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Exterior forms and the mechanics of continuous media

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CONTENTS

INTRODUCTION

In volume IV of the Annales de l'Institut Fourier (1952), I showed the part that is played in the study of the mechanics of rigid systems by the existence of an exterior form Ω_2 of degree two and maximal rank on a $2n+1$-dimensional differentiable manifold V_{2n+1} that is the generator of the equations of motion. It is then natural to pose the question: Do the mechanical equations of continuous media possess a generating exterior form, and what is its topological support?

If one reverts to the most elementary concept of motion in a 3-dimensional Galilean continuous medium (viz., a numerical space ρ^3) then one will observe that this motion is nothing but a map φ of class C^r ($r \geq 2$) of the numerical space $\mathbb{R}^4$ (the product of the numerical space $\mathbb{R}^3$ with the real line) into ρ^3. In relativity, since time is related to the medium, the motion of a relativistic continuous medium will be a map of $\mathbb{R}^4$ into ρ^4. More generally, we are led to envision maps φ of class C^r (viz., *r-maps*) of a space $\mathbb{R}^p$ into a space ρ^n, and then maps from a manifold (V_p) into a manifold (W_n), and then to associate Ehresmann's manifold of first-order jets $J^1(V_p, W_n)$ with these maps. We prove that on $J^1(V_p, W_n)$, with $x = (x^1, x^2, \ldots, x^p) \in (V_p)$ and $\xi = (\xi^1, \xi^2, \ldots, \xi^n) \in (W_n)$, there exists an exterior form Ω_{p+1} of degree $p + 1$ such that the solutions of the associated exterior system $i(\mathbf{X})\, \Omega_{p+1} = 0$ that satisfy $d\xi^\sigma \wedge V_p = 0$, moreover ($V_p$ denotes the volume form on the manifold (V_p)), are solutions of a system of first-order partial differential equations that generalize Hamilton's equations. This system is equivalent to a system of linear second-order partial differential equations. The system of generalized Hamilton equations can be defined once one is given a Pfaff form ω on the manifold $J^1(V_p, W_n)$ and a volume form V_p on (V_p).

For the maps of mechanics, everything comes down to constructing the form Ω_{p+1} on $J^1(V_p, W_n)$. Now, a group G that operates on (V_p, W_n) and, by prolongation, on $J^1(V_p, W_n)$, is associated with mechanics. Since the group G must leave the equations of mechanics invariant, the form Ω_{p+1} must be invariant under G. This property permits us to specify the exterior form that corresponds to the case of Galilean mechanics. It is important to note that the equations of mechanics for undeformable systems are naturally derivable from the form Ω_{p+1} without the intervention of postulates on the internal forces: They are generated by a form Ω_2 of degree two on the manifold of the Lie group of displacements.

What makes this point of view interesting is the fact that it reveals the profound nature of the classical theorems of mechanics, as well as suggesting new research. The only inconvenience is that these theorems must be almost completely re-established, which is why this article is limited to the cases in which one may operate fruitfully. Later articles will discuss new points.

Chapter One

Exterior differential forms on $J^1(V_p, W_n)$.

§ 1. – Notions concerning jets.

Ehresmann has shown (Cogresso di Taormina 1951 and the Colloque de Géometrie Différentielle de Strasbourg 1953) that one may associate a set of jets $J^1(V_p, W_n)$ with the r-maps of a manifold (V_p) to a manifold (W_n) that is defined in the following manner:

Let φ be a map of a neighborhood of $x \in (V_p)$ into (W_n) so x is the *source* of the map and $x = \varphi(x) \in (W_n)$ is the *target*. Consider two admissible local charts g and γ in (V_p) and (W_n), resp., such that when u and y belong to the numerical spaces $\mathbb{R}^p$ and ρ^n, respectively, one has:

$$u \in \mathbb{R}^p: \qquad x = g(u)$$

$$y \in \rho^n: \qquad \xi = \varphi(x) = \gamma(y).$$

The composition of maps that is suggested by the diagram:

$$\begin{tikzcd} V_p \arrow[r, "\varphi"] & W_n \\ \mathbb{R}^p \arrow[u, "g"] \arrow[r, "\bar{\varphi}"] & \rho^n \arrow[u, "\gamma"'] \end{tikzcd}$$

leads us to imagine the restriction $\bar{g}$ of g to a neighborhood U of u such that the composed map $\bar{\varphi} = \gamma^{-1} \cdot \varphi \cdot \bar{g}$ is well-defined. One says that φ is an *r-map* of the point x when the map $\bar{\varphi}$ of U into ρ^n admits continuous partial derivatives with respect to the canonical coordinates of $\mathbb{R}^p$ of each order up to r in a neighborhood of the point u. Let $C_x^r(V_p, W_n)$ be the set of maps (φ, x), where φ is an r-map of the point x. Two elements (φ_1, x), (φ_2, x) of $C_x^r(V_p, W_n)$ are said to be *of the same r-class* if:

1) $\varphi_1(x), = \varphi_2(x)$.

2) The pair of local charts (g, γ) associates two maps $\bar{\varphi}_1$ and $\bar{\varphi}_2$ with φ_1 and φ_2, resp., whose partial derivatives of the same order $\leq r$ take the same value at the point u.

These definitions are independent of the pair of local charts (g, γ) since it suffices to consider another pair (g', γ') that partially covers the latter one.

1. The definition of a jet. – An infinitesimal jet of order r – or *r-jet* – of (V_p) to (W_n) is an r-class of $C_x^r(V_p, W_n)$, where x is the source. $j_x^r \xi$ will denote the r-jet that is determined by the pair $(\xi, x) \in C_x^r(V_p, W_n)$. The set of r-jets of (V_p) to (W_n) with source x is denoted by $J_x^r(V_p, W_n)$. The union of the $J_x^r(V_p, W_n)$ as x varies over V_p is called the *manifold of jets of order r*:

$$\bigcup_{x\in V_p} J_x^r(V_p, W_n) = J^r(V_p, W_n).$$

2. The set of jets of order 1: $J^1(V_p, W_n)$. – The set $J^1(\mathbb{R}^p, \rho^n)$ of jets of order 1 is homeomorphic to the $(np + n + p)$-dimensional manifold $\mathbb{R}^p \times N_{pn}^1 \times \rho^n$, in which a point has the canonical coordinates:

$$(x^1, x^2, \ldots, x^p, u_1^1, \ldots, u_i^\sigma, \ldots, u_p^n, \xi^1, \xi^2, \ldots, \xi^n),$$

in which the ξ^σ denote coordinates in ρ^n, the x^i denote coordinates in $\mathbb{R}^p$, and the $u_i^\sigma = \partial\xi^\sigma / \partial x^i$ denote the partial derivatives of the ξ^σ with respect to the x^i. When one is concerned with $J^1(V_p, W_n)$, the preceding set constitutes a system of local coordinates on that manifold.

3. The kernel of the space of jets of order 1. – We call an element with source 0 and target 0 – i.e., an element with the np canonical coordinates $(0, \ldots, 0, u_1^1, \ldots, u_i^\sigma, \ldots, u_p^n, 0, \ldots, 0)$ – the *kernel* of $J^1(\mathbb{R}^p, \rho^n)$, which we denote by N_{pn}^1.

In order to give some intrinsic significance to the kernel of the set of jets $J^1(V_p, W_n)$, we make the following remarks:

a) The elements of $J^1(\mathbb{R}, W_n)$ with source 0 and target ξ that have the coordinates $(0, \ldots, 0, u^1, \ldots, u^n, \xi^1, \ldots, \xi^n)$ are nothing but the tangent vectors to (W_n), the set of which will be denoted by $T(W_n)$. $T(W_n)$ is a fiber bundle with base (W_n) and fiber $\mathbb{R}^n$.

The elements of $J^1(\mathbb{R}^p, W_n)$ with source 0 and target ξ are nothing but the elements of the product of p examples of the tangent bundle to (W_n) at the point ξ.

$J_{x=0}^1(\mathbb{R}^p, W_n)$ is a fiber bundle that has (W_n) for its base and $\mathbb{R}^{np}$ for its fiber; one denotes this space by $T_p(W_n)$. There exists a canonical projection of $J^1(\mathbb{R}^p, W_n)$ onto $\mathbb{R}^p$, which suggests that $J^1(\mathbb{R}^p, W_n)$ is a fiber bundle that has $\mathbb{R}^p$ for its base, $T_p(W_n)$ for its fiber, and the homogeneous linear group $GL(n)$ of $\mathbb{R}^n$ for its structural group.

If one considers $J^1(V_p, W_n)$ then since there exists a canonical projection A of $J^1(V_p, W_n)$ onto (V_p), $J^1(V_p, W_n)$ is a fiber bundle relative to this projection A that has (V_p) for its base, $T_p(W_n)$ for its fiber, and $GL(n)$ for its structural group.

b) The elements of $J^1(V_p, \rho)$ with target 0 are nothing but the tangent covectors to (V_p), the set of which is notated by $T^*(V_p)$. It is a fiber bundle with base (V_p) and fiber $\mathbb{R}^{p*}$. The elements of $J^1(V_p, \rho^n)$ with target 0 and source x are nothing but the elements of the product of n copies of the dual to the tangent bundle to (V_p) at the point x. $J^1_{\xi=0}(V_p, \rho^n)$ is a fiber bundle with (V_p) for its base and $\mathbb{R}^{pn}$ for its fiber; one denotes this space by $T_n^*(V_p)$. There exists a canonical projection of $J^1(V_p, \rho^n)$ onto ρ^n, which suggests that $J^1(V_p, \rho^n)$ is a fiber manifold with ρ^n for its base, $T_n^*(V_p)$ for its fiber, and the homogeneous linear group $GL(p)$ of $\mathbb{R}^p$ for its structural group.

If one considers $J^1(V_p, W_n)$ then since there exists a canonical projection B of $J^1(V_p, W_n)$ onto (W_n), $J^1(V_p, W_n)$ is a fiber bundle relative to this projection B with (W_n) for its base, $T_n^*(V_p)$ for its fiber, and $GL(p)$ for its structural group.

c) There exists a canonical projection C of $J^1(V_p, W_n)$ onto $(V_p \times W_n)$. $J^1(V_p, W_n)$ is then a fiber manifold relative to this projection C with $(V_p \times W_n)$ for its base, the kernel of $J^1(V_p, W_n)$ for its fiber, and $GL(p) \times GL(n)$ for its structural group. The kernel of $J^1(V_p, W_n)$ is homeomorphic to the homogeneous space $L(\mathbb{R}^p, \rho^n)$ of homogeneous linear maps of $\mathbb{R}^p$ into ρ^n.

Remark. – These considerations extend to $J^k(V_p, W_n)$. (Cf., Ehresmann [3])

4. Coordinate changes in the kernel of $J^k(V_p, W_n)$. Consider a change of local coordinates in (W_n) that is defined by the formulas:

$$\xi^\sigma = \xi^\sigma(\eta^1, \ldots, \eta^\rho, \ldots, \eta^n);$$

we set:

$$\frac{\partial \xi^\sigma}{\partial \eta^\rho} = \alpha^\sigma_\rho .$$

If one considers a change of coordinates in $\mathbb{R}^p$ then the local coordinates for V_p are defined by the formulas:

$$x^i = f^i(y^1, y^2, \ldots, y^p);$$

we set:

$$\frac{\partial x^i}{\partial y^j} = a^i_j .$$

If one considers the changes that are inverse to the preceding ones then we can write, upon inverting the partial derivatives:

$$\frac{\partial \eta^\rho}{\partial \xi^\sigma} = \alpha_\sigma^\rho, \qquad \frac{\partial y^j}{\partial x^i} = a_i^j.$$

If one denotes the new canonical coordinates of the kernel of $J^1(V_p, W_n)$ by $V_j^\rho = \partial\eta^\rho / \partial y^j$ then from the relation:

$$u_i^\sigma = \frac{\partial \xi^\sigma}{\partial x^i} = \frac{\partial \eta^\rho}{\partial \xi^\sigma}\frac{\partial \eta^\rho}{\partial y^j}\frac{\partial y^j}{\partial x^i},$$

it results that:

(1) $$u_i^\sigma = \alpha_\rho^\sigma a_i^j v_j^\rho,$$

a formula that shows that the canonical coordinates of $J^1(V_p, W_n)$ may be identified with the components of a tensor that is constructed from the spaces T_ξ and $T_{x'}^*$, upon denoting the tangent space to (W_n) at the point ξ by T_ξ and the dual to the tangent space to (V_p) at the point x by $T_{x'}^*$; $u_i^\sigma \in T_\xi \otimes T_{x'}^*$.

6 [†]**. Hamiltonian coordinates in the kernel of** $J^1(V_p, W_n)$. In the development that we have seen, it is very advantageous to introduce another coordinate system into the kernel of the jets of order 1 that we call *Hamiltonian coordinates*, due to the role that they play in writing down a certain system of partial differential equations that define a family of maps from (V_p) to (W_n).

We give ourselves a volume form V_p on (V_p). Consider the completely anti-symmetric covariant tensors $p_{\sigma i_1 \cdots i_{p-1}}$ of order p that are constructed on $T^*(W_n)$ and $T^{p-1*}(V_p)$. By means of the unit volume tensor on the manifold (V_p), whose contravariant components are $\delta^{i_1 \cdots i_p}$, one defines a once-covariant, once-contravariant tensor p_σ^i by contraction:

$$\frac{1}{(p-1)!} p_{\sigma i_1 \cdots i_{p-1}} \delta^{i_1 \cdots i_p} = p_\sigma^{i_p}.$$

The tensor p_σ^i is an element of $T_\xi \otimes T_{x'}^*$. If one considers the manifold $J^1(W_n, V_p)$ then the p_i^σ are the canonical coordinates on the kernel of the set of jets of (W_n) into (V_p), which are inverse to the jets of (V_p) into (W_n) since they obey the coordinate transformation rule:

$$p_\sigma^i = \alpha_\sigma^\rho a_j^i q_\rho^j.$$

Now, $J^1(W_n, V_p)$ and $J^1(V_p, W_n)$ are two fiber manifolds that have the same base – namely, $W_n \times V_p$ – and the same fiber – viz., $L(\mathbb{R}^p, \rho^n)$ – which is the space of homogeneous linear maps of $\mathbb{R}^p$ to ρ^n. For this reason, one may take the p_σ^i to be

[†] [DHD: There was no section **5** in the original article.]

coordinates in the kernel of $J^1(V_p, W_n)$. However, it is essential to remark that we have not presented any means of expressing the canonical coordinates u_i^σ as functions of the p_σ^i; the structure that is implied by an exterior form Ω_{p+1} – which will be defined later – will permit us to relate the p_σ^i to the u_i^σ.

