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Investigating the properties of electron and meson spin in the classical approximation

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In this article, the logical development of the Proca and Dirac equations in *h* will be carried out. Equations will be derived that determine the precession of the electron and meson spin for the motion of the particle along the classical path. For small velocities, the electron, as well as the meson, behave like classical particles with mechanical and magnetic moments. The magnetic moment of the electron vanishes for large velocities. The magnetic moment of the meson remains unchanged for arbitrary velocities, but the precession of the meson spin differs from the classical expression.

One can perform the transition from the wave equation to the equation in the geometric approximation with the help of the development in a small parameter, and indeed for an arbitrary type of Maxwell, Schrödinger, Dirac, and Proca equations. In that way, one will get the equations of geometrical optics (in the Maxwell case) or the Jacobi-Hamilton equations of classical mechanics in the zero-order approximation. The equations for the conservation laws (of energy or charge) follow from the equations in the first approximation. However, in the event that the wave field is described by several wave functions, as **Rytov** [**1**, **2**] has shown, we can ascertain the equations of the first approximation from our knowledge of the evolution of the polarization state of the wave field along the ray trajectory. Rytov solved that problem for Maxwell's equations and then proved that in the case of the Dirac equations, the equations in the first approximation must include information about the evolution of the orientation of the electron spin under the motion along the classical path.

1. – The equations of motion of the meson in an external electromagnetic field that is characterized by the scalar potential φ and the vector potential **A** can be written in the following way:

$$\left.\begin{aligned}\frac{h}{c}\frac{\partial\mathbf{F}}{\partial t}+\frac{ie}{c}\varphi\mathbf{F}&=\mu e\mathbf{B}-\frac{1}{\mu c}[\mathbf{P}[\mathbf{P}\mathbf{B}]],\\ \frac{h}{c}\frac{\partial\mathbf{B}}{\partial t}+\frac{ie}{c}\varphi\mathbf{B}&=-\mu e\mathbf{F}-\frac{1}{\mu c}\mathbf{P}\cdot\mathbf{P}\mathbf{F},\end{aligned}\right\}\tag{1}$$

in which $\mathbf{P} = - i\, h\, \nabla - (e / i)\, \mathbf{A}$, and μ is the mass of the meson, while e is the charge, and h means Planck's constant divided by 2π. The vector fields $\mathbf{F}$ and $\mathbf{B}$ that characterize the meson field correspond to the electric field and the vector potential in Maxwell's equations. Corresponding to the general method of the transition to the geometric approximation, we set:

$$\mathbf{B} = \mathbf{h}\, e^{(i/h)S}, \qquad \mathbf{F} = \mathbf{f}\, e^{(i/h)S}, \tag{2}$$

in which S is the real function coordinates and time that is the same for all components of the wave function. Equations (1) will then take on the form:

$$\left.\begin{aligned}&h\mu\frac{\partial\mathbf{f}}{\partial t}-i\,\pi_0\,\mu c\,\mathbf{f}+[\boldsymbol{\pi}[\boldsymbol{\pi}\mathbf{b}]]-\mu^2c^2\,\mathbf{b}-i\,h[\mathrm{rot}[\boldsymbol{\pi}\mathbf{b}]+[\boldsymbol{\pi}\ \mathrm{rot}\ \mathbf{b}])-h^2\ \mathrm{rot}\,\mathrm{rot}\,\mathbf{b}=0,\\ &h\mu\frac{\partial\mathbf{b}}{\partial t}-i\,\pi_0\,\mu c\,\mathbf{b}+\boldsymbol{\pi}\cdot\boldsymbol{\pi}\mathbf{f}-i\,h(\mathrm{grad}\,\boldsymbol{\pi}\mathbf{f}+\boldsymbol{\pi}\ \mathrm{grad}\ \mathbf{f})-h^2\ \mathrm{grad}\,\mathrm{div}\,\mathbf{f}=0,\end{aligned}\right\}\tag{3}$$

in which we have set:

$$\left.\begin{aligned}-\frac{1}{c}\frac{\partial S}{\partial t}-\frac{e}{c}\varphi&=\pi_0,\\ \nabla S-\frac{e}{c}\mathbf{A}&=\boldsymbol{\pi}.\end{aligned}\right\}\tag{4}$$

We consider h to be a small parameter and develop $\mathbf{b}$ and $\mathbf{f}$ into series:

$$\left.\begin{aligned}\mathbf{b}&=\mathbf{b}_0+\frac{h}{i}\mathbf{b}_1+\cdots,\\ \mathbf{f}&=\mathbf{f}_0+\frac{h}{i}\mathbf{f}_1+\cdots\end{aligned}\right\}\tag{5}$$

If one introduces those series into equations (3) and further sets the coefficients of equal powers of h equal to each other then that will yield:

The equations of the zero-order approximation:

$$\left.\begin{aligned}\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,\mathbf{b}_0-(\pi^2+\mu^2c^2)\,\mathbf{b}_0-i\mu c\,\pi_0\,\mathbf{f}_0&=0,\\ i\mu c\,\pi_0\,\mathbf{b}_0-\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,\mathbf{f}_0-\mu^2c^2\,\mathbf{f}_0&=0,\end{aligned}\right\}\tag{6}$$

The equations of the first approximation:

$$\left.\begin{aligned}\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,\mathbf{b}_1-(\pi^2+\mu^2c^2)\,\mathbf{b}_1-i\mu\,c\,\pi_0\,\mathbf{f}_1&=-\operatorname{rot}[\boldsymbol{\pi}\,\mathbf{b}_0]-[\boldsymbol{\pi}\operatorname{rot}\mathbf{b}_0]-i\mu\frac{\partial\mathbf{f}_0}{\partial t},\\ i\mu\,c\,\pi_{\mathbf{u}}\,\mathbf{b}_1-\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,\mathbf{f}_1-\mu^2c^2\,\mathbf{f}_1&=i\mu\frac{\partial\mathbf{b}_0}{\partial t}+\operatorname{grad}\boldsymbol{\pi}\,\mathbf{f}_0+\boldsymbol{\pi}\operatorname{div}\mathbf{f}_0.\end{aligned}\right\}\tag{7}$$

The equations in the higher orders of approximation shall not be of interest to us here.

