \mathfrak{T}_n\,dS\,, \tag{31}$$

and that expresses the law of *the conservation of electromagnetic momentum*.

One can replace the last formula with the corresponding differential equation:

$$\frac{\partial \mathfrak{g}}{\partial t} + \operatorname{div} {}^{2}\mathfrak{T} = 0 \,, \tag{31.a}$$

in which:

$$\mathfrak{g} = \frac{1}{c^2}\mathfrak{K} \tag{31.b}$$

means the spatial density of the electromagnetic momentum. However, one must observe that the equation (31.a) is *not* true in the interior of the isolated electrons.

With the aforementioned interpretation, the electromagnetic momentum (just like the electromagnetic energy) seems to be something substantial that can be localized in space. Since it is a vector quantity, it is impossible to treat it as a type of substance in its own right. However, one can appeal to the usual definition of the mechanical momentum (i.e., product of mass and velocity) and define the vector $\mathfrak{g}$ to be the product of a scalar $\mu$ that has the meaning of an ordinary *mass density*, i.e., an ordinary *inertial mass*, per unit volume, with a vector that should represent the velocity of the motion of that mass.

Let it be recalled that we have performed an analogous decomposition in the previous section on the energy flux vector $\mathfrak{K}$. Upon replacing the corresponding expression (21.b) in (31.b), that will give:

$$\mathfrak{g} = \mu \mathfrak{C} \,, \tag{32}$$

with

$$\mu = \frac{\xi}{c^2} \,. \tag{32.a}$$

The last equation expresses an entirely-general relation between mass density and energy density. It corresponds to the relation (16.b) between the mass and the (electrostatic) energy of an electron that we have presented on the basis of the electromagnetic theory of the mass. In that way, the mass of an electron (or any system of electrons) seems to be a property of its electromagnetic field. Indeed, it must not be localized in the electron itself, i.e., the space where its charge is thought to be localized, but in all of the space over which the electromagnetic field of that charge extends. Furthermore, it follows from the formula (32.a) that it is not electric energy that contributes to the mass, but the magnetic energy. In other words, the mass of an electron must increase with its velocity and take on the value that was calculated above only in the limiting case of very small velocities. It is noteworthy that the factor 4 / 3 that appears in (16.b) is missing from the general formula (32.a). That fact poses a major problem for the electromagnetic theory of mass to which we will return later.

Strictly speaking, the decomposition (32), like the corresponding decomposition (21.b), will take on a proper physical sense only in the wave zone where the magnitude and direction of the velocity $\mathfrak{C}$ coincide with those of the velocity of propagation of electromagnetic waves. In that case, the tensor ${}^{2}\mathfrak{T}$ can also be represented in a form that is completely identical to the form that it

has in ordinary mechanics. Namely, if we have a continuously-distributed body with mass density $\mu$ and flow velocity $\mathfrak{C}$ then the components of the tensor (viz., the current density ${}^2\mathfrak{T}$ of $\mathfrak{g}$) will be expressed in terms of the components of $\mathfrak{C}$ and in the same way that the components of $\mathfrak{g}$ (viz., the current density of $\mu$) are expressed in terms of the components of $\mathfrak{C}$ and the scalar $\mu$. In that case, along with the formulas $g_i = \mu C_i$ ($i$ = 1, 2, 3), the corresponding formulas:

$$T_{ik} = \mu C_i C_k \tag{32.b}$$

must also be true.

Now, it is easy to convince oneself that those formulas will coincide with the generally-valid formula:

$$T_{ik} = \frac{1}{4\pi}[-E_i E_k - H_i H_k + \tfrac{1}{2}(E^2 + H^2)\delta_{ik}] \tag{33}$$

that follows from (30.b) *in the wave zone*. In that way, the $\delta_{ik}$ mean the components of the identity tensor:

$${}^2\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{33.a}$$

such that $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$.

We imagine that, e.g., the first axis is laid in the direction of propagation (or radiation), the second axis is in the direction of the electric field strength, and the third one is in the direction of the magnetic field strength. In that way, according to (22), we will get a right-handed coordinate system relative to which the tensor components (33) can be expressed as follows:

$$\left.\begin{aligned} T_{11} = \frac{1}{8\pi}(E^2 + H^2), \quad T_{22} = \frac{1}{8\pi}(-E^2 + H^2) = 0, \quad T_{11} = \frac{1}{8\pi}(E^2 - H^2) = 0, \\ T_{23} = T_{31} = T_{12} = 0. \end{aligned}\right\} \tag{33.b}$$

That same result can be deduced from formula (32.b) when one understands $\mathfrak{C}$ to mean the velocity vector of the light and considers the relation (32.a) between mass and energy density.

However, it must be emphasized that this agreement if true for *only* the wave zone. In general, a velocity vector $\mathfrak{C}$ that would admit a representation of the tensor components (33) in the form (32.a) *cannot be defined at all*, if only on the grounds that we would get six equations for three unknowns (viz., the components of $\mathfrak{C}$) by setting (32.b) equal to (33).

The aforementioned "substantial representation" of the electromagnetic fields as the carrier of energy and mass will then remain unconditionally valid for only the wave zone. However, a decomposition of the energy flux vector (or the momentum), and especially the "momentum flux tensor" ${}^2\mathfrak{T}$, in analogy with classical mechanics, would otherwise be not just physically absurd, but also mathematically impossible in the last case. It should be noted that in the region of space

where that decomposition is permissible, the electromagnetic energy will prove to be twice as big as the kinetic energy (in the sense of ordinary mechanics) of a mass that moves with the speed of light whose magnitude is given by (32.a).

The two conceptions of the tensor ${}^2\mathfrak{T}$ (as the current density of the electromagnetic momentum and the stress tensor) are obviously completely equivalent to each other. We would like to explain that equivalence with the following simple example: We consider a bundle of light rays, i.e., a laterally bounded wave-train that falls on a material plate and is absorbed by that plate. In that way, the energy and momentum that are carried by the ray-bundle in question will go into the plate, so to speak, where they will then assume the usual form of kinetic energy or heat or an ordinary "mechanical" momentum, resp., that can express itself as a motion of the plate in the direction of light rays. It will then follow that the plate experiences a certain *pressure* from the light rays. That so-called *light pressure* is obviously equal to the momentum that is transferred to the plate per unit time, i.e., (in the event that the pressure is referred to a unit area), it is equal to the product $g\,c$. On the other hand, it must be equal to the component of the stress tensor $T_{11} = \xi$ that was calculated above. However, those quantities are identical since it would follow from (32) and (32.a) that:

$$g = \frac{\xi}{c}.$$

We can then say that the light pressure on an absorbing surface that is normal to the light rays is numerically equal to the electromagnetic energy density.

For an oblique incidence of the rays, the surface force (stress) $\mathfrak{T}_n$ that they exert will not reduce to a simple hydrostatic pressure. According to the known rules of tensor calculus [Introduction, (34)], the projection of that force onto any unit vector $\mathfrak{k}$ is generally expressed by the formula:

$$({}^2\mathfrak{T}\ \mathfrak{n})\ \mathfrak{k} = T_{nk} = \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} T_{\alpha\beta}\, n_\alpha\, k_\beta\ .$$

In the special case where the coordinate system to which the formulas (33.b) refer, we will get:

$$T_{nk} = \xi\, n_1\, k_1 = \xi \cos\,\theta \cos\,\theta', \tag{34}$$

where $\theta$ and $\theta'$ mean the angles between the vectors $\mathfrak{n}$ ($\mathfrak{k}$, resp.) and the light rays. The pressure that acts normally to the surface is then equal to:

$$T_{nn} = \xi \cos^2\,\theta, \tag{34.a}$$

in the case considered.

## § 6. – The translational motion of a Lorentz electron.

The transformation formulas for the work (20) that were cited in the previous two sections, and especially formula (30), allow us to calculate the *self-force* that an arbitrary moving electron exerts in a new way, and therefore to exhibit the exact form of the equations of motion of an electron that is based upon the Lorentz principle of motion. In order to do that, we need to consider only the field that is created by the electron in question in the calculation of the electromagnetic momentum (or energy).

Obviously, one can determined the desired self-force *directly* as the resultant of the elementary forces between the various elements of the electron, just as we did approximately in **§ 2** and **§ 3** [formula (24.b)]. However, the indirect method that we shall pursue ([1]) has the following advantage over the direct one: We can use formulas (26.a) and (26.b), Chap. VI in order to represent the electromagnetic field of the electron at a certain moment $t$ as a series whose individual terms depend upon its velocity $\mathfrak{v}$, acceleration $\mathfrak{w}$, and higher derivatives of $\mathfrak{v}$ with respect to time ([2]). The terms that depend upon only the velocity (viz., first-order terms) are irrelevant in the calculation of the self-force by direct methods. We must then consider the second-order terms, which produce the main part of the self-force, namely, the inertial force. It should be remarked that those terms can include the velocity along with the acceleration. However, in the aforementioned approximation, we have restricted ourselves to the case of small velocities, and correspondingly dealt with only the terms in $\mathfrak{w}$ that are independent of $\mathfrak{v}$. The third-order terms determine the next part of the self-force, which corresponds to radiation damping. Along with $d\,\mathfrak{w} / dt$, they can also include $\mathfrak{v}$ and $\mathfrak{w}$. However, we have assumed that $\mathfrak{v}$ and $\mathfrak{w}$ are relatively small in the calculation of the damping force. Analogous contributions are true for the higher-order terms.

By contrast, due to the appearance of the *time derivatives* of energy and momentum in formulas (20) and (30), in the indirect method, one must already consider the *first*-order terms, which depend upon only the velocity, when calculating the self-force (or the work that it does). Those terms indeed play the main role since differentiating the corresponding terms in the momentum $\mathfrak{G}$ with respect to time will produce terms that depend upon the acceleration linearly, so they will determine the inertial force. Briefly, that relation can be formulated as follows: Terms of order $n$ in the series development of the electromagnetic field allow one to calculate the same ($n^{\text{th}}$-order) part of the self-force by means of the indirect method that can only be determined from the ($n$ + 1)$^{\text{th}}$-order terms using the direct method. Thus, if one deals with the *precise* calculation of the "inertial force," i.e., that part of the self-force that is proportional to the acceleration of the electron, but in which one must also consider its dependency upon the velocity, then it will suffice to evaluate the electromagnetic fields of the electron for a motion at *constant velocity* in order to use the indirect method. In that way, the integrals $\oint K_n\, dS$ and $\oint \mathfrak{T}_n\, dS$ that appear in (20) and (30) will drop out for a surface at infinity (since the electric and magnetic field strengths will drop off

([1]) Which goes back to *Abraham*.

([2]) As long as those derivatives are continuous, and as a result finite. In the opposite case, one must consider the time interval in which they remain finite by itself and correspondingly employ different developments for different parts of the field.

in inverse proportion to the square of the distance under uniform rectilinear motion, just as they do for a state of rest), such that those formulas will take on the following simple forms:

$$\mathfrak{F} = -\frac{d\mathfrak{G}}{dt} \tag{35}$$

and

$$A = \mathfrak{F}\,\mathfrak{v} = -\frac{dW}{dt}. \tag{35.a}$$

The integrals (30.a) and (20.a), which determine $\mathfrak{G}$ and $W$, must obviously extend over all of space then.

In Chapter VI, we showed that the electromagnetic field of a *point-charge de* in a state of uniform rectilinear motion (with less than the speed of light) can be expressed by the formulas:

$$d\,\mathfrak{E} = (1-\beta^2)\,de\frac{\mathfrak{R}}{R^{*3}}, \qquad d\,\mathfrak{H} = \frac{1}{c}\,\mathfrak{v} \times d\,\mathfrak{E} \tag{36}$$

[cf., (13), (13.a), and (13.b), Chap. VI]. We can now treat the infinitely-small element *de* of the electron considered to be a point-charge and determine the resultant field strengths $\mathfrak{E}$, $\mathfrak{H}$ by integrating the expressions (36).

Let it be recalled that the vector $\mathfrak{R}$ means the radius vector of the reference point $P$ considered relative to the *simultaneous* (momentary) position $P^*$ of the element *de*, and that in a rectangular coordinate system $X_1$, $X_2$, $X_3$ whose origin lies at $P^*$, and whose $X_1$-axis falls in the direction of motion, the following formula will be true:

$$R^* = \sqrt{x_1^2 + (1-\beta^2)(x_2^2 + x_3^2)}, \tag{36.a}$$

in which $x_1$, $x_2$, $x_3$ are the components of $\mathfrak{R}$.

