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WAVE MECHANICS

OF THE

ELECTRON AND PHOTON

BY

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TABLE OF CONTENTS

Page

CHAPTER VII

Study of the interaction between an electron and a photon.

INTRODUCTION

The present work is dedicated to the wave mechanics of electrified material corpuscles and photons.

We begin with a *succinct presentation of the theory of gravitational and electromagnetic fields* in the general form that Th. De Donder gave them. The utility of the presentation will rest upon the fact that the important considerations in wave mechanics are based upon the "correspondence principle" (see, pp. ?)

We point out the proof of the variational principle that is attached to the motion of the points of a continuous massive distribution when the tensor of massive stresses has a particular form $\left(\text{viz., } P_{\alpha,\beta}^{\beta} = N \dfrac{\partial \log \lambda}{\partial x^{\alpha}}\right)$. We likewise prove that the points of a continuous distribution of mass and electricity follow trajectories that extremize a certain integral [eq. (55) and (70)].

The *second chapter* relates to the *wave mechanics of charged massive points (without spin)* that are placed in an arbitrary gravitational and electromagnetic field. That theory will permit the study of the properties of material particles that are placed in not just an electromagnetic field, but a gravitational one, as well, when one can neglect the spin and magnetic moment of the corpuscle. We shall show that in order to preserve the analogy between geometrical optics and classical mechanics, as well as the one between physical optics and wave mechanics, one can introduce a fifth independent variable and then reason in a five-dimensional space.

The agreement between wave mechanics and classical mechanics is the same as the one that relates physical optics and geometric optics. One has a Fermat theorem for the rays or bicharacteristic lines that are associated with the fundamental equation that is nothing but the variational principle that defines the trajectories of the material points in Einsteinian mechanics.

One can then eliminate the fifth variable. The equation that one obtains is the relativistic Schrödinger equation; it is likewise deduced from a variational principle. Thanks to that fact, Th. De Donder's correspondence principle will then permit us to define a current and an energy-impulse tensor in a manner that is formally identical to the one that is utilized in the theory of the gravitational field. We link that study to the method of operators and show that the impulse corresponds to the covariant operator $h / 2\pi i\ ()_{,\alpha}$, where $()_{,\alpha}$ is the symbol for the covariant derivative.

In *Chapter III*, we will show that the *second-order Dirac equations* can serve as the basis for a theory of the electron and positron, without having to introduce Dirac's hole hypothesis. Meanwhile, it lacks certain interaction terms that yield the phenomena of materialization and dematerialization.

At the end of that chapter, one will find some remarks on the theory of the interaction between light and the electron.

Chapter IV contains a discussion of the *equations of J. M. Whittaker* and a *presentation of Th. De Donder's theory of the photonic field.* One will see that Whittaker's theory cannot represent a Dirac electron. By contrast, Th. De Donder's theory of the photonic field, which utilizes those equations, will preserve its value for the study of photons, which is natural, since Whittaker's equations have the Maxwellian form.

The following two chapters are dedicated to the *wave mechanics of the photon.* After a brief summary of the ideas that guided L. de Broglie in his research, we shall analyze and discuss the theory and its general principles. We will be mainly interested in the mean-value densities and the mean values. L. de Broglie gave two possible definitions for the mean values. Those two definitions will be equivalent for waves that can be developed into a series of plane waves in all of space; they will not be equivalent in the general case (for a spherical wave, for example). Only one of the definitions should be retained; it is the one that leads to mean-value densities with satisfactory variance properties. As one knows, the agreement between wave mechanics and the classical theories can always come about by the intermediary of mean values. For the theory of the photon, that then comes down to comparing those mean-value densities to the corresponding tensors in Maxwell's theory. That comparison will always be possible, thanks to certain inversion formulas that we have calculated. Indeed, one can form sixteen independent linear combinations of the sixteen functions $\Phi_{\alpha,\beta}$ that define the state of the photon. Those sixteen combinations are the components of tensorial magnitudes, among which, one finds the (complex) Maxwellian electromagnetic potential and the (complex) electromagnetic field.

Conversely, it is then possible to replace the $\Phi_{\alpha,\beta}$ in any formula from the theory of the photon with their expressions as functions of the aforementioned tensorial magnitudes. One can then compare the energy-impulse tensor for the theory of the photon, when written in conformity with general principles of the theory, to the Maxwell energy-impulse tensor, when it is constructed from complex fields. We shall show that those two tensors are not identical, but they differ only by terms that will "generally" be zero when they are integrated over all space. In the theory of the photon, one can likewise write a tensor whose "principal" terms are those of the Maxwell tensor; that tensor will not have the canonical form of a mean-value density.

A particularly interesting case is that of the moments of impulse. As in Dirac's theory, the moment of impulse is the sum of two terms here: viz., the orbital moment and the spin. One sees immediately an important difference between the mean-value densities of the total moment and the moment of the Poynting vector with respect to a point O. Like the orbital moment, the moment of the Poynting vector will depend upon a vector $\overrightarrow{OP}$ (P is the point of application of the Poynting vector). On the contrary, the spin density constitutes an intrinsic moment. However, one can compare that spin to an intrinsic moment – or "momentor" – that was introduced by E. Henriot only a few years ago in Maxwell's theory of the electromagnetic field. Here, I would like to point out that it is my desire to compare the momentor to the spin that was the starting point for this work on mean-value densities in the theory of the photon. The result is that the two tensors – viz., spin and momentor – are not identical, although they are similar. Furthermore, although in Maxwell's theory, we must still discuss the question of knowing when one can employ the moment of the Poynting vector or the momentor, in wave mechanics, we must always take the sum of the orbital moment and the spin. In addition, we shall give a relation between the moment of the Poynting vector and the mean-value density of the total moment, which is a relation that will permit us to find the cases in which the two definitions will lead to the same result. We will then show that the mean value of the proper mass will lead to a formula that is analogous to the one that is obtained in Dirac's theory.

Finally, we shall express some of our reflections on the fundamental equation of the photon, the relativistic significance of the new form that it takes in theory of the photon, the general definition of mean-value densities, and the non-Maxwellian quantities in the theory.

In *Chapter VI*, we shall study diverging, converging, or stationary spherical waves in a deeper manner.

The method that is employed will permit us to obtain the general expression for the functions $\Phi_{\alpha\beta}$ of the spherical waves by a simple calculation of identification; that will then give corresponding expressions for fields, electromagnetic potentials, and non-Maxwellian magnitudes. Those waves, which are characterized by two integer numbers (l, m), will each further depend upon two arbitrary constants whose significance we shall discuss, along with their link to the well-known indeterminacy in the Maxwell's theory. The calculation of the moment of impulse in terms of spherical waves is easy to do in wave mechanics and will yield some general formulas that are in complete agreement with the demands of the law of conservation of moment of impulse for multipolar emissions. In order to agree with the well-known results of Maxwell's theory, we shall deduce some theorems from those general formulas on the flux of moments of impulse across a closed surface that tends to infinity for diverging waves and some theorems on the mean values of the moment of impulse of a stationary wave. For the flux at infinity, only the case of diverging dipolar waves has been calculated in Maxwell's theory. W. Heitler studied stationary waves in a very complete manner. His results agree with our own; however, there are some essential differences between our methods. We point out some differences in detail: We have not annulled the scalar potential *a priori*, and we have introduced two radial functions that are coupled, as in Dirac's theory, and they seem to simplify the formulas. Those formulas are written in such a way that they will remain valid for $l = -m$. We have also let the proper mass of the photon be non-zero. We conclude with some remarks on the non-Maxwellian quantities, which can transport energy, in principle.

Finally, in the *last chapter*, we shall develop a new theory of the interaction between light and matter. We will show that the wave mechanics of the electron and the photon will permit us to study the elementary phenomena that relate to the interaction of a photon and an electron in a simple fashion, and without utilizing the quantum theory of fields.

This work was carried out under the direction of Professor Th. De Donder, who never ceased to lavish me with invaluable advice. I am happy to be able to express my profound gratitude to him here.

I would like to warmly thank Professor L. de Broglie for the benevolent interest with which he examined my research into the theory of the photon, as well as for his comments and criticisms, which have been quite useful to me.

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CHAPTER I

MASSIVE AND ELECTROMAGNETIC FIELDS

1. General theory. – The theory of massive gravitational and electromagnetic fields that forms the basis for this chapter was presented in several works by Th. De Donder ([1]). We shall summarize the general method of that author here.

Space-time is defined by the set of four variables x^1, x^2, x^3, x^4, and the fundamental quadratic form:

$$g_{\alpha\beta}(x^1, \ldots, x^4)\, \delta x^\alpha\, \delta x^\beta, \qquad \alpha, \beta = 1, \ldots, 4, \tag{1}$$

in which $g_{\alpha\beta} = g_{\beta\alpha}$ are ten (real) functions of the $x^1, \ldots, x^4$; they are Einstein's gravitational potential. The form (1) is the square of the interval between two infinitely-close points $(x^1, \ldots, x^4)$ and $(x^1 + \delta x^1, \ldots, x^4 + \delta x^4)$ of space-time. x^1, x^2, x^3 represent the geometric variables, and x^4 represents the temporal variable. The normalized minors of the $g_{\alpha\beta}$ will be denoted by $g^{\alpha\beta}$.

Let $\mathcal{M}^{(g)}$ be a multiplier, which is a function of the $g^{\alpha\beta}$ and their derivatives with respect to the $x^1, \ldots, x^4$. It is the *gravitational characteristic function;* it will be defined later on [form. (11)].

Let $\mathcal{M}$ be another multiplier, which can depend upon physical quantities such as mass and charge densities, velocities, etc., in addition to the $g_{\alpha\beta}$ and their derivatives. It is the *phenomenological characteristic function;* its form will be specified later on.

The main variational principle of gravity is expressed by the variational equation:

$$\delta \iiint\int_\Omega (\mathcal{M}^{(g)} + \mathcal{M})\, \delta x^1 \cdots \delta x^4 = 0, \tag{2}$$

in which Ω is a portion of space-time upon whose boundary the variations must vanish.

When one varies this with respect to $g^{\alpha\beta}$, while leaving the other functions that enter into $\mathcal{M}$ fixed, one will obtain the ten Lagrange equations:

$$\frac{\delta(\mathcal{M}^{(g)} + \mathcal{M})}{\delta g^{\alpha\beta}} = 0, \tag{3}$$

in which ([2]):

([1]) See [**1**a], [**1**b], [**1**c], [**1**d], and [**1**e]. – The bold numerals refer to the *Index of Cited Authors.*

([2]) The derivatives $\partial / \partial g^{\alpha\beta}$ are *partial* derivatives (see [**1**b], pp. 4). One will have, for example:

$$\frac{\partial}{\partial g^{12}} g^{21} = \frac{\partial}{\partial g^{12}} \frac{1}{2}(g^{21} + g^{12}) = \frac{1}{2},$$

while:

$$\frac{d}{dg^{12}} g^{21} = 1.$$

$$\frac{\delta}{\delta g^{\alpha\beta}} = \frac{\partial}{\partial g^{\alpha\beta}} - \frac{d}{dx^{\gamma}}\left(\frac{\partial}{\partial g^{\alpha\beta}_{\ \ ,\gamma}}\right) + \dots \qquad \alpha, \beta, \gamma = 1, \dots, 4, \tag{4}$$

with:

$$g^{\alpha\beta}{}_{,\gamma} = \frac{\partial g^{\alpha\beta}}{\partial x^{\gamma}}. \tag{5}$$

It goes without saying that before the operations that were indicated in (3) are performed, one must give the specific form of $\mathcal{M}$, because $\mathcal{M}$ will depend upon tensors and vectors and rules for raising and lowering their indices can make the gravitational potentials appear or disappear in the functions.

One introduces the notations:

$$\mathcal{T}^{(g)}_{\alpha\beta} \equiv \frac{\delta \mathcal{M}^{(g)}}{\delta g^{\alpha\beta}}, \qquad \mathcal{T}_{\alpha\beta} \equiv -\frac{\delta \mathcal{M}}{\delta g^{\alpha\beta}}. \tag{6}$$

Equations (3) are then written:

$$\mathcal{T}^{(g)}_{\alpha\beta} = \mathcal{T}_{\alpha\beta}. \tag{7}$$

However, by virtue of the Hilbert identities, one will have:

$$\mathcal{T}^{(g)\beta}_{\alpha\ \ \ ,\beta} \equiv 0, \tag{8}$$

in which the symbol $,\beta$ denotes a covariant derivative with respect to x^{β}. Set:

$$\mathcal{F}_{\alpha} \equiv \mathcal{T}^{\beta}_{\alpha,\beta}. \tag{9}$$

One infers the *theorem of the total generalized force or the phenomenological tensor* from (7) and (8):

$$\mathcal{F}_{\alpha} = 0. \tag{10}$$

These equations generalize the equations of dynamics and the conservation of energy. They are consequences of (7).

Since they do not depend upon the form of the $\mathcal{M}^{(g)}$, one can consider them to be necessary relations between the functions that enter into $\mathcal{M}$.

From Th. De Donder, the gravitational characteristic function is:

$$\mathcal{M}^{(g)} \equiv (a + bC)\sqrt{-g}, \tag{11}$$

in which a and b are two universal constants ([3]), and C is the Gaussian curvature invariant that relates to the metric (1); g is the value of the determinant $\| g_{\alpha\beta} \|$. Due to (11), equations (7) will become ([4]):

$$-\tfrac{1}{2}(a + bC)\, g_{\alpha\beta} + b\, G_{\alpha\beta} = \mathcal{T}_{\alpha\beta}\,. \tag{12}$$

The choice of phenomenological characteristic function depends upon the phenomena that one assumes to be realized in space-time. Notably, Th. De Donder studies various important special cases in his *Théorie des Champs gravifiques*. We shall first examine some results that relate to purely massive fields; we will then study the massive electromagnetic field.

2. Massive field. – Let $\mathcal{N}(x^1, \ldots, x^4)$ denote a mass density factor and let $\mathcal{P}_{\alpha\beta}(x^1, \ldots, x^4) = \mathcal{P}_{\beta\alpha}(x^1, \ldots, x^4)$ denote a massive symmetric tensor ([5]).

From Th. De Donder, the characteristic function of the gravitational field that is due to the masses is:

$$\mathcal{M} = -\, g^{\alpha\beta}(\mathcal{N}\, u_\alpha\, u_\beta + \mathcal{P}_{\alpha\beta}), \tag{13}$$

in which u_α are the covariant components of the velocity in space-time; they are functions of the $x^1, \ldots, x^4$. One has:

$$u^\alpha = \frac{dx^\alpha}{ds}\,, \tag{14}$$

in which

$$ds = +\sqrt{g_{\alpha\beta}\, dx^\alpha dx^\beta} \tag{15}$$

denotes the interval between infinitely-close points that are both situated along the trajectory of a massive point. One will deduce from (14) and (15) that:

$$W^2 \equiv g_{\alpha\beta}\, u^\alpha\, u^\beta = 1. \tag{16}$$

Equations (12) become:

$$-\tfrac{1}{2}(a + bC)\, g_{\alpha\beta} + b\, C_{\alpha\beta} = N\, u_\alpha\, u_\beta + P_{\alpha\beta}\,. \tag{17}$$

The theorem of the phenomenological tensor is written:

([3]) a is a cosmic constant. b is related to Einstein's constant K by the relation $K = -\,c^2 / b$, where c is the speed of light *in vacuo* ($K = 1.87 \times 10^{-27}$ cm/g).

([4]) Conforming to the usual notations of tensorial calculus:

$$\mathcal{T}_{\alpha\beta} = \sqrt{-g}\; T_{\alpha\beta}\,.$$

([5]) The physical significance of these functions is specified by some important theorems, and we shall refer to the original paper for them (see [**1**c], especially pp. 19). Here, we shall say only that $\mathcal{N}$ is proportional to the usual mass density and that $\mathcal{P}_{\alpha\beta}$ will provide the massive stresses.

$$\mathcal{F}_\alpha \equiv \mathcal{M}\, u_{\alpha,\,\beta}\, u^\beta + u_\alpha\,(\mathcal{N}\, u^\beta)_{,\beta} + \mathcal{P}^{\beta}_{\alpha,\beta} = 0. \tag{18}$$

Thanks to (16) and (18), one will obtain the continuity equation:

$$\frac{\partial(\mathcal{N}\, u^\beta)}{\partial x^\beta} + u^\alpha\, \mathcal{P}^{\beta}_{\alpha,\beta} = 0; \tag{19}$$

hence:

$$\mathcal{F}_\alpha = -\mathcal{M}\, u_{\alpha,\,\beta}\, u^\beta - u_\alpha\, u^\gamma\, \mathcal{P}^{\beta}_{\gamma,\beta} + \mathcal{P}^{\beta}_{\alpha,\beta} = 0. \tag{20}$$

If one has:

$$u^\alpha \mathcal{P}^{\beta}_{\alpha,\beta} \equiv 0 \tag{21}$$

then the equation of continuity will become:

$$\frac{\partial(\mathcal{N}\, u^\beta)}{\partial x^\beta} = 0. \tag{22}$$

Furthermore, if $\mathcal{P}^{\beta}_{\alpha,\beta} = 0$ then equations (20) will simplify. One will have:

$$u_{\alpha,\,\beta}\, u^\beta = 0 \tag{23}$$

at every point at which $\mathcal{N}$ is non-zero.

Thanks to (14), equations (23) will give the Lagrange equations:

$$\frac{\delta W}{\delta x^\alpha} = 0, \tag{24}$$

in which:

$$\frac{\delta}{\delta x^\alpha} \equiv \frac{\partial}{\partial x^\alpha} - \frac{d}{ds}\left(\frac{\partial}{\partial u^\alpha}\right). \tag{25}$$

Due to (16), equations (24) can then be written:

$$\frac{\delta(\frac{1}{2}W^2)}{\delta x^\alpha} = 0. \tag{24′}$$

It results from (24) that *when one has* (21), the trajectories of any massive point will be such that they extremize:

$$\delta\int W\, ds = 0, \quad \text{or} \qquad \delta\int \tfrac{1}{2}W^2\, ds = 0. \tag{26}$$

(The extremities of the path of integration are assumed to be fixed.)

Remarks. –

a) The four equations (24) are not independent. Indeed, one has:

$$\frac{\delta W}{\delta x^\alpha} u^\alpha \equiv 0,$$

because W is homogeneous of first degree in u^α.

b) In the case of the incoherent fluid, one will have, by hypothesis:

$$\mathcal{P}_{\alpha\beta} \equiv 0.$$

The conditions $\mathcal{P}^\beta_{\alpha,\beta} = 0$ will then be fulfilled, and one will have the variational principle (26). However, the theorem (26) will still be valid when there are massive stresses, provided that they satisfy $\mathcal{P}^\beta_{\alpha,\beta} = 0$.

c) When the conditions (21) are not fulfilled, there will again exist a simple case that corresponds to a variational principle that generalizes (26). Set:

$$P^\beta_{\alpha,\beta} \equiv M_\alpha N \qquad \text{(in which } N = \mathcal{N}\sqrt{-g}\text{)}. \tag{27}$$

Equations (20) can be written ($N \neq 0$):

$$u_{\alpha,\beta} u^\beta + u_\alpha u^\gamma M_\gamma + M_\alpha = 0.$$

However, if one takes (16) into account:

$$-\frac{\delta(\lambda W)}{\delta x^\alpha} = \lambda\left(u_{\alpha,\beta} u^\beta + u_\alpha u^\gamma \frac{\lambda_{,\gamma}}{\lambda} - \frac{\lambda_{,\alpha}}{\lambda}\right).$$

Hence, if the M_α are the partial derivatives of a function of the $x^1, \ldots, x^4$ with respect to x^α – i.e., if:

$$M_\alpha \equiv -\frac{\partial \log \lambda(x^1, \ldots, x^4)}{\partial x^\alpha},$$

then equations (20) will be equivalent to:

$$\frac{\delta(\lambda W)}{\delta x^\alpha} = 0.$$

Hence, one will have the variational principle:

$$\delta \int \lambda W\, ds = 0, \tag{28}$$

which generalizes (26).

d) Equations (17) and (20), along with (16), are the fundamental equations of the theory of massive fields. Any system of functions $g_{\alpha\beta}$, u^α, N, $P_{\alpha\beta}$ that satisfies those equations will define a massive field.

Two types of problems have been studied in particular:

1. Find the gravitational field that is produced by a given mass distribution that is at rest with respect to the observer.

2. Find the motion of a test body in a *given* gravitational field $g_{\alpha\beta}$. We only have equations (24) or (26) in mind here. The *given* or *exterior* gravitational field figures in these equations.

3. Massive electromagnetic field. – From Th. De Donder, the characteristic function will be:

$$\mathcal{M} \equiv -g^{\alpha\beta}\left(\mathcal{N}\, u_\alpha u_\beta + \mathcal{P}_{\alpha\beta} - \frac{\sqrt{-g}}{2}\, g^{\mu\nu} H_{\alpha\mu} H_{\beta\nu}\right) \qquad (\alpha, \beta, \mu, \nu = 1, \ldots, 4), \tag{29}$$

in which $H_{\alpha\beta} = -H_{\beta\alpha}$ are the six covariant components of the electromagnetic force. (Here, we envision only the simple case in which there is no electric or electromagnetic polarization in the medium considered.) The field $H_{\alpha\beta}$ satisfies Maxwell's equations ([1]):

$$\left.\begin{aligned} \frac{\partial \sqrt{-g}\, H^{\alpha\beta}}{\partial x^\beta} &= \sigma u^\alpha, \quad (a) \\ \frac{\partial H_{\overline{\alpha\beta}}}{\partial x^\beta} &= 0, \qquad (b) \end{aligned}\right\} \tag{30}$$

in which $\alpha\beta\overline{\alpha\beta}$ forms an even permutation of the numbers 1, 2, 3, 4. The multiplier or density factor σ represents the charge density in space. The multiplier $\mathcal{N}$ represents a mass density in space-time. The functions σ and $\mathcal{N}$ are coupled to the charge and mass densities by some important relations. [See below, form. (58).]

The $c\, u^\alpha (x^1, \ldots, x^4)$ are the contravariant components of the velocity of electricity. By definition, one has:

$$u^\alpha = \frac{dx^\alpha}{ds}, \tag{31}$$

([1]) Unless stated to the contrary, we shall use the same system of units as H. A. Lorentz. (See **[2]**, pp. 191.)

in which *ds* denotes the interval between two infinitely-close points that are situated along the space-time trajectory of a charged, massive point ($\sigma \neq 0$, $\mathcal{N} \neq 0$).

One will then have:

$$(ds)^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad \text{and} \qquad W^2 \equiv g_{\alpha\beta}\, dx^\alpha\, dx^\beta = 1. \tag{32}$$

Equations (12) are written explicitly as:

$$-\tfrac{1}{2}(a + bC)\, g_{\alpha\beta} + b\, G_{\alpha\beta} = N\, u_\alpha\, u_\beta + P_{\alpha\beta} + \tfrac{1}{4} g_{\alpha\beta} H^{\mu\nu} H_{\mu\nu} - H_\alpha{}^\mu H_{\beta\mu} \tag{33}$$

here.

The electromagnetic tensor:

$$\overset{(e)}{T}{}_{\alpha\beta} \equiv \tfrac{1}{4} g_{\alpha\beta} H^{\mu\nu} H_{\mu\nu} - H_\alpha{}^\mu H_{\beta\mu} \tag{34}$$

appears in (33), whose mixed components will be:

$$\overset{(e)}{T}{}_\alpha^\beta \equiv \tfrac{1}{2}(H^{\overline{\alpha\mu}} H_{\overline{\beta\mu}} - H_{\alpha\mu} H^{\beta\mu}), \tag{34′}$$

by virtue of the identity:

$$\tfrac{1}{4}\delta_\alpha^\beta\; H^{\mu\nu} H_{\mu\nu} \equiv \tfrac{1}{2}(H^{\overline{\alpha\mu}} H_{\overline{\beta\mu}} + H_{\alpha\mu} H^{\beta\mu}).$$

In order to make the variance of $H_{\overline{\alpha\beta}}$ apparent, one introduces the adjoint electromagnetic tensor:

$$H_*^{\alpha\beta} \equiv H_{\overline{\alpha\beta}}\,. \tag{35}$$

We recall the following important identity: If $A_{\alpha\beta}$ are the covariant components of an antisymmetric tensor then one will have:

$$\sqrt{-g}\, A^{\overline{\mu\nu}} \equiv \frac{-g_{\alpha\mu}\, g_{\beta\nu}}{\sqrt{-g}} A_{\overline{\alpha\beta}}\,; \tag{36}$$

hence:

$$\frac{1}{\sqrt{-g}} A_{\overline{\alpha\beta}} \equiv -\sqrt{-g}\; g^{\alpha\mu} g^{\beta\nu} A^{\overline{\mu\nu}}\,. \tag{36′}$$

The electromagnetic tensor can then be written:

$$\overset{(e)}{T}{}_\alpha^\beta \equiv -\tfrac{1}{2}(H_{\alpha\mu} H^{\beta\mu} + H^*_{\alpha\mu} H_*^{\beta\mu}). \tag{37}$$

We now study the theorem of the phenomenological tensor (10). We have:

$$\overset{(e)}{T}{}_{\alpha,\beta}^{\beta} \equiv -\tfrac{1}{2} H_{\alpha\mu} H^{\beta\mu}{}_{,\beta} - \tfrac{1}{2} H^*_{\alpha\mu} H_*^{\beta\mu}{}_{,\beta} - \tfrac{1}{2} H_{\alpha\mu,\beta} H^{\beta\mu}{}_{,\beta} - \tfrac{1}{2} H^*_{\alpha\mu,\beta} H_*^{\beta\mu}\,. \tag{38}$$

We utilize the identities:

$$\left.\begin{aligned} -\tfrac{1}{2}H^{*}_{\alpha\mu,\beta}H^{\beta\mu}_{*} &\equiv \tfrac{1}{2}H_{\alpha\gamma}H^{\gamma\mu}{}_{,\mu}+\tfrac{1}{2}H_{\mu\nu}H^{\mu\nu}{}_{,\alpha}, \\ -\tfrac{1}{2}H_{\alpha\gamma}H^{\gamma\mu}{}_{,\mu} &\equiv \tfrac{1}{2}H^{*}_{\alpha\gamma}H^{\gamma\mu}_{*,\mu}-\tfrac{1}{2}H_{\mu\nu}H^{\mu\nu}{}_{,\alpha}. \end{aligned}\right\} \tag{39}$$

It will result that:

$$\overset{(e)}{T}{}^{\beta}_{\alpha,\beta} \equiv H_{\alpha\mu}H^{\beta\mu}{}_{,\beta}+H^{*}_{\alpha\mu}H^{\mu\beta}_{*\ ,\beta}. \tag{40}$$

Thanks to (40) and (30), the law of total force is written:

$$\mathcal{N}u_{\alpha}u_{\beta}+u_{\alpha}\frac{\partial(\mathcal{N}u^{\beta})}{\partial x^{\beta}}+\mathcal{P}^{\beta}_{\alpha,\beta}+\sigma H_{\alpha\beta}u^{\beta}=0\ . \tag{41}$$

We place ourselves in the case where:

$$u^{\alpha}\,\mathcal{P}^{\beta}_{\alpha,\beta}=0. \tag{42}$$

We then deduce the equation of continuity for mass from (41) and the relation (32) that:

$$\frac{\partial(\mathcal{N}u^{\beta})}{\partial x^{\beta}}=0; \tag{43}$$

hence, (41):

$$\mathcal{N}u_{\alpha,\beta}u^{\beta}+\sigma H_{\alpha\beta}u^{\beta}+\mathcal{P}^{\beta}_{\alpha,\beta}=0. \tag{44}$$

By virtue of (30, *a*), one will also have *the equation of continuity for charge:*

$$\frac{\partial(\sigma u^{\beta})}{\partial x^{\beta}}=0. \tag{45}$$

One infers from (43) and (45) that:

$$\frac{d\varepsilon(x)}{ds}\equiv\frac{\partial\varepsilon(x)}{\partial x^{\alpha}}=0, \quad \text{or} \qquad \varepsilon(x)\equiv\frac{\sigma}{\mathcal{N}}\ . \tag{46}$$

When one has $\mathcal{P}^{\beta}_{\alpha,\beta}=0$, (44) will become:

$$\mathcal{N}u_{\alpha,\beta}u^{\beta}+\sigma H_{\alpha\beta}u^{\beta}=0. \tag{44′}$$

Remark. – Equations (33), (30), (31), (41), and (32) are the fundamental equations of the theory of massive electromagnetic fields. Any system of functions:

$$g_{\alpha\beta}\,,\ H_{\alpha\beta},\ u^{\alpha},\ N,\ P_{\alpha\beta},\ \sigma \tag{47}$$

that satisfies those equations will define a massive, electromagnetic, gravitational field. One can further add the conditions (42); one will then obtain a particular massive and electromagnetic field.

It seems almost pointless to us to remark that the solutions (47) can be obtained only in very simple cases.

As for the massive fields, there are two types of problems that one can envision, above all others:

1. Calculate the gravitational field $g_{\alpha\beta}$ for a given distribution of mass and charge ([**1**.d], Chaps. I and III).

2. Find the motion of a charged particle (i.e., a test body) that is placed in an *external* gravitational and electromagnetic field.

Meanwhile, we shall prove, in full generality, a theorem about the trajectories of the points of a continuous distribution of mass and electricity. That theorem will be a generalization of (26).

4. Equations of the space-time motion of points of a charged mass distribution. – Consider a massive gravitational and electromagnetic field that is defined by the functions:

$$g_{\alpha\beta}, \quad H_{\alpha\beta}, \quad \mathcal{N}, \quad \sigma, \quad u^\alpha;$$

those functions will then satisfy equations (30)-(33), and (44).

If one integrates the equations:

$$\frac{dx^\alpha}{ds} = u^\alpha (x^1, \ldots, x^4) \tag{48}$$

then one will have:

$$x^\alpha = x^\alpha (s, s_0 \,;\, x_0^1, \ldots, x_0^4) \tag{49}$$

for the general solution with the initial conditions:

$$x_0^\alpha \equiv x^\alpha (s, s_0 \,;\, x_0^1, \ldots, x_0^4).$$

Let A_0 denote the point with coordinates x_0^α, and let A denote the point with coordinates:

$$x^\alpha = x^\alpha (s, s_0 \,;\, A_0).$$

As s varies, the point A will describe the trajectory of the point that is at A_0 when $s = s_0$.

By virtue of the invariance relation (46), one will have:

$$\varepsilon(A_0) = \varepsilon(A). \tag{50}$$

We now introduce the electromagnetic potential Φ_α by way of the relations:

$$H_{\alpha\beta} = \frac{\partial \Phi_\alpha}{\partial x^\beta} - \frac{\partial \Phi_\beta}{\partial x^\alpha}. \tag{51}$$

Thanks to (48) and (51), equations (44) give four differential equations:

$$\frac{\delta W}{\delta x^\alpha} + \varepsilon \frac{\delta U}{\delta x^\alpha} = 0, \tag{52}$$

where:

$$U \equiv \Phi_\alpha u^\alpha. \tag{53}$$

The functions (49) will be solutions to the differential equations (52), or even, by virtue of (50), the differential equations:

$$\frac{\delta W}{\delta x^\alpha} + \varepsilon(A_0) \frac{\delta U}{\delta x^\alpha} = 0. \tag{54}$$

Equations (54) can be written:

$$\frac{\delta}{\delta x^\alpha} [W + \varepsilon(A_0)\, U] = 0. \tag{54′}$$

They express the idea that the trajectory of the charged, massive point that passes through A_0 is such that it will extremize:

$$\delta \int_{A_0}^{A} [W + \varepsilon(A_0)\, U]\, ds = 0. \tag{55}$$

(The extremities A_0 and A are assumed to be fixed.)

We remark that the four equations (54′) are not independent. Indeed, one will have the identity:

$$\frac{\delta[W + \varepsilon(A_0)U]}{\delta x^\alpha} u^\alpha \equiv 0. \tag{56}$$

Remark. – Equations (30.b) will be satisfied identically when one assumes (51). Equations (30.a) are nothing but:

$$\frac{\delta}{\delta \Phi_\alpha} [\mathcal{M} + 2\sigma u^\alpha \Phi_\alpha] = 0.$$

Indeed:

$$\frac{\delta \mathcal{M}}{\delta \Phi_\alpha} = \frac{\delta}{\delta \Phi_\alpha}\left(\frac{\sqrt{-g}}{2} g^{\gamma\delta} g^{\mu\nu} H_{\gamma\mu} H_{\delta\nu}\right) = -2\, \frac{\partial \sqrt{-g}\, H^{\alpha\beta}}{\partial x^\beta}.$$

These are the Lagrange equations of the variational principle:

$$\delta\iiint\int\left[\frac{\sqrt{-g}}{4}g^{\gamma\delta}g^{\mu\nu}(\Phi_{\gamma,\mu}-\Phi_{\mu,\gamma})(\Phi_{\delta,\nu}-\Phi_{\nu,\delta})+\sigma u^{\alpha}\Phi_{\alpha}\right]\delta x^1 \dots \delta x^4 = 0,$$

in which the functions being varied are the potentials $\Phi_1, \dots, \Phi_4$.

5. Spatial motion of charged, massive points. – It is sometimes convenient to employ time $t \equiv x^4$ to be the independent variable, instead of s. Set:

$$v^{\alpha} \equiv V\,u^{\alpha}, \qquad V \equiv \sqrt{g_{\alpha\beta}v^{\alpha}v^{\beta}} \qquad (\alpha = 1, 2, 3, 4), \tag{57}$$

$$\rho \equiv s\,/\,V, \qquad \rho_{(m)} \equiv \mathcal{N}\,/\,V. \tag{58}$$

We remark that:

$$v^{i} = \frac{dx^i}{dt}, \qquad v^4 = 1 \qquad (i = 1, 2, 3). \tag{59}$$

The first three equations (52) are identical to the following:

$$\rho_{(m)}\frac{\delta W}{\delta x^i}+\rho\frac{\delta U^*}{\delta x^i} = 0, \tag{60}$$

in which:

$$U^* = \Phi_{\alpha}\,v^{\alpha} \qquad (\alpha = 1, 2, 3, 4) \tag{61}$$

and

$$\frac{\delta}{\delta x^i} \equiv \frac{\partial}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial}{\partial v^i}\right).$$

The fourth equation:

$$\rho_{(m)}\left[\frac{\partial V}{\partial t}-\frac{d}{dt}\left(\frac{\partial V}{\partial v^4}\right)\right]+\rho\left[\frac{\partial U^*}{\partial t}-\frac{d}{dt}\left(\frac{\partial U^*}{\partial v^4}\right)\right] = 0 \tag{62}$$

is a consequence of the three equations (60).

The continuity equations (43) and (45) are written:

$$\frac{\partial\rho_{(m)}v^i}{\partial x^i}+\frac{\partial\rho_{(m)}}{\partial t} = 0, \tag{63}$$

$$\frac{\partial\rho v^i}{\partial x^i}+\frac{\partial\rho}{\partial t} = 0, \tag{64}$$

here.

It then results that:

$$\frac{d}{dt}\int \rho_{(m)}\,\delta x^1\delta x^2\delta x^3 = 0 \qquad (\delta t = 0) \tag{65}$$

and

$$\frac{d}{dt}\int \rho\,\delta x^1\delta x^2\delta x^3 = 0 \qquad (\delta t = 0); \tag{66}$$

in other words:

$$E \equiv \int_D \rho\,\delta x^1\delta x^2\delta x^3 \quad \text{and} \quad M_0\,c^2 \equiv \int_D \rho_{(m)}\,\delta x^1\delta x^2\delta x^3 \tag{67}$$

are two integral invariants of equations (60). E is the total charge that is contained in the volume D at the instant t, and M_0 is the total rest *mass* that is contained in D at the instant t.

On the other hand, one has (46):

$$\frac{d\varepsilon}{dt} = \frac{\partial\varepsilon}{\partial t} + \frac{\partial\varepsilon}{\partial x^i}v^i = 0, \qquad \text{in which} \qquad \varepsilon = \frac{\sigma}{\mathcal{N}} = \frac{\rho}{\rho_{(m)}}. \tag{68}$$

Therefore, ε is a (point-like) invariant of equations (60).

Thanks to (68), we can interpret (55) in space x^1, x^2, x^3. Consider a point P_0 whose coordinates are x_0^i at the instant t_0. ε has the value:

$$\varepsilon(P_0) = \varepsilon(x_0^1, x_0^2, x_0^3, t)$$

at P_0 at the instant t_0.

By arguing as we did in space-time, the motion of a charged, massive point that is found at P_0 at the instant t_0 will be defined at any instant t by the equations:

$$x^i = x^i(t, t_0; x_0^1, x_0^2, x_0^3),$$

in which the $x^i(t, t_0; x_0^1, \ldots, x_0^3)$ are functions of t that satisfy the differential equations:

$$\frac{\delta[V + \varepsilon(P_0)U^*]}{\delta x^i} = 0, \quad i = 1, 2, 3. \tag{69}$$

It will then result that the trajectory and the velocity of the charged material point that passes through P_0 at the instant t_0 are such that they extremize:

$$\delta \int L^*\,dt = 0, \tag{70}$$

with

$$L^* = V + \varepsilon(P_0)\,U^*. \tag{71}$$

6. Point-like particle. –

a) *Equations in space and time.* – Let Δv denote the volume that is occupied by a particle at the instant t, and set:

$$e = \int_{\Delta v} \rho\, \delta x^1 \delta x^2 \delta x^3, \qquad m_0\, c^2 = \int_{\Delta v} \rho_{(m)}\, \delta x^1 \delta x^2 \delta x^3, \tag{72}$$

in conformity with (67).

Multiply (60) by $\delta x^1\ \delta x^2\ \delta x^3$ and integrate; one will get:

$$\int_{\Delta v} \rho_{(m)} \frac{\delta V}{\delta x^i} \delta x^1 \delta x^2 \delta x^3 + \int_{\Delta v} \rho \frac{\delta U^*}{\delta x^i} \delta x^1 \delta x^2 \delta x^3 = 0. \tag{73}$$

The gravitational and electromagnetic fields at a point of Δv decompose into two parts:

1. The "internal" field (which is due to the various points of Δv).

2. The "external" field.

The action of the internal field upon the motion of the particle is assumed to be very small in comparison to the action that is exerted by the external field upon that motion. When one supposes, in addition, that the particle is *point-like* ([1]), one will infer the equations of motion for a point-like particle that is placed in a gravitational field and in an (external) electromagnetic field from (73), namely:

$$\boxed{\frac{\delta V}{\delta x^i} + \frac{e}{m_0 c} \frac{\delta U^*}{\delta x^i} = 0.} \tag{74}$$

The trajectories of the point-like particle are the ones that extremize ([2]):

$$\delta \int L^*\, dt = 0, \tag{75}$$

with

$$L^* = V + \varepsilon\, U^* \qquad (e = e / m_0\, c^2). \tag{76}$$

Equations (74) are valid only when one can neglect the reaction of the internal field on the motion of the particle.

b) *Equations in space-time.* – The variational principle (75) is equivalent to the following one:

([1]) That is, when one lets Δv tend to zero.