The equations of the zero-order approximation (6) can be written in the following form:

$$\left.\begin{aligned}L_1\,\mathbf{b}_0-i\mu\,c\,\pi_0\,\mathbf{f}_0&=0,\\ i\,\mu\,c\,\pi_0\,\mathbf{f}_0+L_2\,\mathbf{f}_0&=0,\end{aligned}\right\}\tag{6′}$$

in which L_1 and L_0 represent the linear vectorial functions:

$$L_1=\boldsymbol{\pi}\cdot\boldsymbol{\pi}-\pi^2c^2,\qquad L_2=-\boldsymbol{\pi}\cdot\boldsymbol{\pi}-\pi^2c^2.$$

Since $L_1\,L_2=\mu^2c^2(\pi^2+\mu^2c^2)$, after eliminating $\mathbf{b}_0$ from (6′), one will have:

$$(\pi^2+\mu^2c^2-\pi_0^2)\,\mathbf{f}_0=0.$$

We then get the condition for the existence of non-zero solutions to the system (6):

$$\pi_0^2=\pi^2+\mu^2c^2,\tag{8}$$

or after substituting (4):

$$\frac{\partial S}{\partial t}+e\varphi+c\sqrt{\mu^2c^2+(\nabla S-\frac{e}{c}\mathbf{A})^2}=0,\tag{8′}$$

i.e., the relativistic Jacobi-Hamilton equation. We can interpret (8′) as the equation that describes the motion of the charge of the meson field along trajectories that are orthogonal to the surfaces of equal action in the event that we can show that the current density points along the velocity that is deduced from equation (8) using the rules of classical mechanics. Since the canonical momentum $\mathbf{p}$ is equal to ∇S, the following expression for the velocity of the particle will be valid:

$$\mathbf{v}=\nabla_{\mathbf{p}}\,c\sqrt{\mu^2c^2+\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)^2}=\frac{c\,\boldsymbol{\pi}}{\pi_0}.\tag{9}$$

The charge and current density that are given by the equations:

$$\rho = i\,e\,(\mathbf{F}\,\mathbf{B}^* - \mathbf{F}^*\,\mathbf{B})\ ,$$

$$\mathbf{j}_0 = -\,\frac{e}{\mu}(\mathbf{F}^*\cdot\mathbf{P}\,\mathbf{F} + [\mathbf{B}^*[\mathbf{P}\,\mathbf{B}]])\ \text{+ complex conjugate functions}$$

have the form:

$$\left.\begin{aligned} &\rho_0 = i\,e\,(\mathbf{f}_0\,\mathbf{b}_0^* - \mathbf{f}_0^*\,\mathbf{b}_0)\,, \\ &\mathbf{j}_0 = -\frac{e}{\mu}(\mathbf{f}_0\cdot\boldsymbol{\pi}\ \mathbf{b}_0^* + [\mathbf{b}_0^*\,[\boldsymbol{\pi}\,\mathbf{b}_0]]) + \text{complex conjugate functions} \end{aligned}\right\} \tag{10}$$

in the zero-order approximation.

After substituting $\mathbf{b}_0$ in (6):

$$\mathbf{b}_0 = -\,\frac{i}{\mu\,c\,\pi_0}(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,\mathbf{f}_0 + \mu^2 c^2\,\mathbf{f}_0)\ ,$$

one will get:

$$\rho_0 = -\,\frac{2\,e}{\mu\,c\,\pi_0}(|\,\boldsymbol{\pi}\,\mathbf{f}_0\,|^2 + \mu^2 c^2\,|\,\mathbf{f}_0\,|^2)\ ,$$

$$\mathbf{j}_0 = -\,\frac{2\,e}{\mu\,c\,\pi_0^2}(|\,\boldsymbol{\pi}\,\mathbf{f}_0\,|^2 + \mu^2 c^2\,|\,\mathbf{f}_0\,|^2)\,\boldsymbol{\pi}\ ,$$

or when one considers (9):

$$\mathbf{j}_0 = \rho_0\,\mathbf{v}, \tag{11}$$

i.e., $\mathbf{v}$ is the velocity of the motion of the charge. This interpretation of equation (8) as an equation that describes the motion of the charge is how it differs from the equation in question for the electron, for which one can introduce the concept of the density of particle number. That difference is obviously based upon the fact that the particles in Proca's theory appear only in the second quantization.

The equations in the first approximation (7) represent dissimilar linear equations with respect to $\mathbf{b}_1$ and $\mathbf{f}_1$. The corresponding similar equations coincide with (6) and thus have solutions that deviate from zero. As a result, the dissimilar equations (7) possess solutions only when the right-hand side is orthogonal to the solutions of system of equations that is adjoint with respect to (6). Since the matrix of the equations (6) is Hermitian, the adjoint system will coincide with the original one. As a result, the right-hand side of (7) must be orthogonal to the solutions of the system (6), i.e.:

$$-\ \mathbf{b}_0^*\left(\operatorname{rot}[\boldsymbol{\pi}\,\mathbf{b}_0] + [\boldsymbol{\pi}\operatorname{rot}\mathbf{b}_0] + i\mu\frac{\partial\mathbf{f}_0}{\partial t}\right) + \mathbf{f}_0^*\left(i\mu\frac{\partial\mathbf{b}_0}{\partial t} + \operatorname{grad}\boldsymbol{\pi}\,\mathbf{f}_0 + \boldsymbol{\pi}\operatorname{div}\mathbf{f}_0\right) = 0\ .$$

When one adds that equation to its complex conjugate and considers (10), one will get:

$$\frac{\partial\rho_0}{\partial t} + \operatorname{div}\mathbf{j}_0 = 0\ ,$$

i.e., the equation of the law of conservation of charge.

2. – In order to deduce the number of linearly-independent solutions to the system of equations, one must determine the rank of the matrix of that system. In order to do that, it will suffice to write the matrix for the special case of $\pi_2 = \pi_3 = 0$:

$$\begin{Bmatrix} -\mu^2c^2 & 0 & 0 & -i\,\mu\,c\,\pi_0 & 0 & 0 \\ 0 & -\pi^2-\mu^2c^2 & 0 & 0 & -i\,\mu\,c\,\pi_0 & 0 \\ 0 & 0 & -\pi^2-\mu^2c^2 & 0 & 0 & -i\,\mu\,c\,\pi_0 \\ i\,\mu\,c\,\pi_0 & 0 & 0 & -\pi^2-\mu^2c^2 & 0 & 0 \\ 0 & i\,\mu\,c\,\pi_0 & 0 & 0 & -\mu^2c^2 & 0 \\ 0 & 0 & i\,\mu\,c\,\pi_0 & 0 & 0 & -\mu^2c^2 \end{Bmatrix}.$$

The rank of that matrix is three, as one can easily infer. Therefore, there are three linearly-independent solutions to the system (6). We choose those solutions in the following way:

$$\left.\begin{array}{lll} \mathbf{f}_{01} = \dfrac{1}{\tau}\mathbf{e}_1\,, & \mathbf{f}_{02} = \mathbf{e}_2\,, & \mathbf{f}_{03} = \mathbf{e}_3\,, \\ \mathbf{b}_{01} = -i\,\mathbf{e}_1, & \mathbf{b}_{02} = -\dfrac{1}{\tau}\mathbf{e}_2\,, & \\ & \mathbf{b}_{03} = -\dfrac{1}{\tau}\mathbf{e}_3\,, & \end{array}\right\} \tag{12}$$

in which:

$$\gamma = \frac{\pi_0}{\mu\,c} = \frac{1}{\sqrt{1-\beta^2}}\,, \qquad \beta = \frac{v}{c}\,, \tag{13}$$

and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ mean unit vectors in directions of the tangent, normal, and binormal to the classical path (i.e., the line of the vector $\boldsymbol{\pi}$).

