"Ueber adjungtirte lineare Differentialausdrücke," J. reine angew. Math. 85 (1878), 185-213.

# **On adjoint linear differential expressions**

(By Herrn G. Frobenius in Zurich)

Translated by D. H. Delphenich

In his investigations into the second variation of simple integrals, **Jacobi** arrived at some theorems on linear differential equations that were derived and further developed by many others (\*), and in particular, by **Hesse** (this journal, Bd. 54, pp. 227). The partially laborious calculations that this proof requires can be averted completely when one defines the differential expression that **Fuchs** (this journal, Bd. 76, pp. 183) cared to call the *adjoint* to another one, not by its formal representation, but by its characteristic property that it makes a certain bilinear differential expression become a complete differential. In that way, one also easily gets the relations between the constants that enter into the reduced form of the second variation that **Clebsch** first gave, which are relations that **Hesse** had exhibited only in a somewhat-complicated form. I will first derive the aforementioned theorems for ordinary differential expressions and then briefly give their application to the conversion of the second variation, and finally extend some of those theorems to partial differential expressions.

#### § 1. – On the composition of differential expressions.

If  $A_0, A_1, ..., A_n$  are well-defined (given) functions, but *u* is an undetermined function of *x* then I will call the expression:

(1) 
$$A_0 u + A_1 D u + A_2 D^2 u + \dots + A_n D^n u$$

a (homogeneous) linear differential expression and denote it by A(u), also more briefly by A. In the latter case, A then means the expression itself, while it is an operation symbol in the former. If one replaces the function u in the expression A(u) by a linear differential expression B(u) then one will again obtain a linear differential expression A(B(u)), which I shall call the *composition* of A and B (in that sequence) and denote it by AB(u), also more briefly by AB (cf., this journal, Bd. 80, pp. 321). Its order is the sum of the orders of A and B.

<sup>(\*)</sup> One finds the older literature in **Hesse** (*loc. cit.*) and the more recent ones in **Mayer**, this journal, Bd. 69, pp. 238. Cf., also, **Horner**, Quarterly Journal, no. 55 (1876, Oct.), pp. 218.

I. The coefficient of the highest derivative of a composed differential expression is the product of the coefficients of the highest derivatives of its parts.

That simple remark implies a series of consequences. A differential expression is called (identically) zero when all of its coefficients are zero. If one composes several differential expressions, none of which are zero, then one will once more obtain a differential expression that does not vanish. That is because from the theorem above, the coefficient of the highest derivative will be non-zero. Therefore, when a composed differential expression vanishes, one of its parts must be zero. If AB = 0 and A is non-zero then B = 0. If ABC = 0 and A and C are both non-zero then B = 0. If AC = BC (or CA = CB) and C is non-zero then A = B. For example, if DA = DB (where D is the derivation symbol) then A = B.

A linear differential expression in the undetermined function u whose coefficients are linear differential expressions in another undetermined function v is called a *bilinear differential expression* and will be denoted by A(u, v). It is called *symmetric* when A(u, v) = A(v, u) and *alternating* when A(u, v) = -A(v, u). The sum of the orders of the derivatives of u and v that are multiplied together in a certain term in A(u, v) will be called the *dimension* of that term.

### § 2. – The definition of adjoint differential expressions.

If A(u) denotes the differential expression (1) then:

(2) 
$$(A_{n}u) - D(A_{n}u) + D^{2}(A_{n}u) - \dots + (-1)^{n}D^{n}(A_{n}u)$$

is called the *adjoint differential expression* to A, and in what follows it will always be denoted by A'(u) or A'. Meanwhile, we will make little use of that definition, but define the adjoint differential expression by one of its characteristic properties that we would like to briefly derive, since it will define the foundation for the following investigations.

We take the starting point to be the easily-verified formula:

(a) 
$$D\sum_{\lambda=0}^{\nu-1} (-1)^{\nu-1-\lambda} D^{\lambda} u \cdot D^{\nu-1-\lambda} v = v D^{\nu} u - u (-1)^{\nu} D^{\nu} v,$$

upon which the method of partial integration is based. The differential expression  $v D^{\nu}u - u (-1)^{\nu} D^{\nu}v$  is then the complete differential of a bilinear differential expression:

$$(\beta) \qquad \qquad P_{\nu}(u, v) = \sum_{\lambda=0}^{\nu-1} (-1)^{\nu-1-\lambda} D^{\lambda} u \cdot D^{\nu-1-\lambda} v,$$

whose terms all have dimension (v - 1) and have the alternating coefficients + 1 and - 1. If one replaces *v* with the product  $A_v v$  in ( $\alpha$ ) then one will get:

(
$$\gamma$$
)  $v(A_{\nu}D^{\nu}u) - u((-1)^{\nu}D^{\nu}(A_{\nu}v)) = DP_{\nu}(u,A_{\nu}v)$ .

If the bilinear differential expression  $P_v(u, A_v v)$ , which has order (v-1) relative to u, as well as v, is arranged in terms of the derivatives of u and v then the highest dimension that occurs will be (v-1), and the terms of that dimension will have alternating coefficients  $+A_v$  and  $-A_v$ .

The differential expression:

(3) 
$$\sum_{\nu=1}^{n} P_{\nu}(u, A_{\nu}v) = \sum_{\nu=1}^{n} \sum_{\lambda=0}^{\nu-1} (-1)^{\nu-1-\lambda} D^{\lambda} u \cdot D^{\nu-1-\lambda}(A_{\nu}v) = \sum_{\lambda,\mu} (-1)^{\mu} D^{\lambda} u \cdot D^{\mu}(A_{\lambda+\mu+1}v)$$

shall be called the *concomitant bilinear differential expression* (\*) to the expression A(u) and it shall be denoted by A(u, v). It has order (n - 1) relative to u and v, and the highest dimension that occurs is likewise *only* (n - 1). The terms of dimension (n - 1) have alternating coefficients +  $A_n$  and  $-A_n$ . The coefficients of A'(u), as well as A(u, v), are linear differential expressions of the coefficients of A(u).

If one sums equation ( $\gamma$ ) over  $\nu$  then one will get the following relation (<sup>\*\*</sup>) between the differential expressions (1), (2), and (3):

(4) 
$$v A(u) - u A'(v) = D A(u, v)$$
.

Conversely, when A(u) and B(u) are two such differential expressions such that v A(u) - u B(v) is the derivative of a bilinear differential expression C(u, v) then B(u) = A'(u) and C(u, v) = A'(u, v). That is because when one subtracts equation (4) from the equation:

$$v A(u) - u B(v) = D C(u, v)$$
,

one will get:

(
$$\delta$$
)  $u(A'(v) - B(v)) = D(C(u, v) - A(u, v)).$ 

If one imagines setting v equal to a certain function then C(u, v) - A(u, v) will be a homogeneous linear differential expression in u. If it were not identically zero, but the highest derivative of u

<sup>(\*)</sup> If the coefficients of *A* are single-valued functions of the complex variables *x* in a certain region then *n* independent integrals  $u_0, u_1, ..., u_{n-1}$  of the differential equation A = 0 will experience a linear substitution with constant coefficients when *x* traverses a closed curve that lies inside of that region. One can then determine *n* independent integrals  $v_0, v_1, ..., v_{n-1}$  of the differential equation A' = 0 such that they experience the transposed substitution along that path, so the quantities  $u_0, u_1, ..., u_{n-1}$  are contragredient (this journal, Bd. 76, pp. 194 and pp. 267). On those grounds, *A'* will be called the associated or adjoint expression to *A*. (That remark implies, with no further discussion, e.g., the theorems that **Jürgens** had derived on the fundamental systems of adjoint differential equations in this journal, Bd. 80, pp. 150.) The expression *A* (*u*, *v*) is then comparable to the algebraic structures that **Aronhold** called "intermediate forms" and **Sylvester** called "concomitants." Therefore, I shall call it the concomitant bilinear differential expression.

<sup>(\*\*)</sup> **Jacobi**, this journal, Bd. 32, pp. 189.

that actually appeared in it were the  $m^{\text{th}}$  then the highest derivative of *u* that would actually appear in its derivative would be the  $(m + 1)^{\text{th}}$ , so from equation ( $\delta$ ), one would have m + 1 = 0, while *m* could still be non-negative.

## § 3. – The reciprocity of adjoint differential expressions.

All of the properties of the adjoint differential expression can be easily read off of the characteristic property (4), by which we can now define it. The theorem of **Lagrange** that the adjoint expression to A' is equal to A follows from it immediately.

Furthermore, if the expression A goes to (the linear differential expression) P, A' goes to Q, and A (u, v) goes to R(u, v) when one introduces a new independent variable x' in place of x then one will have:

$$v P(u) - u Q(v) = \frac{dR(u, v)}{dx'} \frac{dx'}{dx}$$

or when one multiplies that by  $\frac{dx}{dx'}$  and replaces v with  $v\frac{dx'}{dx}$ :

$$v P(u) - u Q\left(v \frac{dx'}{dx}\right) \frac{dx}{dx'} = \frac{d}{dx'} R\left(u, v \frac{dx'}{dx}\right).$$

The adjoint differential expression to P(u) is then  $Q\left(u\frac{dx'}{dx}\right)\frac{dx}{dx'}$ , and the concomitant bilinear

differential expression is  $R\left(u, v \frac{dx'}{dx}\right)$ .