([2]) LEVI-CIVITA refers to the variational principle (75) by the name of the "Einstein-De Donder principle." (See [**3**] and [**1**a], eq. (357).)

$$\delta \int (W + \varepsilon\, U)\, ds = 0. \tag{77}$$

In order to see that, one should refer to paragraphs **4** and **5**.

The Lagrange equations of motion can then be written:

$$\frac{\delta(W + \varepsilon U)}{\delta x^\alpha} = 0, \tag{78}$$

or rather, since $W = 1$:

$$\frac{\delta\left(\frac{1}{2} W^2 + \varepsilon U\right)}{\delta x^\alpha} = 0. \tag{79}$$

We now pass on to Hamilton's equations. The variables conjugate to x^α will be:

$$p_\alpha = \frac{\partial L}{\partial u^\alpha}, \tag{80}$$

in which L is the Lagrangian function:

$$L = \tfrac{1}{2} W^2 + \varepsilon\, U\, ; \tag{81}$$

hence:

$$p_\alpha = u_\alpha + \varepsilon\, \Phi_\alpha\, . \tag{82}$$

The Hamiltonian function is:

$$H\,(x, p) = \tfrac{1}{2} g^{\alpha\beta}\,(p_\alpha - \varepsilon\, \Phi_\alpha)(p_\beta - \varepsilon\, \Phi_\beta)\, . \tag{83}$$

By virtue of (32) and (82):

$$H = \tfrac{1}{2}\, . \tag{84}$$

The canonical – or Hamilton's – equations will then be written:

$$\frac{dx^\alpha}{ds} = \frac{\partial H}{\partial p_\alpha}, \qquad\qquad \frac{dp_\alpha}{ds} = -\frac{\partial H}{\partial x^\alpha}\, . \tag{85}$$

Finally, introduce the Jacobian function $S\,(x^1, \ldots, x^4; s)$ by setting:

$$p^\alpha = \frac{\partial S}{\partial x^\alpha}\, . \tag{86}$$

The Jacobi equation that corresponds to (85) will be:

$$\tfrac{1}{2} g^{\alpha\beta}\left(\frac{\partial S}{\partial x^\alpha} - \varepsilon\Phi_\alpha\right)\left(\frac{\partial S}{\partial x^\beta} - \varepsilon\Phi_\beta\right) + \frac{\partial S}{\partial s} = 0. \tag{87}$$

However, by virtue of (84):

$$\frac{\partial S}{\partial s} = -\frac{1}{2}; \tag{88}$$

hence:

$$S = -\tfrac{1}{2}s + S_0\,(x^1, \dots, x^4). \tag{89}$$

The function S_0 will satisfy the equation:

$$g^{\alpha\beta}\left(\frac{\partial S_0}{\partial x^\alpha} - \varepsilon\Phi_\alpha\right)\left(\frac{\partial S_0}{\partial x^\beta} - \varepsilon\Phi_\beta\right) - 1 = 0. \tag{90}$$

That is Jacobi's fundamental equation that relates to a point-like particle of rest mass m_0 and charge e that is placed in a gravitational field along with an electromagnetic field.

c) **Remark**. – The components of the velocity **v** in the geometric space (x^1, x^2, x^3) are coupled with the components $c\,u^\alpha$ of the "velocity" in space-time by the relations (57):

$$v^i = \frac{u^i}{u^4}. \tag{91}$$

The components of the impulse in the space of (x^1, x^2, x^3) will be nothing but the expressions:

$$-m_0\,c\,p_i\,, \tag{92}$$

in which the p_i are the first three covariant components of the p_α that are defined in (82). Conforming to (92), the expressions $m_0\,c\,p_\alpha$ are the covariant components of the impulse in space-time.

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CHAPTER II

WAVE MECHANICS FOR POINT-LIKE PARTICLES WITHOUT SPIN

1. Homogenization. – As L. de Broglie showed, despite its generality, the relativistic mechanics of material points that was presented in the preceding chapter (§ **6**) constitutes only an approximation that is analogous to the relationship of the geometrical optics approximation to physical optics. Now, geometrical optics is based upon an equation that is *homogeneous* and *quadratic* with respect to the first partial derivatives of a function of four independent variables ($x^1, \ldots, x^4$).

The relativistic mechanics (**I**, § **6**) ([1]) of the material point is summarized by the Jacobi equation [**I**, (90)]. That equation us indeed a first-order partial differential equation, but it is not homogeneous with respect to those derivatives. It is natural to *homogenize* the Jacobi equation [**I**, (90)] in order to compare it to the equation of geometrical optics. That homogenization is accomplished very simply by introducing a fifth variable x^0 in the following manner: Let ([2]):

$$\Omega\,(x^1, \ldots, x^4; x^0) \equiv S_0\,(x^1, \ldots, x^4) - x^0 \tag{1}$$

be a function of the five independent variables ($x^1, \ldots, x^4; x^0$). S_0 will satisfy [**I**, (90)], so Ω will then satisfy the homogeneous, quadratic equation:

$$2H\left(x^\alpha, x^0; \frac{\partial\Omega}{\partial x^\alpha}, \frac{\partial\Omega}{\partial x^0}\right) \equiv g^{\alpha\beta}\left(\frac{\partial\Omega}{\partial x^\alpha} + \varepsilon\Phi_\alpha\frac{\partial\Omega}{\partial x^0}\right)\left(\frac{\partial\Omega}{\partial x^\beta} + \varepsilon\Phi_\beta\frac{\partial\Omega}{\partial x^0}\right) - \left(\frac{\partial\Omega}{\partial x^0}\right)^2, \tag{2}$$

since, by virtue of (1):

$$\frac{\partial\Omega}{\partial x^0} = -1. \tag{3}$$

Equation (2) plays the same role in the wave mechanics of material corpuscles that the equations of wave surfaces do in geometrical optics (see § 3).

2. Variational principle. – Recall the left-hand side of (2) and set:

$$J\,(\chi) \equiv g^{\alpha\beta}\left(\frac{\partial\chi}{\partial x^\alpha} + \varepsilon\Phi_\alpha\frac{\partial\chi}{\partial x^0}\right)\left(\frac{\partial\chi}{\partial x^\beta} + \varepsilon\Phi_\beta\frac{\partial\chi}{\partial x^0}\right) - \left(\frac{\partial\chi}{\partial x^0}\right)^2, \tag{4}$$

([1]) The Roman numeral I always refers to Chapter I.

([2]) One can first write $\Omega = S_0 - \varphi(s)$, in which s will be the fifth variable, and φ will be an arbitrary function. One will then pass to (1) upon then setting $x^0 = \varphi(s)$.

in which $\chi(x^1, \ldots, x^4; x^0)$ is a *real* function of the spatio-temporal variables $x^1, \ldots, x^4$, and the fifth variable x^0 that was introduced in the preceding paragraph. Following Th. De Donder ([1]), we write the fundamental equation of the wave mechanics of the charged material point by means of the following variational principle:

$$\frac{\delta\sqrt{-g}\; J(\chi)}{\delta\chi} = 0, \tag{5}$$

in which:

$$\frac{\delta}{\delta\chi} \equiv \frac{\partial}{\partial\chi} - \sum_{\mu=0,1,\cdots,4} \frac{d}{dx^\mu}\left(\frac{\partial}{\partial\chi_{,\mu}}\right); \qquad \chi_{,\mu} \equiv \frac{\partial\chi}{\partial x^\mu}.$$

Upon making (5) more specific, it become:

$$D\chi \equiv \Box\chi + 2\varepsilon\Phi^\alpha \frac{\partial^2\chi}{\partial x^\alpha\,\partial x^\beta} + (\varepsilon^2\Phi^\alpha\Phi^\beta - 1)\frac{\partial^2\chi}{(\partial x^0)^2} + \frac{1}{\sqrt{-g}}\frac{\partial\Phi^\alpha\sqrt{-g}}{\partial x^\alpha}\cdot\frac{\partial\chi}{\partial x^0} = 0, \tag{6}$$

in which:

$$\Box\chi \equiv \frac{1}{\sqrt{-g}} \sum_{\alpha=1}^{4} \frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\,\sum_\alpha g^{\alpha\beta}\frac{\partial\chi}{\partial x^\alpha}\right). \tag{7}$$

(6) is the *fundamental equation of wave mechanics for point-like particles without spin* in the five independent variables $x^1, \ldots, x^4, x^0$. That equation is linear with respect to the first and second derivatives of the function χ.

3. Waves and rays. – Waves ([2]):

$$\Omega\,(x^1, \ldots, x^4, x^0) = 0 \tag{8}$$

that are compatible with equation (6) must satisfy the equation:

$$g^{\alpha\beta}\left(\frac{\partial\Omega}{\partial x^\alpha} + \varepsilon\Phi_\alpha\frac{\partial\Omega}{\partial x^0}\right)\left(\frac{\partial\Omega}{\partial x^\beta} + \varepsilon\Phi_\beta\frac{\partial\Omega}{\partial x^0}\right) - \left(\frac{\partial\Omega}{\partial x^0}\right)^2 = 0. \tag{9}$$

That is precisely equation (2).

Introduce the function:

$$H\,(x, p) \equiv \tfrac{1}{2}\,[g^{\alpha\beta}\,(p_\alpha + \varepsilon\,\Phi_\alpha\,p_0)\,(p_\beta + \varepsilon\,\Phi_\beta\,p_0) - (p_0)^2], \tag{10}$$

([1]) See [**4**], pp. 79.
([2]) For details, consult [**5**].

that is obtained by replacing $\partial\Omega / x^\alpha$ with the variables p_α and replacing $\partial\Omega / x^0$ with the variable p_0 in $H\left(x^\alpha, x^0; \frac{\partial\Omega}{\partial x^\alpha}, \frac{\partial\Omega}{\partial x^0}\right)$, as it was defined in (2). The characteristic lines of equation (9) are defined by the equations:

$$\frac{dx^\mu}{\dfrac{\partial H}{\partial p_\mu}} = \frac{dp_\mu}{-\dfrac{\partial H}{\partial x^\mu}} = d\theta, \qquad \mu = 0, 1, 2, 3, 4, \tag{11}$$

with the condition:

$$H(x, \mu) = 0; \tag{12}$$

θ is an arbitrary parameter. Equations (11) have the Hamiltonian form; the variables p_μ are canonically associated with the x^μ. Let:

$$\left.\begin{aligned} x^\mu &= X^\mu(\theta), \quad (a) \\ p_\mu &= P_\mu(\theta) \quad\;\; (b) \end{aligned}\right\} \tag{13}$$

be a solution to the system (11) and (12). One will have:

$$H(X(\theta), P(\theta)) = 0$$

identically in θ.

Equations (13.a) are the parametric equations of a *ray* in the five-dimensional space $x^1, \ldots, x^4, x^0$. Conforming to the theory of characteristic surfaces ([1]), that ray will be found entirely upon a wave surface.

4. Extremals. – In this paragraph, the x^μ and the p_μ are no longer forced to be coupled by the relation (12). Equations (1) have the Hamiltonian form. As is well-known, one passes to the Lagrangian form by introducing the Lagrangian variables:

$$W^\mu = \frac{\partial H}{\partial p_\mu}, \qquad \mu = 0, 1, 2, \ldots, 4.$$

Explicitly:

$$\left.\begin{aligned} W^0 &= \varepsilon\,\Phi^\alpha(p_\alpha + \varepsilon\,\Phi_\alpha p_0) - p_0, \\ W^\alpha &= g^{\alpha\beta}(p_\beta + \varepsilon\,\Phi_\beta p_0); \end{aligned}\right\} \tag{14}$$

conversely:

$$\left.\begin{aligned} p_0 &= \varepsilon\,\Phi_\alpha W^\alpha - W^0, \\ p_\alpha &= (g_{\alpha\beta} - \varepsilon^2\,\Phi_\alpha\Phi_\beta)W^\beta + \varepsilon\,\Phi_\alpha W^0. \end{aligned}\right\} \tag{14'}$$

([1]) See [**5**], pp. 21.

The Lagrangian function L, which is a function of the x^μ and W_μ , is defined by:

$$L = - H + p_\mu \frac{\partial H}{\partial p_\mu}\,; \tag{15}$$

explicitly, when one utilizes (14′), it will be:

$$L\,(x, W) \equiv \tfrac{1}{2}(g_{\alpha\beta} - \varepsilon^2\,\Phi_\alpha\,\Phi_\beta) - W^\alpha\,W^\beta + 2\,\varepsilon\,\Phi_\alpha\,W^\alpha - (W^{\,0})^{\,2}. \tag{15′}$$

By virtue of (15):

$$L = H. \tag{16}$$

The Hamiltonian equations (11) are equivalent to the Lagrangian equations:

$$\left.\begin{aligned} &\frac{dx^\mu}{d\theta} = W^\mu, \\ &\frac{\partial L}{\partial x^\mu} - \frac{d}{d\theta}\frac{\partial L}{\partial W^\mu} = 0. \end{aligned}\right\} \tag{17}$$

If:

$$\left.\begin{aligned} x^\mu &= x_e^\mu(\theta), \quad (a) \\ W^\mu &= W_e^\mu(\theta) \quad (b) \end{aligned}\right\} \tag{18}$$

is a solution of (17.*a*, *b*) then equations (18.*a*) will define an *extremal line.* When the functions (18.*a*, *b*) are also such that $L\,[x_e\,(\theta), W_e\,(\theta)] = 0$, the extremal considered will be a ray.

Now let:

$$x^\mu = - x^\mu\,(\theta)$$

be the parametric equation of an arbitrary curve l_{AB} in the space of $(x^0, \ldots, x^4)$ that passes through the points A and B with the coordinates:

$$x_A^\mu = x^\mu\,(\theta_1), \quad x_B^\mu = - x^\mu\,(\theta_2).$$

Set:

$$\dot{x}^\mu \equiv \frac{dx^\mu(\theta)}{d\theta}, \qquad I \equiv \int_{l_{AB}} L(x, \dot{x})\,d\theta\,.$$

The integral is taken from A to B along l_{AB} .

The first variation δI is given in full generality by the formula:

$$\delta I \equiv \int_{\theta_1}^{\theta_2}\left(\frac{\partial L(x,\dot{x})}{\partial x^\mu} - \frac{d}{d\theta}\frac{\partial L(x,\dot{x})}{\partial \dot{x}^\mu}\right)(\delta x^\mu - \dot{x}^\mu\,\delta\theta)\,d\theta + \left[\frac{\partial L(x,\dot{x})}{\partial \dot{x}^\mu}(\delta x^\mu - \dot{x}^\mu\,\delta\theta) + L\,\delta\theta\right]_A^B.$$

If the curve l_{AB} is an extremal then, by (17), the functions x^μ (θ) will satisfy the equations:

$$\frac{\partial L(x,\dot{x})}{\partial x^\mu} - \frac{d}{d\theta}\left(\frac{\partial L(x,\dot{x})}{\partial \dot{x}^\mu}\right) = 0. \tag{19}$$

In that case:

$$\delta I = \left[\frac{\partial L(x,\dot{x})}{\partial \dot{x}^\mu}(\delta x^\mu - \dot{x}^\mu \delta\theta) + L\,\delta\theta\right]_A^B. \tag{20}$$

5. Fermat's theorem ([1]). – Now suppose that the extremal considered in § **4** is a ray $\mathcal{L}_{AB}$. With the notation (13.*a*), one will then have:

$$L\,(X\,(\theta),\ \dot{X}\,(\dot{\theta})\,) \equiv 0.$$

The varied curves that are close to the ray $\mathcal{L}_{AB}$ will likewise be subject to the condition:

$$L\,(x,\ \dot{x}\,) = 0; \tag{21}$$

hence, upon solving for x^0:

$$x^0 = \varepsilon\,\Phi_\alpha\,x^\alpha \pm \sqrt{g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta}\ . \tag{21′}$$

Since L is identically zero for any varied curve, as well as for the ray, the integral I will be identically zero when it is taken along any of those curves. Similarly:

$$\delta I \equiv 0.$$

Now, by virtue of (19) and (20), one will then have:

$$\left[\frac{\partial L(X,\dot{X})}{\partial \dot{X}^\mu}\,\delta x^\mu\right]_A^B = 0 \tag{22}$$

when one takes into account that $L\,(x,\ \dot{x}\,)$ is a homogeneous function (of second degree) in $\dot{x}^0,\ \dot{x}^1,\ \ldots,\ \dot{x}^4$. (21) is a relation between the $(\delta x^\mu)_A$ and the $(\delta x^\mu)_B$.

We remark that since the extremal considered is a ray, if the point A is taken arbitrarily then the point B will no longer be arbitrary. Furthermore, one can give four of the five coordinates of the point B *a priori*. (That will result easily from the remark that was made at the end of § **3**.) The interesting case is then one in which the four space-time coordinates x_B^α $(\alpha = 1, \ldots, 4)$ are taken arbitrarily.

([1]) The proof of Fermat's theorem in the most general case was done by Th. De Donder (see [**5***a*] or [**5***b*], pp. 178-183.) Here, we shall follow the method that was employed by Th. De Donder.

The parametric equations of the ray $\mathcal{L}_{AB}$ considered are assumed to be written as in (13.*a*). The coordinates of the extremities A and B are obtained by giving the values θ_1 and θ_2 to the parameter θ.

In the space-time $x^0 = 0$, the ray $\mathcal{L}_{AB}$ corresponds to the curve whose parametric equations are:

$$x^\alpha = X^\alpha(\theta).$$

Finally, consider the case in which the coordinates $x_A^\alpha = x_P^\alpha$, $x_B^\alpha = x_Q^\alpha$, are fixed; i.e.:

$$(\delta x^\alpha)_A = (\delta x^\alpha)_B = 0, \qquad \alpha = 1, \dots, 4.$$

(21) will then become:

$$\left[\frac{\partial L(X,\dot{X})}{\partial \dot{X}^0}\,\delta x^0\right]_A^B = 0. \tag{23}$$

However, L does not depend upon x^0 explicitly; by virtue of (19), one will then have:

$$\frac{d}{d\theta}\left(\frac{\partial L(x,\dot{x})}{\partial \dot{x}^0}\right) = 0, \qquad \text{so} \qquad \left(\frac{\partial L(X,\dot{X})}{\partial \dot{X}^0}\right)_A \equiv \left(\frac{\partial L(X,\dot{X})}{\partial \dot{X}^0}\right)_B .$$

Upon taking that property into account, (23) will give:

$$(\delta x^0)_A - (\delta x^0)_B = 0;$$

hence:

$$\delta \int_{l_{AB}} \dot{x}^0 d\theta = 0. \tag{24}$$

Due to (21), one will have:

$$\int \dot{x}^0 d\theta = \int (\varepsilon\,\Phi_\alpha \dot{x}^\alpha \pm \sqrt{g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta})\,d\theta$$

for both the ray and the varied path.

The variational equation (24) can then be written:

$$\delta \int_{\mathcal{L}_{AB}} (\varepsilon\,\Phi_\alpha \dot{x}^\alpha \pm \sqrt{g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta})\,d\theta = 0. \tag{24′}$$

The integral whose variation we take does not depend upon θ explicitly; it has the parametric form. In addition, it does not depend upon x^0. If l denotes any of the varied curves in the five-dimensional space, and λ denotes the projection of l on space-time then we will have:

$$\int_l (\varepsilon\,\Phi_\alpha \dot{x}^\alpha \pm \sqrt{g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta})\,d\theta = \int_\lambda (\varepsilon\,\Phi_\alpha \dot{x}^\alpha \pm \sqrt{g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta})\,d\theta$$

$$= \int_\lambda (\varepsilon\,\Phi_\alpha dx^\alpha \pm \sqrt{g_{\alpha\beta}\,dx^\alpha dx^\beta})\,.$$

Thanks to those relations, (24′) will become:

$$\delta \int_{\Lambda_{PQ}} (\varepsilon\,\Phi_\alpha dx^\alpha \pm \sqrt{g_{\alpha\beta}\,dx^\alpha dx^\beta}) = 0. \tag{25}$$

If one takes the + sign then one will recover equation [**I**, (77)] for the motion of a charged, massive point with a charge of (+*e*) in a gravitational field and an electromagnetic field. With the – sign, one will get the equation of motion of a charge (–*e*) that is placed in a gravitational field and an electromagnetic field.

One can formulate that property in the following manner: The projections onto space-time ($x^0 = 0$) of the bicharacteristic lines – or rays – that are associated with the second-order partial differential equation (6) are the trajectories of charged material points with charges + *e* or – *e* of the classical theory.

6. Fundamental equations in space-time. – Let Ψ′ and Ψ″ be two real functions of $x^0, x^1, \ldots, x^4$ that are each required to satisfy the (real) equation (6); One will then have:

$$D\,(\Psi') = 0 \qquad \text{and} \qquad D\,(\Psi'') = 0.$$

Those two equations can be replaced with the single equivalent equation:

$$D\,(\Psi) = 0, \tag{26}$$

in which:

$$\Psi = \Psi' + i\,\Psi'' \qquad\qquad (i = \sqrt{-1}).$$

The coefficients of (26) do not depend upon x^0. We then seek the solutions of the form:

$$\Psi\,(x^0, x^1, \ldots, x^4) = \psi\,(x^1, \ldots, x^4)\; e^{2i\pi\frac{m_0 c}{h}x^0},$$

in which *h* is an (undetermined) constant that has the dimensions of action. The function Ψ thus-defined will be a solution of (26) if $\psi\,(x^1, \ldots, x^4)$ satisfies the equation:

$$\Box\psi - \frac{4i\pi}{h}\frac{e}{c}\Phi^\alpha\frac{\partial\psi}{\partial x^\alpha} + \left(\frac{2i\pi}{h}\right)^2\left(\frac{e^2}{c^2}\Phi^\alpha\Phi_\alpha - m_0^2\,c^2\right)\psi - \frac{2i\pi}{h}\frac{e}{c}\frac{1}{\sqrt{-g}}\frac{\partial\Phi^\alpha\sqrt{-g}}{\partial x^\alpha}\psi = 0. \tag{27}$$

That is the relativistic equation of wave mechanics in space-time. It includes a new fundamental constant *h*. Schrödinger has shown that *h* must be identified with Planck's constant.

Remark. –

a) When one takes into account the complementary Maxwell equation:

$$\frac{\partial\sqrt{-g}\;\Phi^\alpha}{\partial x^\alpha} = 0,$$

equation (27) will become:

$$\Box\psi - \frac{4i\pi}{h}\frac{e}{c}\Phi^\alpha\frac{\partial\psi}{\partial x^\alpha} + \left(\frac{2i\pi}{h}\right)^2\left(\frac{e^2}{c^2}\Phi^\alpha\Phi_\alpha - m_0^2\,c^2\right)\psi = 0. \tag{27′}$$

b) In order to eliminate x^0, one can likewise set:

$$\Psi\,(x^1, \ldots, x^4, x^0) = \varphi\,(x^1, \ldots, x^4)\;e^{-2i\pi\frac{m_0 c}{h}x^0}\,.$$

This Ψ will be a solution of (26) if the complex function $\varphi\,(x^1, \ldots, x^4)$ satisfies:

$$\Box\varphi + \frac{4i\pi}{h}\frac{e}{c}\Phi^\alpha\frac{\partial\varphi}{\partial x^\alpha} + \left(\frac{2i\pi}{h}\right)^2\left(\frac{e^2}{c^2}\Phi^\alpha\Phi_\alpha - m_0^2\,c^2\right)\varphi = 0, \tag{28}$$

when one takes the complementary Maxwell equation into account. One passes from (27′) to (28) by changing i into $-\,i$ in the coefficients of (27′), or also by changing e into $-e$ (and ψ into φ). That remark should be compared with the study that was made in §§ **3** and **4**. Indeed, we have seen that the rays that are described in five-dimensional space correspond in space-time to the trajectories of material points of charge $+\,e$ or $-\,e$. Similarly, here, when the fundamental equation (26) is written in five-dimensional space, it will correspond to the two equations (27) and (28) in space-time, and one passes from (27) to (28) by changing e into $-\,e$. Hence, if ψ refers to the particle (m_0, e) then φ will refer to the particle (m_0, $-\,e$).

7. Other forms of the variational principle. –

a) *In the space of* (x^0, x^1, …, x^4). – Instead of taking the (real) form $J\,(x)$ to be the form that serves as the basis for the variational principle (§ **2**), one can take the (real) function:

$$K \equiv \frac{1}{m_0}g^{\alpha\beta}\left(\frac{\partial\Psi}{\partial x^\alpha} + \varepsilon\,\Phi_\alpha\frac{\partial\Psi}{\partial x^0}\right)\left(\frac{\partial\Psi^*}{\partial x^\beta} + \varepsilon\,\Phi_\beta\frac{\partial\Psi^*}{\partial x^0}\right) - \frac{1}{m_0}\frac{\partial\Psi}{\partial x^0}\frac{\partial\Psi^*}{\partial x^0}\,.$$

One will immediately verify that:

$$\frac{\delta\sqrt{-g}\,K}{\delta\Psi} \equiv -\frac{1}{2m_0}D\Psi^* \qquad \text{and} \qquad \frac{\delta\sqrt{-g}\,K}{\delta\Psi^*} \equiv -\frac{1}{2m_0}D\Psi.$$

The equations:

$$\frac{\delta\sqrt{-g}\,K}{\delta\Psi} = 0, \qquad \frac{\delta\sqrt{-g}\,K}{\delta\Psi^*} = 0$$

are then equivalent to (26) and its conjugate.

b) *In the space-time* (x^0, x^1, …, x^4). – When Ψ has the form $\Psi = \psi\, e^{\frac{2\pi i}{h} m_0 c\, x^0}$, the function K will become:

$$K = \frac{1}{m_0} g^{\alpha\beta}\left(\frac{\partial\psi}{\partial x^\alpha} - \frac{2\pi i\, e}{hc}\Phi_\alpha\psi\right)\left(\frac{\partial\psi^*}{\partial x^\alpha} + \frac{2\pi i\, e}{hc}\Phi_\alpha\psi^*\right) + \left(\frac{2\pi i\, m_0\, c}{h}\right)^2 \psi\psi^* .$$

Set:

$$\boxed{L \equiv \frac{-1}{m_0} g^{\alpha\beta}\left(\frac{h}{2\pi i}\frac{\partial\psi}{\partial x^\alpha} - \frac{e}{c}\Phi_\alpha\psi\right)\left(-\frac{h}{2\pi i}\frac{\partial\psi^*}{\partial x^\beta} - \frac{e}{c}\Phi_\beta\psi^*\right) + m_0\, c^2\psi\psi^*.} \tag{29}$$

Upon setting $\psi_{,\alpha} \equiv \partial\psi / x^\alpha$, and similarly $\psi^\times_{,\alpha} \equiv \partial\psi^\times / x^\alpha$:

$$\frac{\partial L}{\partial\psi^*} = \frac{1}{m_0} g^{\alpha\beta}\left(\frac{h}{2\pi i}\frac{\partial\psi}{\partial x^\beta} - \frac{e}{c}\Phi_\beta\psi\right)\frac{e}{c}\Phi_\alpha + m_0\, c^2\psi ,$$

$$\frac{\partial L}{\partial\psi^*_{,\alpha}} = \frac{1}{m_0} g^{\alpha\beta}\left(\frac{h}{2\pi i}\psi_{,\beta} - \frac{e}{c}\Phi_\beta\psi\right);$$

hence:

$$\frac{4\pi^2 m_0}{h^2}\frac{\delta\sqrt{-g}\;L}{\delta\psi^*}$$
$$\equiv \Box\psi - \frac{4\pi i e}{hc}\Phi^\alpha\psi_{,\alpha} + \left(\frac{2\pi i}{h}\right)^2\left(\frac{e^2}{c^2}\Phi^\alpha\Phi_\alpha - m_0^2\, c^2\right)\psi - \frac{2\pi i e}{hc}\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}\,\Phi^\alpha}{\partial x^\alpha}\psi .$$

It will then result that the equation:

$$\frac{\delta\sqrt{-g}\;L}{\delta\psi^*} = 0 \tag{30}$$

is identical to (27). Similarly, the equation:

$$\frac{\delta\sqrt{-g}\ L}{\delta\psi} = 0 \tag{30′}$$

is identical to the conjugate of (27). The fundamental equation (27) of wave mechanics for a charged, massive particle of mass m_0 and charge $\mp e$, will then be derived from the variational principle:

$$\delta\iiiint\sqrt{-g}\ \ L\,dx^1 \ldots dx^4 = 0.$$

8. Th. De Donder's correspondence principle. – In the beginning of the first paragraph, we briefly recalled how L. de Broglie was led to look for a new mechanics that was based upon an equation that is analogous to the fundamental (second-order partial differential) equation of Fresnel's physical optics. The presentation above showed that it is only in the five-dimensional space that one can speak in full generality of a real wave function $\chi(x^0, x^1, \ldots, x^4)$ in wave mechanics.

Now, as of today, there seems to be no physical meaning to that fifth variable x^0 ; it plays only the role of an auxiliary parameter that will be eliminated in the final analysis. The equation that is actually interesting is therefore not the equation that is written in five-dimensional space, but (27). In general, that equation (27) has complex coefficients, and the wave function $\psi(x^1, \ldots, x^4)$ will generally be complex. It is that complex wave function ψ that represents the state of the particle studied in space-time. In other words, all of the physical properties of the particle will be expressed in terms of ψ.

The theory of gravitational and electromagnetic fields studies certain physical quantities whose very general expressions were given in the first chapter. The *correspondence principle* essentially consists of making any physical quantity of Einstein's theory correspond to an expression as a function of ψ and ψ^*.

The electric current-density $\mathcal{C}^\alpha$ and the energy-impulse tensor $T^{\alpha\beta}$ are two particularly important quantities. In the theory of gravitational an electromagnetic fields, one has [**I**, (6)]:

$$\mathcal{T}_{\alpha\beta} = -\frac{\delta\mathcal{M}}{\delta g^{\alpha\beta}}. \tag{31}$$

and by virtue of [**I**, (29), (51)] and Maxwell's equations:

$$\mathcal{C}^\alpha = -\frac{1}{2}\frac{\delta\mathcal{M}}{\delta\Phi_\alpha}. \tag{32}$$

Following Th. De Donder, we define a vector $\mathcal{C}^\alpha$ and a tensor $\mathcal{T}_{\alpha\beta}$ in wave mechanics in a formally-identical manner. Therefore, we set, by definition:

$$\mathcal{T}_{\alpha\beta} = -\frac{\delta\mathcal{L}}{\delta g^{\alpha\beta}}, \tag{33}$$

$$\mathcal{C}^\alpha = -\frac{1}{2}\frac{\delta \mathcal{L}}{\delta \Phi_\alpha}, \tag{34}$$

in which $\mathcal{L}$ is the Lagrangian function (29), multiplied by $\sqrt{-g}$.

The analogy between (31), (32), (33), and (34) purely formal. Indeed, in (31), (32), $g_{\alpha\beta}$ and Φ_α represent the total gravitational field and the total electromagnetic field, respectively. $\mathcal{T}_{\alpha\beta}$ and $\mathcal{C}^\alpha$ define the total energy-impulse tensor and the total current, resp. On the contrary, $g_{\alpha\beta}$ and Φ_α represent the external fields in which the particle considered has been placed, while $\mathcal{T}_{\alpha\beta}$ and $\mathcal{C}^\alpha$ define an "energy-impulse tensor" and a "current," respectively, that relate to the particle under study, since they are expressed with the aid of the wave function ψ.

Here are the explicit values of the left-hand sides of (33) and (34):

The electric current ($\mathcal{C}^1$, $\mathcal{C}^2$, $\mathcal{C}^3$) and the charge density $\mathcal{C}^4$ are:

$$\mathcal{C}^\alpha = \frac{\sqrt{-g}}{2m_0}\frac{e}{c} g^{\alpha\beta}\left[\psi^*\left(\frac{h}{2\pi i}\psi_{,\beta} - \frac{e}{c}\Phi_\beta \psi\right) + \text{conj.}\right]. \tag{33′}$$

Upon remarking that:

$$\frac{\delta\sqrt{-g}}{\delta g^{\alpha\beta}} = \frac{\partial\sqrt{-g}}{\partial g^{\alpha\beta}} = \frac{-\sqrt{-g}}{2} g_{\alpha\beta},$$

one will obtain the expression for the massive symmetric tensor:

$$\mathcal{T}_{\alpha\beta} = \frac{\sqrt{-g}}{2m_0}\left[\psi^*\left(\frac{h}{2\pi i}\psi_{,\alpha} - \frac{e}{c}\Phi_\alpha \psi\right)\left(-\frac{h}{2\pi i}\psi^*_{,\beta} - \frac{e}{c}\Phi_\alpha \psi^*\right) + \text{conj.}\right] + \frac{g_{\alpha\beta}}{2}\mathcal{L}. \tag{34′}$$

Finally, by correspondence, we write down the Maxwell equations [cf., **I**, (30)]:

$$\frac{\partial \overparen{\sqrt{-g}\; H^{\alpha\beta}}}{\partial x^\beta} = \mathcal{C}^\alpha, \tag{35}$$

in which the $\mathcal{C}^\alpha$ are given by (33′). The arc over the symbols signifies the difference between the fields that are calculated by starting with the equations above and the external fields $g_{\alpha\beta}$, Φ_α, which are given *a priori* and figure in the right-hand sides of (35).

One can study the gravitational problems in an analogous manner by starting from the tensor $T_{\alpha\beta}$ that is given by (34′). Conforming to [**I**, (12)], one will be led to write the equations:

$$\tfrac{1}{2}(a + b\widehat{C})\,\widehat{g}_{\alpha\beta} + b\widehat{G}_{\alpha\beta} = T_{\alpha\beta}. \tag{36}$$

For example, consider the case in which a is zero. Set:

$$\hat{g}_{\alpha\beta} = -\,\delta_{\alpha\beta} + \gamma_{\alpha\beta},$$

in which $\delta_{\alpha\beta} = 0$ when $\alpha \neq \beta$, $\delta_{ii} = +1$, $\delta_{44} = -c^2$, and assume that the products of the γ_{ij}, $\frac{1}{c}\gamma_{i4}$, $\frac{1}{c^2}\gamma_{44}$, taken pair-wise, are negligible compared to 1. One knows that the left-hand sides of (36) will then simplify considerably. Following Einstein's method ([1]), one will have:

$$-\frac{b}{2}\hat{C}\,\hat{g}_{\alpha\beta} + b\,\hat{G}_{\alpha\beta} \approx -\frac{b}{2}\Box\left(\gamma_{\alpha\beta} - \tfrac{1}{2}\delta_{\alpha\beta}\gamma\right),$$

in which:

$$\gamma \equiv \sum_{\alpha}\delta_{\alpha\alpha}\gamma_{\alpha\alpha} \qquad \text{and} \qquad \Box \equiv \sum_{\alpha}\delta_{\alpha\alpha}\frac{\partial^2}{(\partial x^\alpha)^2}.$$

Equations (35) become:

$$\Box\left(\gamma_{\alpha\beta} - \tfrac{1}{2}\delta_{\alpha\beta}\gamma\right) = -\frac{2}{b}T_{\alpha\beta},$$

which are equations that one solves by the theory of retarded potentials.

By reason of the value of $b = -\,4.8 \times 10^{47}$ cm-g / s^2, the fields $\gamma_{\alpha\beta}$ that one obtains have hardly attracted the attention of physicists, up to now.

DIMENSIONS. – Denote the dimension of the quantity 0 by {0}.

To fix ideas, we take $\{x^i\} = \mathrm{L} \equiv$ length, $\{x^4\} = \mathrm{T} \equiv$ time; hence, $\{g\} = \mathrm{L}^2\,\mathrm{T}^{-2}$.

We will then have:

$$\{\mathcal{L}\} = \text{energy} \times \mathrm{LT}^{-1} \times \{\psi^*\,\psi\},$$
$$\{\mathcal{C}^i\} = \{e\,c\}\{\psi^*\,\psi\},$$
$$\{\mathcal{C}^4\} = \{e\}\{\psi^*\,\psi\},$$
$$\{T_4^4\} = \text{energy} \times \{\psi^*\,\psi\}, \text{ etc.}$$

It will result from these formulas and the physical significance of $\{\psi^*\,\psi\}$ that it has the dimensions of the inverse of a volume.

9. Conservation of electricity. –

THEOREM. – *By virtue of the fundamental equations* (30) *and* (30′):

([1]) See [**7**].

$$\boxed{\frac{\partial C^\alpha}{\partial x^\alpha} = 0.} \tag{37}$$

Proof:

Calculate:

$$X \equiv \psi^* \frac{\delta \mathcal{L}}{\delta \psi^*} - \psi \frac{\delta \mathcal{L}}{\delta \psi}.$$

One will first get:

$$X = -\frac{\partial}{\partial x^\alpha}\left(\psi^* \frac{\partial \mathcal{L}}{\partial \psi^*_{,\alpha}} - \psi \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\right) + \psi^* \frac{\partial \mathcal{L}}{\partial \psi^*} - \psi \frac{\partial \mathcal{L}}{\partial \psi} + \psi^*_{,\alpha} \frac{\partial \mathcal{L}}{\partial \psi^*_{,\alpha}} - \psi_{,\alpha} \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}.$$

However:

$$\mathcal{C}^\alpha = \left(\psi^* \frac{\partial \mathcal{L}}{\partial \psi^*_{,\alpha}} - \psi \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\right).$$

In addition:

$$\psi^* \frac{\partial \mathcal{L}}{\partial \psi^*} + \psi^*_{,\alpha} \frac{\partial \mathcal{L}}{\partial \psi^*_{,\alpha}} = \psi \frac{\partial \mathcal{L}}{\partial \psi} + \psi_{,\alpha} \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}},$$

since $\mathcal{C}$ is a bilinear function in ψ, ψ_α, and $\psi^\times$, $\psi^\times_\alpha$. It will then result that:

$$X = -\frac{c}{ei}\frac{\partial \mathcal{C}^\alpha}{\partial x^\alpha}. \tag{38}$$

(37) will result immediately from (38), since X is obviously zero, by virtue of equations (30) and (30′).

That theorem succeeds in justifying the definition (34) of the electric current.

10. Law of impulse and energy. – By virtue of the fundamental equations (30) and (30′), one will have:

$$\boxed{\mathcal{T}^\beta_{\alpha,\beta} + H_{\alpha\beta} C^\beta = 0,} \tag{39}$$

in which , β is the symbol of the covariant derivative with respect to x^β.

Proof. – We apply the generalized Hilbert identities ([1]) to the multiplier $\mathcal{L}$. That will immediately give:

([1]) See [**7**.a] and [**7**.b].

$$-2\mathcal{T}^{\beta}_{\alpha,\beta} + 2\mathcal{C}^{\beta} H_{\beta\alpha} + 2\mathcal{C}^{\beta}_{,\beta}\Phi_{\alpha} + \frac{\delta\mathcal{L}}{\delta\psi}\psi_{,\beta} + \frac{\delta\mathcal{L}}{\delta\psi^*}\psi^*_{,\beta} \equiv 0;$$

hence, one will have the theorem thanks to (37).

