

On polarized electron waves

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I. Generalities about the polarization of electron waves.

A paper by **Landé** ⁽¹⁾ on the polarization of matter waves appeared recently in “Naturwissenschaften” in which he attempted to interpret the most important optical phenomena that correspond to the polarization of light in the context of matter waves. In it, **Landé** discussed the production of “linearly-polarized waves,” as well as the transformation of linear polarization into “circular,” and in general “elliptic,” in which the quotation marks mean that the concept that arises in optics must be given a suitable reinterpretation in terms of the properties of the matter in question.

The following investigation would like to once more discuss and computationally formulate the questions that arise in connection with the representation, verification, and transformation of polarized electron waves. In it, the line of reasoning that was published already from different viewpoints and in other contexts of sub-problems of our situation will be repeated briefly, to the extent that is necessary.

Before we go into the details of the situation, it might be permissible to discuss a closely-related preliminary issue that one often hears in conversation that is also taken from a remark in the aforementioned **Landé** paper: Does it not contradict the theorem that **N. Bohr** formulated and presented in an Appendix to a paper of **N. F. Mott** ⁽²⁾ ⁽³⁾

⁽¹⁾ **A. Landé**, Naturwissensch. **17** (1929), 634.

⁽²⁾ **N. F. Mott**, Proc. Roy. Soc. **124** (1929), 425.

⁽³⁾ Cf., also **F. Knauer**, Zeit. Phys. **59** (1930), 807.

that originated in Copenhagen on the infeasibility of the usual **Stern-Gerlach** experiment with electrons that each prove to be polarized electron waves in their consequences? In fact, this incorrect interpretation has been expressed repeatedly in physical circles. However, in regard to it, one must establish: From the outset, the **Bohr** argument does not intend to assert the impossibility of polarizing electron waves, but rather to show that spin cannot be verified as a property of the electron particle, or more precisely, a *point electron* (*). In order to make that assertion more precise, let us discuss a second one, along with its related distinction.

Any experiment with elementary particles can be explained only with the assistance of the wave and particle nature of matter if the phenomena that are possible for it are to be considered *completely*. For example, in experiments with electron diffraction, on the one hand, it is naturally the wave picture that will be necessary if one wishes to explain the positions of the diffraction maxima, but on the other hand, that will not suffice when one wishes to observe the point-wise darkening of the registration plate that takes the form of a discrete elementary record, in addition to the directional distribution of intensities. The particle picture of that elementary process is also necessary for an understanding of it, moreover. Two different aspects of the same experiment then require that one must consider two contradictory basic pictures – viz., the continuum and the discontinuum – in order to understand it. In that way, one will find that a contradictory exclusion exists that corresponds to the conflict between the two basic pictures: One will always find that *either* one *or* the other of the classical pictures is useful; whenever the first one proves to be useful, the other one will break down. For example, if one would like to explain the gradual decline in the intensity at the edge of a shadow then the particle picture will fail. **Heisenberg**'s uncertainty relation ensures only that, at the very least, that picture will not contradict that decline, but one cannot use that relation to explain it precisely. By contrast, if one would like to explain the discrete collision of an incident elementary particle then, as is known, the wave picture will fail, because it leaves the endpoint indeterminate, it does not yield the necessary concentration of energy, etc., and merely the static interpretation of the wave can avert the direct contradiction, so one can at least speak of a possible conceptual system.

Now, in most experiments, we do not observe the entire phenomenon that occurs, but rather from the outset, we select certain interesting aspects of it to be worthy of observation and freely, but also very arbitrarily, renounce observing the other ones. For example, in most experiments that are concerned with intensity, we pass over establishing the transmitted energy that is produced by elementary processes, which has the consequence that we can get by on the basis of the wave picture by itself. The way that one distinguishes a continuous wave from a volley of projectiles will – intentionally – not be discussed; we will perform the experiment *from the wave standpoint*. On the other hand, there are experiments that are performed *from the particle standpoint*. It is quite clear that the observer (if one is present) will take that attitude with the **Wilson** photographs of α and β particles, scintillation counts, establishing coincidences, *et al.* However, the particle standpoint will also be assumed in a less transparent way – e.g., in the **Stern-Gerlach** experiment in the previously-described implementation – as it was for

(*) The author thanks **E Fues** for his participation in the discussions of the following interpretation of **Bohr**'s theorem that took place in the past Spring at the **Bohr** Institute in Copenhagen. Prof. **N. Bohr** was kind enough to read the draft of this section prior to publication.

the investigation of rays of silver atoms. Namely, one easily sees that this experiment no longer states that, in principle, *two* silver atoms – or if you wish, *a single* one – were found to pass through (naturally, if one ignores the practical impossibility that is due to insufficient intensity). That is connected with the fact that in the interpretation of the experiment, merely a point-to-point mapping by rays is observed, so the standpoint of geometrical optics – viz., the particle standpoint – will be assumed. Nothing will be observed of the diffraction phenomena that occur in addition, since they lie outside the scope of our interest.

Now, according to **Bohr**, such experiments “from the particle standpoint” will never lead to the verification of spin for actual elementary particles – i.e., protons and electrons – because the **Heisenberg** uncertainty of each particle experiment is always larger than the experimental effect that one would expect from spin. It is known that for the **Stern-Gerlach** experiment, that is because if one desires to turn off the prejudicial **Lorentz force** for the electron then one will require an aperture so narrow that diffraction will necessarily appear, and with it, the particle picture will disappear.

By contrast, **Bohr**’s theorem does not at all forbid a partial polarization of the electron wave that passes through the **Stern-Gerlach** apparatus. To begin with, one should understand polarization to mean simply the idea that a polarized electron wave will react with an anisotropic obstacle (e.g., a directed atom – i.e., one that emerges from the **Stern-Gerlach** apparatus) in such a way that it depends upon the angle between the atom and the direction of the ray, as well as upon the azimuth of the location of the atom with respect to the direction of the ray. However, it must be stressed that the latter dependency will not suffice to describe the total polarization of an electron wave, but that (from the work of **Pauli**, **Darwin**, *et al*) a vector that can assume more than just transverse positions will be needed in order to characterize it.

Theoretically, the **Dirac** equation – in contrast to the **Schrödinger** equation – initially yields only totally-polarized states of oscillation. Just as one naturally deals theoretically with light in optics by a subsequent averaging over polarized electromagnetic waves, here, one must describe the experimentally-given “natural” matter waves by means of suitable superpositions of phase-independent solutions of the **Dirac** equation ⁽⁴⁾. One can also use a terminology that has recently become customary and say: The “natural” state of matter is represented theoretically by a *mixture of pure states*; that is, of individually-polarized wave-trains.

II. The Dirac current and its polarization state in vector notation.

We shall first write down **Dirac**’s system of equations:

$$[s + \beta mc + a_1 p + a_2 q + a_3 r] \psi = 0, \tag{1a}$$

$$s = -\frac{\hbar}{i} \frac{1}{c} \frac{\partial}{\partial t} + \frac{e}{c} V, \quad p = \frac{\hbar}{i} \frac{\partial}{\partial x} + \frac{e}{c} A_x, \quad q = \frac{\hbar}{i} \frac{\partial}{\partial y} + \frac{e}{c} A_y, \quad r = \frac{\hbar}{i} \frac{\partial}{\partial z} + \frac{e}{c} A_z,$$

⁽⁴⁾ C. G. Darwin, Proc. Roy. Soc. **120** (1928), 621.