The general solution of the system (6) represents a linear combination of equations (12):

$$\left.\begin{array}{l} \mathbf{f}_0 = \dfrac{a_1}{\gamma}\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3\,, \\ \mathbf{b}_0 = -i\left(a_1\,\mathbf{e}_1 + \dfrac{a_2}{\gamma}\mathbf{e}_2 + \dfrac{a_3}{\gamma}\mathbf{e}_3\right), \end{array}\right\} \tag{15}$$

in which $a_1 = a_1\,(\mathbf{r},\,t)$ are functions of coordinates and time.

The functions a_i are not subject to any restrictions by equations (6). However, the condition of the orthogonality of the right-hand side of (7) with respect to all linearly-independent solutions (12) will imply three differential equations that couple the functions a_i .

In what follows, only the stationary states of the mesons in a constant electromagnetic field will be treated. One then has:

$$S(\mathbf{r}, t) = -W\,t + S_0(\mathbf{r}) , \qquad \frac{\partial b_i}{\partial t} = \frac{\partial \mathbf{f}}{\partial t} = 0 , \qquad \frac{\partial a_i}{\partial t} = 0 .$$

The orthogonality conditions have the following form:

$$\left.\begin{aligned} -i\,\gamma\,\mathbf{e}_1 \operatorname{rot}\,[\boldsymbol{\pi}\,\mathbf{b}_0] + \mathbf{e}_1 \operatorname{grad}\boldsymbol{\pi}\,\mathbf{f}_0 + \boldsymbol{\pi}\operatorname{div}\mathbf{f}_0 &= 0, \\ -i\,\gamma\,\mathbf{e}_2\,(\operatorname{rot}\,[\boldsymbol{\pi}\,\mathbf{b}_0] + [\boldsymbol{\pi}\operatorname{rot}\mathbf{b}_0]) + \gamma^2\mathbf{e}_2 \operatorname{grad}\boldsymbol{\pi}\,\mathbf{f}_0 &= 0, \\ -i\,\gamma\,\mathbf{e}_3\,(\operatorname{rot}\,[\boldsymbol{\pi}\,\mathbf{b}_0] + [\boldsymbol{\pi}\operatorname{rot}\mathbf{b}_0]) + \gamma^2\mathbf{e}_3 \operatorname{grad}\boldsymbol{\pi}\,\mathbf{f}_0 &= 0, \end{aligned}\right\} \tag{15}$$

in which one has replaced $\mathbf{b}_0$ and $\mathbf{f}_0$ with the general solution (14). When one transforms those equations, one must observe the following known relations from differential geometry (e.g., [**4**]):

$$\left.\begin{aligned} &\mathbf{e}_2 \operatorname{rot}\ \mathbf{e}_1 = 0, \\ &\mathbf{e}_3 \operatorname{rot}\ \mathbf{e}_2 = \tau, \\ &\mathbf{e}_3 \operatorname{rot}\mathbf{e}_3 + \mathbf{e}_2 \operatorname{rot}\mathbf{e}_2 = \mathbf{e}_1 \operatorname{rot}\mathbf{e}_1 + \frac{2}{T}\ , \end{aligned}\right\} \tag{16}$$

in which τ represents the curvature of the path and $1 / T$ is its torsion, and the following relations exist between the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\mathbf{e}_1 = [\mathbf{e}_2\,\mathbf{e}_3] , \qquad \mathbf{e}_2 = [\mathbf{e}_3\,\mathbf{e}_1] , \qquad \mathbf{e}_3 = [\mathbf{e}_1\,\mathbf{e}_2] .$$

As a result of the transformations, we will get from equation (15) that:

$$2\boldsymbol{\pi}\,\nabla a_1 + a_1 \operatorname{div}\boldsymbol{\pi} - a_2\,\boldsymbol{\pi}\,\tau - 2a_1\,\pi\frac{\nabla\gamma}{\gamma} - \gamma^2(a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3)\nabla\frac{\pi}{\gamma} = 0 ,$$

$$2\boldsymbol{\pi}\,\nabla a_2 + a_2 \operatorname{div}\boldsymbol{\pi} + a_1\,\boldsymbol{\pi}\,\tau + a_3\,\pi\left(\mathbf{e}_1 \operatorname{rot}\mathbf{e}_1 + \frac{2}{T}\right) + a_1\,\gamma^2\,\mathbf{e}_2\nabla\frac{\pi}{\gamma} - 2\,a_2\,\boldsymbol{\pi}\frac{\nabla\gamma}{\gamma} = 0 ,$$

$$2\boldsymbol{\pi}\,\nabla a_3 + a_3 \operatorname{div}\boldsymbol{\pi} - a_2\,\boldsymbol{\pi}\left(\mathbf{e}_1 \operatorname{rot}\mathbf{e}_1 + \frac{2}{T}\right) + a_1\,\gamma^2\,\mathbf{e}_3\,\nabla\frac{\pi}{\gamma} - 2a_3\,\boldsymbol{\pi}\frac{\nabla\gamma}{\gamma} = 0 .$$