Remark. – As in the case of electric current, the property (39) succeeds in justifying the definition (33) of the energy-impulse tensor.

11. Approximations. –

a) We place ourselves in the case where the gravitational field is that of Minkowski; we will then have:

$$g_{i4} = 0; \qquad g_{44} = c^2.$$

Set $x^4 = t$ and:

$$\psi(x^1, x^2, x^3, t) = \xi(x^1, x^2, x^3, t)\, e^{\frac{2\pi i}{h} m_0 c^2 t}.$$

Suppose that:

$$\frac{h}{2\pi i}\frac{\partial\xi}{\partial t}, \qquad \frac{e}{c}\Phi_4\,\xi, \qquad e\,\Phi_i\,\xi$$

are very small of first order with respect to $m_0\, c^2\, \xi$. Equation (27′) will then give the approximate equation:

$$\Delta\xi - \frac{4\pi i\, m_0}{h}\frac{\partial\xi}{\partial t} - \frac{4\pi i\, e}{hc}\sum_{j=1}^{3}\Phi^i\frac{\partial\xi}{\partial x^j} - \frac{8\pi^2}{h^2} m_0\, e\,\nabla\xi = 0. \tag{40}$$

With the same approximation, the components $\mathcal{C}^{\,i}$ and $\mathcal{C}^{\,4} \equiv \rho$ of the electric current and charge density will become:

$$\mathcal{C}^{\,i} = \frac{e}{2m_0}\sum_{j=1}^{3} g^{ij}\xi^*\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^j} - \frac{e}{c}\Phi_j\right)\xi + \text{conj.}, \qquad \mathcal{C}^{\,4} = \rho = e\,\xi^*\xi. \tag{41}$$

Similarly:

$$T^{\,4i} = \sum_{j=1}^{3}\tfrac{1}{2} g^{ij}\left[\xi^*\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^j} - \frac{e}{c}\Phi_j\right)\xi + \text{conj.}\right] = m_0\,\frac{\mathcal{C}^i}{e},$$

$$T^4_4 = m_0 c^2 \xi^*\xi + \tfrac{1}{2}\left(\xi^*\frac{h}{2\pi i}\frac{\partial\xi}{\partial t} + \text{conj.}\right) - e\nabla\xi^*\xi + \frac{e}{2m_0 c}\sum_j \Phi^j\left(\xi^*\frac{h}{2\pi i}\frac{\partial\xi}{\partial x^j} + \text{conj.}\right). \tag{42}$$

The link between $T^{\,ij}$ and $C^{\,j}$ is indeed consistent with their physical significance.

b) Another particularly-interesting case is that of:

$$\Phi_i = \Phi_4 = 0,$$

as well as:

$$g_{i4} = 0;$$

however:

$$g_{44} = c^2 \left(1+\frac{\gamma_{44}}{c^2}\right).$$

The $(ds)^2$ can then be written:

$$(ds)^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2 \left(1+\frac{\gamma_{44}}{c^2}\right)(dt)^2.$$

If γ_{44} does not depend upon time then one can obtain the stationary solutions:

$$\psi(x^1, x^2, x^3, t) = a\,(x^1, x^2, x^3)\; e^{\frac{2\pi iE}{h}t}.$$

The complex function a must satisfy:

$$g_{ik}\sum_{j=1}^{3}\frac{\partial^2 a}{(\partial x^j)^2}+\frac{1}{2}\sum_j\frac{\partial g_{44}}{\partial x^j}\frac{\partial a}{\partial x^j}+\frac{4\pi^2}{h^2}(E^2 - m_0^2\,c^2 - m_0^2\,c^2\gamma_{44})\,a = 0,$$

or, upon setting:

$$E = \mathcal{E} + m_0\,c,$$

$$\left(1+\frac{\gamma_{44}}{c^2}\right)\sum_{j=1}^{3}\frac{\partial^2 a}{(\partial x^j)^2}+\frac{1}{2c^2}\sum_j\frac{\partial \gamma_{44}}{\partial x^j}\frac{\partial a}{\partial x^j}+\frac{4\pi^2}{h^2}\left(\frac{\mathcal{E}^2}{c^2}+2\mathcal{E}\,m_0 - m_0^2\gamma_{44}\right)a = 0.$$

That equation is rigorous ([1]). Neglect the terms in that equation that include c^2 in the denominator. One will then have the approximate equation:

$$\sum_j\frac{\partial^2 a}{(\partial x^j)^2}+\frac{8\pi^2 m_0}{h^2}\left(\mathcal{E} - m_0\,\frac{\gamma_{44}}{2}\right)a = 0.$$

The approximation that we just made can be called the "Newtonian" approximation. In effect, it consists of considering $c^2 \sim \infty$.

([1]) One will obviously arrive at the same equation no matter what value the electromagnetic field has when the particle under study has an electric charge of zero.

12. Equivalence of the variational principle and the operator principle. – In paragraphs **2** and **5**, we showed how one can pass from the classical relativistic mechanics of a charged material point to the equations of wave mechanics thanks to a variational principle. One can obtain the fundamental equations of wave mechanics in space-time by the following symbolic process:

The Hamiltonian function of the relativistic mechanics of the point (m_0, e) was defined by [**I**, (83)] in Chapter **I**, namely:

$$m_0^2 c^2 H \equiv \frac{1}{2} g^{\alpha\beta}\left(m_0 c\, p_\alpha - \frac{e}{c}\Phi_\alpha\right)\left(m_0 c\, p_\beta - \frac{e}{c}\Phi_\beta\right). \tag{43}$$

In addition, one has the condition [**I**, (84)]:

$$H = \tfrac{1}{2} u^\alpha u_\alpha = \tfrac{1}{2}. \tag{44}$$

Introduce *operators* into (43) by replacing:

$$m_0\, c\, p_\alpha \qquad \text{with} \qquad \frac{h}{2\pi i}(\;)_{,\,\alpha}\,, \tag{45}$$

in which the symbol $(\;)_{,\,\alpha}$ denotes the covariant derivative of the function that is placed inside of the parentheses with respect to x^α. H will then be replaced with the operator:

$$H_{\text{op.}} \equiv \frac{1}{m_0^2 c^2}\frac{1}{2} g^{\alpha\beta}\left[\frac{h}{2\pi i}(\;)_{,\alpha} - \frac{e}{c}\Phi_\alpha\right]\left[\frac{h}{2\pi i}(\;)_{,\beta} - \frac{e}{c}\Phi_\beta\right], \tag{46}$$

and the relation (44) will be replaced by the symbolic equation:

$$H_{\text{op.}} = \tfrac{1}{2}. \tag{47}$$

Equation (27) can then be written:

$$H_{\text{op.}}\psi = \tfrac{1}{2}\psi\,. \tag{48}$$

Indeed, one will have:

$$m_0^2 c^2 H_{\text{op.}}\psi = \frac{1}{2}\left(\frac{h}{2\pi i}\right)^2 g^{\alpha\beta}\psi_{,\alpha\beta} - \frac{he}{2\pi i c}\Phi^\alpha\psi_{,\alpha} - \frac{1}{2}\frac{he}{2\pi i c}\Phi^\alpha_{,\alpha}\psi + \frac{1}{2}\frac{e^2}{c^2}\Phi^\alpha\Phi_\alpha\psi$$

if one takes into account that:

$$g^{\alpha\beta}{}_{,\,\beta} = 0. \tag{49}$$

The equation that is conjugate to (27) is obtained in an analogous manner by replacing p_α with the operator $-\dfrac{h}{2\pi i}()_{,\,\alpha}$ in (47). That equation will also be obtained by starting from the Hamiltonian:

$$H \equiv \frac{1}{2m_0^2 c^2} g^{\alpha\beta}\left(p_\alpha + \frac{e}{c}\Phi_\alpha\right)\left(p_\beta + \frac{e}{c}\Phi_\beta\right),$$

which relates to a particle of mass m_0 and charge – e, and replacing $m_0\, c\, p_\alpha$ with $\dfrac{h}{2\pi i}()_{,\alpha}$, as in the case of a particle with charge + e.

Remark. –

a) One can invert the order of factors in (46) by virtue of (49). That inversion will not be permitted when one does not utilize the invariant method and covariant derivatives.

b) We place ourselves in a Minkowski field and take a system of tri-rectangular coordinates. We will then have:

$$(ds)^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2\,(dt)^2. \tag{50}$$

Due to the physical significance of $m_0\, c\, p_\alpha$, (45) will express the idea that the components ($-\, m_0\, c\, p_j$) of the (spatial) impulse correspond to operators ([1]):

$$-\frac{h}{2\pi i}\frac{\partial}{\partial x^j}, \tag{51}$$

and that the energy $m_0\, c\, p_4$ corresponds to the operator:

$$\frac{h}{2\pi i}\frac{\partial}{\partial t}. \tag{52}$$

However, the (spatial) velocity will correspond to the operator:

$$c^j = \frac{u^i}{u^0} = \frac{m_0 c\, p^j - \frac{e}{c}\Phi^j}{m_0 c\, p^4 - \frac{e}{c}\Phi^4}. \tag{53}$$

([1]) Very often, one can take those operators to have the other sign.

13. Operators and mean values. – At the beginning in wave mechanics, one finds out that the new mechanics is characterized by the fact that it makes an operator correspond to each physical quantity in the classical mechanics of a material point. Here, we shall confine ourselves to establishing a link between those operators and the correspondence that was considered in paragraph ([**8**], pp. 37).

That link is remarkably simple for the current. Indeed, by virtue of (45) and [**I**, (82)] the velocity $c\,u_\alpha$ will correspond to the operator:

$$c\,\overline{u}_\alpha = \frac{1}{m_0}\frac{h}{2\pi i}(\ )_{,\alpha} - \frac{e}{c}\Phi_\alpha\,. \tag{54}$$

The components of the electric current will be associated with the operators $ec\,\overline{u}_\alpha$.

On the other hand, if one uses the operator above then the current (33′) can be written:

$$\mathcal{C}^\alpha = \tfrac{1}{2}\psi^* ec\,\overline{u}^\alpha\psi + \text{conj.} \tag{55}$$

Conforming to the usual terminology of wave mechanics, the four components $\mathcal{C}^\alpha$ of the current that are defined by (33′) we then be the mean-value densities of the operators $ec\,\overline{u}^\alpha$.

For the energy-impulse tensor, Einstein's theory leads to the operator:

$$(T_{\alpha\beta})_{\text{op}} = m_0\,c^2\,\overline{u}_\alpha\,\overline{u}_\beta\,, \tag{56}$$

and thus, the mean-value densities:

$$\psi^* m_0\,c^2\,\overline{u}_\alpha\,\overline{u}_\beta\,\psi\,. \tag{57}$$

These densities are not identical to the components of the operator (34′). However, the calculation shows that by virtue of the fundamental equations, those two tensors (57) and (34′) differ only by terms of the form:

$$\mathcal{X}^{\alpha\beta\gamma}{}_{,\gamma}\,.$$

In the non-relativistic theory, formulas (41) show that the components v^j of the velocity (in rectangular Cartesian coordinates) correspond to the operators:

$$\overline{v}^j = -\frac{1}{m_0}\frac{h}{2\pi i}\frac{\partial}{\partial x^j} - \frac{e}{c}\Phi^j\,.$$

The difficulty (53) is avoided here, since one will have:

$$\frac{1}{m_0}\left(\frac{h}{2\pi i}\frac{\partial}{\partial t}-\frac{e}{c}\Phi_4\right) \sim c^2,$$

up to terms that are very small of first order.

CHAPTER III

WAVE MECHANICS OF THE ELECTRON WITH SPIN. MINKOWSKI FIELD.

1. Notations. – The position of a point in space will be determined at any instant by its right-handed rectangular Cartesian coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$. It is convenient to take $x^4 = ct$, where t is time. The square of the elementary interval will then be:

$$(ds)^2 = \sum_\alpha \varepsilon_\alpha (dx^\alpha)^2 , \qquad \alpha = 1, \ldots, 4, \tag{1}$$

in which:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1, \qquad \varepsilon_4 = +1. \tag{2}$$

The covariant components Φ_α and the contravariant ones Φ^α of the electromagnetic potential are coupled by the relations:

$$\Phi_\alpha = \varepsilon_\alpha \Phi^\alpha.$$

Recall that:

$$H_{\alpha\beta} = \frac{\partial \Phi_\alpha}{\partial x^\beta} - \frac{\partial \Phi_\beta}{\partial x^\alpha} . \tag{3}$$

With vectorial notation in space:

$$\Phi^i = \mathcal{V}_{x^i} , \qquad \Phi^4 = V, \tag{4}$$

in which $\mathcal{V}_{x^i}$ are the components of the vector potential $\boldsymbol{\mathcal{V}}$, and V is the scalar potential. Finally:

$$H_{j4} = H_{x^j} , \qquad H_{kl} = \mathcal{H}_{x^j} \qquad (j, k, l = \text{even permutation of } 1, 2, 3),$$

in which H_{x^j} and $\mathcal{H}_{x^j}$ represent the components of the electric and magnetic field, respectively. For ease of notation, we write:

$$p_\alpha \equiv \frac{h}{2\pi i} \frac{\partial}{\partial x^\alpha} . \tag{5}$$

The symbol p_α will no longer denote the classical impulse, as in the preceding chapter, but the *impulse operator* ([1]). We also write:

$$P_\alpha \equiv p_\alpha - \frac{e}{c}\, \Phi_\alpha . \tag{6}$$

([1]) In many works, this operator is taken with the minus sign.

Recall that e denotes the charge of the electron, taken with its sign; thus, $e = -|e|$.

2. Introduction. – Equations [**II**, (27)] are relativistic; however, they do not take the spin of the electron into account. One knows how Dirac was led to replace the second-order equation [**II**, (27)] by the system of four first-order equations in the four functions $\psi_1(x^1, \ldots, x^4), \ldots, \psi_4(x^1, \ldots, x^4)$:

$$\left(P^4 + \sum_{j=1}^{3} \alpha_j P^j - \alpha_4 m_0 c\right) \psi_\beta = 0, \qquad \beta = 1, \ldots, 4. \tag{7}$$

The α_j, α_4 are the Dirac matrices ([1]):

$$\alpha_1 = \begin{pmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} . & . & . & i \\ . & . & -i & . \\ . & i & . & . \\ -i & . & . & . \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} . & . & 1 & . \\ . & . & . & -1 \\ 1 & . & . & . \\ . & -1 & . & . \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} -1 & . & . & . \\ . & -1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}. \tag{8}$$

In order to make the proper magnetic moment of the electron appear explicitly, Dirac left-multiplied (7) symbolically by the operator:

$$P^i - \sum_j \alpha_j P^j + \alpha_4 m_0 c. \tag{9}$$

He then obtained the second-order equations:

$$[\sum_\alpha \varepsilon_\alpha (P^\alpha)^2 - m_0^2 c^2] + 2 m_0 \alpha_4 \sum_{\alpha\beta} m^{\alpha\beta} H_{\alpha\beta}, \qquad \alpha, \beta = 1, \ldots, 4, \tag{10}$$

in which the $m^{\alpha\beta}$ are the Hermitian operators:

$$m^{j4} = \frac{1}{2}\frac{h}{2\pi}\frac{e}{m_0 c} i\, \alpha_j\, \alpha_4, \qquad m^{kl} = +\frac{1}{2}\frac{h}{2\pi}\frac{e}{m_0 c} i\, \alpha_k\, \alpha_l\, \alpha_4, \qquad j, k, l = 1, 2, 3. \tag{11}$$

The symbol $\sum_{\alpha\beta}$ indicates a summation over the combinations of the numbers 1, …, 4 without repetitions. The term in brackets in (10) is the usual Schrödinger operator, but a new operator has been added to it, namely, $(2m_0\, \alpha_4\, m^{\alpha\beta} H_{\alpha\beta})$.

We point out another process that will lead to a system that is equivalent to (11) when we start from (7). That process does not utilize the new operator (9). We infer from (7) that:

([1]) See [**8**]. Very often, one can take matrices that are expressed with the aid of (8) by $\alpha_1, -\alpha_2, \alpha_3, -\alpha_4$, respectively, instead of these α.

$$\psi_\beta = \frac{\alpha_4}{m_0 c}\left(\sum_j \alpha_j P^j + P^4\right)\psi_\beta\ .$$

Upon substituting this value of ψ_β in (7), one will get:

$$\left(\sum_{j=1}^{3}\alpha_j P^j + P^4\right)\frac{\alpha_4}{m_0 c}\left(\sum_{k=1}^{3}\alpha_k P^k + P^4\right)\psi_\beta = \alpha_4\, m_0\, c\ \psi_\beta\,,$$

or, upon performing the calculations:

$$\left\{\frac{\alpha_4}{2m_0 c}\left[\sum_\alpha \varepsilon_\alpha (P^\alpha)^2 - m_0^2 c^2\right] + \sum_{\alpha\beta} m^{\alpha\beta} H_{\alpha\beta}\right\} = 0. \tag{12}$$

These equations are equivalent to (10). One easily shows ([**8**], pp. 141) that in the non-relativistic approximation the *Hermitian* term:

$$-\sum_{\alpha\beta} m^{\alpha\beta} H_{\alpha\beta} \tag{13}$$

can be considered to correspond to a supplementary potential energy that is due to the action of the external electromagnetic field on the electromagnetic moment of the electron, which is represented by the tensor-operator $m^{\alpha\beta}$.

Remark. – We introduce the notations:

$$u^1 = -\alpha_1, \quad u^2 = -\alpha_2, \quad u^3 = -\alpha_3, \quad u^4 = 1. \tag{14}$$

Here again, the u^α will denote operators from now on. Later on, we shall see that they correspond to the space-time velocity (divided by c). The system (7) will then be written:

$$(P_\alpha u^\alpha)\ \psi = \alpha_4\, m_0\, c\ \psi, \tag{15}$$

if the summation is implied and one writes ψ for ψ_1, ψ_2, ψ_3, ψ_4 .

On the other hand, with the present notations, the Schrödinger equation will be written:

$$\left(P_\alpha \frac{P_\alpha}{m_0\, c}\right)\psi = m_0\, c\ \psi. \tag{16}$$

One will see the difference between the symbolic process that led to the Schrödinger equation and the one that yields the Dirac system upon comparing (15) to (16). Classically (hence, in ordinary numbers), one will have $P_\alpha u^\alpha = P_\alpha P^\alpha / m_0 c$. However, in Dirac's theory, one neatly distinguishes between the *dynamical* operator P_α and the *kinematical* quantity u^α. The operators P_α are defined by (6), while the operators u^α are not equal to $P^\alpha / m_0\, c$, but rather, they are essentially different operators, namely, (14). Here, we note that, conforming to the classical definition of the velocity in space, one will have:

$$v^i = c\frac{u^i}{u_4} = \frac{cu^i}{1} = c\, u^i \tag{17}$$

for the operators that correspond to that velocity; the three components of the spatial velocity operator will then be equal to the three components cu^1, cu^2, cu^3 of the operator (14). In addition, (15) will show that the quantity that takes the form of the rest mass of the electron corresponds to the operator $\alpha_4\, m_0$. Once more, energy corresponds to the operator $\frac{h}{2\pi i}\frac{\partial}{\partial t}$. However, equation (7), namely:

$$H\psi = \frac{h}{2\pi i}\frac{\partial\psi}{\partial t}, \tag{15′}$$

in which:

$$H = -\sum_j c\,\alpha_j P^j + e\,\Phi^4 + \alpha_4\, m_0\, c^2, \tag{15″}$$

expresses the equivalence of the operator H with $\frac{h}{2\pi i}\frac{\partial}{\partial t}$ for any ψ that represents the state of an electron. The operator H is referred to by the name of the "Hamiltonian operator."

3. Second-order Dirac equation. – Conforming to (12), we will use that name to refer to the equation:

$$\mathcal{O}\,\Psi = 0, \tag{18}$$

with

$$\mathcal{O} \equiv \frac{\alpha_4}{m_0}[\sum_\alpha \varepsilon_\alpha (P^\alpha)^2 - m_0^2 c^2] + \sum_\alpha\sum_\beta m^{\alpha\beta} H_{\alpha\beta}\,. \tag{19}$$

Any solution of the system (7) will satisfy (18). The converse is not true: There exist solutions to (18) that do not satisfy (7). We shall show that the function Ψ can represent either the state of an electron or the state of a positron of positive energy without appealing to the "hole hypothesis."

Theorem. –

a) The system of four second-order equations (18) is equivalent to the system of eight first-order equations:

$$\left.\begin{aligned}(\sum_j \alpha_j P^j + P^4)\Psi &= \alpha_4 m_0\, c\chi, \quad (a)\\ (\sum_j \alpha_j P^j + P^4)\chi &= \alpha_4 m_0\, c\Psi. \quad (b)\end{aligned}\right\} \tag{20}$$

In order to see that, it is sufficient to replace the χ in (20.b) with its value in (20.a); one will then obtain the system (18).

b) Set:

$$\psi = \tfrac{1}{2}(\Psi + \chi), \qquad \varphi^* = \tfrac{1}{2}(\Psi - \chi). \tag{21}$$

The system (20) is equivalent to the following one:

$$(P^4 + \sum_j \alpha_j P^j - \alpha_4\, m_0\, c)\ \psi = 0, \tag{22}$$

$$(P^4 + \sum_j \alpha_j P^j + \alpha_4\, m_0\, c)\ \varphi^* = 0. \tag{23}$$

Any solution of (18) will then have the form:

$$\Psi = \psi - \varphi^*. \tag{24}$$

c) Equations (22) are the first-order Dirac equations. Any solution ψ will then represent the state of the electron.

On the other hand, let φ_α be the conjugate functions to $\varphi_\alpha^\times$. By virtue of (23), those φ_α will satisfy the system:

$$\left[\left(p^4 + \frac{e}{c}\Phi^4\right) + \alpha_1^*\left(p^1 + \frac{e}{c}\Phi^1\right) + \alpha_2^*\left(p^2 + \frac{e}{c}\Phi^2\right) + \alpha_3^*\left(p^3 + \frac{e}{c}\Phi^3\right) - \alpha_4^* m_0 c\right] \varphi = 0. \tag{25}$$

One passes from (22) to (25) by changing the sign of the charge and taking the conjugate matrices $\alpha_i^\times$ in (25). The system (25) is a Dirac system for the positron, because the change of matrices $\alpha_i \to \alpha_i^\times$ is not essential. Any solution φ will then represent the state of a positron.

Remark. – If the ψ_β are solutions to (22) then the functions:

$$\varphi_\beta^\times = i\ \alpha_1\ \alpha_2\ \alpha_3\ \psi_\beta \tag{26}$$

will be solutions to (23), as one verifies immediately. Conversely, if the $\varphi_\beta^\times$ satisfy (23) then the functions:

$$\psi_\beta = i\, \alpha_1\, \alpha_2\, \alpha_3\, \varphi_\beta^\times \tag{26′}$$

will satisfy (22). In other words, any solution to (22) will correspond to a solution of (23) by means of (26), and conversely.

4. Variational principles. – Introduce the real function:

$$\mathcal{L} \equiv -\frac{c}{2}(\Psi^* \,\mathcal{O}\, \Psi + \text{conj.}). \tag{27}$$

In (27), we have omitted the indices α in the functions Ψ, $\Psi^\times$, and the summations over those indices, as we shall often do. Set:

$$\overset{\circ}{\mathcal{O}} \equiv \frac{\alpha_4}{m_0}[\sum_\alpha \varepsilon_\alpha (P^\alpha)^2 - m_0^2\, c^2] \tag{28}$$

One will verify that:

$$\Psi^\times \overset{\circ}{\mathcal{O}}\, \Psi \equiv \sum_\alpha \left[(P_\alpha^\times \Psi^*) \frac{\alpha_4}{m_0} (P^\alpha \Psi) - m_0^2\, c^2 \Psi^* \frac{\alpha_4}{m_0} \Psi + \sum_\alpha p_\alpha \left(\Psi^* \frac{\alpha_4}{m_0} P^\alpha \Psi \right) \right]; \tag{29}$$

hence:

$$-\frac{c}{2}\Psi^* \overset{\circ}{\mathcal{O}}\, \Psi = -\frac{c}{2}\Psi\, \overset{\circ}{\mathcal{O}}{}^*\, \Psi^* + \sum_\alpha \frac{\partial \mathcal{K}^\alpha}{\partial x^\alpha},$$

in which:

$$\mathcal{K}^\alpha = \left[-\frac{c}{2}\frac{h}{2\pi i}\left(\Psi^* \frac{\alpha_i}{m_0} P^\alpha \Psi \right) \right] + \text{conj.}$$

Hence:

$$\left.\begin{aligned} \mathcal{L} &= -c\Psi^* \mathcal{O} \Psi - \sum_\alpha \frac{\partial \mathcal{K}^\alpha}{\partial x^\alpha}, \quad (a) \\ &= -c\Psi \mathcal{O} \Psi^* + \sum_\alpha \frac{\partial \mathcal{K}^\alpha}{\partial x^\alpha}, \quad (b) \end{aligned}\right\} \tag{30}$$

and also:

$$\mathcal{L} = -\, c\left[(P_\alpha^* \Psi^*) \frac{\alpha_4}{m_0} (P^\alpha \Psi) - m_0^2 c^2 \Psi^* \frac{\alpha_4}{m_0} \Psi \right] - c\Psi^* m^{\alpha\beta} H_{\alpha\beta} \Psi - \frac{c}{2} p_\alpha p^\alpha \left(\Psi^* \frac{\alpha_4}{m_0} \Psi \right), \tag{31}$$

in which the summation is implied.

Thanks to those formulas, and making use of the identity:

$$\frac{\delta}{\delta\Psi_\alpha}\left(\frac{\partial\mathcal{K}^\beta}{\partial x^\beta}\right) \equiv 0,$$

we will get:

$$\frac{\delta\mathcal{L}}{\delta\Psi_\alpha^*} = -\,c\,\mathcal{O}\,\Psi_\alpha \qquad \text{and} \qquad \frac{\delta\mathcal{L}}{\delta\Psi_\alpha} = -\,c\,\mathcal{O}^*\,\Psi_\alpha^*. \tag{32}$$

It will then result that equations (18) are the Lagrange equations of the variational principle:

$$\delta\iiint\int \mathcal{L}\,dx^1\,dx^2\,dx^3\,dx^4 = 0. \tag{33}$$

Remark. – The first-order Dirac equation is the Lagrange equation for the variational principle:

$$\delta\iiint\int \tfrac{1}{2}\,[\psi^*\,(P_\alpha\,u^\alpha - \alpha_4\,m_0\,c)\,\psi + \text{conj.}]\,dx^1\,dx^2\,dx^3\,dx^4 = 0; \tag{34}$$

similarly, equation (25) is the Lagrange equation for:

$$\delta\iiint\int \tfrac{1}{2}\,[\varphi^*\,(-P_\alpha^*\,(u^\alpha)^* - \alpha_4\,m_0\,c)\,\varphi + \text{conj.}]\,dx^1\,dx^2\,dx^3\,dx^4 = 0. \tag{34′}$$

5. The electric current-density vector. – Since equations (18) are deduced from a variational principle, we can once more define an electric current-density, as in the preceding chapter [**II**, (34)], by the variational derivatives:

$$\mathcal{C}^\alpha \equiv -\frac{1}{2}\frac{\delta\mathcal{L}}{\delta\Phi^\alpha}. \tag{35}$$

That current is the sum of two terms:

$$\overset{\circ}{\mathcal{C}}{}^\alpha \equiv -\frac{1}{2}\frac{\delta}{\delta\Phi_\alpha}\left[-\frac{c}{2}(\Psi^*\,\overset{\circ}{\mathcal{O}}\,\Psi + \text{conj.})\right] \tag{36}$$

and

$$\overset{s}{\mathcal{C}}{}^\alpha \equiv -\frac{1}{2}\frac{\delta}{\delta\Phi_\alpha}\left[-c\,\Psi^*m^{\beta\gamma}\,\Psi\,H_{\beta\gamma}\right]. \tag{37}$$

Explicitly:

$$\overset{\circ}{\mathcal{C}}{}^\alpha = \frac{e}{2}\left(\Psi^*\,\frac{\alpha_4}{m_0}\,P^\alpha\Psi + \text{conj.}\right), \tag{36′}$$

$$\overset{s}{\mathcal{C}}{}^\alpha \equiv \frac{\partial}{\partial x^\beta}\,c\,\Psi^*m^{\alpha\beta}\,\Psi, \tag{37′}$$

and

$$\mathcal{C}^\alpha = \frac{e}{2}\left(\Psi^* \frac{\alpha_4}{m_0} P^\alpha \Psi + \text{conj.}\right) + \frac{\partial}{\partial x^\beta} c\,\Psi^* m^{\alpha\beta}\,\Psi\,. \tag{38}$$

(36′) is the analogue of the current in Schrödinger's theory [**II**, (33′)]. The term (37′) is new. Now, in Maxwell's theory, a polarized medium whose polarization tensor is $\mathcal{M}^{\alpha\beta}$ will give rise to a current $\partial\mathcal{M}^{\alpha\beta} / \partial x^\beta$. The tensor:

$$\mathcal{M}^{\alpha\beta} \equiv c\,\Psi^*\, m^{\alpha\beta}\,\Psi,$$

will then be, by definition, the "mean-value density of the electromagnetic polarization" tensor. That interpretation conforms to the results of § **2**. That expression for the current will transform with the aid of (20) and (21); one will have:

$$\left.\begin{aligned} &\mathcal{C}^j = -ce(\psi^*\alpha_j\psi - \varphi\,\alpha_j\varphi^*), \quad j = 1, 2, 3, \quad (a) \\ &\mathcal{C}^4 = ce(\psi^*\psi - \varphi\varphi^*). \qquad\qquad\qquad\qquad (b) \end{aligned}\right\} \tag{39}$$

In order to see that (39) is indeed equal to (38), the simplest way is to remark that, thanks to (21), (39) can be written:

$$\left.\begin{aligned} &\mathcal{C}^j = -\frac{ce}{2}(\Psi^*\alpha_j\chi + \chi^*\,\alpha_j\Psi), \\ &\mathcal{C}^4 = \frac{ce}{2}(\Psi^*\chi + \chi^*\Psi). \end{aligned}\right\} \tag{40}$$

In order to obtain (38), it will suffice to replace χ and $\chi^\times$ in (40) with their values that one infers from (39).

The electric current-density is then composed of two parts. One of them:

$${}^-\mathcal{C}^{\,j} = -\,c\,e\,\psi^*\alpha_j\,\psi, \qquad {}^-\mathcal{C}^4 = c\,e\,\psi^*\psi \tag{41}$$

is a current-density of negative electrons. The other one:

$${}^+\mathcal{C}^{\,j} = c\,e\,\varphi^*\alpha_j\,\varphi, \qquad {}^+\mathcal{C}^4 = -\,c\,e\,\varphi^*\varphi \tag{41′}$$

is a current-density of positive electrons. With the notations of (14), those formulas will become:

$${}^-\mathcal{C}^\alpha = e\,\psi^* c\,u^\alpha\,\psi, \qquad {}^+\mathcal{C}^\alpha = -\,e\,\psi^*(c\,u^\alpha)^*\,\psi. \tag{42}$$

We also point out that the tensor-operator $m^{\alpha\beta}$ can be written:

$$m^{\alpha\beta} = \frac{1}{2}\frac{h}{2\pi}\frac{e}{c}\frac{\alpha_4}{m_0} i\, u^\alpha u^\beta.$$

Remark. – Thanks to the property (26), one can also define two vectors:

$$\mathcal{A}^1 = - c\, e\, \psi^*\, i\, \alpha_2\, \alpha_3\, \varphi^*, \ldots, \quad \mathcal{A}^4 = c\, e\, \psi^*\, i\, \alpha_1\, \alpha_2\, \alpha_3\, \varphi^*, \tag{43}$$

and

$$\mathcal{D}^1 = - c\, e\, \varphi\, i\, \alpha_2\, \alpha_3\, \psi, \ldots, \quad \mathcal{D}^4 = c\, e\, \varphi\, i\, \alpha_1\, \alpha_2\, \alpha_3\, \psi. \tag{43′}$$

Theorem. – One has:

$$\frac{\partial \overset{s}{\mathcal{C}}{}^\alpha}{\partial x^\alpha} \equiv 0, \tag{44}$$

identically, since $\mathcal{M}^{\alpha\beta} = - \mathcal{M}^{\beta\alpha}$. On the other hand, as in Schrödinger's theory (**II**, § **7**), one shows that:

$$\frac{\partial \overset{\mathrm{o}}{\mathcal{C}}{}^\alpha}{\partial x^\alpha} = 0, \tag{45}$$

by virtue of equations (18). Finally, one will likewise have:

$$\frac{\partial\, {}^-\mathcal{C}^\alpha}{\partial x^\alpha} = 0, \tag{46}$$

by virtue of equations (22), and similarly:

$$\frac{\partial\, {}^+\mathcal{C}^\alpha}{\partial x^\alpha} = 0, \tag{47}$$

by virtue of (23).

There is then conservation of the negative electron current and conservation of the positive electron current.

6. – The energy-impulse tensor. – As in Schrödinger's theory, the tensor $\mathcal{T}^{\alpha\beta}$ will be such that:

$$\frac{\partial \mathcal{T}^{\alpha\beta}}{\partial x^\beta} = + \mathcal{C}^\beta H_{\beta\alpha}. \tag{48}$$

The first three terms in the right-hand side of (48) represent the Lorentz force, up to a factor of c ; recall that:

$$\mathcal{C}^j = \mathcal{C}_{xj}, \qquad \mathcal{C}^4 = \rho\, c.$$

The present theory satisfies only the principle of special relativity. It will therefore not be possible to define the energy-impulse tensor here by a variational derivative, as in [**II**, (33)]. In the case of electric current, we have seen that one of the terms in the current of Dirac's theory corresponds to the Schrödinger current. We can likewise define a term in the energy-impulse tensor here that corresponds to the Schrödinger energy-impulse tensor. It is easy to introduce directly a tensor $\mathcal{T}^{\alpha\beta}$ that will satisfy the condition (48).

The form (40) of the current points to a manner of proceeding for us. Consider the tensor-operator:

$$t^{\alpha\beta} \equiv P^{\alpha} u^{\beta} \tag{49}$$

and form the asymmetric tensor:

$$\mathcal{T}^{\alpha\beta} \equiv \tfrac{1}{2}(\Psi^* P^{\alpha} u^{\beta} \chi + \chi^* P^{\alpha} u^{\beta} \Psi). \tag{50}$$

Theorem. – By virtue of equations (20), one has:

$$\frac{\partial}{\partial x^{\beta}} \frac{1}{2} [\Psi^* P^{\alpha} u^{\beta} \chi + \chi^* P^{\alpha} u^{\beta} \Psi] - \mathcal{C}^{\beta} H_{\beta}. \tag{51}$$

Proof. – Thanks to (20):

$$\begin{aligned}
\chi^* u_{\beta} p^{\beta} \Psi_{,\,\alpha} &= \chi^* \alpha_4 m_0 c^2 \chi_{,\,\alpha} + e\, \chi^* u_{\beta} \Phi^{\beta} \Psi_{,\,\alpha} + e\, \chi^* u^{\beta} \Phi_{\beta,\,\alpha} \Psi, \\
-\chi_{,\,\alpha} u^*_{\beta} (p^{\beta})^* \Psi^* &= -\chi_{,\,\alpha} \alpha_4 m_0 c^2 \chi^* - e\, \chi_{,\,\alpha} u^*_{\beta} \Phi^{\beta} \Psi, \\
-\Psi_{,\,\alpha} (u_{\beta} p^{\beta})^* \Psi^* &= -\Psi_{,\,\alpha} \alpha_4 m_0 c^2 \Psi^* - e\, \Psi_{,\,\alpha} u^*_{\beta} \Phi^{\beta} \chi^*, \\
\Psi^* u_{\beta} p^{\beta} \chi_{,\,\alpha} &= \Psi^* \alpha_4 m_0 c^2 \Psi_{,\,\alpha} + e\, \Psi^* u_{\beta} \Phi^{\beta} \chi_{,\,\alpha} + e\, \Psi^* u^{\beta} \Phi_{\beta,\,\alpha} \chi.
\end{aligned}$$

Add corresponding sides of these four relations and get:

$$\frac{\partial}{\partial x^{\beta}} (\chi^* u^{\beta} p_{\alpha} \Psi + \Psi^* u^{\beta} p_{\alpha} \chi) = e\, (\chi^* u^{\beta} \Phi_{\beta,\,\alpha} \Psi + \Psi^* u^{\beta} \Phi_{\beta,\,\alpha} \chi).$$

The proof is completed by adding the term:

$$-e\, (\chi^* u^{\beta} \Phi_{\alpha,\,\beta} \Psi + \Psi^* u^{\beta} \Phi_{\alpha,\,\beta} \chi)$$

to both sides of the latter relation.

Remarks. –

a) The tensor $\mathcal{T}^{\alpha\beta}$ is defined by the conditions (48) only up to terms of the form $\mathcal{X}^{\alpha\gamma\beta}$ such that:

$$\frac{\partial \mathcal{X}^{\alpha\gamma\beta}}{\partial x^{\gamma}} = 0.$$

There then exist other tensors than (50) that satisfy (48).