(the electron charge is $-e$), and with the matrices that **Dirac** employed, it can be specified by:

$$\begin{aligned}
(s - mc) \psi^1 + (p - i q) \psi^4 + r \psi^3 &= 0, \\
(s - mc) \psi^2 + (p + i q) \psi^3 - r \psi^4 &= 0, \\
(s + mc) \psi^3 + (p - i q) \psi^2 + r \psi^1 &= 0, \\
(s + mc) \psi^4 + (p + i q) \psi^1 - r \psi^2 &= 0;
\end{aligned} \tag{1b}$$

h means **Planck**'s constant, divided by 2π . In order to avoid deviating from the previously-existing eigenfunction indices, the indices that distinguish the four **Dirac** components should always be written above. Function symbols with no upper index will always stand for the whole four-component **Dirac** function. The Z -coordinate in (1b) is distinguished by the choice of matrices. The sign of β (as compared to the sign that **Dirac** used) is chosen such that the 1 and 2 components of ψ will be large in comparison to the 3 and 4 components. The components of the associated Dirac **current** will then read:

$$\begin{aligned}
j_x &= -\bar{\psi} a_1 \psi = -(\bar{\psi}^3 \psi^2 + \bar{\psi}^4 \psi^1 + \bar{\psi}^2 \psi^3 + \bar{\psi}^1 \psi^4), \\
j_y &= -\bar{\psi} a_2 \psi = i(\bar{\psi}^3 \psi^2 - \bar{\psi}^4 \psi^1 - \bar{\psi}^2 \psi^3 + \bar{\psi}^1 \psi^4), \\
j_z &= -\bar{\psi} a_3 \psi = -(\bar{\psi}^3 \psi^2 - \bar{\psi}^4 \psi^1 + \bar{\psi}^2 \psi^3 - \bar{\psi}^1 \psi^4).
\end{aligned} \tag{2a}$$

(We would like to understand the product of two **Dirac** functions of the same variables to always mean the “scalar product”; e.g., $\bar{\psi}\psi$ will then stand for $\bar{\psi}^1\psi^1 + \bar{\psi}^2\psi^2 + \bar{\psi}^3\psi^3 + \bar{\psi}^4\psi^4$). When j is multiplied by $-e c$, that will give the electrical current.

We shall now consider a stationary state whose time dependency is given by $e^{-iEt/h}$ and abbreviate $(E + eV) / c$ by C . ψ^3 and ψ^4 can then be expressed in terms of ψ^1 and ψ^2 in the expression (2a) for the current by means of the last two equations in (1b). A somewhat-laborious intermediate calculation will yield the simple result that:

$$C j = \frac{h}{i} (\bar{\psi}^1 \nabla \psi^1 - \psi^1 \nabla \bar{\psi}^1 + \bar{\psi}^2 \nabla \psi^2 - \psi^2 \nabla \bar{\psi}^2) + 2 \frac{3}{c} \mathfrak{A} (\bar{\psi}^1 \psi^1 + \bar{\psi}^2 \psi^2) - h \text{ rot } \mathfrak{M}, \tag{2b}$$

in which \mathfrak{M} (at first, as merely an order of magnitude) means the vector (⁵):

(⁵) **W. Gordon** [Zeit. Phys. **50** (1928), 630] has already given a more general decomposition of the **Dirac** current. Here, we have distinguished the time coordinate and thus renounced relativistic symmetry, so for us, an electrical component of the *dipole moment* six-tensor will not appear explicitly. The position-dependent factor C will replace it, and we will gain the computational advantage that only ψ^1 and ψ^2 will be relevant, so ψ^3 and ψ^4 will not need to be included, at all. Similarly, one can also express a non-stationary current in terms of only ψ^1 and ψ^2 when one develops it into simply-periodic, time-dependent eigenfunctions without the electric moment appearing explicitly. Such formulas for the current will appear informally in the relativistic formulas of the perturbation calculations in Section 3.

$$\begin{aligned}
\mathfrak{M}_x &= -(\bar{\psi}^1 \psi^2 + \bar{\psi}^2 \psi^1), \\
\mathfrak{M}_y &= i (\bar{\psi}^1 \psi^2 - \bar{\psi}^2 \psi^1), \\
\mathfrak{M}_z &= -(\bar{\psi}^1 \psi^1 - \bar{\psi}^2 \psi^2).
\end{aligned} \tag{2c}$$

That expression for the current is relativistically rigorous, and is completely equivalent to (2a) for stationary states. With the help of the transformation formulas for the ψ that **Darwin** ⁽⁶⁾ gave, one convinces oneself that \mathfrak{M} behaves like a vector under spatial coordinate rotations. The entire current vector (viz., $-ec \mathbf{j}$, together with the density $-e\bar{\psi}\psi$) defines a relativistic four-vector with respect to the **Lorentz** transformation. It is that property in particular that ensures the special choice of the **Dirac** current expression, since the continuity equation alone would allow the creation or annihilation of arbitrary current vortices. All electromagnetic field effects can be derived from the current $-ec \mathbf{j}$ and the electric charge density $-e\bar{\psi}\psi$, to the extent that an outline of quantum electrodynamics is recognizable ⁽⁷⁾.

The first bracket in (2b) represents the sum of two **Schrödinger** currents that correspond to the two degrees of freedom, and that also add without interference by the phase coupling of ψ^1 and ψ^2 . The second one, which is the current component that is proportional to the vector potential and the “rest density” (which is called that because the components $\bar{\psi}^3 \psi^3$ and $\bar{\psi}^4 \psi^4$ in the density $\bar{\psi}\psi$ will vanish for an electron at rest, and are otherwise a relativistic correction in order of magnitude), is known from the **Klein-Gordon** formulation of the **Schrödinger** equation. In the third place in (2b), one finds the rotation of a vector \mathfrak{M} , and from (2c), its direction can be obtained from the phase difference and amplitude ratios of ψ^1 and ψ^2 at each point. The **Dirac** current differs from the **Schrödinger** one by the system of current vortices that is coupled to it, which are expressed by the spin here, even in the non-relativistic approximation. If one would like to read off relativistically-rigorous statements from (2b) then one would have to observe the position dependency of C (since it contains V), in addition. According to **Darwin** ⁽⁶⁾, the direction of \mathfrak{M} is connected with the complex behavior of ψ uniquely by:

$$\frac{\psi^1}{\psi^2} = -e^{-i\omega} \tan \frac{\vartheta}{2}. \tag{3}$$

(ϑ : Inclination with respect to the z -axis. ω : Azimuth around the z -axis)

We shall now abandon relativistic rigor and thus arrive at the possibility of interpreting \mathfrak{M} directly as the magnetic polarization, and in that way as something that characterizes the eigen-direction of the wave that was mentioned in Section 1 ^(*). With that step, we go to the approximation that Darwin used in his first paper ⁽⁶⁾. Like him,

⁽⁶⁾ **C. G. Darwin**, Proc. Roy. Soc. **118** (1928), 654.

⁽⁷⁾ **W. Heisenberg** and **W. Pauli**, Zeit. Phys. **56** (1929).

^(*) As one would expect, in the non-relativistic approximation, the direction of \mathfrak{M} coincides with the direction of the density of mechanical spin impulse moment. (Cf., **Gordon**, *loc. cit.*)

we shall derive the magnetic moment m of a closed system that arises from the current vortices. When we simultaneously replace C with $2mc$, we will obtain, from known formulas of electrodynamics and vector analysis:

$$m = \frac{h}{2mc} \cdot ec \cdot \frac{1}{2c} \int_{\text{Vol.}} [\mathbf{r} [\nabla \mathfrak{M}]] d\tau = \frac{eh}{4mc} \left(\int_{\text{Vol.}} 2\mathfrak{M} d\tau + \int_{\text{Surf.}} (\mathbf{r} \mathfrak{M}) d\mathfrak{f} - \int_{\text{Surf.}} \mathfrak{M} (\mathbf{r} d\mathfrak{f}) \right). \quad (4a)$$

When \mathfrak{M} vanishes faster than $1/r^3$ as one goes outward, only the volume integral will remain. The component $\frac{ech}{2mc} \text{rot } \mathfrak{M}$ of the current will then act externally like a magnetic dipole of strength:

$$m = \frac{eh}{2mc} \int \mathfrak{M} d\tau. \quad (4b)$$

If the eigenfunctions are normalized to 1 then one will get a magnetic spin moment of precisely one **Bohr** magneton. The vanishing of the surface integral in (4a), and therefore the validity of (4b), is probably always guaranteed for discrete states of oscillation. For free electrons, one can demand the vanishing of the surface integral by **Darwin's** (⁴) process with the Ansatz of a wave packet that drops off like an e -function as one goes outward. The **Dirac** current vortex of a closed system will then act externally like a magnetic dipole of strength one **Bohr** magneton.