We now introduce a new rectangular coordinate system $X_1'$, $X_2'$, $X_3'$, with the same origin and the same orientation of the axes, but another yardstick for the first axis (which is parallel to $\mathfrak{v}$), by way of the formulas:

$$x_1 = x_1'\sqrt{1-\beta^2}, \qquad x_2 = x_1', \qquad x_3 = x_1. \tag{37}$$

In that way, the expression for $R^*$ above can be represented in the form:

$$R^* = \sqrt{1-\beta^2}\,R', \tag{37.a}$$

in which:

$$R' = \sqrt{x_1'^2 + x_2'^2 + x_3'^2} \tag{37.b}$$

has the meaning of the distance $P^*P$ in the new coordinate system. Upon substituting those expressions in the formula (36) for $d\,\mathfrak{E}$, we will get:

$$d\,\mathfrak{E} = \frac{de}{\sqrt{1-\beta^2}}\frac{\mathfrak{R}}{R'^3},$$

or in components along the coordinate axis:

$$dE_1 = \frac{de}{\sqrt{1-\beta^2}}\frac{x_1}{R'^3} = de\,\frac{x_1'}{R'^3}, \qquad dE_2 = \frac{de}{\sqrt{1-\beta^2}}\frac{x_2'}{R'^3}, \qquad dE_3 = \frac{de}{\sqrt{1-\beta^2}}\frac{x_3'}{R'^3}. \quad (38)$$

It will then follow that the vector $d\,\mathfrak{E}'$, with the components:

$$dE_1' = dE_1, \qquad dE_2' = dE_2\sqrt{1-\beta^2}, \qquad dE_3' = dE_3\sqrt{1-\beta^2}, \quad (38.a)$$

can be treated as the electric field strength of a point-charge *de* that is at rest in the coordinate system $X'$ at the beginning.

Upon summing over the various elements of the electron charge, (38.a) will give the formulas:

$$E_1 = E_1', \qquad E_2 = \frac{E_2'}{\sqrt{1-\beta^2}}, \qquad E_3 = \frac{E_3'}{\sqrt{1-\beta^2}}. \quad (38.b)$$

The $E_1'$, $E_2'$, $E_3'$ in that mean the components of the resultant field strengths of the electron in question relative to the coordinate system $X'$.

We now remark that according to the transformation formula (37), this coordinate system arises from the original one *X* by a *dilatation in the direction of motion*, and indeed with a ratio of $1:\sqrt{1-\beta^2}$. Therefore, if the electron can be represented as spherical in the coordinate system *X* then it must appear to be an ellipsoid of revolution in the transformed coordinate systems $X'$ whose longitudinal axis is larger than its transverse axis by a ratio of $1:\sqrt{1-\beta^2}$, i.e., larger than the original radius of the sphere *a*. The total charge of the electron will remain unchanged under that in such a way that the charge density $\rho'$ (relative to $X'$) will appear to be reduced in comparison to the true charge density $\rho$ (relative to *X*) by the same ratio.

In the cited explanation, we have tacitly assumed that the electron moves like an absolutely-rigid body (in the sense of ordinary mechanics). Now, that picture, which goes back to *Abraham*, *is by no means the most natural*. Namely, in Chap. VI, **§ 3**, we saw that the interaction between the elements of a system of point-charges in a state of uniform rectilinear motion can be determined from a *convection potential* $\psi$ in the same way that the interaction of the same point-charges in the rest state of the system can be determined from the ordinary scalar potential $\varphi$. We have further seen that the surfaces $\psi$ = const. can be represented by *flattened* ellipsoids of revolution for any

point-charge that moves with constant velocity $v = c\,\beta$, and indeed they are flattened in the direction of motion by the ratio $\sqrt{1-\beta^2}:1$. The ellipsoids will once more appear to be spheres when observed in the transformed coordinate system $X'$.

Now, that raises the following question: Why must the moving electron keep the same form as the one at rest, despite the change in the interaction of its elements that is caused by its motion?

Under the assumption of spherical symmetry, in the rest state, that interaction reduces to a radial stress that will be cancelled in some way by the binding forces that act between the elements of the electron (otherwise, the electron would "explode"). In that way, the surface of the electron coincides with a level surface $\varphi = \text{const}$. The same thing is true for all surfaces of constant charge density whatsoever. Now, it would seem entirely natural to regard that coincidence of the surfaces $\rho = \text{const}$. and the surfaces $\varphi = \text{const}$. as the actual *equilibrium condition* for the electron, which must also determine its external form, in particular.

We would now like to follow *Lorentz* (and *Fitzgerald*) and generalize this *equilibrium principle* to the case of an electron in uniform rectilinear motion by replacing the scalar potential with the convection potential $\psi$ (for $\mathfrak{v} = 0$ and $\psi = \varphi$). We then assume the following: *For a moving electron, the surfaces $\rho = \text{const.}$, and especially the "free" surface $\psi = \text{const.}$, must coincide.*

According to (15.a), Chap. VI, the convection potential of an element *de* of the electron is expressed by the formula:

$$d\psi = de\,\frac{(1-\beta^2)}{R^*}, \tag{39}$$

or from (37.a)

$$d\psi = \frac{\sqrt{1-\beta^2}}{R'}\,de. \tag{39.a}$$

The surfaces $d\psi = \text{const}$. will then appear to be spherical with respect to the coordinate system $X'$. It will then follow from this that for an electron that is *spherical relative to that system*, i.e., for an electron for which the surfaces $\rho' = \text{const}$. are concentric spheres in $X'$, the level surfaces of the resulting convection potential must also be spherical (in $X'$). The integral in that case $\int \frac{de}{R'} = \varphi'$ is identical to the scalar potential of an electron at rest, and as is known, that can be represented in the form $e/r'$ ($r'$ means the distance from the reference point to the center *O* of the electron, as measured in $X'$). Therefore:

$$\psi = \sqrt{1-\beta^2}\,\varphi' = \sqrt{1-\beta^2}\,\frac{e}{r'}. \tag{39.b}$$

However, in order for the electron in question to appear spherically-symmetric relative to the coordinate system $X'$, in reality (i.e., relative to *X*), it must be an ellipsoid of revolution that is flattened by a ratio of $\sqrt{1-\beta^2}:1$. In that way, we will be led to the following so-called *Lorentz*

*contraction hypothesis: An electron in a state of uniform rectilinear motion must contract in the direction of motion by a ratio of* $\sqrt{1-\beta^2}:1$. (Its transverse dimensions remain unchanged by that.)

Our problem (viz., calculating the momentum) will be simplified by that assumption, namely, it will be reduced to an electrostatic problem that was solved before. The spatial density of the electromagnetic momentum:

$$\mathfrak{g} = \frac{1}{4\pi c}\,\mathfrak{E}\times\mathfrak{H}$$

can be initially represented by means of the relation $\mathfrak{H} = \frac{1}{c}\,\mathfrak{v}\times\mathfrak{E}$ in the form:

$$\mathfrak{g} = \frac{1}{4\pi c}[E^2\,\mathfrak{v} - (\mathfrak{v}\mathfrak{E})\mathfrak{E}], \tag{40}$$

or in terms of its components along the coordinate axes $X_1, X_2, X_3$ :

$$g_1 = \frac{1}{4\pi}\frac{v}{c^2}(E_2^2 + E_3^2), \qquad g_2 = -\frac{1}{4\pi}\frac{v}{c^2}E_1E_2, \qquad g_3 == -\frac{1}{4\pi}\frac{v}{c^2}E_1E_3.$$

If we introduce the quantities $E_1'$, $E_2'$, $E_3'$ into that in place of $E_1, E_2, E_3$ using (38.b) then we will have:

$$g_1 = \frac{1}{4\pi c^2}\frac{v}{1-\beta^2}(E_2'^2 + E_3'^2), \qquad g_2 = -\frac{1}{4\pi c^2}\frac{v}{1-\beta^2}E_1'E_2', \qquad g_3 == -\frac{1}{4\pi c^2}\frac{v}{1-\beta^2}E_1'E_3'.$$

In order to calculate the total momentum $\mathfrak{G} = \int \mathfrak{g}\,dV$, we remark that a volume element $dV = dx_1\,dx_2\,dx_3$ in $X$ corresponds to the volume element:

$$dV' = dx_1'\,dx_2'\,dx_3' = \frac{dx_1\,dx_2\,dx_3}{\sqrt{1-\beta^2}}$$

in the coordinate system $X'$, according to (37). We will then have:

$$dV = \sqrt{1-\beta^2}\,dV',$$

and as a result:

$$\left.\begin{aligned} G_1 &= \frac{v}{c^2\sqrt{1-\beta^2}}\frac{1}{4\pi}\int (E_2'^2 + E_3'^2)\,dV', \\ G_2 &= -\frac{v}{c^2\sqrt{1-\beta^2}}\frac{1}{4\pi}\int E_1' E_2'\,dV', \\ G_3 &= -\frac{v}{c^2\sqrt{1-\beta^2}}\frac{1}{4\pi}\int E_1' E_3'\,dV'. \end{aligned}\right\} \tag{40.a}$$

Since the electron is spherically-symmetric in its rest state, and that spherical symmetry will be preserved relative to the comoving coordinate system $X'$ as a result of the *Lorentz* contraction, the field the vectors $E_1'$, $E_2'$, $E_3'$ must be judged to be identical in that coordinate system to the ordinary electrostatic field of that electron in its rest state relative to that coordinate system *X*. It will then follow that:

$$\int E_1' E_2'\,dV' = \int E_1' E_3'\,dV' = 0$$

and

$$\int E_1'^2\,dV' = \int E_2'^2\,dV' = \int E_3'^2\,dV' = \tfrac{1}{3}\int E'^2\,dV' = \tfrac{1}{3}\int E^2\,dV',$$

i.e.:

$$\frac{1}{4\pi}\int (E_2'^2 + E_3'^2)\,dV' = \frac{4}{3}U_0\ .$$

The notation:

$$U_0 = \frac{1}{8\pi}\int E^2\,dV$$

means the usual electric energy of the electron in the rest state. If one appeals to (16.b) in order to define the corresponding *rest mass* of the electron by the formula:

$$m_0 = \frac{4}{3}\frac{U_0}{c^2} \tag{41}$$

then from (40.a), one will have:

$$\mathfrak{G} = \frac{m_0\,\mathfrak{v}}{\sqrt{1-\beta^2}} = m\,\mathfrak{v}\ . \tag{41.a}$$

The quantity:

$$m = \frac{m_0}{\sqrt{1-\beta^2}} \tag{41.b}$$

is the actual mass of the electron at the speed $v = c\,\beta$.

We then see that in contrast to ordinary mechanics, the mass of the electron is not a constant quantity, but depends upon the velocity, and indeed in such a way that it will become infinite as *v*

$\rightarrow c$. It will then follow that the speed of light $c$ represents a limiting speed that can never be attained by the action of finite forces.

In ordinary mechanics, one defines the mass to be the ratio of the external force $\mathfrak{F}^a$ to the acceleration $\mathfrak{w}$. Now, that definition proves to be valid only for low speeds. In fact, if one assumes that the self-force of the electron can be represented by $-d\,\mathfrak{G}\,/\,dt$ then its equations of motion will have the following general form:

$$\frac{d}{dt}(m\,\mathfrak{v}) = \mathfrak{F}^a \tag{42}$$

according to the *Lorentz* principle of motion. We now consider the special case in which the magnitude of the velocity remains constant, and only its direction changes in time (that case would correspond to, e.g., motion in an external magnetic field). $m$ will then remain constant, and the formula (42) will reduce to the form:

$$\frac{m_0}{\sqrt{1-v^2/c^2}}\frac{d\mathfrak{v}}{dt} = \mathfrak{F}^a. \tag{42.a}$$

In the opposite case, when the direction of the velocity stays constant, but its magnitude varies, we will get from (42) and (41.b):

$$\frac{d}{dt}(m\,v) = m_0\frac{d}{dt}\frac{v}{\sqrt{1-v^2/c^2}} = m_0\frac{dv}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}} + \frac{v^2/c^2}{(1-v^2/c^2)^{3/2}}\right) = \frac{m_0}{(1-v^2/c^2)^{3/2}}\frac{dv}{dt},$$

and as a result:

$$\frac{m_0}{(1-v^2/c^2)^{3/2}}\frac{dv}{dt} = F^a. \tag{42.b}$$

The mass, as the ratio of the force to acceleration, proves to be different in the two cases (42.a) and (42.b). One correspondingly calls the quantity $\frac{m_0}{\sqrt{1-\beta^2}}$ the *transverse* mass of the electron and calls $\frac{m_0}{(1-\beta^2)^{3/2}}$ the *longitudinal* one. However, it is more convenient to consider the mass to be a unified quantity that is determined by (41.b) and merely define it to be the coefficient of the velocity vector in the general expression (41.a) for momentum.