Since:

$$\gamma = \frac{\pi_0}{\mu\,c} = \frac{W - e\,\varphi}{\mu\,c^2}$$

and

$$\pi = \mu c \sqrt{\gamma^2 - 1} \ ,$$

one can write:

$$\nabla \gamma = \frac{e}{\mu c^2} \mathbf{E}$$

and

$$\frac{1}{\pi} \nabla \frac{\pi}{\gamma} = \frac{e}{\mu c^2} \frac{\mathbf{E}}{\gamma^2 (\gamma^2 - 1)} \ ,$$

in which $\mathbf{E} = - \nabla \varphi$ means the electric field strength. Furthermore, one has:

$$\left.\begin{aligned} &2\frac{\partial a_1}{\partial s} + \frac{a_1}{v} \operatorname{div} \mathbf{v} - a_2 \gamma \tau - \frac{a_1}{\gamma} E_{01} - \frac{1}{\gamma^2 - 1}(a_2 E_{02} + a_3 E_{03}) = 0, \\ &2\frac{\partial a_2}{\partial s} + \frac{a_2}{v} \operatorname{div} \mathbf{v} + a_1 \gamma \tau + a_3 \left(\mathbf{e}_1 \operatorname{rot} \mathbf{e}_1 + \frac{2}{T} \right) + \frac{a_1}{\gamma^2 - 1} E_{01} - \frac{a_2}{\gamma} E_{01} = 0, \\ &2\frac{\partial a_3}{\partial s} + \frac{a_3}{v} \operatorname{div} \mathbf{v} - a_2 \left(\mathbf{e}_1 \operatorname{rot} \mathbf{e}_1 + \frac{2}{T} \right) + \frac{a_1}{\gamma^2 - 1} E_{03} - \frac{a_3}{\gamma} E_{01} = 0, \end{aligned}\right\} \tag{17}$$

in which $\partial / \partial s = \mathbf{e} \nabla$ represents the derivative with respect to the arc-length of the path and $\mathbf{v}$ represents the velocity of the particle that is determined by equation (9), while:

$$E_{0i} = \frac{e}{\mu c^2} \mathbf{E} \mathbf{e}_i \ .$$

Thua, equations (15) will take on a form that includes only derivatives of the a_i with respect to the arc-length. The remaining derivatives that enter into (15) will be reduced automatically.

3. – The differential equations that determine the evolution of the components of the meson spin along the classical path can be easily exhibited with the help of equations (17). When one develops the correct law of the evolution of spin under the motion of the particle in *h*, one will must consider those equations to be the first term in the development.

The meson spin (or more precisely, the spin momentum) will be determined from the following expression (*):

$$\mathbf{G} = h\,([\mathbf{B}^* \mathbf{F}] + [\mathbf{F}^* \mathbf{B}]) \ . \tag{18}$$

Upon substituting the wave functions of order zero, one will get:

(*) Ig. E. Tamm, "Die Theorie des Mesons" (to be published in the Journal of Physics) [**3**].

$$\mathbf{G}_0 = h\left\{\frac{2i}{\gamma}(a_3\,a_2^* - a_2\,a_3^*)\,\mathbf{e}_1 + \frac{1+\gamma^2}{\gamma^2}\,i\,(a_1\,a_3^* - a_3\,a_1^*)\,\mathbf{e}_2 + \frac{1+\gamma^2}{\gamma^2}\,i\,(a_2\,a_1^* - a_1\,a_2^*)\,\mathbf{e}_3\right\}.$$

We shall now form the derivatives of the components of the vector **G** with respect to the axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ with respect to the arc-length of the path s. If one drops the index zero on G and sets $G_1 = \mathbf{G}\mathbf{e}_1$ then one will have:

$$\left.\begin{aligned}
\frac{\partial G_1}{\partial s} &= \qquad \frac{2i}{\gamma}\left(a_2^*\,\frac{\partial a_3}{\partial s} + a_3\,\frac{\partial a_2^*}{\partial s} - \text{complex conjugate}\right) - \frac{G_1}{\gamma}\,E_{01},\\
\frac{\partial G_2}{\partial s} &= \frac{1+\gamma^2}{\gamma^2}\,i\left(a_1^*\,\frac{\partial a_3}{\partial s} + a_3\,\frac{\partial a_1^*}{\partial s} - \text{complex conjugate}\right) - \frac{2G_2}{\gamma\,(1+\gamma^2)}\,E_{01},\\
\frac{\partial G_3}{\partial s} &= \frac{1+\gamma^2}{\gamma^2}\,i\left(a_1^*\,\frac{\partial a_2}{\partial s} + a_2\,\frac{\partial a_1^*}{\partial s} - \text{complex conjugate}\right) - \frac{2G_3}{\gamma\,(1+\gamma^2)}\,E_{01}.
\end{aligned}\right\} \tag{19}$$

After substituting the derivatives a_i from (17) in (19), one will have:

$$\left.\begin{aligned}
&\frac{\partial G_1}{\partial s} + \frac{G_1}{v}\,\mathrm{div}\,\mathbf{v} = \frac{\gamma^2}{1+\gamma^2}\,\tau\,G_2 + \frac{\gamma}{\gamma^4-1}\left(E_{02}\,G_2 + E_{03}\,G_3\right),\\
&\frac{\partial G_2}{\partial s} + \frac{G_2}{v}\,\mathrm{div}\,\mathbf{v} = -\frac{1+\gamma^2}{4}\,\tau\,G_2 - \frac{1}{2}\left(\mathbf{e}_1\,\mathrm{rot}\,\mathbf{e}_1 + \frac{2}{T}\right)G_2 - \frac{1+\gamma^2}{4\gamma\,(\gamma^2-1)}\,E_{02}G_1 + \frac{\gamma^2-1}{4\gamma\,(\gamma^2-1)}\,E_{01}G_2,\\
&\frac{\partial G_3}{\partial s} + \frac{G_3}{v}\,\mathrm{div}\,\mathbf{v} = \frac{1}{2}\left(\mathbf{e}_1\,\mathrm{rot}\,\mathbf{e}_1 + \frac{2}{T}\right)G_2 - \frac{1+\gamma^2}{4\gamma\,(\gamma^2-1)}\,E_{03}G_1 + \frac{\gamma^2-1}{4\gamma\,(\gamma^2-1)}\,E_{01}G_3.
\end{aligned}\right\} \tag{20}$$

It seems preferable to focus upon another vector **M**:

$$\mathbf{M} = i\,h\,u_\mu\,([\mathbf{B}^*\,\mathbf{B}] + [\mathbf{F}^*\,\mathbf{F}])\,, \tag{21}$$

in which $u_\mu = e\,/\,2\mu\,c$, in addition to the vector **G**.

That vector represents the mean value of the operator $u_\mu\,\tau_1\,\mathbf{s}$ (more precisely, the density, so $\int \mathbf{M}\,dV = \overline{u_\mu\,\tau_1\,\mathbf{s}}$), so in some cases, that operator takes on the role of the operator of magnetic moment (*). If one substitutes the zero-order wave function in (21) then one will have:

$$M_1 = u_\mu\,\frac{1+\gamma^2}{2\gamma}\,G_1\,, \qquad M_2 = u_\mu\,\frac{2\gamma}{1+\gamma^2}\,G_2\,, \qquad M_3 = u_\mu\,\frac{2\gamma}{1+\gamma^2}\,G_3\,. \tag{22}$$

(*) Cf., the foregoing article, pp. 27.