That result only seems to contradict the theorem that **Bohr** presented, which was discussed in Section I. Indeed, the effect of the field of the wave packet takes on a certain well-defined addition that originates in the spin. However, if one attempts to ascribe that effect to a magnetic electron, contrary to the **Bohr** postulate, then one will pose the irresolvable question of where the particle is localized inside of the wave packet. Even when the wave packet expresses the attainable maximum in its position and impulse determination, that inaccuracy will bring with it an even greater inaccuracy in the distribution between the **Biot-Savart** effect and the effect of the spin on the field that is greater than the entire addition that originates in the spin. For that reason, it is impossible to deduce the eigenmoment of a particle from any observation of its field effects. By contrast, the component of the field that originates in the polarization of the wave must be experimentally determinable, since it is, in principle, possible to compare a polarized wave with an unpolarized wave of the same position and direction of propagation.

If one would like to make any statement about the breakdown of magnetic measurements in the interior of travelling waves then one would do best to go back to the expression (2b). Naturally, the field of the vortex part of the current will then have only a static meaning, to the same extent as the field of the **Schrödinger** part. It should therefore not be amazing when the current vortex that the spin describes in a travelling wave takes on an extension that must be surprising when one has no static interpretation for it. For the spherical waves, e.g., if one assumes a polarization in, say, the z -axis ($\psi^1 = 0$, $\psi^1 = e^{ikr}/r$), the spin will imply a circular moment around the z -axis. The density of the current vortex will be (up to a factor):

$$-j_x = \frac{2y}{r^4}, \quad j_y = \frac{-2x}{r^4}, \quad j_z = 0.$$

Its intensity then drops off with $1 / r^3$ as one goes outward, while the outbound current drops off with $1 / r^2$. In Section 4, we will see that the vector \mathfrak{M} takes on an entirely different meaning for travelling waves, namely, a statement about the dependency of intensity distribution upon the direction of \mathfrak{M} for a **Stern-Gerlach** or collision experiment, or more generally, an analyzer experiment. With that justification, we also employ the vector \mathfrak{M} here to characterize the polarization state of a travelling wave in any spatial domain.

In contrast to optics, one must then give the direction of polarization by an arrow and not a line. The common element to light and matter waves is the existence of two independent waves by which the polarization state of the entire oscillation is established in terms of their phase difference and amplitude ratio. By equating the amplitudes and phase means [from (2c), \mathfrak{M} will then vanish], one will then obtain natural waves in both case, and therefore obtain “mixtures” from the “pure states.” In the optical case, if one thinks of just one or the other wave as being stimulated then that will correspond to two mutually-perpendicular directions of polarization. By contrast, in the case of matter waves, $\psi^1 = 0$, as well as $\psi^2 = 0$, will both correspond to a polarization in the z -direction, but with the opposite signs.

Except for that distinction, things point strongly to a fundamental difference between \mathfrak{M} and the optical polarization vector \mathfrak{E} (\mathfrak{H} , resp.). Whereas that will imply the existence of an immediate geometric connection with the vectorially-envisioned process of oscillation, in such a way that for two interfering waves for which \mathfrak{E} and \mathfrak{B} add together, there will exist for \mathfrak{M} either an intuitive relationship to the oscillations of the quantity ψ (which one refers to as having the transformation properties of at most a tensor of half-integer rank, but not a vector) or the capacity to superimpose the quantities \mathfrak{M} , which are quadratic in ψ , for two interfering waves. In that fact, one will find the basis for the reason why previous work on spin has favored another way of characterizing the polarization state (namely, being given the amplitude ratios and phase differences, you might say). It was only when **Dirac**'s theory gave the result that a contribution to the current could be derived directly from \mathfrak{M} ⁽⁶⁾ that an intuitive physical meaning could be given to that idea.

III. A suggestion for the production of polarized electron waves.

Now, how should the polarization of an electron wave be produced, in light of what was said? What we called the **Landé** argument in Section I referred to the possibility of a **Stern-Gerlach** experiment, and it will, in principle, still exist even when it does not, as was assumed there, lead to two separate bundles of rays with total linear polarization, but only to a blurry diffraction pattern with partial polarization at the edges. However, it is not necessary to employ an inhomogeneous magnetic field, since an inhomogeneous

electric field – e.g., a Coulomb field – will also suffice, even though the effect will then have a relativistic order of magnitude, and will then be observable only for fast electrons. That is what **Mott** ⁽²⁾ did quite thoroughly in his Copenhagen paper. From the experiments of **Rupp** ⁽⁸⁾ that were published just now, the latter author seemed to have achieved a polarization in that way, while **Chase** ⁽⁹⁾ could determine only a vague suggestion of the effect.

However, the use of inhomogeneous fields is not the only theoretically-possible way of arriving at polarization, since there is still another one that is probably just as promising. It is the *release of directed electrons from an ensemble of particles that is already otherwise oriented*. One can employ the atoms of a ray that is emanated by the **Stern-Gerlach** experiment; however, it is perhaps unnecessary to appeal to such a complicated experiment. *As Heisenberg's theory of ferromagnetism teaches us, nature already has a very convenient supply of oriented colliding pairs in the form of the free conduction electrons in a magnet* ^(*). The ensemble of particles in question will be depolarized directly or indirectly by such a collision experiment, so polarization will not actually be produced, but only transmitted. One can employ the known methods for inducing it; e.g., photo effect, extraction by high fields, electron collision. This principle for the production of polarized electron waves differs from the aforementioned method of inhomogeneous fields insofar as the effect will also have a non-relativistic order of magnitude when one employs electrical perturbing fields. The electron spin then acts in a manner that is similar to the ortho-para-splitting of spectra or the **Heisenberg** picture for ferromagnetism. It will enter into calculations only indirectly when one fixes the symmetry and orthogonality properties of the waves.

We shall now discuss computationally whether, and to what extent, it is possible to separate the electrons that are present in the atomic bond by perturbations without changing their polarization state. Let a stationary state of oscillation be given in which the direction of m is known; e.g., a hydrogen atom in the ground state that passes through the **Stern-Gerlach** apparatus, and is that once more found outside of the magnetic field. The oscillation function ψ^i , which should include the time factor $e^{-iE_i t/\hbar}$, generally obeys the differential equation:

$$\left[\frac{E_i + eV_0}{c} + \beta mc + a_1 p_0 + a_2 q_0 + a_3 r_0 \right] \psi_i = 0.$$

We now introduce a perturbation:

$$\frac{eV}{c} + a_1 \frac{e}{c} A_x + a_2 \frac{e}{c} A_y + a_3 \frac{e}{c} A_z$$

and, as usual, develop the perturbed eigenfunctions into unperturbed solutions with time-varying coefficients. We will get:

⁽⁸⁾ **E. Rupp**, Naturwiss. **18** (1930), 207.

⁽⁹⁾ **C. T. Chase**, Phys. Rev. **34** (1929), 1069.

^(*) The idea of using ferromagnetic media, not as magnet fields, but as oriented collision pairs, emerged from a conversation between **E. Fues** and **W. Groth**.

$$\frac{\hbar}{i} \frac{1}{c} \frac{da_m}{dt} = \sum_n a_n(t) \left[\int \frac{eV}{c} \bar{\psi}_m \psi_n d\tau + \frac{e}{c} \int (A_x \bar{\psi}_m a_1 \psi_n + A_y \bar{\psi}_m a_2 \psi_n + A_z \bar{\psi}_m a_3 \psi_n) d\tau \right]. \quad (5a)$$

The sum over n also extends over the continuous spectral domain. Scalar products of the **Dirac** functions appear inside the integral, so one must sum over the (omitted) spin indices. Now, the transition current that appears inside the second integral can be expressed similarly in terms of only ψ^1 and ψ^2 , like the stationary current that was considered in Section 2. For the present purposes, it will suffice to initially set:

$$\frac{E_n + eV_0}{c} + mc \sim \frac{E_m + eV_0}{c} + mc = C \quad (\sim 2mc).$$

We then write (5a) as:

$$\frac{\hbar}{i} \frac{1}{c} \frac{da_m}{dt} = \sum_n a_n(t) \left[\frac{e}{c} \int V \rho_{mn} d\tau - \frac{e}{c} \int \mathfrak{A} (j'_{mn} + j''_{mn}) d\tau + \frac{e}{c} \int \frac{\hbar}{C} \mathfrak{A} [\nabla \mathfrak{M}_{mn}] d\tau \right]. \quad (5b)$$