If *B* is a linear differential expression and B' is its adjoint, while B(u, v) is the concomitant bilinear expression then:

$$v B (u) - u B'(v) = D B (u, v)$$

If one replaces v with A'(v) then one will get:

$$A'(v) \cdot B(u) - u B'A'(v) = D B(u, A'(v)) .$$

If one replaces u with B(u) in equation (4) then one will find that:

$$v A B (u) - B(u) \cdot A'(v) = D A (B (u), v).$$

Adding those equations will give:

$$v A B (u) - u B'A'(v) = D [A (B (u), v) + B(u, A'(v))]$$

It then follows from this that:

I. If P = A B then P' = B'A' and:

$$P(u, v) = A(B(u), v) + B(u, A'(v))$$

By repeatedly applying that theorem, one will find the more general result that the adjoint expression to ABC... is equal to ...C'B'A'.

II. If a differential expression is composed of several others then the adjoint expression is composed the adjoints in the opposite order.

Because I will refer to that theorem frequently, I would like to call it the *reciprocity theorem*. (Cf., this journal, Bd. 76, pp. 263 and pp. 277; *ibid.*, Bd. 77, pp. 257; *ibid.*, Bd. 80, pp. 328)

The theorem of **Hesse** (this journal, Bd. 54, pp. 232) is immediately obvious from the characteristic equation (4), as well as from the formal representation (2) and (3):

III. The adjoint linear (the concomitant bilinear) differential expression of a sum is equal to the sum of the adjoint linear (and the concomitant bilinear) differential expressions of the individual summands.

It is easy to determine the adjoint to a differential expression with the help of the reciprocity theorem and **Hesse**'s theorem, no matter what form it might also be given in. The adjoint expression to *a u*, where *a* means a certain function of *x*, is *a u*, and the adjoint expression to *D u* is -Du. Therefore, the adjoint expression to  $a \cdot A(u)$  will be equal to A(au), and that of DA(u) will be equal to -AD(u).

One can define the adjoint to a bilinear differential expression A(u, v) in two different ways. Either one considers v to be an undetermined function and imagines that u is set equal to a welldefined value, or one regards u as undetermined. Let A'(u, v) be the adjoint expression to A(u, v)in the former sense. If one then imagines that u is set equal to a well-defined function in equation (4) and calculates the two sides of the adjoint expressions then one will get the formula:

(5) 
$$v A(u) - A(uv) = -A'(u, Dv)$$

If one takes *u* in that to be an integral  $1 / c_0$  of the differential equation A(u) = 0, and one replaces *v* with  $c_0 v$  then one will find that:

$$A(v) = A'\left(\frac{1}{c_0}, Dc_0 v\right),$$

or when one introduces the notation:

$$A'\left(\frac{1}{c_0},v\right) = A_1(v),$$

one will find that:

$$A(v) = A_1 D(c_0 v).$$

In the same way, one can put the  $(n-1)^{\text{th}}$ -order differential expression  $A_1(v)$  into the form:

$$A_1(v) = A_2 D(c_1 v),$$

in which  $1 / c_0$  is an integral of the differential equation  $A_1(v) = 0$  and  $A_2(v)$  is a differential expression of order (n-2). When one proceeds in that way, one will ultimately put the expression *A* into the form:

(6) 
$$A(u) = c_n D c_{n-1} D c_{n-2} \dots c_2 D c_1 D c_0 u,$$

which is composed of nothing but first-order differential expressions. From the reciprocity theorem, the adjoint expression to that is:

(7) 
$$A'(u) = (-1)^n c_0 D c_1 D c_2 \dots c_{n-2} D c_{n-1} D c_n u.$$

#### § 4. – Differential expressions that are equal to their adjoints.

From the reciprocity theorem, the adjoint differential expression to A'A(u) is equal to A'A(u), and when *a* is a well-defined function of *x*, that of A'aA(u) is equal to A'aA(u) (\*).

Conversely, let P(u) be an arbitrary differential expression of order *m* that is equal to its own adjoint. If the coefficient of the highest derivative in *P* is equal to *p* then the corresponding coefficient in *P'* is equal to  $(-1)^m p$ . Let  $P_0(u)$ ,  $P_1(u)$ , ...,  $P_n(u)$  be n + 1 arbitrarily-chosen differential expression, where  $P_v(u)$  has order *v*. Let *q* be the coefficient of  $D^n u$  in  $P_n(u)$ , and let  $p_n$  be determined by the equation:

$$p=\left(-1\right)^{n}q^{2}p_{n}.$$

The coefficient of  $D^{2n}u$  in the expression  $P'_n p_n P_n(u)$  will then be equal to p, and therefore the expression:

$$P(u) - P'_n p_n P_n(u)$$

<sup>(\*)</sup> The adjoint differential expression to A'DA(u) is -A'DA(u).

will have order at most (2n - 1). However, since it is equal to its adjoint, and as a result will have even order, it can have order at most (2n - 2). (Cf., **Hesse**, this journal, Bd. 54., pp. 234.) Let p' be the coefficient of  $D^{2n-2}u$  in it, while q' is that of  $D^{n-1}u$  in  $P_{n-1}$ , and let:

$$p' = (-1)^{n-1} q'^2 p_{n-1}$$
.

Thus:

$$P(y) - P'_{n} p_{n} P_{n}(u) - P'_{n-1} p_{n-1} P_{n-1}(u)$$

will be a differential expression that is equal to its adjoint and have order at most (2n - 4). When one calculates further, one will ultimately bring *P* into the form (\*):

(8) 
$$P(u) = P'_n p_n P_n(u) + P'_{n-1} p_{n-1} P_{n-1}(u) + \dots + P'_0 p_0 P_0(u)$$

If, e.g.,  $P_v(u) = D^v u$ , so  $P'_v(u) = (-1)^v D^v u$ , then one sees from this that every differential expression that is equal to its own adjoint can be put into the form:

(9) 
$$P(u) = p_0 u - D p_1 D u + D^2 p_2 D^2 u - \dots + (-1)^n D^n p_n D^n u.$$

(Jacobi, this journal, Bd. 17, pp. 71)

Since the time of **Jacobi**, one has, conversely, sought to represent every expression in that form when one would like to prove that it is equal to its own adjoint. However, since that is not usually possible without long-winded calculations, and since that form is only an inessential feature of such differential expressions, as its generalization (8) already shows, I will make no use of it in what follows.

A differential expression that is equal to its own adjoint can be represented in the form (8) or (9) by rational operations and differentiations. By contrast, a more important discovery of **Jacobi** was that one can bring every such differential expression into the form A'A(u) with the help of integrations, or into the form A'a A(u) when it should result in a real form.

**Jacobi**'s proof of that theorem can be represented in the following simple way with the help of the reciprocity theorem:

Let P(u) be an expression of order 2n that is equal to its own adjoint, and let P(u, v) be the concomitant bilinear expression. One then has:

(10) 
$$v P(u) - u P(v) = D P(u, v)$$
.

 $\pm P'_{n} p_{n} D p_{n} P_{n}(u) \pm P'_{n-1} p_{n-1} D p_{n-1} P_{n-1}(u) + \dots \pm P'_{0} p_{0} D p_{0} P_{0}(u)$ 

(Jacobi, this journal, Bd. 32, pp. 196)

<sup>(\*)</sup> Similarly, one can bring every expression that is equal and opposite to its adjoint into the form:

The left-hand side changes sign when one switches *u* with *v*. Therefore, one has:

$$D P (u, v) = D (-P (v, u)),$$

and as a result, from § 1:

$$P(u, v) = -P(v, u).$$

I. If a differential expression is equal to its adjoint then the concomitant bilinear differential expression will be alternating (\*).

Therefore, P(u, v) will vanish when u = v. If one takes the v in equation (10) to be an integral  $1 / c_0$  of the differential equation P(v) = 0, then one will get:

$$P(u) = c_0 D P\left(u, \frac{1}{c_0}\right).$$

Since the differential expression  $P\left(u, \frac{1}{c_0}\right)$  of order (2n-1) vanishes for  $u = 1 / c_0$ , it can (§ 3) be

brought into the form  $P_1 D (c_0 u)$ , where  $P_1 (u)$  is a differential expression of order (2n - 2), and as a result:

$$P(u) = c_0 D P_1 D c_0 u.$$

If one takes the adjoint expression to both sides then, from the reciprocity theorem, one will get:

$$P(u) = c_0 D P_1' D c_0 u,$$

and therefore:

$$c_0 D P_1' D c_0 u = c_0 D P_1 D c_0 u$$

so from § 1:

 $P_1'=P_1.$ 

Now, if  $1 / c_1$  is an integral of the differential equation  $P_1 = 0$  then one will have:

$$P_1(u) = c_1 D P_2 D c_1 u$$
,

in which  $P_2$  is a differential expression of degree (2n - 4) that is equal to its adjoint. When one proceeds further in that way, one will bring the given expression into the form:

$$P(u) = c_0 D c_1 D c_2 \dots D c_{n-1} D c D c_{n-1} \dots c_2 D c_1 D c_0 u.$$

<sup>(\*)</sup> If a differential expression is equal and opposite to its adjoint then the concomitant bilinear differential expression will be symmetric.