Now, since $\gamma = 1/\sqrt{1-\beta^2}$, **M** and **G** will obviously be parallel for small velocities, while for large velocities ($v \sim c$):

$$M_{\|} \gg G_{\|}\, u_\mu\,, \qquad\qquad M_{\perp} \ll G_{\perp}\, u_\mu\,,$$

in which $M_{\|}$ and $M_{\perp}$ are the components that are parallel (perpendicular, resp.) to the direction of motion. While giving consideration to equation (22), we can write equation (20) in the following form:

$$\frac{\partial_r \mathbf{G}}{\partial s} + \mathbf{G}\frac{\operatorname{div}\mathbf{v}}{n} = \frac{\gamma\,\tau}{2u_\mu}[\mathbf{M}\,\mathbf{e}_3] - \tfrac{1}{2}\left(\mathbf{e}_1 \operatorname{rot}\mathbf{e}_1 + \frac{2}{T}\right)[\mathbf{G}\,\mathbf{e}_1] - \frac{1}{2(\gamma^2-1)u_\mu}\big[\mathbf{M}[\mathbf{E}_0\,\mathbf{e}_1]\big] - \frac{\gamma^2-1}{2\gamma^2 u_\mu}E_{01}\mathbf{M}_\perp\ ,$$

in which $\partial_r \mathbf{G} / \partial s$ represents the derivative of **G** with respect to the natural axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. (That means that the derivative is calculated when $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are assumed to be constant.) The transition to Cartesian coordinate systems takes place in the following way:

$$\frac{\partial \mathbf{G}}{\partial s} = \frac{\partial_r \mathbf{G}}{\partial s} + [\boldsymbol{\omega}\,\mathbf{G}]\ ,$$

in which $\boldsymbol{\omega} = \tau\,\mathbf{e}_3 - (1/T)\,\mathbf{e}_1$ represents the Darboux vector, which determines the rotation of the natural triple $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ relative to the Cartesian axes. As a result:

$$\frac{\partial \mathbf{G}}{\partial s} + \mathbf{G}\frac{\operatorname{div}\mathbf{v}}{n} = \tau\left[\frac{\gamma}{2u_\mu}\mathbf{M} - \mathbf{G}, \mathbf{e}_3\right] - \tfrac{1}{2}\mathbf{e}_1 \operatorname{rot}\mathbf{e}_1\cdot[\mathbf{G}\,\mathbf{e}_1] - \frac{1}{2(\gamma^2-1)u_\mu}\big[\mathbf{M}[\mathbf{E}_0\,\mathbf{e}_1]\big] + \frac{\gamma^2-1}{2\gamma^2 u_\mu}E_{01}\mathbf{M}_\perp\,. \quad (23)$$

The sense of the term (div **v**) / v on the left-hand side is easy to recognize. Let V be the "fluid" volume, i.e., a volume that is always bounded by the same particles. As is known, one will then have:

$$\frac{dV}{dt} = \int_V \operatorname{div}\mathbf{v}\, dV\ ,$$

or

$$\operatorname{div}\mathbf{v} = \frac{1}{\delta V}\frac{d\,\delta V}{dt}$$

for an infinitesimal "fluid" volume δV.

Due to the fact that $ds = v\,dt$, one will further have:

$$\frac{\partial \mathbf{G}}{\partial s} = \frac{1}{v}\frac{d\mathbf{G}}{dt}\ ,$$

and therefore:

$$\frac{\partial \mathbf{G}}{\partial s} + \mathbf{G}\frac{\operatorname{div}\mathbf{v}}{n} = \frac{1}{v\,\delta V}\frac{d}{dt}(\mathbf{G}\,\delta V)\ .$$

Obviously, the densities **G** and **M** have no physical meaning, as opposed to the quantities $\mathbf{G}\,\delta V$ and $\mathbf{M}\,\delta V$, which correspond to mechanical and magnetic moment of a continuous number of particle exemplars whose motion is described by the Jacobi-Hamilton equation. We will then multiply equation by δV and understand **G** and **M** to mean the quantities $\mathbf{G}\,\delta V$ and $\mathbf{M}\,\delta V$ in what follows.

Thus, (27) will imply that:

$$\frac{d\mathbf{G}}{dt} = u_\mu \sqrt{1-\beta^2}\,[\mathbf{G}_\mu \mathbf{H}] - \left[\mathbf{M} - u_\mu \sqrt{1-\beta^2}\,\mathbf{G}_\mu \cdot \mathbf{H}_\perp + \frac{\beta^2}{2-\beta^2}\left[\mathbf{M}_\parallel [\beta \mathbf{E}]\right]\right] + \beta^2 E_1 \mathbf{M}_\perp\,, \tag{28}$$

in which:

$$u_\mu = \frac{e}{2\mu c}$$

represents the ratio of the magnetic moment to the mechanical moment of the meson at rest, and the index μ means that the quantity in question refers to the meson.

Before we take up the analysis of formula (28), we would like to perform analogous calculations for the electron.

4. – The geometric approximation for the Dirac equations was investigated by **Pauli** [**5**]. However, Pauli employed the orthogonality conditions only in order to derive the continuity equations and abstained from a detailed examination.

The Dirac equations:

$$\frac{i\,h}{c}\frac{\partial \psi}{\partial t} = (e\,\varphi + \boldsymbol{\alpha}\,\mathbf{P} + \alpha_m\,m\,c)\;\psi$$

lead, after the substitution:

$$\psi = a\,e^{(i/h)\,S}, \quad a = a_0 + \frac{h}{i}a_1 + \cdots,$$

to the following equations:

Equations in the zero-order approximation:

$$-\,\pi_0\,a_0 + \boldsymbol{\pi}\,\boldsymbol{\alpha}\,a_0 + \alpha_m\,m\,c\,a_0 = 0\,. \tag{29}$$

Equations in the first approximation:

$$-\,\pi_0\,a_1 + \boldsymbol{\pi}\,\boldsymbol{\alpha}\,a_1 + \alpha_m\,m\,c\,a_1 = -\,\frac{1}{c}\frac{\partial \psi}{\partial t} - \boldsymbol{\alpha}\,\nabla a_0\,. \tag{30}$$