That means:

$$\begin{aligned} \rho_{mn} &= \bar{\psi}_m^1 \psi_n^1 + \bar{\psi}_m^2 \psi_n^2, \\ j'_{mn} &= \frac{\hbar}{iC} (\bar{\psi}_m^1 \nabla \psi_n^1 - \psi_m^1 \nabla \bar{\psi}_n^1 + \bar{\psi}_m^2 \nabla \psi_n^2 - \psi_m^2 \nabla \bar{\psi}_n^2), \quad (\dagger) \\ j''_{mn} &= \frac{2}{C} \cdot \frac{e}{c} \mathfrak{A}_0 (\bar{\psi}_m^1 \psi_n^1 + \bar{\psi}_m^2 \psi_n^2), \\ \mathfrak{M}_{mn} &: -(\bar{\psi}_m^1 \psi_n^1 + \bar{\psi}_m^2 \psi_n^2), \\ &\quad i (\bar{\psi}_m^1 \psi_n^2 - \bar{\psi}_m^2 \psi_n^1), \\ &\quad -(\bar{\psi}_m^1 \psi_n^1 - \bar{\psi}_m^2 \psi_n^2). \end{aligned} \quad (5c)$$

For better clarity, we write the **Schrödinger** part of the current $-j'_{mn}$ in the form $\mathfrak{S} \rho_{mn}$, in which \mathfrak{S} means a vector operator that makes a density out of the associated current, so e.g.:

$$\mathfrak{S} \bar{\psi}_m^1 \psi_n^1 = \bar{\psi}_m^1 \frac{\partial \psi_n^1}{\partial x} - \bar{\psi}_n^1 \frac{\partial \psi_m^1}{\partial x}.$$

When \mathfrak{M} vanishes on the boundary of the domain of integration, the integral over the third part of the current – viz., the vorticial part – can be brought into a form that exhibits its character as the energy of a spin interaction with the perturbing field. When we put $1/C$ in front of the integral, we will get the more intuitive form for (5b):

([†]) When $\nabla \mathfrak{A} = 0$ and $\bar{\psi}_m \psi_n$ vanishes at infinity, one will have $\int \mathfrak{A} \bar{\psi}_m \nabla \psi_n d\tau = - \int \mathfrak{A} \psi_n \nabla \bar{\psi}_m d\tau$.

$$\frac{h}{i} \frac{1}{c} \frac{da_m}{dt} = \frac{e}{c} \sum_n a_n(t) \left[\int \left(V - \frac{h}{Ci} \mathfrak{A} \mathfrak{S} - \frac{2e}{C} \frac{e}{c} \mathfrak{A} \mathfrak{A}_0 \right) \rho_{mn} d\tau + \frac{h}{C} \int \mathfrak{H} \mathfrak{M}_{mn} d\tau \right]. \quad (5d)$$

We assume that at time $t = 0$, only the states whose m lies along the positive z -axis will be excited. In all of the simple cases in which the direction of \mathfrak{M} is constant, from (4b) and (2c), that will mean that $\psi_n^1 = 0$. With no magnetic field, and in the classical approximation, the amplitudes of ψ^1 and ψ^2 will indeed differ by only a constant. The same thing will be true with relativistic rigor for, e.g., a hydrogen atom in the ground state outside of the magnetic field.

We now ask about the state transitions with opposite spins that the perturbation might produce, so for them, one would have $\psi_m^2 = 0$. Formally, (5c) shows that the transition density ρ_{mn} and the transition current $\mathfrak{S} \rho_{mn}$ will vanish for those transitions. The associated “transition spin” \mathfrak{M}_{mn} has no Z -component, but the X -component is $-\bar{\psi}_m^1 \psi_n^2$ and the Y -component is $i \bar{\psi}_m^1 \psi_n^2$. For that reason, the last perturbing integral in (5d) will equal zero only when the perturbing magnetic field strength \mathfrak{H} lies along the $\pm Z$ -direction. We then see that it can be arranged that (at least, in the non-relativistic approximation) the only oscillations of the spectrum (including the continuous part) that will be excited by the perturbation will have *the same* spin orientation as the initially-excited oscillation. If one produces directed electrons from directed atoms (a magnet, resp.) then one can theoretically attain 100% polarization by the use of polarized light whose magnetic vector oscillates in the direction of magnetization. However, one estimates that, in order of magnitude, the part of the “transition spin” also remains small compared to the “transition current” for an arbitrary position of the light vectors, as long as the wave length of the light is large compared to the **De Broglie** wave length of the emitted photo-electron. For that reason, one can abandon the use of polarized light in practice.

Let the relativistic form of the perturbation equation be given here, despite the fact that we shall not go into a discussion of it, since it can be useful for relativistically-rigorous perturbation calculations that relate to the **Dirac** equation. It reads:

$$\begin{aligned} \frac{h}{i} \frac{1}{c} \frac{da_m}{dt} = & \frac{e}{c} \sum_n a_n \left[\int \left(V - \frac{C_n + C_m}{2C_n C_m} \mathfrak{A} \mathfrak{S} - \frac{C_n + C_m}{C_n C_m} \frac{e}{c} \mathfrak{A} \mathfrak{A}_0 \right) \rho_{mn} d\tau \right. \\ & + \int \frac{C_n + C_m}{2C_n C_m} h \mathfrak{A} [\nabla \mathfrak{M}_{mn}] d\tau + \int V \rho_{mn} d\tau + \int \frac{C_n - C_m}{C_n C_m} \frac{h}{i} \mathfrak{A} \nabla \rho_{mn} d\tau \\ & \left. + \int \frac{e}{ic} \frac{C_n - C_m}{C_n C_m} \mathfrak{A} [\mathfrak{A}_0 \mathfrak{M}_{mn}] d\tau - \int h \frac{C_n - C_m}{2C_n C_m} \mathfrak{A} [\mathfrak{S} \mathfrak{M}_{mn}] d\tau \right]. \end{aligned}$$

This is completely equivalent to (4a). We must now distinguish between $C_n = \frac{E_n + eV_0}{c}$

+ mc and $C_m = \frac{E_n + eV_0}{c} + mc$, and generally consider the position dependency of V_0 .

The integral with $C_n - C_m$ gives the relativistic corrections that appear in comparison to (5b). \mathfrak{S} means the previously-defined current operator, and ρ'_{mn} means the relativistic supplementary density $\bar{\psi}^3 \psi^3 + \bar{\psi}^4 \psi^4$. (That term will not simplify when one rewrites it in terms of ψ^1 and ψ^2 .)

If we revert to the non-relativistic approximation then we will see that purely electric perturbations will extract directed electrons. The – perhaps experimentally-promising – method of releasing the electrons from the magnet by the use of high electric fields can be regarded theoretically most simply with the picture that was used by **Nordheim** and **Fowler** ⁽¹⁰⁾ of waves that are contained in a box and pass through the potentials walls. The question of whether the electrons preserve their polarization when going through the potential jump will be treated in Section VI.

Another possibility consists of atomic electrons being extracted by the field of an approaching electron that is directed toward the oriented atom. However, the polarization of the wave that is emitted from the atom in question will naturally be reduced by the scattered primary electrons that are counted with it. The electron collision also occupies an exceptional place, insofar as the ionization energy does not at all need to be applied, since atomic electrons can appear in the scattered waves by exchange.

IV. A principle for verifying the polarization state of a travelling wave.

For a closer examination of the behavior of electrons colliding with directed atoms, we apply the **Born** collision calculation to our problem, and indeed while considering the exchange effects (**Pauli** principle, resp.), as **Oppenheimer** ⁽¹¹⁾, and in connection with that paper, **Mott** ⁽¹²⁾, have done already. The calculations implement the concluding argument of Section 3; however, they will, at the same time, yield the principle of an “analyzer” for the polarized electron waves. We shall then obtain our results somewhat more simply than the cited authors.