Since the coefficient p of the highest derivative of a composite differential expression P is equal to the product of the coefficients of the highest derivatives in its parts, one will have:

$$p = c_0 c_1 \dots c_{n-1} c c_{n-1} \dots c_1 c_0 = c (c_0 c_1 \dots c_{n-1})^2$$

Therefore, when the coefficients of *P* are all real, and only real integrals are employed in the conversion, *c* will have the same sign as *p*. Let *a* be an arbitrary function that has the same sign as  $(-1)^n p$ , and:

$$c = (-1)^n a c_n^2$$

If one then sets:

(6) 
$$c_n D c_{n-1} \dots c_1 D c_0 u = A(u)$$

then one will have:

(7) 
$$A'(u) = (-1)^n c_0 D c_1 \dots c_{n-1} D c_n (u),$$

and therefore:

$$P(u) = A' a A(u)$$

It would be simplest to set  $a = \pm 1$ , or a = 1 when one is not dealing with real expressions. However, should the coefficient of the highest derivative in A be equal to unity, then one would need to choose:

(12) 
$$a = (-1)^n p$$
.

#### § 5. – New proof of Jacobi's theorem.

It follows from equation (11) that all of the *n* integrals of the differential equation A = 0 must also satisfy the differential equation P = 0. However, it will be shown that they are not *n* arbitrary integrals of P = 0 but must fulfill certain conditions. Nonetheless, it is difficult to find those conditions in their simplest form in the way that was just pursued (\*). For that reason, I would like to derive **Jacobi**'s theorem in a different way that will imply those relations with no further analysis.

If one replaces v with a A(v) in equation (4) then, when one appeals to the notation (11), one will get:

$$a A(v) A(u) - u P(v) = D A(u, a A(v))$$
.

<sup>(\*)</sup> A means for doing that will be suggested in remark I in § 6.

If one switches *u* with *v* and subtracts the new equation from the original one then that will give:

$$v P(u) - u P(v) = D(A(u, a A(v)) - A(u, a A(v))).$$

It not only follows from this that the differential expression P is equal to its own adjoint, but that the concomitant bilinear differential expression will be:

$$P(u, v) = A(u, a A(v)) - A(v, a A(u))$$
.

However, the right-hand side of that equation will vanish when u and v are any two integrals of the differential equation A = 0. As a result, one must have P(u, v) = 0 when u and v both satisfy the differential equation A = 0.

Conversely, now let *P* be any differential expression of order 2n that is equal to its adjoint. The concomitant bilinear differential expression *P* (*u*, *v*) will then be alternating, as was shown above. I will now assert that *n* independent integrals of the differential equation P = 0 can be found such that any two of them will annul the expression *P* (*u*, *v*). Namely, let  $a_0, a_1, ..., a_{2n-1}$  be any 2n independent integrals of P = 0, and let:

$$P(a_{\alpha}, a_{\beta}) = a_{\alpha\beta}$$
.

Since P(u, v) is alternating,  $a_{\alpha\beta} = -a_{\beta\alpha}$  and  $a_{\alpha\alpha} = 0$ . Moreover, since equation (10) implies that:

$$DP(a_{\alpha}, a_{\beta}) = a_{\beta} P(a_{\alpha}) - a_{\alpha} P(a_{\beta}) = 0,$$

 $a_{\alpha\beta}$  will be a constant. Now, let  $x_0, x_1, \dots, x_{2n-1}$  and  $y_0, y_1, \dots, y_{2n-1}$  be arbitrary constants, and let:

$$u = \sum_{\alpha=0}^{2n-1} a_{\alpha} x_{\alpha} , \qquad v = \sum_{\beta=0}^{2n-1} a_{\beta} y_{\beta}$$

be any two integrals of the differential equation P = 0.

$$P(u, v) = \sum_{\alpha, \beta} a_{\alpha\beta} x_{\alpha} y_{\beta} = A$$

will then be an alternating bilinear form. In the next section, I will give various ways of finding n independent sequences of values:

(13) 
$$x_0^{(\nu)}, \quad x_1^{(\nu)}, \quad \dots, \quad x_{2n-1}^{(\nu)} \quad (\nu = 0, 1, \dots, n-1),$$

any two of which will annul the form Z.

$$u_{\nu} = \sum_{\alpha=0}^{2n-1} x_{\alpha}^{(\nu)} a_{\alpha} \qquad (\nu = 0, 1, ..., n-1),$$

which will then be *n* independent integrals of the differential equation P = 0 that pairwise satisfy the equation P(u, v) = 0 (\*).

However, if *n* independent functions  $u_0, u_1, ..., u_{n-1}$  are given then one can define a differential equation A = 0 or order *n* that they satisfy. Should the coefficient of the highest derivative of *A* be equal to unity then one would have:

(14) 
$$A(u) = \sum \pm u_0 u'_1 \cdots u^{(n-1)}_{n-1} u^{(n)} \colon \sum \pm u_0 u'_1 \cdots u^{(n-1)}_{n-1}$$

Furthermore, when all integrals of an  $n^{\text{th}}$ -order differential equation A = 0 satisfy a  $2n^{\text{th}}$ -order differential equation P = 0, one can put P into the form:

$$P = BA$$
,

in which *B* is a differential expression of order *n* (cf., this journal, Bd. 76, pp. 257). If *B'* is the adjoint expression to *B* and *B* (u, v) is the concomitant bilinear differential expression then one will have:

$$v B(u) - u B'(v) = D B(A(u), v)$$

If one switches *u* and *v* and subtracts the new equation from the original one then that will give:

$$v P(u) - u P(v) - A(u)B'(v) + A(v)B'(u) = D[B(A(u), v) - B(A(v), u)],$$

or due to (10):

$$A(v)B'(u) - A(u)B'(v) = D[B(A(u), v) - B(A(v), u) - P(u, v)] = DC(u, v)$$

From the way that it is composed, the expression C(u, v) will vanish when u and v are replaced with any two functions  $u_0, u_1, ..., u_{n-1}$  that each annul A(u) and both annul P(u, v). Its derivative is A(v)B'(u) - A(u)B'(v), so it has order at most n relative to u and v. Therefore, C(u, v) can have order at most (n - 1) relative to u and v (§ 1). However, when (n - 1)<sup>th</sup>-order differential expression C(u, v) vanishes for n independent functions  $v = u_0, u_1, ..., u_{n-1}$ , it must vanish identically. As a result, no matter which well-defined function only also sets v equal to, the expression C(u, v), which has order (n - 1) relative to u, must then vanish identically for  $u = u_0$ ,  $u_1, ..., u_{n-1}$ . Therefore:

$$A(v)B'(u) - A(u)B'(v) = 0$$
,

or

$$\frac{B'(u)}{A(u)} = \frac{B'(v)}{A(v)}$$

<sup>(\*)</sup> In order to define *n* independent functions  $u_0, u_1, ..., u_{n-1}$ , in addition to the n(n-1)/2 equations  $P(u_{\mu}, u_{\nu}) = 0$  ( $\mu, \nu = 0, 1, ..., n-1$ ), one does not need the *n* equations  $P(u_{\nu}) = 0$  ( $\nu = 0, 1, ..., n-1$ ), but only one of them, say  $P(u_0) = 0$ .

is an expression that is independent of the choice of the undetermined function u, i.e., a welldefined function a. However, if:

$$B'(u) = a A(u)$$

then from the reciprocity theorem:

$$B(u) = A'(au),$$

and therefore:

(11) 
$$P(u) = A' a A(u) .$$

If one takes A to be the expression (14), in which the coefficient of the highest derivative is unity. then one must choose a to be the function (12).

#### § 6. – Lemma on alternating bilinear forms.

In order to complete the proof that was just carried out, one still needs to show how one can find n independent sequences of values (13) that pairwise annul the alternating bilinear form:

$$Z = \sum_{\alpha,\beta=0}^{2n-1} a_{\alpha\beta} x_{\alpha} y_{\beta} .$$

It would be simplest to determine them in succession. One assumes that the quantities:

$$x_0^{(0)}, \quad x_1^{(0)}, \quad \dots, \quad x_{2n-1}^{(0)}$$

are arbitrary (but not all zero). Since Z is alternating, it will satisfy the linear equation:

$$\sum_{\alpha} \left( \sum_{\beta} a_{\alpha\beta} x_{\beta}^{(0)} \right) x_{\alpha} = 0 \; .$$

One takes  $x_0^{(0)}$ ,  $x_1^{(0)}$ , ...,  $x_{2n-1}^{(0)}$  to be any second solution of that equation that is independent of the previous one, etc. If one has already determined *m* independent sequences of values

$$x_{0}^{(\mu)}, \qquad x_{1}^{(\mu)}, \qquad \dots, \qquad x_{2n-1}^{(\mu)} \qquad (\mu = 0, 1, \dots, m-1)$$
$$\sum_{\alpha} \left( \sum_{\beta} a_{\alpha\beta} x_{\beta}^{(\mu)} \right) x_{\alpha} = 0$$

that satisfy the *m* linear equations:

then 
$$x_0^{(m)}$$
,  $x_1^{(m)}$ , ...,  $x_{2n-1}^{(m)}$  must be a new solution of those equations that is independent of those *m*. Now, *m* homogeneous linear equations in 2*n* unknowns will have at least  $2n - m$  independent

solutions as long as m < n, so in addition to the *m* ones that are known already, there will be 2n - 2m, so at least 2.