The p_σ^i may be used as coordinates for the kernel of $J^1(V_p, W_n)$. The same thing is true for the completely anti-symmetric tensor of order p that is defined by the contracted product of the p_σ^i with the covariant components of the unit volume tensor on (V_p):

(2) $$p_{\sigma i_1 \cdots i_{p-1}} = \frac{1}{p}\delta_{i_1 \cdots i_p} p_\sigma^i .$$

It is the components of the completely anti-symmetric covariant tensor $p_{\sigma i_1 \cdots i_{p-1}}$ to which we shall give the name of *Hamiltonian coordinates* for the kernel of $J^1(V_p, W_n)$.

Notation. – When one assigns an ordering to the coordinates in (V_p) – e.g., the natural ordering – in order to signify that just the index i has been omitted in $p_{\sigma 12\dots p}$, one writes:

$$p_{\sigma \breve{i}} = \delta_{12\dots p}\; p_\sigma^i .$$

7. Practical coordinates in the kernel of $J^1(V_p, W_n)$. When local coordinates systems have been chosen in (W_n) and (V_p), and when one assigns an ordering to the coordinates in (V_p), moreover, it is convenient to use the p_σ^i as the coordinates in the kernel of the set of jets for the sake of calculations, and they be identified with the components of a mixed tensor that is constructed from the spaces T_ξ^* and T_x .

§ 2. – Differential forms on $J^1(V_p, W_n)$.

1. The vector $\mathbf{p}_\sigma$ and the form $\boldsymbol{\theta}(\mathbf{p}_\sigma)$ V_p . – We denote the contravariant vector with source 0, target 0, and components $(0, \dots, 0, p_\sigma^1, \dots, p_\sigma^i, \dots, p_\sigma^p, 0, \dots 0)$ by $\mathbf{p}_\sigma$. The canonical map A of $J^1(V_p, W_n)$ onto (V_p) that was envisioned before in § 1.3 makes a form V_p' on $J^1(V_p, W_n)$ correspond to the volume form V_p that is defined on (V_p). The infinitesimal transformation operator $\boldsymbol{\theta}(\mathbf{p}_\sigma)$ ([1]) makes a form of degree p on $J^1(V_p, W_n)$ – viz., $\boldsymbol{\theta}(\mathbf{p}_\sigma)V_p'$ – correspond to V_p'.

Therefore, in the particular case where (V_p) is a Riemannian manifold whose fundamental metric tensor is g_{ij} , upon denoting the determinant of g_{ij} by g, the classical volume form V_p has the expression:

$$V_p = \sqrt{g}\;\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^p,$$

([1]) Cf., H. Cartan, Colloque Topologie. Bruxelles 1950.

(3) $$\boldsymbol{\theta}(\mathbf{p}_\sigma)V_p' = \sum_{i=1}^{p}(-1)^{i+1}\sqrt{g}\,dp_\sigma^i \wedge dx^1 \wedge \cdots \wedge d\check{x}^i \wedge \cdots dx^p ,$$

in which the $\vee$ sign placed above the dx^i signifies that this term has been omitted.

Remark. – In the preceding expression, one will observe that d denotes the symbol of *absolute* differentiation.

2. The form $\Phi_{p+1} = \sum_{\sigma=1}^{n} d\xi^\sigma \cdot \boldsymbol{\theta}(\mathbf{p}_\sigma)V_p'$. – The forms $d\xi^\sigma$ on W_n lift to $J^1(V_p, W_n)$ by means of the map B (§ 1.3). When one takes the exterior product of these forms with the n-form $\boldsymbol{\theta}(\mathbf{p}_\sigma)$ and sums over the index σ, this generates a form Φ_{p+1} of degree $p + 1$ whose support is $J^1(V_p, W_n)$. From the considerations that were developed in § 1.6, this closed form may be considered as being generated by a completely anti-symmetric tensor $p_{\sigma i_1 \cdots i_{p-1}}$ that is constructed over the spaces T_ξ^* and $(T_x^*)^{p-1}$.

One then has:

$$\Phi_{p+1} = \frac{-1}{(p-1)!} d(p_{\sigma i_1 \cdots i_{p-1}}) \wedge d\xi^\sigma \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} .$$

This expression for Φ_{p+1} is trivially invariant under a change of local coordinates in $J^1(V_p, W_n)$, which shows that Φ_{p+1} is an intrinsic form on $J^1(V_p, W_n)$.

3. The generating form Ω_{p+1}. – We denote a Pfaff form on $J^1(V_p, W_n)$ by ω. In a practical coordinate system, ω may be written:

$$\omega = \sum_{i,\sigma} X_i^\sigma dp_\sigma^i + X_\sigma d\xi^\sigma .$$

For reasons that will be justified by theorem I, we call the form Ω_{p+1} of degree $p + 1$ that is defined on $J^1(V_p, W_n)$:

(4) $$\Omega_{p+1} = \Phi_{p+1} + \omega \wedge V_p'$$

the *generating form*.

We now express Ω_{p+1} in a system of Hamiltonian coordinates. When all of the indices range from 1 to p, the volume form may be written:

$$V_p' = \frac{1}{p!}\delta_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} ,$$

$$\omega \wedge V_p' = (X_i^\sigma dp_\sigma^i) \wedge \frac{1}{p!}\delta_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} ,$$

$$= \frac{1}{(p-1)!} X_i^\sigma dp_{\sigma i_1 \cdots i_p}^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} ,$$

upon taking into account formula (2), which defines the Hamiltonian coordinates. Hence, the expression for Ω_{p+1} is:

$$\Omega_{p+1} = \frac{1}{(p-1)!} dp^i_{\sigma i_1 \cdots i_{p-1}} \wedge (-d\xi^\sigma \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} + X^\sigma_i dx^{i_1} \wedge \cdots \wedge dx^{i_p}) . \tag{5}$$

This expression for Ω_{p+1} is trivially invariant under a change of Hamiltonian coordinates, if one takes that to mean an arbitrary change of coordinates on (V_p) and (W_n), and a change of Hamiltonian coordinates on the kernel of $J^1(V_p, W_n)$.

Remark. – It is pointless to write down the part of the Pfaff form ω that is on (V_p), since it disappears under exterior multiplication by the volume form.

4. The system of exterior forms associated with Ω_{p+1} . – If **X** is an arbitrary vector field on $J^1(V_p, W_n)$ then one call the system of exterior equations $i(\mathbf{X})\, \Omega_{p+1} = 0$, which is reducible to $np + n + p$ equations, the *associated system of equations* for Ω_{p+1}, where $i(\mathbf{X})$ denotes the anti-derivation of H. Cartan. The Hamiltonian coordinates are useful in theoretical questions – in particular, when one wants to show the intrinsic character of certain forms. When it is a question of performing calculations, it is necessary to use practical coordinates. In such a coordinates system, Ω_{p+1} is written:

$$\begin{aligned}\Omega_{p+1} = {} & \sqrt{g}\,(-1)^{i+1} d\xi^\sigma \wedge dp^i_\sigma \wedge dx^1 \wedge \cdots \wedge d\breve{x}^i \wedge \cdots \wedge dx^p \\ & + \sqrt{g}\,(X^i_\sigma dp^i_\sigma + X_\sigma d\xi^\sigma) \wedge dx^1 \wedge \cdots \wedge dx^p .\end{aligned} \tag{6}$$

In particular, the associated system of equations $i(\mathbf{X})\, \Omega_{p+1}$ may be put into the following form:

np equations of the type:

$$\frac{1}{\sqrt{g}} \frac{\partial \Omega_{p+1}}{\partial (dp^i_\sigma)} = (-1)^i\, dx^\sigma \wedge dx^1 \wedge \ldots \wedge d\breve{x}^i \wedge \ldots \wedge dx^p + X^\sigma_i\, dx^1 \wedge \ldots \wedge dx^p = 0, \tag{7}$$

n equations of the type:

$$\frac{1}{\sqrt{g}} \frac{\partial \Omega_{p+1}}{\partial (d\xi^\sigma)} = \sum_{i=1}^{p} (-1)^{i+1} dp^i_\sigma \wedge dx^1 \wedge \cdots \wedge d\breve{x}^i \wedge \cdots \wedge dx^p + X_\sigma dx^1 \wedge \cdots \wedge dx^p = 0, \tag{8}$$

p equations of the type:

$$\begin{aligned}\frac{1}{\sqrt{g}} \frac{\partial \Omega_{p+1}}{\partial (dx^j)} = {} & \sum_{\sigma=1}^{n} \sum_{i=1}^{p} (-1)^{i+j+1} d\xi^\sigma \wedge dp^i_\sigma \wedge dx^1 \wedge \cdots \wedge d\breve{x}^i \wedge \cdots \wedge d\breve{x}^j \wedge \cdots \wedge dx^p \\ & + (-1)^j\, \omega \wedge dx^1 \wedge \ldots \wedge d\breve{x}^j \wedge \ldots \wedge dx^p = 0.\end{aligned} \tag{9}$$

THEOREM 1. – *If the Jacobian with np elements* $\dfrac{D(\cdots X_i^\sigma \cdots)}{D(\cdots p_\sigma^i \cdots)}$ *is non-zero then the solutions to the system* $i(\mathbf{X})\,\Omega_{p+1} = 0$ *that satisfy the n equations* $d\xi^\sigma \wedge V_p = 0$ *are locally the solutions to a system of linear partial differential equations* (H), *and are equivalent to the solutions to a system of n second-order linear partial differential equations* $S_{2,n}(\xi^\sigma)$ *with respect to the variables* x^i.

We now look for the solutions to the exterior system $i(\mathbf{X})\,\Omega_{p+1} = 0$ such that the ξ^σ are functions of x^i of class C^r, $r \geq 2$.

The *np* equations $\dfrac{\partial \Omega_{p+1}}{\partial (dp_\sigma^i)} = 0$ give:

(10)
$$\frac{\partial \xi^\sigma}{\partial x^i} = X_i^\sigma .$$

The X_i are functions of the local coordinates of $J^1(V_p, W_n)$. If $\dfrac{D(\cdots X_i^\sigma \cdots)}{D(\cdots p_\sigma^i \cdots)} \neq 0$ then equations (10) define the p_σ^i as functions of the $\dfrac{\partial \xi^\sigma}{\partial x^i} = u_i^\sigma$, which are the canonical coordinates of the kernel of $J^1(V_p, W_n)$.

Locally, one thus has:

(11)
$$p_\sigma^i = \varphi_\sigma^i \left(\frac{\partial \xi^\sigma}{\partial x^j}, x^h, \xi^\mu \right),$$

in which h, j are arbitrary numbers from the set (1 to p) and ρ, μ are arbitrary numbers from the set (1 to n). By means of equations (10), the p_σ^i then become the components of n vector fields $\mathbf{p}_\sigma$ ($\sigma = 1$ to n) on (V_p). With respect to the frame at the point x of (V_p), the differential of p_σ^i is, upon denoting the coefficients of the infinitesimal connection on (V_p) by Γ_{hj}^i:

$$dp_\sigma^i = \frac{\partial p_\sigma^i}{\partial x^j} dx^j + \Gamma_{hj}^i p^h dx^j = \frac{D(p_\sigma^i)}{Dx^j} dx^j .$$

The n equations $\dfrac{\partial \Omega_{p+1}}{\partial (d\xi^\sigma)} = 0$ then give, upon using the symbol D to indicate the absolute derivative:

(12)
$$\sum_{i=1}^{p} \frac{Dp_\sigma^i}{Dx^i} + X_\sigma = 0.$$

Upon taking the solution (11) to equations (10) into account, the system (12) then gives rise to a system of linear second-order partial differential equations $S_{2,n}(\xi^\sigma)$.

We now show that the last p equations of the associated system are verified, as consequences of (11) and (12). When one considers the p_σ^i to be functions of the x^i, by the intermediary of the functions ξ^σ and their first-order partial derivatives, the expression $\frac{1}{\sqrt{g}}\frac{\partial\Omega_{p+1}}{\partial(dx^j)}$ is written:

$$\frac{1}{\sqrt{g}}\frac{\partial\Omega_{p+1}}{\partial(dx^j)} = -\sum_{i,\sigma}\det\begin{bmatrix}\dfrac{\partial\xi^\sigma}{\partial x^i} & \dfrac{\partial\xi^\sigma}{\partial x^j}\\ \dfrac{Dp_\sigma^i}{Dx^i} & \dfrac{Dp_\sigma^i}{Dx^j}\end{bmatrix}+\sum_{\sigma=1}^{n}X_\sigma\frac{\partial\xi^\sigma}{\partial x^i}+\sum_{\sigma=1}^{n}\sum_{i=1}^{p}X_i^\sigma\frac{Dp_\sigma^i}{Dx^j}$$

$$=\sum_{\sigma=1}^{n}\frac{\partial\xi^\sigma}{\partial x^i}\left[\sum_{i=1}^{p}\frac{Dp_\sigma^i}{Dx^j}+X_\sigma\right]-\sum_{\sigma,i}^{n}\frac{Dp_\sigma^i}{Dx^j}\left(\frac{\partial\xi^\sigma}{\partial x^i}-X_i^\sigma\right).$$

The right-hand side is therefore zero, as a consequence of equations (10) and (12).