In particular, the *real* tensor:

$$\underset{(r)}{\mathcal{T}}^{\alpha\beta} = \tfrac{1}{2}(\mathcal{T}^{\alpha\beta} + \text{conj.}) \tag{51}$$

obviously satisfies the condition (48), since the right-hand side of (48) is real; one will then have:

$$\frac{\partial \underset{(r)}{\mathcal{T}}^{\alpha\beta}}{\partial x^{\beta}} = \mathcal{C}^{\beta} H_{\beta\alpha}. \tag{52}$$

It is interesting to decompose the asymmetric tensor (51) into its symmetric part:

$$\overset{(s)}{\mathcal{T}}{}^{\alpha\beta} = \tfrac{1}{2}(\underset{(r)}{\mathcal{T}}^{\alpha\beta} + \underset{(r)}{\mathcal{T}}^{\beta\alpha}) \tag{53}$$

and its antisymmetric part:

$$\overset{(a)}{\mathcal{T}}{}^{\alpha\beta} = \tfrac{1}{2}(\underset{(r)}{\mathcal{T}}^{\alpha\beta} - \underset{(r)}{\mathcal{T}}^{\beta\alpha})\,. \tag{54}$$

When one lets Σ'_{γ} denote a summation that is performed over indices γ that are different from α and β, one will easily find that:

$$\begin{aligned}(\underset{(r)}{\mathcal{T}}^{\alpha\beta} - \underset{(r)}{\mathcal{T}}^{\beta\alpha}) &= \tfrac{1}{2}(\Psi^{*}\, u^{\alpha} u^{\beta}\, \alpha_4\, m_0\, c^2\, \Psi + \chi^{*}\, u^{\alpha} u^{\beta}\, \alpha_4\, m_0\, c^2\, \chi) \\ &\quad - \Sigma'_{\gamma}\, \tfrac{1}{2}(\Psi^{*}\, u^{\alpha} u^{\beta} u^{\gamma} P_{\gamma}\chi + \chi^{*}\, u^{\alpha} u^{\beta} u^{\gamma} P_{\gamma}\Psi)\,.\end{aligned}$$

Hence, upon taking into account the Hermiticity of the matrices u^{α} and α_4 :

$$2\,\overset{(a)}{\mathcal{T}}{}^{\alpha\beta} = \Sigma'_{\gamma}\, \frac{1}{2}\frac{h}{2\pi}\frac{\partial}{\partial x^{\gamma}}(\Psi^{*}\, i\, u^{\alpha} u^{\beta} u^{\gamma}\chi + \chi^{*}\, i\, u^{\alpha} u^{\beta} u^{\gamma}\Psi) \qquad (\alpha \neq \beta \neq \gamma). \tag{55}$$

The Hermitian operator:

$$s_{\alpha\beta\gamma} = \frac{h}{4\pi} i\, u^{\alpha} u^{\beta} u^{\gamma} \tag{56}$$

appears in the right-hand side, along with the completely-antisymmetric tensor:

$$S_{\alpha\beta\gamma} = \tfrac{1}{2}(\Psi^{*}\, s_{\alpha\beta\gamma}\,\chi + \chi^{*}\, s_{\alpha\beta\gamma}\,\Psi); \tag{56′}$$

explicitly:

$$s_{123} = \frac{h}{4\pi} i\alpha_1\, \alpha_2\, \alpha_3\,, \quad s_{124} = \frac{h}{4\pi} i\alpha_1\, \alpha_2, \quad s_{234} = \frac{h}{4\pi} i\alpha_2\, \alpha_3\,, \quad s_{314} = \frac{h}{4\pi} i\alpha_3\, \alpha_1\,.$$

One can define a vector whose covariant components are:

$$N_1 = S^{234}, \qquad N_2 = S^{314}, \qquad N_3 = S^{124}, \qquad N_4 = -\,S^{123} \tag{57}$$

with the aid of the four components $S^{\alpha\beta\gamma}$ of the completely-antisymmetric tensor (56′).

The relations (55) are then written:

$$2\overset{(a)}{\mathcal{T}}{}^{\alpha\beta} = \frac{\partial N_{\bar\alpha}}{\partial x^{\bar\beta}} - \frac{\partial N_{\bar\beta}}{\partial x^{\bar\alpha}}, \tag{58}$$

in which $\alpha\beta\overline{\alpha\beta}$ forms an even permutation of the numbers 1, 2, 3, 4. It will then result that:

$$\frac{\partial \overset{(a)}{\mathcal{T}}{}^{\alpha\beta}}{\partial x^\beta} = 0;$$

hence:

$$\frac{\partial \overset{(s)}{\mathcal{T}}{}^{\alpha\beta}}{\partial x^\beta} = \mathcal{C}^\beta H_{\beta\alpha}. \tag{59}$$

The symmetric tensor (53) also satisfies the condition (48) then.

Finally, like the current, the tensor $\mathcal{T}^{\alpha\beta}$ decomposes into two terms:

$$\mathcal{T}^{\alpha\beta} = \psi^* P^\alpha u^\beta \psi - \varphi P^\alpha u^\beta \varphi^*.$$

We then set:

$$^-\mathcal{T}_{\alpha\beta} \equiv \psi^* P_\alpha u_\beta \psi = \psi^* \left(\frac{h}{2\pi i} \frac{\partial}{\partial x^\alpha} - \frac{e}{c} \Phi_\alpha \right) u_\beta \psi, \tag{60}$$

$$^+\mathcal{T}_{\alpha\beta} \equiv -\,\varphi^* (P_\alpha u_\beta)^* \varphi = \varphi^* \left(\frac{h}{2\pi i} \frac{\partial}{\partial x^\alpha} + \frac{e}{c} \Phi_\alpha \right) u_\beta^* \varphi. \tag{61}$$

One has:

$$\frac{\partial\, ^-\mathcal{T}^{\alpha\beta}}{\partial x^\beta} = H^{\alpha\beta}\, ^-\mathcal{C}_\beta \qquad \text{and} \qquad \frac{\partial\, ^+\mathcal{T}^{\alpha\beta}}{\partial x^\beta} = H^{\beta\alpha}\, ^+\mathcal{C}_\beta. \tag{62}$$

The tensor:

$$^-\mathcal{T}^{\alpha\beta} + {}^+\mathcal{T}^{\alpha\beta} \tag{63}$$

will likewise satisfy the conditions (48); i.e., one will have:

$$\frac{\partial (^-\mathcal{T}^{\alpha\beta} + {}^+\mathcal{T}^{\alpha\beta})}{\partial x^\beta} = \mathcal{C}^\beta H_{\alpha\beta}. \tag{64}$$

Remarks. – The presentation in this chapter shows that the wave mechanics that is based upon the second-order Dirac equations constitutes only a juxtaposition of the theories of the electron and the positron. It will permit one to pass to states of negative kinetic energy as long as one confines oneself to only the study of non-interacting electrons and positrons. However, some important phenomena, such as materialization and dematerialization, cannot be explained by it. Indeed, by virtue of the continuity equations, one will have both conservation of the electron number and conservation of the positron number.

Moreover, one knows that in order to study phenomena such as the appearance and disappearance of electron-positron pairs, one will be led to quantize the wave functions ([1]). Thanks to second quantization, one can obtained the desired results, but only by introducing some interaction terms that include the densities (43) and (43′) into the expression for the for the energy artificially.

7. Action of a light wave upon an electron. – Let:

$$H \equiv \sum_{j=1}^{3} c\,\alpha_j \left(\frac{h}{2\pi i}\frac{\partial}{\partial x^j} + \frac{e}{c}\mathcal{V}_{x^j}\right) + e\,V - \alpha_4\, m_0\, c^2 \tag{65}$$

denote the Dirac Hamiltonian operator, in which $\mathcal{V}_{x^j}$ are the components of the magnetic potential, and V is the electric potential of the *non*-luminous Maxwell field (which we assume to be time-independent) in which the electron is placed. Let $f_\gamma^n(x, y, z)$ denote a complete system of wave functions such that:

$$H\, f_\gamma^n = E^n\, f_\gamma^n\,. \tag{66}$$

In order to simplify matters, we suppose that the proper values of (66) are discrete and non-degenerate; the general case will introduce only complications in the notations. The functions f_γ^n are normalized and orthogonal:

$$\iiint_{\mathcal{E}} (f^n)^* f^n\, dx\, dy\, dz = 1; \qquad \iiint_{\mathcal{E}} (f^n)^* f^{n'}\, dx\, dy\, dz = 0 \qquad \text{for } n' \neq n. \tag{67}$$

The integration extends *over the entire space of the variables x, y, z.* The function:

$$\psi_\gamma^n = f_\gamma^n\, e^{\frac{2\pi i}{h}E^n t}$$

represents the state of an electron with energy E^n. The state of an electron that is placed in the field $\mathcal{V}$, V in the absence of a light field will be defined in a general fashion by the sum or series:

$$\psi_\gamma = \sum_n a_n\, \psi_\gamma^n\,, \qquad n = 1, 2, \ldots, \tag{68}$$

([1]) See [**9**] and [**10**].

in which the a_n are constants. If one normalizes ψ then one will have:

$$\sum_n |a_n|^2 = 1. \tag{69}$$

One then knows that, in the convenient language of the calculus of probabilities, $|a_n|^2$ represents the probability for the electron to be found in the state of energy E^n. In the absence of perturbations, these $|a_n|^2$ will remain invariant in the course of time. A Maxwellian light wave will be defined by the electromagnetic potentials $\boldsymbol{\mathcal{V}}^{(l)}(x, y, z, t)$, $V^{(l)}(x, y, z, t)$. Physically, the light wave will have the effect of provoking "transitions" in the state of the electron that will be represented by (68) at a well-defined instant. Mathematically, those transitions will be expressed by the idea that the a_n vary in the course of time. The study of those transitions then consists in the study of the way that the a_n can vary in the course of time under the action of a light wave. In other words, the presence of the light field will transform (68) into:

$$\psi_\gamma = \sum_n a_n(t)\,\psi_\gamma^n\,, \tag{70}$$

and the problem will come down to that of finding the equations that the $a_n(t)$ must satisfy.

With the notations that were introduced above, the variational principle (34) will be written:

$$\delta\iiint \frac{1}{2}\left\{\psi^*\left[H + I - \frac{h}{2\pi i}\frac{\partial}{\partial t}\right]\psi + \text{conj.}\right\} dx\,dy\,dz\,dt, \tag{71}$$

in which I denotes the interaction factor between the electron and the light wave:

$$I \equiv e\left(\sum_{j=1}^{3}\alpha_j \mathcal{V}_{x_j}^{(l)} + V^{(l)}\right). \tag{72}$$

That variational principle provides the equations that the $a_n(t)$ must satisfy. Conforming to (67), the domain of integration that relates to the x, y, z is all of the space $\mathcal{E}$. The variational principle (71) must then be interpreted in the following manner:

One varies the ψ, $\psi^\times$ only by the intermediary of the $a_n(t)$, $a_n^\times(t)$. *The variational equation* (71) will then become:

$$\delta\int \bar{\mathcal{L}}\,dt = 0, \tag{73}$$

in which:

$$\bar{\mathcal{L}} \equiv \iiint_{\mathcal{E}} \mathcal{L}\,dx\,dy\,dz\,, \tag{74}$$

with

$$\mathcal{L} \equiv \tfrac{1}{2}\psi^* \left[H + I - \frac{h}{2\pi i}\frac{\partial}{\partial t} \right] + \text{conj.} \tag{75}$$

If one takes (66) and (67) into account then:

$$\bar{\mathcal{L}} \equiv \tfrac{1}{2}\left[\sum_n \sum_{n'} a_n^* \, I_{n,n'} \, a_{n'} - \sum_n a_n^* \frac{h}{2\pi i}\frac{da_n}{dt} \right] + \text{conj.} \tag{76}$$

in which one has set:

$$I_{n,\,n'} \equiv \iiint_{\mathcal{E}} (\psi_n)^* \, I \, \psi^{n'} \, dx \, dy \, dz \, . \tag{77}$$

The *Lagrange* equations for (73) are:

$$\frac{\partial \bar{\mathcal{L}}}{\partial a_n^*} - \frac{d}{dt}\frac{\partial \bar{\mathcal{L}}}{\partial \dot{a}_n^*} = 0 \tag{78}$$

and the conjugate equations:

$$\frac{\partial \bar{\mathcal{L}}}{\partial a_n} - \frac{d}{dt}\frac{\partial \bar{\mathcal{L}}}{\partial \dot{a}_n} = 0, \tag{79}$$

with

$$\dot{a}_n = \frac{da_n}{dt} \qquad \text{and} \qquad \dot{a}_n^* = \frac{da_n^*}{dt} \, .$$

Explicitly, (78) and (79) are written:

$$\frac{h}{2\pi i}\dot{a}_n = \sum_{n'} L_{n,n'} \, a_{n'} \, , \tag{78$'$}$$

$$-\frac{h}{2\pi i}\dot{a}_n^* = \sum_{n'} a_{n'}^* \, L_{n',n} \, . \tag{79$'$}$$

Those equations are well-known. Here, we have shown how one can obtain them by starting from the general variational principle (34).

As one knows, equations (78) are not sufficient. They do not take into account the reaction of the electron on the field in which it is embedded (see [**1**], § 6). Notably, they do not explain spontaneous emission. In order to likewise obtain that phenomenon, one must appeal to second quantization. Now, it seems that second quantization must appear only when one deals with ensembles of n corpuscles (n > 1), since the commutation relations that are introduced into that theory are based upon statistical considerations ([1]). The phenomenon of spontaneous emission will still be true in the absence of Maxwellian photons, so one must be able to find an explanation for that phenomenon without

([1]) See, for example, [**24**], pp. 229, *et seq.*

utilizing second quantization in a theory of elementary interaction phenomena between a photon and an electron.

In Chapter VII, we will see that the wave mechanics of the photon permits one to envision those problems in a satisfactory fashion.

Remarks. –

a) Equations (78′, 79′) can take the Hamiltonian form. Indeed, set:

$$d \equiv \frac{h}{2\pi i} a_n^*, \tag{80}$$

and introduce the Hamiltonian function:

$$\bar{\mathcal{K}} \equiv \sum_n \sum_{n'} a_n^* I_{n,n'} a_{n'} . \tag{81}$$

One easily verifies that equations (78′, 79′) are nothing but:

$$\left.\begin{aligned} \frac{da_n}{dt} &= \frac{\partial \bar{\mathcal{K}}}{\partial d_n}, \\ \frac{dd_n}{dt} &= -\frac{\partial \bar{\mathcal{K}}}{\partial d_n}. \end{aligned}\right\} \tag{82}$$

b) The presentation in this paragraph constituted an application of the theory of transitions that are caused by a perturbation. In a general manner, let H_0 be the Hamiltonian operator of an arbitrary system, and let *I* be the energy operator of the perturbation that perturbs the system under study; in the case of § **7**, H_0 will be given by (65), and *I* will be given by (72).

Let ψ_n be the complete system of proper functions such that:

$$H_0 \ \psi^n = E^n \ \psi^n. \tag{83}$$

The general solution of the equation:

$$H_0 \ \psi^n = \frac{h}{2\pi i} \frac{\partial}{\partial t} \tag{84}$$

will be:

$$\psi = \sum_n a_n \psi^n , \tag{85}$$

in which the a_n are constants. By hypothesis, the perturbation *I* will have the effect of stimulating *transitions* in the system (85). In other words, the a_n will become functions of time as a result of the perturbation *I*. The problem that is then posed in the study of transitions is not that of studying the functions ψ that define the state of system when one takes the perturbation into account, but only that of determining the functions $a_n(t)$. Those functions $a_n(t)$ must satisfy a variational principle that is analogous to the one that yields the Dirac equation. However, here there is just one independent variable – viz., time t – and the unknown functions that are the ψ_γ in Dirac's theory are the $a_n(t)$ here. By virtue of the study that was made above, it seems reasonable for us to consider the *variational equation:*

$$\delta \int \bar{\mathcal{L}}\, dt = 0 \tag{86}$$

in full generality, with:

$$\bar{\mathcal{L}} \equiv \frac{1}{2}\left[\sum_n \sum_{n'} a_n^* I_{n,n'} a_{n'} - \sum_n a_n^* \frac{h}{2\pi i}\frac{da_n}{dt}\right] + \text{conj.} \tag{87}$$

as the fundamental variational principle of the theory of transitions that are caused by a perturbation I. The $I_{n,n'}$ are the elements of the matrix of the operator I on the system of proper functions ψ^n. In order to be able to write that function $\bar{\mathcal{L}}$, it will suffice to know:

1. The perturbation operator I.

2. The proper functions ψ^n that are defined by (83) and relate to the Hamiltonian operator of the system when one neglects the perturbation.

We will utilize the principle (86) in Chapter VII.

c) The elements $I_{n,n'}$ depend upon time; one has:

$$I_{n,n'} = I_{n,n'}(0)\; e^{\frac{2\pi i}{h}(E^{n'} - E^n)t}, \tag{88}$$

in which:

$$I_{n,n'}(0) = \iiint_{\mathcal{E}} (f^n)^* I\, f^{n'} dx\, dy\, dz\,. \tag{89}$$

Upon setting:

$$A_n = a_n\, e^{\frac{2\pi i}{h}E^n t}, \tag{90}$$

equations (78′) can be written:

$$\sum_{n'} [I_{n,n'}(0) + \delta_{nn'} E^{n'}] A_{n'} = \frac{h}{2\pi i}\frac{dA_n}{dt}\,. \tag{91}$$

Finally, we let **A** denote the "vector" whose components are A_n in the space of proper functions f_n, while $\mathcal{I}$ and $\mathcal{H}_{\text{op}}$ will denote the operators whose corresponding matrices in the space of functions f_n will be $I_{n,n'}(0)$ and $\delta_{n,n'} E^{n'}$, respectively. Equations (91) will then take the form:

$$(\mathcal{H}_{\text{op}} + \mathcal{I})\,\mathbf{A} = \frac{h}{2\pi i}\frac{d\mathbf{A}}{dt}\,. \tag{92}$$

These equations (92) have the usual form for the equations of evolution in wave mechanics.

CHAPTER IV

THE PHOTONIC FIELDS OF TH. DE DONDER-J. M. WHITTAKER

1. Fundamental equations. – One remembers some research that was carried out some years ago ([1]) in order to establish the fundamental equations of wave mechanics in Maxwellian form. In 1928, J. M. Whittaker wrote the fundamental equations of the electron in the following form (the summation sign is implicit):

$$\left.\begin{array}{ll}
\left[\dfrac{\partial\sqrt{-g}\,U^{\alpha\beta}}{\partial x^\beta}\right]=\sqrt{-g}\;g^{\alpha\beta}\left[\dfrac{\partial I}{\partial x^\beta}\right]-\left(\dfrac{2\pi m_0 c}{h}\right)^2\sqrt{-g}\;P^\alpha, & \\
 & (a) \\
\left[\dfrac{\partial U_{\overline{\alpha\beta}}}{\partial x^\beta}\right]=\sqrt{-g}\;g^{\alpha\beta}\left[\dfrac{\partial J}{\partial x^\beta}\right]-\left(\dfrac{2\pi m_0 c}{h}\right)^2\sqrt{-g}\;Q^\alpha, & \\
 & (b) \\
\left[\dfrac{\partial\sqrt{-g}\;P^\alpha}{\partial x^\alpha}\right]=\sqrt{-g}\;I, & \\
\left[\dfrac{\partial\sqrt{-g}\;Q^\alpha}{\partial x^\alpha}\right]=\sqrt{-g}\;J, & \\
U_{\alpha\beta}=\left[\dfrac{\partial P_\alpha}{\partial x^\beta}-\dfrac{\partial P_\beta}{\partial x^\alpha}\right]+\dfrac{g_{\alpha\gamma}g_{\beta\delta}}{\sqrt{-g}}\left[\dfrac{\partial Q_{\bar\gamma}}{\partial x^\delta}-\dfrac{\partial Q_{\bar\delta}}{\partial x^\gamma}\right]. & (c)
\end{array}\right\} \qquad (1)$$

One has set:

$$\left[\frac{\partial}{\partial x^\alpha}\right]=\frac{\partial}{\partial x^\alpha}-\xi_\alpha \qquad \text{and} \qquad \xi_\alpha=\frac{2\pi i e}{hc}\Phi_\alpha,$$

in which Φ_α are, as usual, the four components of the electromagnetic potential in which the electron is embedded. m_0, e are the rest mass and charge of the electron.

Those sixteen equations are linear in the sixteen complex functions:

$$U_{\alpha\beta}, \quad P_\alpha, \quad Q_\alpha, \quad I, \quad J, \tag{2}$$

and their first derivatives. The first eight of them have the Maxwellian form that is characterized by the divergence that enters into the left-hand sides. Equations (1.b) are

([1]) See [**11**], [**12**], and [**13**].

the complementary equations that relate to the potentials P_α, Q_α. Finally, the six equations (1.*c*) are the defining equations of the fields as functions of the potentials.

J. M. Whittaker showed that the particular solutions of those equations are solutions of the Dirac equation. Those particular solutions are obtained by establishing twelve relations between the sixteen functions (2). Four unknown functions ψ_α (α = 1, …, 4) will then remain. However, it is easy to see that by virtue of the invariance of the quantities (2), those twelve relations are not invariant, and that J. M. Whittaker's ψ_α do not vary like a semi-vector. If the equations that the ψ_α satisfy in a particular coordinate system are Dirac's then that will no longer true in another system that is obtained by starting from the first one by a Lorentz transformation. It then results that J. M. Whittaker's equations, to which one adjoins the aforementioned twelve equations, cannot be considered to be the fundamental equations of the wave mechanics of a *Dirac electron.*

2. Variational principle. – The eight Maxwellian equations (1.*a*) will yield a variational principle when one takes (1.*b*) and (1.*c*) into account. Indeed, they are equivalent to the Lagrange equations:

$$\frac{\delta\mathcal{M}}{\delta P_\alpha} = 0 \qquad \text{and} \qquad \frac{\delta\mathcal{M}}{\delta Q_\alpha} = 0, \tag{3}$$

in which

$$\mathcal{M} \equiv \sqrt{-g}\left\{\tfrac{1}{2}U^{\alpha\beta}\,U^*_{\alpha\beta} + (I\,I^* - J\,J^*) - (P^\alpha P^*_\alpha - Q^\alpha Q^*_\alpha)\right\}. \tag{4}$$

3. Correspondence principle. – By virtue of the correspondence principle, the current-density vector will be defined, as in [**II**, (34)], by:

$$\mathcal{C}^\alpha = -\frac{1}{2}\frac{\delta\mathcal{M}}{\delta\Phi_\alpha}, \tag{5}$$

and the energy-impulse tensor will be:

$$\mathcal{T}^{\alpha\beta} = -\frac{1}{2}\frac{\delta\mathcal{M}}{\delta g_{\alpha\beta}}. \tag{6}$$

Explicitly:

$$\mathcal{C}^\alpha = -\frac{2\pi ie}{hc}\left\{U^{\alpha\beta}P^*_\beta + \frac{U_{\overline{\alpha\beta}}}{\sqrt{-g}}Q^*_\beta - J^*Q^\alpha\right\} + \text{conj.} \tag{7}$$

We have no need for the very complicated explicit form for the tensor $\mathcal{T}^{\alpha\beta}$. One can prove that by virtue of equations (1), one will have the continuity equation:

$$\frac{\partial \mathcal{C}^{\alpha}}{\partial x^{\alpha}} = 0 \tag{8}$$

and law of energy-impulse:

$$\frac{\partial \mathcal{T}^{\alpha\beta}}{\partial x^{\beta}} = \mathcal{C}^{\beta} H_{\beta\alpha} . \tag{9}$$

4. Photonic fields. – According to Th. De Donder, wave mechanics is characterized by the study of a new field that he referred to by the name of the *photonic field* ([1]). In his way of looking at things, the general theory of the photonic field applies to not just the electron, but to any corpuscles, charged or not, as well as the photon.

After J. M. Whittaker's article appeared, Th. De Donder considered equations (1) to be the fundamental equations of the photonic field. We just pointed out that those equations cannot serve to define the Dirac electron. However, those equations must be valid for the photon, since they have a Maxwellian form. For the photon, one must set $e = 0$ in (1) and replace m_0 with the mass μ_0 of the photon ([2]). Equations (1) will then become:

$$\left.\begin{aligned}
\frac{\partial \sqrt{-g}\, U^{\alpha\beta}}{\partial x^{\beta}} &= \sqrt{-g}\; g^{\alpha\beta}\,\frac{\partial I}{\partial x^{\beta}} - \left(\frac{2\pi m_0 c}{h}\right)^2 \sqrt{-g}\; P^{\alpha}, \\
\frac{\partial U_{\overline{\alpha\beta}}}{\partial x^{\beta}} &= \sqrt{-g}\; g^{\alpha\beta}\,\frac{\partial J}{\partial x^{\beta}} - \left(\frac{2\pi m_0 c}{h}\right)^2 \sqrt{-g}\; Q^{\alpha}, \\
\frac{\partial \sqrt{-g}\; P^{\alpha}}{\partial x^{\alpha}} &= \sqrt{-g}\; I, \\
\frac{\partial \sqrt{-g}\; Q^{\alpha}}{\partial x^{\alpha}} &= \sqrt{-g}\; J, \\
U_{\alpha\beta} &= \frac{\partial P_{\alpha}}{\partial x^{\beta}} - \frac{\partial P_{\beta}}{\partial x^{\alpha}} + \frac{g_{\alpha\gamma} g_{\beta\delta}}{\sqrt{-g}} \left(\frac{\partial Q_{\bar{\gamma}}}{\partial x^{\delta}} - \frac{\partial Q_{\bar{\delta}}}{\partial x^{\gamma}}\right).
\end{aligned}\right\} \tag{10}$$

Those equations generalize the classical Maxwell equations.

([1]) See [**14**], pp. 8. The author reprised and developed that study in [**15**], pp. 77.

([2]) We can annul μ_0 here with no inconvenience.

CHAPTER V

WAVE MECHANICS OF THE PHOTON.

1. Introduction. – In the beginning, the wave mechanics of material particles was developed by analogy with Fresnel's theory of light. Meanwhile, that wave mechanics was soon constructed in an autonomous fashion by basing it upon a new and characteristic formalism that was analyzed in the preceding chapters.

For reasons of unity, it would be natural to try to establish the theory of the photon and the electron upon the same general principles. Notably, the interaction between light and matter demands to be envisioned from that viewpoint.

Furthermore, for quite some time, certain imperfections have been glimpsed in Maxwell's theory, which does not account for the corpuscular aspect of light. The quantization of fields eliminates that imperfection. It has achieved great successes (fluctuations of black-body radiation, the phenomena of emission and absorption of light atoms, etc.), but only at the expense of new difficulties, such as infinite energy at absolute zero, non-commutation of the vector potential with the divergence of the electric field. Those new difficulties have been eliminated at present only in an artificial manner.

De Broglie has shown that his wave mechanics of the photon has eliminated those contradictions and has yielded the theory of ensembles of photon by second quantization, just as the theory of ensembles of electrons is deduced from the wave mechanics of the electron by second quantization.

We briefly summarize some of the ideas that have guided L. de Broglie in his research:

a) The photon will not be a Dirac corpuscle, since it seems certain that an ensemble of Dirac particles must obey Fermi statistics. As a system of photons that obey Bose statistics, it seems more natural to suppose that a photon is a complex particle that is composed of two Dirac particles (more precisely, a corpuscle and an "anti-corpuscle").

b) In the interaction between light and matter, the essential elementary phenomenon is the photoelectric effect. A photon will disappear in the course of the phenomenon. One seeks to express that fact in the language of wave mechanics in the following form: The photon passes from the state that is represented by a certain number of wave functions Φ_ε to the "annihilated state" $\Phi^{(0)}$. That annihilated state plays a very special role: It represents a photon whose measurable characteristics are all zero.

c) Light can be polarized. In addition, the photoelectric effect is influenced by the polarization of the absorbed light. That indicates that the new theory must possess an element that is capable of attributing a polarization state to each photon.

d) The theory of the electromagnetic field introduced the important notions of the electric and magnetic fields. Those fields are measured only by means of the intermediary of the interaction of matter and light. Conforming to remark *b*) and the general principles of wave mechanics, one must think that those fields appear to be densities of the matrix elements that are associated with the transitions $\Phi \rightarrow \Phi^{(0)}$.

e) The wave function Φ_ε of a photon that has not been annihilated satisfies certain fundamental equations. By virtue of the field equations that were defined above, it is necessary that they must satisfy some equations of Maxwellian type.

The presentation that follows is based upon L. de Broglie's new conception of light. He proved that wave mechanics permits one to simplify the study of the properties of light and make it more precise.

2. Fundamental equations. – The Dirac equations that relate to the electron are written:

$$H\,\psi = \frac{h}{2\pi i}\frac{\partial\psi}{\partial t}, \tag{1}$$

in which H is the Hamiltonian [**III**, (15″)]. Similarly, the first task in the wave mechanics of the photon was to establish equations:

$$H\,\Phi = \frac{h}{2\pi i}\frac{\partial\Phi}{\partial t} \tag{2}$$

that would be analogous to (1) in order to determine the time evolution of the wave functions Φ_ε. If the relativistic wave mechanics of systems of two Dirac corpuscles were known then one would only have to apply it to the case of the photon, while utilizing the hypothesis (§ **1**, *a*), in order to obtain the desired equations. However, that theory of systems is not known. In order to find the fundamental equations of the photon, L. de Broglie appealed to a process that is less rigorous ([**16**], pp. 37), but still partially justified ([**18**], pp. 105). Here, we are content to refer to the original memoirs, and we shall note simply that it results from those works that the coordinates *x*, *y*, *z* that figure in the fundamental equations can be interpreted as the coordinates of the center of gravity of the photon.

The equations of the (un-annihilated) photon are (†):

$$\frac{1}{c}\frac{\partial\Phi_{\alpha\beta}}{\partial t} = \left(\mathcal{A}_1\frac{\partial}{\partial x} + \mathcal{A}_2\frac{\partial}{\partial y} + \mathcal{A}_3\frac{\partial}{\partial z} + \frac{2\pi i\mu_0 c}{h}\mathcal{A}_4\right)\Phi_{\alpha\beta}, \tag{I}$$

$$\frac{1}{c}\frac{\partial\Phi_{\alpha\beta}}{\partial t} = \left(\mathcal{B}_1\frac{\partial}{\partial x} + \mathcal{B}_2\frac{\partial}{\partial y} + \mathcal{B}_3\frac{\partial}{\partial z} + \frac{2\pi i\mu_0 c}{h}\mathcal{B}_4\right)\Phi_{\alpha\beta}, \tag{II}$$

(†) Translator's note: It was not clear in the original that he actually meant $\Phi_{\alpha\beta}$ in both equations.

in which $\Phi_{\alpha\beta} = \Phi_{\alpha\beta}(x, y, z, t)$ are sixteen wave functions. (The coordinate trihedron is tri-rectangular.) $\mathcal{A}_1, \ldots, \mathcal{A}_4, \mathcal{B}_1, \ldots, \mathcal{B}_4$ are eight matrices with sixteen rows and columns that are defined with the aid of the Dirac matrices ([1]):

$$\left.\begin{array}{c}(\mathcal{A}_1)_{\alpha\beta,\gamma\varepsilon} = (\alpha_1)_{\alpha\gamma}\,\delta_{\beta\varepsilon}, \quad (\mathcal{A}_2)_{\alpha\beta,\gamma\varepsilon} = (\alpha_2)_{\alpha\gamma}\,\delta_{\beta\varepsilon}, \quad (\mathcal{A}_3)_{\alpha\beta,\gamma\varepsilon} = (\alpha_3)_{\alpha\gamma}\,\delta_{\beta\varepsilon}, \\ (\mathcal{A}_4)_{\alpha\beta,\gamma\varepsilon} = (\alpha_4)_{\alpha\gamma}\,\delta_{\beta\varepsilon}, \\ (\mathcal{B}_1)_{\alpha\beta,\gamma\varepsilon} = (\alpha_1)_{\beta\varepsilon}\,\delta_{\alpha\gamma}, \quad (\mathcal{B}_2)_{\alpha\beta,\gamma\varepsilon} = -(\alpha_2)_{\beta\varepsilon}\,\delta_{\alpha\gamma}, \quad (\mathcal{B}_3)_{\alpha\beta,\gamma\varepsilon} = (\alpha_3)_{\beta\varepsilon}\,\delta_{\alpha\gamma}, \\ (\mathcal{B}_4)_{\alpha\beta,\gamma\varepsilon} = -(\alpha_4)_{\beta\varepsilon}\,\delta_{\alpha\gamma},\end{array}\right\} \tag{3}$$

($\delta_{\alpha\alpha} = 1$, $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$.) For example:

$$\mathcal{A}_1\,\Phi_{\alpha\beta} = \sum\sum(\mathcal{A}_1)_{\alpha\beta,\,\gamma\delta}\,\Phi_{\gamma\delta}\,.$$

L. de Broglie showed that these equations can be considered to be "the wave equations that correspond to the motion of the center of gravity of the (un-annihilated) photon."

The system (I), (II) is equivalent to the system:

$$(H^{(a)} + H^{(b)})\,\Phi = \frac{h}{2\pi i}\frac{\partial\Phi}{\partial t}, \tag{III}$$

$$(H^{(a)} - H^{(b)})\,\Phi = 0, \tag{IV}$$

in which:

$$\left.\begin{array}{l} H^{(a)} \equiv \dfrac{hc}{4\pi i}\left\{\mathcal{A}_1\dfrac{\partial}{\partial x} + \cdots + \dfrac{2\pi i\mu_0 c}{h}\mathcal{A}_4\right\}, \\ H^{(b)} \equiv \dfrac{hc}{4\pi i}\left\{\mathcal{B}_1\dfrac{\partial}{\partial x} + \cdots + \dfrac{2\pi i\mu_0 c}{h}\mathcal{B}_4\right\}. \end{array}\right\} \tag{4}$$

The sixteen equations (III) are the "evolution equations" that were mentioned in (2), and:

$$H \equiv H^{(a)} + H^{(b)} \tag{5}$$

is the Hamiltonian of the theory of the photon. The sixteen equations (IV), which do not include time derivatives, are the "condition equations." Those thirty-two equations are compatible, they are invariant under any Lorentz transformation, and the $\Phi_{\alpha\beta}$ vary like the products $\psi_\alpha\,\psi_\beta^\times$, in which the ψ_α are Dirac wave functions.

In order to make the significance of the condition equations (IV) somewhat more precise, we shall study the case of a monochromatic wave:

([1]) For example, the matrix A_1 is equal to the direct product of the matrix α_1 and the identity matrix: $A_1 = \alpha_1 \times I$. Similarly, $B_1 = I \times \alpha_1$.

$$\Phi_{\alpha\beta} = \varphi_{\alpha\beta}\, e^{\frac{2\pi i E}{h}t},$$

and start from equations (III). By virtue of (III):

$$H\varphi = E\varphi.$$

If one adds the conditions (IV):

$$H^{(a)} = H^{(b)}$$

then:

$$H^{(a)}\,\varphi = \frac{E}{2}\,\varphi, \qquad H^{(b)}\,\varphi = \frac{E}{2}\,\varphi.$$

The condition equations (IV) will then serve to define some pure states for energy for which the proper values of the operators $H^{(a)}$ and $H^{(b)}$ will be equal.

Now, consider the annihilation solution $\Phi^0_{\alpha\beta}$. The energy and impulse of an annihilated photon must be zero. It will then result that the functions $\Phi^0_{\alpha\beta}$ are constants that satisfy the equations:

$$(\mathcal{A}_4 + \mathcal{B}_4)\,\Phi^0_{\alpha\beta} = 0; \tag{6}$$

hence:

$$\Phi^0_{13} = \Phi^0_{14} = \Phi^0_{23} = \Phi^0_{24} = \Phi^0_{31} = \Phi^0_{41} = \Phi^0_{32} = \Phi^0_{42} = 0. \tag{7}$$

It is natural to suppose, in addition, that this annihilation solution preserves its form in any Lorentz system; i.e., that it is invariant with respect to any Lorentz transformation. That condition can succeed in determining the $\Phi^0_{\alpha\beta}$. One will have:

$$\Phi^0_{\alpha\beta} = (\alpha_4)_{\alpha\beta}, \tag{8}$$

up to a multiplicative constant. Those functions are solutions of (III), but not of (IV). One easily sees that (8) is a solution for which $H^{(a)}$ and $H^{(b)}$ have the proper values $\frac{1}{2}\mu_0 c^2$ and $-\frac{1}{2}\mu_0 c^2$, resp.

Remarks. – Like the Dirac ψ_α, the $\Phi_{\alpha\beta}$ will have the dimension $L^{-3/2}$, where L ≡ length. In order for the annihilation solution to likewise have those dimensions, it will suffice to take:

$$\Phi^0_{\alpha\beta} = \lambda\,(\alpha_4)_{\alpha\beta}, \tag{8′}$$

instead of (8), in which λ has the dimensions of $(\text{length})^{-3/2}$. In practice, that annihilation solution will appear only in the densities of matrix elements that are associated with the disappearance or appearance of a photon. Those densities already include an arbitrary multiplicative constant K, into which the constant λ can always be absorbed.

3. Operators. – The main operators that have been encountered in the theory of the photon up to now have had the form:

$$F^{(a)} + F^{(b)};$$

i.e., they are composed of the sum of a (linear, Hermitian) operator "of type (a)" and a (linear, Hermitian) operator "of type (b)". An operator of type (a) does not act upon the second index of $\Phi_{\alpha\beta}$, while an operator of type (b) does not act upon the first one. Therefore, let $f^{(a)}$, $f^{(b)}$ be the operators of Dirac's theory that correspond to $F^{(a)}$, $F^{(b)}$, respectively. If the summation sign is implicit then one will have:

$$F^{(a)}\,\Phi_{\alpha\beta} = (f^{(a)})_{\alpha\gamma}\,\Phi_{\gamma\beta} \qquad \text{and} \qquad F^{(b)}\,\Phi_{\alpha\beta} = (f^{(b)})_{\beta\gamma}\,\Phi_{\alpha\gamma}\;; \tag{10}$$

for example:

$$\mathcal{A}_2\,\Phi_{\alpha\beta} = (\alpha_2)_{\alpha\gamma}\,\Phi_{\gamma\beta} \qquad \text{and} \qquad \mathcal{B}_2\,\Phi_{\alpha\beta} = (-\,\alpha_2)_{\beta\gamma}\,\Phi_{\alpha\gamma}.$$

More generally, for an operator of the form $F^{(a)} F^{(b)}$, one will have:

$$F^{(a)} F^{(b)}\,\Phi_{\alpha\beta} = (f^{(a)})_{\alpha\gamma}\,(f^{(b)})_{\beta\delta}\,\Phi_{\gamma\delta}. \tag{10$'$}$$

Since the matrices $\mathcal{A}$ satisfy the same commutation relations as the Dirac matrices, one can form a table of sixteen operators with the aid of the $\mathcal{A}$ and the identity matrix that is analogous to the table of sixteen fundamental Dirac operators. One operates upon the $\mathcal{B}$ similarly. One will then have two tables of sixteen fundamental operators ([1]) [of the type (9)], namely:

$$\tfrac{1}{2}(\mathcal{A}_4 \pm \mathcal{B}_4), \tag{11}$$

$$\tfrac{1}{2}(\mathcal{A}_1 \pm \mathcal{B}_1), \quad \tfrac{1}{2}(\mathcal{A}_2 \pm \mathcal{B}_2), \quad \tfrac{1}{2}(\mathcal{A}_3 \pm \mathcal{B}_3), \quad \tfrac{1}{2}(1 \pm 1), \tag{12}$$

$$\left\{\begin{array}{l} \tfrac{1}{2}(i\,\mathcal{A}_1\mathcal{A}_4 \pm i\,\mathcal{B}_1\mathcal{B}_4), \cdots \\ \tfrac{1}{2}(i\,\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4 \pm i\,\mathcal{B}_2\mathcal{B}_3\mathcal{B}_4), \cdots \end{array}\right\} \tag{13}$$

$$\tfrac{1}{2}(s_1^a \pm s_1^b)\,, \;\ldots, \qquad \tfrac{1}{2}(s_4^a \pm s_4^b)\,, \tag{14}$$

$$\tfrac{1}{2}(\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4 \pm \mathcal{B}_1\mathcal{B}_3\mathcal{B}_3\mathcal{B}_4)\,. \tag{15}$$

Those operators play a very important role in the theory. In particular, the study of the operators of the type $\tfrac{1}{2}(\mathcal{A} + \mathcal{B})$ parallels that of the corresponding operators α in Dirac's theory.