We consider the collision of a free electron from a hydrogen atom. The **Dirac** equation for this two-body problem reads:

$$\left[L_1 + L_2 + \frac{e^2}{r_1} + \frac{e^2}{r_2} - \frac{e^2}{r_{12}} + E_n + E_\kappa \right] (\psi_{n\kappa} + \zeta) = 0.$$

L_1 and L_2 are the operators of the force-free **Dirac** equation [multiplied by c ; cf., eq. (1)] that act upon the coordinates (including the spin coordinate), E_n is an eigenvalue in the discrete part of the hydrogen spectrum, and E_κ is the eigenvalue of the traveling wave.

⁽¹⁰⁾ **L. Nordheim** and **R. H. Fowler**, Proc. Roy. Soc. **119** (1928), 173.

⁽¹¹⁾ **J. R. Oppenheimer**, Phys. Rev. **32** (1928), 361.

⁽¹²⁾ **N. F. Mott**, Proc. Roy. Soc. **125** (1929), 222.

$\psi_{n\kappa}$ means the completely (i.e., under simultaneous exchange of the position and spin coordinates) antisymmetric starting function $\varphi_n(1)\psi_\kappa(2) - \varphi_n(2)\psi_\kappa(1)$ (it consists of 16 **Dirac** components), in which φ_n represents a hydrogen eigenfunction, and ψ_κ , a travelling wave. With the abbreviations $\frac{e^2}{r_1} - \frac{e^2}{r_{12}} = V_{12}$ and $\frac{e^2}{r_2} - \frac{e^2}{r_{12}} = V_{21}$ for the perturbing energies, one will get the following differential equation for the perturbing function ζ :

$$V_{21}\varphi_n(1)\psi_\kappa(2) - V_{12}\varphi_n(2)\psi_\kappa(1) + \left[L_1 + L_2 + \frac{e^2}{r_1} + \frac{e^2}{r_2} - \frac{e^2}{r_{12}} + E_n + E_\kappa \right] \zeta = 0.$$

One sees that the function ζ must be antisymmetric in the coordinates, since the term that is free of ζ is antisymmetric, and the differential operator is symmetric. It is now irrelevant whether one arrives at a solution by developing ζ in terms of $\varphi(1)$ with coefficients that depend upon 2 or in terms of $\varphi(2)$ with coefficients that depend upon 1. Both developments must naturally give the same solution ζ that is antisymmetric in the coordinates. One does not see the developments of the antisymmetry in 1 and 2 and its identity without any further assumptions, but one will easily convince oneself qualitatively that terms in the discrete spectrum in the development in $\varphi(2)$ will be contained in the continuous spectrum of the development in $\varphi(1)$, and conversely; the antisymmetry of each individual development will also arise that way⁽¹³⁾.

We choose, say, the development in terms of $\varphi(2)$ with the coefficients $\chi_{(n)}(1)$, and neglect the product of perturbing energy times perturbing function, as usual, and that will give:

$$\begin{aligned} \sum_m \int [V_{12}\varphi_n(2)\psi_\kappa(1) - V_{21}\varphi_n(1)\psi_\kappa(2)] \bar{\varphi}_m(2) d\tau_2 \varphi_m(1) \\ = \sum_m [E_n - E_m + E_\kappa + L_1] \chi_{(m)}(1) \varphi_{(m)}(2), \end{aligned} \quad (6a)$$

so for each coefficient of $\varphi(2)$:

$$\begin{aligned} \int V_{12}\varphi_n(2)\bar{\varphi}_m(2) d\tau_2 \psi_\kappa(1) - \int V_{21}\psi_\kappa(2)\bar{\varphi}_m(2) d\tau_2 \varphi_n(1) \\ = [L_1 + E_\kappa + E_n - E_m] \chi_{(m)}(1). \end{aligned} \quad (6b)$$

We then have the well-known formula of **Born**'s scattering calculation before us, except that here each transition of the atom is associated with two source distributions for the scattered waves [as was first introduced into the **Born** scattering calculation by **Oppenheimer**⁽¹¹⁾], and equations (6b) mean the **Dirac** equations.

The inhomogeneous **Dirac** equation of the form $[E + L] \chi = \rho$ leads rigorously to the **Schrödinger** equations for the 1 and 2 components of χ :

$$[(E^2 - m^2 c^4) + c^2 h^2 \Delta] \chi^1 = (E + mc^2) \rho^1 - c(p - iq) \rho^4 - cr \rho^3,$$

⁽¹³⁾ On this, cf.: **E. Fues**, *Sommerfeldfestschrift: Probleme der modernen Physik*, Leipzig, 1928, 1, Appendix.

$$[(E^2 - m^2 c^4) + c^2 h^2 \Delta] \chi^2 = (E + mc^2) \rho^2 - c(p - iq) \rho^3 - cr \rho^4.$$

Since would like to do further calculations in the non-relativistic approximation here, we can write this as:

$$\left[(E - mc^2) + \frac{h^2}{2m} \Delta \right] \chi^1 = \rho^1, \quad (7)$$

$$\left[(E - mc^2) + \frac{h^2}{2m} \Delta \right] \chi^2 = \rho^2.$$

If we call the **Schrödinger** differential operator (multiplied by a constant) that is found in the square bracket of (7) more briefly $[S_m]$ then from (7), we will get from (6b) that:

$$[S_m] \chi_{(m)}^1(1) = \int V_{12} \varphi_n(2) \bar{\varphi}_m(2) d\tau_2 \psi_\kappa^1(1) - \int V_{21} \psi_\kappa(2) \bar{\varphi}_m(2) d\tau_2 \varphi_n^1(1), \quad (8a)$$

$$[S_m] \chi_{(m)}^2(1) = \int V_{12} \varphi_n(2) \bar{\varphi}_m(2) d\tau_2 \psi_\kappa^2(1) - \int V_{21} \psi_\kappa(2) \bar{\varphi}_m(2) d\tau_2 \varphi_n^2(1).$$

We now consider a collision for which the atom remains in its energy state (up to a possible change in the spin direction), so we set $E_n = E_m$. With no external field, we have two linearly-independent functions that correspond to the two spin attitudes in the atom, which belong to equal terms; we call them φ_α and φ_β . (The indices α and β come from n and m , in particular!) Before the collision, let the eigenfunction φ_α be excited, so we shall consider the two scattered waves that belong to the transitions $\alpha \rightarrow \alpha$ and $\alpha \rightarrow \beta$. When we drop the now-superfluous eigenfunction indices (except for α and β), we will get from (8a) that:

$$[S] \chi^1(1) = \int V_{12} \varphi_\alpha(2) \bar{\varphi}_\alpha(2) d\tau_2 \psi^1(1) - \int V_{21} \psi(2) \bar{\varphi}_\alpha(2) d\tau_2 \varphi_\alpha^1(1)$$

$$[S] \chi^2(1) = \int V_{12} \varphi_\alpha(2) \bar{\varphi}_\alpha(2) d\tau_2 \psi^2(1) - \int V_{21} \psi(2) \bar{\varphi}_\alpha(2) d\tau_2 \varphi_\alpha^2(1)$$

and

$$[S] \chi'^1(1) = - \int V_{21} \psi(2) \bar{\varphi}_\beta(2) d\tau_2 \varphi_\alpha^1(1),$$

$$[S] \chi'^2(1) = - \int V_{21} \psi(2) \bar{\varphi}_\beta(2) d\tau_2 \varphi_\alpha^2(1), \quad (8b)$$

since $\varphi_\alpha \bar{\varphi}_\beta = 0$ when φ_α and φ_β are assigned opposite spin directions. χ' means the scattered wave with the atomic spin flipped that belongs to the atomic transition $\alpha \rightarrow \beta$.