Remarks:

1) It is obvious that the solutions of the equations $i(\mathbf{X})\ \Omega_{p+1} = 0$, $d\xi^\sigma \wedge V_p = 0$ are represented by the solutions to the system $S_{2,n}(\xi^\sigma)$ only locally.

2) It is the system of equations (10) that defines the system of practical coordinates as functions of the coordinates of the kernel of $J^1(V_p, W_n)$. One sees that this system is obtained by equating the coefficient X_i^σ of the Pfaffian form ω with the canonical coordinate u_i^σ:

(13) $$X_i^\sigma(p_\rho^i, x^h, \xi^\mu) = u_i^\sigma.$$

The Latin indices take on all of the values from 1 to p, while the Greek indices take on all of the values from 1 to n.

3) If the Pfaff form ω is homologous to 0 – i.e., $\omega = dE$ – then the form Ω_{p+1} is a closed form: $d\Omega_{p+1} = 0$. Since $d[\boldsymbol{\theta}(\mathbf{p}_\sigma)\ V_p] = 0$, one may write:

$$\Omega_{p+1} = d\left[\sum_{\sigma=1}^{n}\xi^\sigma\boldsymbol{\theta}(\mathbf{p}_\sigma)V_p + E\,V_p\right].$$

The system of equations (10) and (12) then takes on the form:

$$\frac{\partial \xi^{\sigma}}{\partial x^{i}} = \frac{\partial E}{\partial p_{\sigma}^{i}},$$
$$(14) \qquad \sum_{i=1}^{p} \frac{D p_{\sigma}^{i}}{D x^{i}} = -\frac{\partial E}{\partial \xi^{\sigma}}.$$

This form (14) is the generalization of Hamilton's equations, which one obtains for $p = 1$. Geometrically, if one considers the maps of the number line t into (W_n) then the manifold of jets $J^1(t, W_n)$ is associated with these maps, which is a fiber bundle that has the number line t for its base and the tangent bundle $T(W_n)$ to (W_n) for its fiber. The elements $p_1, p_2, \ldots, p_n$ are identified with the components of the co-velocity in (W_n). It is for this reason that we call the system of equations (14) the *generalized Hamiltonian system*.

4) If the p-dimensional number space $\mathbb{R}^p$ is referred to a Cartesian coordinate system, and if the function E is a quadratic form with constant coefficients on $J^1(\mathbb{R}^p, W_n)$ then the system $S_{2,n}(\xi^{\sigma})$ has constant coefficients. Ω_{p+1} is the generating form of this system of second-order partial differential equations with constant coefficients.

5) The np functions p_{σ}^{i} on (V_p) are likewise solutions of a system of partial differential equations that are obtained by demanding that the n forms $\sum_{i=1}^{p} X_i^{\sigma} dx^i$ be closed.

6) If $\dfrac{D(\cdots X_i^{\sigma} \cdots)}{D(\cdots p_{\sigma}^{i} \cdots)} = 0$ then some very diverse circumstances are presented. The system $i(\mathbf{X})\, \Omega_{p+1} = 0$ might not have solutions, or it might admit ones that are constructed by starting with the solutions of a system of first-order partial differential equations (systems of first-order partial differential equations are submanifolds of $J^1(V_p, W_n)$), or further, that it admit solutions only for the jets that relate to the partial maps from a subset of (V_p) into (W_n).

5. In order to familiarize the reader with the preceding notations, we treat several examples that, to simplify, relate to the determination of a numerical function on a manifold (V_p). The manifold $J^1(V_p, \rho)$ is associated with maps of (V_p) to the number line ρ.

a) $p = 2$, $V_2 = \mathbb{R}^2$. The maps of $\mathbb{R}^2$ to ρ are defined by a numerical function of two variables x^1, x^2, which are the rectangular coordinates of a point of $\mathbb{R}^2$. Upon using the practical coordinates p^1, p^2 on $J^1(V_p, \rho)$, the Pfaff form is given by:

$$\omega = p^1\, dx^1 + k\, p^2\, dp^2 + g(x^1, x^2, p^1, p^2, \xi)\, d\xi,$$

in which k denotes a constant. The generating form Ω_3 is:

$$\Omega_3 = d\xi \wedge dp^1 \wedge dp^2 + d\xi \wedge dx^1 \wedge dp^2 + (p^1\, dp^1 + k\, p^2\, dp^2 + g\, d\xi) \wedge dx^1 \wedge dx^2.$$

In particular, the associated system $i(\mathbf{X})\,\Omega_3 = 0$ may be put into the form:

$$\frac{\partial \Omega_3}{\partial (dp^1)} = -\,d\xi \wedge dx^2 + p^1\, dx^1 \wedge dx^2 = 0,$$

$$\frac{\partial \Omega_3}{\partial (dp^2)} = d\xi \wedge dx^1 + k\, p^2\, dx^1 \wedge dx^2 = 0,$$

$$\frac{\partial \Omega_3}{\partial (d\xi)} = dp^1 \wedge dx^2 + dx^1 \wedge dp^2 + g\, dx^1 \wedge dx^2 = 0.$$

From theorem 1, we can dispense with writing down the other equations. The first two solutions of this exterior system such that ξ is a function on $\mathbb{R}^2$ are:

$$\frac{\partial \xi}{\partial x^1} = p^1, \qquad \frac{\partial \xi}{\partial x^2} = k\, p^2,$$

The Jacobian is:

$$\frac{D(X_1, X_2)}{D(p_1, p_2)} = \det\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = k.$$

If $k \neq 0$, p^1 and p^2 are functions of x^1 and x^2, and $p^1 = \dfrac{\partial \xi}{\partial x^1}$, $p^2 = \dfrac{1}{k}\dfrac{\partial \xi}{\partial x^2}$ then the third exterior equation gives:

$$\frac{\partial p^1}{\partial x^1} + \frac{\partial p^2}{\partial x^2} + g = 0,$$

from which, we find that the second-order partial differential equation is equivalent to the preceding three equations:

$$S_{2,1}(\xi): \qquad \frac{\partial^2 \xi}{(\partial x^1)^2} + \frac{1}{k}\frac{\partial^2 \xi}{(\partial x^2)^2} + g\left(x^1, x^2, \frac{\partial \xi}{\partial x^1}, \frac{1}{k}\frac{\partial \xi}{\partial x^2}, \xi\right) = 0,$$

which is of elliptic type if $k > 0$ and hyperbolic if $k < 0$.

If $k = 0$ then the Jacobian is $\dfrac{D(X_1, X_2)}{D(p_1, p_2)} = 0$, and one may no longer say that p^2 is a function of x^1, x^2. A possible solution to the system $i(\mathbf{X})\,\Omega_3 = 0$ is composed of $dp^2 = 0$. One may imagine the form $\tilde{\Omega}_3$ that is induced by Ω_3 on the submanifolds p^2 = constant.

It permits us to determine x as belonging to the class of maps of the number line x^1 to the number line ρ. In this case, $\tilde{\Omega}_3 = \Omega_2 \wedge dx^2$, in which x^2 is regarded as constant and x is determined to be the solution of the characteristic system of Ω_2.

b) Furthermore, take $p = 2$, $V_2 = \mathbb{R}^2$, and let ω have the form:

$$\omega = -\,p^2\, dp^1 + (p^1 + p^2)\, dp^2 + g\, d\xi,$$

in which g denotes an arbitrary function of x^1, x^2, p^1, p^2, ξ. A calculation that is analogous to the preceding one shows that the solutions to the equations $i(\mathbf{X})\,\Omega_3 = 0$ such that ξ is a function on $\mathbb{R}^2$ are locally solutions to the partial differential equations of parabolic type:

$$S_{2,1}(\xi): \qquad \frac{\partial^2 \xi}{(\partial x^1)^2} + g\left(x^1, x^2, \frac{\partial \xi}{\partial x^1} + \frac{\partial \xi}{\partial x^2}, -\frac{\partial \xi}{\partial x^1}, \xi\right) = 0.$$

c) For an arbitrary p, $V_p = \mathbb{R}^p$, which is referred to Cartesian coordinates, we propose to find the form Ω_{p+1} that generates the harmonic forms on $\mathbb{R}^p$; it all comes down to determining ω. Observe that, one the one hand, $\Delta\xi = \operatorname{div} \cdot \mathbf{grad}\ \xi$, and, on the other hand, the system of equations (10) and (12) may be written:

$$\frac{\partial \xi}{\partial x^i} = X_i(p^1, \ldots, p^p, x^1, \ldots, x^p p, x),$$

$$\sum_{i=1}^{p} \frac{\partial p^i}{\partial x^i} = -\,X.$$

If one considers the vector $\mathbf{p}(p^i)$ then the latter equation signifies that $\operatorname{div} \mathbf{p} = -\,X$. One thus obtains a solution by taking $X = 0$, $X_i = p^i$ – i.e., $\omega = p^i\, dp^i$ – or:

$$\omega = \tfrac{1}{2}\, d(\text{Norm}\ \mathbf{p}).$$

A very simple means of obtaining the Laplacian of a function in curvilinear coordinates, which we denote by $q^1, \ldots, q^p$, then results. If the metric tensor is g_{ij} and $\mathbf{p}$ denotes a vector of $\mathbb{R}^p$, with $\text{norm}(\mathbf{p}) = g_{ij}\, p^i\, p^j$, then consider the form:

$$\Omega_{p+1} = d\xi \wedge \theta(\mathbf{p})\, V_p + [d\ \text{norm}(\mathbf{p}) + X\, d\xi] \wedge V_p\,,$$

which is written in terms of curvilinear coordinates:

$$\Omega_{p+1} = \sum (-1)^{i+1} \sqrt{g}\; d\xi \wedge dp^i \wedge dq^1 \wedge \cdots \wedge d\breve{q}^i \wedge \cdots \wedge dq^p + \tfrac{1}{2}\sqrt{g}\, d(g_{ij}\, p^i\, p^j)$$

$$+\sqrt{g}\;X\,d\xi\wedge dq^1\wedge\ldots\wedge dq^p.$$

From the associated system, it results that:

$$\frac{\partial\xi}{\partial x^i}=g_{ij}\,p^j, \qquad X=\frac{-1}{\sqrt{g}}\sum_{i=1}^{p}\frac{\partial(\sqrt{g}\,p^i)}{\partial q^i},$$

namely, that:

$$\Delta\xi=-X=\sum_{i=1}^{p}\frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\,g^{ij}\,\dfrac{\partial\xi}{\partial x^j}\right)}{\partial q^i}.$$

Indeed, it suffices to write the equation $\dfrac{\partial\Omega_{p+1}}{\partial(d\xi)}=0$ in the form:

$$\sum_i(-1)^{i+1}[d(p^i\sqrt{g})-p^i d\sqrt{g}]\wedge dq^1\wedge\cdots\wedge d\breve{q}^i\wedge\cdots\wedge dq^p+X\sqrt{g}\,dq^1\wedge\cdots\wedge dq^p=0,$$

which gives:

$$\sum_{i=1}^{p}\frac{D(p^i\sqrt{g})}{Dq^i}-\sum_{i=1}^{p}p^i\frac{\partial\sqrt{g}}{\partial q^i}+X\sqrt{g}\;=0,$$

which gives the desired result upon remarking that:

$$\Gamma^i_{ji}=\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial q^j}.$$

Now, if one considers a Riemannian manifold (V_p) then a vector $\mathbf{p}$ that is tangent to (V_p) is an element of $J^1(V_p,\rho)$ with target 0. By means of the form Ω_{p+1} and its associated system, one generates a class of functions that are harmonic functions on the open sets of (V_p) in the case where the norm is always positive.

Therefore, on the sphere S, when referred to the coordinates of longitude θ and colatitude φ, the form $V_2=R^2\sin\varphi\,d\theta\wedge d\varphi$, the vector $\mathbf{p}$ has the coordinates of p^1, p^2 with respect to the natural frame, and the norm $N(\mathbf{p})=R^2\,[\sin^2\varphi(p^1)^2+(p^2)^2]$. The form:

$$\Omega_3=R^2\sin\varphi\,d\xi\wedge[dp^1\wedge d\varphi-dp^2\wedge d\theta)+(p^1\,dp^1\sin^2\varphi+p^2\,dp^2)\,R^4\sin\varphi\,d\varphi\wedge d\theta$$

gives the associated equations:

$$\frac{\partial\xi}{\partial\theta}=p^1\sin^2\varphi\,R^2, \qquad \frac{\partial\xi}{\partial\varphi}=\;p^2R^2, \qquad \frac{Dp^1}{D\theta}+\frac{Dp^2}{D\varphi}\;=0.$$

From a preceding remark, one may replace this latter equation by the equation:

$$\frac{\partial(p^1\sqrt{g})}{\partial\theta}+\frac{\partial(p^2\sqrt{g})}{\partial\varphi} = 0,$$

which gives the second-order equation:

$$\frac{1}{\sin^2\theta}\frac{\partial^2\xi}{\partial\theta^2}+\frac{\partial^2\xi}{\partial\varphi^2}+\cot\varphi\frac{\partial\xi}{\partial\varphi} = 0.$$

THEOREM 2. – *If the Pfaff form* ω *on* $J^1(\mathbb{R}^p, \rho^n)$ *has constant coefficients on the kernel then the solution to the associated system* $i(\mathbf{X})\ \Omega_{p+1} = 0$, *which are functions on* $\mathbb{R}^p$, *define a linear map of* $\mathbb{R}^p$ *into* ρ^n.