([1]) We utilize the notations $s_1^{\alpha} = i\,A_2\,A_3$, $s_4^{\alpha} = i\,A_1\,A_2\,A_3$, and analogous notations for the $\mathcal{B}$ matrices.

4. Mean values and matrices. – By analogy with the wave mechanics of the electron, one can be tempted to take:

$$\Phi^* (F^{(a)} + F^{(b)})\,\Phi \tag{16}$$

to be the expression for the mean-value density that corresponds to the operator $F^{(a)} + F^{(b)}$. However, that definition is not admissible, by reason of variance. Indeed, consider, for example, the case of the operators:

$$\left.\begin{array}{ll} u^i_{(a)} = -\mathcal{A}_i, \quad u^4_{(a)} = 1; \qquad u^4_{(b)} = -\mathcal{B}_i, \quad u^4_{(b)} = 1; & i = 1, 2, 3, \\ u^\alpha = \tfrac{1}{2}(u^\alpha_{(a)} + u^\alpha_{(b)}), & \alpha = 1, \cdots, 4. \end{array}\right\} \tag{17}$$

The densities:

$$\Phi^* \, c \, u^\alpha \, \Phi$$

are not the components of a vector; they are the components ($\alpha 4$) of a tensor:

$$\Phi^* \frac{c}{2}(u^\alpha_{(a)} u^\beta_{(b)} + u^\alpha_{(b)} u^\beta_{(a)})\,\Phi\,. \tag{18}$$

On the contrary, the densities:

$$\Phi^* \frac{c}{2}(\mathcal{B}_4\, u^\alpha_{(a)} + \mathcal{A}_4 u^\alpha_{(b)})\,\Phi \tag{19}$$

are the components of a vector ([1]). Generalizing that result, one will be led to take:

$$\Phi^* (\mathcal{B}_4\, F^{(a)} + \mathcal{A}_4\, F^{(b)})\,\Phi \tag{20}$$

to be the definition of the mean-value densities [**17**], instead of (16).

The tensors that are obtained in that fashion will always have the proper variance. The mean values will then be given by the integrals:

$$\int \Phi^* (\mathcal{B}_4\, F^{(a)} + \mathcal{A}_4\, F^{(b)})\,\Phi\; dx\, dy\, dz\,; \tag{21}$$

the integration extends over the entire space of the variables x, y, z.

Similarly, let $\Phi^{(p)}$ (p = 1, 2, …) be a complete system of wave functions. The densities of matrix elements that correspond to the operator will then be:

$$(\Phi^{(p)})^* (\mathcal{B}_4\, F^{(a)} + \mathcal{A}_4\, F^{(b)})\,(\Phi^{(p')}) \tag{22}$$

in the system $\Phi^{(p)}$, and the matrix elements will be:

([1]) As in Chapter IV, the square of the elementary interval will be $(ds)^2 = -(dx^1)^2 - \ldots + (dx^4)^2$ here, and the variance will mean under any Lorentz transformation.

$$\int (\Phi^{(p)})^* (\mathcal{B}_4 F^{(a)} + \mathcal{A}_4 F^{(b)}) (\Phi^{(p')}) \, dx\, dy\, dz. \tag{22′}$$

The reasons that led us to take (20), instead of (16), have been of a purely mathematical order up to now. According to L. de Broglie "It seems possible that the modification in the usual definition of ρ whose necessity we have seen in the theory of the photon is related to the fact that the photon is a complex corpuscle that is subject to Lorentz contraction in its internal structure."

A good relativistic theory of systems of two corpuscles must be able to express that difference logically. In the absence of that theory, we will justify the definition (20) by showing that it leads to some results that are completely satisfactory ([1]).

5. Densities of matrix elements associated with the annihilation of the photon. – L. de Broglie defined five tensorial quantities that were related to the passage of the photon from the state Φ to the annihilated state Φ^0. Those five tensors can be divided into two groups:

a) *Maxwellian quantities* ([2]). Namely, one has a *vector* (viz., the electromagnetic potential):

$$P^i = K\,\Phi^0\, \frac{\mathcal{B}_4\,\mathcal{A}_1 - \mathcal{A}_4\,\mathcal{B}_1}{2}\,\Phi, \;\ldots,\; P^4 = -\,K\,\Phi^0\, \frac{\mathcal{B}_4\,\mathcal{A}_1 - \mathcal{A}_4\,\mathcal{B}_1}{2}\,\Phi, \tag{23}$$

and an *antisymmetric tensor* (viz., the electromagnetic field):

$$\left.\begin{aligned} U^{14} &= K\,\frac{2\pi\mu_0 c}{h}\,\Phi^0\,\frac{\mathcal{B}_4\, i\mathcal{A}_1\mathcal{A}_4 - \mathcal{A}_4\, i\mathcal{B}_1\mathcal{B}_4}{2}\,\Phi, \cdots \\ U^{23} &= K\,\frac{2\pi\mu_0 c}{h}\,\Phi^0\,\frac{\mathcal{B}_4\, s_1^{(a)}\mathcal{A}_4 - \mathcal{A}_4\, s_1^{(b)}\mathcal{B}_4}{2}\,\Phi, \cdots \end{aligned}\right\} \tag{24}$$

b) *Non-Maxwellian quantities*. Namely, one has an *invariant*:

$$I_{(1)} = -\,\Phi^0\,\frac{\mathcal{B}_4\,\mathcal{A}_4 + \mathcal{A}_4\,\mathcal{B}_4}{2}\,\Phi, \tag{25}$$

a *completely-antisymmetric tensor* of rank three, or what amounts to the same thing, a vector with contravariant components:

$$Q^1 = K\,\Phi^0\,\frac{\mathcal{B}_4\, s_1^{(a)} + \mathcal{A}_4\, s_1^{(b)}}{2}\,\Phi, \ldots, \qquad Q^4 = -\,K\,\Phi^0\,\frac{\mathcal{B}_4\, s_4^{(a)} + \mathcal{A}_4\, s_4^{(b)}}{2}\,\Phi, \tag{26}$$

([1]) Except in the case of annihilation solutions (see § 11).

([2]) They are called that because they satisfy equations of Maxwellian type (see § 6). The electromagnetic quantities will be expressed in Heaviside-Lorentz units; one will then have $K = h / 2\pi \sqrt{\mu_0}$.

and a *completely-antisymmetric tensor of rank four*, or what amounts to the same thing, an invariant:

$$I_{(2)} = -\Phi^0 \, \frac{\mathcal{B}_4 \, \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_4 \, \mathcal{B}_1\mathcal{B}_2\mathcal{B}_3\mathcal{B}_4}{2} \, \Phi \, . \tag{27}$$

In vectorial notation, we set:

$$P^1 = \mathcal{V}_x \, , \, \ldots, \, P^4 = V,$$

$$\begin{cases} U^{14} = -H_x, \cdots \\ U^{23} = \mathcal{H}_x, \cdots \end{cases}$$

$$Q^1 = K \, \sigma_x \, , \, \ldots, \, Q^4 = K \, \sigma^4 \, .$$

The $\mathcal{V}_{x^i}$ are the components of the (complex) vector potential $\boldsymbol{\mathcal{V}}$. V is the scalar potential. **H** and $\boldsymbol{\mathcal{H}}$ are the components of the (complex) electric field and the (complex) magnetic field, resp. The explicit form of those expressions, as well as their variance, is easily obtained by remarking that (with an implied summation):

$$I_{(1)} = (\alpha_4)_{\beta\gamma} \, \Phi_{\gamma\beta},$$

$$P^1 = -K \, (\alpha_1)_{\beta\gamma} \, \Phi_{\gamma\beta}, \quad P^2 = -K \, (\alpha_2)_{\beta\gamma} \, \Phi_{\gamma\beta}, \, \ldots,$$

$$U^{14} = -\frac{K\mu_0 c \, 2\pi}{h} (i \; \alpha_1 \; \alpha_4)_{\beta\gamma} \, \Phi_{\gamma\beta}, \, \ldots,$$

$$Q^1 = (i \; \alpha_2 \; \alpha_3)_{\beta\gamma} \, \Phi_{\gamma\beta}, \, \ldots,$$

$$I_{(2)} = (\alpha_1 \; \alpha_2 \; \alpha_3 \; \alpha_4)_{\beta\gamma} \, \Phi_{\gamma\beta} \, .$$

The Maxwellian quantities are provided by operators of the table (11 – 15) of type ($\mathcal{A}$ – $\mathcal{B}$). On the contrary, the non-Maxwellian quantities are densities of matrix elements that correspond to the operators of the table (11 – 15) of type ($\mathcal{A}$ + $\mathcal{B}$). All of the other tensors that are inferred from (11 – 15) by the process above are identically zero.

The sixteen linear combinations (23 – 27) of the sixteen $\Phi_{\alpha\beta}$ are linearly-independent. One can then solve inversely, and express the sixteen $\Phi_{\alpha\beta}$ as functions of the sixteen components of the tensors (23 – 27). Here is the result of that solution:

$$
\left.\begin{aligned}
4\Phi_{11} &= \frac{1}{K}\left(V + \frac{\hbar}{\mu_0 c}\mathcal{H}_z\right) - I_{(1)} - \sigma_z, \\
4\Phi_{22} &= \frac{1}{K}\left(V - \frac{\hbar}{\mu_0 c}\mathcal{H}_z\right) - I_{(1)} + \sigma_z, \\
4\Phi_{33} &= \frac{1}{K}\left(V - \frac{\hbar}{\mu_0 c}\mathcal{H}_z\right) + I_{(1)} - \sigma_z, \\
4\Phi_{44} &= \frac{1}{K}\left(V + \frac{\hbar}{\mu_0 c}\mathcal{H}_z\right) + I_{(1)} + \sigma_z, \\
\\
4\Phi_{13} &= \frac{1}{K}\left(-\mathcal{V}_z + i\frac{\hbar}{\mu_0 c}H_z\right) - iI_{(2)} + \sigma_4, \\
4\Phi_{31} &= \frac{1}{K}\left(-\mathcal{V}_z + i\frac{\hbar}{\mu_0 c}H_z\right) + iI_{(2)} + \sigma_4, \\
4\Phi_{24} &= \frac{1}{K}\left(+\mathcal{V}_z - i\frac{\hbar}{\mu_0 c}H_z\right) - iI_{(2)} + \sigma_4, \\
4\Phi_{42} &= \frac{1}{K}\left(+\mathcal{V}_z + i\frac{\hbar}{\mu_0 c}H_z\right) + iI_{(2)} + \sigma_4,
\end{aligned}\right\} \qquad (28)
$$

$$
\left.\begin{aligned}
4\Phi_{12} &= \frac{\hbar}{K\mu_0 c}\left(\mathcal{H}_x + i\,\mathcal{H}_y\right) - \sigma_x - i\sigma_y, \\
4\Phi_{14} &= \frac{\hbar}{K\mu_0 c}i\left(H_x + i\,H_y\right) - \frac{1}{K}\left(\mathcal{V}_x + i\,\mathcal{V}_y\right), \\
4\Phi_{32} &= \frac{\hbar}{K\mu_0 c}i\left(-H_x - i\,H_y\right) - \frac{1}{K}\left(\mathcal{V}_x + i\,\mathcal{V}_y\right), \\
4\Phi_{34} &= \frac{\hbar}{K\mu_0 c}\left(-\mathcal{H}_x - i\,\mathcal{H}_y\right) - \sigma_x - i\sigma_y, \\
\\
4\Phi_{21} &= \frac{\hbar}{K\mu_0 c}\left(\mathcal{H}_x - i\,\mathcal{H}_y\right) - \sigma_x + i\sigma_y, \\
4\Phi_{23} &= \frac{\hbar}{K\mu_0 c}i\left(H_x - i\,H_y\right) - \frac{1}{K}\left(\mathcal{V}_x - i\,\mathcal{V}_y\right), \\
4\Phi_{41} &= \frac{\hbar}{K\mu_0 c}i\left(-H_x + i\,H_y\right) - \frac{1}{K}\left(\mathcal{V}_x - i\,\mathcal{V}_y\right), \\
4\Phi_{43} &= \frac{\hbar}{K\mu_0 c}\left(-\mathcal{H}_x + i\,\mathcal{H}_y\right) - \sigma_x + i\sigma_y, \\
&(\hbar = h/2\pi).
\end{aligned}\right\} \qquad (28, \text{cont.})
$$

These formulas permit us to replace the wave functions with the components of the five tensorial quantities (23 – 27) in every expression for in theory of the photon. Since the electromagnetic field and potential figure among the five tensors, one must agree that such a transformation is interesting. One can conveniently compare the results of the wave mechanics of the photon to the corresponding results of Maxwell's theory by using that transformation.

Remarks. – We note, in passing, the following theorem: Let $\Phi_{\alpha\beta}$ be a system of sixteen arbitrary functions, and let $\Phi^0_{\alpha\beta}$ be the annihilation solution. One has:

$$\Phi^0\, \mathcal{A}_4\, \mathcal{B}_4\, \Phi = -\,\Phi^0\, \Phi, \qquad \Phi^*\, \mathcal{A}_4\, \mathcal{B}_4\, \Phi^0 = -\,\Phi^0\, \Phi^0. \tag{29}$$

The verification is immediate.

For example, one infers that:

$$P^1 = -\,K\,\Phi^0\, \frac{\mathcal{B}_4\,\mathcal{A}_1 - \mathcal{A}_4\mathcal{B}_1}{2}\,\Phi = -\,K\,\Phi^0\, \frac{\mathcal{A}_4\,\mathcal{A}_1 - \mathcal{B}_4\mathcal{B}_1}{2}\,\Phi\ ;$$

hence:

$$\Phi^0\,(\mathcal{A}_4 + \mathcal{B}_4)\,(\mathcal{A}_1 - \mathcal{B}_1)\,\Phi \equiv 0,$$

which one obtains directly, moreover, upon remarking that $\Phi^0\,(\mathcal{A}_4 + \mathcal{B}_4)\,\Phi \equiv 0$. By contrast:

$$(P^1)^* = K\,\Phi^*\, \frac{\mathcal{B}_4\,\mathcal{A}_1 - \mathcal{A}_4\mathcal{B}_1}{2}\,\Phi^0 = -\,K\,\Phi^*\, \frac{\mathcal{A}_4\,\mathcal{A}_1 - \mathcal{B}_4\mathcal{B}_1}{2}\,\Phi^0\,;$$

hence, upon taking into account the commutation relations between the $\mathcal{A}$ and the $\mathcal{B}$:

$$(P^1)^* = K\,\Phi^*\,(\mathcal{A}_4 + \mathcal{B}_4)\,(\mathcal{A}_1 - \mathcal{B}_1)\,\Phi^0.$$

6. Tensorial form of the equations of the photon. – L. de Broglie showed that the system (I), (II) can be replaced with an equivalent system of thirty-two equations that are obtained by forming thirty-two linearly-independent linear combinations of equations (I), (II) in such a manner as to make the tensorial quantities (23-27) appear.

Those equations divide into two distinct groups: A first group of sixteen relations of Maxwellian form into which only the electromagnetic potentials will enter and a second group of sixteen equations into which only the non-Maxwellian quantities will enter.

In tensorial form, the first group is written:

$$\left.\begin{aligned}
&\frac{\partial\sqrt{-g}\,U^{\alpha\beta}}{\partial x^{\beta}} = -\left(\frac{2\pi\,\mu_0 c}{h}\right)^2 P^{\alpha}, && (a)\\
&\frac{\partial U_{\alpha\beta}}{\partial x^{\beta}} = 0, && (b)\\
&U_{\alpha\beta} = \frac{\partial P_{\alpha}}{\partial x^{\beta}} - \frac{\partial P_{\beta}}{\partial x^{\alpha}}, && (c)\\
&\frac{\partial\sqrt{-g}\,P^{\alpha}}{\partial x^{\alpha}} = 0, && (d)
\end{aligned}\right\} \qquad (30)$$

and when one sets:

$$I = 2\pi\,\frac{K\,\mu_0 c^2}{h}\,I_{(1)}\,, \qquad J = 2\pi\,\frac{K\,\mu_0 c^2}{h}\,I_{(2)}\,,$$

the second group will be:

$$\left.\begin{aligned}
&\frac{\partial I}{\partial x^{\beta}} = 0, && (a)\\
&\frac{\partial J}{\partial x^{\beta}} = \left(\frac{2\pi\,\mu_0 c}{h}\right)^2 Q_{\alpha}, && (b)\\
&\frac{\partial Q_{\alpha}}{\partial x^{\beta}} - \frac{\partial Q_{\beta}}{\partial x^{\alpha}} = 0, && (c)\\
&I = 0 && (d)\\
&\frac{\partial\sqrt{-g}\,Q^{\alpha}}{\partial x^{\alpha}} = J\sqrt{-g}\,. && (e)
\end{aligned}\right\} \qquad (31)$$

When the square of the line element is put into the form $(ds)^2 = -\sum (dx^i)^2 + (dx^4)^2$, one will have $\sqrt{-g} = 1$. However, in the form (30, 31), the equations will be valid in an arbitrary coordinate system for which the square of the elementary interval is:

$$(ds)^2 = g_{\alpha\beta}\,dx^{\alpha}\,dx^{\beta} \qquad \text{and} \qquad g = \|\,g_{\alpha\beta}\,\|.$$

They will likewise be valid in an arbitrary Riemannian space.

Equations (30) justify the terms that were introduced in paragraph (5). The functions $\mathcal{V}$ and V are the vector and scalar potential, respectively, while **H** and $\mathcal{H}$ represent the electric and magnetic field, resp. In vectorial notations, (30), (31) will take the form:

$$\left\{\begin{aligned}
&\operatorname{rot}\mathcal{H} = \frac{1}{c}\frac{\partial H}{\partial t} - \left(\frac{2\pi\,\mu_0 c}{h}\right)^2 \mathcal{V},\\
&\operatorname{div} H = \left(\frac{2\pi\,\mu_0 c}{h}\right)^2 V,
\end{aligned}\right.$$

$$\begin{cases} \operatorname{rot} H = -\dfrac{1}{c}\dfrac{\partial \mathcal{H}}{\partial t}, \\ \operatorname{div} \mathcal{H} = 0, \end{cases}$$

$$\begin{cases} H = -\operatorname{grad} V - \dfrac{1}{c}\dfrac{\partial \mathcal{V}}{\partial t}, \\ \mathcal{H} = \operatorname{rot} \mathcal{V}, \end{cases}$$

$$\operatorname{div} \boldsymbol{\mathcal{V}} + \frac{1}{c}\frac{\partial V}{\partial t} = 0,$$

$$\operatorname{grad} I = 0, \qquad \frac{1}{c}\frac{\partial I}{\partial t} = 0, \qquad I = 0,$$

$$\operatorname{grad} J = -\left(\frac{2\pi \mu_0 c}{h}\right)^2 K\,\boldsymbol{\sigma}, \qquad \frac{1}{c}\frac{\partial J}{\partial t} = \left(\frac{2\pi \mu_0 c}{h}\right)^2 K\,\sigma_4\,,$$

$$\operatorname{rot} K\,\boldsymbol{\sigma} = 0, \qquad \operatorname{grad} \sigma_4 + \frac{1}{c}\frac{\partial \boldsymbol{\sigma}}{\partial t} = 0,$$

$$\operatorname{div} K\,\boldsymbol{\sigma} + \frac{1}{c}\frac{\partial K\sigma_4}{\partial t} = J.$$

Meanwhile, one must remark that in wave mechanics, wave functions are general complex. For example, one must take $e^{\frac{2\pi i}{h}Et}$ for a monochromatic wave, and not $\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}\dfrac{2\pi}{h}Et$. It then results that the electromagnetic fields and potentials that are defined by (23) are complex, in general. That situation is interesting, because complex fields suggest themselves in many fundamental questions. Notably, it is well-known that the quantum theory of fields can be presented in a more satisfactory fashion when one utilizes complex fields. Similarly, they are introduced quite naturally in the theory of the interaction of light and matter. One can say that the electromagnetic fields that intervene in elementary phenomena are complex. It remains to explain how one can effect the passage from those "microscopic fields" to the real "macroscopic fields" of Maxwell's theory. That is a point upon which L. de Broglie has already insisted [**19**].

Remark. –

a) There exist wave functions $\Phi_{\alpha\beta}$ of the photon *in vacuo* such that:

$$I_{(1)} = I_{(2)} = \sigma_x = \sigma_y = \sigma_z = \sigma_4 = 0.$$

We refer to these solutions by the name of *Maxwellian waves.*

b) Equations (30), (31) are nothing but equations [VI, (10)], to which, one adds the conditions:

$$I = 0$$

and

$$\left(\frac{2\pi\,\mu_0 c}{h}\right)^2 Q_\alpha = \frac{\partial J}{\partial x^\alpha}\,.$$

c) There exists another way of grouping the equations of the photon.
Indeed, the system (I), (II) is equivalent to the system:

$$(\mathcal{B}_4\,H^{(a)} + \mathcal{A}_4\,H^{(b)})\,\Phi = \tfrac{1}{2}(\mathcal{A}_4 + \mathcal{B}_4)\frac{h}{2\pi i}\frac{\partial\Phi}{\partial t}\,, \tag{V}$$

$$(\mathcal{B}_4\,H^{(a)} - \mathcal{A}_4\,H^{(b)})\,\Phi = \tfrac{1}{2}(\mathcal{A}_4 - \mathcal{B}_4)\frac{h}{2\pi i}\frac{\partial\Phi}{\partial t}\,. \tag{VI}$$

The first group of sixteen equations (V), when written with the aid of the tensorial quantities, is nothing but the system that is composed of the four Maxwell equations (30.*a*), the six equations (30.*c*) that define the fields as functions of the potentials, the four equations (31.*b*), equation (31.*d*), and equation (31.*c*).
We make two remarks on the subject of the systems (V), (VI):

a) The system V is invariant, as is the system VI. That will result immediately from what we said about (V).

b) The system VI is a consequence of system V. Indeed, the Maxwell equations (30.*b*) are satisfied identically by virtue of (30.*c*). Equations (31.*a*) are always verified due to (31.*d*), and equations (31.*c*) are satisfied by virtue of (31.*b*).

The system V then appears to be the essential system, since it defines the wave functions that relate to the un-annihilated photon in a manner that is as complete as (I), (II).

7. Variational principle. – We have seen that the Dirac equations are the Lagrange equations of the variational principle:

$$\delta\iiiint \tfrac{1}{2}\left\{\psi^*\left(H - \frac{h}{2\pi i}\frac{\partial}{\partial t}\right)\psi + \text{conj.}\right\} dx^1\,dx^2\,dx^3\,dx^4 = 0.$$

The function under the integral sign is the real part of the mean-value density that is associated with the operator:

$$H - \frac{h}{2\pi i}\frac{\partial}{\partial t}.$$

Similarly, in the theory of the photon, the sixteen equations (V) are the Lagrange equations of the variational principle:

$$\delta\iiint\int \tfrac{1}{2}\left\{\Phi^*\left(\mathcal{B}_4 H^{(a)} + \mathcal{A}_4 H^{(b)} - \frac{\mathcal{B}_4 + \mathcal{A}_4}{2}\frac{h}{2\pi i}\frac{\partial}{\partial t}\right)\Phi + \text{conj.}\right\} dx^1\, dx^2\, dx^3\, dx^4 = 0, \tag{33}$$

in which, conforming to the general definition of the mean-value densities, the function under the integral sign is the real part of the mean-value density that is associated with the operator:

$$[H^{(a)} + H^{(b)}] - \frac{h}{2\pi i}\frac{\partial}{\partial t}.$$

The varied functions are the sixteen $\Phi_{\alpha\beta}$.

The system VI is likewise derived from a variational principle. However, since it is a consequence of V, that principle has only secondary interest.

It should be remarked that the variational principle (33) is more satisfying than the one that is used in classical Maxwellian theory.

Indeed, in Maxwell's theory, one obtains only the equations that the electromagnetic potentials must satisfy by a variational principle. The equations that define the fields as functions of the potentials must be added to them. On the contrary, here, the variational principle will provide not only Maxwell equations, but also the defining equations of the fields as functions of the potentials.

8. The current-density vector. – L. de Broglie has shown that the four operators:

$$-\frac{c}{2}(\mathcal{A}_4 + \mathcal{B}_4), \qquad \ldots, \qquad \frac{c}{2}(1 + 1)$$

can be taken to be the components of the current-density operator. Conforming to (20), the quadri-vector "mean-value of current-density" will then be:

$$U^1 \equiv \rho\, u_x = -c\,\Phi^*\,\frac{\mathcal{B}_4\mathcal{A}_1 + \mathcal{A}_4\mathcal{B}_1}{2}\Phi, \quad \ldots, \qquad U^4 \equiv c\rho = c\,\Phi^*\,\frac{\mathcal{B}_4 + \mathcal{A}_4}{2}\Phi\,. \tag{34}$$

Indeed, by virtue of (I and II), the continuity equation:

$$\frac{\partial U^\alpha}{\partial x^\alpha} = 0.$$

We utilize the inversion formulas (28) and give K the value:

$$K = \frac{h}{2\pi\sqrt{\mu_0}}; \tag{35}$$

recall that the Heaviside-Lorentz system of units is the one that was chosen.

One also has:

$$\left.\begin{aligned} \rho\mathbf{u} &= \frac{\pi i}{2h}\left\{[\mathcal{V}^*\mathcal{H}] + V^* H - J^*\mathbf{Q}\right\} + \text{conj.}, \\ \rho &= \frac{\pi i}{2hc}\left\{(\mathcal{V}^* H) - J^* Q_4\right\} + \text{conj.} \end{aligned}\right\} \tag{36}$$

or, in tensorial form:

$$U^\alpha = -\frac{\pi i}{2hc}\{U^{\alpha\beta}P_\beta^* - J^*Q^\alpha\} + \text{conj.} \tag{36$'$}$$

The vector U^α depends upon the electromagnetic potentials explicitly; the same thing will be true for the other mean-value densities. That distinguishes the wave mechanics of the photon from Maxwell's theory. Indeed, in the latter theory, one is accustomed to considering the potentials to be auxiliary functions that appear in the quantities that take on some physical sense only by way of the electric and magnetic fields. The well-known indeterminacy in the potentials will then result, which permits one replace $\mathcal{V}$, V with:

$$\mathcal{V}' = \mathcal{V} + \text{grad}\, f, \qquad V' = V - \frac{1}{c}\frac{\partial f}{\partial t}, \tag{37}$$

such that:

$$\sum \frac{\partial^2 f}{(\partial x)^2} - \frac{1}{c^2}\frac{\partial^2 f}{(\partial t)^2} = 0. \tag{38}$$

The relations (37), (38) will no longer be rigorously valid when one supposes that $\mu_0 \neq 0$, because if $\mathcal{V}$, V are solutions to (30) then that will no longer be true for $\mathcal{V}'$, V'. Meanwhile, an indeterminacy that is similar to (37), (38) persists in the theory of the photon. For example, in the case of a plane wave, one can show ([1]) only that potentials that differ by finite quantities (which will remain finite when one lets μ_0 tend to zero) will correspond to fields that differ by only negligible terms (which will tend to zero with μ_0). The same remark applies to the multipolar waves ([2]). The indeterminacy will be removed only when the negligible terms (which are proportional to μ_0) in the fields lead to measurable consequences. One can say that those potentials define the same fields $\mathbf{H}$, $\mathcal{H}$ in the wave mechanics of the photon "approximately."

Consider two Maxwellian waves ($\mathbf{I} = \mathbf{J} = Q_a = 0$):

$$\Phi_{\alpha\beta} \quad \text{or} \quad P^\alpha, U^{\alpha\beta},$$

([1]) See [**20**].
([2]) See Chapter VI, § 4.

$$\Phi'_{\alpha\beta} \qquad \text{or} \qquad P'^{\alpha}, U'^{\alpha\beta},$$

in which the P'^{α} are coupled to the P^{α} by the relations:

$$P'^{\alpha} = P^{\alpha} + \frac{\partial f}{\partial x^{\alpha}}.$$

One will have:

$$U^{\alpha\beta} \sim U'^{\alpha\beta}$$

"approximately."

Hence, (36′) will become:

$$U'^{\alpha} = U^{\alpha} + \left\{\left[-\frac{2\pi i}{4h}\frac{\partial}{\partial x^{\alpha}}(U^{\alpha\beta} f^*) + \frac{2\pi i}{4h}\frac{\partial U^{\alpha\beta}}{\partial x^{\beta}} f^*\right] + \text{conj.}\right\}. \tag{39}$$

The mean-value densities U'^{α} and U^{α} are then different, in general. Here, we shall confine ourselves to proving that, and we shall return to the question of the determination of the potentials in a later paragraph (VI, § **4**).

Remarks. –

a) In the case of a monochromatic Maxwellian wave, one can always annul the scalar potential. One will then have, for a frequency $\nu = E / h$:

$$H = -\frac{1}{c}\frac{\partial \mathcal{V}}{\partial t} = -\frac{2\pi i\nu}{c}\mathcal{V}, \tag{40}$$

and

$$\left.\begin{aligned} E\rho\mathbf{u} &= \frac{c}{4}[H^*\mathcal{H}] + \text{conj.}, \quad &(a) \\ E\rho &= \tfrac{1}{2}[H]^2. &(b) \end{aligned}\right\} \tag{41}$$

One recognizes the classical expression for the Poynting vector in (41.*a*), while (41.*b*) will be equal to the Maxwellian energy density when $|\mathbf{H}|^2 = |\mathcal{H}|^2$. Meanwhile, one must note that the fields are complex here and depend upon time by the exponential $e^{2\pi i\nu t}$, whereas in the classical Maxwell theory, the fields are real expressions that are sinusoidal functions of time. In Maxwell's theory, the energy density ε will then remain a function of time. Meanwhile, for light, the measurable quantity is not ε, but $\bar{\varepsilon}$; i.e., the mean value of ε over time. The essential quantity will then be $\bar{\varepsilon}$, and that is indeed what one recovers in the theory of the photon.

We note that the choice (35) of the constant K was imposed by the fact that we wished to obtain (41) with good coefficients.

b) The expressions (36′) are close to formulas [IV, (7)]. The vector C^{α} / c in [IV, (7)] differs from the current-density vector (36′) only by the terms:

$$-\frac{\pi i}{2h}\frac{1}{\sqrt{-g}}U_{\overline{\alpha\beta}}\,Q^*_\beta + \text{conj.} = \frac{\partial}{\partial x^\beta}\left[-\frac{\pi i}{2h}(U_{\overline{\alpha\beta}}\,J^*)+\text{conj.}\right],$$

which are space-time divergences. Those terms will be zero when J is constant.

9. The energy-impulse tensor. – That tensor will be defined as it is in Dirac's theory. One makes the space-time impulse vector correspond to the operator:

$$\mathcal{J}^1 = -\frac{h}{2\pi i}\frac{\partial}{\partial x}, \qquad \mathcal{J}^2 = -\frac{h}{2\pi i}\frac{\partial}{\partial y}, \qquad \mathcal{J}^3 = -\frac{h}{2\pi i}\frac{\partial}{\partial z}, \qquad \mathcal{J}^4 = \frac{h}{2\pi i}\frac{1}{c}\frac{\partial}{\partial t}, \tag{42}$$

while the current-density operator corresponds to the operator:

$$c\,u^\alpha = \frac{c}{2}(u^\alpha_{(a)} + u^\alpha_{(b)}). \tag{43}$$

As in [III, (60)], one forms the tensor-operator:

$$t^{\alpha\beta} = c\,\mathcal{J}^\alpha u^\beta \tag{44}$$

from those two operators, and conforming to (20), its mean-value densities will be:

$$T^{\alpha\beta} = \Phi^*\,\mathcal{J}^\alpha\,(\mathcal{B}_4\,u^\beta_{(a)} + \mathcal{A}_4\,u^\beta_{(b)})\,\Phi. \tag{45}$$

That asymmetric tensor is doubly-contravariant.

On the other hand, with the operators (43), one can form the symmetric tensor:

$$M^{*\alpha\beta} = \mu_0\,c^2\,\Phi^*\frac{u^\alpha_{(a)}\,u^\beta_{(b)} + u^\beta_{(a)}\,u^\alpha_{(b)}}{2}\Phi \tag{46}$$

that we spoke of above (18). Furthermore, we have shown [**20**], by utilizing the inversion formulas (28), that this tensor is equal to the classical Maxwell tensor (with complex electromagnetic fields), plus some terms that include only non-Maxwellian quantities and terms that are negligible compared to the Maxwellian terms. The tensors (45) and (46) are coupled by some important relations, namely:

$$T^{\alpha\beta} = M^{\alpha\beta} + \frac{hc}{4\pi}\frac{\partial A^{\alpha\beta\gamma}}{\partial x^\gamma} + \frac{h}{4\pi i}\varepsilon_{(\alpha)}\frac{\partial U^\beta}{\partial x^\alpha}, \tag{47}$$

in which:

$$\varepsilon_{(1)} = \varepsilon_{(2)} = \varepsilon_{(3)} = -1, \quad \varepsilon_{(4)} = +1,$$

and $A^{\alpha\beta\gamma}$ is tensor that is antisymmetric in the extreme indices $\alpha\gamma$.

$$\left.\begin{array}{ll} A^{i\beta\gamma} = -\Phi^*(i\,\mathcal{A}_4\,u^\alpha_{(a)}u^\beta_{(b)}u^\gamma_{(a)} + i\,\mathcal{B}_4\,u^\alpha_{(b)}u^\beta_{(a)}u^\gamma_{(b)})\,\Phi, & i \neq \gamma, \\ A^{i\beta\gamma} = -A^{\beta\gamma i}, & i = 1, 2, 3, \gamma = 1, \ldots, 4. \end{array}\right\} \tag{48}$$

Remarks. – As in Dirac's theory, when one starts from (45), one can define a real tensor:

$$T'^{\alpha\beta} = \tfrac{1}{2}(T^{\alpha\beta} + T^{\beta\alpha}), \tag{45′}$$

and also a real, symmetric tensor:

$$T''^{\alpha\beta} = \tfrac{1}{2}(T'^{\alpha\beta} + T'^{\beta\alpha}). \tag{45″}$$

By virtue of (47), the tensor $T''^{\alpha\beta}$ is likewise different from the Maxwellian tensor $M^{\alpha\beta}$. In the wave mechanics of the photon, the energy-impulse tensor is (45) [or (45′) or (45″)]. It is of the canonical type (20), while the Maxwellian tensor (46) is not. Meanwhile, thanks to (47), one understands how one can start from (45) and recover the results that were obtained in Maxwell's theory by utilizing (46). Moreover, in the case of the monochromatic wave, one will have:

$$c\,T^{4i} = \rho\,\mathbf{u}\,E \qquad \text{and} \qquad T^{44} = \rho\,E. \tag{49}$$

By virtue of (41), one will also recover the Maxwell tensor in the case of just one monochromatic Maxwellian wave whose scalar potential has been annulled.

10. Moments of impulse. – As in Dirac's theory, the spatial vector operator of the "orbital moment of impulse" with respect to the coordinate origin is defined by:

$$\mathbf{m}^0 = [\mathbf{r}, \mathcal{J}], \tag{50}$$

in which $\mathbf{r}$ is the (spatial) vector whose origin is at O and whose extremity is x, y, z.

One knows that $\mathbf{m}^0$ is not a first integral in the theory of the photon. However, the operator:

$$\mathbf{m} = \mathbf{m}^0 + \mathbf{m}^s, \tag{51}$$

in which:

$$m_1^s = \frac{h}{2\pi}\frac{i\mathcal{A}_2\,\mathcal{A}_3 + i\mathcal{B}_2\,\mathcal{B}_3}{2}, \ \ldots, \tag{52}$$

is a first integral. $\mathbf{m}$ is the operator that corresponds to the *total* moment of the photon. $\mathbf{m}^s$ is the *spin* operator. Conforming to the definition (20), the mean values of the orbital moment and spin will then be:

$$\mathbf{M}^0 = \Phi^*\frac{\mathcal{A}_4 + \mathcal{B}_4}{2}\,[\mathbf{r}, \mathcal{J}]\,\Phi, \tag{53}$$

$$\mathbf{N} = \frac{h}{4\pi}\Phi^*(\mathcal{B}_4\,\mathbf{s}_a + \mathcal{A}_4\,\mathbf{s}_b)\,\Phi, \tag{54}$$

respectively, in which:

$$s_x^{(a)} = s_{(a)}^1 = \mathcal{A}_1\,\mathcal{A}_2, \ldots$$
$$s_x^{(b)} = s_{(b)}^1 = \mathcal{B}_1\,\mathcal{B}_2, \ldots$$

The mean-value density of the total moment is obviously:

$$\mathbf{M} = \mathbf{M}^0 + \mathbf{N}. \tag{55}$$

We shall not reproduce the study of these space-time quantities here [**20**]. We shall add only a few words about the mean-value density of spin. N_x, N_y, N_z are the spatial components of a space-time vector (or, what amounts to the same thing, a completely-antisymmetric tensor of rank three):

$$\left.\begin{aligned} N_x = N^1 &= \frac{h}{2\pi}\Phi^*\frac{\mathcal{B}_4\,s_{(a)}^1 + \mathcal{A}_4\,s_{(b)}^1}{2}\Phi, \cdots, \\ N^4 &= -\frac{h}{2\pi}\Phi^*\frac{\mathcal{B}_4\,s_{(a)}^1 + \mathcal{A}_4\,s_{(b)}^1}{2}\Phi. \end{aligned}\right\} \tag{56}$$

Explicitly:

$$\left.\begin{aligned} \frac{2\pi}{h}N_z &= |\Phi_{11}|^2 - |\Phi_{22}|^2 - |\Phi_{33}|^2 + |\Phi_{44}|^2 \\ &\quad - |\Phi_{14}|^2 + |\Phi_{32}|^2 - |\Phi_{41}|^2 + |\Phi_{23}|^2, \\ \frac{2\pi}{h}N_4 &= \tfrac{1}{2}\{(\Phi_{11}^* - \Phi_{33}^*)(\Phi_{31} + \Phi_{13}) + (\Phi_{22}^* - \Phi_{44}^*)(\Phi_{42} + \Phi_{24}) \\ &\quad + (\Phi_{12}^* - \Phi_{34}^*)(\Phi_{32} + \Phi_{11}) + (\Phi_{21}^* - \Phi_{43}^*)(\Phi_{23} + \Phi_{41}) + \text{conj.}\} \end{aligned}\right\} \tag{57}$$

Upon utilizing the inversion formulas (28), we will get:

$$\left.\begin{aligned} \mathbf{N} &= \frac{1}{4c}\{-[\mathcal{V}^* H] + V^*\mathcal{H} + I^*\mathbf{Q} + \text{conj.} \\ N^4 &= \frac{1}{4c}\{+[\mathcal{V}^* H] + I^* Q^4 + \text{conj.} \end{aligned}\right\} \tag{58}$$

or rather, in tensorial form:

$$S^{\alpha\beta\gamma} = -\frac{1}{4c}\{U^{\alpha\beta}P_*^{\gamma} + U^{\beta\gamma}P_*^{\alpha} + U^{\gamma\alpha}P_*^{\beta} + I^* Q^{\delta}\} + \text{conj.} \tag{58′}$$

in which $\alpha\beta\gamma\delta$ is an even permutation of the numbers 1, 2, 3, 4. The tensor (58′) is completely antisymmetric in the indices $\alpha\beta\gamma$.