The conditions for the existence of solutions (29) that are non-zero leads to the Jacobi-Hamilton equations (8), (8′). The rank of the matrix of the system equations (29) is equal to two, such that two linearly-independent solutions of that system exist here. If one now writes the system (29) in its developed form and drops the index zero on *a* then one will have:

$$\begin{aligned}
(-\pi_0 + mc)\, a_1 + \xi^*\, a_4 + \pi_3\, a_3 &= 0 ,\\
(-\pi_0 + mc)\, a_2 + \xi\, a_3 - \pi_3\, a_4 &= 0 ,\\
-(\pi_0 + mc)\, a_3 + \xi^*\, a_2 + \pi_3\, a_1 &= 0 ,\\
-(\pi_0 + mc)\, a_4 + \xi\, a_1 - \pi_3\, a_2 &= 0 ,
\end{aligned}$$

in which $\xi = \pi_1 + i\,\pi_2$ and π_1, π_2, π_3 represent the projections of $\boldsymbol{\pi}$ onto the Cartesian coordinate axes. As one easily recognizes, one can choose the following two linearly-independent solutions:

$$a^{(1)} = \begin{pmatrix} \delta \\ 0 \\ -\pi_3 \\ -\xi \end{pmatrix}, \qquad a^{(2)} = \begin{pmatrix} 0 \\ \delta \\ -\xi^* \\ \pi_3 \end{pmatrix}, \tag{31}$$

in which:

$$\delta = -\pi_0 - mc = -mc\,(\gamma + 1) . \tag{31′}$$

As a general solution of the solution (29), one has:

$$\boldsymbol{a} = c_1\, a^{(1)} + c_2\, a^{(2)},$$

in which $c_i = c_i\,(\mathbf{r})$ represents a function of the coordinates. The matrix of the system (29) is Hermitian and therefore coincides with the adjoint system to the original one. As a result, one can write the orthogonality conditions on the right-hand side of (30) relative to the solutions (31), insofar as one restricts oneself to the stationary states, in the following way:

$$a^{(1)} + \boldsymbol{\alpha}\nabla(c_1\, a^{(1)} + c_2\, a^{(2)}) = 0 \qquad (i = 1, 2) . \tag{32}$$

The matrix $\boldsymbol{\alpha}\nabla$ has the form:

$$\boldsymbol{\alpha}\nabla = \left\{\begin{matrix} 0 & 0 & \frac{\partial}{\partial z} & \Pi^* \\ 0 & 0 & \Pi & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \Pi^* & 0 & 0 \\ \Pi & -\frac{\partial}{\partial z} & 0 & 0 \end{matrix}\right\},$$

in which $\Pi = \partial / \partial x + i\, \partial / \partial y$. If one substitutes the solutions (31) in (32) and further considers the notations:

$$\Pi^* \xi = \frac{\partial \pi_3}{\partial z} = \operatorname{div} \boldsymbol{\pi} + i\, (\operatorname{rot} \boldsymbol{\pi})_3 ,$$

$$\xi\, \Pi^* + \pi_3 \frac{\partial}{\partial z} = \boldsymbol{\pi}\nabla - i\, (\boldsymbol{\pi}\nabla)_3$$

then one will get:

$$2\delta\, \boldsymbol{\pi}\nabla c + c\, \boldsymbol{\pi}\nabla\delta + c\, \delta \operatorname{div} \boldsymbol{\pi} + i\, \delta\, \boldsymbol{\sigma}_0 \operatorname{rot} \boldsymbol{\pi}\; c + i\, \boldsymbol{\sigma}_0\, [\boldsymbol{\pi}\nabla\delta] = 0$$

after some simple transformations, in which σ_0 is the vector of matrices whose components represent the two-rowed Pauli matrices, while one must understand c to mean the two functions c_1 and c_2 . If one substitutes the value δ from (31′) then one will get:

$$2\frac{\partial c}{\partial s} + \eta\, c + i\, \sigma_0\, \mathbf{R}\, c = 0 \tag{33}$$

after some transformations, in which:

$$\left.\begin{aligned} \eta &= \frac{\operatorname{div} \mathbf{v}}{v} + \frac{2\gamma + 1}{\gamma\,(\gamma + 1)} E_{01}, \\ \mathbf{R} &= \operatorname{rot} \mathbf{e}_1 + \frac{1}{\gamma^2 - 1} [\mathbf{E}_0\, \mathbf{e}_1]. \end{aligned}\right\} \tag{33′}$$

Just as in the case of the Proca equations, it is also possible here to derive equation (33), which is an analogue of (17) and is the equation for the precession of spin. The mean value of the electron spin (more precisely, its density):

$$\mathbf{G} = \frac{h}{2}\psi + \boldsymbol{\sigma}\psi ,$$

in which $\boldsymbol{\sigma}$ represents the four-rowed spin matrices, will take the following form:

$$\mathbf{G} = h\,m^2c^2\,[(\gamma+1)\,\tilde{\boldsymbol{\sigma}}_0 + (\gamma^2-1)\,\mathbf{e}_1\cdot\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0]\ , \tag{34}$$

after substituting the wave functions of order zero, in which:

$$\tilde{\boldsymbol{\sigma}}_0 = c + \boldsymbol{\sigma}_0\, c\ .$$

The derivative **G** with respect to the arc-length of the classical path will obviously be:

$$\frac{\partial\mathbf{G}}{\partial s}\,h\,m^2c^2\left\{(\gamma+1)\frac{\partial\tilde{\boldsymbol{\sigma}}_0}{\partial s} + (\gamma^2-1)\frac{\partial}{\partial s}(\mathbf{e}_1\cdot\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0) + \tilde{\boldsymbol{\sigma}}_0\,E_{01} + 2\gamma\,\mathbf{e}_1\cdot\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0\,E_{01}\right\}. \tag{35}$$

The derivative $\tilde{\boldsymbol{\sigma}}_0$ will be determined with the help of equation (30):

$$\frac{\partial}{\partial s}(c + \boldsymbol{\sigma}_0\,c) = -\eta\,\tilde{\boldsymbol{\sigma}}_0 - \tfrac{1}{2}(\tilde{\boldsymbol{\sigma}}_0\cdot\tilde{\boldsymbol{\sigma}}_0\,\mathbf{R} - \tilde{\boldsymbol{\sigma}}_0\,\mathbf{R}\cdot\tilde{\boldsymbol{\sigma}}_0)\ ,$$

or else one will have:

$$\frac{\partial\tilde{\boldsymbol{\sigma}}_0}{\partial s} = -\ \eta\,\tilde{\boldsymbol{\sigma}}_0 - [\tilde{\boldsymbol{\sigma}}_0\,\mathbf{R}]\ , \tag{36}$$

when one considers the commutation relations between the components $\boldsymbol{\sigma}_0$. In a similar way, one will get:

$$\frac{\partial}{\partial s}(\mathbf{e}_1\cdot\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0) = -\eta\,\mathbf{e}_1\cdot\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0 - \mathbf{e}_1\cdot\mathbf{e}_1[\tilde{\boldsymbol{\sigma}}\,\mathbf{R}] + c^\dagger\,\frac{\partial}{\partial s}(\mathbf{e}_1\cdot\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0)\,c\ .$$