Since the C_i^σ are constant, if $\omega = \sum C_i^\sigma dp_\sigma^i$ then the first np of equations (10) give:

$$\frac{\partial\xi^\sigma}{\partial x^i} = C_i^\sigma,$$

from which, $\xi^\sigma = C_i^\sigma x^i + C^\sigma$. This solution satisfies the other equations of the system $i(\mathbf{X})\ \Omega_{p+1} = 0$. If the C^σ are zero then this solution comprises the representative polynomial of $J^1(V_p, W_n)$.

Consequence. – By adding this solution, one may always suppose that the Pfaff form has non-constant coefficients on the kernel of $J^1(V_p, W_n)$.

THEOREM 3. – *A system of partial differential equations* (*H*) *corresponds to any Pfaff form* ω *on* $J^1(V_p, W_n)$, *and conversely.*

Indeed, by Theorem 1, $\omega = X_i^\sigma dp_\sigma^i + X_\sigma dx^\sigma$ corresponds to the system:

$$(H)\qquad \begin{aligned} \frac{\partial\xi^\sigma}{\partial x^i} &= X_i^\sigma, \\ \sum_{i=1}^{p}\frac{Dp_\sigma^i}{Dx^i} &= -X_\sigma, \end{aligned} \qquad \sigma\in (1 \text{ to } n).$$

Conversely, for any given system (*H*), the right-hand sides X_i^σ, X_σ are given functions on $J^1(V_p, W_n)$, and, as a consequence, they correspond to a form $\omega = X_i^\sigma dp_\sigma^i + X_\sigma dx^\sigma$ on $J^1(V_p, W_n)$.

THEOREM 4. – *If the Pfaff form ω is the sum of Pfaff forms ω^1 and ω^2, and the solutions f_1^σ and f_2^σ of the system* $i(\mathbf{X})\,\Omega_{p+1} = 0$, *in the sense of Theorem* 1, *are assumed to exist for ω^1 and ω^2 then the solution that relates to ω is* $f_1 + f_2$.

This proposition is obvious, since it corresponds to the linearity of the system of partial differential equations $S_{2,n}(\xi^\sigma)$. In particular, if the X_σ are functions that are defined only on V_p then the solution is obtained from a sum of solutions that relate to the form $\omega = X_i^\sigma dp_\sigma^i$, and a particular solution to the system (H).

6. The forms Ω_{p+1} . – It may be the case that the Pfaff form ω that is defined on $J^1(V_p, W_n)$ depends upon certain numerical functions $\Phi^1, \dots, \Phi^r$ that are defined on $J^1(V_p, W_n)$, or upon certain operators that depend upon these functions. One will then imagine the form $\Omega_{p+r+1} = d\Phi^1 \wedge \dots \wedge d\Phi^r \wedge \Omega_{p+1}$ that is defined on $D^r \times J^1(V_p, W_n)$, in which, D^r is the product of r number lines, on which, the Φ^i take their values. The argument that was made in the proof of Theorem 1 persists upon replacing the operator $i(\mathbf{X})$ with the operator $i(\Phi^1) \wedge i(\Phi^2) \wedge \dots \wedge i(\Phi^r) \wedge i(\mathbf{X})$. In order to determine the set of functions ξ^σ and the functions Φ^i, one then adjoins the equations that define the functions Φ^i to the associated system. The motion of a perfect fluid constitutes an example of this case, when one takes density, pressure, and heating into account.

§ 3. – Properties of the kernel of $J^1(V_p, W_n)$.

In this section, which is principally concerned with the kernel of $J^1(V_p, W_n)$, we denote the latter by $NJ^1(V_p, W_n)$. There exist certain obvious properties of the kernel of $J^1(V_p, W_n)$ that will be used in the applications. They result from the fact that was pointed out in § 1.3 that $J^1(V_p, W_n)$ may be considered to be a fiber manifold that has $V_p \times W_n$ for its base and fibers that are isomorphic to the space $L(\mathbb{R}^p, \rho^n)$ of homogeneous linear maps of $\mathbb{R}^p$ into ρ^n.

PROPOSITION 1. – *If* $W_n = W_\alpha + W_\beta$ *then* $NJ^1(V_p, W_n)$ *is isomorphic to the product of* $NJ^1(V_p, W_\alpha)$ *with* $NJ^1(V_p, W_\beta)$.

It results that the property is true for $L(\mathbb{R}^p, \rho^{\alpha-\beta})$.

Remark. – This proposition is true for $NJ^1(V_p, W_\alpha \times W_\beta)$. It suffices to recall the definition of the r-jets with their source at x: $J_x^r(V_p, W_n)$.

Consequence. – The form Ω_{p+1} is the sum of two forms: Ω_{p+1}^α on $J^1(V_p, W_\alpha)$ and Ω_{p+1}^β on $J^1(V_p, W_\beta)$.

$$\xi \in W_\alpha: \qquad \Omega^{\alpha}_{p+1} = (-1)^{i+1}\, d\xi^\sigma \wedge dp^i_\sigma \wedge dx^1 \wedge \ldots \wedge d\check{x}^i \wedge \ldots \wedge dx^p + \omega^\alpha \wedge V_p\,,$$
$$\eta \in W_\beta: \qquad \Omega^{\beta}_{p+1} = (-1)^{i+1}\, d\xi^\sigma \wedge dq^i_\sigma \wedge dx^1 \wedge \ldots \wedge d\check{x}^i \wedge \ldots \wedge dx^p + \omega^\beta \wedge V_p\,.$$

Remark. – If ω^α and ω^β are basic forms on $J^1(V_p, W_\alpha)$ and on $J^1(V_p, W_\beta)$, resp., then the system of generalized Hamiltonian equations may be decomposed into two distinct systems.

PROPOSITION 2. *If $V_p = V_h \times V_k$ then $NJ^1(V_p, W_n)$ is isomorphic to the product of $NJ^1(V_h, W_n)$ with $NJ^1(V_k, W_n)$.*

The proof is identical to the proof of Proposition 1.

Remark. – This proposition is not exactly true for $NJ^k(V_h \times V_k, W_n)$ if $r > 1$. Indeed, if one imagines the source (x, y) to be in the kernel of $J^k(V_h \times V_k, W_n)$ then one must deal with composed elements such as $\dfrac{\partial^2 f}{\partial x^i \partial y^j}$ with $i + j = r$.

Consequence. – The form Ω_{h+k+1} may be written:

$$\Omega_{h+k+1} = \Phi_{h+1} \wedge V_k + (-1)^h\, V_h \wedge \Phi_{k+1} + \omega \wedge V_h \wedge V_k\,,$$

an expression in which V_h and V_k denote the two volume forms on the manifolds (V_h) and (V_k), respectively, ω denotes a Pfaff form on $J^1(V_h \times V_k, W_n)$, and Φ_{k+1} is a form of degree $k + 1$ on $J^1(V_p, W_n)$.

Indeed, a vector $\mathbf{p}_\sigma$ in $NJ^1(V_h \times V_k, W_n)$ is the sum of two vectors: $\mathbf{u}_\sigma$, which is in $NJ^1(V_h, W_n)$ and $\mathbf{v}_\sigma$, which is in $NJ^1(V_k, W_n)$: $\mathbf{p}_\sigma = \mathbf{u}_\sigma + \mathbf{v}_\sigma$.

$$\begin{aligned}\boldsymbol{\theta}(\mathbf{p}_\sigma)\, V_h \wedge V_k &= [\boldsymbol{\theta}(\mathbf{p}_\sigma)\, V_h] \wedge V_k + (-1)^h\, V_h \wedge [\boldsymbol{\theta}(\mathbf{p}_\sigma)\, V_k],\\ &= [\boldsymbol{\theta}(\mathbf{u}_\sigma)\, V_h] \wedge V_k + (-1)^h\, V_h \wedge [\boldsymbol{\theta}(\mathbf{v}_\sigma)\, V_k],\end{aligned}$$

from which:

$$\Omega_{h+k+1} = \left[\sum_{\sigma=1}^{n} d\xi^\sigma \wedge \boldsymbol{\theta}(\mathbf{u}_\sigma) V_h\right] \wedge V_k + (-1)^h\, V_h \wedge \left[\sum_{\sigma=1}^{n} d\xi^\sigma \wedge \boldsymbol{\theta}(\mathbf{v}_\sigma) V_k\right] + \omega \wedge V_h \wedge V_k\,,$$

or:

(18) $$\Omega_{h+k+1} = \Phi_{k+1} \wedge V_k + (-1)^h\, V_h \wedge \Phi_{k+1} + \omega \wedge V_h \wedge V_k\,.$$

PROPOSITION 3. – *If one lets (V_p), (V_q), (W_n) be three manifolds then any map f from (V_q) to (W_n) is embedded in a map of $J^1(V_p, V_q)$ into $J^1(V_p, W_n)$.*

We have seen that $J^1(V_p, W_n)$ may be regarded as a fiber bundle that has (V_p) for its base and $T_p(W_n)$ for its fiber. Since the map of (V_q) into W_n may be prolonged to a map of the tangent bundles $T(V_q)$ into $T(W_n)$, the proposition is obvious. In the domain of a coordinate system about the point η of (V_q), the map f makes the point ξ of (W_n) correspond to h by way of the formulas $\xi^\sigma = f^\sigma(\eta^\sigma)$. This map may be prolonged to a map of the bundle of tangent vectors to (V_q), which is denoted by $T(V_q)$, to the bundle $T(W_n)$ by means of the formulas:

$$\xi^\sigma = f^\sigma(h^1, \ldots, \eta^\sigma, \ldots, \eta^q), \qquad u_i^\sigma = \frac{\partial f^\sigma}{\partial \eta^\rho} v_i^\rho$$

which translate into a map of $J^1(V_p, V_q)$ to $J^1(V_p, W_n)$ that we call the *prolongation* of f to the manifolds of jets.

Remark. – If one uses the practical coordinates on the kernel of the jets then it is necessary to consider the inverse map f^{-1} and replace the latter formulas with:

$$p_\sigma^i = \frac{\partial \eta^\rho}{\partial \xi^\sigma} \pi_\rho^i .$$

COROLLARY. – *The form Ω_{p+1}, which is defined on $J^1(V_p, W_n)$, then lifts to the manifold $J^1(V_p, V_q)$. Explicitly, the equations of this lift are:*

$$d\xi^\sigma = \alpha_\rho^\sigma d\eta^\rho , \qquad p_\sigma^i = \alpha_\sigma^\rho \pi_\rho^i ,$$

$$\Phi_{p+1} = \sum (-1)^{i+1} d\xi^\sigma \wedge dp_\sigma^i \wedge dx^1 \wedge \cdots \wedge d\breve{x}^i \wedge \cdots \wedge dx^p ,$$

which becomes:

$$\bar{\Phi}_{p+1} = \sum (-1)^{i+1} d\eta^\sigma \wedge d\pi_\sigma^i \wedge dx^1 \wedge \cdots \wedge d\breve{x}^i \wedge \cdots \wedge dx^p ,$$

while:

$$\omega = X_i^\sigma dp_\sigma^i + X_\sigma d\xi^\sigma \qquad \textit{becomes} \qquad \bar{\omega} = Y_i^\rho d\pi_\rho^i + Y_\rho d\eta^\rho ,$$

from which, the form $\bar{\Omega}_{p+1}$ on $J^1(V_p, V_q)$ becomes:

$$\bar{\Omega}_{p+1} = (-1)^{i+1}\, d\eta^\sigma \wedge d\pi_\rho^i \wedge dx^1 \wedge \ldots \wedge d\breve{x}^i \wedge \ldots \wedge \bar{\omega} \wedge dx^1 \wedge \ldots \wedge dx^p.$$

APPLICATION. – Let $J^1(V_h \times V_k , W_n)$ be the manifold of jets that are associated with the maps of $V_h \times V_k$ to W_n, and let φ be a map of $V_h \times V_k$ to W_n . By means of the map φ, the form Ω_{h+k+1} on $J^1(V_h \times V_k , W_n)$ becomes a form on $J^1(V_h \times V_k , V_h \times V_k)$. If the maps of $V_h \times V_k$ to $V_h \times V_k$ reduce to the maps of (V_k) to (V_h) then the form Ω_{h+k+1} is a form on $V_h \times J^1(V_k, V_k)$. Formula (18), in which Ω_{h+1} is zero, gives:

$$\bar{\Omega}_{h+k+1} = V_h \wedge [(-1)^h\, \Phi_{k+1} + (-1)^h\, \omega \wedge V_k].$$

By summing over a chain of (V_h), this form becomes a form Ω_{k+1} of degree $k + 1$ on $J^1(V_k , V_k)$:

$$\Omega_{k+1} = (-1)^{hk} \int_{C(V_h)} (\Phi_{k+1} + \omega \wedge V_k) \wedge V_k \ .$$

We shall now proceed to apply this result to the generation of the equations of the mechanics of rigid systems, although first we shall begin with the equations of mechanics for continuous media.

CHAPTER II

The mechanics of Galilean continuous media.

§ 1. – The generating form for Galilean continuous media.

We shall show that three of the classical postulates of Galilean mechanics lead to the existence of a generating form for the equations of motion, in the sense of the fundamental theorem of Chapter I.

POSTULATE 1. – *There exists a universal time that is independent of the medium considered and is defined up to an additive constant:*

$$t = \tau + t_0 .$$

Consequence. – The motion of an n-dimensional continuous medium will be defined by the maps of the number space $\mathbb{R}^n \times t$ to an n-dimensional number space ρ^n.