Now, in the non-relativistic approximation, the 1 and 2 components are both solutions of the same **Schrödinger** equation and differ only by constant amplitudes by which the direction of the spin is established⁽⁴⁾. We then write $\psi^1 = A\psi^0$, $\psi^2 = B\psi^0$, $\varphi_\alpha^1 = C\varphi^0$, $\varphi_\alpha^2 = D\varphi^0$, $\varphi_\beta^1 = C'\varphi^0$, $\varphi_\beta^2 = D'\varphi^0$, in which, A, B, C, D, C', D' are complex constants, and ψ^0, φ^0 mean solutions of the **Schrödinger** equation. With that, we write the solutions of (8) while dropping the trivial common constants:

$$\begin{aligned}
\chi^1 &= \frac{e^{ikr}}{r} [(C\bar{C} + D\bar{D})Af - (A\bar{C} + B\bar{D})Cg], \\
\chi^2 &= \frac{e^{ikr}}{r} [(C\bar{C} + D\bar{D})Bf - (A\bar{C} + B\bar{D})Dg], \\
\chi^{1'} &= \frac{e^{ik'r}}{r} [\quad \quad \quad - (A\bar{C}' + B\bar{D}')Cg], \\
\chi^{2'} &= \frac{e^{ik'r}}{r} [\quad \quad \quad - (A\bar{C}' + B\bar{D}')Dg],
\end{aligned} \tag{9a}$$

in which $k = mv / h$ (v : speed of the incident electron) is initially equal to k' , and f and g are defined by the following equations:

$$\begin{aligned}
f \frac{e^{ikr}}{r} &= \int \frac{e^{ikr}}{r} \int V_{12} \varphi^0(2) \bar{\varphi}^0(2) d\tau_2 \psi^0(1) d\tau_1, \\
g \frac{e^{ikr}}{r} &= \int \frac{e^{ikr}}{r} \int V_{12} \psi^0(2) \bar{\varphi}^0(2) d\tau_2 \varphi^0(1) d\tau_1,
\end{aligned} \tag{9b}$$

in which f and g are functions of the scattering angle.

One sees that the primed waves in (9), which arise from an exchange of the atomic electrons, come about only by way of the “exchange integral” g . For $g = 0$, and arbitrary C and D – hence, arbitrary atomic orientations – the outgoing waves will have the same polarization as the incoming ones.

At first, the primed and unprimed waves will be capable of interfering, but for the calculation of the scattering current, one must imagine that the slightest perturbation (perhaps by the geomagnetic field) will provoke the degeneracy $\alpha - \beta$. In fact, we think of the degeneracy, and therefore the precise equality of k and k' , as being omitted. The primed and unprimed χ do not interfere then. In addition, the 1 and 2 components do not interfere with each other, which would emerge from (2b); the total scattering current is then the sum of four incoherent currents.

If one takes A, B, C, D to be phase-independent and of equal absolute value then one will get from (9) the scattering formula for the collision of undirected electron with undirected atoms that **Oppenheimer** ⁽¹⁾ derived by starting with the **Schrödinger** equation, and **Mott** ⁽²⁾ confirmed by a similar calculation to be a solution of the **Dirac** equation in the non-relativistic approximation.

When one employs directed atoms, the scattered wave will be partially polarized in the direction of the atomic magnet. If we set $D = 0, C = 1, D' = 1, C' = 0$ in (9), and thus originally employ atoms that are polarized in the negative Z -direction, then the following outgoing electron waves will result (*):

(*) Here, we give the index 0 in order for f and g to characterize the cancellation of the once-scattered wave by a second scattering process.

$$\left\{ \begin{array}{l} r e^{-ikr} \chi^1 = A_0(f_0 - g_0), \\ r e^{-ikr} \chi^2 = B_0 f_0, \end{array} \right. \quad \left\{ \begin{array}{l} r e^{-ik'r} \chi^{1'} = -B_0 g_0, \\ r e^{-ik'r} \chi^{2'} = 0, \end{array} \right. \quad (10)$$

which will not interfere with each other, due to the lifting of the degeneracy. If the incident wave is unpolarized (so $A_0 = B_0$) and independent of the others in phase then the scattered wave (10) will have a partial polarization in the Z-direction.

The same scattering experiment will also yield a method for verifying the polarization that is possible, in principle. We again imagine an oriented scattered atom for that purpose. Let the direction of its magnetic moment be established by the polar angles $\pi - \vartheta$ and $\omega + \pi$ with respect to the Z-axis before the collision. From (3), one must then give the following values to the complex amplitudes:

$$C = \cos \frac{\vartheta}{2}, \quad C' = \sin \frac{\vartheta}{2}, \quad D = e^{i\omega} \cos \frac{\vartheta}{2}, \quad D' = -e^{i\omega} \cos \frac{\vartheta}{2}. \quad (11)$$

One also subsequently convinces oneself that $\varphi_\alpha \bar{\varphi}_\beta$ will vanish with that, which is a fact that was used in equation (9) already.

The total intensity (without the current vortex) of the four waves in eq. (9) will now be (when we multiply by r^2 , which we omit, since we would always like to approximate the spherical wave by a plane wave here, anyway):

$$I = (A\bar{A} + B\bar{B})(f\bar{f} + g\bar{g}) - \left(A\bar{A} \cos^2 \frac{\vartheta}{2} + B\bar{B} \sin^2 \frac{\vartheta}{2} + A\bar{B} \sin \frac{\vartheta}{2} \cos \frac{\vartheta}{2} e^{i\omega} + \bar{A}B \sin \frac{\vartheta}{2} \cos \frac{\vartheta}{2} e^{-i\omega} \right) (g\bar{f} + \bar{g}f). \quad (12a)$$

If we take, say, the incoming wave to be polarized in the negative Z-direction by choosing $B = 0$, $|A|^2 = 1$ then the scattered intensity I will be:

$$I = (f\bar{f} + g\bar{g} - \frac{1}{2}g\bar{f} - \frac{1}{2}\bar{g}f) - \frac{1}{2}(g\bar{f} + \bar{g}f) \cos \vartheta, \quad (12b)$$

up to a proportionality factor, so for a fixed scattering angle, it will have extrema at $\vartheta = 0$ and $\vartheta = \pi$, i.e., when the atomic magnetic is parallel and anti-parallel to the vector M of the travelling wave, resp.

In that fact, we have the basis for an experiment that allows us, in principle, to give the direction of the polarization vector for arbitrary traveling waves when we can only approximate them by a plane wave in a domain that is large compared to atomic dimensions.

We can now verify the polarization of the waves (10) in that way by a second scattering from an oriented atom. Two incoherent **Dirac** waves that are the result of the first scattering will then fall upon the atom that is oriented in the direction $\pi - \vartheta$, $\omega + \pi$. A second scattering will then yield four incoherent **Dirac** waves, each of which consists of two non-interfering components. The total intensity of the eight waves that are scattered from the two atoms will follow from (10), (11), and (12a):

$$\begin{aligned}
I = & 2(g_0 \bar{g}_0 + f_0 \bar{f}_0 - \frac{1}{2} g_0 \bar{f}_0 - \frac{1}{2} \bar{g}_0 f_0)(g \bar{g} + f \bar{f} - \frac{1}{2} g \bar{f} - \frac{1}{2} \bar{g} f) \\
& - (g_0 \bar{g}_0 - \frac{1}{2} g_0 \bar{f}_0 - \frac{1}{2} \bar{g}_0 f_0)(g \bar{f} + \bar{g} f) \cos \vartheta,
\end{aligned} \tag{13}$$

in which we think of the incoming and the once-scattered wave as having been attenuated. [Confer the paper of **Mott** ⁽¹²⁾ on this, who derived a similar formula by treating two-fold scattering as a three-body problem.]

Corresponding statements about inelastic scattering are included in (10)-(13), except that f and g will have different meanings in those cases. We shall pass over the numerical discussion, since the experiment that is at the basis for our considerations can hardly be put into engineering practice, and the behavior under the employment of a magnet in our model can only be regarded as entirely qualitative.

V. Influencing polarization by a magnetic field.

Finally, let us briefly consider the influencing of polarized electron waves by a homogeneous magnetic field. One can find a generally more qualitative discussion of this in the cited paper of **Landé** ⁽¹⁾. Our goal is to derive the behavior of the polarized electron wave in a magnetic field from the associated solutions of the **Dirac** equation by using the foregoing arguments.