Upon inner multiplying the force $\mathfrak{F}^a$ by $\mathfrak{v}$, we will get the work done per unit time by that force. Now, from (42) and (41.b):

$$A = \mathfrak{F}^a \cdot \mathfrak{v} = \mathfrak{v}\,\frac{d}{dt}(m\,\mathfrak{v}) = \mathfrak{v}^2\frac{dm}{dt} + m\left(\mathfrak{v}\frac{d\mathfrak{v}}{dt}\right) = \frac{m_0\,v^2/c^2}{(1-v^2/c^2)^{3/2}}\frac{d(v^2)}{dt} + \frac{m_0\,v^2/c^2}{2\sqrt{1-v^2/c^2}}\frac{d(v^2)}{dt}$$

$$= \frac{m_0}{2(1-v^2/c^2)^{3/2}} \frac{d(v^2)}{dt} = m_0 c^2 \frac{d}{dt} \frac{1}{\sqrt{1-v^2/c^2}}.$$

We then see that the work done during an arbitrary time interval will be equal to the (algebraic) increase in the quantity:

$$W^* = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} = mc^2, \tag{43}$$

such that the difference:

$$T^* = c^2(m-m_0) = c^2 m_0 \left( \frac{1}{\sqrt{1-v^2/c^2}} - 1 \right), \tag{43.a}$$

which will vanish for $v = 0$, will play the role that *kinetic energy* does in ordinary mechanics. If one develops $(1-v^2/c^2)^{-1/2}$ in powers of $v^2/c^2$ then that will give the series:

$$T^* = \tfrac{1}{2} m_0 v^2 + \tfrac{3}{8} m_0 \frac{v^4}{c^2}, \tag{43.b}$$

whose first term actually agrees with the usual expression for kinetic energy when the mass is constant.

It should be noted that that the kinetic energy of the electron that is defined by (43.a) is *different from* its *magnetic* energy $T = \frac{1}{8}\int H^2 dV$. In fact, by means of the relation $\mathfrak{H} = \frac{1}{c}\,\mathfrak{v} \times \mathfrak{E}$, or in the coordinate representation:

$$H_1 = 0, \qquad H_2 = -\frac{v}{c} E_3, \qquad H_3 = +\frac{v}{c} E_2,$$

we will get from (38.b) that:

$$H^2 = H_1^2 + H_2^2 + H_3^2 = \beta^2 (E_2^2 + E_3^2) = \frac{\beta^2}{1-\beta^2} (E_2'^2 + E_3'^2),$$

and as a result:

$$T = \frac{\beta^2}{\sqrt{1-\beta^2}} \frac{1}{8\pi} \int (E_2'^2 + E_3'^2)\, dV' = \frac{\beta^2}{\sqrt{1-\beta^2}} \frac{2}{3} U_0,$$

i.e.:

$$T = \frac{1}{2} \frac{m_0 v^2}{\sqrt{1-v^2/c^2}} = \tfrac{1}{2} m v^2. \tag{44}$$

We can explain the fact that this quantity does not coincide with the kinetic energy of the electron by saying that the electric energy of a moving electron $U$ is different from the corresponding rest

energy $U_0$ . That is why we should expect that we would have $T^* = T + U - U_0$ . However, that is not the case, either. Rather, from (38.b), we have:

$$U = \frac{1}{8\pi}\int E^2\, dV = \frac{1}{8\pi}\int\left( E_1^2 + \frac{E_2'^2 + E_3'^2}{1-\beta^2}\right)\sqrt{1-\beta^2}\, dV'$$

$$= \frac{1}{8\pi}\sqrt{1-\beta^2}\int E_1'^2\, dV' + \frac{1}{8\pi}\frac{1}{\sqrt{1-\beta^2}}\int (E_2'^2 + E_3'^2)\, dV' = \frac{1}{3}U_0\left(\sqrt{1-\beta^2} + \frac{2}{\sqrt{1-\beta^2}}\right),$$

and as a result:

$$W = U + T = m_0 c^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - \tfrac{1}{4}\sqrt{1-v^2/c^2}\right). \qquad (44.a)$$

The difference $W - U_0$ is not equal to $T^*$ then. In the limit of very high speeds that approach the speed of light, the electromagnetic energy (44.a) will coincide with the quantity $W^*$ that was defined above. By contrast, for $v = 0$, we will get $W^* = c^2 m_0$ and $W = \frac{3}{4} m_0 c^2$ . The last formula is identical to (41) (since $W = U_0$ for $v = 0$). However, from the general relation (32.a) between the mass and energy of the electromagnetic field, we should not consider $U_0$ to be the energy of the electron in the rest state, but the quantity $W_0^* = m_0 c^2$ . The quantity that is determined from the general equation (43), will have the meaning of the total energy of the electron at (constant) velocity $v$, accordingly.

That disagreement seems to be directly linked with the contraction hypothesis that was the basis for our calculations. That is because the assumption that a moving electron is compressed in the direction of motion implies a peculiar dependency of that that field on the velocity of the electron as a whole by way of the general differential equations of the electromagnetic field that cannot be overlooked. We will come back to that question later on. Here, it should be observed that equations (41.b) and (42), which were exhibited on the basis of the *Lorentz* principle of motion and the contraction hypothesis, are in excellent agreement with the facts of experiment in regard to the deflection of freely-moving electrons (e.g., in the form of cathode rays) by electric and magnetic forces. By contrast, the *Abraham* picture of the absolutely-rigid electron leads to a dependency of the mass on velocity that breaks down completely for high speeds. We would not like to go into *Abraham*'s theory here, which is presently of only historical interest. However, we will show immediately that the *Lorentz* formula (41.b) can be derived from completely-general principles without having to propose any specialized pictures regarding the composition of electrons.

Those principles are expressed, first of all, by the formula (32.a) of the *equivalence of mass and energy*, and secondly, the general form of the equations of motion (42), i.e., the possibility of representing the momentum as a product of mass and velocity. Since the work done by the external force $\mathfrak{F}^a$ must be equal to the increase in the energy of the electron, it will follow from the principles above that:

$$\frac{d}{dt}(c^2 m) = \mathfrak{v}\frac{d(m\mathfrak{v})}{dt},$$

i.e., with $v = c\beta$:

$$dm = \tfrac{1}{2} m d(\beta^2) + \beta^2 dm,$$

or

$$\frac{dm}{m} = \frac{1}{2}\frac{d(\beta^2)}{1-\beta^2}.$$

Upon integrating that equation, we will get:

$$m = \frac{m_0}{\sqrt{1-\beta^2}},$$

i.e., the *Lorentz* formula.

From the theory that was presented, one must regard the equation $\frac{d}{dt}(m\mathfrak{v}) = \mathfrak{F}^a$ as the approximate form of a more complicated equation of motion that is exactly valid only in the limiting case of low speeds. Had we further considered the acceleration of the electron in the calculation of $\mathfrak{G}$, then we would get an equation in which the damping force would appear along with the inertial force $-\frac{d}{dt}(m\mathfrak{v})$. However, in order to determine that force precisely, it would be necessary to make certain assumptions about the dependency of the form of the electron upon not only the velocity, but also the acceleration. Nonetheless, such a path to the exhibition of exact equations of motion seems inaccessible from the outset.

## § 7. – The rotational motion of a spherical electron.

As long as the electron can be considered to be a rigid or quasi-rigid body, one can examine not only its translational motion, but also the basically-possible *rotational motion* around a free axis that goes through its center, as well as the change that this rotational motion must experience under the action of external *torques*.

The rotation of a spherically-symmetric electron around any free axis with constant angular velocity $\mathfrak{u}$ obviously means a *stationary* electric current. Therefore, if no change in the charge distribution in the electron appears in that way, and in particular, its form remains the same (which we will always assume in what follows) then its electric field must remain unchanged, and its magnetic field must remain constant in time and proportional to the angular velocity. It will then follow that this angular velocity can be arbitrarily large, and that the linear velocity of the equatorial zone of the electron does not need to be restricted to the speed of light.

In the general case of an arbitrary charge distribution, the magnetic moment of the electron is expressed by the formula:

$$\mathfrak{m} = \tfrac{1}{2}\int \mathfrak{r} \times \rho \frac{\mathfrak{v}}{c} dV .$$

($\mathfrak{r}$ = radius vector of the volume element $dV$ relative to the center $O$.) If one substitutes $\mathfrak{v} = \mathfrak{u} \times \mathfrak{r}$ in that then one will have:

$$\mathfrak{m} = \frac{1}{2c}\int \rho\{\mathfrak{u}\, r^2 - \mathfrak{r}^2 (\mathfrak{u}\,\mathfrak{r})\} dV ,$$

or for a spherically-symmetric distribution of charge, since:

$$\overline{\mathfrak{r}_0 (\mathfrak{u}\,\mathfrak{r}_0)} = \mathfrak{u}\,\overline{\cos^2 (\mathfrak{u}, \mathfrak{r}_0)} = \tfrac{1}{3}\mathfrak{u}$$

[cf., the derivation of (17.a)]:

$$\mathfrak{m} = \frac{1}{3c}\mathfrak{u}\int \rho\, \mathfrak{r}^2\, dV . \tag{45}$$

In the case of a surface charge, in particular, the already-known formula (cf., **§ 7**, Chap. IV) will give:

$$\mathfrak{m} = \frac{e\,a^2}{3c}\mathfrak{u} , \tag{45.a}$$

in which $a$ means the radius of the electron. For a uniform volume distribution of charge ($\rho$ = const.), with $dV = 4\pi r^2\, dr$ , we will get:

$$\mathfrak{m} = \frac{e\,a^2}{5c} . \tag{45.b}$$

For the sake of simplicity, in what follows, we will restrict ourselves to the case of surface charge. In that way, the electric field inside the electron will be equal to zero, and outside of it, it will be identical to the field of a point charge concentrated at $O$, so it will be equal to:

$$\mathfrak{E} = \frac{e}{r^2}\mathfrak{r}_0 . \tag{46}$$

The magnetic field in external space is equivalent to the field of an elementary magnetic dipole at $O$:

$$\mathfrak{H} = \frac{1}{r^2}\{3\,(\mathfrak{m}\,\mathfrak{r}_0)\,\mathfrak{r}_0 - \mathfrak{m}\} . \tag{46.a}$$

By contrast, inside of the electron, a homogeneous field:

$$\mathfrak{H} = \frac{2}{a^3}\,\mathfrak{m} \tag{46.b}$$

will prevail [cf., **§ 7**, Chap. V].

If the electron in question is found in an external magnetic field whose field strength $\mathfrak{H}^a$ inside of the corresponding very-small spatial region can be regarded as constant then it must experience a torque with the moment:

$$\mathfrak{M}^a = \mathfrak{m} \times \mathfrak{H}^a \,. \tag{47}$$

For strong inhomogeneous fields, one must also consider the additional force:

$$\mathfrak{F}^a = (\mathfrak{m}\ \mathrm{grad})\ \mathfrak{H}^a \,. \tag{47.a}$$

We now move on to the consideration of the *internal torque*, which can be required by the non-uniform rotation of the electron. In that way, we will employ an indirect method ([1]) for the calculation of that "self-torque" that is connected with the corresponding method for the determination of the self-force. We would next like to show that such a method can actually be developed.