POSTULATE 2. – *The n-dimensional space $\mathbb{R}^n$ is properly Euclidian.*

Consequence. – There exists a fundamental second-order covariant tensor g_{ij} on $\mathbb{R}^n$ that permits us to define the norm of a vector in $\mathbb{R}^n$ by means of a positive-definite quadratic form and its contravariant components. The image space of $\mathbb{R}^n$ under a map is likewise properly Euclidian; we denote its fundamental tensor by $\gamma_{\sigma\rho}$. The properly Euclidian structures of the space $\mathbb{R}^n$ and ρ^n extend to the kernel of the set $J^1(\mathbb{R}^n, \rho^n)$ of jets of order 1. Since an element of the kernel of this set of jets has p_σ^i for its coordinates, which is identified with a tensor that is constructed over the tangent space T_x at the point x of $\mathbb{R}^n$ and the dual T_ξ^* to the tangent space at the point ξ of ρ^n, the metric tensor on the kernel of $J^1(\mathbb{R}^n, \rho^n)$ is $\gamma^{\sigma\rho} g_{ij}$ for the system of coordinates that is used in the kernel.

Remark. – When one is concerned with the kernel of $J^1(\mathbb{R}^n \times t, \rho^n)$, one will consider a metric tensor $g_{n+1,\, n+1}$ on the number line t, and take the tensor that has the components $\gamma^{\sigma\rho} g_{ij}$, $\gamma^{\sigma\rho} g_{n+1,\, n+1}$ to be the metric tensor on the kernel of the jets.

POSTULATE 3. – *In n-dimensional media, all of the frames that are composed of systems of n orthogonal vectors and animated with a uniform translational motion are equivalent.*

Consequence. – If a point M of $\mathbb{R}^n$, which is the source of a map, has the coordinates $(x^1, x^2, \ldots, x^n)$ with respect to the first frame, and the point μ, which is the target of the map in ρ^n, has the coordinates $(x^1, x^2, \ldots, x^n)$, while M and μ have coordinates $(y^1, y^2, \ldots, y^n)$ and $(\eta^1, \eta^2, \ldots, \eta^n)$, respectively, with respect to a second frame then the formulas:

$$\begin{aligned} \xi^\sigma &= a^\sigma_\rho \eta^\rho + v^\sigma \tau , \\ t &= \tau + t_0 , \\ x^i &= a^i_j y^j \end{aligned} \tag{19}$$

define the passage from the second frame to the first one, in which the matrix $\left\| a^\sigma_\rho \right\|$ = $\left\| a^i_j \right\|$ is an orthogonal matrix, and thus constitute a representation of a group G that one calls the *Galilean group* and which is characteristic of Galilean mechanics.

From the fundamental theorem of Chapter I, a generating form of degree $n + 2$ on $J^1(\mathbb{R}^n \times t, \rho^n)$ is associated with the maps of $\mathbb{R}^n \times t$ to ρ^n, which has the following expression in practical coordinates in the first frame:

$$\Omega_{n+2} = \sum (-1)^{i+1} d\xi^\sigma \wedge dp^i_\sigma \wedge dx^1 \wedge \cdots \wedge d\breve{x}^i \wedge \cdots \wedge dx^n \wedge dt \; + \omega \wedge dx^1 \wedge \ldots \wedge dx^n \wedge dt.$$

Since Postulate 3 implies that there exists no privileged frame in an n-dimensional medium, this implies that the form Ω_{n+2} must be invariant under G. As with Ω_{n+1}, the only element that is not well-defined is ω, but the conditions of the invariance of Ω_{n+2} under G permit us to make ω more specific.

Formulas (19) define a map of $\mathbb{R}^n \times t \times \rho^n \times G$ to $\mathbb{R}^n \times t \times \rho^n$. The set $J^1(\mathbb{R}^n \times t, \rho^n)$ of jets is contained in $J^1(\mathbb{R}^n \times t, \mathbb{R}^n \times t \times \rho^n)$. The form Ω_{n+2}, which is defined on the subset of $J^1(\mathbb{R}^n \times t, \mathbb{R}^n \times \rho^n)$ for which the target is ρ^n, lifts to $J^1(\mathbb{R}^n \times t, \mathbb{R}^n \times t \times \rho^n \times G)$ by means of the map of $\mathbb{R}^n \times t \times \rho^n \times G$ to $\mathbb{R}^n \times t \times \rho^n$. For a fixed $\gamma \in G$, the restriction of Ω_{n+2} to $J^1(\mathbb{R}^n \times t, \mathbb{R}^n \times \rho^n \times \{\gamma\})$ is a form $\Omega_{n+2,\,\gamma}$. The invariance translates into the equality:

$$\Omega_{n+2,\,\gamma} = \Omega_{n+2} .$$

The group G operates on $\mathbb{R}^n \times t \times \rho^n$, and by prolongation to the kernel of the jets, on the set $J^1(\mathbb{R}^n \times t, \rho^n)$. The formulas that translate the prolongation of G to the kernel are, in canonical coordinates:

$$\begin{aligned} u^\sigma_i &= a^\sigma_\rho \underline{a}^j_i v^\rho_j , \\ u^\sigma_{n+1} &= a^\sigma_\rho v^\rho_j + v^\sigma , \end{aligned}$$

and, in practical coordinates:

$$p_\sigma^i = \underline{a}_\sigma^\rho a_j^i \pi_\rho^j ,$$
$$p_\sigma^{n+1} = \underline{a}_\sigma^\rho \pi_\rho^{n+1} + v^\sigma .$$

Invariants of the group G. – It is important to the point out that the Galilean group G is a subgroup of a group G', if one denotes the linear group that corresponds to an arbitrary matrix $\| a_j^i \|$ by G'. The group G' leaves the following n algebraic functions of the p_σ^i invariant:

$$\begin{aligned}
J_1' &= \sum p_i^i && \left(\sum \text{ composed of } n \text{ terms}\right), \\
J_2' &= \sum p_j^i p_i^j && \left(\sum \text{ composed of } n^2 \text{ terms}\right), \\
J_3' &= \sum p_j^i p_k^j p_i^k && \left(\sum \text{ composed of } n^3 \text{ terms}\right), \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
J_r' &= \sum p_{i_2}^{i_1} p_{i_3}^{i_2} p_{i_4}^{i_3} \cdots p_{i_1}^{i_r} && \left(\sum \text{ composed of } n^r \text{ terms}\right), \\
J_n' &= \sum p_{i_2}^{i_1} p_{i_3}^{i_2} p_{i_4}^{i_3} \cdots p_{i_1}^{i_n} && \left(\sum \text{ composed of } n^n \text{ terms}\right).
\end{aligned}$$

We denote the set of these invariants by J'. If one takes the Galilean group G then G leaves not only the set J' invariant, but also the norms of the vectors $\mathbf{p}^i$ $(p_1^i, p_2^i, \cdots, p_n^i)$, and, more generally, the norms of the r-vectors that are constructed from the vectors p^i. We denote the set of invariants of the Galilean group G by J.

The determination of the Pfaff form ω **that makes** Ω_{n+2} **be invariant under** G. – Divide the sum of terms that constitute ω into three partial sums:

1) $$\omega_c = \sum_{\sigma=1}^{n} X_{n+1}^\sigma dp_\sigma^{n+1} .$$

We verify that this sum corresponds to the kinetic part of the motion of a point of $\mathbb{R}^n$.

2) $$\omega_\sigma = \sum_{\sigma=1}^{n} \sum_{i=1}^{n} X_i^\sigma dp_\sigma^i .$$

We verify that if the X_i^σ are functions of the p_σ^i then this sum corresponds to the deformation of the medium.

3) $$\omega_\sigma = - H_\sigma dx^\sigma .$$

This sum corresponds to the elementary work that is done by a force field H_σ when it is applied to a point μ of the medium, and does not include any surface action, as one usually intends of it. The – sign that precedes H_σ arises from the fact that since Ω_{n+2} is defined only up to a multiplicative constant, it is necessary to choose one if we are to recover the equations of classical mechanics.

In order to perform the calculation of $\Omega_{n+2,\,g}$, we divide the terms of Ω_{n+2} into two categories:

a) The ones of the form:

$$\varphi_c = \sum_{\sigma=1}^{n} (-1)^{n+2} d\xi^\sigma \wedge dp_\sigma^{n+1} \wedge \cdots \wedge dx^n + \omega_c \wedge dx^1 \wedge \ldots \wedge dx^n \wedge dt.$$

b) The ones of the form:

$$\varphi_d = \sum_{\sigma=1}^{n} (-1)^{i+1} d\xi^\sigma \wedge dp_i^\sigma \wedge dx^1 \wedge \cdots \wedge d\breve{x}^i \wedge \cdots \wedge dx^n \wedge dt + \omega_d \wedge dx^1 \wedge \ldots \wedge dx^n \wedge dt.$$

We next study the manner by which the φ_c terms are transformed:

$$\varphi_c = [(-1)^{n+2}\, d\xi^\sigma \wedge dp_\sigma^{n+1} + (-1)^n\, \omega_c \wedge dt] \wedge dx^1 \wedge \ldots \wedge dx^n,$$

$$\varphi_c = (-1)^n\, [d\xi^\sigma \wedge dp^{n+1} + \omega_c \wedge dt] \wedge (dx^1 \wedge \ldots \wedge dx^n).$$

Under an orthogonal change of variables $dx^1 \wedge \ldots \wedge dx^n = dy^1 \wedge \ldots \wedge dy^n$, we occupy our time with only the terms:

$$\varphi_2 = \sum_{\sigma=1}^{n} d\xi^\sigma \wedge dp_\sigma^{n+1} + \omega_c \wedge dt.$$

From the formulas for a change of frame:

$$d\xi^\sigma = a_\rho^\sigma d\eta^\rho + v^\sigma d\tau, \qquad dp_\sigma^{n+1} = a_\sigma^\rho d\pi_\rho^{n+1}, \qquad \omega_c = X_{n+1}^\sigma dp_\sigma^{n+1},$$

we have:

$$\varphi_2 = a_\sigma^\rho(\alpha_\rho^\sigma d\eta^\rho + v^\sigma d\tau) \wedge d\pi_\rho^{n+1} + X_{n+1}^\sigma a_\sigma^\rho d\pi_\rho^{n+1} \wedge d\tau,$$

$$\varphi_2 = d\eta^\rho \wedge d\pi_\rho^{n+1} + (X_{n+1}^\sigma - v^\sigma) a_\sigma^\rho d\pi_\rho^{n+1} \wedge d\tau.$$

Now, under a change of frame $X_{n+1}^\sigma = a_\rho^\sigma Y_{n+1}^\rho$. Thus, X_{n+1}^σ may be identified with the components of a contravariant vector. It results from this that:

$$X_{n+1}^\sigma = u_{n+1}^\sigma F,$$

where F denotes a function of the invariants of the set J. Under a frame change:

$$X_{n+1}^\sigma = F(a_\rho^\sigma v_{n+1}^\rho + v^\sigma).$$

In order for φ_2 to be invariant, F can only be a constant equal to unity. It then results that $\omega_c = u^{\sigma}_{n+1} dp^{n+1}_{\sigma}$. Since the group G leaves the norm of the vector $\mathbf{p}^{n+1}$ invariant, ω_c may only be proportional to the differential of that invariant, which has the expression $\sum_{\sigma=1}^{n} (p^{n+1}_{\sigma})^2$ in terms of rectangular coordinates. If we let δ denote something that is constant under G, but that may very well be a function of the coordinates of the point μ and other parameters (e.g., temperature and pressure), then we have:

$$\omega_c = \frac{1}{2\delta} d\left[\sum_{\sigma=1}^{n} (p^{n+1}_{\sigma})^2\right].$$

From the fact that $\omega_c = u^{\sigma}_{n+1} dp^{n+1}_{\sigma}$, it results that $p^{n+1}_{\sigma} = \delta u^{\sigma}_{n+1}$, and as a result, one has, in canonical coordinates:

$$\omega_c = \frac{\delta}{2} d\left(u^{\sigma}_{n+1}\right)^2 = \frac{\delta}{2} d\left(\frac{\partial \xi^{\sigma}}{\partial t}\right)^2.$$

From the mechanical point of view, it is interesting to remark that the number δ is nothing but the density, and the vector $\mathbf{p}^{n+1}$ (whose components are p^{n+1}_{σ}) is nothing but the quantity of motion vector of the point μ.