In the presence of only magnetic forces for ψ^1 and ψ^2 , that will rigorously yield the second-order differential equations:

$$\left[\left(\frac{E}{c} \right)^2 - m^2 c^2 - \frac{e^2}{c^2} \mathfrak{A}^2 \right] \psi^1 + h^2 \Delta \psi^1 - \frac{e}{c} h (H_z \psi^1 - i H_y \psi^2 + H_x \psi^2) - 2 \frac{e}{c} \frac{h}{i} \mathfrak{A} \nabla \psi^1 = 0, \tag{14a}$$

$$\left[\left(\frac{E}{c} \right)^2 - m^2 c^2 - \frac{e^2}{c^2} \mathfrak{A}^2 \right] \psi^2 + h^2 \Delta \psi^2 - \frac{e}{c} h (H_z \psi^2 - i H_y \psi^1 + H_x \psi^1) - 2 \frac{e}{c} \frac{h}{i} \mathfrak{A} \nabla \psi^2 = 0.$$

For a homogeneous field in the Z -direction (so $A_x = -\frac{1}{2} H y$, $A_y = \frac{1}{2} H x$), that will become:

$$\left[\left(\frac{E}{c} \right)^2 - m^2 c^2 - \frac{e^2}{c^2} \frac{H^2}{4} (x^2 + y^2) \right] \psi + h^2 \Delta \psi \mp \frac{e h}{c} H \psi - 2 \frac{e}{c} \frac{h}{i} \mathfrak{A} \nabla \psi = 0, \tag{14b}$$

in which the upper sign will be true for ψ^1 , while the lower one shall be true for ψ^2 .

A solution for this differential equation that corresponds to circular orbits was given by **Alexandroff** ⁽¹⁴⁾. We shall adopt it in a somewhat extended form for spiral orbits when we set:

$$\psi = e^{k \ln(x+iy) - \frac{eH}{2ch} \frac{x^2+y^2}{2} + \frac{i}{h} p z}.$$

⁽¹⁴⁾ **W. Alexandroff**, Zeit. Phys. **56** (1929), 818. Ann. Phys. (Leipzig) **2** (1929), 477.

If one then starts with (14b) then one will get the determining equations for the constants k_1, p_1, k_2, p_2 :

$$\begin{aligned} \left(\frac{E}{c}\right)^2 - m^2 c^2 - 2k_1 \frac{ehH}{c} - p_1^2 - 2\frac{e}{c} hH &= 0, \\ \left(\frac{E}{c}\right)^2 - m^2 c^2 - 2k_2 \frac{ehH}{c} - p_2^2 &= 0. \end{aligned} \tag{15}$$

k is connected with the classical orbital radius a by $k = a^2 \frac{eH}{2ch}$, as one easily confirms by looking for the density maxima. The indeterminacy in k and p still leaves open the division of the available energy into the orbital energy and the energy of translation in the field direction (the inclination of the spiral orbits, resp.). We have established the axis of the cylinder upon which the orbits lie by a choice of zero points for x and y . The solution does not correspond to individual spiral orbits with well-defined starting points, but to a smeared-out band around the field direction, such that the density maxima will be distributed uniformly, and not in spirals on the surface of a cylinder. With the same right by which one always collimates the plane wave into a more-or-less broad ray, here we can, in conclusion, think of an individual spiral orbit as being collimated without having to account for that theoretically, as long as the aperture is large compared to the wave length. The pitch of the spiral must also be very large compared to the wave length, which is always the case in practice, as long as one does not approach the other limiting case of the pure circular orbit.

If we write $r e^{ir}$ for $x + iy$ in our solution then the two eigenfunctions will be:

$$\begin{aligned} \psi^1 &= A r^{k_1} e^{ik_1 \varphi} e^{-\frac{eH}{2ch} \frac{r^2}{2} + \frac{i}{h} p_1 z}, \\ \psi^2 &= A r^{k_2} e^{ik_2 \varphi} e^{-\frac{eH}{2ch} \frac{r^2}{2} + \frac{i}{h} p_2 z}. \end{aligned} \tag{16}$$

When take A and B to be real, we can construct the vector \mathfrak{M} from (2c):

$$\begin{aligned} \mathfrak{M}_x &= -2AB r^{k_1+k_2} e^{-\frac{eH}{2ch} \frac{r^2}{2}} \cdot \cos \alpha, \\ \mathfrak{M}_y &= -2AB r^{k_1+k_2} e^{-\frac{eH}{2ch} \frac{r^2}{2}} \cdot \sin \alpha, \\ \mathfrak{M}_z &= -(A^2 r^{2k_1} - B^2 r^{2k_2}) e^{-\frac{eH}{2ch} \frac{r^2}{2}}, \end{aligned}$$

with the abbreviation:

$$a = (k_2 - k_1) \varphi + \frac{1}{h} (p_2 - p_1) z, \tag{17b}$$

and the condition that follows from (15):

$$p_2^2 - p_1^2 + 2\frac{e}{c}hH(k_2 - k_1 - 1) = 0. \quad (17c)$$

In order to be able to track the behavior of a polarized wave, the direction of polarization must be given in any domain. If we assume, say, that \mathfrak{M} has a definite direction for a definite value of z that is independent of φ then we must choose $k_1 = k_2$, and thus $p_2^2 - p_1^2 = 2\frac{e}{c}hH$. With $p_0 = (p_1 + p_2) / 2$, we write this as $p_2 - p_1 = \frac{ehH}{p_0c}$, and finally:

$$a = \frac{eH}{p_0c} \cdot z. \quad (18)$$

The moment in the Z -direction is independent of z and vanishes for $A = B$. One sees from (17) and (18) that the part of \mathfrak{M} that is perpendicular to the field lines rotates uniformly around a field line with increasing z ; the period amounts to $\lambda = \frac{2\pi p_0c}{eH}$. That is also precisely the distance at which an electron with the impulse component p_0 in the field direction will run through a circular orbit of arbitrary radius once perpendicular to the field lines.

If we now consider the other limiting case – namely, $p_1 = p_2$ – which means that we are given a constant polarization in the z -direction for a well-defined value of r and φ , then from (17b), we will have $k_2 - k_1 = 1$. The spin component perpendicular to the field makes precisely one orbit in the complete circular orbit. The component in the field direction is non-zero, and its magnitude depends upon r . However, since the spin (along with the density) is noticeably non-zero only in a small range of r , one can make the Z -component almost vanish by choosing the constants A and B .

It hardly seems possible to test these statements from **Dirac**'s theory experimentally by means of scattering experiments that involve magnetic fields. If one lets the matter waves travel through a magnetic field between a polarizer and an analyzer then for a fixed position of those two the intensity that is measured behind the analyzer must change periodically with the magnetic field strength.

VI. Polarization by reflection and by passing through a potential jump (added in proof).

In Sec. III, we left open the proof that when electrons are extracted by high fields, they will keep the polarization that they had in the stationary state. We shall append the proof as a special section in connection with the picture that **Nordheim** and **Fowler** used⁽¹⁰⁾, since the question of influencing the polarization state by reflection and refraction requires at least some brief remarks. The fact that no polarization is produced by

reflection from a potential wall – in contrast to optics – was shown in a note by **Frenkel** ⁽¹⁵⁾, and was recently treated even more thoroughly by **Alexandroff** ⁽¹⁶⁾. Neither of the authors examined whether, and to what extent, the state of a given polarized wave would experience a change under reflection and refraction.

For that reason, we first once more consider the passage of a plane wave through a flat potential wall. The incident wave advances perpendicular to the Z -direction, and the potential jump from 0 to V lies in the XZ -plane, so $y = 0$. One sees from (1b) that the components of ψ obey the differential equation:

$$\left[\left(\frac{E + eV}{c} \right) - m^2 c^2 \right] \psi + h^2 \Delta \psi = 0.$$

If we likewise add the reflected wave and choose the constants in the exponent such that the boundary conditions are satisfied, then we will have $A e^{\frac{i}{h}(px+qy)}$ + $B e^{\frac{i}{h}(px-xy)}$ for the solution in the first medium and $F e^{\frac{i}{h}(px+q^*y)}$ for the wave that enters the second medium. The constants obey the equations (*):

$$p^2 + q^2 = \left(\frac{E}{c} \right)^2 - m^2 c^2,$$

$$p^2 + q^{*2} = \left(\frac{E + eV}{c} \right)^2 - m^2 c^2.$$

The 1 and 4 components (2 and 3 components, resp.) are coupled with each other by the last two equations of (1b), and indeed for the plane wave, in such a way that the 3 and 4 components differ from the 1 and 2 components by only a constant factor. With the abbreviations C and C^* for $E / c + mc$ and $(E + eV) / c + mc$, resp., we write:

$$A^3 = -\frac{1}{C}(p - iq) A^2, \quad R^3 = -\frac{1}{C}(p + iq) R^2,$$

$$F^3 = -\frac{1}{C}(p - iq^*) F^2,$$

$$A^4 = -\frac{1}{C}(p + iq) A^1, \quad R^4 = -\frac{1}{C}(p - iq) R^1,$$
(19)

⁽¹⁵⁾ **J. Frenkel**, Comptes rendus **188** (1929), 153.