In order to do that, we consider the moment of a force $d\,\mathfrak{F} = \mathfrak{f}\,dV$ that acts upon the volume element $dV$ relative to any point $O$ (e.g., the center of the electron). From the general formula (30), we can then represent the force $\mathfrak{f}$ per unit volume in the form:

$$\mathfrak{f} = -\,\frac{\partial \mathfrak{g}}{\partial t} - \mathrm{div}\ {}^2\mathfrak{T} \tag{48}$$

[cf., (31.a), where we have set $\mathfrak{f} = 0$]. Upon outer multiplying that equation by $\mathfrak{r}$ (viz., the radius vector to $dV$), we will get the associated moment:

$$\mathfrak{r} \times \mathfrak{f} = -\,\mathfrak{r} \times \frac{\partial \mathfrak{g}}{\partial t} - \mathfrak{r} \times \mathrm{div}\ {}^2\mathfrak{T} \,.$$

Since $\mathfrak{r}$ is independent of $t$, one can set $\mathfrak{r} \times \dfrac{\partial \mathfrak{g}}{\partial t} = \dfrac{\partial}{\partial t}\,\mathfrak{r} \times \mathfrak{g}$ . Moreover, due to the symmetry of the tensor ${}^2\mathfrak{T}$, we will have:

$$(\mathfrak{r} \times \mathrm{div}\ {}^2\mathfrak{T})_1 = x_2\ \mathrm{div}\ T_3 - x_3\ \mathrm{div}\ T_2 = x_2\left(\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3}\right) - x_3\left(\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3}\right)$$

$$= \frac{\partial}{\partial x_1}(x_2\,T_{31}) + \frac{\partial}{\partial x_2}(x_2\,T_{32}) + \frac{\partial}{\partial x_3}(x_2\,T_{33}) - \frac{\partial}{\partial x_1}(x_3\,T_{21}) - \frac{\partial}{\partial x_2}(x_3\,T_{22}) - \frac{\partial}{\partial x_3}(x_3\,T_{23})\,,$$

([1]) which also goes back to *Abraham*.

i.e.:

$$(\mathfrak{r} \times \operatorname{div} {}^2\mathfrak{T})_1 = \operatorname{div} (\mathfrak{r} \times {}^2\mathfrak{T})_1 .$$

In that expression, $\mathfrak{r} \times {}^2\mathfrak{T}$ means an asymmetric tensor ${}^2\mathfrak{N}$ with the components:

$$N_{11} = x_2\, T_{31} - x_3\, T_{21} , \quad N_{22} = x_3\, T_{12} - x_1\, T_{32} , N_{33} = x_1\, T_{32} - x_2\, T_{13} , \quad N_{12} = x_2\, T_{32} - x_3\, T_{22} ,$$
$$N_{21} = x_3\, T_{11} - x_1\, T_{31} , \qquad \text{etc.},$$

or in in general

$$N_{ik} = (\mathfrak{r} \times \mathfrak{T}_k)_i \tag{49}$$

($\mathfrak{T}_k$ is the vector with the components $T_{k1}$, $T_{k2}$, $T_{k3}$ ; cf., Introduction, **§ 20**).

One will then have:

$$\mathfrak{r} \times \mathfrak{f} = - \frac{\partial}{\partial t} (\mathfrak{r} \times \mathfrak{f}) - \operatorname{div} {}^2\mathfrak{N} , \tag{49.a}$$

and as a result, for an arbitrary volume, one will have:

$$\int \mathfrak{r} \times \mathfrak{f}\, dV = - \frac{d}{dt} \int \mathfrak{r} \times \mathfrak{g}\, dV - \int {}^2\mathfrak{N}\, \mathfrak{n}\, dS . \tag{49.b}$$

We can decompose the tensor ${}^2\mathfrak{N}$ into a symmetric and a skew-symmetric one, and then replace the latter with a vector. However, that is unnecessary since we will not need that tensor at all in what follows. Namely, we will only examine the "internal" or "self-moment" part $\mathfrak{M}$, which depends upon the angular acceleration of the electron, i.e., it depends upon only the first derivative of the vector $\mathfrak{u}$ with respect to time (and the translational velocity $\mathfrak{v}$, in addition). According to (49.c), we can calculate the quantities $\mathfrak{g}$ and ${}^2\mathfrak{T}$ for a rotation of the electron with constant angular velocity. However, since the tensor components $T_{ik}$ drop off like the square of the electric and magnetic field strengths in that case, i.e., from (46) and (46.a), they are at least inversely proportional to the fourth power of the distance, and as a result, the components of ${}^2\mathfrak{N}$ will be inversely proportional to the third power, so the surface integral (49.b) will drop out when one displaces the surface $S$ to infinity, and we get the following simple equation:

$$\mathfrak{M} = - \frac{d\mathfrak{I}}{dt} , \tag{50}$$

with

$$\mathfrak{I} = \int \mathfrak{r} \times \mathfrak{g}\, dV . \tag{50.a}$$

The vector $\mathfrak{I}$ is called the *electromagnetic impulse moment* of the electron. It would be defined in the same way for an arbitrary electromagnetic field.

In order to calculate that vector, we remark that $\mathfrak{E}$, and as a result $\mathfrak{g}$, vanishes inside of the electron, while outside of it, we will have:

$$\mathfrak{g} = \frac{1}{4\pi c}\mathfrak{E}\times\mathfrak{H} = \frac{e}{4\pi c}\frac{1}{r^5}\mathfrak{m}\times\mathfrak{r}_0, \tag{51}$$

according to (46) and (46.a). One will then have:

$$\mathfrak{r}\times\mathfrak{g} = \frac{e}{4\pi c}\frac{1}{r^4}[\mathfrak{m}-(\mathfrak{m}\,\mathfrak{r}_0)\,\mathfrak{r}_0],$$

and when one recalls the fact that has been used several times before that the mean value of the vector $(\mathfrak{m}\ \mathfrak{r}_0)\ \mathfrak{r}_0$ over all directions of the unit vector $\mathfrak{r}_0$ is equal to $\frac{1}{3}\mathfrak{m}$:

$$\mathfrak{I} = \frac{e}{4\pi c}\mathfrak{m}\cdot\frac{2}{3}\int_a^\infty\frac{dV}{r^4} = \frac{2e}{3c}\mathfrak{m}\int_a^\infty\frac{dr}{r^2},$$

i.e.:

$$\mathfrak{I} = \frac{2}{3}\frac{e}{c\,a}\mathfrak{m}. \tag{51.a}$$

If one introduces the rest mass $m_0$ of the electron, in place of its radius, according to the formula:

$$m_0 = \frac{2}{3}\frac{e}{c^2 a},$$

which corresponds to the surface charge, then that will give the following relation:

$$\mathfrak{m} = \frac{e}{c\,m_0}\mathfrak{I}. \tag{51.b}$$

An easy calculation will show that for a uniform volume charge, one will have $\mathfrak{I} = \frac{4}{7}\frac{e}{c\,a}\mathfrak{m}$ ([1]), or:

([1]) The magnetic field strength inside of the electron at a distance $r$ from the center is composed of two parts: The part $\mathfrak{H}'$ that is created by the internal ball of radius $r$ is expressed by the formula (46.a), in which the total magnetic moment $\mathfrak{m}$ is replaced with the moment $\mathfrak{m}_r$ of the aforementioned ball. However, from (45.b), that is equal $\frac{1}{5}\frac{e_r r^3}{c}\mathfrak{u}$,

$$\mathfrak{m} = \frac{7}{5}\frac{e}{c\,m_0}\mathfrak{I},$$

since one has $m_0 = \frac{4}{5}\frac{e^2}{c^2 a}$ in that case: When one recalls the formula (45.a), the expression (51.a) for $\mathfrak{I}$ can be written in the form:

$$\mathfrak{I} = \tfrac{1}{3} m_0 a^2 \mathfrak{u}. \tag{52}$$

That corresponds to a "moment of inertia" for the electron of magnitude $\frac{1}{3} m_0 a^2$. By comparison, it should be pointed out that the moment of inertia of a hollow sphere of mass ($m_0$) is twice as large. In order to establish the relation between the ordinary mechanical and the electromagnetic quantities that characterize the rotational motion, we would like to now calculate the magnetic energy of the electron. The part of that energy that is localized externally to it is given by the integral:

$$T' = \frac{1}{8\pi}\int_{r=a}^{\infty}\frac{1}{r^6}\{9(\mathfrak{m}\,\mathfrak{r}_0)^2 r_0^2 - 6(\mathfrak{m}\,\mathfrak{r}_0)^2 + m^2\}dV,$$

or, since $r_0^2 = 1$ and the mean value of $(\mathfrak{m}\ \mathfrak{r}_0)^2$ is equal to $\frac{1}{3}\mathfrak{m}^2$:

$$T' = \frac{1}{8\pi}\cdot 2\mathfrak{m}^2\int_a^r \frac{4\pi r^2\,dr}{r^6} = \frac{\mathfrak{m}^2}{3a^3}.$$

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in which $e_r = e\frac{r^3}{a^3}$ means the charge of that ball. The second part, which originates in the outer shell of the ball, will be represented by the integral:

$$\mathfrak{H}'' = \int_r^a \frac{2\mathfrak{u}}{3c\,r^3}\,r^3\cdot 4\pi\rho\,r^2\,dr = \frac{2\mathfrak{u}}{3c}\,4\pi\rho\int_r^a r\,dr = \frac{e\mathfrak{u}}{a^3}(a^2 - r^2),$$

from (45) and (46.b), such that the sum $\mathfrak{H}' + \mathfrak{H}''$ will have the following value:

$$\mathfrak{H} = \frac{e}{5c\,a^3}\{3(\mathfrak{r}\,\mathfrak{u})\mathfrak{r} - \mathfrak{u}\,r^2 + 5(a^2 - r^2)\mathfrak{u}\} = \frac{1}{a^5}\{3(\mathfrak{r}\,\mathfrak{m})\mathfrak{r} - \mathfrak{u}\,r^2 + (5a^2 - 6r^2)\mathfrak{m}\}.$$

As far as the electric field is concerned, for $r < a$, it will be determined by the formula $\mathfrak{E} = \frac{e}{a^3}\mathfrak{r}$. One can easily calculate from this that the part of the impulse moment that originates in the internal electromagnetic field of the electron is equal to $-\frac{2}{21}\frac{e}{ca}\mathfrak{m}$. The other (external) part is obviously expressed by (52.a). In total, we will then get:

$$\mathfrak{I} = \left(\frac{2}{3} - \frac{2}{21}\right)\frac{e}{ca}\mathfrak{m} = \frac{4}{7}\frac{e}{ca}\mathfrak{m}.$$

The magnetic field inside of the electron contributes an amount equal to:

$$T'' = \frac{1}{8\pi}\left(\frac{2\mathfrak{m}}{a^3}\right)^2 \cdot \frac{4\pi}{3} a^3 = \frac{2}{3}\frac{\mathfrak{m}^2}{a^3}$$

to its energy. As a result, it will follow that:

$$T' + T'' = T = \frac{\mathfrak{m}^2}{a^3}, \tag{52.a}$$

or from (45.a):

$$T = \frac{1}{9}\frac{e^2 a^2}{c^2 a}\mathfrak{u}^2 = \tfrac{1}{2}\left(\tfrac{1}{3} m_0 a^2\right)\mathfrak{u}^2. \tag{52.b}$$

We then get the usual expression for the kinetic energy of a rotating body with the moment of inertia $\frac{1}{3} m_0 a^2$.

According to the *Lorentz* principle of motion, the rotational motion of the electron under the action of an external torque with a moment of $\mathfrak{M}^a$ will be determined from the equation $\mathfrak{M} + \mathfrak{M}^a = 0$, i.e., according to (50), from:

$$\frac{d\mathfrak{I}}{dt} = \mathfrak{M}^a, \tag{53}$$

in complete agreement with ordinary mechanics. We would like to write that equation:

$$\frac{1}{\kappa}\frac{d\mathfrak{m}}{dt} = \mathfrak{M}^a \tag{53.a}$$

and

$$I\frac{d\mathfrak{u}}{dt} = \mathfrak{M}^a, \tag{53.b}$$

with the abbreviations:

$$\kappa = \frac{e}{c\, m_0}, \tag{54}$$

$$I = \tfrac{1}{3} m_0 a^2. \tag{54.a}$$

The coefficient $\kappa$ represent the ratio of the magnetic moment of the electron to its impulse moment, and $I$ means the moment of inertia.