From the foregoing, one has the theorem:

THEOREM 1. – *In Galilean mechanics, the kinetic part ω_c of the Pfaff form ω is one-half the differential of the vis viva δv^2, when δ is the density of matter at the point μ.*

Remark. – When one uses practical coordinates, the general expression for φ_2 in orthogonal coordinates is:

$$\varphi_2 = d\xi^{\sigma} \wedge dp^{n+1}_{\sigma} + \frac{1}{2\delta} d\left[\sum_{\sigma=1}^{n} (p^{n+1}_{\sigma})^2\right].$$

If the axes are not orthogonal then:

$$\varphi_2 = d\xi^{\sigma} \wedge dp^{n+1}_{\sigma} + \frac{1}{2\delta} d(\gamma^{\sigma\rho} g_{n+1,n+1} p^{n+1}_{\sigma} p^{n+1}_{\rho}).$$

Now, let us study the transformation of φ_d :

$$\varphi_d = \sum_{\sigma=1}^{n} (-1)^{i+1} d\xi^{\sigma} \wedge dp^{\sigma}_{i} \wedge dx^1 \wedge \cdots \wedge d\breve{x}^i \wedge \cdots \wedge dx^n \wedge dt + \omega_d \wedge dx^1 \wedge \ldots \wedge dx^n \wedge dt.$$

The presence of the factor $dt = d\tau$ in the exterior products that constitute the terms of φ_d leads to the conservation of only $a_\rho^\sigma d\eta^\rho$ in the expression for $d\xi^\sigma$. If one denotes the minor of the determinant a_j^i relative to the element a_j^i by A_j^i:

$$dx^1 \wedge \ldots \wedge d\breve{x}^i \wedge \ldots \wedge dx^n = (-1)^{i+j'} A_{j'}^i dy^1 \wedge \cdots \wedge dy^{j'} \wedge \cdots \wedge dy^n .$$

If one denotes an element of the inverse matrix to $\left\| a_j^i \right\|$ by $\alpha_i^{j'}$ then $A_{j'}^i$ = det | a | · $\alpha_i^{j'}$. Since the matrix a_j^i is an orthogonal matrix, det | a | = + 1 and $\alpha_i^{j'} = a_j^i$. Under a frame change, $dp_\sigma^i = a_\sigma^\mu a_j^i d\pi_\mu^j$, from which, the transform of the terms of φ_d that do not include ω_d is:

$$\left((-1)^{i+1} a_\rho^\sigma d\eta^\rho (-1)^{i+j'}\right)(a_\sigma^\mu a_j^i d\pi_\mu^j) a_{j'}^i (dy^1 \wedge \cdots \wedge d\breve{y}^i \wedge \cdots \wedge dy^n \wedge d\tau .$$

From the properties of orthogonal matrices, it results that:

$$a_j^i a_{j'}^i = 0 \qquad \text{if } j' \neq j, \qquad\qquad a_j^i a_{j'}^i = 1 \qquad \text{if } j' = j.$$

It results that:

$$\varphi_d = \sum (-1)^{j+1} d\eta^\rho \wedge d\pi_\rho^j \wedge dy^1 \wedge \cdots \wedge d\breve{y}^j \wedge \cdots \wedge dy^n \wedge d\tau + \omega_d \wedge dy^1 \wedge \ldots \wedge dy^n \wedge d\tau.$$

Now, imagine the ω_d part of ω. In $\omega_d = X_i^\sigma dp_\sigma^i$, the X_i^σ are functions of the p_σ^i. Under a change of variables $\omega_d = X_i^\sigma dp_\sigma^i = Y_j^\rho d\pi_\rho^j$, the tensorial quantity:

$$X_i^\sigma = a_\rho^\sigma a_i^j Y_j^\rho$$

shows that the X_i^σ may be identified with the components of a mixed tensor that is constructed over the tangent space to $\mathbb{R}^n$ and its dual. If M_i^σ denotes a mixed tensor that is constructed over $\mathbb{R}^n$ and its dual then one has $X_i^\sigma = M_i^\sigma f$ for $\sigma \neq i$, where f is a function of the invariants of the Galilean group G, and $X_i^\sigma = M_i^\sigma f + g$ for $\sigma = i$, where f and g denote two functions of the invariants of the Galilean group G. The presence of the invariant g in the expression for X_i^σ stems from the fact that one may have terms of the form $\left(\sum_{i=1}^n X_i^i\right)\left(\sum_{i=1}^n dp_i^i\right)$ in ω_d, where each term of the product $\left(\sum_{i=1}^n X_i^i\right)\left(\sum_{i=1}^n dp_i^i\right)$ is an invariant.

In particular, the possible expressions for X_i^σ are:

1) For $\sigma \neq i$: $X_i^\sigma = \gamma^{\sigma\rho} g_{ij}\ p_\sigma^i f$, for $\sigma = i$: $X_i^i = p_i^i f + g$.

2) If one considers the contracted product $C_i^\sigma = \sum_{j=1}^{n} p_j^\sigma p_i^j$, and more generally, the contracted product of r indices $C_i^\sigma = \sum p_{j_1}^\sigma p_{j_2}^{j_1} p_{j_3}^{j_2} \cdots p_i^{j_r}$, then a very general possible expression for X_i^σ consists of a linear function of the C_i^σ, where the coefficients are arbitrary functions of the group G.

Finally, we remark that the application of the fundamental theorem of Chapter I gives the generalized Hamiltonian equations:

$$\frac{\partial \xi^\sigma}{\partial x^i} = X_i^\sigma, \qquad \sum_{i=1}^{n} \frac{\partial p_\sigma^i}{\partial x^i} - X_\sigma = 0.$$

For $i \leq n$, the first n^2 equations, by the nature of their right-hand sides X_i^σ, allow us to study the map of $\mathbb{R}^n$ to ρ^n, and, as a result, the deformation of the medium. This is why $\omega_d = X_i^\sigma dp_\sigma^i$ corresponds to the deformations.

The following theorem results from this:

THEOREM 2. – *In Galilean mechanics, the ω_d part of the Pfaff form ω, which corresponds to the deformations of the medium, is a linear function of the components of a mixed tensor M_i^σ that is constructed over $\mathbb{R}^n$ and its dual, and thus has the coefficients:*

$$X_i^\sigma = M_i^\sigma f + \delta_i^\sigma g,$$

in which δ_i^σ denotes the Kronecker symbol, while f and g are functions of the invariants of the Galilean group.

We have therefore constructed a form Ω_{n+1} that is invariant under the Galilean group G and whose support is the jet manifold $J^1(\mathbb{R}^n \times t, \rho^n)$. From the fundamental theorem I of Chapter I, the system that is associated with Ω_{n+1} leads to second-order partial differential equations. These equations exhibit the characteristics that we would like for the axioms of Galilean mechanics to possess. We then propose to axiomatize the latter in the following manner:

AXIOM. – *The equations of Galilean mechanics on an n-dimensional continuous medium are generated by the system of exterior equations that are associated with the exterior form Ω_{n+1} of degree n + 2, which has its support one the jet manifold $J^1(\mathbb{R}^n \times t, \rho^n)$, and which is invariant under the Galilean group G, a system whose solutions one takes to be functions on $\mathbb{R}^n \times t$ of class C^r ($r \geq 2$).*

Remarks. –

1) The preceding axiom entails that we must have assumed postulates I, II, III.

2) The applications show that one therefore recovers the classical equations, and that one may pose very broad hypotheses regarded the deformations of the medium in another context, moreover. Note that it is the energy-momentum vector that is naturally introduced in the second-order partial differential equations.

3) One will observe that there is no reason to introduce the notions of the energy of the deformation tensor and the constraint tensor in order for us to arrive at these results.

4) A general means of generating the mechanics of an n-dimensional continuous medium (time being one of the local coordinates in the medium) is to consider a form Ω_{n+1} on a manifold V_n that admits a group or pseudo-group of transformations G, a form that has its support on the set of jets of maps from V_n to V_n and is invariant under G.

5) In an arbitrary system of practical coordinates, the expression for the generating form for the motions of the n-dimensional Galilean continuous medium is:

(20) $\Omega_{n+1} =$

$$\sqrt{g}\left[\sum_{i=1}^{n}(-1)^{i+1}d\xi^{\sigma}\wedge dp_{\sigma}^{i}\wedge dx^{1}\wedge\cdots\wedge \breve{dx^{i}}\wedge\cdots\wedge dx^{n}\wedge dt+\omega\wedge dx^{1}\wedge\cdots\wedge dx^{n}\wedge dt\right]$$

with:

$$\omega = X_i^{\sigma}dp_{\sigma}^{i}+d\left(\frac{1}{2\delta}\gamma^{\sigma\rho}g_{n+1,n+1}p_{\sigma}^{n+1}p_{\rho}^{n+1}\right)-H_{\sigma}d\xi^{\sigma}, \qquad g = \det g_{ij}.$$

The functions X_i^{σ} must be determined specifically in each case that one encounters in the applications.

§ 2. – The mechanics of filaments.

One must adapt the preceding theory to the dimensions of the medium in question. A filament is a medium with two of its dimensions being negligible when compared to the third one. The points of the filament are thus defined by means of a variable length s_0, which is not subject to tension.

A. STATICS. – The consideration of the equilibrium state of the filament gives rise to the study of maps from the number line s_0 into the 3-dimensional number space ρ^3. From the general theory, we know that the generating form for the equations of statics for the filament is a form of degree two on the jet manifold $J^1(s_0, \rho^3)$:

$$\Omega_2 = \sum_{\sigma=1}^{2} d\xi^\sigma \wedge dp_\sigma + \omega \wedge ds_0 .$$

Ω_2 must be invariant under the transformations of the Galilean group, so the ω_d part of ω must be invariant, and as a result it must depend upon only the differential of a function of just one invariant I, which is the norm of the vector **p** (whose components are p_1, p_2, p_3), in this case. Let $f(I)$ be this function of:

$$I = \sum_{\sigma=1}^{2} (p_\sigma)^2 ,$$

and let:

$$\omega = 2\frac{\partial f}{\partial I} p_\sigma dp_\sigma - H_\sigma d\xi^\sigma .$$

(H_σ are the components of a force field in ρ^3.)

The characteristic equations for Ω may then be written:

$$-\frac{\partial \xi^\sigma}{\partial s_0} + 2\frac{\partial f}{\partial I} p_\sigma = 0,$$
$$\frac{\partial p_\sigma}{\partial s_0} - H_\sigma = 0.$$

In order to determine the function $f(I)$, it is necessary to make some hypothesis on the manner by which the filament behaves.

a) *Inextensible filament.* – Let s denote the arc length of the profile of the filament in its equilibrium position under the action of the applied forces. The hypothesis of inextensibility translates into $ds^2 = ds_0^2$, from which, $\sum_{\sigma=1}^{2} \left(\frac{\partial \xi^\sigma}{\partial s_0}\right)^2 = 1$. From the first three Hamiltonian equations, it then results that:

$$\left(\frac{\partial f}{\partial I}\right)^2 = \frac{1}{4I} ,$$

so the sign of ω_d must be the same as that of ω_h and $f = -\sqrt{I}$.

The generating form for the equations of statics for the filament is therefore:

$$\Omega_2 = \sum_{\sigma=1}^{2} d\xi^\sigma \wedge dp_\sigma - d\left(\sqrt{p_1^2 + p_2^2 + p_3^2}\right) \wedge ds_0 - H_\sigma dx^\sigma \wedge ds_0 .$$

TENSION. – If δ_0 denotes the linear density of the filament then one calls the product $T = \delta_0\sqrt{I}$ the *tension* at a point. Upon replacing $\sqrt{I}$ with T / δ_0, one obtains:

$$\Omega_2 = \sum_{\sigma=1}^{2} d\xi^\sigma \wedge dp_\sigma - \left(\frac{\delta_0}{T}\sum_{\sigma=1}^{3} p_\sigma dp_\sigma + H_\sigma d\xi^\sigma\right) \wedge ds_0 , \tag{21}$$

so the characteristics equations:

$$\frac{d\xi^\sigma}{ds_0} + p_\sigma \frac{\delta_0}{T} = 0, \qquad \frac{d\xi^\sigma}{ds_0} - H_\sigma = 0$$

take on the classical form upon setting $d\xi^\sigma / ds_0 = u^\sigma$:

$$\frac{d(Tu^\sigma)}{ds_0} + \delta_0 H_\sigma = 0.$$

b) *Extensible filament.* – If one assumes a law of elongation that is proportional to the tension:

$$ds = ds_0 (1 + kT)$$

then, from Hamilton's equations, it results that:

$$\left(\frac{ds}{ds_0}\right)^2 = (1 + kT)^2 = 4I\left(\frac{\partial f}{\partial I}\right)^2 .$$

Since $T = \delta_0\sqrt{I}$, by definition, one has:

$$\frac{\partial f}{\partial I} = \frac{1}{2\sqrt{I}}(1 + k\delta_0\sqrt{I}) ,$$

from which:

$$f = \sqrt{I} + \frac{k\delta_0}{2} I .$$

The generating form Ω_2 then assumes the initial form:

$$\Omega_2 = \sum_{\sigma=1}^{3} d\xi^\sigma \wedge dp^\sigma - df \wedge ds_0 - H_\sigma d\xi^\sigma \wedge ds_0 ,$$

since:

$$df = \frac{1}{2\sqrt{I}}(1 + kT)\, dI , \qquad df \wedge ds_0 = \frac{1}{2\sqrt{I}}(1 + kT)\, dI \wedge ds_0 .$$

Upon taking into account that $ds = (1 + kt)\, ds_0$, one finds that:

$$df \wedge ds_0 = \frac{1}{2\sqrt{I}} dI \wedge ds = \frac{1}{\sqrt{I}} \sum_{\sigma=1}^{3} p_\sigma dp_\sigma \wedge ds ,$$

from which, the new expression for Ω_2 as a function of the differential ds of an arc of the profile of the equilibrium curve becomes:

$$\Omega_2 = \sum_{\sigma=1}^{3} d\xi^\sigma \wedge dp^\sigma - \frac{1}{\sqrt{I}} \sum p_\sigma dp_\sigma \wedge ds - H_\sigma \frac{1}{1+kT} d\xi^\sigma \wedge ds ,$$

which generates the classical equations.