⁽¹⁶⁾ **W. Alexandroff**, Zeit. Phys. **60** (1930), 387.

(*) In this section, we shall employ the symbols p and q for the constants of the plane wave (they correspond to the classical impulse components), and not as operators, as in Section I. We write them out here from (1a).

$$F^4 = -\frac{1}{C^*} (p + iq^*) F^1.$$

As **Klein** ⁽¹⁷⁾ has shown, the only boundary condition that one must impose is the continuity of all four components. One sees from our equations that the demand of continuity of the 3 and 4 components enters in place of the **Schrödinger**'s demand that the first derivative should be continuous. Since we have already chosen p and q suitably, our four boundary conditions $A + R = F$ can be written out as:

$$\begin{aligned} A^1 + R^1 &= F^1, \\ \frac{1}{C} [(p + iq) A^1 + (p - iq) R^1] &= \frac{1}{C^*} (p + iq^*) F^1, \\ A^2 + R^2 &= F^2, \\ \frac{1}{C} [(p - iq) A^2 + (p + iq) R^2] &= \frac{1}{C^*} (p - iq^*) F^2. \end{aligned} \tag{20}$$

With the abbreviations $p' = (C / C^*) p$ and $q' = (C / C^*) q^*$, we can write the result for the transmitted wave as:

$$F^1 = \frac{2iq}{p' - p + i(q' + q)} A^1, \quad F^2 = \frac{-2iq}{p' - p - i(q' + q)} A^2, \tag{21}$$

or, for real q' :

$$F^1 = \rho e^{i\varphi} A^1, \quad F^2 = \rho e^{-i\varphi} A^2, \tag{22a}$$

with:

$$\rho^2 = \frac{4q^2}{(p' - p)^2 + (q' + q)^2}, \quad \tan \varphi = \frac{p' - p}{q' + q} = \frac{p(C - C^*)}{q^* C + q C^*}. \tag{22b}$$

If one gives the polarization of the incident wave by the angles ϑ and ω from eq. (3), and thus sets:

$$A^1 = \sin \frac{\vartheta}{2}, \quad A^2 = -e^{i\omega} \cos \frac{\vartheta}{2},$$

then one will have:

$$F^1 = \rho \sin \frac{\vartheta}{2} e^{i\eta}, \quad F^2 = -\cos \frac{\vartheta}{2} e^{i(\omega - \varphi)}, \tag{22c}$$

and thus gives the polarization of the transmitted wave a different orientation. Namely, one easily convinces oneself with (2c) that a rotation of the vector \mathfrak{M} by 2φ around the physically-distinguished Z -axis has occurred. From (22b), φ will get larger the greater

⁽¹⁷⁾ **O. Klein**, Zeit. Phys. **53** (1929), 157.

that p is in comparison to q , and thus, the more obliquely the wave impinges. However, since $\frac{C - C^*}{C + C^*} \sim \frac{eV}{2mc^2}$, φ has, on the whole, only relativistic order of magnitude.

The reflected wave suffers a rotation of the direction of polarization around the same axis that has the same order of magnitude.

We now consider the passage through a potential jump (whose potential can also be greater than the kinetic energy of the impinging wave), in order to make more certain the already obvious suggestion in the result that was just derived that no changes of the polarization state will enter into consideration in it. If a rotation of \mathfrak{M} occurs that depends upon the angle of incidence of the polarized wave then the transmitted waves would all be polarized differently for the various angles of incidence on the incoming side. If one applies that picture to the extraction of electrons from a magnet by a high field then one would expect a more-or-less incomplete polarization of the outgoing wave.

Inside of the potential jump, $F e^{i(px+q^*y)/h} + G e^{i(px-q^*y)/h}$ will now represent a general solution between the planes $y = 0$ and $y = d$. On the incident side, as before, the solution will be $A e^{i(px+q^*y)/h} + R e^{i(px-q^*y)/h}$ (*), and let the outgoing wave be $B e^{i(px+q^*y)/h}$. The potential will be equal to zero to the left and right of the jump.

With the previous abbreviations, we will get the following 2×4 equations for the 1 and 2 components from the two boundary conditions:

$$\left. \begin{aligned} A^1 + R^1 &= F^1 + G^1, & (p+iq)A^1 + (p-iq)R^1 &= (p'+iq')F^1 + (p'-iq')G^1, \\ F^1 e^{iq^*d/h} + G^1 e^{-iq^*d/h} &= B^1 e^{iqd/h}, & (p'+iq')F^1 e^{iq^*d/h} + (p'-iq')G^1 e^{-iq^*d/h} &= (p'+iq')B^1 e^{iqd/h}, \end{aligned} \right\} \quad (23a)$$

$$\left. \begin{aligned} A^2 + R^2 &= F^2 + G^2, & (p-iq)A^2 + (p+iq)R^2 &= (p'-iq')F^2 + (p'+iq')G^2, \\ F^2 e^{iq^*d/h} + G^2 e^{-iq^*d/h} &= B^2 e^{iqd/h}, & (p'-iq')F^2 e^{iq^*d/h} + (p'+iq')G^2 e^{-iq^*d/h} &= (p'-iq')B^2 e^{iqd/h}. \end{aligned} \right\} \quad (23b)$$

By solving (23a) and (23b), one get the same relationship between the incoming and outgoing amplitudes for both components:

$$B^{1,2} = \frac{4qq' e^{i(q^*-q)d/h}}{(p'-p)^2 + (q'+q)^2 - e^{2iq^*d/h} [(p'-p)^2 + (q'-q)^2]} \cdot A^{1,2}. \quad (24)$$

In contrast to simple refraction, the ratio of the amplitudes and the phases of the 1 and 2 components (i.e., the polarization) then remain unchanged, even for oblique passage through a potential jump with planar walls, regardless of whether q is real or imaginary insider the jump. That is due to the fact that q appears quadratically in the factor that couples the amplitudes, and thus, by double refraction. It is intuitively plausible that the change in polarization that results from the first passage will again be reversed by the second one. By contrast, the reflected wave R will generally experience a change in polarization.

(*) Naturally, the amplitudes R and F have different numerical values in this example than they had before.

One can assume that when the change in polarization is not sharp, as we have assumed here, and occurs in the same way on both sides of the jump, the influencing of the polarization by passing through the inhomogeneity will certainly not be larger than it is for a single refraction at the flat wall. One can still neglect it then in experiments that have the goal of producing polarized matter waves by the extraction of electrons from magnets by means of high fields.

VII. Summary.

We first discussed the question of the possibility of verifying a polarization of matter waves in connection with **Bohr's** theorem of the fundamental unobservability of the electron spin as a property of that particle. In Section II, (following **Darwin**) a vorticial component in the **Dirac** expression for current was interpreted as the effect of a spin moment, and in that way, a vector that would characterize the polarization state of matter waves was obtained that also later proved to be definitive for analyzer experiments. The analogy to and difference from optical polarization was discussed. Section III proposed and established a general principle for the production of polarized matter waves that consisted of breaking loose directed electrons that were found in stationary states without altering their direction of polarization. The conducting electrons of a magnet came into question as a supply of directed electrons. In IV, the possibility of employing the scattering of an electron wave by a directed atom as an "analyzer" was treated mathematically. The behavior of polarized waves in a homogeneous magnetic field, and finally, their behavior under reflection and passing through an electric potential jump, was investigated on the basis of the strict solutions of the **Dirac** equation, using the concepts that were obtained.

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