If the moment $\mathfrak{M}^a$ depends upon an external magnetic field $\mathfrak{H}^a$ then we will get from (47) that:

$$\frac{d\mathfrak{m}}{dt} = \kappa\, \mathfrak{m} \times \mathfrak{H}^a. \tag{55}$$

Inner multiplication of the two sides of that equation by $\mathfrak{m}$ will yield:

$$\frac{d}{dt}\mathfrak{m}^2 = 0 ,$$

i.e., $|\mathfrak{m}|$ = const.

It follows in the same way that for a time-constant external field, one will have:

$$\mathfrak{H}^a \frac{d}{dt}\mathfrak{m} = \frac{d}{dt}(\mathfrak{m}\mathfrak{H}^a) = 0 ,$$

i.e., the angle between the vector $\mathfrak{m}$ and $\mathfrak{H}^a$ also remains constant.

One can easily show that the motion of the electron in this case is composed of the "unperturbed" rotation with an angular velocity $\mathfrak{u}$ and a uniform rotation or *precession* of the vector u around the field direction $\mathfrak{H}^a$ with the angular velocity:

$$\mathfrak{o} = -\kappa\mathfrak{H}^a. \tag{55.a}$$

In fact, the time derivative of the vector $\mathfrak{u}$ must therefore be equal to the linear velocity of a point with the radius vector $\mathfrak{u}$, i.e., $d\mathfrak{u} / dt = \mathfrak{o} \times \mathfrak{u}$. We likewise have $d\mathfrak{m} / dt = \mathfrak{o} \times \mathfrak{m}$ . Due to (55.a), that equation agrees with (55).

## § 8. – Combining the rotational and translational motion.

We would now like to examine the case in which the rotating electron moves with a simultaneous *small translational velocity* $\mathfrak{v}$ ($v \ll c$). In the presence of an external electric field $\mathfrak{E}^a$, it must suffer an additional external torque (and force). We would initially like to determine that based upon the assumption that the rotating electron is *completely equivalent* to a magnetic dipole relative to its magnetic effects. (Below, in Chap. IX, **§ 2**, we will see that this assumption is, in reality, unacceptable. The formulas that follow from it are also false, accordingly.)

According to (12.a), Chap. V, a magnetic pole $\mu$ that moves with a velocity in an external field $\mathfrak{H}^a$, $\mathfrak{E}^a$ will feel a force of $\mu(\mathfrak{H}^a - \frac{1}{c}\mathfrak{v} \times \mathfrak{E}^a)$ . As a result, that will give two equal and opposite forces for the fictitious dipole in question that combine to give a resultant moment of $\mathfrak{l} \times [\mu(\mathfrak{H}^a - \frac{1}{c}\mathfrak{v} \times \mathfrak{E}^a)]$ . The $\mathfrak{l}$ in that means the length of the dipole (as measured from the negative pole $-\mu$ to the positive one $+\mu$). The product $\mu\mathfrak{l}$ is nothing but the magnetic moment of the electron $\mathfrak{m}$. Along with the magnetic torque (47), we will also get an electric one:

$$\mathfrak{H}^a = - \mathfrak{m} \times (\frac{1}{c} \mathfrak{v} \times \mathfrak{E}^a) \, . \tag{57}$$

In a strong inhomogeneous field, one must also consider the additional external force:

$$\mathfrak{J}^a = (\mathfrak{m} \text{ grad}) \, (\mathfrak{E}^a \times \frac{1}{c} \mathfrak{v}) = - \frac{1}{c} \mathfrak{v} \times [(\mathfrak{m} \text{ grad}) \, \mathfrak{E}^a] \, . \tag{57.a}$$

However, additional *internal* forces and torques will appear in the case in question that originate in the electromagnetic field that is produced by the combination of rotation and translation.

We initially consider the magnetic field $\mathfrak{H}^v$ that corresponds to pure translation. For an electron with a constant velocity $\mathfrak{v}$, that field is coupled with the basic electric field of the electron $\mathfrak{E}$ by the general relation that has been used many times before:

$$\mathfrak{H}^v = \frac{1}{c} \mathfrak{v} \times \mathfrak{E} \, . \tag{58}$$

That field vanishes inside of the electron, just like $\mathfrak{E}$, and outside of the electron, according to (46), it will reduce to the magnetic field of a charge $e$ that is concentrated at the center of the electron.

However, that comes down to an additional *electric* field $\mathfrak{E}^m$ that originates in the combination of the two types of motions. That field can be attributed to the fact that the electric current that is defined by the "rotation" of the electron will *lose its stationary character* under translational motion. In fact, the magnetic field strength $\mathfrak{H}$ that is produced by the rotation at any fixed reference point $P$ must then change in time. Obviously, the same thing is true for the associated vector potential $\mathfrak{A}$, which is coupled with $\mathfrak{H}$ by the formula $\mathfrak{H} = \text{rot } \mathfrak{A}$. However, a time-varying vector potential will correspond to an additional electric field:

$$\mathfrak{E}^m = - \frac{1}{c} \frac{\partial \mathfrak{A}}{\partial t} .$$

Since the field strengths and the potentials will remain constant at a comoving reference point, i.e., one that is fixed relative to the electron (when one ignores its rotation), just as in the case of the simple translation [cf., (16.a), Chap. VI], we will have the relation:

$$\frac{d\mathfrak{A}}{dt} = \frac{1}{c} \frac{\partial \mathfrak{A}}{\partial t} + (\mathfrak{v} \text{ grad}) \, \mathfrak{A} = 0 \, ,$$

i.e.:

$$- \frac{1}{c} \frac{\partial \mathfrak{A}}{\partial t} = + \left( \frac{1}{c} \mathfrak{v} \, \text{grad} \right) \mathfrak{A} \, ,$$

or from the identity (27) in the Introduction:

$$-\frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t} = +\ \mathrm{grad}\left(\frac{1}{c}\mathfrak{v}\mathfrak{A}\right) - \frac{1}{c}\mathfrak{v}\times\mathfrak{H}\ .$$

The additional electric field $\mathfrak{E}^m$ is then composed of two parts:

$$\mathfrak{E}' = -\frac{1}{c}\mathfrak{v}\times\mathfrak{H} \tag{58.a}$$

and

$$\mathfrak{E}'' = \mathrm{grad}\,(\frac{1}{c}\,\mathfrak{v}\ \mathfrak{A})\ . \tag{58.b}$$

The formula (58.a) is entirely analogous to (58). It corresponds completely to the "magnetic" differential equations (11.a), (11.b), in Chap. V and is obtained from them in the same way as (58) when one allows the existence of magnetic poles. [The opposite signs in (58) and (58.a) correspond to the opposite signs of $\mathfrak{i}$ and $\mathfrak{i}^*$ in (8.a) and (11.b), Chap. V.] Insofar as the rotating electron can be replaced with a magnetic dipole in regard to its magnetic effects, the translational motion of that dipole must produce the additional field (58.a) in that same way that the translation of the electron charge produced the additional field (58).

However, that symmetric (or "skew-symmetric") relationship between the electric and magnetic fields will be destroyed by the appearance of the second component of $\mathfrak{E}^m$, i.e., the electric field $\mathfrak{E}''$. The latter can be treated as an ordinary electrostatic field that is due to the scalar potential:

$$\varphi'' = -\frac{1}{c}\,\mathfrak{v}\ \mathfrak{A}\ . \tag{59}$$

In the case in question, the vector potential $\mathfrak{A}$ is determined by the formula:

$$\mathfrak{A} = \frac{1}{r^2}\,\mathfrak{m}\times\mathfrak{r}_0 \tag{59.a}$$

outside of the electron ($r > a$) [cf., (27), Chap. III]. By contrast, we will have:

$$\mathfrak{A} = \frac{1}{a^3}\,\mathfrak{m}\times\mathfrak{r} \tag{59.b}$$

inside of it [cf., **§ 7**, Chap. IV]. Thus, since $\mathfrak{v}(\mathfrak{m}\times\mathfrak{r}_0) = \mathfrak{r}_0\,(\mathfrak{v}\times\mathfrak{m})$, we will have:

$$\varphi'' = \frac{1}{r^2}\,\mathfrak{p}\mathfrak{r}_0 \,. \tag{60}$$

for $r > a$, with:

$$\mathfrak{p} = -\frac{1}{c}\,\mathfrak{v} \times \mathfrak{m} \,. \tag{60.a}$$

It will follow that $\varphi''$ (for $r > a$) represents the scalar potential of a dipole with an electric moment $\mathfrak{p}$ that is found at the center of the electron. We get:

$$\varphi'' = \frac{1}{a^3}\,\mathfrak{p}\mathfrak{r} \,. \tag{60.b}$$

for the electron interior in the same way. In Chap. IV, **§ 7**, we saw that the potentials (60) and (60.b) outside (inside, resp.) of a ball with radius $a$ can be generated by a surface charge that vanishes in total and is distributed on the outer surface with a density of:

$$\eta = p \cos\theta \qquad\qquad (\theta \sphericalangle \mathfrak{p}, \mathfrak{r}). \tag{60.c}$$

Later on (Chap. X), we will show that based upon the theory of relativity, in reality, the translational motion of the rotating electron must be accompanied by a small change in the charge distribution on its outer surface (or its interior) that is equivalent to the appearance of a charge distribution that is precisely equal and opposite to (60.c). The field $\mathfrak{E}''$ will be cancelled by that additional charge distribution, or *polarization*, of the electron, and the additional field $\mathfrak{E}^m$ will reduce to its first component $\mathfrak{E}'$.

The complete value of the electromagnetic momentum (per unit volume):

$$\mathfrak{g} = \frac{1}{4\pi c}(\mathfrak{E} + \mathfrak{E}' + \mathfrak{E}'') \times (\mathfrak{H} + \mathfrak{H}^v)$$

is composed of the previously-calculated vector $\mathfrak{g}^{(0)} = \frac{1}{4\pi c}\ \mathfrak{E} \times \mathfrak{H}$, the additional terms of first order in $v / c$:

$$\mathfrak{g}^v = \frac{1}{4\pi c}\ \mathfrak{E} \times \mathfrak{H}^v, \quad \mathfrak{g}' = \frac{1}{4\pi c}\ \mathfrak{E} \times \mathfrak{H}', \quad \mathfrak{g}'' = \frac{1}{4\pi c}\ \mathfrak{E}'' \times \mathfrak{H}\,,$$

and an additional term of second order $\frac{1}{4\pi c}(\mathfrak{E}' + \mathfrak{E}'') \times \mathfrak{H}^v$, which we can neglect.

The term $\mathfrak{g}^v$ corresponds to the motion of pure translation without rotation and was considered already in the previous section in the calculation of the inertial force. We would now like to see what sort of contribution to that force, and as a result to the *electron mass*, is due to $\mathfrak{g}'$ and $\mathfrak{g}''$.

(The term $\mathfrak{g}^{(0)}$, which corresponds to the case of $\mathfrak{v} = 0$, still remains irrelevant then.) From (58.a), we will have:

$$\mathfrak{g}' = \frac{1}{4\pi c}\mathfrak{H} \times (\frac{1}{c}\mathfrak{v} \times \mathfrak{H}) = \frac{1}{4\pi c^2}\left\{\mathfrak{v}H^2 - \mathfrak{H}(\mathfrak{v}\mathfrak{H})\right\} .$$

The first term in the curly brackets will produce the contribution $(2T/c^2)\mathfrak{v}$ when integrated over all of space, where $T = \frac{1}{8\pi}\int H^2 dV$ is the quantity (52.a) that was calculated before (viz., the magnetic energy for $v = 0$). According to (46.a), for $r > a$, we will get the following expression for the second term:

$$-\frac{1}{4\pi c}\frac{1}{r^6}\{9\,(\mathfrak{m}\,\mathfrak{r}_0)^2\,(\mathfrak{v}\,\mathfrak{r}_0)\,\mathfrak{r}_0 - 3\,(\mathfrak{m}\,\mathfrak{v})\,(\mathfrak{m}\,\mathfrak{r}_0)\,\mathfrak{r}_0 - 3\,(\mathfrak{m}\,\mathfrak{r}_0)\,(\mathfrak{v}\,\mathfrak{r}_0)\,\mathfrak{m} + (\mathfrak{m}\,\mathfrak{v})\,\mathfrak{m}\} .$$

In order to calculate the mean value of the quantities that appear in it over all directions of the unit vector $\mathfrak{r}_0$, we next make note of the following formulas, which are true for the rectangular components of $\mathfrak{r}_0$, i.e., for its direction cosines $r_1$, $r_2$, $r_3$ :

$$r_1^2 + r_2^2 + r_3^2 = 1 , \qquad \overline{r_\alpha^2} = \tfrac{1}{3} , \qquad \overline{r_\alpha\, r_\beta} = 0 \qquad \text{(for } \alpha \neq \beta\text{).}$$

Moreover:

$$\overline{\left(\sum_\alpha r_\alpha^2\right)^2} = 1 = 3\overline{r_\alpha^4} + 6\overline{r_\alpha^2\, r_\beta^2}$$

and

$$\overline{r_\alpha\, r_\beta\, r_\gamma\, r_\delta} = 0 ,$$

when the four indices $\alpha, \beta, \gamma, \delta$ are not pair-wise equal. We then get:

$$\overline{r_\alpha^4} = \overline{\cos^4\theta} = \frac{1}{4\pi}\int_0^\pi \cos^4\theta \cdot 2\pi \sin\theta\, d\theta = \frac{1}{2}\int_{-1}^{+1} r_\alpha^4\, dr_\alpha = \frac{1}{5}$$

for the mean value of $\overline{r_\alpha^4}$, and as a result, $\overline{r_\alpha^2\, r_\beta^2} = 1/15$.