B. DYNAMICS. – The Galilean dynamics of filaments comes down to the study of maps of $\mathbb{R}^2 = s_0 \times t$ (s_0 is the number line in which the parameter that fixes the length of the filament at rest takes its value and t is the temporal number line) into ρ^3. From the theoretical study, the generating form for the equations of motion is a form of degree 3 on $J^1(s_0 \times t, \rho^3)$:

$$\text{(22)} \quad \Omega_2 = \sum_{\sigma=1}^{3} d\xi^\sigma \wedge dp_\sigma^1 \wedge dt - \sum_{\sigma=1}^{3} d\xi^\sigma \wedge dp_\sigma^2 \wedge ds_0 - d\sqrt{\left(p_1^1\right)^2 + \left(p_2^1\right)^2 + \left(p_3^1\right)^2} \wedge dt$$

$$+ d\left(\frac{\left(p_1^2\right)^2 + \left(p_2^2\right)^2 + \left(p_3^2\right)^2}{2}\right) \wedge ds_0 \wedge dt - H_\sigma d\xi^\sigma \wedge ds_0 \wedge dt .$$

Example. – Transverse vibrations of stretched filaments. – Assume that the filament extends along Ox and that the tension that is imposed in this direction is T_0. Assume that there is no longitudinal displacement – i.e., $\xi^1 = x$ – so the vibration is defined only along the y-axis, and there is only one unknown function – viz., ξ^2 – that is generated by the associated equations of the form:

$$\Omega_3 = d\xi^\sigma \wedge dp_2^1 \wedge dt - d\xi^\sigma \wedge dp_2^2 \wedge dx - d\sqrt{(T_0/\delta_0)^2 + \left(p_2^1\right)^2} \wedge dx \wedge dt + p_2^2 dp_2^2 \wedge dx \wedge dt$$
$$- H_2\, d\xi^2 \wedge dx \wedge dt.$$

The associated equations give:

$$\frac{\partial \xi^2}{\partial x} + \frac{p_2^1}{\sqrt{(T_0/\delta_0)^2 + (p_2^1)^2}} = 0, \qquad -\frac{\partial \xi^2}{\partial t} + p_2^2 = 0, \qquad \frac{\partial p_2^1}{\partial x} + \frac{\partial p_2^2}{\partial t} - H_2 = 0.$$

If one assumes that T_0 / δ_0 is very large when compared to p_2^1 then one may write $p_2^1 = -\dfrac{T}{\delta_0}\dfrac{\partial\xi^2}{\partial x}$, which gives the classical second-order equation:

$$\frac{T}{\delta_0}\frac{\partial^2\xi^2}{(\partial x)^2} - \frac{\partial^2\xi^2}{(\partial t)^2} = H_2 .$$

Remark. – In the foregoing analysis, we assumed that the space ρ^3 was referred to rectangular axes. Upon taking an arbitrary coordinate system on ρ^3, it suffices to take I to be the invariant $I = \gamma^{\sigma\rho} p_\sigma p_\rho$, where $\gamma^{\sigma\rho}$ denotes the metric tensor on contravariant form.

PERFECT FLUID. – A continuous medium will be said to have *perfect fluid type* if the ω_d part of the Pfaff form ω is identically zero, and if, in addition, there exists a function F of the point μ such that the form $1/\delta\, dF$ (in which δ is the density of the medium at the point μ) appears in ω_d .

The generating form of the equations will then be of degree 5 on the jet manifold $J^1(\mathbb{R}^3 \times t, \rho^3)$:

$$\Omega_5 = (-1)^5\sum_{\sigma=1}^{3} d\xi^\sigma \wedge dp_\sigma^4 \wedge V_3 + \tfrac{1}{2} d\left[\left(p_1^4\right)^2 + \left(p_2^4\right)^2 + \left(p_3^4\right)^2\right]\wedge V_4$$
$$+ \left(\frac{1}{\delta}\frac{\partial(P-\beta\tau)}{\partial\xi^\sigma} - H_\sigma\right) d\xi^\sigma \wedge V_4$$

in which P denotes the pressure, τ, the temperature, H_σ, the components of the force field at the point μ of the medium, β is the dilatation coefficient at constant pressure, and $V_3 = dx \wedge dy \wedge dz$, while $V_4 = V_3 \wedge dt$.

One then deduces from Ω_5 that:

$$\frac{\partial\xi^\sigma}{\partial t} = p_\sigma^4, \qquad \delta\frac{\partial p_\sigma^4}{\partial t} + \frac{\partial(P-\beta\tau)}{\partial\xi^\sigma} - \delta H_\sigma = 0,$$

from which, we deduce the classical equations:

$$\delta\frac{\partial^2\xi^\sigma}{\partial t^2} = -\frac{\partial(P-\beta\tau)}{\partial\xi^\sigma} + \delta H_\sigma .$$

Since the functions P, τ, δ are auxiliary unknown functions, one must add three other equations to the preceding one: In particular, one can add the characteristic equation of the fluid $(P, \tau, \delta) = 0$, and the conservation of mass. This latter condition may be interpreted in the following manner: Let $\mathbf{p}^4$ be the vector whose components are (p_1^4, p_2^4, p_3^4, $p_4^4 = d$) and let $\boldsymbol{\theta}(\mathbf{p}^4)$ denote the infinitesimal transformation operator relative to the field $\mathbf{p}^4$, so:

$$\boldsymbol{\theta}(\mathbf{p}^4)\ V_4 = 0.$$

This suggests that we consider a medium that has four dimensions, along with maps of $\mathbb{R}^4$ into ρ^4. Since the density is variable, one imagines the form [*sic*]:

$$\Omega_5 = \sum_{\sigma=1}^{4} (-1)^5\, d\xi^\sigma \wedge dp_\sigma^4 \wedge V_3 + \frac{1}{2\delta} d\left[\left(p_1^4\right)^2 + \left(p_2^4\right)^2 + \left(p_3^4\right)^2 + \left(p_4^4\right)^2\right] \wedge V_4$$
$$+ \sum_{\sigma=1}^{3}\left(\frac{\partial(P-\beta\tau)}{\partial\xi^\sigma} - H_\sigma\right) d\xi^\sigma \wedge V_4 + \sum_{\sigma=1}^{3}\left(\frac{\partial p_\sigma^4}{\partial x^\sigma} - (?)\right) d\xi^4 \wedge V_4\ .$$

The generalized Hamiltonian equations that one deduces are:

$$\frac{\partial\xi^\sigma}{\partial t} = \frac{p_\sigma^4}{\delta}\frac{\partial p_\sigma^4}{\partial t} + \frac{\partial(P-\beta\tau)}{\partial\xi^\sigma} - \delta H_\sigma = 0 \qquad \sigma = 1, 2, 3,$$

$$\frac{\partial p_\sigma^4}{\partial t} + \sum_{\sigma=1}^{3}\left(\frac{\partial p_\sigma}{\partial \xi^\sigma}\right) = 0.$$

The latter equation translates into the conservation of mass, while the condition $p_4^4 = \delta$ entails that $\partial\xi^4 / \partial t = 1$, from which, $\xi^4 = t$ + constant, in accord with the postulate concerning time in the case of Galilean mechanics.

Remark.- The theory of permanent motions is an immediate consequence of the case where Ω_5 admits the infinitesimal transformation that is associated with the vector $\mathbf{t} = (0, 0, \ldots, 0, t)$, which translates into:

$$\boldsymbol{\theta}(\mathbf{t})\ \Omega_5 = 0.$$

§ 4. – *n*-dimensional isotropic media; the Navier-Stokes equations.

The ω_d part of the Pfaff form ω may be a closed form that is homologous to 0. In this case, there exists a function E on the invariants of the set J for the group G of an isotropic medium. One does not generally know how to determine the function E that corresponds to certain properties of the medium, such as having an invariant volume. That is why one confines oneself to looking for an analysis of limited scope; from Theorem 2 of Chapter I, the interesting terms are of degree at least two. Upon confining ourselves to these second-degree terms, there are only two possible ones: $(J_1)^2$ and J_2, if we denote the invariant $\sum_{i=1}^{n} p_i^i$ by J_1, and the quadratic form for the p_σ^i on the kernel $J^1(\mathbb{R}^n, \rho^n)$ by J_2. Upon denoting two coefficients, which might depend upon the temperature, by α and β, one then has:

$$E = +\tfrac{1}{2}[\alpha(J_1)^2 + \beta J_2].$$

Therefore, in the case of the approximation that amounts to the form that the classical linear approximation takes in this theory, the n-dimensional isotropic media ($n \geq 2$) depend upon only two physical coefficients other than the density.

In an arbitrary system of coordinates the generating form is written:

$$\Omega_{n+2} = \sqrt{g}\Big\{\sum(-1)^{i+1} d\xi^\sigma \wedge dp^i_\sigma \wedge dx^1 \wedge \cdots \wedge d\breve{x}^i \wedge \cdots \wedge dx^{n+1} + \frac{1}{2\delta} d\left[(\gamma^{\rho\sigma} g_{n+1,n+1} p^{n+1}_\sigma p^{n+1}_\sigma) + dE - X_\sigma d\xi^\sigma\right] \wedge V_{n+1}\Big\}, \tag{23}$$

in which we have written $x_{n+1} = t$, $g = \det g_{ij}$.

The first series of generalized Hamilton equations may be written:

$$\frac{\partial \xi^\sigma}{\partial x^i} = \frac{\partial E}{\partial p^i_\sigma}, \qquad \frac{\partial \xi^\sigma}{\partial x^{n+1}} = \frac{1}{\delta}\gamma^{\sigma\rho} g_{n+1,n+1} p^{n+1}_\rho .$$

We calculate them explicitly. In arbitrary coordinates:

$$E = \frac{\alpha}{2}\left[\sum\left(p^i_i\right)\right]^2 + \frac{\beta}{2}\gamma^{\sigma\rho} g_{ij} p^i_\sigma p^j_\rho .$$

Set $\dfrac{\partial \xi^\sigma}{\partial x^i} = u^\sigma_i$.

For $\sigma \neq i$: $\qquad u^\sigma_i = \beta\,\gamma^{\rho\sigma} g_{ij}\, p^j_\rho ,$

For $\sigma = i$: $\qquad u^\sigma_i = \beta\,\gamma^{\rho\sigma} g_{ij}\, p^j_\rho + \alpha \sum_{i=1}^{n} p^i_i$.

In a general manner, upon denoting the Kronecker symbol by δ^σ_i, one has:

$$u^\sigma_i = \beta\,\gamma^{\rho\sigma} g_{ij}\, p^j_\rho + \alpha\, \delta^\sigma_i J_1 .$$

Upon resolving $p^j_\rho = \dfrac{1}{\beta}\gamma_{\sigma\rho} g^{ij}(u^\sigma_i - \alpha\,\delta^\sigma_j J_1)$, one determines $J_1 = \sum_{j=1}^{n} p^i_j$ when one sets $\rho = j$ in the preceding formula:

$$\sum p^i_j = \frac{1}{\beta}\gamma_{\sigma\rho} g^{ij}(u^\sigma_i - \alpha\,\delta^\sigma_j J_1) .$$

Upon setting $\pi^i_\sigma = \gamma_{\sigma j}\, g^{ij}$, one finds:

$$\sum_{j=1}^{n} p_j^i = \frac{1}{\beta}\pi_\sigma^i(u_i^\sigma - \alpha\delta_i^\sigma J_1) = \frac{1}{\beta}\pi_\sigma^i u_i^\sigma - \frac{\alpha}{\beta}J_1 \times \sum \pi_i^i ,$$

thus:

$$J_1\left(\beta + \alpha\sum_{i=1}^{n}\pi_i^i\right) = \sum \pi_\sigma^i u_i^\sigma .$$

If one sets $I_1 = \sum_{i,\sigma}\pi_\sigma^i u_i^\sigma$ then:

$$J_1 = \frac{I_1}{\beta + \alpha\sum \pi_i^i} .$$

If one takes the same coordinate system in $\mathbb{R}^n$ and ρ^n then $\sum \pi_i^i = n$, $J_1 = \dfrac{1}{\beta + n\alpha} I_1$:

$$p_\rho^i = \frac{1}{\beta} g_{\sigma\rho} g^{ij}\left(u_i^\sigma - \alpha\delta_i^\alpha \frac{I_1}{\beta + n\alpha}\right).$$

Upon using the absolute derivative, the generalized Hamiltonian equations:

$$\frac{\partial p_\rho^{n+1}}{\partial t} + \sum_{j=1}^{n}\frac{\partial p_\rho^j}{\partial x^j} - X_\rho = 0$$

become:

$$g_{\sigma\rho}\, g^{n+,n+1}\, \frac{D(\delta u_{n+1}^\sigma)}{Dt} + \frac{1}{\beta} g_{\sigma\rho} g^{ij}\frac{Du_i^\sigma}{Dx^j} - \frac{\alpha}{\beta}\frac{1}{\beta + n\alpha}\frac{DI_1}{Dx^\rho} - X_\rho = 0. \tag{24}$$

In particular, in rectangular axes $g_{\sigma\rho} = 0$ when $\sigma \neq \rho$, while $g_{\rho\rho}\, g^{ij} = 1$.

$$\frac{\partial}{\partial t}\left(\frac{\delta\partial\xi^\rho}{\partial t}\right) + \frac{1}{\beta}\Delta\xi^\rho - \frac{\alpha}{\beta}\frac{1}{\beta + n\alpha}\frac{\partial I_1}{\partial x^\rho} - X_\rho = 0. \tag{25}$$

These are the Navier-Stokes equations, in which Δ denotes the Laplacian for an n-dimensional medium. Upon introducing the Lamé coefficients λ, μ, one finds:

$$\alpha = \frac{1}{\mu}\frac{1}{n + \dfrac{u}{\lambda + \mu}}, \qquad \beta = -\frac{1}{\mu}, \tag{26}$$

and one then puts them into the classical vectorial form:

$$\mu\,\Delta\xi + (\lambda + \mu)\,\mathbf{grad}(\mathrm{div}\,\xi) + \mathbf{F} = \frac{\partial}{\partial t}(\delta\mathbf{v}) .$$

Remark. – The expression for E in canonical coordinates is:

$$E = \frac{1}{2\beta}\gamma_{\sigma\rho}g^{ij}u_i^\sigma u_j^\rho - \frac{\alpha}{\beta+n\alpha}\frac{1}{2\beta}(\gamma_{\sigma\rho}g^{ij}u_i^\sigma u_j^\rho)^2,$$

which gives, upon using the Lamé coefficients and rectangular axes:

$$E = -\frac{\mu}{2}\sum_{i,\sigma}(u_i^\sigma)^2 - \frac{1}{2}(\lambda+\mu)\left(\sum u_i^i\right)^2.$$

§ 5. – The generating form of the equations of the mechanics of rigid systems.