Before presenting a *second* method for determining the quantities (13), I shall prove the theorem that the determinant  $|a_{\alpha\beta}|$  is non-zero. The bilinear differential expression:

$$P(u, v) = \sum p_{\kappa\lambda} D^{\kappa} u D^{\lambda} v$$

has order (2n-1) relative to *u* and *v*. Since it includes no term of dimension higher than (2n-1), one will have  $p_{\kappa\lambda} = 0$  when  $\kappa + \lambda > 2n - 1$ . Therefore, the determinant  $|p_{\kappa\lambda}|$  reduces to a term:

$$|p_{\kappa\lambda}| = (-1)^n p_{2n-1,0} p_{2n-2,1} \dots p_{0,2n-1}$$

or since the terms of dimension (2n-1) alternately equal + p and - p, to:

$$|p_{\kappa\lambda}| = p^{2n}.$$

Moreover:

$$a_{\alpha\beta} = P(a_{\alpha}, a_{\beta}) = \sum_{\kappa,\lambda} p_{\kappa\lambda} a_{\alpha}^{(\kappa)} a_{\beta}^{(\lambda)},$$

and therefore:

$$a_{\alpha\beta}|=|p_{\alpha\beta}|\cdot|a_{\alpha}^{(\kappa)}|^{2}=\left(p^{n}\cdot|a_{\alpha}^{(\kappa)}|\right)^{2}.$$

The determinant of the bilinear form Z is therefore non-zero ( $^*$ ). As a result, one can reduce Z to the form:

$$Z = \sum_{\nu=0}^{n-1} X_{\nu} Y_{\nu}' - X_{\nu}' Y_{\nu}$$

by linear substitutions, in which  $X_{\nu}$ ,  $X'_{\nu}$  (n = 0, 1, ..., n - 1) are 2n independent homogeneous linear functions of  $x_0, x_1, ..., x_{2n-1}$ , and  $Y_{\nu}, Y'_{\nu}$  are the same functions of  $y_0, y_1, ..., y_{2n-1}$ . The quantities (13) will then satisfy the conditions that were posed when they satisfy the *n* linear equations  $X_{\nu} = 0$ , which actually possess 2n - n independent solutions, or more generally (**Clebsch**, this journal, Bd. 55, pp. 344), when the are determined from the *n* equations:

$$X'_{\mu} = \sum_{\nu=0}^{n-1} c_{\mu\nu} X_{\nu} \qquad (\mu = 0, 1, ..., n-1),$$

in which:

$$c_{\mu\nu} = c_{\nu\mu}$$

are n(n+1)/2 arbitrary constants.

<sup>(\*)</sup> It then follows that there are no more than n independent sequences of values (13) that pairwise annul Z. (Cf., this journal, Bd. 82, pp. 256.)

*Thirdly*, and finally, one can also determine *n* integrals  $u_0, u_1, ..., u_{n-1}$  in the following way that satisfy the equation P(u, v) = 0 pairwise. Let x' be a nonsingular point for the differential equation P = 0, and let  $e_{\alpha}$  be the integral of it whose development in the neighborhood of x' begins with:

$$e_{\alpha} = \frac{(x-x')^{\alpha}}{1 \cdot 2 \cdots \alpha} + e_{\alpha,2n} (x-x')^{2n} + \cdots \qquad (\alpha = 0, 1, \dots, 2n-1).$$

 $e_{\alpha}^{(\kappa)}$  (a, k = 0, 1, ..., 2n - 1) is then equal to 0 or 1 for x = x' according to whether  $\alpha$  is different from  $\kappa$  or equal to it, respectively. Since the expression:

$$P(e_{\alpha}, e_{\alpha}) = \sum_{\kappa, \lambda} p_{\kappa\lambda} e_{\alpha}^{(\kappa)} e_{\beta}^{(\lambda)}$$

is constant, it will remain unchanged when the variable x is assigned the value x'. As a result, it will be equal to the value that  $p_{\alpha\beta}$  has for x = x', so it will be zero when  $\alpha + \beta > 2n - 1$ .

Therefore,  $e_n$ ,  $e_{n+1}$ , ...,  $e_{2n-1}$  satisfy the equation P(u, v) = 0 pairwise, and likewise any linear combination of any two of them will. One can then choose  $u_0, u_1, ..., u_{n-1}$  to be any *n* independent integrals whose developments in powers of x - x' do not begin with a power less than *n*.

**Remark I.** – The relations between the constants (13) are the ones that **Clebsch** found, while their simplest form was due to **Hesse**. Due to the significance of the work of **Hesse**, it is perhaps interesting to know precisely the point that he had overlooked.

Let  $u_0 = 1 / c_0$  be an integral, so a multiplier of the differential equation P = 0. One will then have  $P(u) = c_0 D Q_1(u)$ , in which the differential expression  $Q_1(u)$  of order (2n - 1) will vanish for  $u = 1 / c_0$ , so it can be brought into the form  $Q_1(u) = P_1 D c_0 u$ . For the further reduction, one cannot employ an arbitrary second integral  $u_1$  of P = 0, but one for which one also has  $Q_1 = 0$ .  $1/c_1$  $= D c_0 u_1$  will then be an integral of  $P_1 = 0$ , so (§ 4) it will also be a multiplier of  $P_1$ , and as a result,  $Q_1$ , as well. Therefore,  $Q_1 = c_1 D Q_2$  and  $Q_2(u) = P_2 c_1 D c_0 u$ . If  $u_2$  is then an integral of P = 0that is independent of  $u_0$  and  $u_1$  for which one has, at the same time,  $Q_2 = 0$  then  $1/c_2 = D c_1 D c_0 u_2$ will be an integral of  $P_2 = 0$ , so it will also be a multiplier of  $P_2$ , and as a result of  $Q_2$ , as well. Therefore,  $Q_2 = c_2 D Q_2$ , etc.

The equation  $Q_1(u_1) = 0$  is a relation between  $u_0$  and  $u_1$ , the equation  $Q_2(u_2) = 0$  splits into two relations between  $u_0, u_1, u_2, Q_3(u_3) = 0$  splits into three relations between  $u_0, u_1, u_2, u_3$ , etc. (**Hesse**, Bd. 54, pp. 253) The reason why **Hesse** failed in his attempts to perform this lesstransparent form for those relations lies in the fact that he overlooked a peculiar combination of the differential expressions  $Q_1, Q_2, Q_3, ...$  Namely, the expression  $Q_m$  of order (2n - m) can be brought into the form:

$$Q_m(u) = v_0 P(u, u_0) + v_1 P(u, u_1) + \ldots + v_{m-1} P(u, u_{m-1}),$$

in which the ratios of the functions  $v_0, v_1, ..., v_{m-1}$  are already determined in such a way that  $Q_m$  has order only (2n - m), so the coefficients of  $D^{2n-1}u$ ,  $D^{2n-2}u$ , ...,  $D^{2n-m+1}u$  on the right-hand side must vanish. Therefore,  $Q_m$  is equal (up to a factor) to:

$P(u,u_0)$	$P(u,u_1)$	•••	$P(u, u_{m-1})$
$u_0$	$u_1$	•••	$u_{m-1}$
$u_0^{(1)}$	$u_1^{(1)}$	•••	$u_{m-1}^{(1)}$
:	•	·.	:
$u_0^{(m-2)}$	$u_1^{(m-2)}$	•••	$u_{m-1}^{(m-2)}$

and  $v_0, v_1, ..., v_{m-1}$  behave like the functions that I have called the *adjoints* to  $u_0, u_1, ..., u_{m-1}$  (this journal, Bd. 77, pp. 250). As a result (*loc. cit.*, pp. 249), they are mutually independent. Now,  $u_m$  must satisfy the equation  $Q_m(u) = 0$ , or else one must have:

$$v_0 P(u_m, u_0) + v_1 P(u_m, u_1) + \ldots + v_{m-1} P(u_m, u_{m-1}) = 0$$
.

However, since the quantities  $P(u_m, u_\mu)$  are constants, due to the independence of  $v_0, v_1, \ldots, v_{m-1}$ , that equation requires that one must have:

$$P(u_m, u_\mu)$$
  $(\mu = 0, 1, ..., m-1); m = 0, 1, ..., n-1).$ 

That is the form that **Clebsch** gave to the relations.

**Remark II.** – Let the coefficients of the  $m^{\text{th}}$  -order linear differential equation P = 0 be analytic functions that are defined in the neighborhood of a nonsingular point  $x_0$  by convergent series in increasing whole positive powers of  $x - x_0$ . In what follows, a line *L* that starts from  $x_0$  and returns to it shall be called *closed* only when the coefficients of *P* can be simultaneously continued to a surface strip (of finite width) that surrounds that line and returns on itself without losing the character of rational functions, and in that way go back to the original function elements. If *x* traverses such a line *L* then *m* independent integrals  $a_0, a_1, \ldots, a_{m-1}$  of the differential equation P = 0 will be converted into *m* linear combinations  $a'_0, a'_1, \ldots, a'_{m-1}$  with constant coefficients, so it will experience a linear substitution  $a_a = \sum_{\beta} k_{\alpha\beta} a'_{\beta}$  that I called the substitution *K* that *corresponds* 

to the line *L*. ( $H^{-1}KH$ ; cf., this journal, Bd. 84, pp. 21 and 23). A closed line *L* does not correspond to a well-defined substitution, but an entire *class* of similar substitutions. However, such a thing will be determined completely when its characteristic determinant is given and decomposed into elementary divisors. Relative to a linear differential equation, any closed line will then correspond to a certain characteristic function that is decomposed into elementary divisors.