Now, one has:

$$\overline{(\mathfrak{m}\mathfrak{r}_0)(\mathfrak{v}\mathfrak{r}_0)} = \sum_\alpha\sum_\beta m_\alpha v_\beta\, \overline{r_\alpha\, r_\beta} = \tfrac{1}{3}\sum_\alpha m_\alpha v_\alpha = \tfrac{1}{3}(\mathfrak{m}\,\mathfrak{v}) ,$$

$$\overline{(\mathfrak{m}\mathfrak{r}_0)r_\alpha} = \sum_\beta m_\beta\, \overline{r_\alpha\, r_\beta} = \tfrac{1}{3}m_\alpha , \qquad \text{i.e.,} \qquad \overline{(\mathfrak{m}\mathfrak{r}_0)\mathfrak{r}_0} = \tfrac{1}{3}\mathfrak{m} ,$$

$$\overline{(\mathfrak{m}\mathfrak{r}_0)^2(\mathfrak{v}\mathfrak{r}_0)r_\alpha} = \sum_\beta\sum_\gamma\sum_\delta m_\beta\, m_\gamma\, v_\delta\, \overline{r_\alpha\, r_\beta\, r_\gamma\, r_\delta}$$

$$= m_\alpha^2 v_\alpha \overline{r_\alpha^2} + \sum_{\beta\neq\alpha} m_\beta^2 v_\alpha \overline{r_\alpha^2 r_\beta^2} + 2\sum_{\gamma\neq\alpha} m_\alpha m_\gamma v_\gamma \overline{r_\alpha^2 r_\gamma^2}$$
$$= \tfrac{1}{5} m_\alpha^2 v_\alpha + \tfrac{1}{15}(\mathfrak{m}^2 - m_\alpha^2) v_\alpha + \tfrac{2}{15}\{(\mathfrak{m}\mathfrak{v}) - m_\alpha v_\alpha\} m_\alpha$$
$$= \tfrac{1}{15}\{\mathfrak{m}^2 v_\alpha + 2(\mathfrak{m}\mathfrak{v})\, m_\alpha\},$$

i.e.:

$$\overline{(\mathfrak{m}\mathfrak{r}_0)^2 (\mathfrak{v}\mathfrak{r}_0)\mathfrak{r}_0} = \tfrac{1}{15}\{\mathfrak{m}^2\mathfrak{v} + 2(\mathfrak{m}\mathfrak{v})\mathfrak{m}\}\ .$$

Upon introducing that mean value into the expression above, multiplying by $4\pi r^2 dr$, and integrating from $a$ to $\infty$, after an easy calculation, one will get:

$$-\frac{1}{15c^2 a^3}\{3\mathfrak{m}^2\mathfrak{v} + (\mathfrak{m}\mathfrak{v})\mathfrak{m}\}.$$

As a result, one will have:

$$\int_{r>a} \mathfrak{g}'\, dV = \frac{2}{3c^2}\frac{\mathfrak{m}^2}{a^3}\mathfrak{v} - \frac{1}{15c^2 a^3}\{3\mathfrak{m}^2\mathfrak{v} + (\mathfrak{m}\mathfrak{v})\mathfrak{m}\}$$

externally ($r > a$). For the interior of the electron ($r < a$), (46.b) will imply that:

$$\frac{1}{4\pi c^2}(\mathfrak{v}\ \mathfrak{H})\,\mathfrak{H} = \frac{4(\mathfrak{m}\mathfrak{v})}{4\pi c^2 a^6}\mathfrak{m},$$

such that the corresponding contribution to the total momentum $\mathfrak{G}' = \int \mathfrak{g}'\, dV$ will be given by:

$$\frac{4\mathfrak{m}^2}{3c^2 a^3}\mathfrak{v} - \frac{4}{3c^2 a^3}(\mathfrak{m}\mathfrak{v})\mathfrak{m}.$$

Ultimately, one will have:

$$\mathfrak{G}' = \frac{9\mathfrak{m}^2}{5c^2 a^3}\mathfrak{v} - \frac{7(\mathfrak{m}\mathfrak{v})}{5c^2 a^3}\mathfrak{m}. \tag{61}$$

We then see that the additional momentum in question is indeed proportional to the velocity, *but not parallel to it*, in general. The first component of $\mathfrak{G}'$, which is parallel to the vector $\mathfrak{v}$, can be treated as an ordinary momentum that is simply added to the momentum $\mathfrak{G}^0 = m_0\,\mathfrak{v}$ that originates in the charge of the electron. In that way, we will get an additional "magnetic" mass:

$$m' = \frac{9\mathfrak{m}^2}{5c^2 a^3} = \frac{9}{5}\frac{T^2}{c^2}, \tag{61.a}$$

which does not need to be small in comparison to the "electric" mass:

$$m_0 = \frac{2}{3}\frac{e^2}{c^2 a} = \frac{4}{3}\frac{U}{c^2}$$

(see below, **§ 8**).

It should be noted that the second component of $\mathfrak{G}'$, which is parallel to the vector $\mathfrak{m}$, will vanish only when the magnetic moment of the electron is perpendicular to its translational velocity.

The momentum $\mathfrak{G}'' = \int \mathfrak{g}''\, dV$ is quite simple to determine. Indeed, from (60), we have:

$$\mathfrak{G}'' = \frac{1}{r^3}\{3\,(\mathfrak{p}\ \mathfrak{r}_0)\,\mathfrak{r}_0 - \mathfrak{p}\}$$

for $r > a$), and a result:

$$\mathfrak{g}'' = \frac{1}{4\pi c}\frac{1}{r^6}[3\,(\mathfrak{p}\ \mathfrak{r}_0)\,\mathfrak{r}_0 - \mathfrak{p}] \times [3\,(\mathfrak{m}\ \mathfrak{r}_0)\,\mathfrak{r}_0 - \mathfrak{m}]$$

$$= -\frac{1}{4\pi c}\frac{1}{r^6}\{3\,(\mathfrak{p}\ \mathfrak{r}_0)\,(\mathfrak{r}_0 \times \mathfrak{m}) + 3\,(\mathfrak{m}\ \mathfrak{r}_0)\,(\mathfrak{p} \times \mathfrak{r}_0) - (\mathfrak{p} \times \mathfrak{m})]\,.$$

In order to calculate the mean value of the bracketed expression, we (scalar) multiply it by and arbitrary fixed vector $\mathfrak{l}$ and write the product in the form:

$$3\,(\mathfrak{p}\ \mathfrak{r}_0)\,[(\mathfrak{m} \times \mathfrak{l})\,\mathfrak{r}_0] + 3\,(\mathfrak{m}\ \mathfrak{r}_0)\,[(\mathfrak{l} \times \mathfrak{p})\,\mathfrak{r}_0] - (\mathfrak{p} \times \mathfrak{m})\,\mathfrak{l}\,.$$

Its mean value is given by the formula $\overline{(\mathfrak{m}\,\mathfrak{r}_0)(\mathfrak{v}\,\mathfrak{r}_0)} = \frac{1}{3}\,\mathfrak{m}\ \mathfrak{v}$ (see above):

$$\mathfrak{p}\,(\mathfrak{m} \times \mathfrak{l}) + \mathfrak{m}\,(\mathfrak{l} \times \mathfrak{p}) - \mathfrak{l}\,(\mathfrak{p} \times \mathfrak{m}) = \mathfrak{l}\,(\mathfrak{p} \times \mathfrak{m})\,.$$

One will then have:

$$\overline{\mathfrak{g}''} = -\frac{1}{4\pi c}\frac{1}{r^6}\ \mathfrak{p} \times \mathfrak{m}\,,$$

and for all of the exterior space:

$$\int \mathfrak{g}''\, dV = \int \overline{\mathfrak{g}''} \cdot 4\pi r^2 dV = \frac{1}{3c\,a^3}\ \mathfrak{m} \times \mathfrak{p}\,.$$

According to (60.b) and (46.b), for the interior of the electron ($r < a$), we have:

$$\mathfrak{E}'' = -\frac{1}{a^3}\,\mathfrak{p}\,, \qquad \mathfrak{g}'' = \frac{1}{4\pi c\,a^6}\,\mathfrak{m} \times \mathfrak{p}\,, \qquad \int \mathfrak{g}''\, dV = \frac{1}{3c\,a^3}\ \mathfrak{m} \times \mathfrak{p}\,,$$

so in total:

$$\mathfrak{G}'' = \frac{2}{3c\,a^3}\,\mathfrak{m}\times\mathfrak{p}\,,$$

or from (60.a):

$$\mathfrak{G}'' = -\frac{2\,\mathfrak{m}^2}{3c^2\,a^3}\,\mathfrak{v} + \frac{2}{3c^2\,a^3}\,(\mathfrak{m}\,\mathfrak{v})\,\mathfrak{m}\,. \tag{62}$$

Upon adding (61) and (62), we will get:

$$\mathfrak{G}^m = \frac{17}{15}\frac{\mathfrak{m}^2}{c^2\,a^3}\,\mathfrak{v} - \frac{11}{15}\frac{(\mathfrak{m}\,\mathfrak{v})}{c^2\,a^3}\,\mathfrak{m}\,. \tag{62.a}$$

However, $\mathfrak{G}''$ will, in fact be compensated precisely by the momentum that originates in the "polarization" of the electron, such that the additional "magnetic" momentum and "magnetic" mass will be expressed by the formulas (61) and (61.a).

The remarkable fact that the momentum of a rotating – or "magnetic" – electron is not parallel to the translational velocity corresponds to self-torque $\mathfrak{M}$ that is generated by that velocity and strives to align the magnetic axis perpendicular to the direction of motion. As a result of that alignment, the momentum will be in the direction as the velocity, and the vector $\mathfrak{M}$ will vanish.