If one then considers that $\partial\mathbf{e}_1 / \partial s = \tau\,\mathbf{e}_2$ then one will have:

$$\frac{\partial}{\partial s}(\mathbf{e}_1\cdot\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0) = \tau\,(\mathbf{e}_1\cdot\mathbf{e}_2\,\tilde{\boldsymbol{\sigma}}_0 - \mathbf{e}_2\cdot\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0)\ ,$$

and therefore:

$$\frac{\partial}{\partial s}(\mathbf{e}_1\cdot\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0) = -\eta\,\mathbf{e}_1\cdot\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0 - \mathbf{e}_1\cdot\mathbf{e}_1[\tilde{\boldsymbol{\sigma}}\,\mathbf{R}] + \tau\,(\mathbf{e}_1\cdot\tilde{\boldsymbol{\sigma}}_0\,\mathbf{e}_2 + \mathbf{e}_2\cdot\tilde{\boldsymbol{\sigma}}_0\,\mathbf{e}_1)\,. \tag{37}$$

One finally gets from (34) that:

$$\mathbf{e}_1\,\tilde{\boldsymbol{\sigma}}_0 = \frac{G_1}{h\,m^2c^2\,\gamma\,(\gamma+1)}\ , \qquad \tilde{\boldsymbol{\sigma}}_0 = \frac{1}{h\,m^2c^2\,\gamma\,(\gamma+1)}\left(\mathbf{G} - \frac{\gamma-1}{\gamma}\mathbf{e}_1\,G_1\right). \tag{38}$$

Upon substituting (36), (37), and (38) in (35), one will get:

$$\frac{\partial \mathbf{G}}{\partial s}$$
$$= -\ \eta\,\mathbf{G} - [\mathbf{G}\,\mathbf{R}] + \frac{\gamma-1}{\gamma}[\mathbf{e}_1\,\mathbf{R}]\mathbf{G}_1 + (\gamma-1)\mathbf{G}[\mathbf{e}_1\,\mathbf{R}]\cdot\mathbf{e}_1 + \tau(\gamma-1)\left(\mathbf{e}_1\,G_2 + \frac{1}{\gamma}\mathbf{e}_2\,G_1\right) - \frac{1}{\gamma+1}E_{01}\,\mathbf{G} - \frac{1}{\gamma}\,\mathbf{e}_1 E_{01}\,G_1\ .$$

If one introduces the values of η and **R** from (33) here and further replaces rot $\mathbf{e}_1$ and τ from (24) and (26) then one will get:

$$\frac{\partial \mathbf{G}}{\partial s} + \mathbf{G}\,\frac{\operatorname{div}\mathbf{v}}{c} = \frac{1}{\sqrt{\gamma^2-1}}[\mathbf{G}\,\mathbf{H}_0] + \frac{1}{\gamma}[\mathbf{e}_1[\mathbf{E}_0\,\mathbf{G}]]$$

after performing some simple transformations. We now convert the left-hand side of this equation in the same way as in equation (27) and write **G** instead of **G** δV from now on. Furthermore, we replace γ, $\mathbf{E}_0$, **H** with their values in (13) and (25), and we will ultimately get:

$$\frac{d\mathbf{G}}{dt} = u_e\sqrt{1-\beta^2}\,[\mathbf{G}_e\,\mathbf{H}] + u_e\sqrt{1-\beta^2}\,[\boldsymbol{\beta}[\mathbf{E}\,\mathbf{G}_e]]\ ,$$

in which $u_e = e / mc$ is the ratio of the magnetic and mechanical moments of the electron at rest, and the index e says that the quantity in question refers to the electron.

5. – Equations (28) and (39), which determine the precession of the meson and electron spin in an external electromagnetic field, were obtained as the result of a logical development of the equations of motion in h, and therefore represent the classical approximations to the exact laws of spin variation. We then recognize that neither the electron nor the meson can be regarded as classical rotating balls with a certain magnetic moment, even in this approximation, and that equations (28) and (39) exhibit a series of peculiarities that characterize the gyroscopic properties of mesons and electrons.

We would next like to focus upon the case of small velocities. If β^2 can be neglected with respect 1 then (39) will imply that:

$$\frac{d\mathbf{G}}{dt} = u_e\,[\mathbf{G}_e\,\mathbf{H}] + u_e\,[\boldsymbol{\beta}\,[\mathbf{E}\,\mathbf{G}_e]] = u_e\,[\mathbf{G}_e\,\mathbf{H}] + \frac{u_e}{2}[\mathbf{G}_e\,[\mathbf{E}\boldsymbol{\beta}]] - u_e\,\mathbf{N}\ , \tag{40}$$

in which:

$$\mathbf{N} = \mathbf{G}_e\cdot\mathbf{E}\,\boldsymbol{\beta} - \tfrac{1}{2}\boldsymbol{\beta}\cdot\mathbf{E}\,\mathbf{G}_e - \tfrac{1}{2}\,\mathbf{E}\cdot\boldsymbol{\beta}\,\mathbf{G}_e\ .$$

If one assumes that the magnetic field is small then the electric field essentially governs the translational motion of the electron:

$$m\,\frac{d\mathbf{v}}{dt} = e\,\mathbf{E}\,.$$

If one substitutes **E** in the expression for **N** then one will get:

$$-\,u_e\,\mathbf{N} = -\,\frac{1}{2c^2}\left(\frac{d}{dt}\,[\mathbf{v}[\mathbf{G}_e\mathbf{v}]]-\left[-\mathbf{v}\left[\frac{d\mathbf{G}_e}{dt}\,\mathbf{v}\right]\right]\right).$$

If $\beta^2 \ll 1$ then the last term can be neglected, and it will then follow from (40) that:

$$\frac{d\mathbf{G}_e}{dt} = u_e\left[\mathbf{G}_e\,\mathbf{H}+\tfrac{1}{2}[\mathbf{E}\boldsymbol{\beta}]\right]-\frac{1}{2c^2}\frac{d}{dt}[\mathbf{v}[\mathbf{G}_e\mathbf{v}]]\,.$$

If the motion is periodic then one will have:

$$\overline{\frac{d\mathbf{G}_e}{dt}} = \overline{u_e\left[\mathbf{G}_e\,\mathbf{H}+\tfrac{1}{2}[\mathbf{E}\boldsymbol{\beta}]\right]}\,, \tag{41}$$

as long as one takes the mean over a period. The coefficient 1/2 in the effective field [**E** $\boldsymbol{\beta}$] represents nothing but the "Thomas factor" [**6, 7**], which appears here as a result of the accelerated motion of the electron. Thus, the electron behaves like a classical rotating particle for small velocities.