We now let P(u) be a differential expression that is equal to its adjoint expression and appeal to the same notations as above. Along the line L, the integral  $u = \sum x_{\alpha} a_{\alpha}$  will go to  $u' = \sum x'_{\alpha} a_{\alpha}$ 

and the integral  $v = \sum y_{\alpha} a_{\alpha}$  will go to  $v' = \sum y'_{\alpha} a_{\alpha}$ . The 2*n* constants  $x'_{\alpha}$  are linear functions of the constants  $x_{\alpha}$ , and the linear substitution by which the quantities  $x_{\alpha}$  go to  $x'_{\alpha}$  is contragredient to the one that the 2*n* integrals  $a_{\alpha}$  experience along the line *L*. Namely, if  $a_{\alpha} = \sum_{\alpha} k_{\alpha\beta} a'_{\beta}$  then  $x'_{\alpha}$ 

$$= \sum_{\alpha} k_{\alpha\beta} x_{\alpha} \text{ and likewise } y'_{\beta} = \sum_{\alpha} k_{\alpha\beta} y_{\alpha} \text{ . The bilinear expression:}$$
$$P(u, v) = \sum_{\alpha} a_{\alpha\beta} x_{\alpha} y_{\beta} = Z \quad \text{will now go to} \quad P(u', v') = \sum_{\alpha} a_{\alpha\beta} x'_{\alpha} y'_{\beta} = Z'.$$

However, since P(u, v) is a quantity that is independent of x, it can experience no change along the line L, and as a result, one must have Z = Z'. The linear substitution that takes the quantities  $x_{\alpha}$  to  $x'_{\alpha}$  then has the property of transforming an alternating bilinear form with cogredient variables with a non-vanishing determinant into itself. As a result (this journal, Bd. 84, pp. 41), the elementary divisors of its characteristic function will be pairwise of equal degree and vanish for reciprocal values, except for the ones that are zero for the values  $\pm 1$ . However, the ones that have an odd exponent must be likewise pairwise present among the latter. The characteristic determinant of the contragredient substitution K must have the same properties then (*loc. cit.*, pp. 25 and 21). That also follows from the fact that the substitution K will transform the reciprocal form to Z into itself.

The elementary divisors of the characteristic function that corresponds to a closed path relative to a differential equation that is equal [equal and opposite, resp.] to itself must be pairwise of equal degree and vanish for reciprocal values, with the exception of the ones that are zero for the values  $\pm 1$  and have an even [off, resp.] exponents.

#### § 7. – Conversion of a determinant.

Since the determinant of the functions  $u_0, u_1, ..., u_{n-1}$  plays an important role in the calculus of variations, we would like to examine it somewhat more thoroughly.

Let  $b_{\alpha\beta}$  be the coefficient of  $a_{\beta\alpha}$  in the non-vanishing determinant  $|a_{\alpha\beta}|$  of degree 2*n*, divided by the entire determinant, and let:

$$Y = \sum b_{\alpha\beta} x_{\alpha} y_{\beta}$$

be the adjoint form of Z, so it is likewise an alternating form. Furthermore, let:

(15) 
$$y_0^{(\nu)}, y_1^{(\nu)}, ..., y_{2n-1}^{(\nu)}$$
  $(\nu = 0, 1, ..., n-1)$ 

be n independent sequences of values that annul Y pairwise. I shall then assert that any two solutions of the n independent linear equations:

(\rho) 
$$\sum_{\alpha=0}^{2n-1} y_{\alpha}^{(\nu)} x_{\alpha} = 0 \qquad (\nu = 0, 1, ..., n-1)$$

will annul Z. That is because, due to the equations:

(\$\alpha\$) 
$$\sum_{\alpha,\beta} b_{\alpha\beta} y_{\alpha}^{(\mu)} y_{\beta}^{(\nu)} = 0 \qquad (\mu,\nu=0,\,1,\,...,\,n-1),$$

the expressions:

(
$$\sigma$$
)  $x_{\alpha}^{(\mu)} = \sum_{\beta} b_{\alpha\beta} y_{\beta}^{(\mu)}$  ( $\alpha = 0, 1, ..., 2n - 1; \mu = 0, 1, ..., n - 1$ )

will be *n* independent solutions of equations ( $\rho$ ), since one has:

(
$$\beta$$
)  $\sum_{\beta} x_{\alpha}^{(\mu)} y_{\beta}^{(\nu)} = 0$   $(\mu, \nu = 0, 1, ..., n-1).$ 

However, solving equations ( $\sigma$ ) will give:

(7) 
$$y_{\alpha}^{(\nu)} = \sum_{\beta} a_{\alpha\beta} x_{\beta}^{(\nu)} .$$

If one multiplies that equation by  $x_{\alpha}^{(\mu)}$  and sums over  $\alpha$  then one will get:

(
$$\gamma$$
) 
$$\sum_{\alpha,\beta} a_{\alpha\beta} x_{\alpha}^{(\mu)} x_{\beta}^{(\nu)} = 0$$

due to ( $\beta$ ). Since the *n* sequences of values  $x_{\alpha}^{(\nu)}$  that are defined by equations ( $\sigma$ ) are independent, and since the *n* independent linear equations ( $\rho$ ) between the 2*n* unknowns have no more than 2*n* – *n* independent solutions, every solution of them will be a linear combination of the *n* solutions ( $\sigma$ ). However, since the *n* sequences of values ( $\sigma$ ) annul the form *Z* pairwise, as formulas ( $\gamma$ ) show, any two linear combinations of them will possess the same property, since *Z* is alternating. Therefore, the *n* sequences of values (13) will also annul the form *Z* pairwise when it is not given by the formulas ( $\sigma$ ) but is defined by any *n* independent solutions of equations ( $\rho$ ), i.e., equations ( $\alpha$ ) and ( $\beta$ ) always imply equations ( $\gamma$ ) [and even without appealing to equations ( $\sigma$ ) or ( $\tau$ )]. Equations ( $\alpha$ ) likewise follow from equations ( $\gamma$ ) and ( $\beta$ ) [but equations ( $\beta$ ) do not necessarily follow from ( $\alpha$ ) and ( $\gamma$ )]. I have called two systems of quantities (13) and (15) that are coupled together by equations ( $\beta$ ) *adjoint* (this journal, Bd. 82, pp. 238) and showed that the determinant of degree *n* of the one of them ( $x_{\alpha}^{(\nu)}$ ) is equal to the complementary determinant of degree *n* of the other one ( $y_{\alpha}^{(\nu)}$ ), up to a common non-zero factor. I. If n independent sequences of values annul an alternating bilinear form in 2n variable pairs with non-vanishing determinant pairwise then the n sequences of quantities that are adjoint to them will annul the adjoint form pairwise.

From the extended multiplication theorem, the determinant of degree *n* of the quantities:

(
$$\delta$$
)  $u_{\mu}^{(\nu)} = \sum_{\alpha=0}^{2n-1} x_{\alpha}^{(\mu)} a_{\alpha}^{(\nu)}$   $(\mu, \nu=0, 1, ..., n-1)$ 

can be decomposed into a sum of products of each determinant of degree *n* of the system  $a_{\alpha}^{(\nu)}$  and the corresponding one for the system  $x_{\alpha}^{(\nu)}$ . From **Laplace**'s determinant theorem, the determinant:

can be decomposed into a sum of products of a determinant of degree *n* of the system  $a_{\alpha}^{(\nu)}$  and the complementary determinant of the system  $y_{\alpha}^{(\nu)}$ . However, since the determinants of degree *n* of the system  $x_{\alpha}^{(\nu)}$  behave like the complements of the system  $y_{\alpha}^{(\nu)}$ , the determinant  $|u_{\mu}^{(\nu)}|$  of degree *n* will be equal to the determinant (16) of degree 2*n*, up to a constant factor. One will arrive at the same result when one multiplies (16) by a determinant of degree 2*n* whose first *n* rows are defined by the quantities  $x_{\alpha}^{(\nu)}$ , and whose last *n* rows are defined by the arbitrary constants  $c_{\alpha}^{(\nu)}$ . It is especially remarkable that the constants (15) that enter into (16) do not need to go back to the constants (13) by equation ( $\beta$ ) under that conversion, but can obviously be defined such that they pairwise annul the alternating bilinear form *Y*.

One will obtain a system of quantities (15) in an especially simple way when one chooses  $u_0$ ,  $u_1, \ldots, u_{n-1}$  to be the special system of n independent integrals that was mentioned at the end of § 6, none of which includes a power less than n in its development in powers of x - x'. Namely, if one sets x = x' in ( $\delta$ ) then the left-hand side of that equation will vanish, and as a result one will satisfy equations (b) when one takes  $y_{\alpha}^{(v)}$  to have the values that  $a_{\alpha}^{(v)}$  assume for x = x' (**Mayer**, this journal, Bd. 79, pp. 257). The n sequences of values thus-obtained are independent, because when all of determinants of degree n of those quantities  $y_{\alpha}^{(v)}$  vanish, the determinant  $|a_{\alpha}^{(v)}|$  of degree 2n will vanish for x = x', so x' will be an (essential or inessential) singular value for the differential equation P = 0.