In fact, from the general formula (50), one will have:

$$\mathfrak{M} = -\frac{d\mathfrak{I}}{dt}\,.$$

It should be recalled that the radius vector $\mathfrak{r}$ that enters into the expression (50.a) for $\mathfrak{I}$ must refer to a *fixed* point. If one would then like to refer the differentiation with respect to time to the comoving reference point (volume element) – in which $\mathfrak{g}$ is regarded as a constant quantity – then one must set $d\,\mathfrak{r}\,/\,dt = \mathfrak{v}$. One will then have:

$$\mathfrak{M} = -\int \mathfrak{v}\times\mathfrak{g}\,dV = -\,\mathfrak{v}\times\int \mathfrak{g}\,dV\,,$$

i.e.:

$$\mathfrak{M} = \mathfrak{G}\times\mathfrak{v}\,. \tag{63}$$

That formula shows that $\mathfrak{M}$ will be non-zero only when $\mathfrak{G}$ has a component that is not parallel to $\mathfrak{v}$. If one substitutes $\mathfrak{G} = \mathfrak{G}''$ in it then, from (61), that will give:

$$\mathfrak{M}' = \frac{7}{5}\frac{(\mathfrak{m}\,\mathfrak{v})}{c^2\,a^3}\,(\mathfrak{v}\times\mathfrak{m})\,. \tag{63.a}$$

If the expression (62.a) were correct then the complete value of $\mathfrak{M}$ would be equal to:

$$\mathfrak{M} = \frac{11}{15}\frac{(\mathfrak{m}\mathfrak{v})}{c^2 a^3}(\mathfrak{v}\times\mathfrak{m}). \tag{63.b}$$

## § 9. – The magneton.

The most recent advances in the field of the optical and magnetic properties of the atom have removed all doubt from the oft-expressed suspicion ([1]) that electrons rotate around their own axis (or, more precisely, *they have a magnetic moment*), or at least for negative electrons ([2]). The theory of spherical, surface-charged electrons that was proposed implied the exact value of the ratio $\kappa = e/m_0 c$ of the magnetic moment to the angular moment of the negative electrons in a remarkable way. It should be noted that this value is twice as large as the one that corresponds to a translational motion of the electron around a fixed point. In fact, we have seen in Chap. III, **§ 9** that the magnetic moment of a linear electric current relative to any point is composed of the contributions:

$$\mathfrak{m} = \frac{e}{2c}\mathfrak{r}\times\mathfrak{v} \tag{64}$$

that originate in the individual electrons. On the other hand, the corresponding mechanical moment is expressed by the formula:

$$\kappa' = \frac{e}{2c\,m_0}. \tag{64.b}$$

If the electron moves around a fixed point $O$ to which it is attracted by a central electric field, as is found approximately in, e.g., atoms, then the magnitude and direction of its mechanical angular momentum must remain constant. That will follow immediately from the equations of motion $\frac{d}{dt}(\mathfrak{m}\mathfrak{v}) = \mathfrak{F}^a$ upon outer multiplying it by the radius vector of the electron $\mathfrak{r}$. That is because the vectors $\mathfrak{F}^a$ and $\mathfrak{r}$ are parallel to each other, so one will have:

$$\mathfrak{r}\times\frac{d}{dt}(\mathfrak{m}\mathfrak{v}) = \mathfrak{r}\times\mathfrak{F}^a = 0,$$

and as a result, since:

$$\frac{d}{dt}(\mathfrak{r}\times\mathfrak{m}\mathfrak{v}) = \mathfrak{r}\times\frac{d}{dt}(\mathfrak{m}\mathfrak{v}) + \mathfrak{v}\times m\,\mathfrak{v} = \mathfrak{r}\times\frac{d}{dt}(\mathfrak{m}\mathfrak{v}),$$

([1]) Especially by *Abraham* and *A. H. Compton*.

([2]) One has *G. Uhlenbeck* and *S. Goudsmit* (1926) to thank for being the first to observe the fact that the magnetic electron in an electric field must feel an additional force [see below, formula (56.a)] that is responsible for its previously-unexplained optical and magnetic properties.

one will have:

$$\mathfrak{r} \times m\,\mathfrak{v} = \mathfrak{J} = \text{const.}$$

It would emerge from the derivation that was just given that this theorem will also be true when one considers the dependency of the mass on the velocity ([1]). In what follows, we would like to restrict ourselves to small velocities and set $m = m_0 = \text{const}$. It will then follow from the theorem above that the magnetic moment of electron, or expressed more precisely, the *electronic orbit* (around the center of attraction $O$), must remain constant.

At distances that are large compared to the dimensions of the orbit, a magnetic field will be generated by the orbital motion of the electron whose mean over a time interval that includes several orbital periods (but can be, at the same time, very small in comparison to the ordinary unit of time) will be identical to the field of a stationary elementary current, or an elementary magnet (magnetic dipole) at the point $O$. That is why one cares to refer to an electron that orbits in an atom as a *magneton* ([2]). However, it would more correct to speak of a *double magneton* since a rotation of the electron around its own axis will appear along with the orbital motion (just like what happens with the motion of the Earth around the Sun!), that will have a magnetic moment $\mathfrak{m}$ as a consequence. The electron equivalent in its magnetic effects to a system of *two* elementary magnets with the moments $\mathfrak{m}$ and $\mathfrak{m}'$ (obviously, one can think of the latter as being combined into a resultant moment $\mathfrak{m} + \mathfrak{m}'$).

That equivalence refers to not only the actions that are exerted upon the electron, but also the actions that the electron itself experiences in an external magnetic field $\mathfrak{H}^a$. That question was resolved above (**§ 6**) for the case of the rotational motion. As far as the orbital motion is concerned, one can also characterize the change in the vector $\mathfrak{m}'$ that is produced by the external field by the equation:

$$\overline{\frac{d\mathfrak{m}'}{dt}} = \kappa'\,\mathfrak{m}' \times \mathfrak{H}^a\,, \tag{65}$$

which is entirely analogous to (55). The overbar over the time derivative of $\mathfrak{m}'$ in that means that one is not dealing with the exact "instantaneous" value of that derivative, but with a *mean value* in the same sense as in the determination of the magnetic field that is produced by the orbital motion of the electron. The *precise* formula for the change in velocity of the angular momentum $\mathfrak{J}' = (1/\kappa')\,\mathfrak{m}'$, obviously reads:

$$\frac{d\mathfrak{J}'}{dt} = \mathfrak{M}^a,$$

in which:

$$\mathfrak{M}^a = \mathfrak{r}' \times \frac{e}{c}(\mathfrak{v} \times \mathfrak{H}^a) \tag{65.a}$$

([1]) It then represents the generalization of the usual "law of areas" in classical mechanics.
([2]) That terminology was introduced by *P. Weiss*.

is the moment of the magnetic force that acts on the electron. That moment must suffer certain fluctuations during each orbit that will mutually cancel in the mean and remain trivial for the mean variation of $\mathfrak{J}'$. The latter will then be determined exclusively by the *mean value* of $\mathfrak{M}^a$ over an orbital period (or several orbits when the motion is not strictly periodic). For sufficiently-weak external fields (i.e., in the first approximation), that mean value $\mathfrak{M}^a$ is identical, in practice, to the one that relates to unperturbed motion (but in the presence of $\mathfrak{H}^a$).

Now, we point out that the mean value of the *time derivative* of any quantity that experiences no systematic (i.e., monotone) change under unperturbed motion, but merely oscillates about a certain mean value, must vanish as long we ignore the perturbation. In particular, we will then have ([1]):

$$\overline{\frac{d}{dt}[\mathfrak{r}\times(\mathfrak{r}\times\mathfrak{H}^a)]} = 0\,,$$

or, since one has $\frac{d}{dt}[\mathfrak{r}\times(\mathfrak{r}\times\mathfrak{H}^a)] = \mathfrak{r}\times(\mathfrak{v}\times\mathfrak{H}^a)+\mathfrak{v}\times(\mathfrak{r}\times\mathfrak{H}^a)$ :

$$\overline{\mathfrak{r}\times(\mathfrak{v}\times\mathfrak{H}^a)}+\overline{\mathfrak{v}\times(\mathfrak{r}\times\mathfrak{H}^a)} = 0\,. \tag{65.b}$$

When one recalls the identity:

$$\mathfrak{r}\times(\mathfrak{v}\times\mathfrak{H}^a)+\mathfrak{H}^a\times(\mathfrak{r}\times\mathfrak{v})+\mathfrak{v}\times(\mathfrak{H}^a\times\mathfrak{r}) = 0 \tag{65.c}$$

[cf., Introduction (5.a)], that will give:

$$\overline{\mathfrak{r}\times(\mathfrak{v}\times\mathfrak{H}^a)} = \tfrac{1}{2}(\mathfrak{r}\times\mathfrak{v})\times\mathfrak{H}^a\,. \tag{65.d}$$

We have omitted the overbar on the right-hand side of that equation since the vector $\mathfrak{r}\times\mathfrak{v}$ will remain constant in the unperturbed motion.

According to (65.a) and (65.d), we will then get:

$$\overline{\mathfrak{M}^a} = \frac{e}{2c}(\mathfrak{r}\times\mathfrak{v})\times\mathfrak{H}^a = \mathfrak{m}'\times\mathfrak{H}^a, \tag{66}$$

and ultimately:

$$\overline{\frac{d\mathfrak{J}'}{dt}} = \mathfrak{m}'\times\mathfrak{H}^a,$$

which agrees with (65).

([1]) In order to avoid any misunderstandings, we would like to emphasize the fact that the following mean values represent a *zeroth-order* approximation (relative to $\mathfrak{H}^a$), while it is the *first* order of approximation that is suggested in (65).

According to that equation, the perturbation of the orbital motion that is caused by the external field reduces to a *precession* of the vector $\mathfrak{m}'$ around the field direction with an angular velocity:

$$\mathfrak{o}' = -\,\kappa'\,\mathfrak{H}^a \tag{66.a}$$

(viz., the so-called *Larmor precession*) ([1]).

That angular velocity is half as big as the angular velocity $\mathfrak{o}$ with which the vector $\mathfrak{m}$, i.e., the magnetic axis of the electron, must precess around the same field direction.

It should be remarked that such an independent precession of the two vectors $\mathfrak{m}$ and $\mathfrak{m}'$ is, in reality, impossible since they are, so to speak, coupled to each other by the action of the central electric field that keeps the electron in the vicinity of the point *O*. If one assumes, e.g., that the electron experiences moment (57) and the force (57.a) during its orbital motion, and one sets $\mathfrak{F}^a = e\,\mathfrak{E}^a = e\,f\cdot\mathfrak{r}$, in which *f* means a scalar function of the distance, so from (57):

$$\mathfrak{M}^a = f\,\frac{e}{c}\,\mathfrak{m}\times(\mathfrak{r}\times\mathfrak{v}) = 2\,f\,\mathfrak{m}\times\mathfrak{m}'. \tag{67}$$

That moment obviously corresponds to a virtual magnetic field of strength:

$$\mathfrak{H}' = 2\,f\,\mathfrak{m}' = \frac{2|\,\mathfrak{E}^a\,|}{r}\,\mathfrak{m}' \tag{67.a}$$

that we can formally attribute to the magneton that accompanies the orbital motion of the electron. Due to the fact that $(\mathfrak{m}\ \mathrm{grad})\,f\,\mathfrak{r} = \mathfrak{m}\,f + \mathfrak{r}\,(\mathfrak{m}\ \mathrm{grad}\,f)$, the additional force (57.a) can be written in the form:

$$-\,e\,f\,\frac{1}{c}\,\mathfrak{v}\times\mathfrak{m} - \left(\frac{1}{c}\,\mathfrak{v}\times\mathfrak{r}\right)(\mathfrak{m}\ \mathrm{grad}\,f) = +\,e\,f\,\mathfrak{p} + 2\,(\mathfrak{m}\ \mathrm{grad}\,f)\cdot\mathfrak{m}',$$

in which $\mathfrak{p}$ means the dipole moment that is defined by (60.a). We get the following expression for the moment of that force that causes the change in the vector $\mathfrak{m}'$:

$$\mathfrak{M}'^a = e\,f\,\mathfrak{r}\times\mathfrak{v} + 2\,(\mathfrak{m}\ \mathrm{grad}\,f)\,(\mathfrak{r}\times\mathfrak{m})\,. \tag{67.a}$$

That expression is quite different from (57). We will see below (Chap. IX, **§ 2**) that in reality, both of them *are false* [just like the fundamental formulas (57) and (47.a)]. The additional force and torque that are required by the translational motion of the rotating electron in an electric field originate, not in the fictitious magnetic dipole, but in an *electric polarization* of the electron, i.e., of the corresponding electric dipole (with a moment of $-\,\mathfrak{p}$) and have an entirely symmetric

([1]) After *J. Larmor*.

interaction between $\mathfrak{m}$ and $\mathfrak{m}'$ as a consequence, such that the resultant angular momentum will remain constant.

Along with the external magnetic and electric torque, the rotating electron must experience a self-torque that is given by (63.a) [(63.b), resp.]. However, that torque is proportional to the square of the ratio $v / c$, and for that reason, it can be neglected for small translational velocities (which we have restricted ourselves to from the outset).

Experimentally, the "internal" magnetic moment of the negative electron amounts to about:

$$| \mathfrak{m} | = 10^{-20}.$$

The total magnetic moment of the atom, which is composed vectorially from the moments $\mathfrak{m}$ and $\mathfrak{m}'$ of the individual electrons, is always a whole-number multiple of the cited value.