What is the tensor in Maxwell's theory that corresponds to (58′)?

E. Henriot was the first to draw attention to quantities of that kind just a few years ago ([1]). By a deeper analysis of couples and moments of electromagnetic impulse, E. Henriot was led to define a third-order tensor ($M^{\alpha\beta\gamma}$) in the classical Maxwell theory that represented the moments and flux of electromagnetic moments of the second kind, or "momentors." That tensor, which is antisymmetric in the extreme indices α and β, appears quite naturally in the expression for the law of conservation of moments of impulse in Maxwell's theory. It resembles the tensor (58′), but it is not identical to it, as one sees immediately upon remarking that $M^{\alpha\beta\gamma}$ is not completely antisymmetric. It is easy to write down the tensor that corresponds to $M^{\alpha\beta\gamma}$ in the theory of the photon. That tensor, which is nothing but (48), does not have the canonical type (20).

From a general standpoint, one is now in possession of two types of moments of electromagnetic impulse in Maxwell's theory: The moments "of the first kind," which are calculated by means of the Poynting vector, and E. Henriot's moments "of the second kind," or "momentors." The question is then to know what the links that exist between those two types of moments. In what cases does one employ one of them or the other one? The momentor leads to some good results for the monochromatic plane wave. For the electric dipole wave, Sommerfeld showed that one obtains the exact formula for the flux at infinity by utilizing the moment of the first kind. The momentor, in its present form, does not yield satisfactory results for that dipole wave.

On the contrary, in the theory of the photon, the general theory shows that the total moment is always the sum of two terms: The orbital moment and the spin. Moreover, we have shown [**20**] that this definition does yield good results in the case of electric dipole and quadripolar waves, as well as in the case of the plane wave. In the next chapter, that study will be make more precise, as well as generalized and envisioned in a new manner.

11. The operator $(\mathcal{A}_4 + \mathcal{B}_4) / 2$. – The physical quantity that the operator $\alpha_4\, m_0$ in the Dirac equation corresponds to is "proper mass." In the case of a monochromatic plane wave, the integral:

$$\int^t \overline{m}_0 c\, dt = m_0\, c \int^t dt \iiint_{\mathcal{E}} \psi^*\, \alpha_4\, \psi\, dx\, dy\, dz \tag{59}$$

is nothing but the action integral:

$$\int^t m_0 c \sqrt{1-\frac{v^2}{c^2}}\, dt$$

of relativistic dynamics ([**8**], pp. 223). The integration over the spatial variables x, y, z in (59) extends to the entire space $\mathcal{E}$ of those variables.

([1]) See [**11**].

Similarly, the operator $\mu_0 (\mathcal{A}_4 + \mathcal{B}_4) / 2$ corresponds to the physical quantity of the "proper mass of the photon" here. Conforming to the definition (20), the mean-value density that relates to the operator is $(\mathcal{A}_4 + \mathcal{B}_4) / 2$:

$$\Omega_{(1)} = \Phi^* \mathcal{B}_4 \mathcal{A}_4 \Phi. \tag{60}$$

It is an invariant. Explicitly:

$$\left.\begin{aligned} \Omega_{(1)} \equiv -|\Phi_{11}|^2 + |\Phi_{22}|^2 - |\Phi_{33}|^2 - |\Phi_{44}|^2 \\ + |\Phi_{13}|^2 + |\Phi_{24}|^2 + |\Phi_{31}|^2 + |\Phi_{42}|^2 \\ - |\Phi_{12}|^2 - |\Phi_{31}|^2 + |\Phi_{14}|^2 + |\Phi_{32}|^2 \\ - |\Phi_{21}|^2 - |\Phi_{43}|^2 - |\Phi_{41}|^2 + |\Phi_{23}|^2, \end{aligned}\right\} \tag{61}$$

or rather, upon utilizing the inverse formulas (28):

$\mu_0 \, c^2 \, \Omega_{(1)} =$

$$\frac{1}{4}\left\{|H|^2 - |\mathcal{H}|^2 + \left(\frac{2\pi\mu_0 c}{h}\right)^2 (|\mathcal{V}|^2 - |V|^2) - \left(\frac{2\pi\mu_0 c}{h}\right)^2 (|Q|^2 - |Q^4|^2) - (|I|^2 - |J|^2)\right\}. \tag{61′}$$

The mean value of μ_0 will then be:

$$\bar{\mu}_0 = \mu_0 \iiint_{\mathcal{E}} \Omega_{(1)} dx\, dy\, dz\,,$$

in which the integration is performed over the entire domain $\mathcal{E}$ of the variables x, y, z.

Now consider the case of a monochromatic plane wave that is Maxwellian, to simplify things. The solution will then depend upon three constants a, b, c; upon setting:

$$\left.\begin{array}{l} P = e^{\frac{2\pi i}{h}(Et - pz)}, \quad \Delta c = E + \mu_0 c^2, \\ \mathcal{V}_x = (b-c)\dfrac{hc}{\pi\Delta}P, \quad \mathcal{V}_y = i(b+a)\dfrac{hc}{\pi\Delta}P, \quad \mathcal{V}_z = d\,\dfrac{hc}{\pi\Delta}P, \\ V = d\,\dfrac{hc}{\pi\Delta}\cdot\dfrac{pc}{E}P, \\ \text{one will have:} \\ \left\{\begin{array}{l} H_x = -2i\dfrac{E}{\Delta}(b-a)P, \quad H_y = 2\dfrac{E}{\Delta}(b+a)P, \qquad H_z = -2id\,\dfrac{E}{\Delta}\dfrac{\mu_0 c^2}{\Delta}, \\ \mathcal{H}_x = -2\dfrac{pc}{\Delta}(b+a)P, \quad \mathcal{H}_y = -2i\dfrac{pc}{\Delta}(b-a)P, \quad \mathcal{H}_y = 0. \end{array}\right. \end{array}\right\} \tag{63}$$

E and p are coupled by the condition:

$$E^2 = p^2\,c^2 + \mu_0^2\,c^4\,. \tag{64}$$

One easily finds that for that plane wave:

$$\Omega_{(1)} = \rho\,\frac{\mu_0 c^2}{E}\,, \tag{65}$$

in which ρ is the value of the density (36) in the case of the wave (63).

Due to (64), (65) can be written:

$$\Omega_{(1)} = \rho\sqrt{1 - \frac{p^2c^2}{E^2}}\,. \tag{66}$$

Integrate this over all of the space $\mathcal{E}$, while taking into account the normalization condition:

$$\iiint_{\mathcal{E}} \rho\;dx\,dy\,dz = 1.$$

One will then get:

$$\bar{\mu}_0 = \mu_0\sqrt{1 - \frac{p^2c^2}{E^2}}$$

and

$$\int^t \bar{\mu}_0 c\,dt = \int^t \mu_0 c\sqrt{1 - \frac{p^2c^2}{E^2}}\;dt\,, \tag{67}$$

which is indeed the action integral for the photon that is analogous to (59′), since one easily sees that the speed of the photon is:

$$c = \frac{pc^2}{E}.$$

Remarks. – Up to terms in $\left(\frac{2\pi\mu_0 c}{h}\right)^2$, the right-hand side of (61) is nothing but the transposition of the Lagrangian function that was utilized in Maxwell's theory (I, § 4, remark) into complex fields.

The Lagrange equations of the variational problem:

$$\delta\iiint\int \Omega_{(1)}\, dx\, dy\, dz\, dt = 0 \tag{68}$$

are

$$\frac{\delta\Omega_{(1)}}{\delta P_\alpha} = 0 \qquad \text{and} \qquad \frac{\delta\Omega_{(1)}}{\delta P_\alpha^*} = 0.$$

These are precisely Maxwell's equations (30) and their conjugates.

12. On the annihilation solution. – In the case of the annihilation solution, the definition (20) gives:

$$\begin{aligned} \rho &= 0, \\ \Omega_{(1)} &= -1, \end{aligned} \tag{69}$$

which are results that are obviously unacceptable; it is good to reflect upon that fact.

We obtained (69) by starting with the definition (20). It seems that the necessity of taking (20), instead of (16), comes from a relativistic effect which must unavoidably be taken into account if the photon is to move with a velocity that is close to c. However, the annihilation solution represents the state of a photon whose energy, impulse, mass, and spin are zero. Relativistic effects are therefore no longer relevant. We are thus led to think that it is more natural to return to the definition (16) for the annihilation solution.

With that, one will find that the probability density of presence is:

$$\delta \equiv \Phi^{(0)}\,\Phi^{(0)} = 4\lambda^2,$$

and the mean-value density of proper mass is:

$$m \equiv \mu_0\, \Phi^{(0)}\, \frac{\mathcal{A}_4 + \mathcal{B}_4}{2}\, \Phi^{(0)} = 0.$$

This time, the results are acceptable. The normalization of the wave $\Phi^{(0)}$ is performed, as usual, by expressing that:

$$\iiint_{\mathcal{E}} \delta\; dx\, dy\, dz = 1;$$

hence, upon letting v denote the volume of the space $\mathcal{E}$ of the variables x, y, z:

$$4\lambda^2\, v = 1. \tag{70}$$

Ordinarily, the volume v is infinite. One can avoid that difficulty, as in the case of the monochromatic plane wave, by first bounding the space $\mathcal{E}$, and then letting it increase indefinitely in turn.

Remark. – The following remark will confirm the ideas that were proposed above in regard to the definitions (20) and (16):

Let Φ be a monochromatic, plane, Maxwellian wave that represents a photon with velocity zero. That wave will be nothing but (63) when one sets $p = 0$. For example, one will then find that:

$$\left.\begin{array}{c} \mu_0 \Phi^* \mathcal{B}_4 \mathcal{A}_4 \Phi = \mu_0 \Phi^* \dfrac{\mathcal{B}_4 + \mathcal{A}_4}{2} \Phi, \\[2ex] \Phi^* \dfrac{\mathcal{B}_4 + \mathcal{A}_4}{2} \Phi = \Phi^* \Phi, \\[2ex] \Phi^* \dfrac{\mathcal{B}_4 + \mathcal{A}_4}{2} \dfrac{h}{2\pi i} \dfrac{\partial \Phi}{\partial t} = \Phi^* \dfrac{h}{2\pi i} \dfrac{\partial \Phi}{\partial t}, \\[3ex] \Phi^* \dfrac{\mathcal{B}_4\, \mathbf{s}^{(a)} + \mathcal{A}_4\, \mathbf{s}^{(b)}}{2} \dfrac{h}{2\pi i} \dfrac{\partial \Phi}{\partial t} = \Phi^* \dfrac{\mathbf{s}^{(a)} + \mathbf{s}^{(b)}}{2} \dfrac{\partial \Phi}{\partial t}. \end{array}\right\} \tag{71}$$

The left-hand sides of these equalities represent the mean-value densities of mass, presence, energy, and spin, respectively, which are densities that one calculates with the aid of formula (20). On the contrary, the right-hand sides of these equalities represent the mean-value densities of mass, presence, energy, and spin that are calculated by adopting the definition (16). The relations (71) then express the idea that the definitions (20) and (16) will be equivalent when one is dealing with Maxwellian photons of velocity zero.

13. Some important matrix elements. – Let Φ^p (p = 1, 2, …) denote a complete system of wave functions that satisfy the photon equations (I), (II). The annihilation function will always be represented by Φ^0. Let $\mathcal{A}$ be any of the sixteen matrices of type (a) that figure in the table (11) – (15), while $\mathcal{B}$ is the matrix of type (b) that corresponds to $\mathcal{A}$. The densities of elements that correspond to the operators:

$$\tfrac{1}{2}(\mathcal{A} + \mathcal{B}) \tag{72}$$

are

$$\left.\begin{array}{lr}(\Phi^p)^* \dfrac{\mathcal{B}_4\mathcal{A}+\mathcal{A}_4\mathcal{B}}{2}\Phi^{p'}, & (a) \\ \Phi^0 \dfrac{\mathcal{B}_4\mathcal{A}+\mathcal{A}_4\mathcal{B}}{2}\Phi^p, & (b) \\ (\Phi^p)^* \dfrac{\mathcal{B}_4\mathcal{A}+\mathcal{A}_4\mathcal{B}}{2}\Phi^0. & (c)\end{array}\right\} \tag{73}$$

The element $\Phi^0 \to \Phi^0$ was examined in the preceding paragraph. Thanks to the inversion formulas, one can write those expressions as functions of the tensorial quantities (23-27). One will then see (cf., §§ 7, 9, 10) that the elements (73.*a*) that are linked with the transition $p' \to p$ are not zero. By contrast, the elements (73.*b*, *c*) will be identically zero for:

$$\begin{aligned}\mathcal{A} = {} & \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, 1, \\ & i\mathcal{A}_1\mathcal{A}_4, \ldots, i\mathcal{A}_2\mathcal{A}_3, \ldots \\ & \mathcal{A}_4 .\end{aligned}$$

Apart from $\mathcal{A}_4$, these matrices are the ones that appear in the definitions of the Maxwellian quantities. For the other matrices of type (72) in the table (11-15), the elements (73.*b*) will define non-Maxwellian quantities $\sigma_1, \ldots, \sigma_4, I_{(2)}$ that are associated with the state *p*.

Things are different for the operators:

$$\tfrac{1}{2}(\mathcal{A}-\mathcal{B}).$$

Here, we shall consider only the case of the operators (12), which we shall appeal to in what follows. Upon utilizing the notations (17), the densities of matrix elements here will be:

$$\left.\begin{array}{lr}(\Phi^p)^* \dfrac{\mathcal{B}_4 u^\alpha_{(a)} - \mathcal{A}_4 u^\alpha_{(b)}}{2}\Phi^{p'}, & (a) \\ \Phi^0 \dfrac{\mathcal{B}_4 u^\alpha_{(a)} - \mathcal{A}_4 u^\alpha_{(b)}}{2}\Phi^p, & (b) \\ (\Phi^p)^* \dfrac{\mathcal{B}_4 u^\alpha_{(a)} - \mathcal{A}_4 u^\alpha_{(b)}}{2}\Phi^0. & (c)\end{array}\right\} \tag{74}$$

We remark that the operator $\frac{1}{2}(u^4_{(a)} - u^4_{(b)}) - \frac{1}{2}(1-1)$ corresponds to a density $\Phi^* (\mathcal{B}_4 - \mathcal{A}_4)\, \Phi / 2$ that is not identically zero. That element is necessary for reasons of variance, as we shall see. Upon using the inversion formula, one will find that:

$$\left.\begin{aligned}
&\Phi^* \frac{\mathcal{B}_4 \boldsymbol{\mathcal{A}} - \mathcal{A}_4 \boldsymbol{\mathcal{B}}}{2} \Phi \\
&= \frac{2\pi\sqrt{\mu_0}}{h}\frac{1}{4}\left\{\boldsymbol{\mathcal{V}}^* I_{(1)} + \boldsymbol{\mathcal{H}}^* \sigma_4 \frac{h}{2\pi\mu_0 c} - [\mathbf{H}^*\boldsymbol{\sigma}]\frac{h}{2\pi\mu_0 c} + I_{(1)}^* \boldsymbol{\mathcal{V}} + \sigma_4^* \boldsymbol{\mathcal{H}} \frac{h}{2\pi\mu_0 c} + [\boldsymbol{\sigma}^* \mathbf{H}]\frac{h}{2\pi\mu_0 c}\right\}, \\
&\Phi^* \frac{\mathcal{B}_4 - \mathcal{A}_4}{2}\Phi = \frac{2\pi\sqrt{\mu_0}}{h}\frac{1}{4}\left\{-V^* I_{(1)} - (\boldsymbol{\mathcal{H}}^*\boldsymbol{\sigma})\frac{h}{2\pi\mu_0 c} - I_{(1)}^* V + (\boldsymbol{\sigma}^*\boldsymbol{\mathcal{H}})\frac{h}{2\pi\mu_0 c}\right\}.
\end{aligned}\right\} \tag{75}$$

It will then result that:

1. The elements (74.*a*) will be zero if the non-Maxwellian quantities that are linked with the states p' and p are zero.

2. The elements (74.*b*) will be equal to:

$$\left.\begin{aligned}
-\Phi^* \frac{\mathcal{B}_4 \boldsymbol{\mathcal{A}} - \mathcal{A}_4 \boldsymbol{\mathcal{B}}}{2}\Phi^p &= -\frac{2\pi\sqrt{\mu_0}}{h}\boldsymbol{\mathcal{V}}^p, \\
+\Phi^* \frac{\mathcal{B}_4 - \mathcal{A}_4}{2}\Phi &= \frac{2\pi\sqrt{\mu_0}}{h} V^p.
\end{aligned}\right\} \tag{76}$$

The elements (74.*c*) are the conjugate expressions to (76).

Those densities play an important role in the theory of the interaction of light and matter.

14. Non-Maxwellian quantities. – The wave mechanics of the photon permits one to have (un-annihilated) states that correspond to zero electromagnetic fields and potentials. One will then have:

$$\left.\begin{gathered}
\Phi_{11} = -\Phi_{22} = \Phi_{33} = -\Phi_{44}, \\
\Phi_{13} = \Phi_{24}, \quad \Phi_{31} = \Phi_{42}, \quad \Phi_{12} = \Phi_{34}, \quad \Phi_{21} = \Phi_{43}, \\
\Phi_{14} = \Phi_{32} = \Phi_{23} = \Phi_{41} = 0.
\end{gathered}\right\} \tag{77}$$

Meanwhile, the photons that are represented by such functions can transport energy. For example, consider the case of the monochromatic plane wave that propagates along *Oz*. With the notations that were introduced above, that non-Maxwellian wave will be defined by ([**17**], pp. 51):

$$\left.\begin{gathered}
I_{(1)} = \sigma_x = \sigma_y = 0, \\
\sigma_z = i\,p\,c\,d'P, \quad \sigma_4 = i\,e\,d'P, \\
I_{(2)} = \mu_0 c\,d'P.
\end{gathered}\right\} \tag{78}$$

(d' is an arbitrary constant.) Only the components $\Phi_{\alpha\beta}$ of spin zero are non-zero. The functions (78) then define a photon state of zero spin, but energy E. The calculation of mean values will then give:

$$N_x = N_y = N_z = N_4 = 0,$$

and

$$\rho = -\frac{\pi i}{2hc} I_{(2)}\,\sigma_4\,, \qquad \rho\,u_z = -\frac{\pi i}{2hc} I_{(2)}\,\sigma_z\,,$$

so:

$$u_z = \frac{pc^2}{E},$$

which is indeed the relativistic expression for velocity as a function of the quantity of motion and energy.

Spherical waves give rise to some remarkable analogies (see VI, § 2, Remark *b*). At present, no physical manifestation of these non-Maxwellian photon states is known. It is possible that these non-Maxwellian states do not exist *in vacuo*, and that the non-Maxwellian quantities are non-zero only in matter. The question remains open.

APPENDIX

A general method for transforming expressions $\Phi^* \mathcal{B}\mathcal{A}\,\Phi$.

Let $\mathcal{A}^p$ denote any of the sixteen matrices of type (*a*) that figure in the table (11-15), and let $\mathcal{B}^p$ denote the corresponding matrices of type (*b*). One has:

$$\left.\begin{aligned} (\mathcal{A}^p)_{\beta\gamma,\lambda\varepsilon} &= \alpha^p_{\beta\lambda}\,\delta_{\gamma\varepsilon}, \\ (\mathcal{B}^p)_{\beta\gamma,\lambda\varepsilon} &= \beta^p_{\gamma\varepsilon}\,\delta_{\beta\lambda}. \end{aligned}\right\} \tag{86}$$

α^p is any of the sixteen Dirac matrices, while $\alpha^p_{\beta\lambda}$ is the element that is found on row β and column λ of the matrix α^p. By virtue of the definitions (3) of the matrices $\mathcal{B}^p$, the matrices β^p will be equal to $\pm\,\alpha^p$. W. Pauli proved the following general theorem:

$$\sum_{p=1}^{16} \alpha^p_{\rho\sigma}\alpha^p_{\bar\rho\bar\sigma} = 4\,\delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma}\,, \qquad \rho,\ \sigma,\ \bar\rho\,,\ \bar\sigma = 1, \ldots, 4. \tag{89}$$

Thanks to that theorem, we shall prove some interesting properties of the bilinear quantities in $\Phi^\times_{\alpha\beta}$ and $\Phi'_{\alpha\beta}$, where $\Phi_{\alpha\beta}$ and $\Phi'_{\alpha\beta}$ are two systems of wave functions. Like theorem (87), these properties are independent of the special form of the α matrices. It is based solely upon the definitions (86) of the matrices $\mathcal{A}$ and $\mathcal{B}$ and the commutation relations:

$$\alpha_\beta\,\alpha_\gamma + \alpha_\gamma\,\alpha_\beta = 2\;\delta_{\beta\gamma}. \tag{88}$$

By virtue of (87), one will have:

$$\sum_p \alpha^p_{\rho\sigma}(\alpha^q\alpha^p\alpha^r)_{\bar\rho\bar\sigma} = 4\alpha^q_{\bar\rho\sigma}\alpha^r_{\rho\bar\sigma}, \qquad p, q, r = 1, \ldots, 16. \tag{89}$$

Multiply both sides of (89) by $\Phi^\times_{\rho\sigma}\;\Phi'_{\bar\rho\bar\sigma}$ and sum over $\rho, \sigma, \bar\rho, \bar\sigma$. If the summation sign is implicit then one will get:

$$[\Phi^*_{\rho\sigma}\alpha^p_{\rho\sigma}][(\alpha^q\alpha^p\alpha^r)_{\bar\rho\bar\sigma}\Phi'_{\bar\sigma\bar\rho}] = 4\,\Phi^*_{\rho\sigma}\alpha^q_{\bar\rho\sigma}\alpha^r_{\rho\bar\sigma}\Phi'_{\bar\sigma\bar\rho}. \tag{90}$$

These identities can be written with the aid of the matrices $\mathcal{A}$ and $\mathcal{B}$ and the annihilation solution $\Phi^0 = (\alpha_4)$:

$$\sum_p \Phi^*\mathcal{B}_4\,\mathcal{A}^p\Phi^0\cdot\Phi^0\mathcal{B}_4\,\mathcal{A}^q\mathcal{A}^p\mathcal{A}^r\Phi' = \pm\,4\;\Phi^*\,(\mathcal{B}^q)^*\,\mathcal{A}^r\,\Phi'. \tag{91}$$

In the right-hand side of (91), one takes the + sign when $\alpha^q = +\beta^q$ and the – sign when $\alpha^q = -\beta^q$.

In the same way, if one starts with the identities:

$$\sum_{p=1}^{16}\beta^p_{\rho\sigma}\beta^p_{\bar\rho\bar\sigma} = 4\,\delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma}$$

then one can show that:

$$\sum_p \Phi^*\mathcal{A}_4\,\mathcal{B}^p\Phi^0\cdot\Phi^0\mathcal{A}_4\,\mathcal{B}^q\mathcal{B}^p\mathcal{B}^r\Phi' = \pm\,4\;\Phi^*\,(\mathcal{B}^q)^*\,\mathcal{A}^r\,\Phi'. \tag{92}$$

In the right-hand side of (92), one takes the + sign when $\alpha^r = +\beta^r$ and the – sign when $\alpha^r = -\beta^r$.

One can deduce from these very general identities, in particular, the expression for the mean-value densities that are associated with the operators in the table (11-15) as functions of the tensorial quantities in the theory of the photon. In order to see that, it will suffice to remark that, up to constant factors, the expressions:

$$\Phi^0\mathcal{A}_4\,\mathcal{B}^q\,\Phi \qquad \text{and} \qquad \Phi^*\mathcal{B}_4\,\mathcal{A}^p\,\Phi^0$$

are equal to the aforementioned tensorial quantities and their conjugates. Finally, one sets $\mathcal{A}^q = \mathcal{A}_4$ and $\mathcal{B}^q = \mathcal{B}_4$ in (91); hence:

$$\sum_p \Phi^*\mathcal{B}_4\,\mathcal{A}^p\Phi^0\cdot\Phi^0\mathcal{B}_4\,\mathcal{A}_4\mathcal{A}^p\mathcal{A}^r\Phi' = \pm\,4\;\Phi^*\mathcal{B}_4\,\mathcal{A}^r\,\Phi'. \tag{93}$$

Similarly, upon setting $\mathcal{A}^r = \mathcal{A}_4$ and $\mathcal{B}^r = \mathcal{B}_4$ in (92), one will have:

$$\sum_p \Phi^* \mathcal{A}_4 \mathcal{B}^p \Phi^0 \cdot \Phi^0 \mathcal{A}_4 \mathcal{B}^q \mathcal{B}^p \mathcal{B}_4 \Phi' = \mp 4\, \Phi^* \mathcal{B}^q \mathcal{A}_4 \Phi'. \tag{94}$$

In the right-hand side of (94), one takes the – sign when $\mathcal{B}^q = (\mathcal{B}^q)^*$ and the + sign when $\mathcal{B}^q = -(\mathcal{B}^q)^*$.

Thanks to the identities (93) and (94), all of the mean-value densities that are associated with the operators in the table (11-15) can be expressed as functions of the tensorial quantities in the theory of the photon *with no numerical specialization of the Dirac α matrices.*

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CHAPTER VI

SPHERICAL WAVES IN THE THEORY OF THE PHOTON

1. Classical theory. – The problem of multipolar waves has been the object of numerous works in the classical Maxwell theory. We are mainly interested in the question of the moments of impulse that are transported by those waves.

For a *divergent* electric dipolar wave, Sommerfeld especially has calculated the flux of the moment of electromagnetic impulse across a closed surface that tends to infinity; he used the classical expressions:

$$\mathbf{M}\,(P) = [\mathbf{r} \cdot \mathbf{G}\,(P)], \qquad\qquad \mathbf{G} = \frac{1}{c}[\mathbf{H} \cdot \boldsymbol{\mathcal{H}}]. \tag{1}$$

($\mathbf{M}$ = density of the moment of electromagnetic impulse with respect to the point O, $\mathbf{r} \equiv O\mathbf{P}$) $\mathbf{G}$ is the electromagnetic impulse density vector. One can likewise employ the densities of impulse flux that are give by the components T^{ij} (i, j = 1, 2, 3) of the Maxwell tensor.

The same method can apply to the case of a diverging multipolar wave of any order. To our knowledge, those calculations, which are quite complicated, have not been done yet.

W. Heitler ([1]) has made a deep study of *stationary* multipolar waves that are contained inside a sphere with perfectly-reflecting walls. These stationary l-polar waves are the sum of a divergent l-polar wave and a convergent l-polar wave; they no longer have a pole at the origin (see § **2**). W. Heitler obtained a classification of the stationary multipolar waves. They are characterized by two "quantum numbers" l, m; the system thus-formed is complete and orthogonal. In addition, for an l-polar wave (l, m) of frequency ν and total energy U, the total angular moment will be:

$$M_z = m\frac{U}{2\pi\nu}, \qquad M_x = 0, \qquad M_y = 0.$$

(Oz is the spherical coordinate axis.)

The theory of the photon permits one to envision those questions in terms of the general principles of wave mechanics, and to generalize their meaning and make them more precise from the macroscopic viewpoint.

2. Spherical waves. – Spherical waves can be obtained in L. de Broglie's theory of light in a manner that is analogous to the one that is used in Dirac's theory of the electron.

In the case of a monochromatic wave:

([1]) See [**22**].

$$\Phi_{\beta\gamma}(x, y, z, t) = \varphi_{\beta\gamma}(x, y, z)\; e^{\frac{2\pi i W}{h}t},$$

the equations [(I), Chap. V] are written:

$$\sum_{\varepsilon}\left\{\frac{2\pi i}{h}\left[\frac{W}{c}\delta_{\beta\varepsilon} - \mu_0 c\,(\alpha_4)_{\beta\varepsilon}\right] - \frac{\partial}{\partial x}(\alpha_1)_{\beta\varepsilon} - \frac{\partial}{\partial y}(\alpha_2)_{\beta\varepsilon} - \frac{\partial}{\partial z}(\alpha_3)_{\beta\varepsilon}\right\}\varphi_{\varepsilon\gamma} = 0, \tag{I}$$

and the equations [(II), Chap. V] are written:

$$\sum_{\varepsilon}\left\{\frac{2\pi i}{h}\left[\frac{W}{c}\delta_{\gamma\varepsilon} + \mu_0 c\,(\alpha_4)_{\gamma\varepsilon}\right] - \frac{\partial}{\partial x}(\alpha_1)_{\gamma\varepsilon} - \frac{\partial}{\partial y}(\alpha_2)_{\gamma\varepsilon} - \frac{\partial}{\partial z}(\alpha_3)_{\gamma\varepsilon}\right\}\varphi_{\varepsilon\beta} = 0. \tag{II}$$

The system (I) will then be composed of the four Dirac systems that are obtained by setting $\gamma = 1, 2, 3, 4$ in (I).

One passes from (I) to (II) formally by changing i into $-i$, W into $-W$, and $\varphi_{\varepsilon\gamma}$ into $\varphi_{\beta\varepsilon}$. When one appeals to the known results of Dirac's theory, one can write down a solution $\Phi^{\mathrm{I}}_{\beta\gamma}$ of system (I) and a solution $\Phi^{\mathrm{II}}_{\beta\gamma}$ of system (II) and then identify $\Phi^{\mathrm{I}}_{\beta\gamma} \equiv \Phi^{\mathrm{II}}_{\beta\gamma}$.

A. THE SOLUTION $\Phi^{\mathrm{I}} = \Phi^{\mathrm{I}}\; e^{\frac{2\pi i w}{h}t}$. – The formulas that were obtained by C. G. Darwin in the solution of the Dirac equations ([1]) permit us to write down two types of solutions of equations (I) directly:

$$\left.\begin{array}{ll}
\text{Type } (+) & \text{Type } (-) \\
\varphi_{1\gamma} = i\,F^l_{+\gamma}\,Y^m_{l+1}, & i\,(l-m)F^l_{-\gamma}\,Y^m_{l-1}, \\
\varphi_{2\gamma} = -i\,F^l_{+\gamma}\,Y^{m-1}_{l+1}, & i\,(l+m-1)F^l_{-\gamma}\,Y^{m-1}_{l-1}, \\
\varphi_{3\gamma} = (l-m+1)\,G^l_{+\gamma}\,Y^m_l, & G^l_{-\gamma}\,Y^m_l, \\
\varphi_{4\gamma} = (l+m)\,G^l_{+\gamma}\,Y^{m-1}_l, & -G^l_{-\gamma}\,Y^{m-1}_l, \\
l = 0, 1, 2, \cdots & l = 1, 2, \ldots \\
m = -l, -(l-1), \cdots, (l+1), & m = -l, -(l-1), \cdots, l.
\end{array}\right\} \tag{3}$$

$F^l_{+\gamma}$, $G^l_{+\gamma}$, $F^l_{-\gamma}$, $G^l_{-\gamma}$ are radial functions; for any index γ, they will satisfy the equations:

([1]) See [**8**], pp. 231, *et seq.*

$$\left.\begin{aligned}
\frac{2\pi}{h}\left(\frac{W}{c}+\mu_0 c\right)F_+^l+\dot{G}_+^l-\frac{l}{r}G_+^l&=0,\\
-\frac{2\pi}{h}\left(\frac{W}{c}+\mu_0 c\right)G_+^l+\dot{F}_+^l+\frac{l+2}{r}F_+^l&=0,\\
\frac{2\pi}{h}\left(\frac{W}{c}+\mu_0 c\right)F_-^l+\dot{G}_-^l+\frac{l+1}{r}G_-^l&=0,\\
-\frac{2\pi}{h}\left(\frac{W}{c}+\mu_0 c\right)G_-^l+\dot{F}_-^l-\frac{l-2}{r}F_-^l&=0.
\end{aligned}\right\}\tag{4}$$

(The dot above the functions F and G indicates a derivative with respect to r.)

The $Y_l^m = Y_l^m(\theta, \varphi)$ are Laplace's spherical functions:

$$\left.\begin{gathered}
Y_l^m(\theta,\varphi)=(l-m)!\,e^{im\varphi}\frac{d^{l+m}}{(d\cos\theta)^{l+m}}\left(\frac{\cos^2\theta-1}{2^l\cdot l!}\right)r\cdot\sin^m\theta\\
l=0,1,2,\cdots\\
m=-l,-(l-1),\cdots,+l.
\end{gathered}\right\}\tag{5}$$

(r, θ, φ are the spherical coordinates about the z-axis.)

Each of the two systems of functions (3a and b) satisfies (I). However, in view of the identification that was mentioned above, one must consider a more general solution, namely:

$$\left.\begin{aligned}
\varphi_{1\beta}^1&=Y_{l+1}^{m+\varepsilon_\beta}\,i\,F_{+\beta}^l+Y_l^{m+\varepsilon_\beta}[i\,F_{+\beta}^{l-1}+i\,(l+1-m-\varepsilon_\beta)F_{-\beta}^{l+1}]+Y_{l-1}^{m+\varepsilon_\beta}i\,(l-m-\varepsilon_\beta)F_{-\beta}^l,\\
\varphi_{2\beta}^1&=Y_{l+1}^{m-1+\varepsilon_\beta}(-i\,F_{+\beta}^l)+Y_l^{m+\varepsilon_\beta}[-i\,F_{+\beta}^{l-1}+i\,(l+m+\varepsilon_\beta)F_{-\beta}^{l+1}]+Y_{l-1}^{m-1+\varepsilon_\beta}i\,(l+m+\varepsilon_\beta-1)F_{-\beta}^l,\\
\varphi_{3\beta}^1&=Y_{l+1}^{m+\varepsilon_\beta}\,i\,G_{+\beta}^{l+1}+Y_l^{m+\varepsilon_\beta}[G_{+\beta}^l+(l-m-\varepsilon_\beta+1)G_{+\beta}^l]+Y_{l-1}^{m+\varepsilon_\beta}\,(l-m-\varepsilon_\beta)G_{+\beta}^{l-1},\\
\varphi_{4\beta}^1&=Y_{l+1}^{m-1+\varepsilon_\beta}(-G_{-\beta}^{l+1})+Y_l^{m-1+\varepsilon_\beta}[-G_{+\beta}^l+(l+m+\varepsilon_\beta)G_{+\beta}^l]+Y_{l-1}^{m-1+\varepsilon_\beta}i\,(l+m+\varepsilon_\beta-1)G_{-\beta}^{l-1}.
\end{aligned}\right\}\tag{6}$$

The ε_β are undetermined whole numbers; we immediately point out that the identification $\varphi^{\mathrm{I}} = \varphi^{\mathrm{II}}$ will show that $\varepsilon_1 = \varepsilon_3 = 0$, $\varepsilon_2 = \varepsilon_4 = 1$.

We remark that by virtue of (4), the functions ([1]):

$$F_+^{l-1},\quad G_+^l,\qquad F_-^{l+1},\quad G_-^l \tag{7}$$

all satisfy the same second-order differential equation:

([1]) The functions F_+^{-1} and G_-^0 that one obtains by setting $l = 0$ in (7) are set to zero identically.

$$\ddot{f}_l + \frac{2}{r}\,\dot{f}_r + \left[\frac{4\pi^2}{h^2}\left(\frac{W^2}{c^2} - \mu_0^2 c^2\right) - \frac{l(l+1)}{r^2}\right] f_l = 0 \tag{8}$$

for any value of the index β.

Such an equation (8) has two linearly-independent solutions; one of them is everywhere finite, while the other one has a pole at the origin.

Conforming to the theory of retarded potentials and the series developments in which the multipolar waves appear, one takes:

$$f_l^d(kr) = \frac{C_l}{i\sqrt{kr}}\left\{J_{l+\frac{1}{2}}(kr) + i(-1)^l J_{-l-\frac{1}{2}}(kr)\right\} \qquad \left(h = \frac{2\pi\nu}{c}\right) \tag{9}$$

for a diverging spherical wave, in which the C_l are arbitrary constants (which are determined by the normalization conditions), and $J_{l+\frac{1}{2}}$, $J_{-l-\frac{1}{2}}$ are the well-known Bessel functions.

For a converging spherical wave, the radial function is deduced from (9) by changing i into $-i$ (1); hence:

$$f_l^c(kr) = \frac{D_l\, e^{ip}}{-i\sqrt{kr}}\left\{J_{l+\frac{1}{2}}(kr) - i(-1)^l J_{-l-\frac{1}{2}}(kr)\right\}, \tag{10}$$

in which D_l are arbitrary constants, and e^{ip} is a phase factor that we have introduced as a result of the calculations above. Moreover, the angular distributions of the functions $\varphi_{\alpha\beta}$ are obviously the same in the two cases of diverging and converging waves.

Finally, consider a wave that is composed of the superposition of two spherical waves with the same center, one of which $\Phi^d = \varphi^d\, e^{2\pi i\nu t}$ is diverging, and the other of which $\Phi^c = \varphi^c\, e^{2\pi i\nu t}$ is converging, and they have the same frequency, the same amplitude ($C_l = D_l$), the same angular distribution, and are p out of phase; hence, one will write $p = \pi$ in (10). By virtue of (9) and (10), the radial function will then be:

$$f_l^s(kr) = \frac{C_l}{i\sqrt{kr}} J_{l+\frac{1}{2}}(kr)\,. \tag{11}$$

The waves thus-obtained are stationary. They no longer have a pole at the origin, since the radial function is (11). That is the case that was envisioned by W. Heitler (*loc. cit.*) (2).

Depending upon whether one is dealing with a diverging, converging, or stationary spherical wave, the functions (7) will be proportional to (9), (10), or (11), resp., no matter what the value of the index β that the functions (7) are affected with. The proportionality factors will naturally depend upon the indices $+\beta$, $-\beta$, as well as t.

(1) See, for example, [**23**], pp. 84.

(2) When these waves are contained in a sphere of finite radius, the frequencies ν must satisfy a relation that is of no interest to us here. (See [**22**].)