### § 8. – On the second variation of simple integrals.

Let:

$$F = \int_{x_0}^{x_1} f(x, y, y^{(1)}, \dots, y^{(m)}) dx$$

be a well-defined integral that includes an unknown function y and its derivatives up to order m, while *u* is an undetermined function of *x*,  $\varepsilon$  is a very small number, and:

$$\int_{x_0}^{x_1} f(x, y + \varepsilon u, y^{(1)} + \varepsilon u^{(1)}, \dots, y^{(m)} + \varepsilon u^{(m)}) dx = F + \varepsilon G + \frac{\varepsilon^2}{1 \cdot 2} H + \cdots,$$

so:

$$G = \int_{x_0}^{x_1} R(u) \, dx \,, \qquad H = \int_{x_0}^{x_1} S(u, u) \, dx \,, \qquad \dots,$$

in which:

$$R(u) = \sum_{v} \frac{\partial f}{\partial y^{(v)}} D^{v} u, \qquad S(u, v) = \sum_{\mu, v} \frac{\partial^{2} f}{\partial y^{(\mu)} \partial y^{(v)}} D^{\mu} u D^{v} v, \qquad \dots$$

(Cf., Hesse, loc. cit., pp. 227-229.) If R'(u) is the adjoint expression to the linear differential expression R(u), and R(u, v) is the concomitant bilinear expression then:

$$v R(u) - u R'(v) = D R(u, v),$$

and therefore, for v = 1:

$$R(u) = u R'(1) + D R(u, 1),$$

/

in which:

(17) 
$$R'(1) = \sum (-1)^{\nu} D^{\nu} \left( \frac{\partial f}{\partial y^{(\nu)}} \right).$$

Therefore:

$$G = \int_{x_0}^{x_1} R(u) \, dx = \int_{x_0}^{x_1} dR(u, 1) + \int_{x_0}^{x_1} u \, R'(1) \, dx$$

In the calculus of variations, one then infers that the integral F can be a maximum or a minimum for only those functions y that satisfy the differential equation R'(1) = 0 or:

(17\*) 
$$\frac{\partial f}{\partial y} - D \frac{\partial f}{\partial y^{(1)}} + D^2 \frac{\partial f}{\partial y^{(2)}} + \dots + (-1)^m D^m \frac{\partial f}{\partial y^{(m)}} = 0.$$

In the following, *y* shall then be understood to mean the function that is determined by that differential equation and the auxiliary conditions of the problem. (**Hesse**, *loc. cit.*, pp. 229)

If one considers the functions v in the *symmetric* bilinear differential expression S(u, v) to be undetermined then one lets S'(u, v) be the adjoint expression and lets S(u; v, w) be the concomitant bilinear differential expression in v and w. Both expressions are also linear with respect to u (§ 2). If one then switches u and v in the equation:

(
$$\alpha$$
)  $w S(u, v) - v S'(u, w) = D S(u; v, w)$ 

and subtracts the original equation from the new one then since S(u, v) = S(v, u), one will get:

$$v S'(u, w) - u S'(v, w) = D [S (v; u, w) - S (u; v, w)].$$

It will then follow from this that when one sets *w* equal to a certain function and considers *u* to be undetermined, the expression S'(u, w) will be identical to its adjoint (\*). (Cf., the laborious proof in **Hesse**, *loc. cit.*, pp. 233-239.)

I. If S(u, v) is a symmetric bilinear differential expression and S'(u, v) is its adjoint, when v is considered to be undetermined, then the expression S'(u, v) will be identical to its adjoint when u is regarded as undetermined.

If one sets w = 1 then the expression:

(18) 
$$S'(u,1) = P(u) = \sum_{\mu,\nu} (-1)^{\nu} D^{\nu} \left( \frac{\partial^2 f}{\partial y^{(\mu)} \partial y^{(\nu)}} D^{\mu} u \right)$$

will be equal to its own adjoint (\*\*). It is therefore of even order (2n) and can be brought (§ 5) into the form:

(11) P(u) = A'a A(u).

If one sets v = a A(u) in the equation:

<sup>(\*)</sup> If  $S(u_1, u_2, ..., u_k)$  is a differential expression that is linear (and homogeneous) relative to each of the undetermined functions  $u_1, u_2, ..., u_k$  then one can consider  $u_{\alpha}$  along to be undetermined and calculate the adjoint expression  $S_{\alpha}(u_1, u_2, ..., u_k)$ . One again considers only  $u_{\beta}$  to be undetermined in that and then determines the adjoint expression  $S_{\alpha\beta}$ , etc. The relations that exist between the differential expressions  $S_{\alpha}$ ,  $S_{\alpha\beta}$ ,  $S_{\alpha\beta\gamma}$ , ... can be found by means of the same process that was used to derive the special theorem above.

<sup>(\*\*)</sup> Since the adjoint expression to  $D^{\nu}u$  is equal to  $(-1)^{\nu}D^{\nu}u$ , that of  $(-1)^{\nu}D^{\nu}(f_{\mu\nu}D^{\mu}u)$  will be equal to  $(-1)^{\mu}D^{\mu}(f_{\mu\nu}D^{\nu}u)$ , from the reciprocity theorem. From that, one can once more infer that the expression  $(-1)^{\nu}D^{\nu}(f_{\mu\nu}D^{\mu}u)$  is equal to its own adjoint, assuming that  $f_{\mu\nu} = f_{\nu\mu}$ .

(4) 
$$v A(u) - u A'(v) = D A(u, v)$$

then one will get:

$$a (A (u))^2 - u P (u) = D A (u, a A(u)).$$

If one further sets u = v and w = 1 in equation ( $\alpha$ ) then since S'(u, 1) is denoted by P(u), that will give:

$$S(u, u) - u P(u) = D S(u; u, 1)$$
.

Upon subtracting both equations, one will ultimately find that:

$$S(u, u) = D[A(u; u, 1) - A(u, a A(u))] + a(A(u))^{2},$$

and from that, one will get Jacobi's conversion of the second variation:

(19) 
$$H = \int_{x_0}^{x_1} S(u,u) \, dx = \int_{x_0}^{x_1} d\left[S(u;u,1) - A(u,aA(u))\right] \, dx + \int_{x_0}^{x_1} a\left(A(u)\right)^2 \, dx \, dx$$

If y means an undetermined function then (17) will be a nonlinear differential expression whose order is equal to at most 2m. If one sets y equal to a function of two independent variables x and h in it and differentiates with respect to h then one will get:

(20) 
$$\frac{\partial R'(1)}{\partial h} = \sum_{\mu,\nu} (-1)^{\nu} D^{\nu} \left( \frac{\partial^2 f}{\partial y^{(\mu)} \partial y^{(\nu)}} \frac{\partial y^{(\mu)}}{\partial h} \right) = P \left( \frac{\partial y}{\partial h} \right).$$

Since the order of the expression P(u) is equal to 2n, it will follow that R'(1) is also of order 2n with respect to y [cf., the examples of **Spitzer**, Sitzungsberichte der Wiener Akademie (1854), pp. 1014, and *ibid*. (1855), pp. 41].

#### II. The order of the differential equation $(17^*)$ is always an even number.

If y is the general integral of it with the 2n arbitrary constants  $h_0$ ,  $h_1$ , ...,  $h_{2n-1}$  then the 2n functions:

$$a_{\alpha} = \frac{\partial y}{\partial h_{\alpha}} \qquad (\alpha = 0, 1, ..., 2n - 1)$$

will be linearly independent (**Jacobi**, this journal, Bd. 23, pp. 55 and 56) and will satisfy the linear differential equation P(u) = 0 due to the identity (20). If one combines *n* independent integrals  $u_0$ ,  $u_1, ..., u_{n-1}$  of it (§ 6) that annul the concomitant bilinear differential expression P(u, v) [= S(v; u, 1) - S(u; v, 1)] pairwise then the expression A(u) will be given by equation (14), and the

function *a* will be given by (12). If  $\partial^2 f / \partial y^{(m)^2}$  is non-zero then the degree of the differential equation (17<sup>\*</sup>) will be equal to 2*m*, and:

$$a = \frac{\partial^2 f}{\partial y^{(m)2}}$$

## § 9. – On adjoint linear partial differential expressions.

The relationships between adjoint linear ordinary differential equations, to the extent that they are not concerned with their integrals, can be extended to partial differential equations.