The motions of rigid systems in an n-dimensional medium are maps from $\mathbb{R}^n \times t$ to ρ^n that preserve the lengths and orientations of the figures. Locally, one thus has:

$$d\sigma^2 = \sum_{\sigma=1}^{n}(d\xi^\sigma)^2 = ds^2 = \sum_{\sigma=1}^{n}(dx^i)^2 .$$

Upon setting $\frac{\partial \xi^\sigma}{\partial x^i} = u_i^\sigma$, the system of equations results:

$$\sum_{\sigma=1}^{n}\left(u_i^\sigma\right)^2 = 1, \qquad \sum_{\sigma=1}^{n}u_i^\sigma u_i^\sigma = 0.$$

In order to solve these equations, we first interpret them geometrically. At a point μ with the coordinates (ξ^σ), consider the natural frame (μ, $\mathbf{e}_i$) that is composed of the point μ and n vectors $\mathbf{e}_i$ that have coordinates $\left(u_i^\sigma\right)$ with σ varying from 1 to n. Since this frame is orthonormal, the corresponding metric tensor is $\gamma_{\sigma\rho} = 0$ when $\sigma \neq \rho$ and $\gamma_{\sigma\sigma} = 0$ when $\sigma = \rho$. It results from this that the Christoffel symbols are zero. Thus, the n vectors $\mathbf{e}_i$ have fixed directions, the u_i are constants, and:

$$u_i^\sigma = m_i^\sigma .$$

The n^2 elements m_i^σ are the elements of an orthogonal matrix.

In order to characterize the $\omega_d = X_i^\sigma dp_\sigma^i$ part of the Pfaff form ω on the kernel of $J^1(\mathbb{R}^n \times t, \rho^n)$ that would correspond to a motion of the medium, in the course of which, the distances remain invariant, we remark that the resulting generalized Hamilton equations are $\frac{\partial \xi^\sigma}{\partial x^i} = u_i^\sigma = X_i^\sigma$; hence, $\omega d = m_i^\sigma dp_\sigma^i$. This gives us the theorem:

THEOREM 3. – *If the motion of a continuous medium is such that the mutual distances between these points remain invariant then the* ω_d *part of the Pfaff form reduces to a form with constant coefficients* m_i^σ *, the set of whose elements comprises an orthogonal matrix.*

Remark. – By virtue of Theorem 2 in Chapter I, one may further say that $\omega_d = 0$, modulo a Pfaff form with constant coefficients.

If V_D denotes the manifold of the group of displacements of an n-dimensional space then the family of maps in question is comprised of the maps of $\mathbb{R}^n \times t$ into ρ^n that are defined by the formula:

$$\xi^\sigma = m_i^\sigma x^i + \eta^\sigma,$$

in which $\left\| m_i^\sigma \right\|$ denotes an orthonormal matrix.

For the map of $\mathbb{R}^n \times V_D$ to ρ^n, the general form Ω_{n+2} that is defined on $J^1(\mathbb{R}^n \times V_D$, $\rho^n)$ lifts to $J^1(\mathbb{R}^n \times t, \mathbb{R}^n \times V_D)$. Since the maps of $\mathbb{R}^n$ to $\mathbb{R}^n$ reduce to the identity, the form Ω_{n+2} becomes a form on $\mathbb{R}^n \times J^1(t, V_D)$. When this form is summed over a domain of $\mathbb{R}^n$, it generates a form Ω_2 of degree two whose support is $J^1(t, V_D)$. Now, upon denoting the tangent space to V_D by $T(V_D)$, $J^1(t, V_D)$ is homeomorphic to $t \times T(V_D)$. One may thus the preceding result as:

THEOREM 4. – *For a rigid system, the generating form of the equations of motion is a form of degree two whose support is the product of the number line t by the tangent space to the manifold of the group of displacements.* $V_D = \mathbb{R}^n \times (SO_n)$, *in which* SO_n *denotes the group of rotations in* $\mathbb{R}^n$.

COROLLARY. – *In the three-dimensional space of a system of solids, the generating form for the equations of motion will be a form of degree two on the product of the number line t with the tangent spaces to the manifold of the displacement group.*

Remark. – The case of a material point is a particular case of the preceding discussion: If the rigid system in animated with a translational motion then V_D reduces to $\mathbb{R}^3$. The motion of the system reduces to that of a material point whose equations of motion are generated by a form of degree two on the seven-dimensional manifold $t \times T(\mathbb{R}^3)$.

Consequences. – It is very important to remark that the equations of the mechanics of a point particle, and the equations of the mechanics of rigid systems are naturally deduced from the equations of continuous media, while the converse inference necessitates that we must make an appeal to a postulate concerning the forces that one calls "internal," and which are completely unknown.

§ 6. – The role of symmetric tensors.

One may be very surprised to find that in order to treat the theory of the motion of continuous media, we have not needed to introduce the notion of constraints. We shall now examine the consequences of the introduction of a twice-covariant symmetric tensor T^{ij} to account for the motion of an n-dimensional continuous medium. Let α be a covariant vector field on $\mathbb{R}^n$ with components α_i. The contracted product $T^{ij}\,\alpha_j = f^i$ is a contravariant vector $\mathbf{f}$. If D is an arbitrary domain of $\mathbb{R}^n$ then the flux of $\mathbf{f}$ that traverses the frontier of D, which we denote by ∂D, is:

$$\int_{\partial D} i(\mathbf{f})V_n\,,$$

upon denoting the volume form on $\mathbb{R}^n$ by V_n.

For any field α that leaves the ds^2 of the medium invariant – any field that verifies the Killing equations – one writes that the sum of the effects of the forces that act on the closed volume D and on its frontier is zero. If $\mathbf{X}$ denotes the force field that exists at every point of the medium then one has:

$$\int_{\partial D} i(\mathbf{f})V_n + \int_D (\mathbf{X}\cdot\alpha)V_n = 0,$$

a condition which is further written:

$$\int_{\partial D} \boldsymbol{\theta}(\mathbf{f})V_n + \int_D (\mathbf{X}\cdot\alpha)V_n = 0,$$

which gives, upon using Cartesian coordinates:

$$\int_D \left(\sum_{i=1}^{n}\frac{\partial T^{ij}\alpha_j}{\partial x^i} + X^j\alpha_j\right)V_n = 0.$$

Since the preceding integral is zero for uniform fields α, in particular, one must have:

$$\sum_{i=1}^{n}\frac{\partial T^{ij}}{\partial x^i} + X^j = 0.$$

A comparison is called for between this equation and the analogous equation for the generalized Hamiltonian system:

$$\sum_{i=1}^{n} \frac{\partial p_\sigma^i}{\partial x^i} - X_\sigma = 0.$$

As the latter does not have the same tensorial character as the former, we use the twice-contravariant metric tensor $\gamma^{\sigma\rho}$ to put p_σ^i and X_σ into contravariant form:

$$p^{i\rho} = \gamma^{\sigma\rho}\, p_\sigma^i, \qquad X^\sigma = \gamma^{\sigma\rho} X_\sigma .$$

Upon using rectangular axes, one obtains:

$$\sum_{i=1}^{n} \frac{\partial p^{i\rho}}{\partial x^i} - X^\rho = 0.$$

Upon equating the index j with ρ, one deduces that the components T^{ij} of the symmetric tensor are solutions to the system of partial differential equations:

$$\sum_{i=1}^{n} \frac{\partial T^{ij}}{\partial x^i} = -\sum_{i=1}^{n} \frac{\partial p^{ij}}{\partial x^i}, \tag{27}$$

where the p^{ij} that figure in the right-hand side are functions of the x^h, $\dfrac{\partial \xi^\sigma}{\partial x^i}$ that are deduced from the generalized Hamiltonian equations:

$$\frac{\partial \xi^\sigma}{\partial x^i} = X_i^\sigma(\cdots, x^h, \cdots, p_\rho^i, \cdots), \qquad p^{ij} = \gamma^{j\rho} p_\rho^i .$$

Since the system (27) is linear one obtains the general solution by adding the solution to the system without the right-hand side:

$$\sum_{i=1}^{n} \frac{\partial S^{ij}}{\partial x^i} = 0 \qquad\qquad j = 1, \ldots, n$$

to the particular solution of the system with the right-hand side.

One solves this latter system in the following manner: If one is given n arbitrary constants β_j then upon multiplying the equation of rank j by β_j and summing over the j, one obtains:

$$\sum_{i=1}^{n} \beta_j \frac{\partial S^{ij}}{\partial x^i} = 0.$$

Upon setting $\| S^{ij} \| \times \left\| \beta_j^i \right\| = \varphi^j$, one has the equation $\displaystyle\sum_{i=1}^{n} \frac{\partial x^i}{\partial \varphi^i} = 0$, which expresses that:

$$T^{jj} = \frac{-2}{\beta} g^{ij} u^i_j + \frac{\alpha}{\beta} \frac{1}{\beta + n\alpha} I_1 + \frac{1}{\beta} \int \sum_{i=1}^{n} \frac{\partial (g^{ij} u^i_j)}{\partial x^i} dx^j .$$

Now:

$$\sum_{i=1}^{n} \frac{\partial (g^{ij} u^i_j)}{\partial x^i} = g^{ij} \sum_{i=1}^{n} \frac{\partial^2 \xi^i}{\partial x^i \partial x^j} ,$$

$$\int \sum_{i=1}^{n} \frac{\partial (g^{ij} u^i_j)}{\partial x^i} dx^j = g^{ij} \int \sum_{i=1}^{n} \frac{\partial^2 \xi^i}{\partial x^i \partial x^j} dx^j = g^{ij} \left(\sum_{i=1}^{n} \frac{\partial \xi^\sigma}{\partial x^i} + C \right) = g^{jj} (I_1 + C),$$

upon denoting a constant by C.

Finally:

$$T^{ii} = -\left[\frac{-2}{\beta} g^{jj} u^k_k + \frac{\alpha}{\beta} \frac{1}{\beta + n\alpha} I_1 + \frac{1}{\beta} (I_1 + C) \right] g^{ii} .$$

Upon introducing the Lamé coefficients, these formulas may be written:

For $i \neq j$: $T^{ii} = +\mu (g^{ii} u^j_i + g^{jj} u^i_j)$,

For $i = j$: $T^{jj} = 2\mu g^{ii} u^j_i + g^{jj} [(\lambda + \mu) - \mu] I_1 - \mu C = 2\mu g^{jj} u^j_j + g^{jj} (\lambda I_1 - \mu C)$.

If one introduces the displacement vector $\mathbf{v}$ in $\mathbb{R}^n$ and $\xi = \mathbf{x} + \mathbf{v}$ for $i \neq j$, while $u^j_i = v^j_i$ for $j = 1$, $u^j_j = v^j_j + 1$ then preceding formula give:

$$T^{ij} = \mu \left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} \right),$$

$$T^{jj} = \mu \frac{\partial v^i}{\partial x^j} + \lambda \sum_{i=1}^{n} \frac{\partial v^j}{\partial x^i} + 2\mu - 3\lambda - \mu C .$$

These are the classical formulas if one takes the constant $C = \dfrac{2\mu + 3\lambda}{\mu}$.

The role of the deformation tensor. – We conclude our study of the role of the symmetric tensors with that of the deformation tensor. If the metric tensor on ρ^n is $\gamma_{\rho\sigma}$ then $d\sigma^2 = \gamma_{\rho\sigma}\, d\xi^\sigma\, d\xi^\rho$, and if the metric tensor on $\mathbb{R}^n$ is g_{ij}, so $ds^2 = g_{ij}\, dx^i\, dx^j$, then one calls the tensor e_{ij}, which is such that $d\sigma^2 - ds^2 = e_{ij}\, dx^i\, dx^j$ the deformation tensor, and the expression for the deformation tensor as a function of the canonical coordinates on $J^1(\mathbb{R}^n, \rho^n)$ becomes:

$$e_{ij} = \gamma_{\sigma\rho} u_i^\sigma u_j^\rho - g_{ij} .$$

By means of the generalized Hamiltonian equations:

$$\frac{\partial \xi^\sigma}{\partial x^i} = X_i^\sigma(\cdots, p_\sigma^j, \cdots, \xi^\sigma, \cdots, x^h, \cdots),$$

one obtains the expression for the deformation tensor as a function of the practical coordinates:

$$e_{ij} = \gamma_{\sigma\rho} X_i^\sigma X_j^\rho - g_{ij} .$$

Thus, when the form ω_d is known, one immediately knows the deformation tensor as a function of the coordinates on $J^1(\mathbb{R}^n, \rho^n)$. However, if one knows, conversely, the deformation tensor on $\mathbb{R}^n$ – i.e., its expression as a function of the variables x^i – then it is important to remark that this does not suffice to reconstitute the ω_d part of the Pfaff form on the set $J^1(\mathbb{R}^n, \rho^n)$ of jets.

BIBLIOGRAPHY

[1] H. CARTAN, *Colloque de Topologie sur les Espaces Fibrés*, *Bruxelles*, 1950, Masson & Co., Paris, 1951, pp. 15 to 17.

[2] C. EHRESMANN, “Les prolongements d’une variété différentiable. I. Calcul de jets,” *C. R. Acad. Paris* **233** (1951), pp. 598; “II. L’espace des jets d’order r de V_n dans V_n,” *C. R. Acad. Paris* é (1951), pp. 77.

[3] C. EHRESMANN, “Les prolongements d’une variété différentiable,” *Atti IV Cogresso Unione mat. Italiana*, Taormina Ott., 1951.

[4] C. EHRESMANN, “Introduction à la theorie des structures infinitésimales et des pseudo-groupes de Lie,” (*Colloque intern. Géometrie différentielle*, *C. N. R. S.*, 1953, pp. 97-110.)

[5] F. GALLISOT, “Les formes extérieure et la mechanique des milieux continus,” *C. R. Acad. Paris* **244** (1957), pp. 2347.

[6] P. LIBERMANN, “Sur le problème d’equivalence de certain structures infinitésimales,” *Thése Ann. di Matmatica*, XXXVI, 1954. “Sur les pseudo-groupes de Lie,” *Colloque de Topologie de Strasbourg* (April, 1954).

[7] A. LICHNEROWICZ, *Théorie Globale des Connexiones et des groupes d’holonomie*, *Consiglio Nazionale delle Ricerche*, Paris, Dunod, 1956.