B. THE SOLUTION $\Phi^{\mathrm{II}} = \varphi^{\mathrm{II}}\, e^{2\pi i\nu t}$. – The solution Φ^{II}, which will be identified with Φ^{I}, is easily obtained by noting that one passes from (I) to (II) formally by changing i and $-i$ and W into $-W$. The two solutions (3) correspond to the following two solutions here, upon using the property:

$$(Y_l^m)^* = (-1)^m\, Y_l^{-m};$$

$$\left.\begin{array}{ll}
\text{Type } (+) & \text{Type } (-) \\
\varphi_{\alpha 1} = -i\,H_{+\alpha}^l\,Y_{l+1}^m, & -i(l+m)H_{-\alpha}^l\,Y_{l-1}^m, \\
\varphi_{\alpha 2} = -i\,H_{+\alpha}^l\,Y_{l+1}^{m+1}, & i(l-m-1)H_{-\alpha}^l\,Y_{l-1}^{m+1}, \\
\varphi_{\alpha 3} = (l+m+1)\,I_{+\alpha}^l\,Y_l^m, & I_{-\alpha}^l\,Y_l^m, \\
\varphi_{\alpha 4} = -(l-m)\,I_{+\alpha}^l\,Y_l^{m+1}, & I_{-\alpha}^l\,Y_l^m, \\
l = 0,1,2,\cdots & l = 1,2,\cdots \\
m = -(l+1),-l,\cdots,+l & m = -l,-(l-1),\cdots,(l-1).
\end{array}\right\} \tag{13}$$

$H_{+\alpha}^l$, $I_{+\alpha}^l$, $H_{-\alpha}^l$, $I_{-\alpha}^l$ are radial functions. For any index α, they will satisfy the equations:

$$\left.\begin{aligned}
\frac{2\pi}{h}\left(-\frac{W}{c}+\mu_0 c\right)H_+^l + \dot{I}_+^l - \frac{l}{r}I_+^l &= 0, \\
-\frac{2\pi}{h}\left(-\frac{W}{c}-\mu_0 c\right)I_+^l + \dot{H}_+^l + \frac{l+2}{r}H_+^l &= 0, \\
\frac{2\pi}{h}\left(-\frac{W}{c}+\mu_0 c\right)H_-^l + \dot{I}_-^l + \frac{l+1}{r}I_-^l &= 0, \\
-\frac{2\pi}{h}\left(-\frac{W}{c}-\mu_0 c\right)I_-^l + \dot{H}_-^l - \frac{l-1}{r}H_-^l &= 0.
\end{aligned}\right\} \tag{14}$$

By virtue of (14), the functions:

$$H_+^{l-1},\ I_+^l,\ I_-^l,\quad H_-^{l+1} \tag{15}$$

likewise satisfy equation (8).

In view of the aforementioned identification, one can consider a more general solution, namely:

$$\left.\begin{aligned}
\varphi_{\alpha 1} &= Y_{l+1}^{m+\lambda_\alpha}(-i\,H_{+\alpha}^l) + Y_l^{m+\lambda_\alpha}[-i\,H_{+\alpha}^{l-1} - i\,(l+1+m+\lambda_\alpha)H_{-\alpha}^{l+1}] + Y_{l-1}^{m+\lambda_\alpha}[-i\,(l+m+\lambda_\alpha)H_{-\alpha}^l], \\
\varphi_{\alpha 2} &= Y_{l+1}^{m+\lambda_\alpha+1}(-i\,H_{+\alpha}^l) + Y_l^{m+\lambda_\alpha+1}[-i\,H_{+\alpha}^{l-1} + i\,(l-m-\lambda_\alpha)H_{-\alpha}^{l+1}] + Y_{l-1}^{m+\lambda_\alpha+1}[+i\,(l-m-\lambda_\alpha-1)H_{-\alpha}^l], \\
\varphi_{\alpha 3} &= Y_{l+1}^{m+\lambda_\alpha}\, I_{-\alpha}^{l+1} + Y_l^{m+\lambda_\alpha}\,[I_{-\alpha}^l + (l+1+m+\lambda_\alpha)I_{+\beta}^l] + Y_{l-1}^{m+\lambda_\alpha}\,(l+m+\lambda_\alpha)I_{+\alpha}^{l-1}, \\
\varphi_{\alpha 4} &= Y_{l+1}^{m+\lambda_\alpha+1} I_{-\alpha}^{l+1} + Y_l^{m+\lambda_\alpha+1}[I_{-\alpha}^l - (l-m-\lambda_\alpha)I_{+\alpha}^l] + Y_{l-1}^{m+\lambda_\alpha+1} i[-(l-m-\lambda_\alpha-1)I_{+\alpha}^{l-1}].
\end{aligned}\right\} \tag{16}$$

The λ_α are undetermined whole numbers. We see that the identification $\varphi^{\mathrm{I}} = \varphi^{\mathrm{II}}$ will give $\lambda_1 = \lambda_3 = 0$, $\lambda_2 = \lambda_4 = -1$.

C. IDENTIFICATION. – One can distinguish two cases that we shall refer to by the names electrical and magnetic spherical waves, resp., conforming to the usual terminology ([1]):

1. *Electrical spherical wave.* One annuls:

$$\left.\begin{aligned} F_{+1}^{l} = F_{+2}^{l} = F_{-1}^{l} = F_{-2}^{l} = 0, \\ F_{+3}^{l-1} = F_{+1}^{l-1} = F_{-3}^{l+1} = F_{-1}^{l+1} = 0. \end{aligned}\right\} \tag{17}$$

We then remark that when $F_{+\alpha}^{l} = F_{+\beta}^{l}$, we will also have $G_{+\alpha}^{l} = G_{+\beta}^{l}$, by virtue of (4).

Upon setting $\varepsilon_1 = 0$, one will immediately get:

$$\varepsilon_1 = \varepsilon_3 = \lambda_1 = \lambda_3 = 0; \quad \varepsilon_2 = \varepsilon_4 = 1; \qquad \lambda_2 = \lambda_4 = 0\,. \tag{18}$$

As a result, upon identifying the φ_{13}, φ_{14}, φ_{23}, and φ_{31}, φ_{32}, φ_{41}:

$$\left.\begin{aligned} i\,F_{+3}^{l} = i\,F_{+4}^{l} = I_{-1}^{l+1} = I_{-2}^{l+1} \\ i\,(l-m)G_{-3}^{l} = -i\,(l+m)G_{-4}^{l} = (l+m)H_{+1}^{l-1} = (l-m)H_{+2}^{l-1}, \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} F_{-1}^{l+1} = F_{+2}^{l+1} = -iI_{+3}^{l} = iI_{+4}^{l} \\ (l-m)F_{+1}^{l-1} = -(l+m)F_{+2}^{l+1} = -i\,(l+m)I_{-3}^{l} = -i\,(l-m)I_{-4}^{l}, \end{aligned}\right\} \tag{20}$$

and upon identifying the φ_{11}, φ_{12}, φ_{21}, φ_{22}, and taking (19) and (20) into account:

$$\left.\begin{aligned} i\,F_{+1}^{l-1} + i\,(l-m+1)F_{-1}^{l+1} &= -G_{-4}^{l} + (l+m+1)G_{+3}^{l}, && (a) \\ i\,F_{+2}^{l-1} + \quad i\,(l-m)F_{-1}^{l+1} &= -G_{-4}^{l} - (l-m)G_{+3}^{l}, && (b) \\ -i\,F_{+1}^{l-1} + \quad i\,(l+m)F_{-1}^{l+1} &= \quad G_{-3}^{l} - (l+m)G_{+3}^{l}, && (c) \\ -i\,F_{+2}^{l-1} + i\,(l+m+1)F_{-1}^{l+1} &= \quad G_{-3}^{l} + (l-m+1)G_{+3}^{l}. && (d) \end{aligned}\right\} \tag{21}$$

In addition, it results from (19) and (20) that:

$$\left.\begin{aligned} \varphi_{13} = -\varphi_{24}; \quad \varphi_{31} = -\varphi_{41}; \\ \varphi_{11} = \varphi_{44}; \quad \varphi_{22} = \varphi_{33}; \quad \varphi_{12} = -\varphi_{34}; \quad \varphi_{21} = -\varphi_{43}. \end{aligned}\right\} \tag{22}$$

These relations (22) express the idea that:

([1]) Cf., the end of § **4**.

$$I_{(1)} = I_{(2)} = \sigma_x = \sigma_y = \sigma_z = \sigma_4 = 0\ ; \tag{22′}$$

i.e., that the six non-Maxwellian quantities are zero for an electric spherical wave. In order to verify (22′), one refers to the inversion formulas [V, (28)]. Equations (21.*a*, *b*, *c*, *d*) are not independent. If one adds corresponding sides of (*a*), – (*b*), and (*c*) then one will get (*d*). Moreover, for $l \neq -m$, (*b*) will be a consequence of (*c*), and for $l = -m$, equation (*c*) will be satisfied identically, since by virtue of (19) and (20):

$$\left.\begin{aligned}(l-m)F_{+1}^{l-1} &= -(l-m)F_{+2}^{l-1}, \quad (a)\\ (l-m)G_{-3}^{l} &= -(l+m)G_{-4}^{l}. \quad (b)\end{aligned}\right\} \tag{23}$$

Finally, all of the functions F_+, F_-, G_+, G_-, that were introduced in (6) can be expressed as functions of two of the other ones; in other words, the $\varphi_{\alpha\beta}$ again include two arbitrary constants. We suppose that these constants are contained implicitly in G_{+3}^{l} and G_{-3}^{l} for $l \neq -m$, and in G_{+3}^{l} and G_{-4}^{l} for $l = -m$. In order to consider these two cases simultaneously, we shall leave G_{+3}^{l} and G_{-3}^{l} in these formulas, as well as G_{-4}^{l}. It will always be easy to take the condition (23.*b*) into account. The solution of equations (21) will then give:

$$\left.\begin{aligned}i\,(2l+1)F_{+1}^{l-1} &= -G_{-3}^{l} + 2(l+m)(l+1)G_{+3}^{l},\\ i\,(2l+1)F_{+2}^{l-1} &= -G_{-4}^{l} - 2(l-m)(l+1)G_{+3}^{l},\\ i\,(2l+1)F_{-1}^{l+1} &= G_{-3}^{l} - G_{-4}^{l} + G_{+3}^{l}.\end{aligned}\right\} \tag{24}$$

The other functions:

$$G_{+1}^{l-1}, \quad G_{+2}^{l-1}, \quad G_{-1}^{l+1}, \quad F_{+3}^{l}, \quad F_{-3}^{l}, \quad F_{-4}^{l} \tag{25}$$

are likewise expressed as functions of the G_{+3}^{l}, G_{-3}^{l}, G_{-4}^{l}, thanks to equations (4).

Here are the explicit values of the functions $\varphi_{\alpha\beta}$.

$$\left.\begin{aligned}&\left\{\begin{aligned}\varphi_{11} = \varphi_{44} &= Y_l^m[-G_{-4}^{l} + (l+m+1)G_{-3}^{l}],\\ \varphi_{22} = \varphi_{33} &= Y_l^m[G_{-3}^{l} + (l-m+1)G_{+3}^{l}],\\ -\varphi_{24} = \varphi_{13} &= Y_{l+1}^m\, i\,F_{+3}^{l} + Y_{l-1}^m\, i\,(l-m)G_{+1}^{l-1},\\ -\varphi_{42} = \varphi_{31} &= Y_{l+1}^m\, G_{-1}^{l} + Y_{l-1}^m (l-m)G_{+1}^{l},\end{aligned}\right.\\ &\left\{\begin{aligned}-\varphi_{12} = \varphi_{34} &= Y_{l+1}^m[G_{-4}^{l} + (l-m)G_{+3}^{l}],\\ \varphi_{14} &= Y_{l+1}^{m+1} i\,F_{+3}^{l} + Y_{l-1}^{m+1} i(l-m-1)F_{-4}^{l},\\ \varphi_{32} &= Y_{l+1}^{m+1}G_{-1}^{l+1} + Y_{l-1}^{m+1}(l-m-1)G_{+2}^{l-1},\end{aligned}\right.\\ &\left\{\begin{aligned}-\varphi_{34} = \varphi_{21} &= Y_l^{m-1}[G_{-3}^{l} - (l+m)G_{+3}^{l}],\\ \varphi_{23} &= Y_{l+1}^{m-1}(-i\,F_{+3}^{l}) + Y_{l-1}^{m-1} i(l+m-1)F_{-3}^{l},\\ \varphi_{41} &= Y_{l+1}^{m-1}(-G_{-1}^{l+1}) + Y_{l-1}^{m-1}(l+m-1)G_{+1}^{l-1}.\end{aligned}\right.\end{aligned}\right\} \tag{26}$$

It is pointless to replace the functions (25) in these formulas with their expressions as functions of G_{+3}^l, G_{-3}^l, G_{-4}^l that one infers from equations (4).

The electromagnetic fields and potentials that correspond to (26) are deduced immediately from (26) and [V, (28)]. Upon absorbing the multiplicative constant $4\pi K\,\mu_0 c/h$ into the two arbitrary constants that were pointed out above and not writing out the temporal factor $e^{\frac{2\pi i w}{h}t}$, one will have the electric field:

$$H_x + i\,H_y = \frac{1}{i}Y_{l+1}^{m+1}(i\,F_{+3}^l - G_{-1}^{l+1}) + \frac{1}{i}Y_{l-1}^{m+1}(l-m-1)(i\,F_{-1}^l - G_{+2}^{l-1}),$$

$$H_x - i\,H_y = \frac{1}{i}Y_{l+1}^{m+1}(-i\,F_{+3}^l + G_{-1}^{l+1}) + \frac{1}{i}Y_{l-1}^{m-1}(l+m-1)(i\,F_{-3}^l - G_{+1}^{l-1}),$$

$$H_z = \frac{1}{i}Y_{l+1}^{m}(i\,F_{+3}^l - G_{-1}^{l+1}) + \frac{1}{i}Y_{l-1}^{m}(l-m)(i\,F_{-3}^l - G_{+1}^{l-1}),$$

the magnetic field:

$$\left.\begin{aligned} \mathcal{H}_x + i\mathcal{H}_y &= Y_l^{m+1}\,2(-G_{-4}^l - (l-m)G_{+3}^l), \\ \mathcal{H}_x - i\mathcal{H}_y &= Y_l^{m-1}\,2(G_{-3}^l - (l+m)G_{+2}^l), \\ \mathcal{H}_z &= Y_l^{m}\,2(-G_{-4}^l - G_{-3}^l + 2mG_{+3}^l), \end{aligned}\right\} \tag{27}$$

the vector potential:

$$\mathcal{V}_x + i\,\mathcal{V}_y = -\frac{h}{2\pi\mu_0 c}\{Y_{l+1}^{m+1}(i\,F_{+3}^l + G_{-1}^{l+1}) + Y_{l-1}^{m+1}(l-m-1)(i\,F_{-4}^l + G_{+2}^{l-1})\},$$

$$\mathcal{V}_x - i\,\mathcal{V}_y = -\frac{h}{2\pi\mu_0 c}\{Y_{l+1}^{m+1}(-i\,F_{+3}^l - G_{-1}^{l+1}) + Y_{l-1}^{m-1}(l+m-1)(i\,F_{-3}^l + G_{+1}^{l-1})\},$$

$$\mathcal{V}_z = -\frac{h}{2\pi\mu_0 c}\{Y_{l+1}^{m}(i\,F_{+3}^l + G_{-1}^{l+1}) + Y_{l-1}^{m}(l-m)(i\,F_{-3}^l + G_{+2}^{l-1})\},$$

the scalar potential:

$$V = \frac{h}{2\pi\mu_0 c}Y_l^m\{-G_{-1}^l + G_{-3}^l + 2(l+1)G_{+3}^l\}.$$

Any spherical wave is characterized by two whole numbers *l* and *m*; *l* can take all of the positive whole number values ([1]):

([1]) The case where $l = 0$ will be considered later on (end of § **4**).

$$l = 1, 2, \ldots, \tag{28}$$

and for a given value of l, m can take any of the $(2l + 1)$ values:

$$m = -l, -(l-1), \ldots, +l. \tag{29}$$

An "electric spherical wave" of frequency $\nu = W / h$ will then have the form:

$$\overset{e}{\Phi}{}^{lm}_{\alpha\beta} = \overset{e}{\varphi}{}^{lm}_{\alpha\beta}\, e^{2\pi i\nu t}, \tag{30}$$

in which the $\overset{e}{\varphi}{}^{lm}_{\alpha\beta}$ are defined by (26). In addition, that wave will be qualified by diverging, converging, or stationary according to whether one uses (9), (10), or (11). However, it is important to point out that this wave still includes two arbitrary constants. We shall return to that point later on (§ **4**).

2. *Magnetic spherical wave.* One annuls:

$$\left.\begin{aligned} F^l_{+3} &= F^l_{+4} = F^l_{-3} = F^l_{-3} = 0, \\ F^{l-1}_{+1} &= F^{l-1}_{+2} = F^{l+1}_{-4} = F^{l+1}_{-2} = 0. \end{aligned}\right\} \tag{31}$$

Upon following the same method as in 1., one will first find (18); as a result:

$$\left.\begin{aligned} i\,F^l_{+1} &= i\,F^l_{+2} = -i\,H^l_{+1} = i\,H^l_{+2}, \\ G^{l+1}_{-3} &= G^{l+1}_{-4} = I^{l+1}_{-3} = -I^{l+1}_{-4}, \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} i\,(l-m)F^l_{-1} &= -i\,(l+m)F^l_{-2} = -i\,(l+m)H^l_{-1} = -i\,(l-m)H^l_{-2}, \\ (l-m)G^{l-1}_{+3} &= -(l+m)G^{l-1}_{+4} = (l+m)I^{l-1}_{+3} = (l-m)I^{l-1}_{+4}, \end{aligned}\right\} \tag{33}$$

and

$$\left.\begin{aligned} i\,F^{l-1}_{+3} + i\,(l-m)F^{l+1}_{-3} &= \tau[-G^l_{-2} + (l+m+1)G^l_{+1}], && (a) \\ i\,F^{l-1}_{+4} + i\,(l-m)F^{l+1}_{-3} &= \tau[-G^l_{-2} - (l-m)G^l_{+1}], && (b) \\ -i\,F^{l-1}_{+3} + i\,(l+m)F^{l+1}_{-3} &= \tau[G^l_{-1} - (l+m)G^l_{+1}], && (c) \\ -i\,F^{l-1}_{+4} + i\,(l+m+1)F^{l+1}_{-3} &= \tau[G^l_{-1} + (l-m+1)G^l_{+1}], && (d) \end{aligned}\right\} \tag{34}$$

in which we have set:

$$\tau = \frac{W - \mu_0 c^2}{W + \mu_0 c^2}, \tag{35}$$

to simplify the writing.

It results from (32), (33), and (34) that:

$$\left.\begin{array}{lr}\varphi_{11}=-\varphi_{22}, \quad \varphi_{33}=-\varphi_{44}, & (a) \\ \varphi_{13}=\tau\,\varphi_{42}, \quad \varphi_{14}=-\tau\,\varphi_{32}, \quad \varphi_{23}=-\tau\,\varphi_{41}, \quad \varphi_{24}=\tau\,\varphi_{31}. & (b)\end{array}\right\} \tag{36}$$

Upon utilizing the inversion formulas, the six relations (36) can be written:

$$\left.\begin{array}{lr}I_{(1)}=0, & (a) \\ V=0, & (b)\end{array}\right\} \tag{35$'$}$$

$$\left.\begin{array}{lr}H_x=-\dfrac{2\pi iv}{c}\mathcal{V}_x, \quad H_y=-\dfrac{2\pi iv}{c}\mathcal{V}_y, \quad H_z=-\dfrac{2\pi iv}{c}\mathcal{V}_z, & (a) \\ \sigma_4=i\dfrac{W}{\mu_0c^2}I_{(2)}. & (b)\end{array}\right\} \tag{36$'$}$$

One must obviously have (35′.*a*) and (36′.*b*) due to the equations of the photon. On the other hand, the condition (35′.*b*) implies (36′.*b*) by virtue of the defining equations of the fields as functions of the potentials.

Equations (34) are formally the same as (21).

The four equations (34) are therefore not independent: (*d*) is a consequence of (*a*), (*b*), (*c*), and one can make the same remarks about (34.*b*, *c*) that one makes about (21.*b*, *c*). Two arbitrary constants ultimately remain. However, for the same reasons as in 1., we still leave the three symbols G_{+1}^l, G_{-1}^l, and G_{-3}^l in the formulas. Upon solving (34), one will get (cf. 24):

$$\begin{aligned}
i\,(2l+1)\;F_{+3}^{l-1} &= \tau\,[-\,G_{-1}^l+2\,(l+m)\,(l+1)\;G_{+1}^l\,],\\
i\,(2l+1)\;F_{+4}^{l-1} &= \tau\,[-\,G_{-2}^l+2\,(l-m)\,(l+1)\;G_{+1}^l\,],\\
i\,(2l+1)\;F_{-3}^{l+1} &= \tau\,(G_{-1}^l-G_{-2}^l+G_{+1}^l\,).
\end{aligned}$$

Finally, we give the explicit value of $\varphi_{\alpha\beta}$:

$$\left.\begin{aligned}
-\varphi_{22}=\varphi_{11} &= Y_{l+1}^m\,i\,F_{+1}^l+Y_{l-1}^m\,i\,(l-m)F_{-1}^l,\\
-\varphi_{44}=\varphi_{31} &= Y_{l+1}^m\,G_{-3}^l+Y_{l-1}^m\,(l-m)G_{+3}^{l-1},\\
\tau\,\varphi_{42}=\varphi_{13} &= Y_l^m\,\tau[-G_{-2}^l+(l+m+1)G_{+1}^l],\\
\tau\,\varphi_{31}=\varphi_{24} &= Y_l^m\,\tau[\,G_{-1}^l+(l-m+1)G_{+1}^l],\\
\varphi_{12} &= Y_{l+1}^{m+1}\,i\;F_{-1}^l+Y_{l-1}^{m+1}i\,(l-m-1)F_{-2}^l,\\
\varphi_{34} &= Y_{l+1}^{m+1}\,G_{-3}^{l+1}+Y_{l-1}^{m+1}i\,(l-m-1)G_{+1}^{l-1},\\
-\tau\,\varphi_{32}=\varphi_{14} &= Y_l^{m+1}\,\tau[-G_{-2}^l-(l-m)G_{+1}^l,\\
-\tau\,\varphi_{41}=\varphi_{23} &= Y_l^{m-1}\,\tau[G_{-1}^l-(l+m)G_{+1}^l],\\
\varphi_{21} &= Y_{l+1}^{m-1}\,(-i\;F_{+1}^l)+Y_{l-1}^{m-1}i\,(l+m-1)F_{-1}^l,\\
\varphi_{43} &= Y_{l+1}^{m-1}\,(-G_{+3}^l)+Y_{l-1}^{m-1}(l+m-1)G_{+3}^{l-1}.
\end{aligned}\right\} \tag{38}$$

One immediately infers the expressions for the tensorial quantities from (38) and [V, (28)]. Upon absorbing the multiplicative constant $K\,4\pi\mu_0\,c\,/\,h$ into the two arbitrary constants, one will get the electric field:

$$\left.\begin{aligned} H_x + iH_y &= \frac{1}{i}\frac{2W}{W+\mu_0 c^2}[-G_{-2}^{l} - (l-m)G_{-2}^{l}]Y_l^{m+1}, \\ H_x - iH_y &= \frac{1}{i}\frac{2W}{W+\mu_0 c^2}[G_{-1}^{l} - (l+m)G_{+1}^{l}]Y_l^{m-1}, \\ H_z &= \frac{1}{i}\frac{2W}{W+\mu_0 c^2}Y_l^{m}(-G_{-2}^{l} - G_{-1}^{l} + 2m\,G_{+3}^{l-1}), \end{aligned}\right\} \tag{39.a}$$

the magnetic field:

$$\left.\begin{aligned} \mathcal{H}_x + i\mathcal{H}_y &= Y_{l+1}^{m+1}(iF_{+1}^{l} - G_{-3}^{l+1}) + Y_{l-1}^{m+1}(l-m-1)(iF_{-2}^{l} - G_{+1}^{l+1}), \\ \mathcal{H}_x - i\mathcal{H}_y &= Y_{l+1}^{m-1}(-iF_{+1}^{l} + G_{-3}^{l+1}) + Y_{l-1}^{m-1}(l+m-1)(iF_{-1}^{l} - G_{+1}^{l+1}), \\ \mathcal{H}_z &= Y_{l+1}^{m-1}(iF_{+1}^{l} - G_{-3}^{l+1}) + Y_{l-1}^{m}(l-m)(iF_{-1}^{l} - G_{+3}^{l-1}). \end{aligned}\right\} \tag{39.b}$$

Recall that the vector potential is given by (36′.*a*):

$$\mathcal{V}_x = -\frac{c}{2\pi i\,\nu}H_x\,, \qquad \mathcal{V}_y = -\frac{c}{2\pi i\,\nu}H_y\,, \qquad \mathcal{V}_z = -\frac{c}{2\pi i\,\nu}H_z\,, \tag{39.c}$$

and that the scalar potential is zero:

$$V = 0. \tag{39.d}$$

The non-Maxwellian quantities are not necessarily zero. One has:

$$\left.\begin{aligned} K(\sigma_x + i\sigma_y) &= -\frac{h}{2\pi\mu_0 c}\{Y_{l+1}^{m+1}(iF_{+1}^{l} + G_{-1}^{l+1}) + Y_{l-1}^{m+1}(l-m-1)(iF_{-2}^{l} + G_{+1}^{l+1})\}, \\ K(\sigma_x - i\sigma_y) &= -\frac{h}{2\pi\mu_0 c}\{Y_{l+1}^{m-1}(-iF_{+1}^{l} - G_{-3}^{l+1}) + Y_{l-1}^{m-1}(l+m-1)(iF_{-1}^{l} + G_{+3}^{l-1})\}, \\ K\sigma_z &= -\frac{h}{2\pi\mu_0 c}\{Y_{l+1}^{m}(iF_{+1}^{l} + G_{-3}^{l+1}) + Y_{l-1}^{m}(l-m)(iF_{-1}^{l} + G_{+3}^{l-1})\}. \end{aligned}\right\} \tag{39.e}$$

Recall that σ_4 is expressed as a function of $I_{(2)}$ by means of (36′.*b*). Finally:

$$K\,I_{(2)} = \frac{hc}{2\pi(W+\mu_0 c^2)}Y_{l+1}^{m}\{-G_{-2}^{l} + G_{-1}^{l} + 2(l+1)\,G_{+1}^{l}\}. \tag{39.f}$$

Any wave (38) or (39) is characterized by the two numbers l and m. l can take all positive integer values ([1]):

$$l = 1, 2, \ldots, \tag{40}$$

whereas, for a given value of l, m can take any of the $(2l + 1)$ integer values:

$$m = -l, \ldots, +l. \tag{41}$$

A "magnetic spherical wave" will then be represented by the notation:

$$\overset{\circ}{\Phi}{}^{lm} = \overset{\circ}{\varphi}{}^{lm}\, e^{2\pi i\nu t}, \tag{42}$$

in which the $\overset{\circ}{\varphi}{}^{lm}$ are defined in (38). That wave (42) will be diverging, converging, or stationary according to whether one uses (9), (10), or (11), resp. It still includes two arbitrary constants, which will be examined in paragraph **4**.

Remarks. –

a) Give all functions that relate to electric spherical waves an overhead ε and give all functions that relate to magnetic spherical functions an overhead μ. One effortlessly sees that one can write:

$$\overset{\varepsilon}{G}{}^{l}_{+3} = \overset{\mu}{G}{}^{l}_{+1}, \qquad \overset{\varepsilon}{G}{}^{l}_{-3} = \overset{\mu}{G}{}^{l}_{-1}, \qquad \overset{\varepsilon}{G}{}^{l}_{-1} = \overset{\mu}{G}{}^{l}_{-2};$$

hence, one likewise has:

$$\overset{\varepsilon}{F}{}^{l-1}_{+1} = \tau\,\overset{\mu}{F}{}^{l-1}_{+3}, \qquad \overset{\varepsilon}{F}{}^{l-1}_{+2} = \tau\,\overset{\mu}{F}{}^{l-1}_{+1}, \qquad \overset{\varepsilon}{F}{}^{l+1}_{-1} = \tau\,\overset{\mu}{F}{}^{l+1}_{-3}.$$

It will then result that, up to terms of order $\mu_0\, c^2 / W$:

$$\mathbf{H}_{(\varepsilon)} \approx -i\,\mathcal{H}_{(\varepsilon)},$$
$$\mathcal{H}_{(\varepsilon)} \approx i\,\mathbf{H}_{(\varepsilon)}.$$

b) It might seem that the Maxwellian quantities (39.*a* – *d*) are necessarily associated with the expressions (39.*e* – *f*) for the non-Maxwellian quantities. That link is only due to the method of solution of the fundamental equations that we have adopted. It is obvious that the wave that is defined by the fields and potentials (39.*a* – *d*) and using:

$$\sigma_x = \sigma_y = \sigma_z = \sigma_4 = I_{(2)} = 0$$

in place of (39.*e* – *f*) will likewise satisfy the equations of the photon.

([1]) The case in which $l = 0$ will be considered later on (end of § **4**).

Similarly, the wave that is defined by:

$$\mathbf{H} = 0, \qquad \mathcal{H} = 0, \qquad \mathcal{V} = 0, \qquad V = 0,$$

and whose non-Maxwellian quantities are given by (39.*e*, *f*), is likewise a solution of equations (I), (II). In that case, one can makes some remarks that are analogous to the ones in paragraph **12** of Chapter V. The non-Maxwellian photon that is represented by (39.*e*, *f*) will transport an energy W and a moment of impulse whose projection onto the z-axis will be $-\, mh / 2\pi$ and whose length-squared will be equal to $l\,(l + 1)\, h^2 / 4\pi^2$. (See § **3**).

3. Moments of impulse. –In the theory of the photon, the total moment of impulse operator is a vector whose projection into the z-axis is:

$$m_z = m_z^0 + m_z^s ,$$

in which:

$$m_z^0 = -\frac{h}{2\pi i}\frac{\partial}{\partial\varphi},$$

$$m_z^s = \frac{h}{4\pi}(i\, \mathcal{A}_1\, \mathcal{A}_2 + i\, \mathcal{B}_1\, \mathcal{B}_2).$$

The calculation of $m_z \Phi_{\alpha\beta}^{lm}$ is extremely simple. One immediately finds that:

$$\boxed{m_z \Phi_{\alpha\beta}^{lm} = -m\frac{h}{2\pi}\Phi_{\alpha\beta}^{lm}} \tag{44}$$

for a electric or magnetic spherical wave.

The photon whose state is represented by the wave functions $\overset{\varepsilon}{\Phi}{}_{\alpha\beta}^{lm}$ or $\overset{\mu}{\Phi}{}_{\alpha\beta}^{lm}$ will then transport a moment of impulse along the z-axis that is equal to $-\, mh / 2\pi$.

On the other hand, the operator m_z does not commute with the other components m_x and m_y . Conforming to the general principles of wave mechanics, that must say that one cannot simultaneously determine two of the components m_x, m_y, m_z of the total moment of impulse vector for a photon. When one of the components is fixed, which is the case that we suppose, there will no longer exist relations that are capable of determining the other components.

By contrast, m^2 does commute with m_z . We calculate $m^2\, \Phi^{lm}$ and show that we will have:

$$\boxed{m^2\Phi^{lm} = \frac{h^2}{4\pi^2}l(l+1)\Phi^{lm},} \tag{45}$$

in any case.

First, here are some general relations: By virtue of the definition of the vector **m**:

$$m^2 = (m_0)^2 + \frac{h}{2\pi}(\mathbf{m}_0 \cdot \mathbf{s}_a + \mathbf{s}_b) + \frac{h^2}{4\pi^2}(\mathbf{s}_a\ \mathbf{s}_b) + \frac{3}{2}\frac{h^2}{4\pi^2}. \tag{46}$$

Thanks to equations (4):

$$(m_0)^2 G_-^l = \frac{h^2}{4\pi^2}\, l\,(l+1)\, G_-^l ; \tag{47}$$

the same relation exists for F_-^{l+1}, F_+^{l-1}, and G_+^l. Upon using the explicit form of the matrices s_a and s_b, one will have:

$$\left.\begin{aligned}(\mathbf{m}_0, \mathbf{s}_a + \mathbf{s}_b)\,\Phi_{\alpha\beta} &= (m_x^0 + i m_y^0)(\delta_{\alpha 1}\Phi_{\alpha 1} - \delta_{2\beta}\Phi_{\alpha 1} + \delta_{\alpha 3}\Phi_{4\beta} - \delta_{1\beta}\Phi_{\alpha 3}) \\ &+ (m_x^0 - i m_y^0)(\delta_{\alpha 2}\Phi_{1\beta} - \delta_{1\beta}\Phi_{\alpha 2} + \delta_{\alpha 4}\Phi_{3\beta} - \delta_{2\beta}\Phi_{\alpha 4}) + m_z^0[(-1)^\beta - (-1)^\alpha]\Phi_{\alpha\beta}\end{aligned}\right\} \tag{48}$$

and

$$\left.\begin{aligned}\tfrac{1}{2}(\mathbf{s}_a\ \mathbf{s}_b)\,\Phi_{\alpha\beta} &= -\delta_{\beta 2}(\delta_{\alpha 2}\Phi_{11} + \delta_{\alpha 4}\Phi_{31}) - \delta_{\beta 4}(\delta_{\alpha 2}\Phi_{13} + \delta_{\alpha 4}\Phi_{33}) \\ &- \delta_{\beta 1}(\delta_{\alpha 1}\Phi_{22} + \delta_{\alpha 3}\Phi_{42}) - \delta_{\beta 3}(\delta_{\alpha 1}\Phi_{24} + \delta_{\alpha 3}\Phi_{44}) - \tfrac{1}{2}(-1)^{\alpha+\beta}\Phi_{\alpha\beta},\end{aligned}\right\} \tag{49}$$

in which $\delta_{\alpha\alpha} = 1$ and $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$. One remarks that:

$$\left.\begin{aligned}m_x^0 + i m_y^0 &= -\frac{h}{2\pi i} ri\left\{-\sin\theta\cdot e^{i\varphi}\frac{\partial}{\partial z} + \cos\theta\cdot\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right\}, \\ m_x^0 - i m_y^0 &= -\frac{h}{2\pi i} ri\left\{-\sin\theta\cdot e^{-i\varphi}\frac{\partial}{\partial z} + \cos\theta\cdot\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right\}.\end{aligned}\right\} \tag{50}$$

Finally, set:

$$\xi_l(f) \equiv \dot f - \frac{l}{r} f, \qquad \eta_l(f) = \dot f + \frac{l+1}{r} f, \tag{51}$$

in which $f \equiv f(r)$ is a function of r. One has ([1]), upon writing ξ_l for $\xi_l(r)$:

$$\left.\begin{aligned}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f\, Y_l^m &= \frac{1}{2l+1}\{\xi_l\, Y_{l+1}^{m+1} - (l-m)(l-m-1)\,\eta_l\, Y_{l-1}^{m-1}\}, \\ \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f\, Y_l^m &= \frac{1}{2l+1}\{-\xi_l\, Y_{l+1}^{m-1} + (l+m)(l+m-1)\,\eta_l\, Y_{l-1}^{m-1}\}, \\ \frac{\partial}{\partial z} f\, Y_l^m &= \frac{1}{2l+1}\{\xi_l\, Y_{l+1}^m + (l+m)(l-m)\,\eta_l\, Y_{l-1}^m\}.\end{aligned}\right\} \tag{52}$$

Note the following relations:

([1]) See [**8**], pp. 232.

$$\left.\begin{aligned} Y_{l+1}^{m-1} + (l+m-1)(l-m+1)Y_{l-1}^{m-1} &= (2l+1)\cos\theta\, Y_l^{m-1}, \\ Y_{l+1}^{m} + (l-m+1)(l-m)Y_{l-1}^{m} &= (2l+1)\sin\theta\, e^{i\varphi} Y_l^{m-1}, \\ -Y_{l+1}^{m} + (l+m+1)(l+m)Y_{l-1}^{m} &= (2l+1)\sin\theta\, e^{-i\varphi} Y_l^{m+1}. \end{aligned}\right\} \tag{53}$$

These few formulas will permit one to perform the calculation of the $m^2\,\Phi^{lm}$ for an electric or magnetic spherical wave with no difficulty. For example, one can calculate $m^2\,\overset{\circ}{\Phi}{}_{11}^{lm}$. One finds from (47) that:

$$m^2\,\overset{\circ}{\Phi}{}_{11}^{lm} = \frac{h^2}{4\pi^2}\,l(l+1)\,\overset{\circ}{\Phi}{}_{11}^{lm}. \tag{54}$$

Then, upon denoting the radial functions of $\overset{\circ}{\Phi}{}_{11}^{lm}$, $\overset{\circ}{\Phi}{}_{12}^{lm}$, … by f_{11}, f_{12}, …, resp., one will have:

$$(\mathbf{m}_0\cdot\mathbf{s}_a+\mathbf{s}_b)\,\overset{\varepsilon}{\varphi}{}_{11}^{lm} = -\frac{h}{2\pi}Y_l^m\,[-f_{21}\,(l-m+1)+f_{12}\,(l+m+1)]$$

$$\tfrac{1}{2}(\mathbf{s}_a\,\mathbf{s}_b)\,\overset{\varepsilon}{\varphi}{}_{11}^{lm} = -\frac{h}{2\pi}Y_l^m\left(f_{22}+\tfrac{1}{2}f_{11}\right);$$

hence, upon replacing the $f_{\alpha\beta}$ with their values:

$$\left[\frac{h}{2\pi}(\mathbf{m}_0\cdot\mathbf{s}_a+\mathbf{s}_b)+\frac{1}{2}\frac{h^2}{4\pi^2}(\mathbf{s}_a\,\mathbf{s}_b)\right]\overset{\varepsilon}{\varphi}{}_{11}^{lm} = -\frac{h^2}{4\pi^2}\frac{3}{2}\overset{\varepsilon}{\varphi}{}_{11}^{lm}. \tag{55}$$

(46), (54), and (55) give:

$$m^2\,\overset{\varepsilon}{\Phi}{}_{11}^{lm} = \frac{h^2}{4\pi^2}\,l(l+1)\,\overset{\varepsilon}{\Phi}{}_{11}^{lm}.$$

The calculations will have the same form for the other components whose indices "are of the same pair" (12, 21, 22, 33, 34, 43, 44) (†). One will have some simplifications for the components whose indices "have different pairs" (13, 14, 23, 24, 31, 41, 32, 43). For example, for Φ_{14}:

$$(m_0)^2\,\overset{\varepsilon}{\Phi}{}_{14}^{lm} = \frac{h^2}{4\pi^2}(l+1)(l+2)\,\overset{\varepsilon}{\Phi}{}_{14}^{lm},$$

$$(\mathbf{m}_0\cdot\mathbf{s}_a+\mathbf{s}_b)\,\overset{\varepsilon}{\Phi}{}_{14}^{lm} = -\frac{h^2}{4\pi^2}\,2(l+2)\,\overset{\varepsilon}{\Phi}{}_{14}^{lm},$$

$$\tfrac{1}{2}(\mathbf{s}_a\cdot\mathbf{s}_b)\,\overset{\varepsilon}{\Phi}{}_{14}^{lm} = -\frac{1}{2}(-1)^{1+4}\,\overset{\varepsilon}{\Phi}{}_{14}^{lm} = \frac{1}{2}\overset{\varepsilon}{\Phi}{}_{14}^{lm};$$

hence, one further infers that:

(†) Translators note: The "pairs" in question are 12 and 34, apparently.

$$(m_0)^2 \overset{\varepsilon}{\Phi}{}^{lm}_{14} = \frac{h^2}{4\pi^2} l(l+1) \overset{\varepsilon}{\Phi}{}^{lm}_{14} .$$

One will encounter the same calculations as above in the case of a magnetic spherical wave.