If *u* and *v* are two functions of  $x_1, ..., x_n$  then from § 2, ( $\alpha$ ):

$$v\frac{\partial^{v_{\alpha}} u}{\partial x_{\alpha}^{v_{\alpha}}} - u(-1)^{v_{\alpha}}\frac{\partial^{v_{\alpha}} v}{\partial x_{\alpha}^{v_{\alpha}}} = \frac{\partial P}{\partial x_{\alpha}},$$

in which:

$$P = \sum_{\lambda=0}^{\nu_{\alpha}-1} (-1)^{\nu_{\alpha}-1-\lambda} \frac{\partial^{\lambda} u}{\partial x_{\alpha}^{\lambda}} \frac{\partial^{\nu_{\alpha}-1-\lambda} v}{\partial x_{\alpha}^{\nu_{\alpha}-1-\lambda}}$$

is a bilinear differential expression in *u* and *v*. If one replaces:

*u* with 
$$\frac{\partial^{v_{\alpha+1}+\cdots+v_n}v}{\partial x_{\alpha+1}^{v_{\alpha+1}}\cdots\partial x_{\alpha-1}^{v_{\alpha-1}}}$$

in that equation and

$$v$$
 with  $(-1)^{v_1+\cdots+v_{\alpha-1}} \frac{\partial^{v_1+\cdots+v_{\alpha-1}}v}{\partial x_1^{v_1}\cdots\partial x_{\alpha-1}^{v_{\alpha-1}}}$ 

then one will get:

$$(-1)^{v_1+\cdots+v_{\alpha-1}} \frac{\partial^{v_1+\cdots+v_{\alpha-1}} v}{\partial x_1^{v_1}\cdots\partial x_{\alpha-1}^{v_{\alpha-1}}} \frac{\partial^{v_\alpha+\cdots+v_n} u}{\partial x_\alpha^{v_\alpha}\cdots\partial x_n^{v_n}} - (-1)^{v_1+\cdots+v_\alpha} \frac{\partial^{v_1+\cdots+v_1} v}{\partial x_1^{v_1}\cdots\partial x_\alpha^{v_\alpha}} \frac{\partial^{v_{\alpha+1}+\cdots+v_n} u}{\partial x_{\alpha+1}^{v_{\alpha+1}}\cdots\partial x_n^{v_n}} = \frac{\partial Q_\alpha}{\partial x_\alpha}.$$

If one takes  $\alpha$  to be the numbers 1, 2, ..., *n* in succession and adds up the equations in question then that will give:

$$v \frac{\partial^{\nu_1 + \dots + \nu_n} u}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}} - u (-1)^{\nu_1 + \dots + \nu_n} \frac{\partial^{\nu_1 + \dots + \nu_n} v}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}} = \frac{\partial Q_1}{\partial x_1} + \dots + \frac{\partial Q_n}{\partial x_n} .$$

If one finally replaces *v* with the product of *v* and a well-defined function  $A_{\nu_1 \cdots \nu_n}$  then one will find that:

$$v\left(A_{\nu_{1}\cdots\nu_{n}}\frac{\partial^{\nu_{1}+\cdots+\nu_{n}}u}{\partial x_{1}^{\nu_{1}}\cdots\partial x_{n}^{\nu_{n}}}\right)-u\left((-1)^{\nu_{1}+\cdots+\nu_{n}}\frac{\partial^{\nu_{1}+\cdots+\nu_{n}}(A_{\nu_{1}\cdots\nu_{n}}\nu)}{\partial x_{1}^{\nu_{1}}\cdots\partial x_{n}^{\nu_{n}}}\right)=\frac{\partial R_{1}}{\partial x_{1}}+\cdots+\frac{\partial R_{n}}{\partial x_{n}},$$

in which  $R_1, ..., R_n$  are bilinear differential expressions in u and v. If:

$$A(u) = \sum A_{\nu_1 \cdots \nu_n} \frac{\partial^{\nu_1 \cdots \nu_n} u}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}}$$

is a linear (partial) differential expression then:

$$A'(u) = \sum (-1)^{\nu_1 + \dots + \nu_n} \frac{\partial^{\nu_1 + \dots + \nu_n} (A_{\nu_1 \cdots \nu_n} u)}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}}$$

is called the *adjoint* differential expression. When one sums the last formula, one will find the following relation between those two differential expressions:

(21) 
$$v A(u) - u A'(v) = \sum \frac{\partial A_v(u, v)}{\partial x_v},$$

in which  $A_v(u, v)$  (n = 1, 2, ..., n) are *n* bilinear differential expressions in *u* and *v*.

The meaning of that equation is based upon the fact that one can infer from it that A'(u) is conversely the adjoint expression to A(u). That is all the more remarkable since the bilinear expressions  $A_v(u, v)$  are not at all well-defined but can assume many different forms. The proof of that assertion is based upon the lemma:

I. If  $A_1, A_2, ..., A_n$  are linear differential expressions in u then the coefficient of u in the adjoint expression to

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \dots + \frac{\partial A_n}{\partial x_n}$$

will be zero.

If *a* is a well-defined function of  $x_1, ..., x_n$  then:

$$\frac{\partial}{\partial x_1} \left( a \frac{\partial^{v_1 + v_2 + \dots + v_n} u}{\partial x_1^{v_1} \partial x_2^{v_2} \cdots \partial x_n^{v_n}} \right) = \frac{\partial a}{\partial x_1} \frac{\partial^{v_1 + v_2 + \dots + v_n} u}{\partial x_1^{v_1} \partial x_2^{v_2} \cdots \partial x_n^{v_n}} + a \frac{\partial^{v_1 + 1 + v_2 + \dots + v_n} u}{\partial x_1^{v_1 + 1} \partial x_2^{v_2} \cdots \partial x_n^{v_n}} \right)$$

The adjoint differential expression to that is:

$$(-1)^{\nu_1+\cdots+\nu_n}\left[\frac{\partial^{\nu_1+\nu_2+\cdots+\nu_n}\left(\frac{\partial a}{\partial x_1}u\right)}{\partial x_1^{\nu_1}\partial x_2^{\nu_2}\cdots\partial x_n^{\nu_n}}-\frac{\partial^{\nu_1+1+\nu_2+\cdots+\nu_n}au}{\partial x_1^{\nu_1+1}\partial x_2^{\nu_2}\cdots\partial x_n^{\nu_n}}\right],$$

which then vanishes for u = 1. If one arranges that according to the derivatives of u then the coefficient of u will be equal to zero. The assertion above will then follow from the remark that the adjoint expression to a sum is a sum of the adjoint expressions to the summands.

Now let *A* (*u*) and *B* (*u*) be two linear differential expressions, and let *n* bilinear differential expressions  $C_v(u, v)$  (v = 1, 2, ..., n) be determined in such a way that:

$$v A(u) - u B(v) = \sum \frac{\partial C_v}{\partial x_v}.$$

If one subtracts equation (21) from that then one will get:

$$u(A'(v)-B(v)) = \sum \frac{\partial(C_v-A_v)}{\partial x_v}.$$

If one imagines that v is set equal to a well-defined function then both sides of that equation will be linear differential expressions in u. The adjoint expression to the left-hand side is u(A'(v) - B(v)). The coefficient of u in the adjoint expression to the right-hand side is equal to zero. As a result, B(v) = A'(v).

We can then define the adjoint expression to another one from now on by the characteristic equation (21). It follows immediately from this that the adjoint expression to A'(u) is equal to A(u).

#### § 10. – The reciprocity theorem.

If one replaces u with B(u) in the equation:

(21) 
$$vA(u) - uA'(v) = \sum \frac{\partial A_v(u,v)}{\partial x_v}$$

then one will get:

$$v A B(u) - B(u) A'(v) = \sum \frac{\partial A_v (B(u), v)}{\partial x_v}$$

If one further replaces v with A'(v) in the equation:

$$v B(u) - u B'(v) = \sum \frac{\partial B_v(u,v)}{\partial x_v}$$

then one will get:

$$A'(v)B(u) - uB'A'(v) = \sum \frac{\partial B_v(u, A'(v))}{\partial x_v}$$

Upon adding both equations, one will find that:

$$vAB(u) - uB'A'(v) = \sum \frac{\partial [A_v(B(u),v) + B_v(u,A(v))]}{\partial x_v}.$$

As a result, B'A' is the adjoint expression to AB.

I. If a differential expression is a composition of several others then the adjoint expression will be a composition of the adjoints in the opposite sequence.

The adjoint expression to  $\frac{\partial A(u)}{\partial x_{\alpha}}$  is then  $-A'\left(\frac{\partial u}{\partial x_{\alpha}}\right)$ , from which it will be clear that the

coefficient of *u* will vanish in it.

In equation (21), one imagines that u has been set equal to a well-defined function and takes the adjoint expression to both sides. If one lets  $A'_{v}(u,v)$  denote the adjoint expression to  $A_{v}(u,v)$ , when the v in it is considered to be an undetermined function, then one will get:

(22) 
$$vA(u) - A(uv) = -\sum A'_{\nu}\left(u, \frac{\partial v}{\partial x_{\nu}}\right).$$

Conclusions that are similar to the ones that were inferred from (5) can be inferred from the latter equation. Here, we would like to use it only to determine the most general form of the bilinear expressions  $A_v(u, v)$ . If one arranges the bilinear differential expression A(u v) - v A(u), which vanishes for v = 1, in derivatives of v then that might give:

$$A(uv) - vA(u) = \sum_{v} P_{v}(u) \frac{\partial v}{\partial x_{v}} + \sum_{v,\alpha} P_{v\alpha}(u) \frac{\partial^{2} v}{\partial x_{v} \partial x_{\alpha}} + \sum_{v,\alpha,\beta} P_{v\alpha\beta}(u) \frac{\partial^{3} v}{\partial x_{v} \partial x_{\alpha} \partial x_{\beta}} + \cdots,$$

in which the coefficients are linear differential expressions in *u*. Indeed, when *v* and *a* are different,  $P_{v\alpha} + P_{\alpha v}$  means the coefficients of  $\frac{\partial^2 v}{\partial x_v \partial x_\alpha}$ , when decomposed into two summands in any way,  $P_{\alpha vv} + P_{v\alpha v} + P_{vv\alpha}$  means the coefficients of  $\frac{\partial^3 v}{\partial x_v^2 \partial x_\alpha}$ , when split into three parts arbitrarily, etc.