If one substitutes it in the formula (45) then that will give the equatorial velocity of a negative electron with $a = 10^{-13}$ cm and $e = 4.77 \times 10^{-10}$ as roughly $10^{13}$ cm / sec. As was mentioned before, that fact does not need to be regarded as an objection to the theory that was proposed.

However, a more fundamental objection will be raised when we estimate the additional mass that originates in the rotation of the electron. According to (61.a), that "magnetic" mass has an order of magnitude of $\frac{\mathfrak{m}^2}{c^2 a^3}$. With $| \mathfrak{m} | \approx 10^{-20}$ and $a \approx 10^{-13}$, it must be equal to around $10^{-21}$. However, it actually amounts to only $10^{-27}$, which is *a million times smaller!* One should recall that the "radius" of the electron was calculated from precisely the latter value under the assumption that the mass of the electron was purely-electrical, i.e., it was determined from the formula $m_0 = \frac{2}{3}\frac{e^2}{c^2 a}$. By contrast, if one would like to treat that mass as mainly magnetic in origin then that would give a value for the electron radius of:

$$a \approx \sqrt[3]{\frac{\mathfrak{m}^2}{c^2 m_0}} \approx 10^{-11}.$$

However, that value is certainly too large. From the known results of *Rutherford* regarding the size of the atomic nucleus, the usual value ($10^{-13}$) is already considered to be too large.

## § 10. – Critique of the theory of extended electrons.

In this chapter, we have treated the electron as a spatially-extended structure with a surface or volume charge and exhibited its equations of motion on the basis of the *Lorentzian* principle, and indeed in agreement with experiments, on the whole.

We must now review that picture critically and ask ourselves whether it cannot be replaced with a different one, and in particular, with the much-simpler picture of the point-like electron.

Let us first make some epistemological remarks. Since electrons are the ultimate indestructible and indivisible elements from which material bodies are composed, it would seem somewhat naïve to once more picture them as small rigid or quasi-rigid bodies. The question of the structure of material bodies is the question of how they are composed of smaller, discrete mutually-inseparable particles. In that way, one will arrive at molecules, atoms, and ultimately electrons. However, since electrons are not further divisible, it would seem senseless to speak of the "elements" of one and the same electron. The *charge* of the electron (just like its mass in the usual conception of classical mechanics) is not a form of matter, but a *property*. However, since that property is invariant, and additive, in addition (under the composition of several electrons), that will give the possibility of treating that property as the *representative* of the corresponding thing (in that spirit, we have frequently referred to a charged particle as merely *a charge*) and localize it in the same spatial region. In that way, one can just as well imagine a point, a line, a surface, or a volume. To the extent that one is dealing with an interaction between *different* electrons, the conception of the electrons as point charges is the simplest and most natural. In that way, the other one will be used as a tool, as we did in, e.g., the derivation of the electromagnetic field of a moving point-charge in Chap. VI, **§ 1** and **2**.

The inducement to treat the electric charge as a continuously-distributed substance with a finite volume density first comes from the possibility of replacing the integral laws that determine the action-at-a-distance between different electrons with differential equations (which imply a fictitious "local action"). However, the decisive basis for introducing spatially-extended electrons lies in the fact that it is only in that way that one will arrive at the general study of energy that was presented in the previous chapter. Since it is therefore entirely irrelevant *from a purely-mathematical* standpoint whether one regards the charge of the electron as a volume charge or a surface charge (see **§ 1**), one cares to consider those two possibilities to be physically equivalent. By contrast, the line and point conceptions of the electron were rejected because they proved to be mathematically untenable in the sense of the aforementioned formulation of the concept of energy and its related concepts. However, it is clear that from a purely-geometric standpoint the electron with a volume charge is no less different from an electron with a surface charge than the latter is from an electron with a line charge.

We have therefore not allowed ourselves to be guided by epistemological principles or physical facts, but by a mathematical formalism that is coupled with the differential form of the fundamental physical laws.

We have extended that formalism by a very essential physical principle, namely, the *Lorentz principle of motion.* The meaning of that principle for the "field theory" of electromagnetic energy and momentum consists of the fact that it will imply the *conservation laws* for those quantities in the sense of equations (21) and (31) or the corresponding differential equations:

$$\frac{\partial \xi}{\partial t} + \operatorname{div} \mathfrak{K} = 0 , \qquad \frac{\partial \mathfrak{g}}{\partial t} + \operatorname{div}^2 \mathfrak{T} = 0 \qquad \left( \mathfrak{g} = \frac{1}{c^2} \mathfrak{K} \right). \tag{68}$$

Those equations are certainly quite convenient for determining the motion of an electron, but not at all necessary. One can solve the same problem by a direct method, namely, when one calculates the force that the electron exerts on itself and sets it equal to minus the external force

that originates in the other electrons. In that way, one must treat the elements of the electron as point charges and define its self-force to be the resultant of their reciprocal actions.

The actual physical sense of introducing the spatially-extended electrons with volume or surface charges in place of point-like ones consists of just the idea that it will bring about the possibility of exhibiting their equations of motion theoretically with the help of *Lorentz's* principle. Energy, momentum, etc., are considered to be only *auxiliary constructions* that will ease the solution of that problem. The *basic concepts* of electrodynamics, like those of classical mechanics, are *force* and *motion*. The basic problems are, *firstly*, to determine the interactions of the different electrons as function of their motion (and configuration), and secondly, to determine the change in that motion (and configuration) as a function of their interactions. We have solved the first problem completely in the previous chapter. We know the electromagnetic field that is generated by a *given motion* of the electron in question and the force that acts on another electron that is found in that field. However, in order to get a closed system of equations that will allow us to follow the history of the system of electrons considered, we must further determine the motion of an electron under the action of *given external forces*. *One* possibility for solving that problem on the basis of rational foundations consists of "blowing up" the electrons to spheres or other spatially-extended structures with surface or volume charges, and introducing *self-forces* that will be compensated by external forces.

However, that is not the *only* possibility, nor is it even the simplest or most reasonable one. In nature, we can observe only "external" forces that act on an electron on the part of the other ones. The "self-force" represents a metaphysical fiction, due to the aforementioned fact that it is epistemologically and physically meaningless to subdivide an electron into further "elements."

Nonetheless, one can always demand that the theory of self-forces should treat not only the motion, but also the equilibrium of an electron; but that is not at all the case. The *Lorentz* principle demands the vanishing of the total force for the *whole* electron, but not for its individual "parts." No equilibrium exists in the interior of the electron. The electrical force of repulsion between its "elements" remains uncompensated, at least by forces of an electric nature. Obviously, one can suspect that those forces are compensated by forces of a different nature, e.g., "elastic" ones. However, that oft-expressed, and partially-developed suspicion means a connecting bridge to electrodynamics as a closed physical theory. I am of the opinion that this problem, just like all problems and complications that are coupled with the subdivision of the electron into elements, should be considered to be only an *apparent* problem of the same type as many other scholastic problems that the philosophers and theologians of the Middle Ages had addressed ([1]).

*Either* there are no individual electrons at all, but only a continuous distribution of current and charge density over all of space, so it would then make no sense to speak of any sort of equations of motion. The role of such equations of motion could be played by the principle of the conservation of the electromagnetic energy and momentum that was expressed by equations (68). However, in that way, they must be valid in all of space, and not just "outside of the electrons."

*Or* the electrons are discrete indivisible material particles, and it would then seem simplest and most-natural to regard them as mere force centers with no extension in space, i.e., as material *points*, as *Leibniz* and *Boskowitsch* said before in regard to the ultimate elements of matter.

([1]) For example, recall the famous problem of the number of angels that can dance on the head of a pin.

In that way, all of the problems and complications that relate to the structure of the electron vanish. However, at the same time, it would seem as though all of the study of energy that was developed in this chapter will also vanish, and therefore the electromagnetic theory of the mass and moment of inertia of the electron, as well, i.e., theory of its translational and rotational motion.

We will see below (Chap. X) that the equations of motion that were found above can be exhibited in connection with the known Ansätze of classical mechanics on the basis of a completely-general formal principle, namely, *Einstein's* relativity principle. However, in that way, it is necessary to define the mass (or more precisely, the rest mass) of an electron and its magnetic moment as *primary* properties that are independent of charge. Accordingly, one must not attribute the energy, momentum, and angular momentum of the electron to its electromagnetic field but consider it to be its own *mechanical* attribute that is not directly coupled with the aforementioned field. The quantities $\frac{1}{8\pi}\int (E^2 + H^2)\, dV$, etc., are physically meaningful for an individual electron only when it can be subdivided into infinitely-smaller elements. In that way, it will represent the *mutual* energy, etc., of those elements relative to each other. However, if one considers the electron to be a largely indivisible thing then that aforementioned volume integral will lose all of its physical sense. In the case of point-like electrons, they would then be infinite.

However, along with the mechanical energy and momentum of the individual electrons, one must also give some attention to their *mutual* energy and momentum. In so doing, those "mutual" quantities cannot be defined directly as functions of the relative configuration and velocity of the electrons that are thought to be in a state of interaction due to the delayed character of electromagnetic action-at-a-distance. Nonetheless, one can probably represent them as integrals over all space of quantities that will be obtained from the total energy density $\xi = \frac{1}{8\pi}(E^2 + H^2)$, etc., when one subtracts the quantities from it that correspond to the proper energy, etc., of the individual electrons. The resultant "mutual" quantities ([1]) must obviously be expressed by products of the elementary electric and magnetic field strengths $\mathfrak{E}^a$, $\mathfrak{H}^a$ that originate in the *different* electrons. For the volume density of the mutual electromagnetic energy, we get, e.g. [cf., (3), **§ 1**]:

$$\xi^* = \frac{1}{8\pi}\sum\sum_{\alpha\neq\beta} (\mathfrak{E}_\alpha \mathfrak{E}_\beta + \mathfrak{H}_\alpha \mathfrak{H}_\beta), \tag{69}$$

and for the density of the mutual electromagnetic momentum:

$$\mathfrak{g}^* = \frac{1}{4\pi c}\sum\sum_{\alpha\neq\beta} \mathfrak{E}_\alpha \times \mathfrak{H}_\beta . \tag{69.a}$$

Accordingly, we must also replace the density of the electromagnetic energy current $\mathfrak{K} = c^2\,\mathfrak{g} = \frac{c}{8\pi}\mathfrak{E} \times \mathfrak{H}$ and the tensor density of the momentum flux [2]$\mathfrak{T}$ with the mutual "remainder" that is

([1]) In what follows, we will denote that remainder by an asterisk (*).

obtained by subtracting the terms that originate in the individual electrons. If one replaces the individual *electromagnetic* energy and momentum of those electrons with the corresponding *mechanical* quantities $W_\alpha$ and $\mathfrak{G}_\alpha$ then the conservation laws for the total energy and total momentum can be written in the form:

$$-\frac{d}{dt}\left(\sum_\alpha W_\alpha + W^*\right) = \oint K_n^*\, dS\,, \tag{70}$$

$$-\frac{d}{dt}\left(\sum_\alpha \mathfrak{G}_\alpha + \mathfrak{G}^*\right) = \oint \mathfrak{T}_n^*\, dS\,, \tag{70.a}$$

in which:

$$W^* = \int \xi^* dV\,, \qquad \mathfrak{G}^* = \int \mathfrak{g}^* dV \tag{70.b}$$

mean the mutual electromagnetic energy and momentum, resp., that are included in the volume $V$. In so doing, the volume $V$ will remain arbitrary. However, it is preferable to choose it to be large enough that the surface $S$ lies completely in the wave zone, and as a result, it will include all of the electrons in question. Nonetheless, the introduction of those quantities can be useful only when one expresses them explicitly in terms of the kinematical elements of the electrons in question, i.e., their configuration, velocities, accelerations, etc., and indeed for the same time-point at which the entire system is considered. One can do that by means of the series development that was cited in Chap. VI, **§ 5**. However, we shall not go into the details of that.

It should be pointed out that equations (70) and (70.a) are incomplete in comparison to the corresponding equations for the complete electromagnetic energy and momentum, and indeed in the sense that terms that would correspond to the radiation of energy and momentum by the individual electrons are missing from then. Obviously, that lacuna must be filled by introducing additional, non-conservative terms into the equations of motion.