Formulas (44) and (45) express the following fundamental *theorem: A photon that is represented by a wave* $\Phi^{lm}_{\alpha\beta}$ *will transport a moment of impulse whose projection onto the z-axis will be equal to* – *mh* / 2π, *and whose length-squared will be equal to l* (l + 1) h^2 / $4\pi^2$.

We can deduce some theorems about the flux and mean values of the moment of impulse from these general results.

First, consider a diverging spherical wave [1]. Multiply both sides of (44) on the left by $c\,\Phi^{\times}_{\alpha\beta}$ $(\mathcal{A}_4 + \mathcal{B}_4) / 2$, sum over the indices α and β, and integrate over a closed surface that surrounds the origin and tends to infinity. One will get:

$$c\oint_S \Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} m_z \Phi \cdot dS = c\oint_S \Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} \Phi \cdot dS \cdot (-m)\frac{h}{2\pi} . \tag{56}$$

(S is the closed surface around the origin that tends to infinity.) The integral:

$$c\oint_S \Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} \Phi \cdot dS$$

represents the number of photons that cross the surface S per unit time [**20**]. When multiplied by $h\nu$, where ν is the frequency of the wave considered, it will be equal to the energy flux Σ that crosses S per unit time. On the other hand:

$$c\oint_S \Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} m_z \Phi \cdot dS = c\oint_S \Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} m_z^0 \Phi \cdot dS + c\frac{h}{2\pi}\oint_S \Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} \frac{(s_z^a + s_z^b)}{2} \Phi \cdot dS . \tag{57}$$

The first integral on the right-hand side of (57) is the flux of orbital moment that crosses S per unit time. Finally:

$$\frac{h}{2\pi}\Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} \cdot \frac{s_z^a + s_z^b}{2} \Phi = \frac{h}{2\pi}\Phi^* \frac{\mathcal{B}_4\, s_z^a + \mathcal{A}_4\, s_z^b}{2} \Phi + \frac{h}{2\pi}\Phi^* \frac{\mathcal{A}_4 - \mathcal{B}_4}{2} \cdot \frac{s_z^a - s_z^b}{2} \Phi . \tag{58}$$

The first term on the right-hand side of (58) is the mean-value density of spin, conforming to the definition [V, (20)] of mean-value densities. Upon using the inversion formulas [V, (28)], the second term is written explicitly as:

([1]) For ease of notation, in what follows, we shall suppress the indices l and m that the symbols $\Phi_{\alpha\beta}$ are affected with.

$$\frac{h}{2\pi}\Phi^*\frac{\mathcal{A}_4-\mathcal{B}_4}{2}\cdot\frac{s_z^a-s_z^b}{2}\Phi = \frac{1}{4c}(-V^*H_z - c\,\sigma_z^* I_{(1)}) + \text{conj.} \tag{59}$$

(59) is zero for a wave whose scalar potential is zero, since $I_{(1)} = 0$ for an un-annihilated photon. We place ourselves in the case where:

$$V = 0. \tag{60}$$

The expression:

$$c\oint_S \Phi^*\frac{\mathcal{A}_4+\mathcal{B}_4}{2}m_z\Phi\cdot dS = c\oint_S \Phi^*\frac{\mathcal{A}_4+\mathcal{B}_4}{2}m_z^0\Phi\cdot dS + c\frac{h}{2\pi}\oint_S \Phi^*\frac{\mathcal{B}_4 s_z^a+\mathcal{A}_4 s_z^b}{2}\Phi\cdot dS$$

will then represent the flux $\mathcal{M}_z$ of the total moment of impulse along the z-axis that crosses S per unit time, and the relation (56) can be written:

$$\frac{\mathcal{M}_z}{\Sigma} = \frac{-m}{2\pi\nu}. \tag{61.a}$$

On the other hand, upon using (50), (52), and (53), one will find that:

$$\mathcal{M}_x = \mathcal{M}_y = 0. \tag{61.b}$$

In the case of an electric spherical wave, the annulling of the scalar potential will fix one of the two arbitrary constants. The remaining constant is then a multiplicative constant that will determine the intensity of the wave; it will be fixed by a normalization condition. In the next paragraph, we shall see that the real part of the electromagnetic field thus-specified will be that of a classical electric l-pole wave, up to terms of order $\mu_0 c^2 / W$. For an electric dipole wave ($l = 1$, $m = -1$, 0, or $+1$), (61) is nothing but the well-known Abraham-Sommerfeld formula ([1]).

In the case of a magnetic spherical wave, one can fix one of the two arbitrary constants in such a manner as to obtain the electromagnetic fields of a classical magnetic multipolar wave (see § **4**). Formulas (61) then generalize the Abraham-Sommerfeld formula to the case of electric and magnetic multipole waves.

We remark, in passing, that in order to arrive at (61.a, b), the condition (60) can be replaced with the less-restrictive hypothesis that:

$$\oint_S (V^*\mathcal{H}_z + \text{conj.})\,dS = 0. \tag{62}$$

Now consider the case of a stationary spherical wave. Multiply both sides of (44) on the left by $\Phi^\times(\mathcal{A}_4 + \mathcal{B}_4) / 2$, and sum over the indices α and β. That will give:

([1]) In [**20**], we have obtained formulas (61.a, b) in the cases of electric dipole and quadrupole waves by direct calculation.

$$\oint_{\mathcal{V}} \Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} m_z \, \Phi \cdot dx\,dy\,dz = \oint_{\mathcal{V}} \Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} \Phi \cdot dx\,dy\,dz \cdot (-m) \frac{h}{2\pi}, \tag{63}$$

in which the integration extends over the domain $\mathcal{V}$ of the entire sphere in which one finds that stationary. The integral:

$$N \equiv \oint_{\mathcal{V}} \Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} \Phi \cdot dx\,dy\,dz$$

expresses the total number of photons that are situated in the sphere; the product:

$$U = N\,h\nu$$

will then be the total energy in the wave Φ. When one takes (60) into account, the left-hand side of (63) will become the value M_z of the projection onto the z-axis of the total (orbital and spin) moment of impulse of the wave Φ. The relation (63) will then be written:

$$\frac{M_z}{U} = \frac{-m}{2\pi\nu}. \tag{64}$$

On the other hand, one will find that:

$$M_x = M_y = 0. \tag{65}$$

W. Heitler obtained that result (2) in the classical Maxwell theory [**22**] for a stationary electric or magnetic multipolar wave (see, end of § **4**).

Some analogous theorems can be written down concerning the means value of the square of the length of the total moment that is transported by a "multipolar photon." The operator m^2 does not have the canonical type (20), since it includes products ($s_a\ s_b$). However, by virtue of the relations (45), m^2 is equivalent to the canonical operator $l\,(l + 1)\,h^2 / 4\pi^2$ in the case of spherical waves.

4. On the complete determination of spherical waves. Multipolar waves. – The spherical waves that were defined in paragraph **2** still include two arbitrary constants whose role we shall now study more closely. We say that a spherical wave is completely determined if the expressions for $\Phi_{\alpha\beta}^{lm}$ are given up to a multiplicative constant. They will always be present, due to the fact that the equations of the photon are homogeneous in the $\Phi_{\alpha\beta}$ and their derivatives: If $\Phi_{\alpha\beta}$ is a solution then $C\,\Phi_{\alpha\beta}$ will also be one (C = constant). That constant will be fixed by a normalization condition. For example, for a stationary spherical wave, one can write:

$$\oint_{\mathcal{V}} \Phi^* \frac{\mathcal{A}_4 + \mathcal{B}_4}{2} \Phi \cdot dx\,dy\,dz = 1.$$

For a diverging or converging wave, for example, one can write that the flux of photons that crosses a closed surface S that surrounds the origin and tends to infinity is equal to one photon per unit time. However, we are not interested in the normalization of the wave here.

We shall show that, up to terms of order $\mu_0 c^2 / W$ (in which, $W = h\nu$), the electric and magnetic fields of an electric and magnetic spherical wave of frequency ν will depend upon the same single constant.

First, consider the case of an *electric spherical wave*. Introduce the notations:

$$G_{+3}^l = a\,g, \qquad G_{-3}^l = b'\,g, \qquad G_{-4}^l = b''\,g,$$

in which a, b', b'' are three "finite" constants; i.e., they do not tend to zero when one makes $\mu_0 \to 0$. By virtue of (23), b' and b'' are coupled by the relation:

$$(l - m)\,b' = -(l + m)\,b''\,.$$

Set:

$$f_+ = -\frac{h}{2\pi(W - \mu_0 c^2)}\left(\dot{g} - \frac{l}{r}g\right), \qquad f_- = -\frac{h}{2\pi(W + \mu_0 c^2)}\left(\dot{g} - \frac{l+1}{r}g\right),$$

$$\alpha = \frac{2\mu_0 c^2}{(2l+1)(W - \mu_0 c^2)}, \qquad C = \frac{b'' - b' + 2la}{2l+1}\,.$$

One will then have:

$$i\,F_{+3}^l - G_{-1}^{l+1} = i\,C\,f_+ + i\alpha\,(b'' - b' - a)\,f_+\,,$$

$$i\,F_{-3}^l - G_{+1}^{l-1} = -i\,\frac{l+1}{l}(l + m)\,C\,f_- + i\alpha\,[b' - 2a\,(l + 1)(l + m)]\,f_-\,,$$

$$i\,F_{-4}^l - G_{+2}^{l-1} = i\,\frac{l+1}{l}(l - m)\,C\,f_- + i\alpha\,[b'' + 2a\,(l + 1)(l - m)]\,f_-\,,$$

$$G_{-4}^l + (l - m)G_{+3}^l = \frac{2l+1}{2l}(l - m)\,C\,g\,,$$

$$G_{-3}^l - (l + m)G_{+3}^l = -\frac{2l+1}{2l}(l + m)\,C\,g\,,$$

$$-\,G_{-4}^l - G_{-3}^l + 2mG_{+3}^l = \frac{2l+1}{l}\,m\,C\,g\,.$$

Hence, up to terms of order $\mu_0 c^2 / W$ (or α):

$$\left.\begin{aligned} H_x + iH_y &\simeq C\left[f_+ Y_{l+1}^{m+1} + \frac{l+1}{l}(l-m)(l-m-1)\, f_- Y_{l-1}^{m+1}\right], \\ H_x - iH_y &\simeq C\left[-f_+ Y_{l+1}^{m-1} - \frac{l+1}{l}(l+m)(l+m-1)\, f_- Y_{l-1}^{m-1}\right], \\ H_z &\simeq C\left[f_+ Y_{l+1}^{m} - \frac{l+1}{l}(l+m)(l-m)\, f_- Y_{l-1}^{m}\right], \end{aligned}\right\} \qquad (66.a)$$

$$\left.\begin{aligned} \mathcal{H}_x + i\mathcal{H}_y &\simeq -C\frac{2l+1}{l}(l-m)\, g\, Y_l^{m+1}, \\ \mathcal{H}_x - i\mathcal{H}_y &\simeq -C\frac{2l+1}{l}(l+m)\, g\, Y_l^{m-1}, \\ \mathcal{H}_z &\simeq C\frac{2l+1}{l}\, m\, g\, Y_l^{m} \end{aligned}\right\} \qquad (66.b)$$

It results from these formulas that in order to have "finite" electromagnetic" fields (i.e., ones with terms other than ones that are proportional to α), one must suppose that $C \neq 0$. In addition, any relation that establishes a link between the two arbitrary constants in order to completely determine the wave $\Phi_{\alpha\beta}^{lm}$ will change only the terms in the expressions for the electromagnetic fields that are proportional to α; the right-hand sides of (66) will remain the same. However, the potentials $\mathcal{V}$, V can vary by finite quantities. In order to see that, take an example. First set:

$$-b'' + b' + 2(l+1)\, a = 0,$$

which expresses the idea that the scalar potential is zero. The radial function $(iF_{+3}^{l} + G_{-1}^{l+1})\, h / 2\pi\mu_0 c$ that appears in the vector potential will then have a "finite" coefficient; indeed, it is equal to:

$$\frac{hc}{\pi i\,(W - \mu_0 c^2)}\, a\, f_+ .$$

Moreover, one will see that with the condition (67), one will have:

$$\mathbf{H} = -\frac{2\pi i W}{hc}\, \mathcal{V}.$$

Now set:

$$a + \frac{W + \mu_0 c^2}{W - \mu_0 c^2}\,\frac{b' - b'' + a}{2l+1} = 0,$$

which expresses the idea that $iF_{+3}^{l} + G_{-1}^{l+1}$ is identically zero. In that case, the scalar potential will no longer be zero; it will be equal to:

$$\frac{hc}{\pi(W+\mu_0c^2)}(2l + 1)\, a\, g.$$

However, the fields (66) are the same in both cases. A certain indeterminacy in the potentials then persists in the theory of the photon that corresponds to the well-known indeterminacy in Maxwell's theory. That indeterminacy will be removed when the terms in the fields that are proportional to $\mu_0\, c^2 / W$ lead to measurable consequences. We have seen that in order to obtain the theorems (69) and (64), we had to impose the condition $V = 0$. That condition determined an electric spherical wave $\Phi_{\alpha\beta}^{lm}$ completely, but it is not also certain that it is essential. Indeed, the theorem on the proper values (44), which is more fundamental than (61) and (64), does not require the annulment of the scalar potential. Rather, it seems that it is only by a study of the phenomena of the production of electromagnetic waves and the interaction between photons and electrons that these problems can be solved in a satisfactory manner.

The case of a magnetic spherical wave is treated in an analogous fashion. Thanks to the correspondence that exists between the expressions for the electromagnetic fields of electric and magnetic spherical waves, it is no longer necessary to redo the calculations. Here, one necessarily has $V = 0$. Therefore, not only the fields, but also the vector potential, will depend upon a single constant C, up to terms of order $\mu_0\, c^2 / W$.

Multipole waves. – The spherical waves for which the constant C is non-zero are referred to in particular by the name of *multipole waves*. That term is justified by the fact that the real parts of the fields thus-obtained will be the multipole fields of Maxwell's theory, up to terms of order $\mu_0\, c^2 / W$.

We point out that the expressions that are obtained here from multipole waves differ from W. Heitler's expressions not only by the consideration of the wave functions $\Phi_{\alpha\beta}$ and non-Maxwellian quantities and the proper mass μ_0 of the photon, but also by the use of the radial functions F and G, which seem to simplify the formulas. Finally, unlike Heitler, we do not annul the scalar potential.

Remark. – The case in which $C = 0$ merits attention. For an electric spherical wave, the hypothesis $C = 0$ defines a zero magnetic field, an electric field that is proportional to $\mu_0\, c^2 / W$, a vector potential, and a scalar potential that is proportional to $h / \mu_0\, c$. The products of a component of the field with a component of the potential will then be "finite." It will result, in particular, that the density ρ will be finite (i.e., non-zero as $\mu_0 \to 0$). Inn other words, the fields of order $\mu_0\, c^2 / W$ correspond to a "finite" current-density vector. The states of the photon that are characterized by $C = 0$ must then be in the same class as the non-Maxwellian states whose existence *in vacuo* is doubtful. Note that the case in which $l = 0$, which was not considered up to now, has that type, because the two constants b' and b'' will be zero then.

CHAPTER VII

STUDY OF THE INTERACTION BETWEEN AN ELECTRON AND A PHOTON

1. Wave functions of the electron and the photon. – The coordinates of the electron and the photon with respect to an inertial trihedron (called fixed) are $X^1 \equiv X$, $Y^1 \equiv Y$, $Z^1 \equiv Z$ and $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$, respectively. The clock that is attached to that inertial trihedron will display the time τ.

The Dirac Hamiltonian operator, which was defined in § **7** of Chapter IV [IV, (65)], is now written:

$$\mathrm{H}^{\mathrm{d}} \equiv \sum_{j=1}^{3} c\,\alpha_j \left(\frac{h}{2\pi i}\frac{\partial}{\partial X^j} + \frac{e}{c}\mathcal{V}_j \right) + e\,V + \alpha_4\, m_0\, c^2. \tag{1}$$

Recall that V and $\mathcal{V}$ are the electric and magnetic potentials, resp., of the *non-*luminous Maxwellian field in which the electron is found. We shall consider only the case in which the non-luminous field is independent of time τ ; $\mathcal{V}_j$ and V will then be functions of X, Y, Z.

As in [IV, (66)], the complete system of the functions $f^n\,(X, Y, Z)$ will be such that:

$$\mathrm{H}^{\mathrm{d}} f_\gamma^n = E^n f_\gamma^n \qquad \begin{matrix} n = 1, 2, \\ \gamma = 1, \dots, 4, \end{matrix} \tag{2}$$

with

$$\left.\begin{aligned} \iiint_{\mathcal{E}} \left(\sum_\gamma (f_\gamma^n)^* f_\gamma^n \right) dX\,dY\,dX &= 1, \\ \iiint_{\mathcal{E}} \left(\sum_\gamma (f_\gamma^n)^* f_\gamma^{n'} \right) dX\,dY\,dX &= 0, \quad \text{for } n' \neq n. \end{aligned}\right\} \tag{3}$$

The function:

$$\psi^n_{(X,Y,Z,\tau)} = f^n_{(X,Y,Z)}\, e^{\frac{2\pi i}{h}E^n \tau} \tag{4}$$

will then represent the state of an electron of energy E^n. In a general fashion, the state of the electron that is placed in the field $\mathcal{V}$, V will be defined by the sum or series:

$$\psi = \sum_n a_n \psi^n , \tag{5}$$

in which the a_n are constants. We normalize them, so:

$$\sum_n |\,a_n\,|^2 = 1. \tag{6}$$

For ease of notation, we likewise write the complete system of functions $g^p_{\alpha\beta}(x, y, z)$ of the photon in the form of a denumerable sequence ([1]). By definition, the $g^p_{\alpha\beta}$ satisfy:

$$\left.\begin{aligned} &(\mathrm{H}^{(a)}+\mathrm{H}^{(b)})g^p_{\alpha\beta} = E^p g^p_{\alpha\beta}, \qquad && p = 1, 2, \ldots \\ & && \alpha, \beta = 1, \ldots, 4 \\ &(\mathrm{H}^{(a)}-\mathrm{H}^{(b)})g^p_{\alpha\beta} = 0. && \end{aligned}\right\} \tag{7}$$

Those functions are normalized and orthogonal:

$$\left.\begin{aligned} &\iiint_{\mathcal{E}}\left(\sum_\alpha\sum_\beta (g^p_{\alpha\beta})^* \frac{\mathcal{A}_4+\mathcal{B}_4}{2} g^p_{\alpha\beta}\right) dx\,dy\,dz = 1, \\ &\iiint_{\mathcal{E}}\left(\sum_\alpha\sum_\beta (g^p_{\alpha\beta})^* \frac{\mathcal{A}_4+\mathcal{B}_4}{2} g^{p'}_{\alpha\beta}\right) dx\,dy\,dz = 0, \quad \text{for } p' \neq p; \end{aligned}\right\} \tag{8}$$

the integration extends over the entire space $\mathcal{E}$ of the variables x, y, z.

The functions:

$$\Phi^p_{\alpha\beta}(x, y, z, \tau) \equiv g^p_{\alpha\beta}(x, y, z)\; e^{\frac{2\pi i}{h}E^p\tau} \tag{9}$$

then represent the state of an un-annihilated photon of energy E^p.

The annihilation solution is [V, (8′)]:

$$\Phi^{(0)}_{\alpha\beta} = \lambda\,(\alpha_4)_{\alpha\beta}\,. \tag{9′}$$

As we explained already in [V, (8′)], the constant λ is introduced for dimensional reasons, but it plays no essential role. One has [V, (6)]:

$$(\mathrm{H}^{(a)} + \mathrm{H}^{(b)})\,\Phi^{(0)}_{\alpha\beta} = 0. \tag{10}$$

In a general fashion, the state of the photon will be defined by the sum or series:

$$\Phi_{\alpha\beta} = \sum_p b_p\,\Phi^p_{\alpha\beta} + b_0\,\Phi^{(0)}_{\alpha\beta}, \qquad p = 1, 2, \ldots, \tag{11}$$

in which the b_p, b_0 are *constants.* We also write:

$$\Phi_{\alpha\beta} = \sum_\mu b_\mu\,\Phi^\mu_{\alpha\beta}, \qquad \mu = 1, 2, \ldots \tag{12}$$

([1]) To that end, one must be able to suppose that the photon is found in a finite volume; for example, in a cube of volume L^3. That is a hypothesis that one frequently makes. (See, for example, [**25**], pp. 41.)

We normalize them; hence:

$$\sum_{\mu} | b_\mu |^2 = 1. \tag{13}$$

Having done that, the state of an electron of energy E^n and a photon of energy E^p, when envisioned simultaneously, but not taking the interaction into account, will be represented by the products:

$$\Psi^{pn}_{\alpha\beta\gamma}(x, y, z; X, Y, Z; \tau) = \Phi^{p}_{\alpha\beta}(x, y, z; \tau)\ \psi^{n}_{\gamma}(X, Y, Z; \tau)\,. \tag{14}$$

When the photon is annihilated:

$$\Psi^{pn}_{\alpha\beta\gamma}(X, Y, Z; \tau) = \Phi^{(0)}_{\alpha\beta}\ \psi^{n}_{\gamma}(X, Y, Z; \tau)\,. \tag{15}$$

In a general fashion, when one neglects the interaction, the state of the electron-photon system will then be defined by:

$$\Psi_{\alpha\beta\gamma} = \sum_p \sum_n c_{pn} \Phi^{p}_{\alpha\beta} \psi^{n}_{\gamma} + \sum_n c_{0n} \Phi^{(0)}_{\alpha\beta} \psi^{n}_{\gamma}, \qquad p, n = 1, 2, \ldots, \tag{16}$$

or

$$\Psi_{\alpha\beta\gamma} = \sum_\mu \sum_n c_{\mu n} \Phi^{\mu}_{\alpha\beta} \psi^{n}_{\gamma}, \qquad \mu = 1, 2, \ldots, \tag{16$'$}$$

in which the $c_{\mu n}$ are *constants.* More especially, one can suppose that:

$$\Psi_{\alpha\beta\gamma} = \left(\sum_\mu b_\mu \Phi^{\mu}_{\alpha\beta}\right)\left(\sum_n a_n \psi^{n}_{\gamma}\right); \tag{17}$$

hence:

$$c_{\mu n} = b_\mu\, a_n \tag{18}$$

and

$$| c_{\mu n} |^2 = | b_\mu |^2\, | a_n |^2. \tag{18$'$}$$

By virtue of (6), (13), and (18), one will have:

$$\sum_\mu \sum_n | c_n |^2 = 1. \tag{19}$$

In the convenient language of the calculation of probabilities, (18′) expresses the idea that the probability of finding the electron-photon in the state (n, μ) is equal to the product of the probabilities of finding the electron in the state (n) and the photon in the state (μ). It is natural to assume that this property must be realized, which will lead one to accept that the $\Psi_{\alpha\beta\gamma}$ must have the form (17). However, (18) will not be used in what follows.

2. Transitions caused by interactions. – In Chapter IV, § **7**, we saw that the study of the interaction between light and electrons consists essentially of the study of the electron transitions that are caused by the light wave. That problem is now transformed into the *study of the interaction between a photon and an electron*. This time, the perturbation that is due to the interaction will have the effect of stimulating *transitions in the electron-photon system* (16). The general form of the argument will then remain the same as in IV, § **7**; see remark *b*. *The* $c_{\mu n}$ *become functions of time* τ that must satisfy the variational equation:

$$\delta\int\frac{1}{2}\left\{\sum_{\mu}\sum_{n}\sum_{\mu'}\sum_{n'}c_{\mu n}^{*}\left(I_{\mu n,\mu' n'}-\delta_{\mu n,\mu' n'}\frac{h}{2\pi i}\frac{d}{d\tau}\right)c_{\mu' n'}+\text{conj.}\right\}d\tau=0, \tag{20}$$

as in IV, (86), in which $\delta_{\mu n,\,\mu' n'}$ is equal to unity for $\mu = \mu'$ and $n = n'$, and it is zero in all other cases.

In (20), I is the interaction energy operator, and $I_{\mu n,\,\mu' n'}$ are the elements of the matrix of that operator in the system of proper functions (14).

It remains for us to determine the operator I. We suppose that this operator has the form:

$$I = I^{(a)} + I^{(b)}, \tag{21}$$

like all of the main operators of the theory of the photon.

In Dirac's theory, where the light field is a Maxwellian electromagnetic field, we have seen [IV, (72)] that the interaction is represented by:

$$e\left(\sum_{j=1}^{3}\alpha_j\,\mathcal{V}^{(l)}+V^{(l)}\right), \tag{22}$$

in which $\mathcal{V}^{(l)}$ and $V^{(l)}$ are the vector and scalar potentials, resp., of the Maxwell field considered.

The interaction operator of the theory that is being presented is obtained by replacing the electromagnetic potentials in (22) with operators. Those operators are not the matrices:

$$-K\,\frac{\mathcal{A}_4+\mathcal{B}_4}{2}, \qquad K\cdot\frac{1+1}{2} \tag{23}$$

that L. de Broglie associated with the potentials; we shall see the reason for that later. We shall show that one must take [1]:

[1] $K' = K / \lambda$; see formula (9′).

$$\left.\begin{aligned} I^{(a)} &\equiv e\left[\sum_{j=1}^{3}\alpha_j\cdot\frac{K'}{2}\frac{\mathcal{A}_j}{2}-\frac{K'}{2}\frac{1}{2}\right]\delta(x-X)\,\delta(y-Y)\,\delta(z-Z), && (a)\\ I^{(b)} &\equiv e\left[\sum_{j=1}^{3}\alpha_j\left(-\frac{K'}{2}\frac{\mathcal{B}_j}{2}\right)+\frac{K'}{2}\frac{1}{2}\right]\delta(x-X)\,\delta(y-Y)\,\delta(z-Z). && (b)\end{aligned}\right\} \tag{24}$$

The product $\delta(x-X)\ \delta(y-Y)\ \delta(z-Z)$ of three Dirac functions expresses the fact that the interaction will take place only when the coordinates X, Y, Z of the electron and those x, y, z of the photon are identical at the same instant τ. The matrices (23) lead to an operator $I^{(b)}$ of the opposite sign to (24.*b*). Recall that according to L. de Broglie, the photon is composed of the fusion of a Dirac corpuscle and its corresponding anti-corpuscle; $I^{(a)}$ relates to the Dirac corpuscle, and $I^{(b)}$ relates to the anti-corpuscle. One can make the change of the sign that is introduced when one passes from (24.*a*) to (24.*b*) more plausible by some considerations that are based upon those remarks. However, the necessity of the change of sign is seen clearly in the expressions for the elements of the matrix $I_{\mu n,\,\mu' n'}$.

Conforming to the general definition [V, (20)] of the matrix elements of the theory of the photon, one has:

$$I_{pn,\,p'n'} = \iiint_{\mathcal{E}} dX\ dY\ dZ \iiint_{\mathcal{E}} dx\,dy\,dz\cdot(\psi^n\Phi^p)^*\,\frac{\mathcal{B}_4 I^{(a)}+\mathcal{A}_4 I^{(b)}}{2}\,\Phi^{p'}\psi^{n'}, \tag{25}$$

$$I_{0n,\,p'n'} = \iiint_{\mathcal{E}} dX\ dY\ dZ \iiint_{\mathcal{E}} dx\,dy\,dz\cdot(\psi^n\Phi^{(0)})^*\,\frac{\mathcal{B}_4 I^{(a)}+\mathcal{A}_4 I^{(b)}}{2}\,\Phi^{p'}\psi^{n'}, \tag{26}$$

$$I_{pn,\,0n'} = \iiint_{\mathcal{E}} dX\ dY\ dZ \iiint_{\mathcal{E}} dx\,dy\,dz\cdot(\psi^n\Phi^p)^*\,\frac{\mathcal{B}_4 I^{(a)}+\mathcal{A}_4 I^{(b)}}{2}\,\Phi^{(0)}\psi^{n'}. \tag{27}$$

As we are currently doing, we have omitted the indices α, β, γ and the summation signs over those indices from formulas (25), (26), and (27).

In addition, one has:

$$I_{0n,\,0n'} = 0, \tag{28}$$

with one or the other of the two definitions [V, (16)] or [V, (20)].

By virtue of the definitions (24) of $I^{(a)}$ and $I^{(b)}$, and upon using the definition [V, (23)] of the electromagnetic potentials in the theory of the photon, one will get:

$$I_{0n,\,p'n'} = \frac{e}{2}\iiint_{\mathcal{E}} dX\ dY\ dZ\left[\sum_j(\psi^n)^*\alpha_j\psi^{n'}\cdot\mathcal{V}_j^{p'}+(\psi^n)^*\psi^{n'}\cdot V^{p'}\right], \tag{26′}$$

$$I_{pn,\,0n'} = \frac{e}{2}\iiint_{\mathcal{E}} dX\ dY\ dZ\left[\sum_j(\mathcal{V}_j^{p})^*\cdot(\psi^n)^*\alpha_j\psi^{n'}+(V^{p'})^*(\psi^n)^*\psi^{n'}\right]. \tag{26′}$$

One now sees why we have taken the expressions (24) for $I^{(a)}$ and $I^{(b)}$. Thanks to (24), one will recover some matrix elements in (26′), (27′) that are comparable to the matrix elements $I_{n,\,n'}$ of the classical Dirac theory [IV, (77)]. Nevertheless, the elements of Chapter IV, § 7 are not identical to (26′) and (27′). They differ in two ways: In [IV, (77)], the potentials were *real*, whereas here they are complex; in addition, one will note the presence of the factor 1/2 in (26′) and (27′), which did not figure in [IV, (77)]. The meaning of those two differences is shown by the example in which the real wave is a monochromatic, plane, Maxwellian wave that is polarized rectilinearly. One can write:

$$\mathcal{V}_3 = \mathcal{V}_z = a \cos \omega\left(t - \frac{x}{c}\right), \tag{29}$$

or, what amounts to the same thing:

$$\mathcal{V}_z = \frac{a}{2}\left[e^{i\omega(t-x/c)} + e^{-i\omega(t-x/c)}\right]. \tag{29′}$$

Now, the action of the wave on the electron decomposes into precisely two parts, as is shown by (29′). In the phenomenon of absorption, only the term:

$$\frac{a}{2}e^{i\omega(t-x/c)}$$

intervenes.

In the theory of the photon, the complex potential that corresponds to (29) is:

$$\frac{a}{2}e^{i\omega(t-x/c)}\,.$$

In order to see that it is $I_{0n,\,p'n'}$ that plays the role here of the $I_{n,n'}$ in the classical theory [IV, (77)], it will suffice to compare equations (37) (see below) with equations [IV, (78′)]; one will then understand the necessity of the factor 1/2 in (26′).

The matrices $I_{\mu\,n,\,p'n'}$ are defined completely by (25), (26), (27), and (24). Now take the *variational principle* (20); i.e.:

$$\delta\int \bar{\mathcal{L}}\, d\tau = 0, \tag{30}$$

with:

$$\bar{\mathcal{L}} \equiv \frac{1}{2}\left[\sum_{\mu}\sum_{n}\sum_{p'}\sum_{n'} c^*_{\mu n}\left(I_{\mu n,\mu' n'} - \delta_{\mu n,\mu' n'}\frac{h}{2\pi i}\frac{d}{d\tau}\right)c_{\mu' n'} + \text{conj.}\right]. \tag{31}$$

The *Lagrange* equations of (30) are:

$$\frac{\partial\bar{\mathcal{L}}}{\partial c^*_{\mu n}} - \frac{d}{d\tau}\frac{\partial\bar{\mathcal{L}}}{\partial \dot{c}^*_{\mu n}} = 0, \tag{32}$$

with

$$\dot{c}^*_{\mu n} = \frac{dc^*_{\mu n}}{d\tau},$$

and the conjugate equations:

$$\frac{\partial \bar{\mathcal{L}}}{\partial c^*_{\mu n}} - \frac{d}{d\tau}\frac{\partial \bar{\mathcal{L}}}{\partial \dot{c}^*_{\mu n}} = 0, \tag{33}$$

with

$$\dot{c}_{\mu n} = \frac{dc_{\mu n}}{d\tau}.$$

Explicitly, equations (32) are written:

$$\boxed{\frac{h}{2\pi i}\dot{c}_{\mu n} = \sum_{\mu'}\sum_{n'} I_{\mu n,\mu' n'}\, c_{\mu' n'}.} \tag{34}$$

Equations (34) imply the *electron and photon transitions* that are stimulated by their interaction; they are represented by the operator I.

3. Conservation law. – By virtue of equations (34) and their conjugates, one has:

$$\frac{d}{d\tau}\left(\sum_{\mu}\sum_{n} | c_{\mu n} |^2\right) = 0, \tag{35}$$

The verification is immediate.

Thanks to (35), if the condition (19) is satisfied at an initial instant then it will be satisfied at any instant. That condition has a simple physical meaning: It means that the electron-photon system will certainly be found in one of the states ($\mu\, n$) at any instant.

4. Case of Maxwellian waves. – At this point, the most interesting case is the one in which the waves Φ^p are Maxwellian; i.e., all non-Maxwellian quantities that are associated with the Φ^p are zero. Upon referring to [V, (75)], one will then have:

$$I_{\mu n, p'n'} = 0. \tag{36}$$

All of the matrix elements that correspond to the passage of the photon from the state p' to an arbitrary (un-annihilated) state p are then zero. Equations (34) then become:

$$\left.\begin{aligned} &\frac{h}{2\pi i}\dot{c}_{pn} = \sum_{n'} I_{pn,0n'}\, c_{0n'}, && (a)\\ &\frac{h}{2\pi i}\dot{c}_{0n} = \sum_{p'}\sum_{n'} I_{0n,p'n'}\, c_{p'n'}. && (b) \end{aligned}\right\} \tag{37}$$

These equations account for the elementary phenomena that consist of the interaction of a Maxwellian photon and an electron. They are known, but they were obtained by methods that are sharply different from ours, in that they are more complicated and utilize the quantum theory of the electromagnetic field ([1]). In Chapter IV, § 7, we made some remarks on the subject of those methods.

([1]) For example, see [**24**], pp. 237, *et seq.*, and [**25**]. The case of a system of *n* photons (both annihilated and not) in interaction with an electron is treated by using the wave mechanics of a system of photons. One will recover the results of those other theories when one makes *n* tend to infinity.

INDEX OF CITED AUTHORS

1. TH. DE DONDER:

   a. *La Gravifique Einsteinienne*, Gauthier-Villars, Paris, 1921.

   b. “Introduction à la Gravifique Einsteinienne,” Mém. des Sciences math., Gauthier-Villars, Paris, 1925, fasc. VIII.

   c. “Théorie des Champs gravifiques,” *ibid.*, 1926, fasc. XIV.

   d. “Application de la Gravifique Einsteinienne,” *ibid.*, 1930, fasc. XLIII.

   e. “Application de la Gravifique à l’électrodynamique des corps en mouvement,” *ibid.*, 1932, fasc. LVIII.

2. – *Théorie mathématique de l’électricité*, Gauthier-Villars, Paris, 1925.

3. LEVI-CIVITA, “Some mathematical aspects of the new Mechanics,” Bull. Am. Math. Soc. **34** (1933), pp. 551.

4. TH DE DONDER, “Sur les extrémales décrites par les électrons et les particles électrisées,” Bull. Acad. roy. de Belg., Cl. d. Sc. **13** (1927), pp. 79.

5. J. VAN MIEGHEM, “Contribution à la théorie du principe des ondes enveloppes de Huygens,” Acad. roy. de Belg., Cl. d. Sc., Mémoire in 8^0, (2) **15** (1936).

   TH. DE DONDER:

5a. – *Volume jubilaire du professeur M. Brillouin*, 1935.

5b. – *Théorie invariantive du calcul des variations*, new edition, Gauthier-Villars, Paris, 1936.

6. – “Le principe de correspondence déduit de la Gravifique et de la Mécanique ondulatoire,” Bull. Acad. roy. de Belg., Cl. d. Sc. **13** (1927), 504-509. – See also
   O. KLEIN, Zeit. Phys. **41** (1927), Heft 6-7,
   L. ROSENFELD, Bull. Acad. roy. de Belg., Cl. d. Sc. **13** (1937).

7. A. EINSTEIN, *The Meaning of Relativity.*

   TH. DE DONDER:

7a. – C. R. Acad. Sci. Paris 11 and 25 June 1923 and 23 July 1923.

7b. – Bull. Acad. roy. de Belg., Cl. d. Sc., 7 April 1924.

8. L. DE BROGLIE, *L’électron magnétique*, Hermann, Paris, 1934.

9. W. PAULI, "Théorie quantique relativiste des particules obéissant à la statistique d'Einstein-Bose," Ann. Inst. H. Poincaré **6** (1936), fasc. II.

10. D. IWANENKO and SOKOLOW, "Bemerkungen zur zweiten Quantisierung der Dirac-Gleichung," Phys. Zeit. d. Sowjetunion **2** (1937), pp. 590.

11. C. G. DARWIN, Proc. Roy. Soc. London A **118** (1928), pp. 654.

12. TH. DE DONDER,

– "Le champ photonique," Bull. Acad. roy. de Belg., Cl. d. Sc., **14** (1928), pp. 307.

– "Le champ photonique et la généralisation relativiste de la Mécanique ondulatoire," C. R. Acad. Sci. Paris **188** (1929), pp. 150.

13. J. M. WHITTAKER, Proc. Roy. Soc. London A **121** (1928), pp. 543.

14. TH. DE DONDER, Bull. Acad. roy. de Belg., Cl. d. Sc., **15** (2 February 1929) and 4 January 1930, pp. 8.

15. – Ann. Inst. Henri Poincaré, v. I, fasc. 2, pp. 77.

16. L. DE BROGLIE:

– "Une nouvelle conception de la lumière," Act. Scient. et Indust. 181, XIII, 1934.

17. – "Nouvelles recherches sur la lumière," *ibid.*, 411, XX, 1936.

18a. – C. R. Acad. Sci. Paris, August 1936.

18b. – Ann. Soc. Scient., Brussels, (1) **57** (1937), pp. 105.

19. – "La quantification des champs dans la théorie du photon," C. R. Acad. Sci. Paris, 9 August 1937, pp. 25.

20. J. GÉHÉNIAU, "Contribution à la Théorie de la Lumière de L. de Broglie," Exposés de Physique Théorique, Hermann et Cie, Paris, 1939.

21. E. HENRIOT, "Les couples de radiation et les moments électro-magnétiques," Mémorial des Sciences physiques, 1936, fasc. XXX.

22. W. HEITLER, "On the radiation emitted by a multipole and its angular momentum," Proc. Camb. Phil. Soc. **32** (1936), pp. 112.

23. E. U. CONDON and G. H. SHORTLEY, *The theory of atomic spectra*, Cambridge, 1935.

24. P. A. M. DIRAC, *Quantum Mechanics*, 2nd ed., Oxford, 1935.

25. W. HEITLER, *Quantum Theory of Radiation*, Oxford, 1936.