One does not need to conclude the development above with the terms of highest order that actually occur either, but one can add arbitrarily many higher-order terms, in which the various coefficients

of the same derivative have a zero sum. Equation (22) will then be fulfilled identically when one sets:

(23) 
$$A_{\nu}(u, v) = P_{\nu}(u)v - \sum_{\alpha} \frac{\partial (P_{\nu\alpha}(u) \cdot v)}{\partial x_{\alpha}} + \sum_{\alpha,\beta} \frac{\partial^{2} (P_{\nu\alpha\beta}(u) \cdot v)}{\partial x_{\alpha} \partial x_{\beta}} - \cdots$$

However, if one takes the adjoint expressions relative to v on both sides of (22) then one will again obtain equation (21). Moreover, since any arbitrary bilinear differential expression can be written in the form (23), one needs only to traverse the given path backwards in order to convince oneself that (23) is the most general form for the n bilinear differential expressions that enter into equation (21).

#### § 11. – Principle of the last multiplier.

I shall now turn to the question of what form the relations between adjoint differential expressions might take when new variables are introduced in place of the independent ones. To that end, I shall nee **Jacobi**'s formula, which lies at the basis for the principle of the last multiplier. I would then like to communicate a brief digression into a new derivation of the formula.

When the (total) linear differential expression:

$$a_1\,dx_1+\ldots+a_n\,dx_n=\sum a\,dx$$

whose coefficients are functions of  $x_1, ..., x_n$ , goes to  $\sum a' dx'$  by introducing *n* new independent variables, at the same time, the bilinear differential expression:

(a) 
$$\delta\left(\sum a\,dx\right) - d\left(\sum a\,\delta x\right) = \sum_{\alpha,\beta} \left(\frac{\partial a_{\alpha}}{\partial x_{\beta}} - \frac{\partial a_{\beta}}{\partial x_{\alpha}}\right) dx_{\alpha}\,\delta x_{\beta}$$

will be converted into  $\sum \left( \frac{\partial a'_{\alpha}}{\partial x'_{\beta}} - \frac{\partial a'_{\beta}}{\partial x'_{\alpha}} \right) dx'_{\alpha} \, \delta x'_{\beta}$ , and will then be called the *bilinear covariant* of

the linear differential expression (this journal, Bd. 70, pp. 73; *ibid.*, Bd. 82, pp. 235). The differentials of the original variables are coupled with those of the new ones by the linear equations:

$$dx_{\alpha} = \sum_{\beta} \frac{\partial x_{\alpha}}{\partial x'_{\beta}} dx'_{\beta} \qquad (\alpha = 1, 2, ..., n)$$

whose determinant is:

$$\left|\frac{\partial x_{\alpha}}{\partial x_{\beta}'}\right| = D \,.$$

The n-2 independent differential expressions:

$$a_{\mu 1} dx_1 + \ldots + a_{\mu n} dx_n$$
 might go to  $a'_{\mu 1} dx'_1 + \cdots + a'_{\mu n} dx'_n$  ( $\mu = 1, 2, \ldots, n-2$ )

under that substitution (Cf., Bd. 82, pp. 279). Let u and v be two undetermined functions of  $x_1, ..., x_n$ , and let:

$$W = \begin{vmatrix} \frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u}{\partial x_n} \\ \frac{\partial v}{\partial x_1} & \cdots & \frac{\partial v}{\partial x_n} \\ a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n-2,1} & \cdots & a_{n-2,n} \end{vmatrix} = \sum A_{\alpha\beta} \frac{\partial u}{\partial x_{\alpha}} \frac{\partial v}{\partial x_{\beta}}$$

Since  $A_{\alpha\beta} = -A_{\beta\alpha}$  and  $A_{\alpha\alpha} = 0$ , *W* will be an alternating bilinear partial differential expression. If *W*' denotes the analogous determinant that is defined by the transformed functions *u*, *v* and the coefficients of the transformed forms then:

$$W' = WD$$
.

Therefore, *W* will be a *contravariant* of the (n-2) differential expressions in question (**Christoffel**, this journal, Bd. 70, pp. 64). However, if ( $\alpha$ ) is a covariant and *W* is a contravariant of a system of forms then (**Aronhold**, this journal, Bd. 62, pp. 339):

$$(\beta) \qquad \qquad J = \frac{1}{2} \sum_{\alpha,\beta} A_{\alpha\beta} \left( \frac{\partial a_{\alpha}}{\partial x_{\beta}} - \frac{\partial a_{\beta}}{\partial x_{\alpha}} \right) = \sum A_{\alpha\beta} \frac{\partial a_{\alpha}}{\partial x_{\beta}} = -\sum A_{\alpha\beta} \frac{\partial a_{\beta}}{\partial x_{\alpha}}$$

will be an invariant of it, and when J' is composed of the coefficients of the transformed forms in that same way that was given for J, one will have:

$$J' = JD$$
.

We now consider a system of n - 1 independent differential expressions:

(
$$\gamma$$
)  $a_{\mu 1} dx_1 + \ldots + a_{\mu n} dx_n$   $(\mu = 1, 2, \ldots, n-1).$ 

They have a linear contravariant:

$$U = \begin{vmatrix} \frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u}{\partial x_1} \\ a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n} \end{vmatrix} = \sum A_{\alpha} \frac{\partial u}{\partial x_{\alpha}},$$

which will be called the partial differential expression that is *adjoint* to it (this journal, Bd. 82, pp. 268). It is coupled with the analogous contravariant of the transformed system by the equation:

$$(\delta) \qquad \qquad U' = UD \,.$$

Furthermore, from ( $\beta$ ), the n - 1 expressions:

$$J_{\mu} = \sum_{\alpha,\beta} \frac{\partial A_{\alpha}}{\partial a_{\mu\beta}} \frac{\partial a_{\mu\beta}}{\partial x_{\alpha}} \qquad (\mu = 1, 2, ..., n-1)$$

will be invariants of the system of forms  $(\gamma)$ , and as a result, their sum:

$$J = \sum_{\mu} J_{\mu} = \sum_{\alpha} \sum_{\mu,\beta} \frac{\partial A_{\alpha}}{\partial a_{\mu\beta}} \frac{\partial a_{\mu\beta}}{\partial x_{\alpha}} = \sum_{\alpha} \frac{\partial A_{\alpha}}{\partial x_{\alpha}}$$

will also be an invariant of it that is coupled with the same invariant in the transformed system by the equation:

$$(\varepsilon) J' = J D .$$

However, by means of the substitution that was applied, one will now have:

$$U = A_1 \frac{\partial u}{\partial x_1} + \dots + A_n \frac{\partial u}{\partial x_n} = A_1' \frac{\partial u}{\partial x_1'} + \dots + A_n' \frac{\partial u}{\partial x_n'},$$

so from ( $\delta$ ):

$$U' = (A'_1 D) \frac{\partial u}{\partial x'_1} + \dots + (A'_n D) \frac{\partial u}{\partial x'_n} ,$$

and therefore:

$$J'=\sum\frac{\partial(A'_{\alpha}D)}{\partial x'_{\alpha}},$$

so from ( $\varepsilon$ ):

(24) 
$$\sum \frac{\partial (A'_{\alpha} D)}{\partial x'_{\alpha}} = D \sum \frac{\partial A_{\alpha}}{\partial x_{\alpha}} .$$

(**Jacobi**, this journal, Bd. 27, pp. 243. The derivation above is a generalization of the method by which **Jacobi** based a special case of the theorem above in *loc. cit.*, pp. 203.)

If  $A_1, ..., A_n$  are linear differential expressions in an undetermined function, or several linear differential expressions in several undetermined functions, then:

$$A_{\alpha} = \sum_{\beta} A_{\beta} \frac{\partial x'_{\alpha}}{\partial x_{\beta}} \qquad (a = 1, 2, ..., n)$$

will also be linear differential expressions.

Now, the differential expressions A(u), A'(u),  $A_v(u, v)$  that enter into equation (21) might go to P(u), Q(u), and  $R_v(u, v)$  by the introduction of new variables. One would then have:

$$v P(u) - u Q(v) = \sum \frac{\partial R_v(u, v)}{\partial x_v} = \frac{1}{D} \sum \frac{\partial (R'_v(u, v) \cdot D)}{\partial x'_v},$$

in which  $R'_{\nu}(u, v)$  are bilinear differential expressions, and the determinant *D* of the substitution is a well-defined function of  $x'_1$ , ...,  $x'_n$ . If one multiplies by *D* and replaces *v* with *v* / *D* then one will get:

$$v P(u) - u\left(Q\left(\frac{v}{D}\right) \cdot D\right) = \sum \frac{\partial}{\partial x'_{v}} \left(R'_{v}\left(u, \frac{v}{D}\right) \cdot D\right).$$

As a result,  $Q\left(\frac{v}{D}\right) \cdot D$  will be the adjoint expression to P(u).

Zurich, in February 1877.