For $\beta^2 \ll 1$, since $\mathbf{M} = u_\mu\,\mathbf{G}$ (up to β^2 precisely), equation (28) will give the expression:

$$\frac{d\mathbf{G}_\mu}{dt} = u_\mu\,[\mathbf{G}_\mu\,\mathbf{H}]\,,$$

i.e., the precession of the meson spin in an electric field is equal to zero. It is easy to show that this is likewise a classical result, and that difference between this and the electron is a simple consequence of another u-value. In fact, according to Thomas, the additional precession that appears thanks to the accelerated motion of the particle amounts to:

$$\boldsymbol{\omega}_{\mathrm{T}} = \frac{1}{2c^2}[\mathbf{v}\,\mathbf{a}]\,,$$

in which **a** means the acceleration. If one sets $\mathbf{a} = (e\,/\,m)\,\mathbf{E}$ here (while neglecting the effect of the magnetic field) then:

$$\boldsymbol{\omega}_{\Gamma} = -\,u_{kl}\,[\mathbf{E}\,\boldsymbol{\beta}]\,,\qquad \omega_{\mathrm{T}} = -\,u_{kl}\,[\mathbf{E}\,\boldsymbol{\beta}]\,,$$

in which $u_{kl} = e / 2mc$ represents the classical quotient of the magnetic and mechanical moments. Therefore, the equation for the precession of the classical particle with the given magnitude u will have the form:

$$\frac{d\mathbf{G}}{dt} = u\left\{[\mathbf{G}\,\mathbf{H}] + \left(1 - \frac{u_{kl}}{u}\right)[\mathbf{G}\,[\mathbf{E}\,\boldsymbol{\beta}]]\right\} .$$

Since $u_e = 2\,u_{kl}$ for the electron, the Thomas factor will be equal to 1/2 here. However, $u_\mu = u_{kl}$ for the meson, and therefore the Thomas factor will be equal to zero then, which agrees with (42).

We shall now turn to the case of large velocities. It emerges from (39) that the magnetic moment of the electron will vanish for large velocities, i.e., it will be:

$$\mathfrak{M} = \mathfrak{M}_0\sqrt{1-\beta^2} ,$$

where $\mathfrak{M}_0 = u_e\,G_e$. One observes that the same dependency of the magnetic moment of the electron on the velocity also appears in the treatment of the energy eigenvalues of the electron in a magnetic field (*).

The character of the precession that is described by equation (28) can be explained most easily in the special case of a magnetic field that is perpendicular to the direction of motion. One will then have:

$$\frac{d\mathbf{G}_\mu}{dt} = 2u_\mu\sqrt{1-\beta^2}\,[\mathbf{G}_\mu\,\mathbf{H}] - [\mathbf{M}\,\mathbf{H}] ,$$

or after passing to the natural coordinate system:

$$\frac{d_r\mathbf{G}_\mu}{dt} = -\,[\mathbf{M}\,\mathbf{H}] .$$

If one writes that in the form of an equation in coordinates then that will give:

$$\begin{aligned}
\frac{d_r\mathbf{G}_1}{dt} &= -\;u_\mu\,\frac{2\sqrt{1-\beta^2}}{2-\beta^2}\,G_2\,H ,\\
\frac{d_r\mathbf{G}_2}{dt} &= \quad u_\mu\,\frac{2-\beta^2}{2\sqrt{1-\beta^2}}\,G_1\,H ,\\
\frac{d_r\mathbf{G}_3}{dt} &= 0 .
\end{aligned}$$

As one easily recognizes, the projection **G** onto the field direction keeps its magnitude, and the end of the vector **G** describes an ellipse with respect to the natural axes whose axis ratio is:

(*) Cf., the foregoing article, pp. 27, formula (33′).

$$\frac{a_{\parallel}}{a_{\perp}} = \frac{2\sqrt{1-\beta^2}}{2-\beta^2} ,$$

which makes the direction of motion opposite to the direction that corresponds to a positive magnetic moment, and the angular velocity is equal to $u_\mu H$, moreover. That elliptical precession must be combined with the uniform rotational motion of the natural axes with the angular velocity:

$$2u_\mu \sqrt{1-\beta^2}\, H .$$

For small velocities, the latter motion overlaps the rotational moment relative to the natural axes completely such that the resulting angular motion will result in the positive direction. For a velocity of $v = \sqrt{3}\, c / 2$, the angular velocity of the natural axes will be equal to the angular velocity of the vector **G** relative to the natural axes, and therefore the end of the vector **G** will describe a line. For extremely large velocities, one can neglect the angular motion in the natural axes, and one will then have:

$$\frac{d\mathbf{G}_\mu}{dt} = - [\mathbf{M}\ \mathbf{H}] .$$

In the general case of an arbitrary field, one has:

$$\frac{d\mathbf{G}_\mu}{dt} = - [\mathbf{M}, \mathbf{H}_\perp + [\mathbf{E}\ \boldsymbol{\beta}]] . \tag{43}$$

for large velocity.

It emerges from this equation that for large velocities **M** actually plays the role of the magnetic moment, which makes the magnetic moment possess a negative sign. That consequence is linked with certain caveats. In fact, the vectors **G** and **M** are not parallel, and therefore cannot speak of the sign of their quotient, but only the sense of rotation of the vector **G**. Those directions of rotation are opposite for equations (42) and (43), but there is no velocity that lies between them for which the derivative $d\mathbf{G}_\mu / dt$ will be equal to zero, i.e., the magnetic moment will not be equal to zero for any velocity.

It is difficult to compare equation (43) with any classical particle model, since the motion of a classical rotating particle for large velocities has not been researched enough. At any rate, one must observe the following fact here: When one goes from the derivative with respect to t to the derivative with respect to proper time, the effective field $(H_\perp + [\mathbf{E}\boldsymbol{\beta}]) / \sqrt{1-\beta^2}$ will be on the right-hand side, and therefore the equation also seems reasonable from the classical standpoint.

For a planar motion of the electron in a magnetic field, when we employ the natural axes, we will get from (39) that:

$$\frac{d_r \mathbf{G}_e}{dt} = 0 ,$$

i.e., the electron spin remains immobile relative to the natural triple. However, in the presence of an electric field, the rotational velocity of the triple does not coincide with the second term of formula (39) such that the precession with respect to the natural axes will also deviate from zero in this case.

We therefore arrive at the following result: For small velocities, the electron, as well as the meson, behave like classical rotating particles in regard to their gyroscopic properties. The magnetic moment of the meson does not vanish for large velocities, but its behavior exhibits some peculiarities [e.g., the negative sign in equation (43), the non-parallelism of **G** and **M**].

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